

From Möbius inversion to renormalisation

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This paper traces a straight line from classical Möbius inversion to Hopf-algebraic perturbative renormalisation. This line, which is logical but not entirely historical, consists of just a few main abstraction steps, and some intermediate steps dwelled upon for mathematical pleasure. The paper is largely expository, but contains many new perspectives on well-known results. For example, the equivalence between the Bogoliubov recursion and the Atkinson formula is exhibited as a direct generalisation of the equivalence between the Weisner–Rota recursion and the Hall–Leroux formula for Möbius inversion.

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Introduction

The goal of the present contribution is to explain the connection between Möbius inversion and renormalisation theory, through a sequence of straightforward generalisations. The flavour of renormalisation concerned is the BPHZ renormalisation of perturbative quantum field theories, introduced by Bogoliubov, Parasiuk, Hepp and Zimmermann (1955–1969), and more specifically its Hopf-algebraic interpretation discovered by Kreimer [32] in 1998. Subsequent work of Connes, Kreimer [9, 10], Ebrahimi-Fard, Guo, Manchon and others [18, 17], distilled the BPHZ construction into a piece of abstract algebra, involving characters of a Hopf algebra with values in a Rota-Baxter algebra. It has important connections with disparate subjects in pure mathematics, such as multiple zeta values, numerical integration, and stochastic analysis. The construction itself can be viewed from various perspectives, such as that of Birkhoff decomposition and the Riemann–Hilbert problem [9, 10], the Baker–Campbell–Hausdorff formula and Lie theory [18], or the abstract viewpoint of filtered non-commutative Rota–Baxter algebras [17]. There are excellent surveys of these developments,

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such as Ebrahimi-Fard–Kreimer [19] (focusing on physical motivation), Manchon [36] (generous with mathematical preliminaries on coalgebras and Hopf algebras), and the longer survey of Figueroa and Gracia-Bondía [22] (particularly relevant in the present context for exploiting also the combinatorial viewpoint of incidence algebras).

The present expository paper derives the construction as a direct generalisation of classical Möbius inversion: after the abstraction steps from the classical Möbius function via incidence algebras to abstract Möbius inversion, the remaining step is just to add a Rota–Baxter operator to the formulae. This is very close in spirit to Kreimer’s original contribution [32], where the counter-term was staged as a twisted antipode, but the explicit interpretation in terms of Möbius inversion seems not to have been made before, and in any case deserves to be more widely known. The perspective is attractive for its simplicity, and leads to clean and elementary proofs (and slightly more general results—bialgebras rather than Hopf algebras). For ampler perspectives and deeper connections to various areas of mathematics, we refer to the bibliography and the pointers given along the way.

Before starting from scratch with Möbius inversion in classical number theory (§2), it is appropriate to begin in §1 by indicating more precisely where we are going, with a brief introduction to BPHZ renormalisation from an abstract viewpoint. After Möbius inversion for arithmetic functions in §2, we move to Möbius inversion in incidence algebras in §3; we deal with both posets and Möbius categories. In §4 we establish the abstract Möbius inversion principle, for general filtered coalgebras with the property that the zeroth piece is spanned by group-like elements. This is inspired by recent work on Möbius inversion in homotopical contexts. Finally in §6 we add a Rota–Baxter operator to the abstract Möbius inversion formulae. This yields directly the Bogoliubov recursion of renormalisation, and simultaneously the Atkinson formula.

1. Hopf-algebraic BPHZ renormalisation

Perturbative quantum field theory is concerned with expanding the scattering matrix into a sum over graphs. The Feynman rules assign to each graph of the theory an amplitude. Unfortunately, for many graphs with loops (non-zero first Betti number), the corresponding amplitude is given by a divergent integral. Renormalisation is the task of extracting meaningful finite values from these infinities.

In the (modern account of the) BPHZ approach, the first step consists in introducing a formal parameter, the *regularisation parameter* ε , in such

a way that the amplitudes no longer take values directly in the complex numbers but rather in the ring of Laurent series $\mathbb{C}[\varepsilon^{-1}, \varepsilon]$. The amplitudes are now well defined: the divergences are expressed by series with a pole at $\varepsilon = 0$. The next step is to subtract counter-terms for ‘divergent’ graphs. The *minimal subtraction scheme* aims simply to subtract the pole part, but the naive attempt—just subtracting the pole part for a given graph—turns out to be too brutal, destroying important physical features of the Feynman rules. The problem can be localised to the fact that a divergent graph may itself have divergent subgraphs, and these sub-divergences should be sorted out first, before attempting to determine the counter-term for the graph as a whole. In the end, the correct procedure, found by Bogoliubov and Parasiuk [5] and fine-tuned and proved valid by Hepp [29], is a rather intricate recursive over-counting/under-counting procedure, of a flavour not unfamiliar to combinatorists. The development culminated with Zimmermann [48] finding a closed formula for the counter-term, the famous *forest formula*, instead of a recursion.¹ One crucial property is that the renormalised Feynman rule remains a character, just like the unrenormalised Feynman rule, expressing the fundamental principle that the amplitude of two independent processes is the product of their amplitudes. The renormalised Feynman rule assigns to every graph a power series without pole part, and the desired finite amplitude can finally be obtained by setting ε to 0. This procedure, called BPHZ renormalisation, is described in many textbooks on quantum field theory and renormalisation (e.g. [8], [40]).

