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Picard-Fuchs equations of families of QM abelian surfaces

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We describe an algorithm for computing the Picard-Fuchs equation for a family of twists of a fixed elliptic surface. We then apply this algorithm to obtain the equations for several examples, which come from families of Kummer surfaces over Shimura curves, as studied in our previous work. We use this to find correspondences between the parameter spaces of our families and Shimura curves.

1. Introduction

Let $\pi : X \to \mathbb{P}^1$ be a family of complex algebraic varieties. As $s \in \mathbb{P}^1$ varies, the periods of the fibers X_s, i.e., integrals of holomorphically varying differential forms against a topologically constant family of homology classes, satisfy a certain differential equation, known as the *Picard-Fuchs* equation, whose coefficients are rational functions. These equations and their power series solutions are interesting in several respects.

Many of the previously studied examples were families of elliptic curves with some extra structure. Here we study families of abelian surfaces with quaternionic multiplication over Shimura curves.

Let B be a division quaternion algebra over the field $\mathbb{Q}$ of rational numbers, which is indefinite in the sense that $B \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathrm{M}_{2\times 2}(\mathbb{R})$. Let $\mathcal{M}$ be a maximal order in B. Then, the group Γ of norm one elements in $\mathcal{M}$ embeds in $\mathrm{SL}_2(\mathbb{R})$, via the embedding of B in $\mathrm{M}_{2\times 2}(\mathbb{R})$, as a discrete congruence subgroup. The quotients of the complex upper half plane $\mathcal{H}$ by Γ, and more generally by congruence subgroups $\Gamma' \subset \Gamma$, are algebraizable as moduli spaces of abelian surfaces whose endomorphism algebras are certain orders in $\mathcal{M}$ (so called quaternionic multiplication, or QM, abelian surfaces), together with some extra structures [11, Chapter 7]. Under mild assumptions these quotients, known as *Shimura curves*, carry a universal family of such abelian surfaces. Since equations for K3 surfaces are easier to write down than for abelian surfaces, it is natural to consider instead the universal family of the associated Kummer surfaces. These are obtained by taking the fiber by fiber quotient by multiplication by ± 1, and blowing up the resulting singularities at the two-torsion points. The universal families of these Kummer surfaces, which we call QM Kummer surfaces, have the further advantage of existing in more cases than the universal families of the corresponding abelian surfaces.

In trying to write explicit equations for QM Kummer surfaces, we were led to study in [3] families of quadratic twists of a fixed elliptic surface (see Section 3). We identified 11 such families explicitly related to QM Kummer surfaces. Each of these is a family of varieties over $\mathbb{P}^1$ for which we know that the generic fiber is isogenous, in a possibly complicated way, to a Kummer surface associated with a QM abelian surface. Consequently, there is a correspondence between the base spaces for these families, and Shimura curves. In [3] we carried out a detailed analysis of this correspondence, which was highly involved and required a delicate study of finite discriminant forms.

The main new ingredient in the present work is Algorithm 2, which computes the Picard-Fuchs equation for a family of twists of a fixed elliptic surface, together with Theorem 4.3, that proves its validity. A nice feature of our algorithm is that it only requires knowledge of the Picard-Fuchs equation of the elliptic surface which we twist. As input to our algorithm we thus need a method for computing the Picard-Fuchs equation for an elliptic fibration. We describe an algorithm for doing this borrowed from a MAPLE script of F. Beukers. We furnish a proof that this algorithm works since we have found no record of this in the literature.

After describing the algorithm and proving our main result, Theorem 4.3, we apply the algorithm to the study of the families of QM Kummer surfaces described above. In all of the examples we expect the resulting Picard-Fuchs equation to be of degree 3, and furthermore to be the symmetric square (see Section 5) of a degree 2 equation. This turns out indeed to be the case and we list the resulting degree 2 differential equations.

On the Shimura curve side, these degree 2 equations have been studied by Elkies in [5]. This suggests using these Picard-Fuchs equations to discover and verify the correspondences (often isomorphisms) between the bases of the families we study and the related Shimura curve. We describe a method for doing this and apply it in all the examples. We note that while this method falls short of a rigorous proof of the existence of a correspondence between the underlying moduli problems, it is far easier than the analysis carried out in [3]. In fact, we already used the results of the present work in [3] to exclude some potential correspondences. Furthermore, we have one example (no. 4 in Section 8) where the methods of this paper show the consistency of our conjectured isomorphism between the parameter spaces, even though we still do not have a proof of this isomorphism.

2. The Picard-Fuchs equation

We briefly recall the Picard-Fuchs differential equation for a family of varieties over a curve. For further details see for example [10].

Let C be a smooth complex analytic curve and let V/C be a local system of $\mathbb{C}$-vector spaces of dimension n over C. The locally free rank n $\mathcal{O}_C$-module $\mathcal{V} := V \otimes_{\mathbb{C}} \mathcal{O}_C$ carries a canonical connection ∇ defined by the condition that it vanishes on sections of V. We fix a meromorphic vector field d/dt on C, e.g., the one associated with a rational parameter t if $C = \mathbb{P}^1$. Contracting ∇ along d/dt gives the *covariant derivative* operator

$$\nabla_{d/dt} : \mathcal{V} \to \mathcal{V}.$$

Let α be a meromorphic section of $\mathcal{V}$. Since $\mathcal{V}$ has rank n, there is going to be a relation

$$\sum_{i=0}^{m} a_i (\nabla_{d/dt})^i \alpha = 0$$

with $m \leq n$ and meromorphic functions a_i on C. We may normalize this by insisting that $a_m = 1$.

Suppose $\gamma \in V^*(U)$, for some open $U \in C$, where V^* is the dual of V. The evaluation of γ on α, which we suggestively write as $\int_\gamma \alpha$, is a meromorphic function on U and is called a *period* of α. Since ∇ vanishes on sections of V it follows easily that the period $y = \int_\gamma \alpha$ satisfies the differential equation

$$\frac{d^m}{dt^m} y + \sum_{i=0}^{m-1} a_i \frac{d^i}{dt^i} y = 0,$$

which is called the *Picard-Fuchs* equation associated with α.

When V comes from geometry, a bit more can be said. Suppose that $\pi : X \to C$ is a smooth projective family of algebraic varieties, and that V is the family of cohomology groups

$$V = \mathbb{R}^l \pi_* \mathbb{C}$$

For some non-negative integer l. In this case, $\mathcal{V}$ is canonically identifies with the vector bundle of *de Rham* cohomology groups,

$$\mathcal{V} \cong \mathbb{R}^l \pi_* \Omega^\bullet_{X/C},$$

and the connection ∇ is identified with the *Gauss-Manin* connection on the latter vector bundle. If C and π are algebraic, it follows easily that we may take α to be an algebraic (meromorphic) section of $\mathcal{V}$ and that then the coefficients a_i in the Picard-Fuchs equation will be rational functions on C. We will call this a Picard-Fuchs equation associated with the H^l of the family.

In geometric situations we may further take α to be a section in the sub-bundle $\pi_*\Omega^l_{X/C}$ and we may take γ to be a family of Homology classes, so that the associated period is now indeed the integral $\int_\gamma \alpha$.

In our applications, it will always be the case that the sub-bundle $\pi_*\Omega^l_{X/C}$ will be of rank 1. Thus, α is determined up to a product by a rational function. The Picard-Fuchs equation is in some sense unique then, since we may recover easily the equation associated with such a product from the equation for α (see also Section 6 for how to remove the remaining ambiguity).

We can also consider Picard-Fuchs equations associated with sub-local systems $V \subset \mathbb{R}^l\pi_*\mathbb{C}$ provided our chosen α resides in $V \otimes \mathcal{O}_C$.

