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An odd variant of multiple zeta values

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For positive integers $i_1, \dots, i_k$ with $i_1 > 1$, we define the multiple t-value $t(i_1, \dots, i_k)$ as the sum of those terms of the usual infinite series for the multiple zeta value $\zeta(i_1, \dots, i_k)$ with odd denominators. Multiple t-values can be written as rational linear combinations of the alternating or "colored" multiple zeta values. Using known results for colored multiple zeta values, we obtain tables of multiple t-values through weight 7, suggesting some interesting conjectures, including one that the dimension of the rational vector space generated by weight-n multiple t-values has dimension equal to the nth Fibonacci number. Like the multiple zeta values, the multiple t-values can be multiplied according to the rules of the harmonic algebra. Using this fact, we obtain explicit formulas for multiple t-values with repeated arguments analogous to those known for multiple zeta values. We express the generating function of the height one multiple t-values $t(n, 1, \dots, 1)$ in terms of a generalized hypergeometric function. We also define alternating multiple t-values and prove some results about them.

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1. Introduction

In the past few decades the multiple zeta values

$$\zeta(i_1, \dots, i_k) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{i_1} n_2^{i_2} \cdots n_k^{i_k}}$$

have appeared prominently in both number theory and physics. In this paper we consider the related quantities

$$t(i_1, \dots, i_k) = \sum_{\substack{n_1 > n_2 > \cdots > n_k \geq 1 \\ n_i \text{ odd}}} \frac{1}{n_1^{i_1} n_2^{i_2} \cdots n_k^{i_k}}, \tag{1.1}$$

which we call multiple t-values. In both these definitions $i_1, i_2, \dots, i_k$ are positive integers with $i_1 > 1$; we call k the "depth" and $i_1 + \cdots + i_k$ the "weight." Our study reveals that multiple t-values have remarkable parallels to, and contrasts with, multiple zeta values.

Our notation is adapted from N. Nielsen [24], who wrote t_n for $t(n)$ and gave the formula

$$\sum_{i=1}^{n-1} t(2i)t(2n-2i) = \frac{2n-1}{2} t(2n). \tag{1.2}$$

This may be compared with the formula (also given in [24])

$$\sum_{i=1}^{n-1} \zeta(2i)\zeta(2n-2i) = \frac{2n+1}{2} \zeta(2n). \tag{1.3}$$

Of course

$$\zeta(i) = \sum_{n\ge 1} \frac{1}{n^i} = \sum_{n \text{ odd}} \frac{1}{n^i} + \sum_{n \text{ even}} \frac{1}{n^i} = t(i) + \frac{1}{2^i}\zeta(i),$$

so that $t(i) = (1-2^{-i})\zeta(i)$ and the classical formula

$$\zeta(2n) = \frac{(-1)^{n-1} B_{2n} (2\pi)^{2n}}{2(2n)!}$$

is paralleled by

$$t(2n) = \frac{(-1)^{n-1} B_{2n} (2^{2n}-1)\pi^{2n}}{2(2n)!}. \tag{1.4}$$

Eq. (1.4) can be expressed by the generating function

$$\sum_{n=1}^{\infty} t(2n) x^{2n-1} = \frac{\pi x}{4} \tan\left(\frac{\pi x}{2}\right), \tag{1.5}$$

from which Nielsen's formula (1.2) follows easily by differentiating both sides of Eq. (1.5) and then comparing that to the result of squaring both sides.

The multiplication of multiple t-values as series works just like the multiplication of multiple zeta values as series, so that we have, for example,

$$t(2)t(3,1,1) = t(2,3,1,1) + t(5,1,1) + t(3,2,1,1) + t(3,3,1) + t(3,1,2,1) + t(3,1,3) + t(3,1,1,2).$$

Consequently (paralleling [11, Thm. 2.2]), any symmetric sum of multiple t-values, e.g., $t(3,2,2)+t(2,3,2)+t(2,2,3)$, is a rational polynomial in ordinary t-values, e.g.,

$$t(3,2,2)+t(2,3,2)+t(2,2,3)=\frac{1}{2}t(2)^2t(3)-\frac{1}{2}t(3)t(4)-t(2)t(5)+t(7).$$

In particular, any multiple t-value with all its arguments equal to the same integer k is a rational polynomial in the t-values $t(k), t(2k), t(3k), \ldots$. In view of Eq. (1.4) above, this means that, when k is even, a multiple t-value of the form $t(k,k,\ldots,k)$ (with n repetitions) is a rational multiple of π^{nk}, just as with multiple zeta values. For example, the well-known identities

$$\zeta(\{2\}_n)=\frac{\pi^{2n}}{(2n+1)!},\quad \zeta(\{4\}_n)=\frac{2^{2n+1}\pi^{4n}}{(4n+2)!},\quad \zeta(\{6\}_n)=\frac{6(2\pi)^{6n}}{(6n+3)!}$$

(where $\{k\}_n$ means k repeated n times) have multiple t-value counterparts

$$t(\{2\}_n)=\frac{\pi^{2n}}{2^{2n}(2n)!},\quad t(\{4\}_n)=\frac{\pi^{4n}}{2^{2n}(4n)!},\quad t(\{6\}_n)=\frac{3\pi^{6n}}{4(6n)!}. \tag{1.6}$$

As with multiple zeta values, one can define t-star values by

$$t^\star(i_1,\ldots,i_k)=\sum_{\substack{n_1\ge n_2\ge\cdots\ge n_k\ge 1\\ n_i \text{ odd}}}\frac{1}{n_1^{i_1}n_2^{i_2}\cdots n_k^{i_k}} \tag{1.7}$$

for positive integers $i_1,\ldots,i_k$ with $i_1>1$. S. Muneta [21] gave an identity expressing $\zeta^\star(\{2m\}_n)$ as π^{2mn} times a rational polynomial in Bernoulli numbers; similarly, in Theorem 3.7 below we give an identity expressing $t^\star(\{2m\}_n)$ as π^{2mn} times a rational polynomial in Euler numbers. The case $m=1$ is

$$t^\star(\{2\}_n)=\frac{\pi^{2n}}{2^{2n}(2n)!}(-1)^nE_{2n}.$$

Despite the parallels, the algebra of multiple t-values is quite different in some ways from the algebra of multiple zeta values. Both the duality theorem and the "double shuffle relations" [16] of multiple zeta values are missing for multiple t-values. The difference already appears in weight 3: while for multiple zeta values there is the famous identity $\zeta(2,1)=\zeta(3)$, for multiple t-values one has instead

$$t(2,1)=-\frac{1}{2}t(3)+t(2)\log 2. \tag{1.8}$$

Nevertheless, as we show in §4, any multiple t-value is a rational linear combination of ordinary and alternating (or "colored") multiple zeta values, the latter being forms such as

$$\zeta(\bar{3},1)=\sum_{i>j\geq 1}\frac{(-1)^i}{i^3 j}.$$

Existing tables of such values [3, 4] can be used to give formulas for multiple t-values in terms of alternating multiple zeta values; in Appendix A we give such formulas through weight 7. As with multiple zeta values, all known relations of multiple t-values are homogeneous by weight.

Examination of the tables leads to several conjectures, which are presented in §2 below. Conjecture 2.1 asserts that the algebra of multiple t-values admits a weight-decreasing derivation, which can easily be seen not to exist in the case of multiple zeta values. Conjecture 2.2 states that the rational vector space generated by the multiple t-values of weight n has dimension equal to the nth Fibonacci number. (Conjecture 2.3, due to B. Saha [26], gives this a concrete form by proposing a basis for the weight-n multiple t-values.) This compares to the well-known conjecture that the dimension of the rational vector space of weight-n multiple zeta values is the nth Padovan number.

In §3 we prove the analogue for multiple t-values of the symmetric sum theorem for multiple zeta values [11]. This implies the the results for repeated arguments given in Eqs. (1.6) above. In §4 we show how any multiple t-value can be written as a sum of alternating multiple zeta values.

The "height one" multiple zeta values $\zeta(n,1,\dots,1)$ are rational polynomials in the ordinary zeta values $\zeta(i)$, $i\geq 2$; indeed this follows from the generating-function identity [5, 12]

$$\sum_{i,j\geq 1}\zeta(i+1,\{1\}_{j-1})x^i y^j = 1-\frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)}. \tag{1.9}$$

It is already apparent from

$$t(3,1)=-\frac{37}{60}t(4)-\frac{1}{2}\zeta(\bar{3},1)+t(3)\log 2$$

that multiple t-values of height one are more complicated. Nevertheless, in §5 we express

$$H(x,y)=\sum_{i,j\geq 1}t(i+1,\{1\}_{j-1})x^i y^j$$

as the value of a generalized hypergeometric function (Theorem 5.1 below). In contrast with the generating function (1.9) for height one multiple zeta values, which is symmetric in x and y, $H(x,y)$ is very far from symmetric; e.g., the series

$$t(2)+t(2,1)+t(2,1,1)+t(2,1,1,1)+\cdots$$

converges (and in fact converges to twice Catalan's constant; see Eq. (5.1) below), while

$$t(2)+t(3)+t(3)+t(4)+\cdots,$$

like the corresponding series of multiple zeta values, diverges.

In §6 we define alternating multiple t-values in a manner analogous to alternating multiple zeta values. Then one can write a formula for $t(\bar{n},\ldots,\bar{n})$ in terms of the t-values of even integers and the values of the Dirichlet beta function at odd integers, and in fact there are explicit formulas

$$t(\{\bar{1}\}_k)=(-1)^{\lfloor\frac{k+1}{2}\rfloor}\frac{\pi^k}{2^{2k}k!}\quad\text{and}\quad t(\{\bar{3}\}_k)=(-1)^{\lfloor\frac{k+1}{2}\rfloor}\frac{3\pi^{3k}}{2^{3k+1}(3k)!}.$$

The multiple t-value (1.1) can be written in terms of the Hurwitz multiple zeta function

$$\zeta(i_1,\ldots,i_k;a_1,\ldots,a_k)=\sum_{n_1>\cdots>n_k\geq 1}\frac{1}{(n_1+a_1)^{i_1}\cdots(n_k+a_k)^{i_k}}$$

discussed in [22]; taking $a_1=\cdots=a_k=-\frac{1}{2}$, we have

$$t(i_1,\ldots,i_k)=2^{i_1+\cdots+i_k}\zeta(i_1,\ldots,i_k;-\tfrac{1}{2},\ldots,-\tfrac{1}{2}). \tag{1.10}$$

Double t-values $t(n,m)$ are referred to as "double zeta values of level 2" in [17] and [23], where they are written as $\zeta^{\mathbf{oo}}(m,n)$ and $\zeta^{\mathbf{o}}(m,n)$ respectively.

2. Conjectures on the algebra of multiple t-values

Let $\mathfrak{H}^1$ be the underlying rational vector space of the noncommutative polynomial algebra $\mathbb{Q}\langle z_1,z_2,\ldots\rangle$, and let $\mathfrak{H}^0$ be the subalgebra of of $\mathfrak{H}^1$ generated by 1 and those monomials that start with z_i, $i>1$. From [12] we have the following result.

Theorem 2.1. *The rational vector space* $\mathfrak{H}^1$ *with the product* $*$ *defined recursively by* $w_1 * 1 = 1 * w_1 = w_1$ *and*

$$z_i w_1 * z_j w_2 = z_i(w_1 * z_j w_2) + z_j(z_i w_1 * w_2) + z_{i+j}(w_1 * w_2) \tag{2.1}$$

for all monomials w_1, w_2 *of* $\mathfrak{H}^1$ *is a commutative algebra. Further,* $\mathfrak{H}^1$ *is a polynomial algebra,* $\mathfrak{H}^0$ *is a subalgebra, and* $\mathfrak{H}^1 = \mathfrak{H}^0[z_1]$.

We recall that the quasi-symmetric functions QSym are the set of formal power series f in $x_1, x_2, \dots$ of bounded degree such that, for any sequence $i_1 < i_2 < \cdots < i_p$, the coefficient of

$$x_{i_1}^{n_1} x_{i_2}^{n_2} \cdots x_{i_p}^{n_p}$$

in f is the same as the coefficient in f of

$$x_1^{n_1} x_2^{n_2} \cdots x_p^{n_p}.$$

Then QSym is an algebra, and it contains the symmetric functions Sym as a proper subalgebra. As a vector space, QSym is generated by the monomial quasi-symmetric functions

$$M_{n_1,\dots,n_p} = \sum_{i_1 < \cdots < i_p} x_{i_1}^{n_1} \cdots x_{i_p}^{n_p}.$$

In [12] the following result is proven.

Theorem 2.2. $(\mathfrak{H}^1, *)$ *is isomorphic to the algebra* QSym *of quasi-symmetric functions via the map that sends* $z_{n_1} \cdots z_{n_p}$ *to* $M_{n_1,\dots,n_p}$.

