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Integrality of the LMOV invariants for framed unknot

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The Labastida-Marinõ-Ooguri-Vafa (LMOV) invariants are the open string BPS invariants which are expected to be integers based on the string duality conjecture from M-theory. Several explicit formulae of LMOV invariants for framed unknot have been obtained in the literature. In this paper, we present a unified method to deal with the integrality of such explicit formulae. Furthermore, we also prove the integrality of certain LMOV invariants for framed unknot in higher genera.

1. Introduction

Topological string amplitude is the generating function of Gromov-Witten invariants which are usually rational numbers according to their definitions [12]. In 1998, Gopakumar and Vafa [9, 10] found that topological string amplitude is also the generating function of a series of integer-valued invariants related to BPS counting in M-theory. Later, Ooguri and Vafa [26] extended the above result to open string case, the corresponding integer-valued invariants are named as OV invariants. Furthermore, the OV invariants are further refined by Labasitida, Mariño and Vafa in [20], then the resulted invariants are called LMOV invariants [21]. An expanded physicist's reconsideration of the GV and LMOV can be found in [4]. We refer to [31] for a brief review of the applications of these integrality structures of topological strings in mathematics.

The open string LMOV invariants have been studied in many papers, such as [18, 19, 20, 21, 3, 8, 16, 22, 23, 24, 13, 14]. Based on the large N duality of topological string and Chern-Simons theory [27, 11, 26], the open string LMOV invariants can be approached by investigating the colored HOMFLYPT invariants of the dual knots. For a knot $\mathcal{K}$, we use the notation $n_{\mu,g,Q}(\mathcal{K})$ to denote the LMOV invariants of genus g with a boundary type μ which is a partition of a positive integer, where Q is a parameter describing the dependence of the relative homology class of the dual Calabi-Yau geometry of the knot $\mathcal{K}$. We refer to Section 2 for more detailed definition of $n_{\mu,g,Q}(\mathcal{K})$.

In particular, when the knot $\mathcal{K}$ is a framed unknot U_τ with framing $\tau \in \mathbb{Z}$, we have different ways to compute its LMOV invariants according to string dualities which have been proved in this situation [17, 28, 5]. Then one can obtain several explicit formulae [8, 22] for the genus zero LMOV invariants $n_{\mu,0,Q}(U_\tau)$ of framed unknot U_τ. It turns out that these explicit formulae are certain combinations of the Möbius function and binomial numbers. Based on the integrality conjecture for LMOV invariants, these formulae will give integers. However, such an argument is not so obvious, a rigorous proof is required.

In this paper, we present a straightforward way to prove the integrality of these formulae. We use the notation $n_{m,l}(\tau)$ to denote the LMOV invariants $n_{(m),0,l-\frac{m}{2}}(U_\tau)$ of the framed unknot U_τ of genus 0, where $m \geq 1$ and $l \geq 0$ are two integers. We have the following explicit formula [25, 22] for $n_{m,l}(\tau)$.

For $b \geq 0$ and $a \in \mathbb{Z}$, we introduce the notation $\binom{a}{b}$ which is defined as follows

$$\binom{a}{b} = \begin{cases} 1, & b = 0, \\ \binom{a}{b}, & b \geq 1 \text{ and } a \geq 0, \\ (-1)^b \binom{-a+b-1}{b}, & b \geq 1 \text{ and } a < 0. \end{cases}$$

We define

$$c_{m,l}(\tau) = -\frac{(-1)^{m\tau+m+l}}{m^2}\binom{m}{l}\binom{m\tau+l-1}{m-1},$$

then

$$n_{m,l}(\tau) = \sum_{d|m,d|l} \frac{\mu(d)}{d^2} c_{\frac{m}{d},\frac{l}{d}}(\tau), \tag{1}$$

where $\mu(d)$ denotes the Möbius functions.

In Section 3, we prove that

Theorem 1.1. *For any $\tau \in \mathbb{Z}$, $m \geq 1, l \geq 0$, we have $n_{m,l}(\tau) \in \mathbb{Z}$.*

Remark 1.2. In fact, such form of the formula (1) is very general. For example, if we take some special values of l or τ, it will give the formulae in

[8] (cf. the formulae (1.4) and (1.5) in [8]):

(2)
$$b^-_{K_p,r} = -\frac{1}{r^2}\sum_{d|r}\mu(\frac{r}{d})\binom{3d-1}{d-1}, \qquad b^+_{K_p,r} = \frac{1}{r^2}\sum_{d|r}\mu(\frac{r}{d})\binom{(2|p|+1)d-1}{d-1}$$

for $p \leq -1$ and

(3)
$$\begin{aligned} b^-_{K_p,r} &= -\frac{1}{r^2}\sum_{d|r}\mu(\frac{r}{d})(-1)^{d+1}\binom{2d-1}{d-1}, \\ b^+_{K_p,r} &= \frac{1}{r^2}\sum_{d|r}\mu(\frac{r}{d})(-1)^{d}\binom{(2p+2)d-1}{d-1} \end{aligned}$$

for $p \geq 2$. The above formulae (2) and (3) are referred as the extremal BPS invariants of twist knots in [8]. Therefore, Theorem 1.1 implies the integrality of formulae (2) and (3) immediately. Moreover, the integrality of another special case of the formula (1) was also proved in [30].

Then, denoted by $n_{(m_1,m_2)}(\tau)$ the LMOV invariants $n_{(m_1,m_2),0,\frac{m_1+m_2}{2}}(U_\tau)$ of the framed unknot U_τ with $\mu = (m_1, m_2)$, $g = 0$ and $Q = \frac{m_1+m_2}{2}$, where $m_1 \geq m_2 \geq 1$, we obtain the following formula

(4)
$$\begin{aligned} n_{(m_1,m_2)}(\tau) = \frac{1}{m_1+m_2} & \sum_{d|m_1,d|m_2} \mu(d)(-1)^{(m_1+m_2)(\tau+1)/d} \\ & \cdot \binom{(m_1\tau+m_1)/d-1}{m_1/d}\binom{(m_2\tau+m_2)/d}{m_2/d}. \end{aligned}$$

From this expression, we know that $n_{(m_1,m_2)}(\tau) = n_{(m_2,m_1)}(\tau)$. With the similar method, in Section 4, we prove that

Theorem 1.3. *For any $\tau \in \mathbb{Z}$ and $m_1, m_2 \geq 1$, then $n_{(m_1,m_2)}(\tau) \in \mathbb{Z}$.*

Next, let $n_{m,g,Q}(\tau)$ be the LMOV invariants $n_{(m),g,Q}(U_\tau)$ of higher genus g with boundary condition $\mu = (m)$. We define the following generating function

$$g_m(q,a) = \sum_{g\geq 0}\sum_{Q\in\mathbb{Z}/2} n_{m,g,Q}(\tau) z^{2g-2} a^Q$$

where $z = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$.