Kreimer’s seminal discovery [32] is that the combinatorics in this procedure is encoded in a Hopf algebra of graphs H . As a vector space, H is spanned by all 1PI graphs of the given quantum field theory. The multiplication in H is given by taking disjoint union of graphs. The comultiplication $\Delta : H \rightarrow H \otimes H$ is given on connected 1PI graphs Γ by

$$\Delta(\Gamma) = \sum_{\gamma \subset \Gamma} \gamma \otimes \Gamma/\gamma,$$

where the sum is over all (superficially divergent) 1PI subgraphs γ (possibly not connected), and the quotient graph Γ/γ is obtained by contracting each connected component of γ to a vertex (the residue of γ). Altogether, H is a Hopf algebra, graded by loop number. The regularised Feynman rules are

¹Important as it is, the forest formula is not dealt with in the present exposition, as it is not clear how it relates to *general* Möbius inversion; but see [22] and [37] for important insight in this direction for certain special classes of Hopf algebras.

characters $\phi : H \rightarrow A$ with values in $A = \mathbb{C}[\varepsilon^{-1}, \varepsilon]$. The *BPHZ counter-term* ϕ_- is given by the recursive formula (for $\deg \Gamma > 0$)

$$\phi_-(\Gamma) = -R[\phi(\Gamma) + \sum_{\substack{\gamma \subset \Gamma \\ \gamma \neq \emptyset, \gamma \neq \Gamma}} \phi_-(\gamma) \phi(\Gamma/\gamma)],$$

or more conceptually:

$$\phi_- = e - R[\phi_- * (\phi - e)],$$

where $*$ is convolution of linear maps $H \rightarrow A$ (see 4.2 below for the definition), where e is the neutral element for convolution, and $R : A \rightarrow A$ is the idempotent linear operator that to a Laurent series assigns its pole part. The recursion is well founded, thanks to the grading of H : the convolution refers to taking out subgraphs via Δ , and the arguments to ϕ_- in the convolution are graphs with strictly fewer loops than the input to ϕ_- on the left-hand side of the equation, since $(\phi - e)$ vanishes on graphs without loops.

The renormalised Feynman rule is finally given in terms of convolution² as

$$\phi_+ := \phi_- * \phi.$$

It takes values in $\text{Ker } R = \mathbb{C}[[\varepsilon]]$, so that it makes sense finally to set $\varepsilon = 0$ to obtain a finite amplitude for each graph. The crucial fact that ϕ_- and ϕ_+ are again characters turns out to be a consequence of a special property of the operator R , namely the equation

$$(1) \quad R(x \cdot y) + R(x) \cdot R(y) = R(R(x) \cdot y + x \cdot R(y)),$$

which is to say that R is a *Rota–Baxter operator*.³

²That ϕ_+ can be written as a convolution was realised by Connes and Kreimer [10] (thus exhibiting the renormalisation procedure as an instance of the general mathematical construction called Birkhoff decomposition). Previously ϕ_+ was computed via an auxiliary construction known as Bogoliubov’s preparation map.

³Kreimer himself did isolate conditions on R ensuring that ϕ_- and ϕ_+ are again characters, but it was Brouder who observed that these conditions can be formulated as a single “multiplicativity constraint”, namely (1) (see [33], footnote 4); Connes and Kreimer [10] referred to this multiplicativity constraint. Ebrahimi-Fard then pointed out that this constraint is the Rota–Baxter equation (the first published mention being [15]), and started to import results and methods from this mathematical theory.

The abstraction of these discoveries is the purely algebraic result that for any graded Hopf algebra H and for any commutative algebra A with a Rota–Baxter operator R like this, the same procedure works to transform a character $\phi : H \rightarrow A$ into another character ϕ_- such that the convolution $\phi_+ := \phi_- * \phi$ takes values in the kernel of R (the abstraction of the property of being pole free). This is the result we will arrive at in Section 6, from the standpoint of Möbius inversion.

It must be stressed that this neat little piece of algebra is only a minor aspect of perturbative renormalisation, as it does not account for the analytic (or number-theoretic) aspects of Feynman amplitudes, e.g. the computation of the individual integrals. The merit of the Hopf-algebraic approach is rather to separate out the combinatorics from the analysis, and explain it in a conceptual way. It is also worth remembering that assigning a renormalised amplitude to every graph is not the end of the story, because there are infinitely many graphs (their number even grows factorially in the number of loops), and in general the sum of all these finite amplitudes will still be a divergent series in the coupling constant. New techniques are being applied to tackle this problem, such as resurgence theory (see for example [13]). The present contribution deliberately ignores all these analytic aspects.

2. The classical Möbius function

2.1. Arithmetic functions and Dirichlet series. Write

$$\mathbb{N}^\times = \{1, 2, 3, \dots\}$$

for the set of positive natural numbers. An *arithmetic function* is just a function

$$f : \mathbb{N}^\times \rightarrow \mathbb{C}$$

(meant to encode some arithmetic feature of each number n). To each arithmetic function f one associates a *Dirichlet series*

$$F(s) = \sum_{n \geq 1} \frac{f(n)}{n^s},$$

thought of as a function defined on some open set of the complex plane. The study of arithmetic functions in terms of their associated Dirichlet series is a central topic in analytic number theory [1].

2.2. The zeta function. A fundamental example is the *zeta function*

$$\begin{aligned}\zeta : \mathbb{N}^\times &\longrightarrow \mathbb{C} \\ n &\longmapsto 1.\end{aligned}$$

The associated Dirichlet series is the *Riemann zeta function*

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}.$$

2.3. Classical Möbius inversion.⁴ The classical Möbius inversion principle for arithmetic functions says that

$$\begin{aligned}\text{if } f(n) &= \sum_{d|n} g(d) \\ \text{then } g(n) &= \sum_{d|n} f(d)\mu(n/d),\end{aligned}$$

where μ is the *Möbius function*⁵

$$(2) \quad \mu(n) = \begin{cases} 0 & \text{if } n \text{ contains a square factor} \\ (-1)^r & \text{if } n \text{ is the product of } r \text{ distinct primes.} \end{cases}$$

2.4. Example: Euler's totient function. *Euler's totient function* is by definition

$$\varphi(n) := \#\{1 \leq k \leq n \mid (k, n) = 1\}.$$

It is not difficult to see that we have the relation

$$n = \sum_{d|n} \varphi(d),$$

so by Möbius inversion we get a formula for φ :

$$\varphi(n) = \sum_{d|n} d \mu(n/d).$$

⁴This is due to Möbius [39], see Hardy and Wright [28], Thm. 266.

⁵According to Hardy and Wright [28] (notes to Ch. XVI), the Möbius function occurs implicitly in the work of Euler as early as 1748.