Henceforth we will identify $\mathfrak{H}^0$ with the subalgebra QSym^0 of QSym generated by $M_{n_1,\dots,n_p}$ with $n_1 > 1$. Let $\mathfrak{T}$ be the subspace of $\mathbb{R}$ generated over the rationals by 1 and the multiple t-values. Theorem 3.1 below gives a homomorphism $\theta : \mathfrak{H}^0 \to \mathfrak{T}$, which we can also regard as a homomorphism from QSym^0 to $\mathfrak{T}$. Now in [14] it is shown that the linear map $A_- : \mathrm{QSym} \to \mathrm{QSym}$ with $A_-(1) = 0$ and

$$A_-(M_{(i_1,\dots,i_k)}) = \begin{cases} M_{(i_1,\dots,i_{k-1})}, & \text{if } i_k = 1, \\ 0, & \text{otherwise,} \end{cases}$$

is a derivation. It is also evident that $A_-(\mathrm{QSym}^0) \subset \mathrm{QSym}^0$. We make the following conjecture.

Conjecture 2.1. *The algebra $\mathcal{T}$ admits a derivation d such that $d\theta = \theta A_-$.*

By Corollary 4.1 below, 2^k times any multiple t-value of depth k is a signed sum of alternating multiple zeta values. Now the Multiple Zeta Value Data Mine [4] gives formulas for all alternating multiple zeta values through weight 12 as rational linear combinations of what are believed to be basis elements (all the relations there are proved, but it is possible that the "basis" is really just a spanning set, as undiscovered rational relations may exist). Using this resource, we have expressed multiple t-values through weight 7 as rational linear combinations of products of (1) ordinary t-values $t(n)$, $n \geq 2$; (2) $\log 2$; and (3) selected alternating multiple zeta values. The particular alternating multiple zeta values used are taken from the conjectural basis used in [4]; through weight 7 they are

$$\zeta(\bar{3},1),\ \zeta(\bar{3},1,1),\ \zeta(\bar{5},1),\ \zeta(\bar{3},1,1,1),\ \zeta(\bar{5},1,1),\ \zeta(\bar{3},3,1),\ \zeta(\bar{3},1,1,1,1).$$

These formulas are listed in the Appendix A below. For these formulas, d is formal differentiation with respect to $\log 2$. For example,

$$t(2,3,1) = -\frac{2}{21}t(6) - \frac{3}{196}t(3)^2 - \frac{1}{2}t(2)\zeta(\bar{3},1) + \frac{1}{4}\zeta(\bar{5},1) - \frac{1}{2}t(5)\log 2 + \frac{4}{7}t(2)t(3)\log 2$$

and

$$dt(2,3,1) = -\frac{1}{2}t(5) + \frac{4}{7}t(2)t(3) = t(2,3).$$

We note that if $\mathcal{Z}$ denotes the $\mathbb{Q}$-subalgebra of $\mathbb{R}$ spanned by 1 and the multiple zeta values, then no derivation d of $\mathcal{Z}$ with $d\zeta = \zeta A_-$ can exist. For if there were such a d, we would have

$$\zeta(2) = \zeta A_-(M_{(2,1)}) = d\zeta(2,1) = d\zeta(3) = \zeta A_-(M_{(3)}) = 0.$$

If we write $\mathcal{Z}_n$ for the $\mathbb{Q}$-subspace of $\mathcal{Z}$ spanned by multiple zeta values of weight n, then it is generally believed that $\dim \mathcal{Z}_n = P_n$, where P_n is the nth Padovan number (i.e., $P_1 = 0$, $P_2 = P_3 = 1$, and $P_n = P_{n-2} + P_{n-3}$ for $n > 3$). It is unlikely that this will be proved soon, because of the difficulty of showing, for example, that $\zeta(3)^2$ is not a rational multiple of $\zeta(6)$. Nevertheless, it has been known for some time that $\dim \mathcal{Z}_n$ is *at most* P_n, and indeed F. Brown [6] proved that the set of cardinality P_n suggested by the author in [12] – the multiple zeta values of weight n whose exponent strings have only 2's and 3's – spans $\mathcal{Z}_n$. If $\mathcal{T}_n$ denotes the $\mathbb{Q}$-subspace of $\mathcal{T}$ spanned by all multiple t-values of weight n, we offer the following conjecture.

Conjecture 2.2. *For $n \geq 2$, $\dim \mathcal{T}_n = F_n$, the nth Fibonacci number (defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$).*

From Appendix A it follows that $\dim \mathcal{T}_n \leq F_n$ for $n \leq 7$. If we let $\mathcal{A}_n$ be the $\mathbb{Q}$-subspace of $\mathbb{R}$ generated by all the alternating multiple zeta values of weight n, then it is a long-standing conjecture that $\dim \mathcal{A}_n = F_{n+1}$, and it is known that $\dim \mathcal{A}_n \leq F_{n+1}$ (see [28, Thm. 13.2.1]).

We note that Conjecture 2.1 implies the surjectivity of $d : \mathcal{T}_{n+1} \to \mathcal{T}_n$, since any multiple t-value $t(i_1, \dots, i_k) \in \mathcal{T}_n$ would be the image under d of $t(i_1, \dots, i_k, 1) \in \mathcal{T}_{n+1}$. Hence Conjectures 2.1 and 2.2 together imply that $\dim(\mathcal{T}_n \cap \ker d) = F_{n-2}$ for $n \geq 3$. A more explicit form of this comes from the following conjecture of B. Saha [26], which is similar to the author's conjecture on $\mathcal{Z}_n$ in [12].

Conjecture 2.3 (B. Saha)**.** *For $n \geq 2$, $\mathcal{T}_n$ has basis*

$$C_n = \{t(a_1 + 1, a_2, \dots, a_r) \mid a_1 + \cdots + a_r = n - 1,\ a_i \in \{1, 2\}\}.$$

Conjecture 2.3 is consistent with the relations in Appendix A; in fact, in Appendix B we express all multiple t-values through weight 7 in terms of elements of C_n. Note that for $n \geq 3$, C_n is the union of disjoint subsets $C_n^{(j)} = \{t(a_1 + 1, a_2, \dots, a_r) \in C_n \mid a_r = j\}$, $j = 1, 2$; further,

$$t(a_1 + 1, a_2, \dots, a_r) \in C_n^{(j)} \quad \text{implies} \quad t(a_1 + 1, a_2, \dots, a_{r-1}) \in C_{n-j}.$$

This gives an inductive proof that $|C_n| = F_n$, but note also that $C_n \cap \ker d = C_n^{(2)}$.

The relation between the $\mathbb{Q}$-subalgebras $\mathcal{T}$ and $\mathcal{Z}$ of $\mathbb{R}$ is far from clear. Since $\zeta(n)$ is a rational multiple of $t(n)$ for $n \geq 2$, any rational polynomial in the ordinary zeta values is in $\mathcal{T}$. In particular, $\mathcal{Z}_n \subseteq \mathcal{T}_n$ for $n \leq 7$. M. Kaneko and H. Tsumura [18] make the following conjecture, which implies that $\mathcal{Z} \subset \mathcal{T}$.

Conjecture 2.4 (M. Kaneko and H. Tsumura)**.** *A basis for $\mathcal{Z}_n$ is given by the set of elements $t(2)^k t(n_1, \dots, n_r)$ with all the n_i odd and at least 3, and $n_1 + \cdots + n_r = n - 2k$.*

We remark that the number of elements in Kaneko and Tsumura's conjectural basis for $\mathcal{Z}_n$ is the coefficient of t^n in

$$\frac{1}{1-t^2} \cdot \frac{1}{1 - t^3 - t^5 - t^7 - \cdots} = \frac{1}{1-t^2} \cdot \frac{1}{1 - \frac{t^3}{1-t^2}} = \frac{1}{1 - t^2 - t^3},$$

and this latter function is well-known as the generating function of the Padovan numbers P_n mentioned above.

3. Multiple t-values as homomorphic images

The following result for multiple t-values parallels the result [12, Thm. 4.2] for multiple zeta values.

Theorem 3.1. *There is an algebra homomorphism $\theta : \mathfrak{H}^0 \to \mathbb{R}$ sending 1 to 1 and $z_{i_1} \cdots z_{i_k}$ to $t(i_1, \dots, i_k)$ for all positive-integer strings $(i_1, \dots, i_k)$ with $i_1 > 1$.*

Proof. The point is that the recursive rule (2.1) for the words in the z's corresponds to the rules for multiplying the $t(i_1, \dots, i_k)$, e.g.,

$$t(2)t(3,1) = t(2,3,1) + t(3,2,1) + t(3,1,2) + t(5,1) + t(3,3). \qquad \square$$

The next result corresponds to the symmetric-sum theorem [11, Thm. 2.2] for multiple zeta values.

Theorem 3.2. *Let $i_1, \dots, i_k$ be integers all 2 or greater. If the symmetric group S_k acts on strings of length k by permutation, then*

$$\sum_{\sigma \in S_k} t(\sigma \cdot (i_1, \dots, i_k)) = \sum_{\mathcal{B}=\{B_1,\dots,B_l\} \in \Pi_k} (-1)^{k-l} c(\mathcal{B}) \prod_{s=1}^{l} t\left(\sum_{j \in B_s} i_j\right),$$

where Π_k is the set of partitions of the set $\{1, 2, \dots k\}$ and

$$c(\mathcal{B}) = (\operatorname{card} B_1 - 1)!(\operatorname{card} B_2 - 1)! \cdots (\operatorname{card} B_l - 1)!$$

for a partition $\mathcal{B} \in \Pi_k$ with blocks $B_1, \dots, B_l$.

Proof. The following identity holds in QSym [15, Thm. 2.3]:

$$\sum_{\sigma \in S_k} M_{\sigma \cdot I} = \sum_{\mathcal{B}=\{B_1,\dots,B_l\} \in \Pi_k} (-1)^{k-l} c(\mathcal{B}) M_{(b_1)} M_{(b_2)} \cdots M_{(b_l)},$$

where $I = (i_1, \dots, i_k)$ and $b_s = \sum_{j \in B_s} i_j$. Apply the homomorphism θ to obtain the conclusion. $\square$

If we take $i_1 = \cdots = i_k = n$ in this result, we get an expression for multiple t-values of repeated arguments. With a little work, we can state it in terms of integer rather than set partitions.

Corollary 3.1. *If $n \geq 2$, then*

$$t(\{n\}_k) = \sum_{\lambda \vdash k} \frac{(-1)^{k-\ell(\lambda)}}{m_1(\lambda)! 1^{m_1(\lambda)} m_2(\lambda)! 2^{m_2(\lambda)} \cdots} \prod_{j=1}^{\ell(\lambda)} t(n\lambda_j),$$

where $\ell(\lambda)$ is the number of parts of the partition λ and $m_i(\lambda)$ is the multiplicity of i in λ.

Proof. Set $i_1 = \cdots = i_k = n$ in Theorem 3.2 to get

$$k! t(\{n\}_k) = \sum_{\substack{\text{part. } \{B_1, \dots, B_l\} \\ \text{of } \{1, \dots, k\}}} (-1)^{k-l} (\lambda_1 - 1)! \cdots (\lambda_l - 1)! t(n\lambda_1) \cdots t(n\lambda_l),$$

where we write λ_i for card B_i. Now the number of partitions $\{B_1, \dots, B_l\}$ of the set $\{1, \dots, k\}$ corresponding to a partition $\lambda = (\lambda_1, \dots, \lambda_l)$ of k is

$$\frac{1}{m_1(\lambda) m_2(\lambda) \cdots} \binom{k}{\lambda_1} \binom{k - \lambda_1}{\lambda_2} \cdots = \frac{1}{m_1(\lambda) m_2(\lambda) \cdots} \frac{k!}{\lambda_1! \lambda_2! \cdots \lambda_l!},$$

so

$$t(\{n\}_k) = \sum_{\lambda \vdash k} \frac{(-1)^{k-l} (\lambda - 1)! \cdots (\lambda_l - 1)!}{m_1(\lambda) m_2(\lambda) \cdots \lambda_1! \cdots \lambda_l!} t(n\lambda_1) \cdots t(n\lambda_l) =$$

$$\sum_{\lambda \vdash k} \frac{(-1)^{k-l}}{m_1(\lambda)! 1^{m_1(\lambda)} m_2(\lambda)! 2^{m_2(\lambda)} \cdots} t(n\lambda_1) \cdots t(n\lambda_l),$$

and the result follows. □

An alternative way to express the preceding result is as follows. Let $P_k(x_1, \dots, x_k)$ be the polynomial that expresses the kth elementary symmetric function e_k in terms of the power sums $p_1, \dots, p_k$, i.e.,

$$e_k = P_k(p_1, p_2, \dots, p_k) \tag{3.1}$$

(cf. [20, Eqs. (2.14′)]). Then

$$t(\{n\}_k) = P_k(t(n), t(2n), \dots, t(kn)). \tag{3.2}$$

If n is even, Corollary 3.1 and Eq. (1.4) imply that $t(\{n\}_k)$ is a rational multiple of π^{nk}. As we see below, for small even values of n there are effective formulas for this rational multiple.