Let

$$\mathcal{Z}_m(q,a) = (-1)^{m\tau} \sum_{|\nu|=m} \frac{1}{\mathfrak{z}_\nu} \frac{\{m\nu\tau\}}{\{m\}\{m\tau\}} \frac{\{\nu\}_a}{\{\nu\}}$$

where $\mathfrak{z}_\nu = |Aut(\nu)| \prod_{i=1}^{l(\nu)} \nu_i$ and $\{m\}$ denotes the quantum integer, see Section 2 for introduction of the above notations.

By the definition of LMOV invariants in Section 2, we obtain the following expression

$$g_m(q,a) = \sum_{d|m} \mu(d) \mathcal{Z}_{m/d}(q^d, a^d).$$

In Section 5, we prove that

Theorem 1.4. *For any $m \geq 1$, we have $g_m(q,a) \in z^{-2}\mathbb{Z}[z^2, a^{\pm\frac{1}{2}}]$, where $z = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$.*

Therefore, Theorem 1.4 implies that $n_{m,g,Q}(\tau) \in \mathbb{Z}$ and moreover $n_{m,g,Q}(\tau)$ vanishes for large g and Q.

2. LMOV invariants

2.1. Basic notations

We first introduce some basic notations. A partition λ is a finite sequence of positive integers $(\lambda_1, \lambda_2, ..)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots$. The length of λ is the total number of parts in λ and denoted by $l(\lambda)$. The weight of λ is defined by $|\lambda| = \sum_{i=1}^{l(\lambda)} \lambda_i$. The automorphism group of λ, denoted by $\mathrm{Aut}(\lambda)$, contains all the permutations that permute parts of λ by keeping it as a partition. Obviously, $\mathrm{Aut}(\lambda)$ has the order $|\mathrm{Aut}(\lambda)| = \prod_{i=1}^{l(\lambda)} m_i(\lambda)!$ where $m_i(\lambda)$ denotes the number of times that i occurs in λ. Define $\mathfrak{z}_\lambda = |\mathrm{Aut}(\lambda)| \prod_{i=1}^{\lambda} \lambda_i$.

In the following, we will use the notation $\mathcal{P}_+$ to denote the set of all the partitions of positive integers. Let 0 be the partition of 0, i.e. the empty partition. Define $\mathcal{P} = \mathcal{P}_+ \cup \{0\}$.

The power sum symmetric function of infinite variables $\mathbf{x} = (x_1, .., x_N, ..)$ is defined by $p_n(\mathbf{x}) = \sum_i x_i^n$. Given a partition λ, we define $p_\lambda(\mathbf{x}) =$

$\prod_{j=1}^{l(\lambda)} p_{\lambda_j}(\mathbf{x})$. The Schur function $s_\lambda(\mathbf{x})$ is determined by the Frobenius formula

$$s_\lambda(\mathbf{x}) = \sum_\mu \frac{\chi_\lambda(\mu)}{\mathfrak{z}_\mu} p_\mu(\mathbf{x}),$$

where χ_λ is the character of the irreducible representation of the symmetric group $S_{|\lambda|}$ corresponding to λ, we have $\chi_\lambda(\mu) = 0$ if $|\mu| \neq |\lambda|$. The orthogonality of character formula gives

$$\sum_\lambda \frac{\chi_\lambda(\mu)\chi_\lambda(\nu)}{\mathfrak{z}_\mu} = \delta_{\mu\nu}.$$

Let $n \in \mathbb{Z}$ and λ, μ, ν denote the partitions. We introduce the following notations

$$\{n\}_x = x^{\frac{n}{2}} - x^{-\frac{n}{2}}, \ \{\mu\}_x = \prod_{i=1}^{l(\mu)} \{\mu_i\}_x.$$

In particular, let $\{n\} = \{n\}_q$ and $\{\mu\} = \{\mu\}_q$.

2.2. LMOV invariants for framed knots

Although the LMOV invariants are determined by the integrality structure of topological open string partition function. Based on the large N duality of topological string and Chern-Simons theory [11, 26], one can also introduce the LMOV invariants through Chern-Simons theory of links/knots [18, 19, 20, 21].

Given a partition λ, we let $\kappa_\lambda = \sum_{i=1}^{l(\lambda)} \lambda_i(\lambda_i - 2i + 1)$. Let $\mathcal{K}_\tau$ be a knot with framing $\tau \in \mathbb{Z}$. The framed colored HOMFLYPT invariant of $\mathcal{K}_\tau$ is defined as follows

$$\mathcal{H}_\lambda(\mathcal{K}_\tau; q, a) = (-1)^{|\lambda|\tau} q^{\frac{\kappa_\lambda \tau}{2}} W_\lambda(\mathcal{K}_\tau; q, a),$$

where $W_\lambda(\mathcal{K}_\tau; q, a)$ is the ordinary (framing-independent) colored HOMFLYPT invariant of $\mathcal{K}_\tau$, we refer to [29] for the concrete definition of $W_\lambda(\mathcal{K}_\tau; q, a)$.