2.5. Dirichlet convolution. A conceptual account of the Möbius inversion principle is given in terms of *Dirichlet convolution* for arithmetic functions:

$$(f * g)(n) := \sum_{i \cdot j = n} f(i)g(j),$$

which corresponds precisely to (pointwise) product of Dirichlet series. The neutral element for this convolution product is the arithmetic function

$$\varepsilon(n) := \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{else.} \end{cases}$$

Now the Möbius inversion principle reads more conceptually

$$f = g * \zeta \quad \Rightarrow \quad g = f * \mu,$$

and the content is this:

Proposition 2.6. *The Möbius function is the convolution inverse of the zeta function.*

2.7. Example (continued). Let ι denote the arithmetic function $\iota(n) = n$. Its associated Dirichlet series is

$$\sum_{n \geq 1} \frac{n}{n^s} = \zeta(s - 1).$$

Restating the Möbius inversion formula for Euler's totient φ in terms of Dirichlet convolution yields

$$\iota = \varphi * \zeta \quad \Rightarrow \quad \varphi = \iota * \mu,$$

so that the Dirichlet series associated to φ is

$$\frac{\zeta(s - 1)}{\zeta(s)}.$$

3. Incidence algebras

In the 1930s, Möbius inversion was applied in group theory by Weisner [47] and independently by Hall [27].⁶ Both were motivated by the lattice of sub-

⁶Hall defined and computed *Eulerian functions* of groups using Möbius inversion in subgroup lattices. For cyclic groups, this recovers Euler's totient function.

groups of a finite group, but found it worth developing the theory more generally; Weisner for complete lattices, Hall for finite posets.

In the 1960s, Rota [41] systematised the theory extensively, in the setting of locally finite posets, and made Möbius inversion a central tool in enumerative combinatorics. The setting of posets is now widely considered the natural context for Möbius inversion (see for example Stanley’s book [45]). Cartier and Foata [7] developed the theory for monoids with the finite-decomposition property, and Leroux [35] unified these contexts in the general notion of Möbius category, reviewed below. More recently, Lawvere and Menni [34] and Gálvez, Kock, and Tonks [24, 25, 26] took Leroux’s ideas further into category theory and homotopy theory.⁷ These abstract developments were crucial for distilling out the perspectives of the present contribution.

We briefly recall the notions of incidence algebras and Möbius inversion for posets and Möbius categories. All proofs will be deferred to the abstract setting of Section 4. Throughout, \mathbb{k} denotes a ground field, ‘linear’ means \mathbb{k} -linear, and \otimes is short for $\otimes_{\mathbb{k}}$.

3.1. The incidence (co)algebra of a locally finite poset. A poset (P, \leq) is called *locally finite* if all its intervals $[x, y] := \{z \in P : x \leq z \leq y\}$ are finite. The free vector space C_P on the set of intervals becomes a coalgebra

⁷Lawvere and Menni [34] gave an ‘objective’ version of Leroux’s theory: this means working with the combinatorial objects themselves instead of the vector spaces they span. The classical theory is obtained by taking cardinality. One advantage of this approach—beyond making all proofs natively bijective—is that one can eliminate finiteness conditions, if just one refrains from taking cardinality: the constructions work the same with infinite sets, and at this level, Möbius inversion works for *any* category, not just Möbius categories. More recently, Gálvez, Kock, and Tonks [24, 25, 26] discovered that simplicial objects more general than categories admit incidence algebras and Möbius inversion, and passed to the homotopical context of simplicial ∞ -groupoids. Where categories express the general ability to compose, their notion of *decomposition space* expresses the general ability to decompose, in an appropriate manner so as to induce a coassociative incidence coalgebra, and an attendant Möbius inversion principle. There are plenty of examples in combinatorics of coalgebras and bialgebras which are the incidence coalgebra of a decomposition space but not of a category or a poset. An example relevant to the present context is the Connes–Kreimer Hopf algebra of rooted trees [9], which is the incidence bialgebra of a decomposition space but not directly of a category [24, Ex. 0.1]. (Various quotient constructions are possible, though, such as those given by Dür [14] and Kaufmann–Ward [30].)

$(C_P, \Delta, \varepsilon)$ with comultiplication $\Delta : C_P \rightarrow C_P \otimes C_P$ defined by

$$\Delta([x, y]) := \sum_{z \in [x, y]} [x, z] \otimes [z, y]$$

and counit $\varepsilon : C_P \rightarrow \mathbb{k}$ defined as

$$\varepsilon([x, y]) := \begin{cases} 1 & \text{if } x = y \\ 0 & \text{else.} \end{cases}$$

The *incidence algebra* of P is the convolution algebra of C_P (with values in the ground field); the multiplication is thus given by

$$(\alpha * \beta)([x, y]) := \sum_{z \in [x, y]} \alpha([x, z])\beta([z, y]),$$

and the unit is ε .

3.2. The zeta function. The *zeta function* is defined as

$$\begin{aligned} \zeta : C_P &\longrightarrow \mathbb{k} \\ [x, y] &\longmapsto 1. \end{aligned}$$

(Note that this function is constant on the set of intervals, but of course not constant on the vector space spanned by the intervals.)

Theorem 3.3 (Rota [41]⁸). *For any locally finite poset, the zeta function is convolution invertible; its inverse, called the Möbius function $\mu := \zeta^{-1}$, is given by the recursive formula*

$$\mu([x, y]) = \begin{cases} 1 & \text{if } x = y \\ - \sum_{\substack{z \in [x, y] \\ z \neq y}} \mu([x, z]) & \text{if } x < y. \end{cases}$$

This is a recursive definition by length of intervals, well founded because of the condition $z \neq y$.

⁸The result was essentially proved already by Weisner [47] (but only for complete lattices) and by Hall [27] (but only for finite posets).