We now describe the local systems considered in this work. Let $\mathcal{H}$ be the complex upper half plane. Let $\pi^u : E^u \to \mathcal{H}$ be the universal family of elliptic curves, whose fiber over $\tau \in \mathcal{H}$ is

$$E^u_\tau = \mathbb{C}/\mathbb{Z}\langle 1, \tau\rangle.$$

We consider the resulting local system

$$\mathcal{S}h := \mathbb{R}^1\pi_*\mathbb{C},$$

which has a constant fiber $\mathbb{C}^2$. See [12, § 12] for a detailed discussion. Note that $\pi^u_*\Omega^1_{E^u/\mathcal{H}}$ has the section dz, where z is the standard coordinate on $\mathbb{C}$, whose associated periods are 1 and τ, hence its Picard-Fuchs equation is $y'' = 0$.

Let $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ be a discrete group, acting on $\mathcal{H}$ by fractional linear transformations. It acts on $\mathcal{S}h$ via the standard representation of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{C}^2$. When $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is a subgroup of finite index not containing $-\mathrm{Id}$, the quotient $X_\Gamma := \Gamma\backslash\mathcal{H}$ has a family $E_\Gamma := \Gamma\backslash E^u$ of elliptic curves above it, and both are algebraizable. Moreover, the quotient $\Gamma\backslash\mathcal{S}h$ is a local system on X_Γ, isomorphic to $\mathbb{R}^1\pi^\Gamma_*\mathbb{C}$, with π^Γ the induced projection.

Let B be an indefinite rational quaternion algebra and let $\Gamma \subset B^\times$ be as in the introduction. Let $\pi^u : A^u \to X_\Gamma$ be the associated universal family of abelian surfaces with quaternionic multiplication (ignoring for the time being the finite number of singular fibers over the elliptic points). Analytically, it can be written as the quotient of $\mathbb{C}^2 \times \mathcal{H}$, by the semi-direct product of Γ and the additive group of a lattice in B. Then (see [2]) the local system $\mathbb{R}^2\pi^u_*\mathbb{C}$ splits as a direct sum of a 3-dimensional constant local system and a system isomorphic to the symmetric square, $\mathrm{Symm}^2(\mathcal{S}h)$, of $\mathcal{S}h$.

Recall that the Kummer surface associated to an abelian surface A is the resolution of singularities of the quotient $A/\pm 1$ obtained by blowing up the 16 singularities at the 2-torsion points. For any family $\pi^A : A \to X$ of abelian surfaces, let $\pi^S : S = \mathrm{Kummer}(A) \to X$ be the associated family of Kummer surfaces. Then, the local system $\mathbb{R}^2\pi^S_*\mathbb{C}$ splits as a sum of $\mathbb{R}^2\pi^A_*\mathbb{C}$ and a 16-dimensional system coming from the blowups of the singularities at the 2-torsion points. This 16-dimensional system is trivialized by the finite covering given by the 2-torsion points. In particular, when $A = A^u$ is the universal family over a Shimura curve we see that $\mathbb{R}^2\pi^S_*\mathbb{C}$ splits as a sum of $\mathrm{Symm}^2(\mathcal{S}h)$ and a 19-dimensional system, trivialized by a finite covering. Furthermore, we have an inclusion of sheaves, $\pi^S_*\Omega^2_{S/X} \subset \mathrm{Symm}^2(\mathcal{S}h) \otimes \mathcal{O}_{X_\Gamma}$. Consequently the Picard-Fuchs equation satisfied by the periods of a relative 2-form ω on S is going to be of degree 3, and will be the symmetric square of a Picard-Fuchs equation of degree 2 associated with the local system $\mathcal{S}h$ (see Section 5 for symmetric squares of equations).

3. Elliptic surfaces and their Picard-Fuchs equations

An elliptic surface, which we shall only consider over $\mathbb{P}^1$, is a smooth and connected compact complex algebraic surface E, together with a surjective morphism $\pi : E \to \mathbb{P}^1$, such that the generic fiber is a curve of genus 1. We will always assume that the fibration is relatively minimal and has a given section, denoted 0.

For all but a finite number of points $s \in \mathbb{P}^1$, the fiber $E_s = \pi^{-1}(s)$ is an elliptic curve. The singular locus $\Sigma = \Sigma(E)$ of the fibration is the (finite) subset of $\mathbb{P}^1$ over which the fibers are singular (namely, π is not everywhere smooth). Kodaira [9] classified all possible types of singular fibers (see also [1, Chapter V.7]).

The generic fiber of an elliptic surface may be given by a Weierstrass equation of the form $y^2 = f(x)$, where $f(x) = ax^3 + bx^2 + cx + d$ and a,b,c,d are rational functions of the parameter t on $\mathbb{P}^1$.

Given two distinct points α and β in $\mathbb{P}^1$, the quadratic twist $E_{\alpha,\beta}$ at these points can be described in two ways. Algebraically, if E has Weierstrass equation $y^2 = f(x)$ and α and β are finite points, then $E_{\alpha,\beta}$ has the

Weierstrass equation

$$\frac{t-\alpha}{t-\beta} y^2 = f(x). \tag{3.1}$$

Analytically, $E_{\alpha,\beta}$ can be described as follows. Take the double cover $B' \to \mathbb{P}^1$ ramified at α and β and let E' be the pullback surface. Now quotient E' by the transformation which identifies the two fibers above each fiber of E with sign -1 and resolve the evident singularities.

Definition 3.1. Let $E \to \mathbb{P}^1$ be an elliptic surface as above with $s \in \Sigma = \Sigma(E)$. For $\lambda \in \mathbb{P}^1 - \Sigma$ let $E_{s,\lambda}$ be the twisted family at s and at λ. These surfaces vary in a family $\mathcal{TW}_s(E)$ over the λ-*line* $\mathbb{P}^1(\lambda) - \Sigma$.

The local system $\mathbb{R}^1\pi_*\mathbb{C}$ over $\mathbb{P}^1 - \Sigma$ has dimension 2. Its dual is the *homological invariant* (tensored with $\mathbb{C}$) associated by Kodaira to the elliptic surface, and we denote it by F.

A Picard-Fuchs equation for the H^1 of a general elliptic surface E, corresponding to the invariant differential $\omega = dx/y$, can be computed using Algorithm 1. It is taken from a MAPLE script of F. Beukers (see Section 9). We failed to find it documented anywhere so we give a short proof that it indeed works. Note that in this algorithm the quantities q_i and c_j are initially variables but eventually get assigned values which are rational functions in t.

Algorithm 1: Computing a Picard-Fuchs equation for an elliptic surface

Input: An elliptic surface given by a Weierstrass equation $y^2 = ax^3 + bx^2 + cx + d$, with a, b, c, d rational functions of t

Output: The Picard-Fuchs equation $y'' + c_1 y' + c_2 y = 0$ satisfied by the periods of the invariant differential $\omega = dx/y$

$f \leftarrow ax^3 + bx^2 + cx + d \ (f = f(t,x))$;
$f_t \leftarrow \frac{\partial f}{\partial t}$;
$f_{tt} \leftarrow \frac{\partial f_t}{\partial t}$;
$f_x \leftarrow \frac{\partial f}{\partial x}$;
$q \leftarrow q_4 x^4 + q_3 x^3 + q_2 x^2 + q_1 x + q_0$;
$q_x \leftarrow \frac{\partial q}{\partial x}$;
$e \leftarrow \frac{-f_{tt}\cdot f}{2} + \frac{3f_t^2}{4} - c_1 \frac{f_t \cdot f}{2} + c_2 f^2 + \frac{3 f_x \cdot q}{2} - f \cdot q_x$;
$C \leftarrow \mathtt{COEFFICIENTS}(e, x)$;
$(c_1, c_2, q_0, q_1, q_2, q_3, q_4) \leftarrow \mathtt{SOLVE}(C = 0)$;

We note that the equations to be solved in the last step of the algorithm are in fact 7 linear equations in the 7 variables c_i and q_j, with coefficients which are rational functions in t, so solving them is just linear algebra.