As shown in [12], the homomorphism $\zeta : \mathfrak{H}^0 \to \mathbb{R}$ can be extended to $\mathfrak{H}^1$ by defining $\zeta(z_1) = \gamma$ (Euler's constant). This extension has the property that it sends the generating function $H(x)$ of the complete symmetric functions to $\Gamma(1-x)$. For θ we have the following.

Theorem 3.3. *The homomorphism* $\theta : \mathfrak{H}^0 \to \mathbb{R}$ *can be extended to a homomorphism* $\theta : \mathfrak{H}^1 \to \mathbb{R}$ *such that*

$$\theta(H(x)) = \pi^{-\frac{1}{2}} e^{-\frac{\gamma x}{2}} \Gamma\left(\frac{1-x}{2}\right).$$

Proof. In view of Theorem 2.1 above, it suffices to define $\theta(z_1)$, which we set equal to $\log 2$. Then because

$$-\frac{1}{2}\psi\left(\frac{1}{2}-\frac{x}{2}\right) = \frac{\gamma}{2} + \log 2 + \sum_{i\geq 2} t(i) x^{i-1} \tag{3.3}$$

(for which see [10, Eqs. (8.370, 8.373)]), where ψ is the logarithmic derivative of the gamma function, we have

$$\theta(P(x)) = -\frac{\gamma}{2} - \frac{1}{2}\psi\left(\frac{1-x}{2}\right),$$

where $P(x) = \sum_{i\geq 1} p_i x^{i-1}$. Since $P(x)$ is the logarithmic derivative of $H(x)$, the conclusion follows. Cf. [7, Eq. (0.7b)]. □

If we extend the notation $t(i_1, \ldots, i_k)$ to all strings of positive integers $i_1, \ldots, i_k$ by letting

$$t(i_1, \ldots, i_k) = \theta(M_{i_k, \ldots, i_1}),$$

then Theorem 3.2 and Corollary 3.1 are true without restrictions on the positive integers involved. For example, we then have

$$t(\{1\}_k) = \sum_{\lambda \vdash k} \frac{(-1)^{k-l}}{m_1(\lambda)! 1^{m_1(\lambda)} m_2(\lambda)! 2^{m_2(\lambda)} \cdots} t(\lambda_1) \cdots t(\lambda_l). \tag{3.4}$$

We note that $M_{1,\ldots,1}$ (n repetitions of 1) is the elementary symmetric function e_n, so

$$1+\sum_{k=1}^{\infty} t(\{1\}_k)x^k = \theta E(x) = \theta\left(\frac{1}{H(-x)}\right) = \sqrt{\pi}e^{-\frac{\gamma x}{2}}\Gamma\left(\frac{1+x}{2}\right)^{-1}$$

by Theorem 3.3.

We now return to multiple t-values of the form $t(n,n,\dots,n)$ for $n \geq 2$.

Theorem 3.4. *For $n \geq 2$, the generating function*

$$T_n(x) = 1+\sum_{k=1}^{\infty} t(\{n\}_k)x^{kn}$$

is given by

$$T_n(x) = \frac{Z_n(x)}{Z_n\left(\frac{x}{2}\right)},$$

where $Z_n(x)$ is the corresponding generating function for multiple zeta values:

$$Z_n(x) = 1+\sum_{k=1}^{\infty} \zeta(\{n\}_k)x^{kn}.$$

Proof. We start by noting that

$$H(x)^{-1} = \exp\left(\int_0^x P(t)dt\right) = \exp\left(\sum_{n\geq 1}\frac{p_n x^n}{n}\right),$$

so that the generating function $E(x) = \sum_{n\geq 0} e_n x^n$ of the elementary symmetric functions is

$$E(x) = H(-x)^{-1} = \exp\left(\sum_{n\geq 1}\frac{(-1)^{n-1}p_n x^n}{n}\right).$$

Now $t(\{n\}_k)$ is the image under θ of the symmetric function $\mathcal{P}_n(e_k)$, where $\mathcal{P}_n$: QSym $\to$ QSym takes any monomial quasi-symmetric function $M_{t_1,t_2,\dots,t_p}$ to $M_{nt_1,\dots,nt_p}$. Then $T_n(x)$ is the image under $\theta\mathcal{P}_n$ of $E(x^n)$, and so can be written

$$\exp\left(\sum_{k\geq 1}\frac{(-1)^{k-1}t(nk)x^{kn}}{k}\right) = \exp\left(\sum_{k\geq 1}\frac{(-1)^{k-1}(1-2^{-nk})\zeta(nk)x^{kn}}{k}\right)$$

$$= \exp\left(\sum_{k\geq 1}\frac{(-1)^{k-1}\zeta(nk)x^{kn}}{k}\right)\exp\left(-\sum_{k\geq 1}\frac{(-1)^{k-1}\zeta(nk)x^{kn}}{k2^{kn}}\right) = \frac{Z_n(x)}{Z_n\left(\frac{x}{2}\right)}.$$

□

From this result we can deduce the identities in Eqs. (1.6) above, using the known results about multiple zeta values. From [5] we have

$$Z_{2m}(x) = \frac{1}{(i\pi x)^m}\prod_{j=1}^{m}\sin(e^{\frac{(2j-1)\pi i}{2m}}\pi x),$$

so it follows from Theorem 3.4 that

$$T_{2m}(x) = \prod_{j=1}^{m}\cos\left(e^{\frac{(2j-1)\pi i}{2m}}\frac{\pi x}{2}\right). \tag{3.5}$$

Hence

$$\begin{aligned}
T_2(x) &= \cos\left(e^{\frac{\pi i}{2}}\frac{\pi x}{2}\right) = \cosh\left(\frac{\pi x}{2}\right)\\
T_4(x) &= \cos\left(e^{\frac{\pi i}{4}}\frac{\pi x}{2}\right)\cos\left(e^{\frac{3\pi i}{4}}\frac{\pi x}{2}\right) = \frac{1}{2}\left[\cos\left(\frac{\pi x}{\sqrt{2}}\right) + \cosh\left(\frac{\pi x}{\sqrt{2}}\right)\right]\\
T_6(x) &= \cos\left(e^{\frac{\pi i}{6}}\frac{\pi x}{2}\right)\cos\left(e^{\frac{\pi i}{2}}\frac{\pi x}{2}\right)\cos\left(e^{\frac{5\pi i}{6}}\frac{\pi x}{2}\right) =\\
&\quad \frac{1}{4}\left[1 + \cos\left(e^{\frac{\pi i}{6}}\pi x\right) + \cos\left(e^{\frac{\pi i}{2}}\pi x\right) + \cos\left(e^{\frac{5\pi i}{6}}\pi x\right)\right],
\end{aligned}$$

from which Eqs. (1.6) follow.

For $m = 4$ we have

$$\begin{aligned}
T_8(x) = \prod_{j=1}^{4}\cos\left(e^{\frac{(2j-1)\pi i}{8}}\frac{\pi x}{2}\right) =&\\
&\frac{1}{8}\left[\Phi(\alpha\pi x) + \Phi(\beta\pi x) + \Phi(e^{\frac{\pi i}{4}}\alpha\pi x) + \Phi(e^{\frac{\pi i}{4}}\beta\pi x)\right],
\end{aligned}$$

where $\alpha = \sqrt{1+\frac{1}{\sqrt{2}}}$, $\beta = \sqrt{1-\frac{1}{\sqrt{2}}}$, and $\Phi(x) = \cos x + \cosh x$. From this follows

$$t(\{8\}_k) = \frac{\pi^{8k}}{2^{2k+1}(8k)!}[(3+2\sqrt{2})^{2k} + (3-2\sqrt{2})^{2k}]. \tag{3.6}$$

An equivalent identity is given by C-L. Chung [8]. Eq. (3.6) may be compared to the corresponding formula from [5]:

$$\zeta(\{8\}_k) = \frac{8(2\pi)^{8k}}{2^{2k+1}(8k+4)!}[(3+2\sqrt{2})^{2k+1} + (3-2\sqrt{2})^{2k+1}].$$

For $m = 5$ we have

$$T_{10}(x) = \prod_{j=1}^{5} \cos\left(\rho^{2j-1}\frac{\pi x}{2}\right) = \frac{1}{16}\left[1 + \sum_{j=1}^{5}\left[\cos(\rho^{2j-1}\pi x) + \cos(\rho^{2j-1}\sigma\pi x) + \cos(\rho^{2j-1}\tau\pi x)\right]\right],$$

where $\rho = e^{\frac{\pi i}{10}}$, $\sigma = \frac{1}{2}(\sqrt{5}-1)$, and $\tau = \frac{1}{2}(\sqrt{5}+1)$. From this it follows that

$$t(\{10\}_k) = \frac{5\pi^{10k}(L_{10k}+1)}{16(10k)!}, \tag{3.7}$$

where L_n is the nth Lucas number. This corresponds to the result of [5] that

$$\zeta(\{10\}_k) = \frac{10(2\pi)^{10k}(L_{10k+5}+1)}{(10k+5)!}.$$

For $m = 6$ we have

$$T_{12}(x) = \prod_{j=1}^{6} \cos\left(e^{\frac{(2j-1)\pi i}{12}}\frac{\pi x}{2}\right) = \frac{1}{16}\left[1 + \Phi(\xi\pi x) + \Phi(\xi^3\pi x) + \Phi(\xi^5\pi x)\right] + \frac{1}{32}\sum_{j=0}^{2}\left[\Phi(\xi^{2j}\sqrt{2}\pi x) + \Phi(\xi^{2j}\gamma\pi x) + \Phi(\xi^{2j}\delta\pi x)\right],$$

where $\xi = e^{\frac{\pi i}{12}}$, $\gamma = \sqrt{2+\sqrt{3}}$, $\delta = \sqrt{2-\sqrt{3}}$, and Φ is as above. From this follows

$$t(\{12\}_k) = \frac{3\pi^{12k}}{16(12k)!}[(2+\sqrt{3})^{6k} + (2-\sqrt{3})^{6k} + 2^{6k} + (-1)^k 2]. \tag{3.8}$$

This may be compared to the corresponding result in [5]:

$$\zeta(\{12\}_k) = \frac{12(2\pi)^{12k}}{(12k+6)!}[(2+\sqrt{3})^{6k+3} + (2-\sqrt{3})^{6k+3} + 2^{6k+3}].$$

Z. Shen and L. Jia [27, Thm. 2] establish a general result for $t(\{2m\}_n)$ that is somewhat less explicit than the cases given here.

Multiple t-star values are defined by Eq. (1.7) above. We have the following result, which corresponds to [11, Thm. 2.1]. The notation is as in Theorem 3.2 above.

Theorem 3.5. *Let $i_1, \ldots, i_k$ be integers all 2 or greater. If the symmetric group S_k acts on strings of length k by permutation, then*

$$\sum_{\sigma \in S_k} t^\star(\sigma \cdot (i_1, \ldots, i_k)) = \sum_{\mathcal{B}=\{B_1,\ldots,B_l\} \in \Pi_k} c(\mathcal{B}) \prod_{s=1}^{l} t\left(\sum_{j \in B_s} i_j\right).$$

Proof. See [15, Thm. 4.1]. □

If we take $i_1 = i_2 = \cdots = i_k = n$ in this theorem we get a formula for $t^\star(\{n\}_k)$, $n \geq 2$, comparable to Corollary 3.1 above.

Corollary 3.2. *If $n \geq 2$, then*

$$t^\star(\{n\}_k) = \sum_{\lambda \vdash n} \frac{1}{m_1(\lambda)1^{m_1(\lambda)} m_2(\lambda)2^{m_2(\lambda)} \cdots} \prod_{j=1}^{\ell(\lambda)} t(n\lambda_j).$$

As with Corollary 3.1, this result has an expression involving the symmetric functions. Let $Q_k(x_1, \ldots, x_k)$ be the polynomial expressing the complete symmetric function h_k in terms of power sums $p_1, p_2, \ldots, p_k$. Then

$$t^\star(\{n\}_k) = Q_k(t(n), t(2n), \ldots, t(kn)).$$

We can extend $t^\star$ to any string of positive integers using Theorem 3.3. Then there is an analogue of Eq. (3.4) for $t^\star$.

$$t^\star(\{1\}_k) = \sum_{\lambda \vdash k} \frac{1}{m_1(\lambda)!1^{m_1(\lambda)} m_2(\lambda)!2^{m_2(\lambda)} \cdots} t(\lambda_1) \cdots t(\lambda_l).$$

We can also express the numbers $t^\star(\{n\}_k)$ using generating functions. Here we have a result similar to Theorem 3.4.