Let $\mathcal{Z}_\mu(\mathcal{K}_\tau; q, a) = \sum_\lambda \chi_\lambda(\mu)\mathcal{H}_\lambda(\mathcal{K}_\tau; q, a)$, the Chern-Simons partition function of $\mathcal{K}_\tau$ is defined by

$$Z_{CS}^{(S^3, \mathcal{K}_\tau)}(q, a, \mathbf{x}) = \sum_{\lambda \in \mathcal{P}} \mathcal{H}_\lambda(\mathcal{K}_\tau; q, a) s_\lambda(\mathbf{x}) = \sum_{\mu \in \mathcal{P}} \frac{\mathcal{Z}_\mu(\mathcal{K}_\tau; q, a)}{\mathfrak{z}_\mu} p_\mu(\mathbf{x}). \tag{5}$$

Then we define the functions $f_\lambda(\mathcal{K}_\tau; q, a)$ by

$$Z_{CS}^{(S^3,\mathcal{K}_\tau)}(q,a,\mathbf{x}) = \exp\left(\sum_{d=1}^{\infty}\frac{1}{d}\sum_{\lambda\in\mathcal{P}_+} f_\lambda(\mathcal{K}_\tau; q^d, a^d)s_\lambda(\mathbf{x}^d)\right).$$

Let $\hat{f}_\mu(\mathcal{K}_\tau; q, a) = \sum_\lambda f_\lambda(\mathcal{K}_\tau; q, a)M_{\lambda\mu}(q)^{-1}$, where

$$M_{\lambda\mu}(q) = \sum_\mu \frac{\chi_\lambda(C_\nu)\chi_\mu(C_\nu)}{\mathfrak{z}_\nu}\prod_{j=1}^{l(\nu)}(q^{\nu_j/2} - q^{-\nu_j/2}).$$

Denote $z = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$, then the LMOV conjecture for framed knot $\mathcal{K}_\tau$ stated that [25], for any $\mu \in \mathcal{P}_+$, there are integers $N_{\mu,g,Q}(\mathcal{K}_\tau)$ such that

$$\hat{f}_\mu(\mathcal{K}_\tau; q, a) = \sum_{g\geq 0}\sum_{Q\in\mathbb{Z}/2} N_{\mu,g,Q}(\mathcal{K}_\tau)z^{2g-2}a^Q \in z^{-2}\mathbb{Z}[z^2, a^{\pm\frac{1}{2}}].$$

Therefore,

$$\begin{aligned} g_\mu(\mathcal{K}_\tau; q, a) &= \sum_\nu \chi_\nu(\mu)\hat{f}_\nu(\mathcal{K}_\tau; q, a) \\ &= \sum_{g\geq 0}\sum_{Q\in\mathbb{Z}/2} n_{\mu,g,Q}(\mathcal{K}_\tau)z^{2g-2}a^Q \in z^{-2}\mathbb{Z}[z^2, a^{\pm\frac{1}{2}}]. \end{aligned}$$

where $n_{\mu,g,Q}(\mathcal{K}_\tau) = \sum_\nu \chi_\nu(\mu)N_{\nu,g,Q}(\mathcal{K}_\tau)$. These conjectural integers $n_{\mu,g,Q}(\mathcal{K}_\tau)$ (and $N_{\nu,g,Q}(\mathcal{K}_\tau)$) are referred to as the LMOV invariants in this paper. We refer to [8, 16, 13, 14] for another slightly different introduction of LMOV invariants which are referred to as OV invariants in [30]. The integrality of certain OV invariants for a large family of knots/links have been proved in [13, 14] recently by using the knots-quivers correspondence.

In order to get an explicit expression for $g_\mu(\mathcal{K}_\tau; q, a)$, we introduce $F_\mu(\mathcal{K}_\tau; q, a)$ through the following expansion formula

$$\log(Z_{CS}^{(S^3,\mathcal{K}_\tau)}(q,a,\mathbf{x})) = \sum_{\mu\in\mathcal{P}_+} F_\mu(\mathcal{K}_\tau; q, a)p_\mu(\mathbf{x}).$$

Then, by formula (5), we have

$$F_\mu(\mathcal{K}_\tau; q, a) = \sum_{n\geq 1}\sum_{\cup_{i=1}^n \nu^i = \mu}\frac{(-1)^{n-1}}{n}\prod_{i=1}^n \frac{\mathcal{Z}_{\nu^i}(\mathcal{K}_\tau; q, a)}{\mathfrak{z}_{\nu^i}}.$$

Remark 2.1. For two partitions ν^1 and ν^2, the notation $\nu^1 \cup \nu^2$ denotes the new partition obtained by combining all the parts in ν^1, ν^2. For example, if $\mu = (2,2,1)$, then the list of all the pairs (ν^1, ν^2) such that $\nu^1 \cup \nu^2 = (2,2,1)$ is

$$(\nu^1 = (2), \nu^2 = (2,1)),\ (\nu^1 = (2,1), \nu^2 = (2)),$$
$$(\nu^1 = (1), \nu^2 = (2,2)),\ (\nu^1 = (2,2), \nu^2 = (1)).$$

Finally, by using the Möbius inversion formula, we obtain

$$g_\mu(\mathcal{K}_\tau; q, a) = \mathfrak{z}_\mu \frac{1}{\{\mu\}} \sum_{d|\mu} \frac{\mu(d)}{d} F_{\mu/d}(\mathcal{K}_\tau; q^d, a^d). \tag{6}$$

Remark 2.2. In the above discussion, for the sake of brevity, we only consider the case of a framed knot, actually, the LMOV invariants can be defined for any framed link.

2.3. LMOV invariants for framed unknot U_τ

In the following, we only consider the case of a framed unknot U_τ. In this situation, the large N duality of topological string and Chern-Simons theory [25] has been proved in [17, 28]. Therefore, we can also compute the LMOV invariants for framed unknot U_τ through the open topological string theory.

We denote by $n_{m,l}(\tau)$ the LMOV invariants $n_{(m),0,l-\frac{m}{2}}(U_\tau)$ of the framed unknot U_τ, where $g = 0$ and $m \geq 1$, $l \geq 0$. According to the computations shown in [25] (or cf. pages 15–16 in [22]), the explicit closed formula for $n_{m,l}(\tau)$ is given by formula (1). The integrality of $n_{m,l}(\tau)$ is given by Theorem 1.1.

Then, denoted by $n_{(m_1,m_2)}(\tau)$ the LMOV invariants $n_{(m_1,m_2),0,\frac{m_1+m_2}{2}}(U_\tau)$ of the framed unknot U_τ with $\mu = (m_1, m_2)$, $g = 0$ and $Q = \frac{m_1+m_2}{2}$, where $m_1 \geq m_2 \geq 1$. By using the computations in open topological string theory (cf. pages 18–19 in [22]), we obtain the explicit closed formula for $n_{(m_1,m_2)}(\tau)$ which is given by formula (4). The integrality of $n_{(m_1,m_2)}(\tau)$ is given by Theorem 1.3.