Proposition 3.2. *Algorithm 1 gives the Picard-Fuchs equation for H^1 of the elliptic surface E.*

Proof. We express y in terms of x as $y = f(x)^{1/2}$. Applying the covariant Gauss-Manin differentiation with respect to t amounts to differentiating (after eliminating y) with respect to t. On the invariant differential $\omega = f(x)^{-1/2}dx$ we find

$$\nabla_{d/dt}\omega = -\frac{1}{2}f^{-\frac{3}{2}}f_t dx$$
$$\nabla^2_{d/dt}\omega = \left(\frac{3}{4}f^{-\frac{5}{2}}f_t^2 - \frac{1}{2}f^{-\frac{3}{2}}f_{tt}\right)dx.$$

Now we may write the general differential operator of degree 2 applied to ω,

$$\begin{aligned}\nabla^2_{d/dt}\omega &+ c_1(t)\nabla_{d/dt}\omega + c_2(t)\omega \\ = &\left(\frac{3}{4}f^{-\frac{5}{2}}f_t^2 - \frac{1}{2}f^{-\frac{3}{2}}f_{tt} - \frac{1}{2}c_1 f^{-\frac{3}{2}}f_t + c_2 f^{-\frac{1}{2}}\right)dx.\end{aligned} \tag{3.2}$$

For the appropriately chosen c_1 and c_2 this will give a trivial de Rham cohomology class on $E/\mathbb{P}^1$. Reduction theory (see for example [8]) tells us that it is going to be the (relative) differential of a rational function of the form $q(x)/y^n$ and examining the poles at the 2-torsion points shows that one can take $n = 3$ and q a polynomial of degree at most 4. This is given by

$$d\frac{q(x)}{y^3} = d\frac{q(x)}{f^{3/2}} = \left(-\frac{3}{2}q \cdot f^{-\frac{5}{2}} \cdot f_x + q_x f^{-\frac{3}{2}}\right)dx. \tag{3.3}$$

To find the Picard-Fuchs equation, we equate (3.2) to (3.3), multiply by $f^{5/2}$ to clear denominators. This gives the quantity e in the algorithm. Then we simply solve $e = 0$, identically in x, expressing the c_i's and q_j's in terms of t. □

To illustrate the algorithm, we consider the elliptic surface which is number 9 on our list. It has equation $y^2 = f$ with $f = 4x^3 - 3t^2(9t^2 - 8)x - t^2(27t^4 - 36t^2 + 8)$. Then $f_t = -2t(8 + 81t^4 - 24x + 18t^2(3x - 4))$ and $f_{tt} = -16 - 810t^4 + 48x - 108t^2(3x - 4)$, $f_x = 3(8t^2 - 9t^4 + 4x^2)$. We consider a general degree 4 polynomial $q = q_0 + q_1x + q_2x^2 + q_3x^3 + q_4x^4$ and its derivative

$q_x = q_1 + 2q_2x + 3q_3x^2 + 4q_4x^3$. Then, the equation obtained from equating (3.2) to (3.3) multiplied by $f^5/2$ becomes

$$\begin{aligned}
&16c_2x^6 + ((216t^3 - 96t)c_1 + (192t^2 - 216t^4)c_2 + 648t^2 - 96)x^4\\
&+ ((324t^5 - 288t^3 + 32)t + (-216t^6 + 288t^4 - 64t^2)c_2\\
&+ 1620t^4 - 864t^2 + 32)x^3 + ((-1458t^7 + 1944t^5 - 576t^3)c_1\\
&+ (729t^8 - 1296t^6 + 576t^4)c_2 + 4374t^6 - 3240t^4 + 1152t^2)x^2\\
&+ \big((-3645t^9 + 6480t^7 - 3240t^5 + 384t^3)c_1\\
&\quad + (1458t^{10} - 3240t^8 + 2160t^6 - 384t^4)c_2\\
&\quad + 10935t^8 - 12960t^6 + 5400t^4 - 768t^2\big)x\\
&+ (-2187t^{11} + 4860t^9 - 3456t^7 + 864t^5 - 64t^3)c_1\\
&+ (729t^{12} - 1944t^{10} + 1728t^8 - 576t^6 + 64t^4)c_2\\
&+ 8748t^{10} - 14580t^8 + 8208t^6 - 1440t^4 + 128t^2\\
= &-2q_4x^6 - 6q_3x^5 + \left(-10q_2 + \left(-\frac{135}{2}t^4 + 60t^2\right)q_4\right)x^4\\
&+ \left(-14q_1 + \left(-\frac{81}{2}t^4 + 36t^2\right)q_3 + (-108t^6 + 144t^4 - 32t^2)q_4\right)x^3\\
&+ \left(-18q_0 + \left(-\frac{27}{2}t^4 + 12t^2\right)q_2 + (-81t^6 + 108t^4 - 24t^2)q_3\right)x^2\\
&+ \left(\left(\frac{27}{2}t^4 - 12t^2\right)q_1 + (-54t^6 + 72t^4 - 16t^2)q_2\right)x\\
&+ \left(\frac{81}{2}t^4 - 36t^2\right)q_0 + (-27t^6 + 36t^4 - 8t^2)q_1
\end{aligned}$$

We equate the coefficients of the respective powers of x and solve the resulting equations to get the unique solution

$$\begin{aligned}
q_0 &= \frac{32t^2}{3(t^2-1)}\\
q_1 &= -4\frac{-16 + 180t^2 - 378t^4 + 243t^6}{9(t^2-1)}\\
q_2 &= -2\frac{40 - 99t^2 + 81t^4}{3(t^2-1)}\\
q_3 &= 0\\
q_4 &= -8\frac{1+9t^2}{9t^2(t^2-1)}\\
c_1 &= \frac{3t^2-1}{t(t^2-1)}, \qquad c_2 = \frac{9t^2+1}{9t^2(t^2-1)}.
\end{aligned}$$

4. The Picard-Fuchs equation for a family of twists

In this section we prove our main theorem, Theorem 4.3, which describes the differential equation satisfied by the periods of the H^2 of the family of twists $\mathcal{TW}_s(E)$, described in Definition 3.1, of a fixed elliptic surface E. We will in fact show that the periods for this H^2 can be recovered from the periods for H^1 of E and the differential equation can be recovered solely based on a Picard-Fuchs equation for H^1 of E.

To simplify the notation, we assume that $s = 0$, i.e., that the twists are at 0 and a varying point. Recall from the description following (3.1) that $E_{0,\lambda}$ can be obtained from E as follows: One takes a double covering $\pi_\lambda : B' \to \mathbb{P}^1$ which is ramified exactly over 0 and λ. Let $d_\lambda : B' \to B'$ be the deck transformation of the covering. One considers the pullback $\pi_\lambda^* E$ and takes the quotient $\pi_\lambda^* E/D_\lambda$ where D_λ is the map $(s, e) \mapsto (d_\lambda(s), -e)$, i.e., the map that identifies the fibers at s and $d_\lambda(s)$ but via the map -1. The result may have singularities at the fixed points 0 and λ of d_λ and resolving them one obtains $E_{0,\lambda}$. We henceforth ease notation and write E_λ for $E_{0,\lambda}$.