Theorem 3.6. *For integers $n \geq 2$,*

$$1 + \sum_{k=1}^{\infty} t^\star(\{n\}_k) x^{kn} = \frac{Z_n(e^{\frac{\pi i}{n}} \frac{x}{2})}{Z_n(e^{\frac{\pi i}{n}} x)}.$$

Proof. We note that $t^\star(\{n\}_k)$ is the image under $\theta\mathcal{P}_n$ of h_k, so that

$$1 + \sum_{k=1}^{\infty} t^\star(\{n\}_k) x^{kn} = \theta\mathcal{P}_n(H(x)) = \exp\left(\sum_{k\geq 1} \frac{t(nk)x^{kn}}{k}\right)$$
$$= \exp\left(\sum_{k\geq 1} \frac{(1-2^{-nk})\zeta(nk)x^{kn}}{k}\right).$$

Now proceed as in the proof of Theorem 3.4. □

In the case where n is an even integer $2m$, we can express $t^\star(\{n\}_k)$ as a rational multiple of π^{nk}. Similar results have been proved by Shen and Jia [27], and by Chung [8].

Theorem 3.7. *For positive integers m and k,*

$$t^\star(\{2m\}_k) = \frac{(-1)^{mk}}{(2mk)!} \left(\frac{\pi}{2}\right)^{2mk} \sum_{\substack{n_1+\cdots+n_m=mk \\ n_j\geq 0}} \binom{2mk}{2n_1 \cdots 2n_m} \prod_{j=1}^{m} e^{\frac{2\pi i}{m}(j-1)n_j} E_{2n_j}.$$

Proof. We follow Muneta's proof [21] of the corresponding result for multiple zeta-star values. Using the infinite product for cosine, we have

$$\sec\left(\pi e^{\frac{\pi i k}{m}} x\right) = \prod_{h=1}^{\infty} \left(1 - \frac{e^{\frac{2\pi i k}{m}}(2x)^2}{(2h-1)^2}\right)^{-1}.$$

From this follows

$$\prod_{k=0}^{m-1} \sec\left(\pi e^{\frac{\pi i k}{m}} x\right) = \prod_{h=1}^{\infty} \left(1 - \frac{(2x)^{2m}}{(2h-1)^{2m}}\right)^{-1} \tag{3.9}$$

since

$$(1 - e^{\frac{2\pi i}{m}} u)(1 - e^{\frac{4\pi i}{m}} u) \cdots (1 - e^{\frac{2(m-1)\pi i}{m}} u) = 1 - u^m.$$

Now the right-hand side of Eq. (3.9) can be expanded as

$$1+\sum_{h_1\ge 1}\frac{(2x)^{2m}}{(2h_1-1)^{2m}}+\sum_{h_1\ge h_2\ge 1}\frac{(2x)^{4m}}{(2h_1-1)^{2m}(2h_2-1)^{2m}}+\cdots = 1+\sum_{k=1}^{\infty} t^\star(\{2m\}_k)(2x)^{2km}. \tag{3.10}$$

On the other hand, using the Maclaurin series for secant we can expand the left-hand side of Eq. (3.9) as

$$\prod_{k=0}^{m-1}\left(\sum_{j=0}^{\infty}\frac{(-1)^j E_{2j}}{(2j)!}\pi^{2j}e^{\frac{2\pi ikj}{m}}x^{2j}\right) = \sum_{n=0}^{\infty}\sum_{j_0+\cdots+j_{m-1}=nm}\frac{(-1)^{nm}E_{j_0}E_{j_1}\cdots E_{j_{m-1}}}{(2j_0)!(2j_1)!\cdots(2j_{m-1})!}e^{\frac{2\pi i}{m}[j_1+2j_2+\cdots+(m-1)j_{m-1}]}(\pi x)^{2mn},$$

where we have used the fact that only powers of x^{2m} appear in the expansion. The latter expression can be written as

$$(-1)^{nm}\sum_{n=0}^{\infty}\frac{(\pi x)^{2nm}}{(2nm)!}\sum_{n_1+\cdots+n_m=nm}\binom{2nm}{2n_1\cdots 2n_m}\prod_{j=1}^{m}e^{\frac{2\pi i}{m}(j-1)n_j}E_{2n_j},$$

and comparing coefficients with Eq. (3.10) gives the conclusion. □

4. Multiple *t*-values and alternating multiple zeta values

Following [13], let $\mathfrak{E}_2$ be the underlying rational vector space of the non-commutative polynomial algebra on generators $z_{n,p}$, $n\in\{1,2,\dots\}$ and $p\in\{0,1\}$, with the product $*$ defined recursively by

$$aw_1 * bw_2 = a(w_1 * bw_2) + b(aw_1 * w_2) + (a\circ b)(w_1 * w_2) \tag{4.1}$$

for words w_1, w_2 and letters a, b. Here the operation $\circ$ is given by

$$z_{n_1,p_1}\circ z_{n_2,p_2} = z_{n_1+n_2,p_1+p_2},$$

where addition in the second subscript is taken mod 2. Then if $\mathfrak{E}_2^0$ is the subspace generated by all words that do not begin with $z_{1,0}$, $(\mathfrak{E}_2^0,*)$ is a subalgebra of $(\mathfrak{E}_2,*)$. There is a homomorphism $Z:\mathfrak{E}_2^0\to\mathbb{R}$ defined by

$$Z(z_{n_1,p_1}\cdots z_{n_k,p_k}) = \sum_{m_1>\cdots>m_k\ge 1}\frac{(-1)^{m_1p_1+\cdots+m_kp_k}}{m_1^{n_1}\cdots m_k^{n_k}}.$$

The series on the right-hand side of the preceding equation is an alternating or "colored" multiple zeta value. The usual notation for multiple zeta values can be extended to such quantities by using an upper bar, e.g., $\zeta(\bar{3}, 2, \bar{1})$ denotes $Z(z_{3,1}z_{2,0}z_{1,1})$.

For any $u \in \mathfrak{E}_2$ and monomial w of $\mathfrak{E}_2$, let $\operatorname{coeff}_u(w)$ denote the coefficient of w in u. Call an element $u \in \mathfrak{E}_2$ totally symmetric if

$$\operatorname{coeff}_u(z_{n_1,p_1} \cdots z_{n_k,p_k}) = \operatorname{coeff}_u(z_{n_1,0} \cdots z_{n_k,0})$$

for any monomial $z_{n_1,p_1} \cdots z_{n_k,p_k}$ of $\mathfrak{E}_2$, and totally antisymmetric if

$$\operatorname{coeff}_u(z_{n_1,p_1} \cdots z_{n_k,p_k}) = (-1)^{p_1+\cdots+p_k} \operatorname{coeff}_u(z_{n_1,0} \cdots z_{n_k,0}).$$

Let $\mathfrak{E}_2^S$ be the vector space of totally symmetric elements of $\mathfrak{E}_2$, and $\mathfrak{E}_2^A$ the vector space of totally antisymmetric elements.

Theorem 4.1. *$\mathfrak{E}_2^S$ and $\mathfrak{E}_2^A$ are subalgebras of $(\mathfrak{E}_2, *)$. Further, both are isomorphic to $(\mathfrak{H}^1, *)$.*

Proof. We use the result of [13] that $\mathfrak{E}_2$ is isomorphic to a subring of the power series ring $\mathbb{Q}[[t_1, t_2, \ldots]]$ via the map $\phi : \mathfrak{E}_2 \to \mathbb{Q}[[t_1, t_2, \ldots]]$ given by

$$\phi(z_{n_1,p_1} \cdots z_{n_k,p_k}) = \sum_{m_1 > \cdots > m_k \geq 1} (-1)^{m_1 p_1 + \cdots + m_k p_k} t_{m_1}^{n_1} \cdots t_{m_k}^{n_k}.$$

Now let $u \in \mathfrak{E}_2^S$, and consider $\phi(u) \in \mathbb{Q}[[t_1, t_2, \ldots]]$. For any monomial $z_{n_1,p_1} \cdots z_{n_k,p_k}$ occurring in u,

$$\begin{aligned} \operatorname{coeff}_{\phi(u)}(t_{m_1}^{n_1} \cdots t_{m_k}^{n_k}) &= \sum_{p_1=0}^{1} \cdots \sum_{p_k=0}^{1} (-1)^{m_1 p_1 + \cdots + m_k p_k} \operatorname{coeff}_u(z_{n_1,0} \cdots z_{n_k,0}) \\ &= \prod_{i=1}^{k} (1 + (-1)^{m_i}) \operatorname{coeff}_u(z_{n_1,0} \cdots z_{n_k,0}) \\ &= \begin{cases} 2^k \operatorname{coeff}_u(z_{n_1,0} \cdots z_{n_k,0}), & \text{if all the } m_i \text{ are even,} \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Thus ϕ sends $\mathfrak{E}_2^S$ to the subring of $\mathbb{Q}[[t_1, t_2, \ldots]]$ generated by the power series

$$\sum_{m_1>m_2>\cdots>m_k\geq 1,\ m_i \text{ even}} t_{m_1}^{n_1}\cdots t_{m_k}^{n_k}.$$

This is evidently isomorphic to QSym.

Now suppose $u \in \mathfrak{E}_2^A$. For any monomial $z_{n_1,p_1}\cdots z_{n_k,p_k}$ occurring in u, we have

$$\begin{aligned}
\operatorname{coeff}_{\phi(u)}(t_{m_1}^{n_1}\cdots t_{m_k}^{n_k}) &= \sum_{p_1=0}^{1}\cdots\sum_{p_k=0}^{1}(-1)^{m_1p_1+\cdots+m_kp_k}\operatorname{coeff}_u(z_{n_1,p_1}\cdots z_{n_k,p_k}) \\
&= \sum_{p_1=0}^{1}\cdots\sum_{p_k=0}^{1}(-1)^{(m_1+1)p_1+\cdots+(m_k+1)p_k}\operatorname{coeff}_u(z_{n_1,0}\cdots z_{n_k,0}) \\
&= \prod_{i=1}^{k}(1+(-1)^{m_i+1})\operatorname{coeff}_u(z_{n_1,0}\cdots z_{n_k,0}) \\
&= \begin{cases} 2^k\operatorname{coeff}_u(z_{n_1,0}\cdots z_{n_k,0}), & \text{if all the } m_i \text{ are odd},\\ 0, & \text{otherwise.}\end{cases}
\end{aligned}$$

Thus ϕ sends $\mathfrak{E}_2^A$ to the subring of $\mathbb{Q}[[t_1,t_2,\ldots]]$ generated by power series

$$\sum_{m_1>m_2>\cdots>m_k\geq 1,\ m_i \text{ odd}} t_{m_1}^{n_1}\cdots t_{m_k}^{n_k},$$

which is also isomorphic to QSym. □

We can define functions $S:\mathfrak{H}^1\to\mathfrak{E}_2^S$ and $A:\mathfrak{H}^1\to\mathfrak{E}_2^A$ by

$$\begin{aligned}
S(z_{n_1}\cdots z_{n_k}) &= \sum_{p_1,\ldots,p_k\in\{0,1\}} z_{n_1,p_1}\cdots z_{n_k,p_k}\\
A(z_{n_1}\cdots z_{n_k}) &= \sum_{p_1,\ldots,p_k\in\{0,1\}} (-1)^{p_1+\cdots+p_k} z_{n_1,p_1}\cdots z_{n_k,p_k}.
\end{aligned}$$

By the preceding proof we have

$$\phi S(z_{n_1}\cdots z_{n_k}) = 2^k \sum_{m_1>\cdots>m_k\geq 1,\ m_i \text{ even}} t_{m_1}^{n_1}\cdots t_{m_k}^{n_k} \tag{4.2}$$

and

$$\phi A(z_{n_1}\cdots z_{n_k}) = 2^k \sum_{m_1>\cdots>m_k\geq 1,\ m_i \text{ odd}} t_{m_1}^{n_1}\cdots t_{m_k}^{n_k}. \tag{4.3}$$

Now consider the homomorphism $\mathrm{ev} : \mathbb{Q}[[t_1, t_2, \dots]] \to \mathbb{R}$ sending t_j to $\frac{1}{j}$. Of course this doesn't make sense on all of $\mathbb{Q}[[t_1, t_2, \dots]]$, but by Eq. (4.3) it does send $\phi A(z_{n_1} \cdots z_{n_k})$ to $2^k t(n_k, \dots, n_1)$ if $n_k > 1$. Hence we have the following.

Corollary 4.1. *For positive integers $a_1, \dots, a_k$ with $a_1 \geq 2$,*

$$t(a_1, \dots, a_k) = \frac{1}{2^k} \sum_{\epsilon_1, \dots, \epsilon_k = \pm 1} \epsilon_1 \cdots \epsilon_k \zeta(\epsilon_1 \diamond a_1, \dots, \epsilon_k \diamond a_k),$$

where the sum is over the 2^k k-tuples $(\epsilon_1, \dots, \epsilon_k)$ with each $\epsilon_i \in \{1, -1\}$, and $\diamond$ is defined by $1 \diamond i = i$ and $-1 \diamond i = \bar{i}$.