Corollary 3.4. *We have*

$$f = g * \zeta \quad \Rightarrow \quad g = f * \mu.$$

In other words,

$$\begin{aligned} \text{if} \quad f([x, y]) &= \sum_{z \in [x, y]} g([x, z]) \\ \text{then} \quad g([x, y]) &= \sum_{z \in [x, y]} f([x, z]) \mu([z, y]). \end{aligned}$$

In fact, Rota proved more:

Theorem 3.5 (Rota [41]). $\phi : C_P \rightarrow \mathbb{k}$ is convolution invertible provided $\phi([x, x]) = 1$ for all $x \in P$; the convolution inverse ψ is determined by the recursive formula

$$\psi([x, y]) = \begin{cases} 1 & \text{if } x = y \\ - \sum_{\substack{z \in [x, y] \\ z \neq y}} \psi([x, z]) \phi([z, y]) & \text{if } x < y. \end{cases}$$

The recursion can be written more compactly as

$$\boxed{\psi = \varepsilon - \psi * (\phi - \varepsilon)}$$

Indeed, subtracting ε from ϕ inside the sum expresses the fact that we don't want the last summand ($z = y$), and adding the term ε outside the sum expresses the first case ($x = y$).

3.6. Möbius categories (Leroux). Leroux [35] introduced the common generalisation of locally finite posets and Cartier–Foata monoids: *Möbius categories*. Recall that posets and monoids are special cases of categories: a poset can be considered as a category whose objects are the elements of the poset, and in which there is an arrow from x to y if and only if $x \leq y$ in the poset. This means that arrows now play the role of intervals, and splitting intervals becomes factorisation of arrows. A monoid can be considered as a category with only one object, the arrows being then the monoid elements, composed by the monoid multiplication. The finiteness conditions for posets and monoids now generalise as follows. A category is *Möbius* if every arrow admits only finitely many non-trivial decompositions (of any length). For a

Möbius category \mathcal{C} , the set of arrows \mathcal{C}_1 forms a linear basis of its *incidence coalgebra*, where the comultiplication of an arrow is the set of all its (length-2) factorisations

$$\Delta(f) := \sum_{boa=f} a \otimes b,$$

immediately generalising the comultiplication of intervals of a poset. The counit $\varepsilon : \mathcal{C}_1 \rightarrow \mathbb{k}$ sends identity arrows to 1 and all other arrows to 0 (again exactly as the case of intervals in a poset). The *incidence algebra* is the convolution algebra of this coalgebra, i.e. the algebra of maps $\mathcal{C}_1 \rightarrow \mathbb{k}$ with multiplication given by convolution: $(\alpha * \beta)(f) := \sum_{boa=f} \alpha(a)\beta(b)$. Note that ε is the neutral element for convolution.

An example of a Möbius category which is neither a poset nor a monoid is given by the free category on a quiver. Its incidence algebra is then the path algebra of the quiver [12].

The *zeta function* is the function $\mathcal{C}_1 \rightarrow \mathbb{k}$ sending every arrow to 1.

Theorem 3.7 (Content–Lemay–Leroux [11]). *For \mathcal{C} a Möbius category, the zeta function is convolution invertible with inverse*

$$\mu = \Phi_{\text{even}} - \Phi_{\text{odd}}.$$

Here $\Phi_{\text{even}}(f)$ is the number of even-length chains of arrows composing to f (not allowing identity arrows), and similarly for Φ_{odd} .

This alternating-sum formula goes back to Hall [27], and was also exploited by Cartier and Foata [7].⁹ We shall give a slick proof of it in the abstract setting of the next section, where we shall also relate it to Rota’s recursive formula

$$(3) \quad \mu = \varepsilon - \mu * (\zeta - \varepsilon),$$

valid in any Möbius category.

3.8. Example. In the incidence algebra of the monoid $(\mathbb{N}, +, 0)$, the Möbius function is

$$(4) \quad \mu(n) = \begin{cases} 1 & \text{if } n = 0 \\ -1 & \text{if } n = 1 \\ 0 & \text{if } n > 1. \end{cases}$$

⁹It is important also because of its relation to Euler characteristic. For example, for a finite poset P with a minimal and a maximal element added, the alternating-sum formula for the Möbius function coincides with the usual formula for Euler characteristic of the order complex of P (see Stanley [45]).

This is easily proved by checking that this function satisfies the recursion of the general formula (3).

Hence the inversion principle says in this case

$$\begin{aligned} \text{if } f(n) &= \sum_{k \leq n} g(k) \\ \text{then } g(n) &= f(n) - f(n-1). \end{aligned}$$

In other words, convolution with the Möbius function is Newton's (backward) finite-difference operator. So convolution with μ acts as 'differentiation' while convolution with ζ acts as 'integration'. If we interpret the sequences $a : \mathbb{N} \rightarrow \mathbb{k}$ as formal power series, then the zeta function is the geometric series, while the Möbius function is $1 - x$, as follows from (4).

3.9. Example. For the monoid \mathbb{N}^\times , the incidence algebra is the classical algebra of arithmetic functions under Dirichlet convolution, recovering the classical Möbius inversion principle as in Section 2. Again, the closed formula (2) for the Möbius function can be established easily by simply showing that it satisfies the general recursive formula. A better proof explores the fact that the incidence algebra of a product (of Möbius categories) is the tensor product of the incidence algebras, and that the Möbius function of a product is the tensor product of Möbius functions. Now it follows from unique factorisation of primes that \mathbb{N}^\times is the (weak¹⁰) product

$$\mathbb{N}^\times \simeq \prod_p (\mathbb{N}, +)$$

identifying a number $n = \prod_p p^{r_p} \in \mathbb{N}^\times$ with the infinite vector $(r_2, r_3, r_5, \dots) \in \prod_p \mathbb{N}$. The classical formula (2) for the Möbius function now follows as the product of infinitely many copies of the Möbius function in (4).¹¹

3.10. Example: powersets—the inclusion-exclusion principle. Let X be a fixed finite set, and consider the powerset of X , i.e. the set $\mathcal{P}(X)$ of all subsets of X . It is a poset under the inclusion relation \subset . An interval in $\mathcal{P}(X)$ is given by a pair of nested subsets of X , say $T \subset S$. If the cardinality

¹⁰Weak means that only finitely many factors are allowed to be non-trivial.