We now write a particular homology class $\Gamma_\lambda \in H_2(E_\lambda, \mathbb{C})$. We will obtain Γ_λ by modifying a fixed homology class $\Gamma' \in H_2(E, \mathbb{C})$. In fact, we take Γ' in $H_1(\mathbb{P}^1 - \Sigma, F)$, where F is the homological invariant (see Section 3). An element of $H_1(\mathbb{P}^1 - \Sigma, F)$ consists of a formal sum $\sum(\gamma_i, x_i)$ where γ_i are paths in $\mathbb{P}^1 - \Sigma$ and x_i is a section of F_{γ_i}, in such a way that the obvious boundary map vanishes. Write the one form on the elliptic surface E, dx/y, as a family of differential forms ω_t and consider the function G_i on the path γ_i given at a point t by $G_i(t) = \int_{x_i(t)} \omega_t$.

Having fixed Γ' we can write a family of 2-homology classes Γ_λ in $H_2(E_\lambda, \mathbb{C})$ as follows. Each path γ_i can be pulled back to B'. If one of the pullbacks is δ_i then the other one is $d_\lambda(\delta_i)$. The section x_i pulls back to both of these lifts. Since we identify the fiber at s with the fiber at $d_\lambda(s)$ via the map -1 and since -1 acts as -1 on the first homology it is clear that $\Gamma'_\lambda := \sum(\delta_i, x_i) + \sum(d_\lambda(\delta_i), -x_i)$ descends to the required homology class $\Gamma_\lambda \in H_2(E_\lambda, \mathbb{C})$.

Let us now write explicitly a period associated with this homology class. We first need to choose a holomorphic differential 2-form η_λ on E_λ. We have a 2-form on E, $\eta = \omega_t \wedge dt$. We can write an affine model for B', the double cover of $\mathbb{P}^1$ ramified at 0 and λ, as $s^2 = t(t - \lambda)$. The form $s^{-1} \cdot \pi_\lambda^*$ has the right behavior with respect to deck transformations and therefore descends to the required form η_λ on E_λ. Note that multiplying by s^{-1} eliminates the zeroes that dt acquires from the ramified cover. We now compute the period

$\int_{\Gamma_\lambda} \eta_\lambda$. An easy computation shows this is equal to

$$\begin{aligned}
&\int_{\Gamma'_\lambda} s^{-1}\pi_\lambda^*\eta \\
&= \sum_i \left(\int_{\delta_i} s^{-1}G_i(\pi_\lambda(s))\pi_\lambda^* dt + \int_{d_\lambda \delta_i} s^{-1}(-G_i(\pi_\lambda(s)))\pi_\lambda^* dt \right) \\
&= 2\sum_i \int_{\delta_i} s^{-1}G_i(\pi_\lambda(s))\pi_\lambda^* dt \\
&= 2\sum_i \int_{\gamma_i} (t(t-\lambda))^{-1/2}G_i(t)dt.
\end{aligned}$$

Dividing by 2 we get the period

$$u(\lambda) := \sum \int_{\gamma_i} (t(t-\lambda))^{-1/2}G_i(t)dt.$$

Our goal is now to compute a differential equation satisfied by u. In doing so, we will only use the fact that the G_i satisfy the Picard-Fuchs equation for the elliptic family E, an equation of degree 2 of the form

$$y'' + b_1(t)y' + b_2(t)y = 0, \tag{4.1}$$

with b_i rational functions of t. The computation is inspired by the computation in [4, 2.10].

Lemma 4.1. *Suppose $y = y(t)$ satisfies (4.1). Then, for a fixed λ, the function $z = z_\lambda(t) := (t(t-\lambda))^{-1/2}y$ satisfies the equation*

$$z'' + \alpha_\lambda(t)z' + \beta_\lambda(t)z = 0$$

(derivatives with respect to t) with

$$\begin{aligned}
\alpha_\lambda(t) &= b_1(t) + \frac{2t-\lambda}{t(t-\lambda)} \\
\beta_\lambda(t) &= b_2(t) + b_1(t)\frac{2t-\lambda}{2t(t-\lambda)} - \frac{\lambda^2}{4t^2(t-\lambda)^2}.
\end{aligned}$$

Proof. A straightforward computation. □

Suppose now that we are given two rational functions $p(t) = p_\lambda(t)$ and $q(t) = q_\lambda(t)$. We have, deriving with respect to t,

$$(pz + qz')' = p'z + (p + q')z' + qz''.$$

If we impose the relation

$$p + q' = \alpha_\lambda q, \tag{4.2}$$

then we can write

$$p'z + (p + q')z' + qz'' = p'z + q(z'' + \alpha_\lambda z') = p'z - q\beta_\lambda z,$$

by using the differential equation for z. The relation (4.2) gives $p = \alpha q - q'$ so that $p' = \alpha' q + q'\alpha - q''$ and we finally end up with the relation

$$(pz + qz')' = (\alpha' q + q'\alpha - q'' - q\beta)z.$$

Now, we can do the following: We have $u(\lambda) = \sum \int_{\gamma_i} z_\lambda(t)dt$. Since z depends on λ only through division by $\sqrt{t - \lambda}$, we easily get by differentiating n times with respect to λ inside the integral sign,

$$\frac{d^n u}{d\lambda^n} = \frac{1 \cdot 3 \cdot \cdots \cdot (2n-1)}{2^n} \sum_i \int_{\gamma_i} \frac{z}{(t-\lambda)^n} dt. \tag{4.3}$$

Lemma 4.2. *There is a choice for q such that we may expand $\alpha' q + q'\alpha - q'' - q\beta$ as a polynomial in $(t - \lambda)^{-1}$ with coefficients which are rational functions of λ,*

$$\alpha' q + q'\alpha - q'' - q\beta = \sum_n c_n(\lambda)(t - \lambda)^{-n}. \tag{4.4}$$

Proof. First let q_0 be the least common multiple of the denominators of α and β as rational functions of t. Then, $\alpha' q_0 + q_0'\alpha - q_0'' - q_0\beta$ is a polynomial in t and can therefore also be written as a polynomial in $t - \lambda$, with coefficients which are rational functions of λ. Suppose that this polynomial has degree m. Then, we may simply take $q = q_0(t - \lambda)^{-m}$. $\square$

We may modify the paths γ_i to homotopic paths making sure that $\sqrt{t - \lambda}$ is single valued on each path. Also, the sums of the monodromies of the G_i

around the paths γ_i is 0 because Γ' is closed. Thus, we have,

$$\begin{aligned}
0 &= \sum_i \int_{\gamma_i} \frac{d}{dt}\left(p_\lambda z_\lambda + q_\lambda \frac{d}{dt} z_\lambda\right) dt \\
&= \sum_i \int_{\gamma_i} \left(\frac{d\alpha_\lambda}{dt} q_\lambda + \frac{dq_\lambda}{dt}\alpha_\lambda - \frac{d^2 q_\lambda}{dt^2} - q_\lambda \beta_\lambda\right) z_\lambda dt \\
&= \sum_i \int_{\gamma_i} \sum_n c_n(\lambda)(t-\lambda)^{-n} z_\lambda dt \\
&= \sum_n c_n(\lambda) \sum_i \int_{\gamma_i} \frac{z_\lambda}{(t-\lambda)^n} dt \\
&= \sum_n \tilde{c}_n(\lambda) \frac{d^n u}{d\lambda^n},
\end{aligned}$$

by (4.3), with

$$\tilde{c}_n(\lambda) = \frac{2^n}{1 \cdot 3 \cdot \dots \cdot (2n-1)} c_n(\lambda).$$

We have therefore proved the following.