For double t-values this is

$$t(a, b) = \frac{1}{4}(\zeta(a, b) - \zeta(a, \bar{b}) - \zeta(\bar{a}, b) + \zeta(\bar{a}, \bar{b})), \tag{4.4}$$

as stated in [17] and [23]. The preceding result expresses a t-value of depth k as a sum of 2^k alternating multiple zeta values. Actually one can do somewhat better: it is possible to write such a multiple t-value as a sum of 2^{k-1} alternating multiple zeta values as follows. We require a bit of additional notation. For $p \leq k$, let $L_p\zeta(i_1, \dots, i_k)$ be the sum of all $\binom{k}{p}$ alterating multiple zeta values in which the upper bar is applied to exactly p of the positive integers i_j, e.g.,

$$L_2\zeta(i_1, i_2, i_3) = \zeta(\bar{i}_1, \bar{i}_2, i_3) + \zeta(\bar{i}_1, i_2, \bar{i}_3) + \zeta(i_1, \bar{i}_2, \bar{i}_3).$$

Then we have the following result.

Corollary 4.2. *For positive integers $a_1, \dots, a_k$ with $a_1 \geq 2$,*

$$t(a_1, \dots, a_k) = \left(\frac{1}{2^{k-1}} - \frac{1}{2^{a_1 + \cdots + a_k}} \right) \zeta(a_1, \dots, a_k) + \frac{1}{2^{k-1}} \sum_{2 \leq p \leq k \text{ even}} L_p\zeta(a_1, \dots, a_k).$$

Proof. Using the notation just introduced, the previous corollary can be stated

$$2^{-k} \sum_{p=0}^{k} (-1)^p L_p\zeta(a_1, \dots, a_k) = t(a_1, \dots, a_k). \tag{4.5}$$

Now apply ev to Eq. (4.2) above to get

$$2^{-k}\sum_{p=0}^{k} L_p\zeta(a_1,\ldots,a_k) = \sum_{\substack{n_1>\cdots>n_k\geq 1\\ n_i \text{ even}}} \frac{1}{n_1^{a_1}\cdots n_k^{a_k}} = 2^{-a_1-\cdots-a_k}\zeta(a_1,\ldots,a_k),$$

which when added to Eq. (4.5) gives the conclusion. □

We note that already in the case $k=2$, Eq. (4.4) can be replaced by

$$t(a,b) = \left(\frac{1}{2} - \frac{1}{2^{a+b}}\right)\zeta(a,b) + \frac{1}{2}\zeta(\bar{a},\bar{b}).$$

The function $Z : \mathfrak{E}_2^0 \to \mathbb{R}$ can be written as the composition $\mathrm{ev}\,\phi R$, where R sends $z_{n_1,p_1}\cdots z_{n_k,p_k}$ to $z_{n_k,p_k}\cdots z_{n_1,p_1}$. (Note that ev makes sense on $\phi R(\mathfrak{E}_2^0)$.) This gives us the following result.

Theorem 4.2. *The image $Z(\mathfrak{E}_2^S \cap \mathfrak{E}_2^0) \subset \mathbb{R}$ is the set $\mathcal{Z}$ of rational linear combinations of multiple zeta values, and the image $Z(\mathfrak{E}_2^A \cap \mathfrak{E}_2^0) \subset \mathbb{R}$ is the set $\mathcal{T}$ of rational linear combinations of multiple t-values.*

5. A generating function

In this section we obtain a formula for the generating function of height one multiple t-values, i.e.,

$$H(x,y) = \sum_{i,j\geq 1} t(i+1,\{1\}_{j-1})x^i y^j = \sum_{i,j\geq 1} \theta(z_{i+1}z_1^{j-1})x^i y^j.$$

To this end we introduce some functions indexed by words as follows. For a nonempty word $w = z_{p_1}z_{p_2}\cdots z_{p_k}$ of $\mathfrak{H}^1$ and $r\in\mathbb{C}$, define

$$\mathcal{L}_r(w) = \sum_{j_1>\cdots>j_k\geq 0} \frac{r^{2j_1+1}}{(2j_1+1)^{p_1}\cdots(2j_k+1)^{p_k}}.$$

Then $\mathcal{L}_r(w)$ converges if $|r|<1$, and $\mathcal{L}_1(z_{p_1}\cdots z_{p_k}) = t(p_1,\ldots,p_k)$ if $p_1>1$. We have the following result.

Lemma 5.1. *For $k>1$,*

$$\frac{d}{dr}\mathcal{L}_r(z_{p_1}z_{p_2}\cdots z_{p_k}) = \begin{cases} \frac{1}{r}\mathcal{L}_r(z_{p_1-1}z_{p_2}\cdots z_{p_k}), & \text{if } p_1>1,\\ \frac{r}{1-r^2}\mathcal{L}_r(z_{p_2}\cdots z_{p_k}), & \text{if } p_1=1.\end{cases}$$

In the case $k=1$, we have

$$\frac{d}{dr}\mathcal{L}_r(z_{p_1}) = \begin{cases} \frac{1}{r}\mathcal{L}_r(z_{p_1-1}), & \text{if } p_1 > 1, \\ \frac{1}{1-r^2}, & \text{if } p_1 = 1. \end{cases}$$

Proof. Evidently

$$\frac{d}{dr}\mathcal{L}_r(z_{p_1}z_{p_2}\cdots z_{p_k}) = \sum_{j_1>\cdots>j_k\ge 0} \frac{r^{2j_1}}{(2j_1+1)^{p_1-1}\cdots(2j_k+1)^{p_k}},$$

which is clearly $\frac{1}{r}\mathcal{L}_r(z_{p_1-1}z_{p_2}\cdots z_{p_k})$ if $p_1>1$. Otherwise, it's

$$\sum_{j_2>j_3>\cdots>j_k\ge 0} \frac{r^{2(j_2+1)}+r^{2(j_2+2)}+\cdots}{(2j_2+1)^{p_2}\cdots(2j_k+1)^{p_k}} = \frac{r}{1-r^2}\mathcal{L}_r(z_{p_2}\cdots z_{p_k}).$$

The case $k=1$ follows since

$$\mathcal{L}_r(z_1) = r + \frac{r^3}{3} + \frac{r^5}{5} + \cdots = \int_0^r \frac{1}{1-t^2}dt. \qquad \square$$

We now obtain our formula for $H(x,y)$.

Theorem 5.1.

$$H(x,y) = {}_3F_2\left[\begin{matrix} \frac{1+y}{2}, \frac{1-x}{2}, 1 \\ \frac{3}{2}, \frac{3-x}{2} \end{matrix}; 1\right].$$

Proof. Define generating functions

$$H_r(x,y) = \sum_{i,j\ge 1} \mathcal{L}_r(z_{i+1}z_1^{j-1})x^iy^j$$

and

$$G_r(y) = \sum_{n\ge 1}\mathcal{L}_r(z_1^n)y^n = \mathcal{L}_r(z_1)y + \mathcal{L}_r(z_1^2)y^2 + \mathcal{L}_r(z_1^3)y^3 + \cdots.$$

By Lemma 5.1, the derivative of G_r with respect to r is

$$\frac{1}{1-r^2}y + \frac{r}{1-r^2}\mathcal{L}_r(z_1)y^2 + \frac{r}{1-r^2}\mathcal{L}_r(z_1^2)y^3 + \cdots = \frac{y}{1-r^2} + \frac{ry}{1-r^2}G_r(y).$$

Hence

$$\frac{dG_r}{dr} - \frac{ry}{1-r^2}G_r = \frac{y}{1-r^2},$$

or

$$\frac{d}{dr}\left((1-r^2)^{\frac{y}{2}}G_r\right) = \frac{y}{1-r^2}(1-r^2)^{\frac{y}{2}} = y(1-r^2)^{\frac{y}{2}-1},$$

and thus

$$G_r = (1-r^2)^{-\frac{y}{2}}\int_0^r y(1-s^2)^{\frac{y}{2}-1}ds = y(1-r^2)^{-\frac{y}{2}}\frac{r}{2}\int_0^1 (1-r^2u)^{\frac{y}{2}-1}u^{-\frac{1}{2}}du$$
$$= ry(1-r^2)^{-\frac{y}{2}}{}_2F_1\left[\begin{matrix}1-\frac{y}{2},\frac{1}{2}\\ \frac{3}{2}\end{matrix};r^2\right],$$

where we used Euler's integral formula for ${}_2F_1$. Then G_r can be written (via [25, 15.16.1])

$$ry\,{}_2F_1\left[\begin{matrix}\frac{y}{2},\frac{1}{2}\\ \frac{1}{2}\end{matrix};r^2\right]{}_2F_1\left[\begin{matrix}1-\frac{y}{2},\frac{1}{2}\\ \frac{3}{2}\end{matrix};r^2\right] = y\sum_{s=0}^{\infty}\frac{\left(\frac{y}{2}+\frac{1}{2}\right)\cdots\left(\frac{y}{2}-\frac{1}{2}+s\right)}{\left(\frac{1}{2}+1\right)\cdots\left(\frac{1}{2}+s\right)}r^{2s+1}.$$

Again using Lemma 5.1,

$$\frac{dH_r}{dr} = \frac{1}{r}\sum_{i,j\geq 1}\mathcal{L}_r(z_iz_1^{j-1})x^iy^j = \frac{x}{r}G_r + \frac{x}{r}H_r$$

or

$$\frac{dH_r}{dr} - \frac{x}{r}H_r = xy\sum_{s=0}^{\infty}\frac{\left(\frac{y}{2}+\frac{1}{2}\right)\cdots\left(\frac{y}{2}-\frac{1}{2}+s\right)}{\left(\frac{1}{2}+1\right)\cdots\left(\frac{1}{2}+s\right)}r^{2s}.$$

It follows that

$$\frac{d}{dr}(r^{-x}H_r) = xy\sum_{s=0}^{\infty}\frac{\left(\frac{y}{2}+\frac{1}{2}\right)\cdots\left(\frac{y}{2}-\frac{1}{2}+s\right)}{\left(\frac{1}{2}+1\right)\cdots\left(\frac{1}{2}+s\right)}r^{2s-x}$$

and thus

$$H_r = xy\sum_{s=0}^{\infty}\frac{\left(\frac{y}{2}+\frac{1}{2}\right)\cdots\left(\frac{y}{2}-\frac{1}{2}+s\right)}{\left(\frac{1}{2}+1\right)\cdots\left(\frac{1}{2}+s\right)}\frac{r^{2s+1}}{2s-x+1} =$$
$$\frac{xyr}{1-x}\sum_{s=0}^{\infty}\frac{\left(\frac{1}{2}+\frac{y}{2}\right)_s\left(\frac{1}{2}-\frac{x}{2}\right)_s}{\left(\frac{3}{2}\right)_s\left(\frac{3}{2}-\frac{x}{2}\right)_s}r^{2s} = \frac{xyr}{1-x}{}_3F_2\left[\begin{matrix}\frac{1+y}{2},\frac{1-x}{2},1\\ \frac{3}{2},\frac{3-x}{2}\end{matrix};r^2\right]$$

where $(x)_n$ means $x(x+1)\cdots(x+n-1)$. Now set $r=1$ to obtain the conclusion. □

The coefficient of xy in the definition of $H(x,y)$ is $t(2)$, while from Theorem 5.1 it can be seen to be ${}_3F_2\left(\frac{1}{2},\frac{1}{2},1;\frac{3}{2},\frac{3}{2};1\right)$, so these must be equal. In fact this equality generalizes, but first we need a lemma. (As above, $\{a\}_n$ means n repetitions of a.)

Lemma 5.2. *If*

$$\frac{x}{1-x}{}_3F_2\left[\begin{matrix}a, b, \frac{1-x}{2} \\ c, \frac{3-x}{2}\end{matrix};1\right] = \sum_{j=1}^{\infty} p_j x^j,$$

then

$$p_n = {}_{n+2}F_{n+1}\left[\begin{matrix}a, b, \left\{\frac{1}{2}\right\}_n \\ c, \left\{\frac{3}{2}\right\}_n\end{matrix};1\right].$$

Proof. Note that

$$\begin{aligned}\frac{x}{1-x}{}_3F_2\left[\begin{matrix}a, b, \frac{1-x}{2} \\ c, \frac{3-x}{2}\end{matrix};1\right] &= \frac{x}{1-x}\sum_{s=0}^{\infty}\frac{(a)_s(b)_s\left(\frac{1-x}{2}\right)_s}{(c)_s\left(\frac{3-x}{2}\right)_s}\frac{1}{s!} \\ &= x\sum_{s=0}^{\infty}\frac{(a)_s(b)_s}{(c)_s s!}\frac{1}{2s+1-x},\end{aligned}$$

whose nth derivative is seen to be

$$n!\sum_{s=0}^{\infty}\frac{(a)_s(b)_s}{(c)_s s!}\frac{1}{(2s+1-x)^n} + n!x\sum_{s=0}^{\infty}\frac{(a)_s(b)_s}{(c)_s s!}\frac{1}{(2s+1-x)^{n+1}}.$$

Then

$$p_n = \frac{1}{n!}\frac{d^n}{dx^n}\bigg|_{x=0}\frac{x}{1-x}{}_3F_2\left[\begin{matrix}a, b, \frac{1-x}{2} \\ c, \frac{3-x}{2}\end{matrix};1\right] = \sum_{s=0}^{\infty}\frac{(a)_s(b)_s}{(c)_s s!}\frac{1}{(2s+1)^n},$$

and the conclusion follows since $\left(\frac{1}{2}\right)_s / \left(\frac{3}{2}\right)_s = \frac{1}{2s+1}$. □

Now we can deduce the following corollary of Theorem 5.1.