¹¹This fact also gives a nice proof of the Euler product expansion (see [28, Thm. 280])

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - \frac{1}{p^s}}$$

from 1737 (almost a hundred years earlier than Dirichlet and Möbius).

of X is n , then clearly $\mathcal{P}(X)$ is isomorphic as a poset to $\mathbf{2}^n$, where $\mathbf{2}$ denotes the 2-element poset $[0, 1] \subset (\mathbb{N}, \leq)$, so it follows from (4) and the product rule that the Möbius function on $\mathcal{P}(X)$ is given by

$$\mu(T \subset S) = (-1)^{|S-T|}.$$

This is the inclusion-exclusion principle. As an example of this, consider the problem of counting *derangements*, i.e. permutations without fixpoints. Since every permutation of a set S determines a subset T of points which are actually moved, we can write

$$\text{perm}(S) = \sum_{T \subset S} \text{der}(T)$$

(with the evident notation). Hence by Möbius inversion, we find the formula for derangements

$$\text{der}(S) = \sum_{T \subset S} (-1)^{|S-T|} \text{perm}(T),$$

which is a typical inclusion-exclusion formula.

4. Abstract Möbius inversion

For background on coalgebras, bialgebras and Hopf algebras, a standard reference is Sweedler [46]. The little background needed here is amply covered also in [36].

4.1. Coalgebras. Let (C, Δ, ε) be a filtered coalgebra. Recall that a *filtration* of a coalgebra is an increasing sequence of sub-coalgebras

$$C_0 \subset C_1 \subset C_2 \subset \cdots = C$$

such that

$$\Delta(C_n) \subset \sum_{p+q=n} C_p \otimes C_q,$$

and recall that an element $x \in C$ is *group-like* when $\Delta(x) = x \otimes x$; this implies $\varepsilon(x) = 1$. It follows that group-like elements are always of filtration degree zero. We make the following standing assumption (see [31]):

(5) *We assume that C_0 is spanned by group-like elements.*

4.2. Convolution algebras. The various convolution products seen so far are all instances of the following general notion. If (C, Δ, ε) is a coalgebra and (A, m, u) is an algebra, then the space of linear maps $\text{Lin}(C, A)$ becomes an algebra under the convolution product: for $\alpha, \beta \in \text{Lin}(C, A)$, define $\alpha * \beta$ to be the composite

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{\alpha \otimes \beta} A \otimes A \xrightarrow{m} A,$$

that is, in Sweedler notation [46]:

$$(\alpha * \beta)(x) = \sum_{(x)} \alpha(x_{(1)})\beta(x_{(2)}).$$

The unit for the convolution product is

$$e := u \circ \varepsilon.$$

Theorem 4.3.¹² *If $\phi \in \text{Lin}(C, A)$ sends all group-like elements to 1, then ϕ is convolution invertible. The inverse ψ is given by the recursive formula*

$$(6) \quad \psi = e - \psi * (\phi - e).$$

We shall give a slick proof consisting mostly of definitions.¹³

4.4. Main Construction. Put $\psi_0 := e$ and

$$\psi_{n+1} := \psi_n * (\phi - e).$$

Put also

$$\psi_{\text{even}} := \sum_{n \text{ even}} \psi_n \quad \text{and} \quad \psi_{\text{odd}} := \sum_{n \text{ odd}} \psi_n.$$

Finally put

$$\boxed{\psi := \psi_{\text{even}} - \psi_{\text{odd}}}$$

¹²I do not know of any reference for this result. It may be new, but is in any case a straightforward abstraction of the theorems of Rota and Content–Lemay–Leroux already quoted, once the degree-zero condition (5) has been identified [31].

¹³The proof ingredients go a long way back. The even-odd splitting was first used by Hall [27] for complete lattices, then by Cartier–Foata [7] for monoids, and by Content–Lemay–Leroux [11] for Möbius categories, and further exploited in [34] and [25]. The recursive formula goes back to Weisner [47]. The combined proof is inspired by [6].

In other words, $\psi = \sum_{n \geq 0} (-1)^n (\phi - e)^{*n}$. This is an infinite sum of functions, but it is nevertheless well defined, because for every input, only finitely many terms in the sum are non-zero. Indeed, given an element $x \in C$ of filtration degree r , then for $n > r$ the n -fold convolution power of $(\phi - e)$ involves the n -fold comultiplication of x , and since $n > r$ at least one of these factors must be of degree 0, and hence is killed by $(\phi - e)$, thanks to the standing assumption (5).

Now from $\psi_{n+1} = \psi_n * (\phi - e)$ we get

$$\psi_{\text{odd}} = \psi_{\text{even}} * (\phi - e) \quad \text{and} \quad \psi_{\text{even}} = e + \psi_{\text{odd}} * (\phi - e),$$

and subtracting these two equations we arrive finally at the formula

$$\boxed{\psi = e - \psi * (\phi - e)}$$

of the theorem.

Proof of Theorem 4.3. First of all, the recursive formula is meaningful: since ϕ sends group-like elements to 1, it agrees with e in filtration degree 0 (thanks to the standing assumption (5)). Therefore, in the convolution product on the right-hand side, ψ is only evaluated on elements of filtration degree strictly less than the element given on the left-hand side. Rearranging terms gives $\psi * \phi = e$, showing that ψ is an inverse on the left.

All the arguments can be repeated with $\psi_{n+1} = (\phi - e) * \psi_n$ (instead of $\psi_{n+1} = \psi_n * (\phi - e)$), arriving at the right-sided formula $\psi = e - (\phi - e) * \psi$, and rearrangement of the terms now shows that ψ is also an inverse on the right. \square

5. Möbius inversion in bialgebras

Suppose now that B is a bialgebra, still assumed to be filtered, and still assumed to have B_0 spanned by group-like elements. Recall that a bialgebra is simultaneously a coalgebra and an algebra, enjoying in particular the compatibility

$$(7) \quad \Delta(x \cdot y) = \Delta(x) \cdot \Delta(y) \quad \forall x, y.$$

For the target algebra A , we must now assume it is commutative. This is used in the proof of Lemma 5.2 below, and the rest of the paper depends on that lemma.