Theorem 4.3. *Let E be an elliptic surface whose periods satisfy the differential equation* (4.1)*. Then, algorithm 2 computes a differential equation with polynomial coefficients satisfied by a non-trivial period for H^2 of the family $\mathcal{TW}_0(E)$.*

To demonstrate algorithm 2, we continue with the example that we used to demonstrate algorithm 1, resulting in a Picard-Fuchs equation $y'' + b_1 y' + b_2 y = 0$ with $b_1 = \frac{(1-3t^2)}{(t-t^3)}$, $b_2 = \frac{(1+9t^2)}{(9t^2(t^2-1))}$. We compute the quantities

$$\begin{aligned}
\alpha &= b_1 + \frac{2t-\lambda}{t(t-\lambda)} = \frac{2\lambda - 3t - 4\lambda t^2 + 5t^3}{t(t-\lambda)(t^2-1)} \\
\beta &= b_2 + b_1 \frac{2t-\lambda}{2t(t-\lambda)} - \frac{\lambda^2}{4t^2(t-\lambda)^2} \\
&= \frac{16t^2(9t^2-2) + \lambda^2(81t^2-5) + \lambda(46t - 234t^3)}{36(\lambda-t)^2 t^2 (t^2-1)}.
\end{aligned}$$

The denominators of α and β are respectively $t(t-\lambda)(t^2-1)$ and $36(\lambda - t)^2 t^2 (t^2-1)$ with a least common multiple of $q_0 = -36(\lambda-t)^2 t^2 (t^2-1)$. It

---

Algorithm 2: Computing a differential equation for periods of $\mathcal{TW}_0(E)$

Input: A Picard-Fuchs equation $y'' + b_1(t)y' + b_2(t)y = 0$ for an elliptic surface E

Output: A vector $\tilde{c}$ such that a Picard-Fuchs equation for the family of twists $\mathcal{TW}_0(E)$ is given by $\sum_n \tilde{c}_n(\lambda)\frac{d^n u}{d\lambda^n}$

$\alpha \leftarrow b_1(t) + \frac{2t-\lambda}{t(t-\lambda)}$;

$\beta \leftarrow b_2(t) + b_1(t)\frac{2t-\lambda}{2t(t-\lambda)} - \frac{\lambda^2}{4t^2(t-\lambda)^2}$;

$q_0 \leftarrow$ `LCM(DENOMINATOR(`α`),DENOMINATOR(`β`));`

$pol_0 \leftarrow \alpha' q_0 + q_0'\alpha - q_0'' - q_0\beta$ (derivatives w.r.t. t);

$m \leftarrow$ `DEG(`pol_0`);`

$q \leftarrow q_0(t-\lambda)^{-m}$;

$pol \leftarrow (\alpha' q + q'\alpha - q'' - q\beta)_{t\leftarrow s+\lambda}$ (derivatives w.r.t. t);

$c_n(\lambda) \leftarrow$ `COEFFICIENT(`s^{-n} *in pol*`);`

$\tilde{c}_n(\lambda) \leftarrow \frac{2^n}{1\cdot 3\cdot\cdots\cdot(2n-1)} c_n(\lambda)$;

---

then suffices to take $m = 3$ and obtain

$$q = q_0(\lambda - t)^{-3} = \frac{36t^2(t^2-1)}{\lambda - t}.$$

Now we compute

$$\alpha' q + q'\alpha - q'' - q\beta = \frac{32t^2 + \lambda^2(5 - 81t^2) + \lambda(98t - 54t^3)}{(\lambda - t)^3}$$

and changing variables $t = s + \lambda$ we get

$$\begin{aligned} pol = {} & 54\lambda - 135\lambda^2 s^{-3} + 135\lambda^4 s^{-3} \\ & - 162\lambda s^{-2} + 324\lambda^3 s^{-2} - 32 s^{-1} + 243\lambda^2 s^{-1}. \end{aligned}$$

We consider the coefficients of s^{-n} for $n = 0, 1, 2, 3$, and multiply them by 1, 2, 4/3, 8/15 respectively to obtain the coefficients of the Picard-Fuchs equation for the family of twists: $c_3 y''' + c_2 y'' + c_1 y' + c_0 y = 0$ with

$$c_3 = 72\lambda^2(\lambda^2 - 1), \quad c_2 = 216\lambda(2\lambda^2 - 1), \quad c_1 = 486\lambda^2 - 64, \quad c_0 = 54\lambda.$$

5. K3 surfaces

In [3] we studied a particular class of elliptic fibrations. Out of the list of elliptic fibrations with 4 singular fibers compiled by Herfurtner [6], we picked out the ones for which the twists give K3 surfaces, generically with Picard number 19. These K3 surfaces are then isogenous to Kummer surfaces associated with Abelian surfaces whose isogeny algebra is a rational quaternion algebra. We further picked out only the examples in which the quaternion algebra in question is indefinite. There are 11 examples, which we list below (Table 1, see also [3, Table 1]). The method for deciding which families of twists correspond to quaternion algebras and the method for determining the discriminant of the associated algebra are detailed in [3, Proposition 2 and Lemma 2].

As discussed in Section 2, for each of the examples above, the resulting Picard-Fuchs equation should be of degree 3 and should be a symmetric square of a degree 2 equation, which is a Picard-Fuchs equation for the Shimura local system descended to the base. In this section we verify that this is indeed the case, and we compute the degree 2 equations.

Symmetric squares of differential equations are considered, for example in [10, Example 6.5.2]. Given a differential equation of degree 2, $y'' + ay' + by = 0$, one looks for the equations satisfied by $z = y^2$. The result is

$$\begin{aligned} &z''' + \alpha z'' + \beta z' + \gamma z = 0 \\ &\text{with } \alpha = 3a,\ \beta = 4b + 2a^2 + a',\ \gamma = 4ab + 2b'. \end{aligned} \tag{5.1}$$

If we are given a differential equation of degree 3, we can check if it is a symmetric square of one of degree 2 and find the "square root" as described in Algorithm 3.

Algorithm 3: Taking the square root of a degree 3 differential equation

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Input: A differential equation $z''' + \alpha z'' + \beta z' + \gamma z = 0$
Output: A differential equation $y'' + ay' + by = 0$ whose symmetric square equals the input equation, if it exists

$a \leftarrow \alpha/3$;
$b \leftarrow (\beta - 2a^2 - a')/4$;
$c \leftarrow \gamma - 4ab - 2b'$;
if $c \neq 0$ the equation is not a square

Not surprisingly, in all 11 examples, the resulting differential equation turns out to be the symmetric square of an equation of degree 2. As an example, consider the differential equation obtained in Section 4 for the twists in example 9. We first normalize the equation to get the equation $z''' + \alpha z'' + \beta z' + \gamma z = 0$ with

$$\alpha = 3\frac{(2\lambda^2-1)}{\lambda(\lambda^2-1)}, \quad \beta = \frac{243\lambda^2-32}{36\lambda^2(\lambda^2-1)}, \quad \gamma = \frac{3\lambda}{4\lambda^2(\lambda^2-1)},$$

and then apply the algorithm to get

$$a = \frac{(2\lambda^2-1)}{\lambda(\lambda^2-1)}, \quad b = \frac{4+27\lambda^2}{144\lambda^2(\lambda^2-1)},$$

and $c = 0$, so that this is indeed the square of the equation

$$y'' + \frac{(2\lambda^2-1)}{\lambda(\lambda^2-1)}y' + \frac{4+27\lambda^2}{144\lambda^2(\lambda^2-1)}y = 0.$$

In Table 1 below we list the examples together with the resulting equations of degree 2 (one can recover the degree 3 equation using (5.1)).

The first column is the example number, which is the same as in Table 1 in [3]. The second column gives the types of singular fibers for the based elliptic surface and their locations and the third column gives the coefficients of the degree 2 equation. The final column gives the expected discriminant for the associated quaternion algebra. In the table $\gamma = \frac{-\left(1+\sqrt{-2}\right)^4}{3}$ and $\delta = \frac{\left(1+\sqrt{-7}\right)^7}{512}$. Conjugates for such elements are over $\mathbb{Q}$.