Corollary 5.1. *For $n \geq 2$, $t(n) = {}_{n+1}F_n\left(1, \left\{\frac{1}{2}\right\}_n; \left\{\frac{3}{2}\right\}_n; 1\right)$.*

Proof. Theorem 5.1 implies that

$$t(n) = \text{coefficient of } x^{n-1}y \text{ in } H(x,y) = \text{coefficient of } x^{n-1} \text{ in } \frac{x}{1-x}\,{}_3F_2\left[\begin{matrix}1, \frac{1}{2}, \frac{1-x}{2} \\ \frac{3}{2}, \frac{3-x}{2}\end{matrix}; 1\right].$$

Now apply Lemma 5.2 with $a = 1$, $b = \frac{1}{2}$, and $c = \frac{3}{2}$. □

Note also that

$$t(2, \{1\}_{n-1}) = \text{coefficient of } xy^n \text{ in } H(x,y) = \text{coefficient of } y^{n-1} \text{ in } {}_3F_2\left[\begin{matrix}1, \frac{1+y}{2}, \frac{1}{2} \\ \frac{3}{2}, \frac{3}{2}\end{matrix}; 1\right],$$

and setting $y = 1$ in Theorem 5.1 gives

$$\sum_{j=1}^{\infty} t(n, \{1\}_{j-1}) = \text{coefficient of } x^{n-1} \text{ in } \frac{x}{1-x}\,{}_3F_2\left[\begin{matrix}1, 1, \frac{1-x}{2} \\ \frac{3}{2}, \frac{3-x}{2}\end{matrix}; 1\right].$$

Applying Lemma 5.2 with $a = b = 1$ and $c = \frac{3}{2}$ to the latter gives the following.

Corollary 5.2. *For $n \geq 2$,*

$$\sum_{j=1}^{\infty} t(n, \{1\}_{j-1}) = {}_{n+1}F_n\left[\begin{matrix}1, 1, \left\{\frac{1}{2}\right\}_{n-1} \\ \left\{\frac{3}{2}\right\}_n\end{matrix}; 1\right].$$

In particular, for $n = 2$ we have

$$\sum_{j=1}^{\infty} t(2, \{1\}_{j-1}) = {}_3F_2\left[\begin{matrix}1, 1, \frac{1}{2} \\ \frac{3}{2}, \frac{3}{2}\end{matrix}; 1\right] = 2G, \tag{5.1}$$

where $G = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^2}$ is Catalan's constant (see [19, Eq. (7.4.4.183a)]).

Conjecture 2.1 above implies that $H(x,y) = e^{y \log 2} A(x,y)$, for $A(x,y)$ not depending on $\log 2$. Using the tables of Appendix A, it appears that

$$A(x,y) = t(2)xy + t(3)x^2y - \frac{t(3)}{2}xy^2 + t(4)x^3y - \left(\frac{37}{60}t(4) + \frac{1}{2}\zeta(\bar{3},1)\right)x^2y^2 + \left(\frac{11}{60}t(4) + \frac{1}{4}\zeta(\bar{3},1)\right)xy^3 + \cdots$$

through degree 4.

6. Alternating multiple t-values

We can define alternating multiple t-values in a way similar to alternating multiple zeta values: for the algebra $\mathfrak{E}_2^0$ defined in §3, there is a homomorphism $T : \mathfrak{E}_2^0 \to \mathbb{R}$ defined by

$$T(z_{n_1,p_1} \cdots z_{n_k,p_k}) = \sum_{m_1 > \cdots > m_k \geq 1} \frac{(-1)^{m_1 p_1 + \cdots + m_k p_k}}{(2m_1-1)^{n_1} \cdots (2m_k-1)^{n_k}}.$$

We write, e.g., $t(\bar{3},2,\bar{1})$ for $T(z_{3,1}z_{2,0}z_{1,1})$. Then $t(\bar{n}) = -\beta(n)$, where β is the Dirichlet beta function

$$\beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}, \ \mathrm{Re}(s) > 0.$$

In particular, $t(\bar{1}) = -\frac{\pi}{4}$ and $t(\bar{2}) = -G$. From the integral representation

$$t(\bar{1},1) = \int_0^1 \frac{x \tan^{-1} x}{1+x^2} dx$$

we have $t(\bar{1},1) = \frac{G}{2} - \frac{\pi}{8}\log 2$.

For any positive integer n, define a $\mathbb{Q}$-linear map $\hat{\mathcal{P}}_n : \mathfrak{H}^1 \to \mathfrak{E}_2^0$ by $\hat{\mathcal{P}}_n(1) = 1$ and

$$\hat{\mathcal{P}}_n(z_{i_1} \cdots z_{i_k}) = z_{ni_1,i_1} \cdots z_{ni_k,i_k},$$

where we recall that the second subscript is to be considered mod 2.

Proposition 6.1. *$\hat{\mathcal{P}}_n$ is a homomorphism.*

Proof. The key point to check is that the definition of $\hat{\mathcal{P}}_n$ is compatible with the recursive definitions (2.1) and (4.1) of the products in $\mathfrak{H}^1$ and $\mathfrak{E}_2$ respectively. Using induction on word length, we have, for words w, v of $\mathfrak{H}^1$,

$$\begin{aligned}
\hat{\mathcal{P}}_n(z_i w * z_j v) &= \hat{\mathcal{P}}_n(z_i(w * z_j v) + z_j(z_i w * v) + z_{i+j}(w * v)) \\
&= z_{ni,i}(\hat{\mathcal{P}}_n(w) * \hat{\mathcal{P}}_n(z_j v)) + z_{nj,j}(\hat{\mathcal{P}}_n(z_i w) * \hat{\mathcal{P}}_n(v)) \\
&+ z_{ni+nj,i+j}(\hat{\mathcal{P}}_n(w) * \hat{\mathcal{P}}_n(v)) \\
&= z_{ni,i}(\hat{\mathcal{P}}_n(w) * z_{nj,j}\hat{\mathcal{P}}_n(v)) + z_{nj,j}(z_{ni,i}\hat{\mathcal{P}}_n(w) * \hat{\mathcal{P}}_n(v)) \\
&+ z_{ni+nj,i+j}(\hat{\mathcal{P}}_n(w) * \hat{\mathcal{P}}_n(v)) \\
&= z_{ni,i}\hat{\mathcal{P}}_n(w) * z_{nj,j}\hat{\mathcal{P}}_n(v) \\
&= \hat{\mathcal{P}}_n(z_i w) * \hat{\mathcal{P}}_n(z_j v).
\end{aligned}$$

□

Then we have a homomorphism $\hat{\theta}_n = T\hat{\mathcal{P}}_n : \mathfrak{H}^1 \to \mathbb{R}$, and applying it to Eq. (3.1) above gives an analogue of Eq. (3.2) for alternating multiple t-values, i.e.,

$$t(\{\bar{n}\}_k) = P_k(t(\bar{n}), t(2n), t(\overline{3n}), t(4n), \dots),$$

and this holds for all positive integers n. For example, $t(\bar{1}, \bar{1}) = -\frac{\pi^2}{32}$, $t(\bar{2}, \bar{2}) = \frac{G^2}{2} - \frac{\pi^4}{192}$, and $t(\bar{3}, \bar{3}) = -\frac{\pi^6}{30720}$. In fact, we have an explicit formula for the generating function of the values $t(\{\bar{n}\}_k)$ when n is odd, providing a counterpart to Eq. (3.5) above (see also [5, Eq. (35)] for the alternating multiple zeta values).

Theorem 6.1. *For nonnegative integers m,*

$$1 + \sum_{k=1}^{\infty} t(\{\overline{2m+1}\}_k) x^{(2m+1)k} = \prod_{j=0}^{2m} \left(1 - (-1)^j \sin\left(e^{\frac{\pi j i}{2m+1}} \frac{\pi x}{2}\right)\right)^{\frac{1}{2}}.$$

Proof. As in §2, P, H, and E are the generating functions of power-sum, complete, and elementary symmetric functions respectively. Starting with

$$\frac{\pi}{4} \tan \frac{\pi x}{2} - \frac{\pi}{4} \sec \frac{\pi x}{2} = \sum_{k \geq 0 \text{ even}}^{\infty} t(\overline{k+1}) x^k + \sum_{k \geq 1 \text{ odd}}^{\infty} t(k+1) x^k$$

(which follows from Eq. (1.5) above and [2, Eq. (23.2.22)]) we have

$$\frac{\pi}{4} \sum_{j=0}^{2m} \left[e^{\frac{2\pi j i}{2m+1}} \tan\left(e^{\frac{2\pi j i}{2m+1}} \frac{\pi x}{2}\right) - e^{\frac{2\pi j i}{2m+1}} \sec\left(e^{\frac{2\pi j i}{2m+1}} \frac{\pi x}{2}\right) \right] =$$

$$(2m+1) \left[\sum_{k \geq 1 \text{ odd}}^{\infty} t(\overline{(2m+1)k}) x^{(2m+1)k-1} + \sum_{k \geq 2 \text{ even}}^{\infty} t((2m+1)k) x^{(2m+1)k-1} \right]$$

or

$$\frac{\pi}{4} \sum_{j=0}^{2m} \left[\eta^j \tan\left(\eta^j \frac{\pi x}{2}\right) - (-1)^j \eta^j \sec\left(\eta^j \frac{\pi x}{2}\right) \right] = (2m+1) x^{2m} \hat{\theta}_{2m+1} P(x^{2m+1})$$

for $\eta = e^{\frac{\pi i}{2m+1}}$. The integral of the left-hand side is

$$\frac{1}{2}\sum_{j=0}^{2m}\left[\log\left(\sec\left(\eta^j\frac{\pi x}{2}\right)\right)-(-1)^j\log\left(\sec\left(\eta^j\frac{\pi x}{2}\right)+\tan\left(\eta^j\frac{\pi x}{2}\right)\right)\right]$$
$$=\log\prod_{j=0}^{2m}\left(1+(-1)^j\sin\left(\eta^j\frac{\pi x}{2}\right)\right)^{-\frac{1}{2}}.$$

Since

$$\frac{d}{dx}\log H(x^{2m+1})=\frac{H'(x^{2m+1})}{H(x^{2m+1})}(2m+1)x^{2m}=(2m+1)x^{2m}P(x^{2m+1})$$

we have

$$\hat{\theta}_{2m+1}H(x^{2m+1})=\prod_{j=0}^{2m}\left(1+(-1)^j\sin\left(e^{\frac{\pi j i}{2m+1}}\frac{\pi x}{2}\right)\right)^{-\frac{1}{2}},$$

and the conclusion follows using $E(t)=H(-t)^{-1}$. □

Our result for $t(\{\bar{1}\}_k)$ is as follows (cf. [5, Eq. (62)] for $\zeta(\{\bar{1}\}_k)$).

Corollary 6.1. *For all positive integers k,*

$$t(\{\bar{1}\}_k)=(-1)^{\lfloor\frac{k+1}{2}\rfloor}\frac{\pi^k}{2^{2k}k!}.$$

Proof. From the preceding result

$$1+\sum_{k=1}^{\infty}t(\{\bar{1}\}_k)x^k=\sqrt{1-\sin\frac{\pi x}{2}}$$

so it suffices to show that

$$\sqrt{1-\sin z}=\sum_{n=0}^{\infty}(-1)^{\lfloor\frac{n+1}{2}\rfloor}\frac{z^n}{2^n n!}.$$

This can be seen by writing the left-hand side as

$$\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{2^{2n}(2n)!}-\sum_{n=1}^{\infty}\frac{(-1)^n z^{2n-1}}{2^{2n-1}(2n-1)!}=\cos\frac{z}{2}-\sin\frac{z}{2} \tag{6.1}$$

and then noting that the right-hand side of Eq. (6.1) squares to $1-\sin z$. □

We also have the following formula for $t(\{3\}_k)$.