Let $n_{m,g,Q}(\tau)$ be the LMOV invariants $n_{(m),g,Q}(U_\tau)$ of the framed unknot U_τ. We consider the following generating function

$$g_m(q,a) = \sum_{g\geq 0} \sum_{Q \in \mathbb{Z}/2} n_{m,g,Q}(\tau) z^{2g-2} a^Q$$

which can be computed by using formula (6). Considering the following function

$$\phi_{\mu,\nu}(x)=\sum_{\lambda}\chi_{\lambda}(\mu)\chi_{\lambda}(\nu)x^{\kappa_{\lambda}}.$$

By Lemma 5.1 in [3], for $d\in\mathbb{Z}_{+}$, we have

$$\phi_{(d),\nu}(x)=\frac{\{d\nu\}_{x^2}}{\{d\}_{x^2}}. \tag{7}$$

By the expression of colored HOMFLYPT invariant for unknot (cf. formula (4.6) in [21])

$$W_{\lambda}(U;q,a)=\sum_{\nu}\frac{\chi_{\lambda}(\nu)}{\mathfrak{z}_{\nu}}\frac{\{\nu\}_a}{\{\nu\}},$$

we obtain

$$\begin{aligned}\mathcal{Z}_{\mu}(U_{\tau};q,a)&=\sum_{\lambda}\chi_{\lambda}(\mu)\mathcal{H}_{\lambda}(U_{\tau};q,a)\\&=(-1)^{|\mu|\tau}\sum_{\lambda}\chi_{\lambda}(\mu)q^{\frac{\kappa_{\lambda}\tau}{2}}\sum_{\nu}\frac{\chi_{\lambda}(\nu)}{\mathfrak{z}_{\nu}}\frac{\{\nu\}_a}{\{\nu\}}\\&=(-1)^{|\mu|\tau}\sum_{\nu}\frac{1}{\mathfrak{z}_{\nu}}\phi_{\mu,\nu}(q^{\frac{\tau}{2}})\frac{\{\nu\}_a}{\{\nu\}}.\end{aligned}$$

In particular, for $\mu=(m)$ with $m\geq 1$, formula (7) implies that

$$\mathcal{Z}_{(m)}(U_{\tau};q,a)=(-1)^{m\tau}\sum_{|\nu|=m}\frac{1}{\mathfrak{z}_{\nu}}\frac{\{m\nu\tau\}}{\{m\tau\}}\frac{\{\nu\}_a}{\{\nu\}}.$$

For brevity, if we let

$$\mathcal{Z}_{m}(q,a)=\frac{1}{\{m\}}\mathcal{Z}_{(m)}(U_{\tau};q,a)=(-1)^{m\tau}\sum_{|\nu|=m}\frac{1}{\mathfrak{z}_{\nu}}\frac{\{m\nu\tau\}}{\{m\}\{m\tau\}}\frac{\{\nu\}_a}{\{\nu\}},$$

then formula (6) gives

$$g_m(q,a)=\sum_{d|m}\mu(d)\mathcal{Z}_{m/d}(q^d,a^d). \tag{8}$$

Theorem 1.4 shows that $g_m(q,a)\in z^{-2}\mathbb{Z}[z^2,a^{\pm\frac{1}{2}}]$ for any $m\geq 1$.

3. Proof of the Theorem 1.1

For nonnegative integer n and prime number p, we introduce the following function

$$f_p(n) = \prod_{i=1, p\nmid i}^{n} i = \frac{n!}{p^{[n/p]}[n/p]!}. \tag{9}$$

Given a positive integer k, throughout this paper, we use the notation $p^k \mid\mid n$ to denote that p^k divides n, but p^{k+1} does not.

Before giving the proof of Theorem 1.1, we first establish several useful lemmas.

Lemma 3.1. *For odd prime numbers p and $\alpha \geq 1$ or for $p = 2$, $\alpha \geq 2$, we have $p^{2\alpha} \mid f_p(p^\alpha n) - f_p(p^\alpha)^n$. For $p = 2, \alpha = 1$, $f_2(2n) \equiv (-1)^{[n/2]} \pmod 4$.*

Proof. With $\alpha \geq 2$ or $p > 2$, $p^{\alpha-1}(p-1)$ is even,

$$\begin{aligned}
&f_p(p^\alpha n) - f_p(p^\alpha(n-1))f_p(p^\alpha)\\
&= f_p(p^\alpha(n-1)) \left(\prod_{i=1,p\nmid i}^{p^\alpha} (p^\alpha(n-1)+i) - f_p(p^\alpha) \right)\\
&\equiv p^\alpha(n-1) f_p(p^\alpha(n-1)) f_p(p^\alpha) \left(\sum_{i=1,p\nmid i}^{p^\alpha} \frac{1}{i} \right) \pmod{p^{2\alpha}}\\
&\equiv p^\alpha(n-1) f_p(p^\alpha(n-1)) f_p(p^\alpha) \left(\sum_{i=1,p\nmid i}^{[p^\alpha/2]} (\frac{1}{i} + \frac{1}{p^\alpha - i}) \right) \pmod{p^{2\alpha}}\\
&\equiv p^\alpha(n-1) f_p(p^\alpha(n-1)) f_p(p^\alpha) \left(\sum_{i=1,p\nmid i}^{[p^\alpha/2]} \frac{p^\alpha}{i(p^\alpha - i)} \right) \equiv 0, \pmod{p^{2\alpha}}
\end{aligned}$$

Thus the first part of the Lemma 3.1 is proved by induction. For $p = 2, \alpha = 1$, the formula is straightforward. □

Lemma 3.2. *For odd prime number p and $m = p^\alpha a, l = p^\beta b$, $p \nmid a, p \nmid b$, $\alpha \geq 1, \beta \geq 0$, we have*

$$p^{2\alpha} \mid \binom{m}{l}\binom{m\tau + l - 1}{m-1} - \binom{\frac{m}{p}}{\frac{l}{p}}\binom{\frac{m\tau+l}{p} - 1}{\frac{m}{p} - 1}$$

where for $\beta = 0$, the second term is defined to be zero.