5.1. Multiplicativity. Call a linear map $\phi : B \rightarrow A$ *multiplicative*¹⁴ if it preserves multiplication:

$$\phi(x \cdot y) = \phi(x) \cdot \phi(y) \quad \forall x, y.$$

Lemma 5.2. *The convolution of two multiplicative functions is again a multiplicative function. In particular, multiplicative functions form a monoid.*

Proof. This follows immediately from the bialgebra axiom (7): by expansion in Sweedler notation we have on one hand

$$\begin{aligned} (\alpha * \beta)(xy) &\stackrel{(7)}{=} \sum_{(x),(y)} \alpha(x_{(1)}y_{(1)})\beta(x_{(2)}y_{(2)}) \\ &= \sum_{(x),(y)} \alpha(x_{(1)})\alpha(y_{(1)})\beta(x_{(2)})\beta(y_{(2)}), \end{aligned}$$

(assuming that α and β are multiplicative), and on the other hand

$$(\alpha * \beta)(x)(\alpha * \beta)(y) = \sum_{(x),(y)} \alpha(x_{(1)})\beta(x_{(2)})\alpha(y_{(1)})\beta(y_{(2)}).$$

Since A is assumed commutative, these two expressions are equal. \square

Note that multiplicative functions do not form a linear subspace, as the sum of two multiplicative functions is rarely multiplicative.

Lemma 5.3. *As before, assume $\phi : B \rightarrow A$ sends group-like elements to 1 and is multiplicative. Then for any $\alpha : B \rightarrow A$ multiplicative we have*

$$(\alpha * \phi')(xy) = (\alpha * \phi')(x)\alpha(y) + \alpha(x)(\alpha * \phi')(y) + (\alpha * \phi')(x)(\alpha * \phi')(y),$$

where for short we use the temporary notation $\phi' := (\phi - e)$.

Proof. This is simply linearity: substitute $\phi - e$ for ϕ' , expand both sides of the equation, and use multiplicativity of α , ϕ , and $\alpha * \phi$ (thanks to Lemma 5.2). \square

¹⁴Note: in number theory, for arithmetic functions $\alpha : \mathbb{N}^\times \rightarrow \mathbb{C}$, the word ‘multiplicative’ is used for something else, namely the condition $\alpha(mn) = \alpha(m)\alpha(n)$ for all m and n relatively prime. The notions are not directly related, because \mathbb{N}^\times is not a bialgebra for the usual multiplication: for example, $\Delta(2 \cdot 2)$ has three terms whereas $\Delta(2)\Delta(2)$ has four terms.

Proposition 5.4. *Suppose ϕ sends group-like elements to 1, and let ψ denote its convolution inverse. If ϕ is multiplicative, then so is ψ .*

Proof. The proof goes by induction on the degree of xy . If both x and y are group-like (i.e. degree 0), it is clear that $\psi(xy) = 1 = \psi(x)\psi(y)$. Now for the induction step. We use the shorthand notation $\phi' := \phi - e$. First use the recursive formula (6):

$$\psi(xy) = -(\psi * \phi')(xy).$$

By induction, the ψ inside the convolution is multiplicative, because its arguments are all of lower degree (the only case of equal degree in the left-hand tensor factor corresponds to degree 0 in the right-hand tensor factor, which is killed by $\phi' = \phi - e$), so we can apply Lemma 5.3:

$$= -(\psi * \phi')(x)\psi(y) - \psi(x)(\psi * \phi')(y) - (\psi * \phi')(x)(\psi * \phi')(y),$$

and then the recursive equation (6) backwards (four times):

$$= \psi(x)\psi(y) + \psi(x)\psi(y) - \psi(x)\psi(y) = \psi(x)\psi(y). \quad \square$$

5.5. Antipodes for bialgebras. Recall that a filtered bialgebra B is *connected* if B_0 is spanned by the unit, and that any connected bialgebras is Hopf [46]. We shall call B *not-quite-connected* [31] in the situation where B_0 is spanned by group-like elements.¹⁵ A notion of antipode for not-quite-connected bialgebras was introduced recently by Carlier and Kock [6]. It specialises to the usual antipode in the case of a connected bialgebra, and in any case it still serves to compute the Möbius function as $\mu = \zeta \circ S$ as for Hopf algebras.

In fact, the antipode S itself is an example of abstract Möbius inversion, as we now proceed to explain. The idea is simply that one can use the bialgebra B itself as algebra of values, and invoke abstract Möbius inversion in $\text{Lin}(B, B)$. The identity $B \rightarrow B$ does not in general admit a convolution inverse, because it does not send all group-like elements to 1. But if we just fix that artificially then we can give it as input to the general construction, and the outcome will be the antipode S in the sense of [6].

¹⁵Related notions of *almost connected* appear in the literature for Feynman categories [30] and for cooperads [23], in both cases with constructions of bialgebras and Hopf algebras as motivation.

To this end, we need to choose B_+ , a linear complement to $B_0 \subset B$, and we need to choose it inside $\text{Ker } \varepsilon$. (Note that if the filtration is actually a grading, then B_+ is canonical, namely the span of all homogeneous elements of positive degree. In practice, B is often of combinatorial nature and a basis is already given.) Define the linear operator $T : B \rightarrow B$ by

$$T(x) = \begin{cases} 1 & \text{if } x \text{ group-like} \\ x & \text{if } x \in B_+. \end{cases}$$

Now apply the Main Construction 4.4, writing S instead of ψ :

$$S_0 := e, \quad S_{n+1} := S_n * (T - e), \quad S := S_{\text{even}} - S_{\text{odd}},$$

arriving at the recursion

$$S = e - S * (T - e).$$

By the general Möbius inversion Theorem 4.3, S is the convolution inverse to T . (Note that so far the arguments do not involve multiplicativity—neither T nor S are multiplicative for general not-quite-connected bialgebras.) But the neat feature of this S is that it can invert ‘anything multiplicative’, by precomposition. Precisely:

Proposition 5.6. *Suppose $\phi : B \rightarrow A$ takes group-like elements to 1, and let ψ denote its convolution inverse. If ϕ is multiplicative, then*

$$\psi = \phi \circ S.$$

Proof. We calculate

$$\phi *_A (\phi \circ S) \stackrel{(1)}{=} (\phi \circ T) *_A (\phi \circ S) \stackrel{(2)}{=} \phi \circ (T *_B S) \stackrel{(3)}{=} \phi \circ (\eta_B \circ \varepsilon) \stackrel{(4)}{=} \eta_A \circ \varepsilon = e.$$

Here (1) holds because ϕ takes all group-like elements to 1, and T only replaces general group-like elements by the particular group-like element 1. Step (2) follows immediately from the assumption that ϕ is multiplicative. Step (3) is the fact that S is convolution inverse to T , and step (4) is the fact that ϕ is unital. \square

It is obviously an important property that for multiplicative functions, Möbius inversion can be given uniformly by precomposition with the antipode. For this reason, algebraic combinatorics gradually shifted emphasis from Möbius inversion to antipodes [44]—when they are available. However, we shall see that it is Möbius inversion that generalises to renormalisation, not the antipode.

6. Direct-sum decomposition and renormalisation

Coming back to the case of a coalgebra C , the Möbius inversion principle says that for every linear function $\phi : C \rightarrow A$ (taking value 1 on the group-like elements) there exists another linear function $\psi : C \rightarrow A$ that convolves it to the neutral e .

Sometimes one is interested in less drastic transformations. For example, given a linear subspace $K \subset A$, is it possible to convolve ϕ into K ? This question is precisely what BPHZ renormalisation answers: in this case, A is an algebra of ‘amplitudes’, K is a subalgebra of ‘finite amplitudes’, and the result of convolving a map $\phi : C \rightarrow A$ into K is *renormalisation*. In detail the set-up is the following.

6.1. A decomposition problem. Suppose we have a decomposition

$$A = A_+ \oplus A_-$$

of A into a direct sum of vector spaces. Let $R : A \rightarrow A$ denote projection¹⁶ onto A_- relatively to this direct-sum decomposition, so that $A_+ = \text{Ker } R$.

Given $\phi : C \rightarrow A$ (sending group-like elements to 1) find another $\psi : C \rightarrow A$ such that $\psi * \phi$ takes values in A_+ , or at least maps $\text{Ker } \varepsilon$ to A_+ . In other words, find ψ such that $R(\psi * \phi)(x) = 0$, for all $x \in \text{Ker } \varepsilon$.

This problem can be approached precisely as in the Möbius inversion case (which is the case where $A_- = A$ and R is the identity map). The only change required is to define a modified convolution product $*_R$ on $\text{Lin}(C, A)$, defined as¹⁷

$$\alpha *_R \beta := R(\alpha * \beta).$$

Note that $*_R$ is generally neither associative nor unital, but none of these two properties are needed in the following main construction.

6.2. Main Construction. Put $\psi_0 := e$ and

$$\psi_{n+1} := \psi_n *_R (\phi - e).$$

(Note that since $*_R$ is not associative, the parentheses cannot be omitted when expanding. They must be pushed left, as for example in $\psi_2 = (\psi_0 *_R$

¹⁶In 6.5 below we shall impose the Rota–Baxter axiom.

¹⁷This modified convolution product should not be confused with the so-called *double product* in the non-commutative algebra $\text{Lin}(C, A)$, defined as $\alpha *_R \beta := R(\alpha) * \beta + \alpha * R(\beta) + \alpha * \beta$. The double product (in the case where R is Rota–Baxter) plays a role in Lie-theoretic aspects of renormalisation [18].

$(\phi - e) *_{\mathcal{R}} (\phi - e)$, which is different from $\psi_0 *_{\mathcal{R}} ((\phi - e) *_{\mathcal{R}} (\phi - e))$.) As in the classical case, put

$$\psi_{\text{even}} := \sum_{n \text{ even}} \psi_n \quad \text{and} \quad \psi_{\text{odd}} := \sum_{n \text{ odd}} \psi_n,$$

and finally

$$(8) \quad \boxed{\psi := \psi_{\text{even}} - \psi_{\text{odd}}}$$

Just as in the classical case, these are locally finite sums. This is a consequence of the filtration of C —the argument is not affected by the fact that the convolution has been modified.

Now from $\psi_{n+1} = \psi_n *_{\mathcal{R}} (\phi - e)$ we get

$$\psi_{\text{odd}} = \psi_{\text{even}} *_{\mathcal{R}} (\phi - e) \quad \text{and} \quad \psi_{\text{even}} = \psi_0 + \psi_{\text{odd}} *_{\mathcal{R}} (\phi - e),$$

and subtracting these two equations we arrive finally at the formula

$$(9) \quad \boxed{\psi = e - \psi *_{\mathcal{R}} (\phi - e)}$$

Lemma 6.3. *ψ sends group-like elements to 1.*

Proof. This is clear from (9) since $\phi - e$ kills group-like elements. □

Lemma 6.4. *For all $x \in \text{Ker } \varepsilon$, we have*

- (i) $\psi(x) \in \text{Im } R = A_-$
- (ii) $(\psi * \phi)(x) \in \text{Ker } R = A_+$.

If we assume $1 \in A_+$, then (ii) holds for all $x \in C$.

Proof. Assuming $x \in \text{Ker } \varepsilon$, the first statement is obvious from (9). It follows that we have $R(\psi(x)) = \psi(x)$. Rearranging the terms of the recursive equation (9), we see that

$$\psi *_{\mathcal{R}} \phi = e - \psi + \psi *_{\mathcal{R}} e = e - \psi + R(\psi).$$

For $x \in \text{Ker } \varepsilon$, the right-hand side is zero, whence the second statement. If not $x \in \text{Ker } \varepsilon$ then we can assume x group-like, and then $(\psi * \phi)(x) = 1$ by Lemma 6.3. So then (ii) follows from the alternative assumption $1 \in A_+$. □

6.5. Bialgebra case, Rota–Baxter equation, and multiplicativity.

For a bialgebra instead of coalgebra, it is natural to demand that ψ be multiplicative, provided ϕ is so. To achieve this, it turns out one should just demand the direct-sum decomposition $A = A_+ \oplus A_-$ to be multiplicative, in the sense that both A_+ and A_- are subalgebras (although not unital subalgebras).