Corollary 6.2. *For all positive integers k,*

$$t(\{\bar{3}\}_k) = (-1)^{\lfloor \frac{k+1}{2} \rfloor} \frac{3\pi^{3k}}{2^{3k+1}(3k)!}.$$

Proof. In view of Theorem 6.1 it suffices to show

$$\sqrt{(1-\sin x)(1+\sin(\omega x))(1-\sin(\omega^2 x))} = \frac{1}{2}\left[\cos x + \cos(\omega x) + \cos(\omega^2 x) - 1 + \sin x - \sin(\omega x) + \sin(\omega^2 x)\right]$$

for $\omega = e^{\frac{\pi i}{3}}$. To show this, first express the right-hand side as

$$2\cos\frac{x}{2}\cos\frac{\omega x}{2}\cos\frac{\omega^2 x}{2} - 1 + 2\sin\frac{x}{2}\sin\frac{\omega x}{2}\sin\frac{\omega^2 x}{2}$$

and square it. Now use double-angle formulas for cosine and sine to rewrite the result in terms of sine and cosine of x, ηx, and $\eta^2 x$, and compare to the square of the left-hand side: after cancellation, the two can be seen to be the same using the addition formula for cosine and the equality $1 + \omega^2 = \omega$. □

Appendix A. Multiple t-values of weight ≤ 7

$$t(2,1) = -\frac{1}{2}t(3) + t(2)\log 2$$

$$\begin{aligned}
t(3,1) &= -\frac{37}{60}t(4) - \frac{1}{2}\zeta(\bar{3},1) + t(3)\log 2 \\
t(2,2) &= \frac{1}{4}t(4) \\
t(2,1,1) &= \frac{11}{60}t(4) + \frac{1}{4}\zeta(\bar{3},1) - \frac{1}{2}t(3)\log 2 + \frac{1}{2}t(2)\log^2 2
\end{aligned}$$

$$\begin{aligned}
t(4,1) &= -\frac{1}{2}t(5) - \frac{1}{7}t(2)t(3) + t(4)\log 2 \\
t(3,2) &= -\frac{1}{2}t(5) + \frac{3}{7}t(2)t(3) \\
t(2,3) &= -\frac{1}{2}t(5) + \frac{4}{7}t(2)t(3) \\
t(3,1,1) &= -\frac{23}{248}t(5) + \frac{5}{21}t(2)t(3) - \frac{1}{2}\zeta(\bar{3},1,1) - \frac{37}{60}t(4)\log 2
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{2}\zeta(\bar{3},1)\log 2 + \frac{1}{2}t(3)\log^2 2\\
t(2,2,1) &= \frac{1}{8}t(5) - \frac{3}{14}t(2)t(3) + \frac{1}{4}t(4)\log 2\\
t(2,1,2) &= \frac{3}{4}t(5) - \frac{1}{2}t(2)t(3)\\
t(2,1,1,1) &= -\frac{35}{248}t(5) + \frac{2}{21}t(2)t(3) + \frac{1}{4}\zeta(\bar{3},1,1) + \frac{11}{60}t(4)\log 2\\
&+ \frac{1}{4}\zeta(\bar{3},1)\log 2 - \frac{1}{4}t(3)\log^2 2 + \frac{1}{6}t(2)\log^3 2
\end{aligned}
$$

$$
\begin{aligned}
t(5,1) &= -\frac{73}{84}t(6) + \frac{17}{98}t(3)^2 - \frac{1}{2}\zeta(\bar{5},1) + t(5)\log 2\\
t(4,2) &= -\frac{1}{7}t(6) + \frac{1}{7}t(3)^2\\
t(3,3) &= -\frac{1}{2}t(6) + \frac{1}{2}t(3)^2\\
t(2,4) &= \frac{11}{28}t(6) - \frac{1}{7}t(3)^2\\
t(4,1,1) &= \frac{45}{112}t(6) - \frac{31}{196}t(3)^2 + \frac{1}{4}\zeta(\bar{5},1) - \frac{1}{2}t(5)\log 2 - \frac{1}{7}t(2)t(3)\log 2\\
&+ \frac{1}{2}t(4)\log^2 2\\
t(3,2,1) &= \frac{37}{112}t(6) - \frac{33}{98}t(3)^2 + \frac{1}{4}\zeta(\bar{5},1) - \frac{1}{2}t(5)\log 2 + \frac{3}{7}t(2)t(3)\log 2\\
t(3,1,2) &= \frac{61}{168}t(6) - \frac{9}{28}t(3)^2\\
t(2,3,1) &= -\frac{2}{21}t(6) - \frac{3}{196}t(3)^2 - \frac{1}{2}t(2)\zeta(\bar{3},1) + \frac{1}{4}\zeta(\bar{5},1) - \frac{1}{2}t(5)\log 2\\
&+ \frac{4}{7}t(2)t(3)\log 2\\
t(2,1,3) &= \frac{27}{112}t(6) - \frac{5}{28}t(3)^2 + \frac{1}{2}t(2)\zeta(\bar{3},1)\\
t(2,2,2) &= \frac{1}{48}t(6)\\
t(3,1,1,1) &= \frac{5}{56}t(6) - \frac{27}{196}t(3)^2 + \frac{9}{16}\zeta(\bar{5},1) - \frac{1}{2}\zeta(\bar{3},1,1,1)\\
&- \frac{23}{248}t(5)\log 2 + \frac{5}{21}t(2)t(3)\log 2 - \frac{1}{2}\zeta(\bar{3},1,1)\log 2\\
&- \frac{37}{120}t(4)\log^2 2 - \frac{1}{4}\zeta(\bar{3},1)\log^2 2 + \frac{1}{6}t(3)\log^3 2
\end{aligned}
$$

$$t(2,2,1,1) = \frac{23}{336}t(6) - \frac{2}{49}t(3)^2 + \frac{1}{4}t(2)\zeta(\bar{3},1) - \frac{1}{16}\zeta(\bar{5},1) + \frac{1}{8}t(5)\log 2 - \frac{3}{14}t(2)t(3)\log 2 + \frac{1}{8}t(4)\log^2 2$$

$$t(2,1,2,1) = -\frac{11}{28}t(6) + \frac{121}{392}t(3)^2 - \frac{3}{8}\zeta(\bar{5},1) + \frac{3}{4}t(5)\log 2 - \frac{1}{2}t(2)t(3)\log 2$$

$$t(2,1,1,2) = -\frac{1}{16}t(6) + \frac{1}{8}t(3)^2 - \frac{1}{4}t(2)\zeta(\bar{3},1)$$

$$t(2,1,1,1,1) = -\frac{3}{112}t(6) + \frac{2}{49}t(3)^2 - \frac{3}{16}\zeta(\bar{5},1) + \frac{1}{4}\zeta(\bar{3},1,1,1) - \frac{35}{248}t(5)\log 2 + \frac{2}{21}t(2)t(3)\log 2 + \frac{1}{4}\zeta(\bar{3},1,1)\log 2 + \frac{11}{120}t(4)\log^2 2 + \frac{1}{8}\zeta(\bar{3},1)\log^2 2 - \frac{1}{12}t(3)\log^3 2 + \frac{1}{24}t(2)\log^4 2$$

$$t(6,1) = -\frac{1}{2}t(7) - \frac{1}{7}t(3)t(4) - \frac{1}{31}t(2)t(5) + t(6)\log 2$$

$$t(5,2) = -\frac{1}{2}t(7) + \frac{2}{7}t(3)t(4) + \frac{5}{31}t(2)t(5)$$

$$t(4,3) = -\frac{1}{2}t(7) + \frac{6}{7}t(3)t(4) - \frac{10}{31}t(2)t(5)$$

$$t(3,4) = -\frac{1}{2}t(7) + \frac{1}{7}t(3)t(4) + \frac{10}{31}t(2)t(5)$$

$$t(2,5) = -\frac{1}{2}t(7) - \frac{2}{7}t(3)t(4) + \frac{26}{31}t(2)t(5)$$

$$t(5,1,1) = \frac{682}{2159}t(7) - \frac{429}{2380}t(3)t(4) + \frac{146}{1581}t(2)t(5) + \frac{2}{17}\zeta(\bar{5},1,1) - \frac{7}{68}\zeta(\bar{3},3,1) - \frac{73}{84}t(6)\log 2 + \frac{17}{98}t(3)^2\log 2 - \frac{1}{2}\zeta(\bar{5},1)\log 2 + \frac{1}{2}t(5)\log^2 2$$

$$t(4,2,1) = \frac{19887}{34544}t(7) - \frac{1033}{1785}t(3)t(4) + \frac{27}{1054}t(2)t(5) + \frac{3}{17}\zeta(\bar{5},1,1) - \frac{1}{34}\zeta(\bar{3},3,1) - \frac{1}{7}t(6)\log 2 + \frac{1}{7}t(3)^2\log 2$$

$$t(4,1,2) = \frac{5}{8}t(7) - \frac{19}{28}t(3)t(4) + \frac{5}{62}t(2)t(5)$$

$$\begin{aligned}
t(3,3,1) &= -\frac{20229}{17272}t(7) + \frac{56}{85}t(3)t(4) + \frac{407}{1054}t(2)t(5) - \frac{1}{2}t(3)\zeta(\bar{3},1)\\
&\quad - \frac{21}{34}\zeta(\bar{5},1,1) + \frac{7}{68}\zeta(\bar{3},3,1) - \frac{1}{2}t(6)\log 2 + \frac{1}{2}t(3)^2\log 2\\
t(3,1,3) &= \frac{28865}{8636}t(7) - \frac{1973}{1020}t(3)t(4) - \frac{560}{527}t(2)t(5) + \frac{1}{2}t(3)\zeta(\bar{3},1)\\
&\quad + \frac{21}{17}\zeta(\bar{5},1,1) - \frac{7}{34}\zeta(\bar{3},3,1)\\
t(3,2,2) &= \frac{3}{16}t(7) + \frac{3}{28}t(3)t(4) - \frac{15}{62}t(2)t(5)\\
t(2,3,2) &= \frac{5}{8}t(7) - \frac{1}{2}t(2)t(5)\\
t(2,2,3) &= \frac{3}{16}t(7) + \frac{1}{7}t(3)t(4) - \frac{8}{31}t(2)t(5)\\
t(2,4,1) &= -\frac{6933}{34544}t(7) + \frac{2347}{7140}t(3)t(4) - \frac{265}{1054}t(2)t(5) - \frac{3}{17}\zeta(\bar{5},1,1)\\
&\quad + \frac{1}{34}\zeta(\bar{3},3,1) + \frac{11}{28}t(6)\log 2 - \frac{1}{7}t(3)^2\log 2\\
t(2,1,4) &= \frac{5}{8}t(7) + \frac{5}{28}t(3)t(4) - \frac{18}{31}t(2)t(5)
\end{aligned}$$

$$\begin{aligned}
t(4,1,1,1) &= -\frac{35117}{69088}t(7) + \frac{1801}{3570}t(3)t(4) - \frac{106}{1581}t(2)t(5) - \frac{5}{34}\zeta(\bar{5},1,1)\\
&\quad + \frac{9}{136}\zeta(\bar{3},3,1) + \frac{45}{112}t(6)\log 2 - \frac{31}{196}t(3)^2\log 2 + \frac{1}{4}\zeta(\bar{5},1)\log 2\\
&\quad - \frac{1}{4}t(5)\log^2 2 - \frac{1}{14}t(2)t(3)\log^2 2 + \frac{1}{6}t(4)\log^3 2\\
t(3,2,1,1) &= \frac{2373}{4064}t(7) - \frac{51}{140}t(3)t(4) - \frac{1}{6}t(2)t(5) + \frac{1}{4}t(3)\zeta(\bar{3},1)\\
&\quad + \frac{1}{4}\zeta(\bar{5},1,1) + \frac{37}{112}t(6)\log 2 - \frac{33}{98}t(3)^2\log 2 + \frac{1}{4}\zeta(\bar{5},1)\log 2\\
&\quad - \frac{1}{4}t(5)\log^2 2 + \frac{3}{14}t(2)t(3)\log^2 2\\
t(3,1,2,1) &= -\frac{77617}{69088}t(7) + \frac{3903}{4760}t(3)t(4) + \frac{207}{1054}t(2)t(5) - \frac{27}{68}\zeta(\bar{5},1,1)\\
&\quad + \frac{9}{136}\zeta(\bar{3},3,1) + \frac{61}{168}t(6)\log 2 - \frac{9}{28}t(3)^2\log 2\\
t(3,1,1,2) &= -\frac{126255}{69088}t(7) + \frac{13301}{14280}t(3)t(4) + \frac{365}{527}t(2)t(5) - \frac{1}{4}t(3)\zeta(\bar{3},1)\\
&\quad - \frac{21}{34}\zeta(\bar{5},1,1) + \frac{7}{68}\zeta(\bar{3},3,1)
\end{aligned}$$