Proof.

$$
\begin{aligned}
&\binom{m}{l}\binom{m\tau+l-1}{m-1}-\binom{\frac{m}{p}}{\frac{l}{p}}\binom{\frac{m\tau+l}{p}-1}{\frac{m}{p}-1}\\
&=\binom{\frac{m}{p}}{\frac{l}{p}}\binom{\frac{m\tau+l}{p}-1}{\frac{m}{p}-1}\left(\frac{f_p(m)}{f_p(l)f_p(m-l)}\cdot\frac{f_p(m\tau+l)}{f_p(m)f_p(m(\tau-1)+l)}-1\right) \qquad (10)
\end{aligned}
$$

Write $\binom{\frac{m}{p}}{\frac{l}{p}}=\frac{m}{l}\binom{\frac{m}{p}-1}{\frac{l}{p}-1}$ and $\binom{\frac{m\tau+l}{p}-1}{\frac{m}{p}-1}=\frac{m}{m\tau+l}\binom{\frac{m\tau+l}{p}}{\frac{m}{p}}$, both are divisible by $p^{\max(\alpha-\beta,0)}$. Each element of $\{m,l,m-l,m\tau+l,m(\tau-1)+l\}$ is divisible by $p^{\min(\alpha,\beta)}$, so by Lemma 3.1,

$$
\frac{f_p(m)}{f_p(l)f_p(m-l)}\cdot\frac{f_p(m\tau+l)}{f_p(m)f_p(m(\tau-1)+l)}-1 \qquad (11)
$$

is divisible by $p^{2\min(\alpha,\beta)}$ (including the case $\beta=0$) in p-adic number field. Thus (10) is divisible by $p^{2\max(\alpha-\beta,0)+2\min(\alpha,\beta)}=p^{2\alpha}$. $\square$

Lemma 3.3. *For $m=2^\alpha a, l=2^\beta b, \alpha\geq 1, \beta\geq 0$,*

$$
2^{2\alpha}\mid(-1)^{m\tau+m+l}\binom{m}{l}\binom{m\tau+l-1}{m-1}-(-1)^{\frac{m\tau+m+l}{2}}\binom{\frac{m}{2}}{\frac{l}{2}}\binom{\frac{m\tau+l}{2}-1}{\frac{m}{2}-1},
$$

where the second term is set to zero for $\beta=0$.

Proof. For the case $\alpha\geq 2,\beta\geq 2$, both $m\tau+m+l$ and $(m\tau+m+l)/2$ are even, the Lemma is proved as in Lemma 3.2. For the case $\beta=0$, both $\binom{m}{l}$ and $\binom{m\tau+l-1}{m-1}$ are divisible by 2^α, and the Lemma is also proved. For remaining cases $\alpha>\beta=1$ or $\beta\geq\alpha=1$, we compute similarly as (10),

$$
\begin{aligned}
&(-1)^{m\tau+m+l}\binom{m}{l}\binom{m\tau+l-1}{m-1}-(-1)^{\frac{m\tau+m+l}{2}}\binom{\frac{m}{2}}{\frac{l}{2}}\binom{\frac{m\tau+l}{2}-1}{\frac{m}{2}-1}\\
&=\binom{\frac{m}{2}}{\frac{l}{2}}\binom{\frac{m\tau+l}{2}-1}{\frac{m}{2}-1}\\
&\quad\times\left(\frac{f_2(m)}{f_2(l)f_2(m-l)}\cdot\frac{f_2(m\tau+l)}{f_2(m(\tau-1)+l)f_2(m)}-(-1)^{\frac{m\tau+m+l}{2}}\right) \qquad (12)
\end{aligned}
$$

Both $\binom{\frac{m}{2}}{\frac{l}{2}}$ and $\binom{\frac{m\tau+l}{2}-1}{\frac{m}{2}-1}$ are divisible by $2^{\alpha-1}$, it suffices to prove that the third factor is divisible by 4, which is, by Lemma 3.1,

$$
(-1)^{[\frac{l}{4}]+[\frac{m-l}{4}]+[\frac{m\tau+l}{4}]+[\frac{m(\tau-1)+l}{4}]}-(-1)^{\frac{m\tau+m+l}{2}}. \pmod 4
$$

It is divisible by 4 if

$$[\frac{l}{4}]+[\frac{m-l}{4}]+[\frac{m\tau+l}{4}]+[\frac{m(\tau-1)+l}{4}]+\frac{m\tau+m+l}{2} \tag{13}$$

is even. Parity of (13) depends only on $\tau \pmod 2$. For $\tau=1$, (13) reduces to $[l/4]+[(m-l)/4]+[(m+l)/4]+[l/4]+l/2$. For $\tau=0$, it reduces to $[l/4]+[(m-l)/4]+[l/4]+[(l-m)/4]+(m+l)/2$. Both are obviously even.

□

Now, we can finish the proof of Theorem 1.1.

Proof. For a prime number $p \mid m$, write $m=p^{\alpha}a, p\nmid a$.

$$\begin{aligned}
n_{m,l}(\tau) &= \sum_{d|m,d|l}\frac{\mu(d)}{d^2}c_{\frac{m}{d},\frac{l}{d}}(\tau)\\
&= \frac{1}{m^2}\sum_{d|m,d|l}\mu(d)(-1)^{\frac{m\tau+m+l}{d}}\binom{\frac{m}{d}}{\frac{l}{d}}\binom{\frac{m\tau+l}{d}-1}{\frac{m}{d}-1}\\
&= \frac{1}{m^2}\sum_{d|m,d|l,p\nmid d}\mu(d)\left((-1)^{\frac{m\tau+m+l}{d}}\binom{\frac{m}{d}}{\frac{l}{d}}\binom{\frac{m\tau+l}{d}-1}{\frac{m}{d}-1}\right.\\
&\qquad\left.-(-1)^{\frac{m\tau+m+l}{dp}}\binom{\frac{m}{dp}}{\frac{l}{dp}}\binom{\frac{m\tau+l}{dp}-1}{\frac{m}{dp}-1}\right)
\end{aligned} \tag{14}$$

where for $pd\nmid l$, the second term of (14) is understood to be zero. For odd prime number p, $\frac{m\tau+m+l}{d}$ and $\frac{m\tau+m+l}{dp}$ have the same parity. Since $p^{\alpha} \mid\mid \frac{m}{d}$, $p^{2\alpha}$ divides the summand in (14) by Lemma 3.2. For $p=2$, it is divisible by $2^{2\alpha}$ by Lemma 3.3.

$p^{2\alpha}$ divides the sum in (14) for every $p^{\alpha} \mid\mid m$, thus $n_{m,l}$ is an integer. □

4. Proof of the Theorem 1.3

We introduce the following lemma first.