6. Schwarzian derivatives

Our goal in the rest of this work is to compare the differential equations that we obtained in the previous section to those obtained by Elkies in [5]. One essential problem is that the equation depends in an essential way on the choice of the section of the de Rham bundle. Even if, as is the case for us, the choice is between different sections of a line bundle, it still means that the periods could be multiplied by an arbitrary rational function. To compare two differential equations it is best to compare quantities which are invariant with respect to such scaling.

This can be done as follows for equations of degree 2. Consider the quotient of two independent solutions. This is invariant with respect to scaling. It depends of course on the choice of the two solutions, but only up to a

1	I_1, I_1, I_8, II $\gamma, \bar{\gamma}, \infty, 0$ $\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{35}{36}$	$a = \frac{27-21\lambda+6\lambda^2}{27\lambda-14\lambda^2+3\lambda^3},\ b = \frac{3(-1-6\lambda+3\lambda^2)}{16\lambda^2(27-14\lambda+3\lambda^2)}$	6
2	I_1, I_2, I_7, II $\frac{-9}{4}, \frac{-8}{9}, \infty, 0$ $\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{35}{36}$	$a = \frac{144+339\lambda+144\lambda^2}{144\lambda+226\lambda^2+72\lambda^3},\ b = \frac{-2+36\lambda+27\lambda^2}{4\lambda^2(72+113\lambda+36\lambda^2)}$	6
3	I_1, I_4, I_5, II $-10, 0, \infty, \frac{1}{8}$ $\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{35}{36}$	$a = \frac{-5+119\lambda+16\lambda^2}{\lambda(-10+79\lambda+8\lambda^2)},\ b = \frac{6(-1+7\lambda+2\lambda^2)}{(1-8\lambda)^2\lambda(10+\lambda)}$	15
4	I_2, I_3, I_5, II $\frac{-5}{9}, 0, \infty, 3$ $\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{35}{36}$	$a = \frac{15+39\lambda-36\lambda^2}{30\lambda+44\lambda^2-18\lambda^3},\ b = \frac{-23-246\lambda+81\lambda^2}{48(-3+\lambda)^2\lambda(5+9\lambda)}$	10
5	I_1, I_1, I_7, III $\delta, \bar{\delta}, \infty, 0$ $\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{15}{16}$	$a = \frac{64+39\lambda+16\lambda^2}{64\lambda+26\lambda^2+8\lambda^3},\ b = \frac{-2+4\lambda+3\lambda^2}{4\lambda^2(32+13\lambda+4\lambda^2)}$	14
6	I_1, I_2, I_6, III $4, 1, \infty, 0$ $\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{15}{16}$	$a = \frac{8-15\lambda+4\lambda^2}{2\lambda(4-5\lambda+\lambda^2)},\ b = \frac{-1-6\lambda+3\lambda^2}{16\lambda^2(4-5\lambda+\lambda^2)}$	6
7	I_1, I_3, I_5, III $\frac{-25}{3}, 0, \infty, \frac{1}{5}$ $\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{15}{16}$	$a = \frac{25-369\lambda-60\lambda^2}{50\lambda-244\lambda^2-30\lambda^3},\ b = \frac{-167+630\lambda+225\lambda^2}{16(1-5\lambda)^2\lambda(25+3\lambda)}$	6
8	I_2, I_3, I_4, III $\frac{-1}{3}, 0, \infty, 1$ $\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{15}{16}$	$a = \frac{1+3\lambda-12\lambda^2}{2\lambda+4\lambda^2-6\lambda^3},\ b = \frac{-1-9\lambda+9\lambda^2}{16(-1+\lambda)^2(\lambda+3\lambda^2)}$	6
9	I_1, I_1, I_6, IV $1, -1, \infty, 0$ $\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{8}{9}$	$a = \frac{1-2\lambda^2}{\lambda-\lambda^3},\ b = \frac{4+27\lambda^2}{144\lambda^2(-1+\lambda^2)}$	6
10	I_1, I_2, I_5, IV $\frac{-27}{4}, \frac{-1}{2}, \infty, 0$ $\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{8}{9}$	$a = \frac{27+87\lambda+16\lambda^2}{27\lambda+58\lambda^2+8\lambda^3},\ b = \frac{-3+16\lambda+6\lambda^2}{4\lambda^2(27+58\lambda+8\lambda^2)}$	10
11	I_3, I_3, I_2, IV $\infty, 0, -1, 1$ $\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{8}{9}$	$a = \frac{-1+\lambda+4\lambda^2}{2(-\lambda+\lambda^3)},\ b = \frac{-13-22\lambda+27\lambda^2}{144(-1+\lambda)^2(\lambda+\lambda^2)}$	6

Table 1: Twists of elliptic surfaces.

fractional linear transformation. Applying the Schwarzian derivative, to be recalled next, removes this ambiguity. Our reference for this material is [5] (see also [7]).

Definition 6.1. The Schwarzian derivative of a function $z = z(\zeta)$ with respect to the parameter ζ is the function [5, (13)]

$$S_\zeta(z) = \frac{2z'z''' - 3(z'')^2}{(z')^2}$$

where derivatives are with respect to ζ.

We recall the following relevant results

Proposition 6.2. 1) *If z_1 is obtained from z by a fractional linear transformation, then $S_\zeta(z_1) = S_\zeta(z)$.*

2) *If z is the quotient of a basis of solutions to the differential equation $y'' + ay' + by = 0$, derivative taken with respect to ζ, then the Schwarzian derivative of z, which is independent of the choice of solutions by the first part, is given by [5, (17)]*

$$S_\zeta(z) = 4b - a^2 - 2a'.$$

This gives our required invariant. To describe the dependency of the Schwarzian derivative on the parameter ζ, it is better to replace $S_\zeta(z)$ by the quadratic differential

$$\sigma_\zeta(z) = S_\zeta(z)(d\zeta)^2. \tag{6.1}$$

Suppose now that ζ is a function of η. Then we have the formula [5, (14)]

$$S_\eta(z) = \left(\frac{d\zeta}{d\eta}\right)^2 S_\zeta(z) + S_\eta(\zeta)$$

and thus

$$\sigma_\eta(z) = \sigma_\zeta(z) + \sigma_\eta(\zeta). \tag{6.2}$$

Suppose now that we have a local system over a curve X, giving rise, using a local parameter ζ, to a Picard-Fuchs equation of degree 2. Let z be the quotient of two basis elements of solutions, giving rise to a sigma invariant $\sigma_\zeta(z)$; It depends only on the local system and the parameter ζ.

Let us pull back the system to a curve Y with local parameter η, so that ζ is a function of η. The periods remain the same, so the corresponding quotient of solutions is still z, pulled back to Y, and we may write the associated sigma invariant with respect to η, $\sigma_\eta(z)$, which is determined from the sigma invariant on X using (6.2). Thus, if we suspect that a local system on Y is the pull back of the local system on X via a given morphism we can use sigma invariants and (6.2) to confirm (but not prove!) this suspicion.

Suppose now that we want, in the situation above, to guess the formula for a morphism that will pull back a given system on X to a given system on Y. We will derive from the sigma invariant a certain residue which will have a very simple behavior with respect to morphisms and will make it easy to guess such a morphism.

A quadratic differential $\sigma = f(\zeta)(d\zeta)^2$ has a well defined residue, denoted $\operatorname{res}(\sigma)$, which is the coefficient of ζ^{-2} in f. With a change of variable $\zeta = \zeta(\eta)$ which has order n the residue is multiplied by a factor of n^2.