$$\begin{aligned}
t(2,3,1,1) &= -\frac{8297}{69088}t(7) + \frac{152}{357}t(3)t(4) - \frac{2921}{12648}t(2)t(5) - \frac{1}{2}t(2)\zeta(\bar{3},1,1) \\
&\quad + \frac{1}{34}\zeta(\bar{5},1,1) + \frac{5}{136}\zeta(\bar{3},3,1) - \frac{2}{21}t(6)\log 2 - \frac{3}{196}t(3)^2\log 2 \\
&\quad - \frac{1}{2}t(2)\zeta(\bar{3},1)\log 2 + \frac{1}{4}\zeta(\bar{5},1)\log 2 - \frac{1}{4}t(5)\log^2 2 \\
&\quad + \frac{2}{7}t(2)t(3)\log^2 2 \\
t(2,1,3,1) &= \frac{43073}{69088}t(7) - \frac{11813}{7140}t(3)t(4) + \frac{1813}{2108}t(2)t(5) + t(2)\zeta(\bar{3},1,1) \\
&\quad + \frac{1}{4}t(3)\zeta(\bar{3},1) + \frac{27}{68}\zeta(\bar{5},1,1) - \frac{9}{136}\zeta(\bar{3},3,1) + \frac{27}{112}t(6)\log 2 \\
&\quad - \frac{5}{28}t(3)^2\log 2 + \frac{1}{2}t(2)\zeta(\bar{3},1)\log 2 \\
t(2,1,1,3) &= -\frac{126255}{69088}t(7) + \frac{2587}{1785}t(3)t(4) + \frac{1169}{4216}t(2)t(5) - \frac{1}{2}t(2)\zeta(\bar{3},1,1) \\
&\quad - \frac{1}{4}t(3)\zeta(\bar{3},1) - \frac{21}{34}\zeta(\bar{5},1,1) + \frac{7}{68}\zeta(\bar{3},3,1) \\
t(2,2,2,1) &= -\frac{1}{32}t(7) - \frac{3}{56}t(3)t(4) + \frac{15}{248}t(2)t(5) + \frac{1}{48}t(6)\log 2 \\
t(2,2,1,2) &= -\frac{15}{32}t(7) - \frac{3}{56}t(3)t(4) + \frac{53}{124}t(2)t(5) \\
t(2,1,2,2) &= -\frac{15}{32}t(7) - \frac{1}{14}t(3)t(4) + \frac{71}{248}t(2)t(5)
\end{aligned}$$

$$\begin{aligned}
t(3,1,1,1,1) &= \frac{15729}{17272}t(7) - \frac{1259}{2380}t(3)t(4) - \frac{431}{1581}t(2)t(5) + \frac{93}{272}\zeta(\bar{5},1,1) \\
&\quad + \frac{5}{136}\zeta(\bar{3},3,1) - \frac{1}{2}\zeta(\bar{3},1,1,1,1) + \frac{5}{56}t(6)\log 2 \\
&\quad - \frac{27}{196}t(3)^2\log 2 + \frac{9}{16}\zeta(\bar{5},1)\log 2 - \frac{1}{2}\zeta(\bar{3},1,1,1)\log 2 \\
&\quad - \frac{23}{496}t(5)\log^2 2 + \frac{5}{42}t(2)t(3)\log^2 2 - \frac{1}{4}\zeta(\bar{3},1,1)\log^2 2 \\
&\quad - \frac{37}{360}t(4)\log^3 2 - \frac{1}{12}\zeta(\bar{3},1)\log^3 2 + \frac{1}{24}t(3)\log^4 2 \\
t(2,2,1,1,1) &= \frac{5}{2032}t(7) - \frac{2}{21}t(3)t(4) + \frac{55}{744}t(2)t(5) + \frac{1}{4}t(2)\zeta(\bar{3},1,1) \\
&\quad - \frac{1}{16}\zeta(\bar{5},1,1) + \frac{23}{336}t(6)\log 2 - \frac{2}{49}t(3)^2\log 2 \\
&\quad + \frac{1}{4}t(2)\zeta(\bar{3},1)\log 2 - \frac{1}{16}\zeta(\bar{5},1)\log 2 + \frac{1}{16}t(5)\log^2 2
\end{aligned}$$

$$
\begin{aligned}
&- \frac{3}{28}t(2)t(3)\log^2 2 + \frac{1}{24}t(4)\log^3 2\\
t(2,1,2,1,1) =& \frac{341}{2032}t(7) + \frac{2}{7}t(3)t(4) - \frac{89}{248}t(2)t(5) - \frac{1}{4}t(2)\zeta(\bar{3},1,1)\\
&- \frac{1}{8}t(3)\zeta(\bar{3},1) - \frac{1}{16}\zeta(\bar{3},3,1) - \frac{11}{28}t(6)\log 2 + \frac{121}{392}t(3)^2\log 2\\
&- \frac{3}{8}\zeta(\bar{5},1)\log 2 + \frac{3}{8}t(5)\log^2 2 - \frac{1}{4}t(2)t(3)\log^2 2\\
t(2,1,1,2,1) =& \frac{13353}{34544}t(7) + \frac{419}{14280}t(3)t(4) - \frac{1529}{4216}t(2)t(5)\\
&- \frac{1}{4}t(2)\zeta(\bar{3},1,1) + \frac{21}{136}\zeta(\bar{5},1,1) - \frac{7}{272}\zeta(\bar{3},3,1)\\
&- \frac{1}{16}t(6)\log 2 + \frac{1}{8}t(3)^2\log 2 - \frac{1}{4}t(2)\zeta(\bar{3},1)\log 2\\
t(2,1,1,1,2) =& \frac{21989}{17272}t(7) - \frac{10093}{14280}t(3)t(4) - \frac{1715}{4216}t(2)t(5)\\
&+ \frac{1}{4}t(2)\zeta(\bar{3},1,1) + \frac{1}{8}t(3)\zeta(\bar{3},1) + \frac{21}{68}\zeta(\bar{5},1,1) - \frac{7}{136}\zeta(\bar{3},3,1)\\
t(2,1,1,1,1,1) =& -\frac{2407}{4064}t(7) + \frac{7}{30}t(3)t(4) + \frac{26}{93}t(2)t(5) - \frac{3}{16}\zeta(\bar{5},1,1)\\
&+ \frac{1}{4}\zeta(\bar{3},1,1,1,1) - \frac{3}{112}t(6)\log 2 + \frac{2}{49}t(3)^2\log 2\\
&- \frac{3}{16}\zeta(\bar{5},1)\log 2 + \frac{1}{4}\zeta(\bar{3},1,1,1)\log 2 - \frac{35}{496}t(5)\log^2 2\\
&+ \frac{1}{21}t(2)t(3)\log^2 2 + \frac{1}{8}\zeta(\bar{3},1,1)\log^2 2 + \frac{11}{360}t(4)\log^3 2\\
&+ \frac{1}{24}\zeta(\bar{3},1)\log^3 2 - \frac{1}{48}t(3)\log^4 2 + \frac{1}{120}t(2)\log^5 2
\end{aligned}
$$

Appendix B. Multiple *t*-values in terms of Saha elements

$$t(4) = 4t(2,2)$$

$$
\begin{aligned}
t(5) &= 7t(3,2) + 6t(2,1,2)\\
t(4,1) &= \frac{1}{2}t(3,2) - t(2,1,2) + 4t(2,2,1)\\
t(2,3) &= \frac{5}{2}t(3,2) + t(2,1,2)
\end{aligned}
$$

$$
\begin{aligned}
t(6) &= 48t(2,2,2)\\
t(5,1) &= -\frac{19}{9}t(3,1,2) - \frac{25}{9}t(2,2,2) + 7t(3,2,1) + 6t(2,1,2,1)
\end{aligned}
$$

$$
\begin{aligned}
t(4,2) &= -\frac{4}{9}t(3,1,2) + \frac{8}{9}t(2,2,2) \\
t(3,3) &= -\frac{14}{9}t(3,1,2) + \frac{28}{9}t(2,2,2) \\
t(2,4) &= \frac{4}{9}t(3,1,2) + \frac{100}{9}t(2,2,2) \\
t(4,1,1) &= \frac{1}{18}t(3,1,2) - \frac{173}{18}t(2,2,2) + 4t(2,1,1,2) + \frac{1}{2}t(3,2,1) \\
&\quad - t(2,1,2,1) + 4t(2,2,1,1) \\
t(2,3,1) &= -\frac{5}{6}t(3,1,2) - \frac{29}{6}t(2,2,2) + 2t(2,1,1,2) + \frac{5}{2}t(3,2,1) \\
&\quad + t(2,1,2,1) \\
t(2,1,3) &= -\frac{2}{9}t(3,1,2) + \frac{85}{9}t(2,2,2) - 2t(2,1,1,2)
\end{aligned}
$$

$$
\begin{aligned}
t(7) &= -\frac{211}{87}t(3,2,2) + \frac{62}{29}t(2,2,1,2) - \frac{152}{29}t(2,1,2,2) \\
t(6,1) &= \frac{4723}{174}t(3,2,2) + \frac{233}{29}t(2,2,1,2) + \frac{20}{29}t(2,1,2,2) + 48t(2,2,2,1) \\
t(5,2) &= \frac{1397}{348}t(3,2,2) + \frac{151}{58}t(2,2,1,2) + \frac{2}{29}t(2,1,2,2) \\
t(4,3) &= \frac{1861}{174}t(3,2,2) + \frac{151}{29}t(2,2,1,2) + \frac{4}{29}t(2,1,2,2) \\
t(3,4) &= \frac{797}{348}t(3,2,2) + \frac{127}{58}t(2,2,1,2) - \frac{6}{29}t(2,1,2,2) \\
t(2,5) &= -\frac{509}{174}t(3,3,2) + \frac{33}{29}t(2,2,1,2) - \frac{36}{29}t(2,1,2,2) \\
t(5,1,1) &= \frac{10405}{4176}t(3,2,2) + \frac{3119}{696}t(2,2,1,2) + \frac{167}{174}t(2,1,2,2) + 7t(3,1,1,2) \\
&\quad + 6t(2,1,1,1,2) - \frac{19}{9}t(3,1,2,1) - \frac{25}{9}t(2,2,2,1) + 7t(3,2,1,1) \\
&\quad + 6t(2,1,2,1,1) \\
t(4,2,1) &= -\frac{8735}{4176}t(3,2,2) - \frac{709}{696}t(2,2,1,2) - \frac{7}{174}t(2,1,2,2) - \frac{4}{9}t(3,1,2,1) \\
&\quad + \frac{8}{9}t(2,2,2,1) \\
t(4,1,2) &= -\frac{12235}{1392}t(3,2,2) - \frac{1081}{232}t(2,2,1,2) - \frac{11}{58}t(2,1,2,2) \\
t(3,3,1) &= \frac{1745}{1044}t(3,2,2) - \frac{197}{174}t(2,2,1,2) + \frac{2}{87}t(2,1,2,2) + 2t(3,1,1,2)
\end{aligned}
$$

$$-\frac{14}{9}t(3,1,2,1)+\frac{28}{9}t(2,2,2,1)$$

$$t(3,1,3)=-\frac{595}{1392}t(3,2,2)+\frac{127}{232}t(2,2,1,2)-\frac{3}{58}t(2,1,2,2)-2t(3,1,1,2)$$

$$t(2,3,2)=-\frac{295}{348}t(3,2,2)-\frac{93}{58}t(2,2,1,2)-\frac{2}{29}t(2,1,2,2)$$

$$t(2,2,3)=\frac{649}{464}t(3,2,2)+\frac{57}{232}t(2,2,1,2)-\frac{5}{58}t(2,1,2,2)$$

$$t(2,4,1)=\frac{11887}{2088}t(3,2,2)+\frac{269}{348}t(2,2,1,2)+\frac{11}{87}t(2,1,2,2)+\frac{4}{9}t(3,1,2,1)+\frac{100}{9}t(2,2,2,1)$$

$$t(2,1,4)=\frac{55}{48}t(3,2,2)-\frac{3}{8}t(2,2,1,2)-\frac{1}{2}t(2,1,2,2)$$

$$t(4,1,1,1)=-\frac{11503}{2088}t(3,2,2)-\frac{539}{348}t(2,2,1,2)-\frac{25}{174}t(2,1,2,2)+\frac{1}{2}t(3,1,1,2)-t(2,1,1,1,2)+\frac{1}{18}t(3,1,2,1)-\frac{173}{18}t(2,2,2,1)+4t(2,1,1,2,1)+\frac{1}{2}t(3,2,1,1)-t(2,1,2,1,1)+4t(2,2,1,1,1)$$

$$t(2,3,1,1)=-\frac{1355}{696}t(3,2,2)-\frac{31}{116}t(2,2,1,2)+\frac{9}{58}t(2,1,2,2)+\frac{5}{2}t(3,1,1,2)+t(2,1,1,1,2)-\frac{5}{6}t(3,1,2,1)-\frac{29}{6}t(2,2,2,1)+2t(2,1,1,2,1)+\frac{5}{2}t(3,2,1,1)+t(2,1,2,1,1)$$

$$t(2,1,3,1)=\frac{7787}{1044}t(3,2,2)+\frac{859}{174}t(2,2,1,2)+\frac{35}{87}t(2,1,2,2)+2t(2,1,1,1,2)-\frac{2}{9}t(3,1,2,1)+\frac{85}{9}t(2,2,2,1)-2t(2,1,1,2,1)$$

$$t(2,1,1,3)=-\frac{113}{174}t(3,2,2)-\frac{37}{29}t(2,2,1,2)-\frac{15}{29}t(2,1,2,2)-2t(2,1,1,1,2)$$

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