Lemma 4.1. *If $p^{\beta} \mid\mid (a,b), p^{\alpha} \mid a+b$, then $p^{\alpha-\beta}$ divides*

$$\binom{a\tau+a-1}{a}\binom{b\tau+b}{b}.$$

Proof. Power of prime p in $n!$ is $\sum_{k=1}^{\infty}[\frac{n}{p^k}]$. Apply this to the binomial coefficients to find that the power of p in $\binom{a\tau+a-1}{a}\binom{b\tau+b}{b}$ is

$$\begin{aligned}
&\sum_{i=1}^{\infty}\Big([\frac{a\tau+a-1}{p^i}]+[\frac{b\tau+b}{p^i}]\Big)-\Big([\frac{a\tau-1}{p^i}]+[\frac{b\tau}{p^i}]\Big)-\Big([\frac{a}{p^i}]+[\frac{b}{p^i}]\Big)\\
&\geq\sum_{i=1}^{\alpha}\Big((\frac{(a+b)(\tau+1)}{p^i}-1)-(\frac{(a+b)\tau}{p^i}-1)\Big)-\sum_{i=1}^{\beta}(\frac{a+b}{p^i})\\
&\quad-\sum_{i=\beta+1}^{\alpha}(\frac{a+b}{p^i}-1)\\
&=\alpha-\beta
\end{aligned}$$

where we use the fact that for $k \mid m+n+1, k>1$, $[m/k]+[n/k]=(m+n+1)/k-1$ and for $k \mid m+n, k\nmid m$, $[m/k]+[n/k]=(m+n)/k-1$. $\square$

Recall the definition of the function $f_p(n)$ given by formula (9). It is obvious that

$$f_p(p^\alpha k)\equiv f_p(p^\alpha)^k\equiv(-1)^k \pmod{p^\alpha} \tag{15}$$

Now, we can finish the proof of Theorem 1.3:

Proof. By definition,

$$\begin{aligned}
n_{(m_1,m_2)}(\tau)=&\frac{1}{m_1+m_2}\sum_{d|m_1,d|m_2}\mu(d)(-1)^{(m_1+m_2)(\tau+1)/d}\\
&\cdot\binom{(m_1\tau+m_1)/d-1}{m_1/d}\binom{(m_2\tau+m_2)/d}{m_2/d}
\end{aligned}\tag{16}$$

Let p be any prime divisor of m_1+m_2, $p^\alpha \mid\mid m_1+m_2$. We will prove p^α divides the summation in (16), thus m_1+m_2 also divides and n_{m_1,m_2} are integers.

If $p\nmid m_1$, each summand in (16) corresponds to $p\nmid d$, so $p^\alpha \mid (m_1+m_2)/d$ and $p\nmid m_1/d$. By Lemma 4.1 applied to $a=m_1/d, b=m_2/d$, p^α divides each summand and thus the summation.

If $p^\beta \mid\mid m_1, \beta\geq 1$, consider two summands in (16) corresponding to d and pd such that $pd \mid (m_1,m_2), \mu(pd)\neq 0$. When p is an odd prime or $\alpha\geq 2$, the sign $(-1)^{(m_1+m_2)(\tau+1)/d}$ and $(-1)^{(m_1+m_2)(\tau+1)/(pd)}$ are equal. When $p=2, \alpha=1$, modulo 2 the sign is irrelevant. Write $a=m_1/d, b=m_2/d$, then $p^\alpha \mid a+b, p^\beta \mid\mid a$.

$$
\begin{aligned}
&\binom{a\tau+a-1}{a}\binom{b\tau+b}{b}-\binom{(a\tau+a)/p-1}{a/p}\binom{(b\tau+b)/p}{b/p}\\
&=\binom{(a\tau+a)/p-1}{a/p}\binom{(b\tau+b)/p}{b/p}\left(\frac{f_p(a\tau+a)f_p(b\tau+b)}{f_p(a\tau)f_p(a)f_p(b\tau)f_p(b)}-1\right)\\
&=\binom{(a\tau+a)/p-1}{a/p}\binom{(b\tau+b)/p}{b/p}\\
&\quad\times\frac{f_p(a\tau+a)f_p(b\tau+b)-f_p(a\tau)f_p(a)f_p(b\tau)f_p(b)}{f_p(a\tau)f_p(a)f_p(b\tau)f_p(b)} \qquad (17)
\end{aligned}
$$

The term $\binom{(a\tau+a)/p-1}{a/p}\binom{(b\tau+b)/p}{b/p}$ is divisible by $p^{\alpha-\beta}$ by Lemma 4.1. The numerator of the fraction term in (17) is divisible by p^β by (15), and the denominator is not divisible by p. We proved that p^α divides (17), take summation over d, we get that p^α divides the summation in (16). This is true for any $p \mid m_1+m_2$, thus $n_{(m_1,m_2)}(\tau)$ is an integer. $\square$

5. Proof of the Theorem 1.4

We establish several lemmas first.

Lemma 5.1. *Suppose k is a positive integer, then the number*

$$c_m(k,y)=\sum_{|\lambda|=m}\frac{1}{\mathfrak{z}_\lambda}k^{l(\lambda)}\{\lambda\}_{y^2}$$

is equal to the coefficient of t^m in $(\frac{1-t/y}{1-ty})^k$.

Proof. Suppose the number of i's in the partition λ is $a_i, i=1,\cdots$. Then

$$
\begin{aligned}
c_m(k,y) &= \sum_{\sum ia_i=m}\prod_i\frac{1}{a_i!i^{a_i}}k^{a_i}(y^i-y^{-i})^{a_i}\\
&=\left[\prod_{i=1}^\infty\left(\sum_{j=0}^\infty t^{ij}\frac{1}{j!i^j}k^j(y^i-y^{-i})^j\right)\right]_{t^m}\\
&=\left[\prod_{i=1}^\infty\exp(t^ik(y^i-y^{-i})/i)\right]_{t^m}\\
&=\left[\exp(k\ln(1-ty)^{-1}+k\ln(1-t/y))\right]_{t^m}\\
&=\left[(\frac{1-t/y}{1-ty})^k\right]_{t^m}
\end{aligned}
$$

$\square$

Lemma 5.2. *Let* $R = \mathbb{Q}[q^{\pm 1/2}, a^{\pm 1/2}]$. *Then*

$$\{m\}\{m\tau\}g_m(q,a) = \sum_{d|m}\sum_{|\mu|=m/d} \frac{\mu(d)(-1)^{m\tau/d}}{\mathfrak{z}_\mu}\frac{\{m\mu\tau\}}{\{d\mu\}}\{d\mu\}_a \tag{18}$$

is divisible by $\{m\tau\}\{m\}/\{1\}^2$ *in* R.