Lemma 6.6 (Atkinson [2]). *To give such a subalgebra decomposition $A = A_+ \oplus A_-$ is equivalent to giving an idempotent linear operator $R : A \rightarrow A$ satisfying the Rota–Baxter equation.*¹⁸

$$(10) \quad R(x \cdot y) + R(x) \cdot R(y) = R(R(x) \cdot y + x \cdot R(y)) \quad \forall x, y.$$

Proof. This check is direct: given R , it follows directly from the Rota–Baxter equation that both $A_+ := \text{Ker}(R)$ and $A_- := \text{Im}(R)$ are subalgebras (closed under multiplication). Conversely, given a subalgebra decomposition $A = A_+ \oplus A_-$, it is easy to check the Rota–Baxter equation. \square

Proposition 6.7. *Suppose ϕ sends group-like elements to 1. If ϕ is multiplicative, then so is ψ .*

Proof. We use the shorthand notation $\phi' := \phi - e$. We calculate on one hand

$$\begin{aligned} \psi(xy) &= -R[(\psi * \phi')(xy)] \\ &= -R[(\psi * \phi')(x)\psi(y) + \psi(x)(\psi * \phi')(y) + (\psi * \phi')(x)(\psi * \phi')(y)], \end{aligned}$$

by induction on $\text{deg}(xy)$, using Lemma 5.3, exactly as in the proof of Proposition 5.4 in the classical Möbius case.

On the other hand, we compute (using (9) twice):

$$\psi(x)\psi(y) = (-R[(\psi * \phi')(x)]) (-R[(\psi * \phi')(y)]).$$

Now apply the Rota–Baxter identity (10):

$$= -R[(\psi * \phi')(x) \cdot (\psi * \phi')(y) - R[(\psi * \phi')(x)] \cdot (\psi * \phi')(y) - (\psi * \phi')(x) \cdot R[(\psi * \phi')(y)]]$$

¹⁸The equation is more generally written $\theta R(xy) + R(x)R(y) = R(R(x)y + xR(y))$ for a fixed scalar weight θ , in order to accommodate the $\theta = 0$ case, which is the equation satisfied by integration by parts. The equation relevant presently is thus the weight-1 Rota–Baxter equation, according to the classical convention. More recent sources (including [16], [20], [21]) tend to use the opposite convention, where the $\theta R(xy)$ -term is on the other side of the equation, and the weight relevant to BPHZ recursion is thus instead called weight -1 .

and use (9) backwards twice:

$$= -R [(\psi * \phi')(x) \cdot (\psi * \phi')(y) + \psi(x) \cdot (\psi * \phi')(y) + (\psi * \phi')(x) \cdot \psi(y)],$$

which agrees with the computation of $\psi(xy)$. \square

6.8. Non-concluding historical remarks. Equation (9) is the abstract BPHZ recursion of Section 1, often called the Bogoliubov recursion. The Hall–Leroux style even-odd formula

$$\psi = \psi_{\text{even}} - \psi_{\text{odd}}$$

of Equation (8) features less prominently in renormalisation theory, see [19] and [20]. Expanded, it says

$$\psi = \sum_{n \geq 0} (-1)^n \psi_n = \sum_{n \geq 0} (-1)^n R(R(R(\cdots * \phi') * \phi') * \phi'),$$

where the n th term of the sum has n applications of R and n convolution factors, and where as usual we use the shorthand $\phi' := \phi - e$. This is the solution of Atkinson [2] to the factorisation problem posed by R . Atkinson actually uses the abstract form of the ‘Bogoliubov’ recursion in his Second Proof [2], in a way similar to the proofs above. The equivalence between Atkinson’s formula and the Bogoliubov recursion has been exploited further in the context of renormalisation and Lie theory by Ebrahimi-Fard, Manchon and Patras [21]. It is striking that it comes about from the two aspects of general Möbius inversion.

* * *

Frederick Atkinson spent the first part of his mathematical life working in analytic number theory, contributing in particular to the theory of arithmetic functions and Dirichlet series. His 1949 paper with Cherwell [3] (cited in Hardy and Wright [28]), is about average values of arithmetic functions related by Möbius inversion. In the 1950s his interests shifted to functional analysis and operator theory, which was the context for his interest in Baxter’s work, leading to his 1963 paper [2] already mentioned. For more information about Atkinson’s life and work, see [38].

The notion of Rota–Baxter algebra had been introduced by Glen Baxter [4] in fluctuation theory of sums of random variables in 1960. Rota, Cartier, Foata, and others realised the usefulness of the notion (at the time

called *Baxter algebras*¹⁹) also in algebra and combinatorics, notably in the theory of symmetric functions, and Rota [42] used elementary category theory to unify several results by establishing them in the free Rota–Baxter algebra. For a glimpse into the extensive theory of Rota–Baxter algebras, with emphasis on their use in renormalisation, see [16, 17, 18].

Gian-Carlo Rota was a main character both in the development of Möbius inversion and in the development of Rota–Baxter algebras, in both cases making these constructions into general tools. Naturally, he also combined these two toolboxes: for example, in his 1969 proof of the so-called Bohnenblust–Spitzer identity (see also [21]) in the free (and hence in every) Rota–Baxter algebra [43], a key point is showing that the signs in that formula arise from the Möbius function of the partition lattice [41]. Rota did not have the idea of entangling the Rota–Baxter operator with the recursions of Möbius inversion itself, though. From the ahistorical viewpoint of the present contribution, this is what Bogoliubov [5] and Atkinson [2] achieved—without having the general theory of Möbius inversion at their disposal.

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¹⁹The renaming to *Rota–Baxter algebra* occurred in [15], marking also the first connection between this subject and renormalisation.

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