Suppose now that $\zeta = \eta^n$. Then, by (6.1),

$$\begin{aligned} S_\eta(\zeta) &= 2\frac{n(n-1)(n-2)\eta^{n-3}}{n\eta^{n-1}} - 3\left(\frac{n(n-1)\eta^{n-2}}{n\eta^{n-1}}\right)^2 \\ &= \eta^{-2}\left(2(n-1)(n-2) - 3(n-1)^2\right) \\ &= \eta^{-2}\left(2(n^2-3n+2) - 3(n^2-2n+1)\right) \\ &= \eta^{-2}(1-n^2), \end{aligned}$$

and therefore $\sigma_\eta(\zeta) = (1-n^2)(d\eta/\eta)^2$. Therefore, using (6.2), if the residue of $\sigma_\zeta(z)$ is α then the residue of $\sigma_\eta(z)$ is

$$n^2\alpha + (1-n^2) = 1 - n^2(1-\alpha). \tag{6.3}$$

This leads to the following.

Definition 6.3. The Schwarzian residue of a quadratic differential σ is given by the formula

$$\operatorname{res}_s(\sigma) = 1 - \operatorname{res}(\sigma).$$

By (6.3) we have

Proposition 6.4. *if $\zeta = \zeta(\eta)$ is a change of variables of degree n then*

$$\operatorname{res}_S \sigma_\eta(z) = n^2 \operatorname{res}_S \sigma_\zeta(z).$$

We observe in particular that the Schwarzian residue of the sigma invariant of a local system is independent of parameter and under pullback it will multiply by n^2 at a point where the ramification index is n. Note that at points where the differential is holomorphic the Schwarzian residue is 1 and not 0.

In our examples, the local system pulls back to the Shimura local system $\mathcal{S}h$ over the upper half plane $\mathcal{H}$. Recall that the Picard-Fuchs equation with respect to the standard parameter on $\mathcal{H}$ of this system is simply $y'' = 0$ and the sigma invariant is 0 so the Schwarzian residue is 1 at every point. Thus, for our systems, the Schwarzian residue at every point is always $1/n^2$ for some positive integer n, which we call the *Schwarzian index* at the point. It is, of course, just the ellipticity index of the point.

In the following table we list for the examples we have the sigma invariant (with respect to the parameter λ, neglecting the $d\lambda^2$ term, and the Schwarzian indices at points when it is bigger than 1 (where the original fibration had a singular fiber).

7. The results of Elkies

In [5] Elkies computes certain differential equations associated with Shimura curves. While this is not stated explicitly, these are exactly the Picard-Fuchs equations associated with the Shimura local system descended to the Shimura curve because the quotient of the two solutions gives the coordinate τ on the upper half plane, just as for the Shimura local system, as in Section 2.

We briefly list the types of Shimura curves considered. For more information one may consult [5] or [3] (our notation is consistent with the latter reference). For each discriminant D (always the product of an even number of primes) the Shimura curve V_D is the quotient of the upper half plane by the group Γ of norm one elements in a maximal order in a quaternion algebra of discriminant D (see the introduction). For each prime $p|D$ it carries a modular involution w_p and these involutions commute with each other so that we can also set $w_n = \prod_{p|n} w_p$ for $n|D$. We let V_D^* be the quotient of V_D by the group generated by all its modular involutions. Finally, for a prime p which does not divide D there is a modular curve $V_{D,p}$, which corresponds to an additional "$\Gamma_0(p)$" structure, This retains all of the previous involutions but has an additional involution w_p.

In table 3 we give, for each relevant curve, the equation that Elkies finds, the associated sigma invariant and the Schwarzian indices at the relevant

1	$\gamma, \bar{\gamma}, \infty, 0$ $2, 2, 2, 6$	$\frac{3(945-652\lambda+142\lambda^2-60\lambda^3+9\lambda^4)}{4\lambda^2(27-14\lambda+3\lambda^2)^2}$
2	$\frac{-9}{4}, \frac{-8}{9}, \infty, 0$ $2, 2, 2, 6$	$\frac{20160+42008\lambda+41331\lambda^2+17388\lambda^3+3888\lambda^4}{4\lambda^2(72+113\lambda+36\lambda^2)^2}$
3	$-10, 0, \infty, \frac{1}{8}$ $2, 2, 2, 6$	$\frac{3(25-210\lambda+2179\lambda^2+216\lambda^3+16\lambda^4)}{(1-8\lambda)^2\lambda^2(10+\lambda)^2}$
4	$\frac{-5}{9}, 0, \infty, 3$ $2, 2, 2, 6$	$\frac{2025+4295\lambda+9156\lambda^2-1809\lambda^3+729\lambda^4}{12(-3+\lambda)^2\lambda^2(5+9\lambda)^2}$
5	$\delta, \bar{\delta}, \infty, 0$ $2, 2, 2, 4$	$\frac{3840+2072\lambda+43\lambda^2+220\lambda^3+48\lambda^4}{4\lambda^2(32+13\lambda+4\lambda^2)^2}$
6	$4, 1, \infty, 0$ $2, 2, 2, 4$	$\frac{3(20-33\lambda+28\lambda^2-7\lambda^3+\lambda^4)}{4\lambda^2(4-5\lambda+\lambda^2)^2}$
7	$\frac{-25}{3}, 0, \infty, \frac{1}{5}$ $2, 2, 2, 4$	$\frac{15(125-675\lambda+4244\lambda^2+501\lambda^3+45\lambda^4)}{4(1-5\lambda)^2\lambda^2(25+3\lambda)^2}$
8	$\frac{-1}{3}, 0, \infty, 1$ $2, 2, 2, 4$	$\frac{3(1+3\lambda+13\lambda^2-6\lambda^3+9\lambda^4)}{4(-1+\lambda)^2(\lambda+3\lambda^2)^2}$
9	$1, -1, \infty, 0$ $2, 2, 2, 3$	$\frac{32+49\lambda^2+27\lambda^4}{36\lambda^2(-1+\lambda^2)^2}$
10	$\frac{-27}{4}, \frac{-1}{2}, \infty, 0$ $2, 2, 2, 3$	$\frac{648+1824\lambda+3157\lambda^2+476\lambda^3+48\lambda^4}{\lambda^2(27+58\lambda+8\lambda^2)^2}$
11	$\infty, 0, -1, 1$ $2, 2, 2, 3$	$\frac{27+5\lambda+64\lambda^2+5\lambda^3+27\lambda^4}{36\lambda^2(-1+\lambda^2)^2}$

Table 2: Sigma invariants and Schwarzian indices.

points. The equations of Elkies are in non-normalized form $ay'' + by' + cy = 0$, so one needs to normalize first before computing the sigma invariant.

Here δ_1 is a solution to the equation $16t^2 + 13t + 8 = 0$.

For discriminant 6 Elkies does not write down the equation explicitly, though he gives a recipe to discover one of 4 possible equations. As he notes, however, since there are only 3 elliptic points, the sigma invariant is uniquely determined by the indices of ellipticity. Suppose that these are at $t = 0, 1, \infty$. The most general form of σ is

$$\sigma = \left(\frac{a}{t^2} + \frac{b}{(t-1)^2} + \frac{c}{t} + \frac{d}{t-1}\right)(dt)^2$$

V_{10}^*	$t(t-2)(t-27)y''$ $+\frac{10t^2-203t+216}{6}y'$ $+(\frac{7t}{144}-\frac{7}{18})y=0$ $\sigma = \frac{10368-7296\,t+3157\,t^2-119\,t^3+3\,t^4}{4\,(-27+t)^2\,(-2+t)^2\,t^2}$	$27, 2, \infty, 0$ $2, 2, 2, 3$
V_{14}^*	$t(16t^2+13t+8)y''$ $+(24t^2+13t+4)y'$ $+(\frac{3}{4}t+\frac{3}{16})y=0$ $\sigma = \frac{192+440\,t+43\,t^2+1036\,t^3+960\,t^4}{4\,t^2\,(8+13\,t+16\,t^2)^2}$	$\delta_1, \bar{\delta}_1, 0, \infty$ $2, 2, 2, 4$
V_{15}^*	$(t-81)(t-1)ty''$ $+(\frac{3t^2}{2}-82t+\frac{81}{2})y'$ $+(\frac{t}{18}-\frac{1}{2})y=0$ $\sigma = \frac{35t^4-3680t^3+244242t^2-244944t+177147}{36(t-81)^2(t-1)^2t^2}$	$1, 81, 0, \infty$ $2, 2, 2, 6$