Proof. By the definition (8) of $g_m(q,a)$, we have the formula (18). It is clear that

$$\{m\}\{m\tau\}g_m(q,a) \in R.$$

Denote $\Phi_n(q) = \prod_{d|n}(q^d-1)^{\mu(n/d)}$ to be the n-th cyclotomic polynomial, which is irreducible over R. Then $q^n - 1 = \prod_{d|n}\Phi_d(q)$, and

$$\begin{aligned}\{m\}\{m\tau\} &= q^{-\frac{m+m\tau}{2}} \prod_{m_1|m}\Phi_{m_1}(q)\prod_{m_1|m\tau}\Phi_{m_1}(q) \\ &= q^{-\frac{m+m\tau}{2}} \prod_{m_1|m}\Phi_{m_1}(q)^2 \prod_{m_1|m\tau, m_1\nmid m}\Phi_{m_1}(q)\end{aligned} \tag{19}$$

(i) For $m_1 \mid m\tau, m_1 \nmid m$, and any $|\mu| = m/d$, at least one of $d\mu_i$'s are not divisible by m_1, thus $\{m\mu_i\tau\}/\{d\mu_i\}$ is divisible by $\Phi_{m_1}(q)$. So $\Phi_{m_1}(q)$ divides $\{m\}\{m\tau\}g_m(q,a)$.

(ii) For $m_1 \mid m$ and any $|\mu| = m/d$, if not all $d\mu_i$ are divisible by m_1, then at least two of them are not divisible. Then two of corresponding $\{m\mu_i\tau\}/\{d\mu_i\}$ are divisible by $\Phi_{m_1}(q)$.

We consider modulo $\{m_1\}^2$ in the ring R. It is easy to see, for $a, b \geq 1$,

$$\frac{\{abm_1\}}{\{bm_1\}} \equiv a\left(\frac{q^{m_1/2}+q^{-m_1/2}}{2}\right)^{(a-1)b} \pmod{\{m_1\}^2}$$

We write $x = (q^{m_1/2}+q^{-m_1/2})/2$, then $x^2 \equiv 1 \pmod{\{m_1\}^2}$.

Then modulo $\Phi_{m_1}(q)^2$, we have

$$\begin{aligned}&\{m\}\{m\tau\}g_m(q,a) \\ &\equiv \sum_{d|m}\sum_{|\mu|=m/d, m_1|d\mu} \frac{\mu(d)(-1)^{m\tau/d}}{\mathfrak{z}_\mu}\frac{\{m\mu\tau\}}{\{d\mu\}}\{d\mu\}_a \\ &\equiv \sum_{d|m}\sum_{|\mu|=m/d, m_1|d\mu} \frac{\mu(d)(-1)^{m\tau/d}}{\mathfrak{z}_\mu}\left(\frac{m\tau}{d}\right)^{l(\mu)} x^{(m|\mu|\tau - d|\mu|)/m_1}\{d\mu\}_a\end{aligned}$$

$$\equiv \sum_{d|m} \sum_{|\lambda|=m/\mathrm{lcm}(d,m_1)} \mu(d)(-1)^{m\tau/d} x^{\frac{m}{m_1}(\frac{m\tau}{d}-1)}$$
$$\cdot \frac{1}{\mathfrak{z}_\lambda} \left(\frac{m\tau}{\mathrm{lcm}(d,m_1)}\right)^{l(\lambda)} \{\lambda\}_{a^{\mathrm{lcm}(d,m_1)}}$$
$$\equiv \sum_{d|m} \mu(d)(-1)^{m\tau/d} x^{\frac{m}{m_1}(\frac{m\tau}{d}-1)}$$
$$\times \left[\left(\frac{1-t^{\mathrm{lcm}(d,m_1)}a^{-\mathrm{lcm}(d,m_1)/2}}{1-t^{\mathrm{lcm}(d,m_1)}a^{\mathrm{lcm}(d,m_1)/2}}\right)^{m\tau/\mathrm{lcm}(d,m_1)}\right]_{t^m} \tag{20}$$

- For the cases m_1 with an odd prime factor p, or $p = 2$ divides m_1 and $4 \mid m$, or $p = 2$ divides m_1 and $2 \mid \tau$: Consider those d with $\mu(d) \neq 0$ and $p \nmid d$, we have $\mathrm{lcm}(d, m_1) = \mathrm{lcm}(pd, m_1)$ and parity of $m\tau/d$ equals parity of $m\tau/(pd)$, but $\mu(d) = -\mu(pd)$. Thus two terms in (20) corresponding to d and pd canceled.
- For the remaining case $2 \,\|\, m, m_1 = 2, 2 \nmid \tau$: $\Phi_{m_1}(q)^2 = (q^{1/2} + q^{-1/2})^2 = 2x + 2$. Coefficients of x in (20) equals sum of terms corresponds to odd $d \mid m, \mu(d) \neq 0$, while constant term coefficient equals to sum of terms corresponds to $2d \mid m, \mu(2d) \neq 0$. The coefficients of term for d and $2d$ match, so (20) is divisible by $x + 1$.

In summary, we have proved that for $m_1 \mid m\tau, m_1 \nmid m$, $\Phi_{m_1}(q)$ divides $\{m\}\{m\tau\}g_m(q,a)$; for $m_1 \mid m, m_1 \neq 1$, $\Phi_{m_1}(q)^2$ divides $\{m\}\{m\tau\}g_m(q,a)$. By (19), the lemma is proved. □

Lemma 5.3. *For any integer $m \geq 1$, we have*

$$g_m(q,a) \in z^{-2}\mathbb{Q}[z^2, a^{\pm\frac{1}{2}}].$$

Proof. By Lemma 5.2, we have

$$f(q,a) := z^2 g_m(q,a) = \frac{\{1\}^2}{\{m\}\{m\tau\}} \sum_{d|m} \sum_{|\mu|=m/d} \frac{\mu(d)(-1)^{m\tau/d}}{\mathfrak{z}_\mu} \frac{\{m\mu\tau\}}{\{d\mu\}} \{d\mu\}_a$$
$$\in \mathbb{Q}[q^{\pm\frac{1}{2}}, a^{\pm 1}].$$

As a function of q, it is clear $f(q,a)$ admits $f(q,a) = f(q^{-1},a)$. Furthermore, for any $d|m$ and $|\mu| = m/d$, we have

$$m|\mu|\tau - d|\mu| - m\tau - m \equiv m^2\tau/d - m\tau = m\tau(m/d - 1) \equiv 0 \pmod 2,$$

which implies $f(q,a) = f(-q,a)$. Therefore, $f(q,a) = z^2 g_m(q,a) \in \mathbb{Q}[z^2, a^{\pm\frac{1}{2}}]$. The lemma is proved. $\square$

Lemma 5.4. *For any $\tau \in \mathbb{Z}$, we have*

$$\{m\}\{m\tau\}\mathcal{Z}_m(q,a) \in \mathbb{Z}[q^{\pm\frac{1}{2}}, a^{\pm\frac{1}{2}}]. \tag{21}$$

Proof. Since

$$\begin{aligned}(-1)^{m\tau}\{m\}\{m\tau\}\mathcal{Z}_m(q,a) &= \sum_{|\mu|=m} \frac{\{m\tau\mu\}}{\mathfrak{z}_\mu\{\mu\}}\{\mu\}_a \\ &= \sum_{\sum_{j\geq 1} jk_j = m} \frac{\prod_{j\geq 1}(\{m\tau j\}\{j\}_a)^{k_j}}{\prod_{j\geq 1} j^{k_j} k_j!},\end{aligned}$$

we construct a generating function

$$\begin{aligned} f(x) &= \sum_{n\geq 0} x^n \sum_{\sum_{j\geq 1} jk_j = n} \frac{\prod_{j\geq 1}(\{m\tau j\}\{j\}_a)^{k_j}}{\prod_{j\geq 1} j^{k_j} k_j!} \\ &= \sum_{n\geq 0} \sum_{\sum_{j\geq 1} jk_j = n} \frac{\prod_{j\geq 1}(\{m\tau j\}\{j\}_a x^j)^{k_j}}{\prod_{j\geq 1} j^{k_j} k_j!} \\ &= \exp\left(\sum_{j\geq 1} \frac{\{m\tau j\}\{j\}_a x^j}{j\{j\}}\right), \end{aligned} \tag{22}$$

Then $(-1)^{m\tau}\{m\}\{m\tau\}\mathcal{Z}_m(q,a) = [f(x)]_{x^m}$.

For $\tau = 0$, it is the trivial case.

For $\tau \geq 1$, we use the expansion $\frac{\{m\tau j\}}{\{j\}} = \sum_{k=0}^{m\tau-1} q^{\frac{j(m\tau-2k-1)}{2}}$, then

$$\begin{aligned} f(x) &= \exp\left(\sum_{k\geq 0}^{m\tau-1}\sum_{j\geq 1}\left(\frac{(q^{\frac{m\tau-1-2k}{2}} a^{\frac{1}{2}} x)^j}{j} - \frac{(q^{\frac{m\tau-1-2k}{2}} a^{-\frac{1}{2}} x)^j}{j}\right)\right) \\ &= \exp\left(\sum_{k\geq 0}^{m\tau-1} \log \frac{1+q^{\frac{m\tau-1-2k}{2}} a^{-\frac{1}{2}} x}{1+q^{\frac{m\tau-1-2k}{2}} a^{\frac{1}{2}} x}\right) \\ &= \prod_{k=0}^{m\tau-1} \frac{1+q^{\frac{m\tau-1-2k}{2}} a^{-\frac{1}{2}} x}{1+q^{\frac{m\tau-1-2k}{2}} a^{\frac{1}{2}} x}. \end{aligned}$$

We introduce the q-binomial coefficients defined by

$$\binom{m}{r}_q = \frac{(1-q^m)(1-q^{m-1})\cdots(1-q^{m-r+1})}{(1-q)(1-q^2)\cdots(1-q^r)}$$

for $r \leq m$, and in particular $\binom{m}{0}_q = 1$. The q-binomial coefficients $\binom{m}{r}_q \in \mathbb{Z}[q]$ (see Chapter 2 of [15] for q-binomial coefficients). There are analogs of the binomial formula, and of Newton's generalized version of it for negative integer exponents,

$$\prod_{k=0}^{n-1}(1+q^k t) = \sum_{k=0}^{n} q^{\frac{k(k-1)}{2}} \binom{n}{k}_q t^k$$
$$\prod_{k=0}^{n-1}\frac{1}{(1-q^k t)} = \sum_{k=0}^{\infty} \binom{n+k-1}{k}_q t^k.$$

Therefore, the coefficient $[f(x)]_{x^m}$ of x^m in $f(x)$ is given by

$$\sum_{j+k=m} (-1)^k q^{\frac{j(j-1)-(m\tau-1)m}{2}} a^{\frac{k-j}{2}} \binom{m\tau}{j}_q \binom{m\tau+k-1}{k}_q,$$

which lies in the ring $\mathbb{Z}[q^{\pm\frac{1}{2}}, a^{\pm\frac{1}{2}}]$ by the integrality of Gaussian binomial.

For the case $\tau \leq -1$, we write $\{m\tau j\} = -\{-m\tau j\}$ in the formula (22), then the similar computations give the formula (21). □

Now, we can finish the proof of Theorem 1.4 as follow:

Proof. Lemma 5.3 implies that there exist rational numbers $n_{m,g,Q}(\tau)$, such that

$$z^2 g_m(q,a) = \sum_{g\geq 0}\sum_{Q} n_{m,g,Q}(\tau) z^{2g} a^Q \in \mathbb{Q}[z^2, a^{\pm\frac{1}{2}}].$$

So we only need to show $n_{m,g,Q}(\tau)$ are integers. By lemma 5.4 and the formula (8) for $g_m(q,a)$, we have

$$\{m\}\{m\tau\} z^2 g_m(q,a) \in \mathbb{Z}[q^{\pm\frac{1}{2}}, a^{\pm\frac{1}{2}}],$$

which is equivalent to

$$(q^{\frac{m}{2}} - q^{-\frac{m}{2}})(q^{\frac{m\tau}{2}} - q^{-\frac{m\tau}{2}}) \sum_{g\geq 0}\sum_{Q} n_{m,g,Q}(\tau)(q^{1/2}-q^{-1/2})^{2g} a^Q$$

$$\in \mathbb{Z}[q^{\pm\frac{1}{2}}, a^{\pm\frac{1}{2}}].$$

So it is easy to get the contradiction if we assume there exists $n_{m,g,Q}(\tau)$ which is not an integer. $\square$

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