Table 3: Elkies's list of differential equations.

and one has the condition $c + d = 0$ to avoid a pole of order 3 at ∞. The residues are a, b and $a + b + d$ at 0, 1 and ∞ respectively, from which all the coefficients are easily determined. In the case at hand, Elkies chooses the coordinate t so that the indices are $2, 4, 6$ at 0, 1 and ∞ respectively. This gives

$$\sigma = \left(\frac{3}{4t^2} + \frac{15}{16(t-1)^2} + \frac{103}{144t} - \frac{103}{144(t-1)}\right)(dt)^2.$$

8. comparison with the results of Elkies

In this section we compare Elkies's list with the list of differential equations we obtained in Section 5. Let us explain more precisely what we mean by this. In each of our 11 examples we have a family of K3 surfaces over $\mathbb{P}^1$. In their relative H^2's we have identified a certain rank 3 sub-local system and we know that is is the symmetric square of another, rank 2, local system. Finally, we computed a Picard-Fuchs equation for this rank 2 system. On the other hand, over each Shimura curve V there is a family of abelian surfaces with quaternionic multiplication and in its relative H^1 there is a certain rank 2 local system. Elkies implicitly computes a Picard-Fuchs equation for these systems.

Algebraic Geometry [3] tells us that for each of the 11 examples there is a Shimura curve V such that every K3 surface appearing as a fiber of the family is related to a Kummer surface of a QM abelian surface appearing in the universal family above V. This gives rise to a correspondence between the projective line at the base of the family of K3 surfaces and V, which is of the form $V \leftarrow X \rightarrow \mathbb{P}^1$.

Algebraic Geometry further tells us that the rank 2 local system will be preserved under the correspondence, so that the pullbacks of the two systems to X will be isomorphic. Having only this information at hand, in addition to the computed sigma invariants of the systems on both V and $\mathbb{P}^1$, the transformation formula (6.2) and the behavior of the Schwarzian indices, we attempt in this section to find the required correspondence, or at least a list of possible correspondences. We are satisfied if we find a correspondence such that the pullbacks of the two sigma invariants to X are the same. This is not a proof that the correspondence is the correct one, a fact which then needs to be established by more precise means [3, Section 8]. It can nevertheless be used to exclude certain possible correspondences (see [3, Subsection 8.2]). In what follows we indicate our guess for the correspondence, point that it is indeed compatible with the sigma invariants (which can be checked directly using (6.2)) and note if we have identified the correspondence on the level of moduli problems in [3].

No. 10 - Corresponds to discriminant 10. The correspondence has to carry the special points of the fibration at $\lambda = -27/4, -1/2, \infty, 0$ with respective Schwarzian indices $2, 2, 2, 3$ to the special points $t = 27, 2, \infty, 0$ with the same respective indices for the equation that Elkies finds for $V_{10}/(w_5, w_2)$. It is trivial to guess the relation $t = -4\lambda$. A change of variables for the sigma invariants confirms this. It can be proved rigorously (see [3, Subsection 8.3]) that the λ-line is isomorphic to $V_{14}/(w_2, w_5)$

No. 5 - Corresponds to discriminant 14. The correspondence has to carry the special points of the fibration at $\lambda = \delta, \bar{\delta}, \infty, 0$ with respective Schwarzian indices $2, 2, 2, 4$ to the special points $t = \delta_1, \bar{\delta}_1, 0, \infty$ with the same respective indices for the equation that Elkies finds for $V_{14}/(w_7, w_2)$. Since δ is a solution of the equation $4x^2 + 13x + 32 = 0$ it is easy to guess the relation $t = 2/\lambda$. A change of variables for the sigma invariants confirms this. It can be proved rigorously (see [3, Subsection 8.1]) that the λ-line is indeed isomorphic to the Shimura curve $V_{14}/(w_2, w_7)$.

No. 3 - Corresponds to discriminant 15. The correspondence has to carry the special points $t = 1, 81, 0, \infty$ with respective Schwarzian indices $2, 2, 2, 6$ for the equation that Elkies finds for $V_{15}/(w_5, w_3)$ to the special points of the fibration at $\lambda = -10, 0, \infty, 1/8$ with the same indices. This can be done with the change of variables $\lambda = \frac{t-81}{8t}$ and a change of variables for the sigma invariants confirms this. It can be proved rigorously (see [3, Lemma 10]).

No. 4 - Corresponds to discriminant 10. In this case we speculated (but could not prove) that the λ-line was the curve $V_{10,3}/\langle w_2, w_5, w_3\rangle$. Here we show this is consistent with the Picard-Fuchs equations. According to Elkies, the curve $V_{10,3}/\langle w_2, w_5\rangle$ is a degree 4 cover of V_{10}^*, rational with a coordinate x such that

$$t = \frac{(-6+6\,x)^3}{(1+x)^2\,(17-10\,x+9\,x^2)}$$

(this is equation (57) in [5] but the 7 there should be corrected to 17, as for example in the computation between equations (59) and (60)). From the expression

$$\frac{6^3}{9\tau+8} \text{ with } \tau = \frac{(3x^2+5)^2}{9(x-1)^3}$$

which is also from (57) in [5], it is easy to see that the map $x \to t$ sends $1, \infty, -1, 5$ to $0, 0, \infty, 2$ with multiplicities $3, 1, 2, 2$ respectively, $\pm\sqrt{-5/3}$ to 27 with multiplicity 2, the two roots of $9x^2 - 10x + 5 = 0$ to 2 with multiplicity 1, and the two roots of $9x^2 - 10x + 17 = 0$ to ∞ with multiplicity 1. Thus, the elliptic points for $V_{10,3}/\langle w_2, w_5\rangle$ are going to be at $x = \infty$ with multiplicity 3 and at the roots of the equations $9x^2 - 10x + 5 = 0$ and $9x^2 - 10x + 17 = 0$ with multiplicity 2. The involution w_3 is given by Elkies, just after (57), to be $w_3(x) = \frac{10}{9} - x$ and so a coordinate on the quotient is given by

$$\zeta = 9\left(x - \frac{5}{9}\right)^2 = 9x^2 - 10x + \frac{25}{9}.$$

We see that the elliptic points will map to $\zeta = \infty, -20/9, -128/9$, so these will be elliptic of degree $6, 2, 2$, and in addition the ramification point 0 is elliptic of degree 2. We can map ζ to λ with the correct orders of ramification by

$$\lambda = 3 - \frac{128}{3\zeta + \frac{128}{3}} = 3 - \frac{128}{3(9x^2 - 10x + 17)}.$$

This is confirmed by the matching of the sigma invariants.

Other examples correspond to discriminant 6. Some of them are directly interrelated. Consider examples number 6 and 8. The special points are $\lambda_1 = 4, 1, \infty, 0$ and $\lambda_2 = -1/3, 0, \infty, 1$ respectively with the same indices. There is a finite number of ways to carry one set to the other preserving the indices, and testing each one using the sigma invariants we get the correct transformation $\lambda_1 = 1 - 1/\lambda_2$. It turns out (see [3, Subsection 8.7]) that making this change of variable makes the two base elliptic fibrations isogenous.

Consider next examples 9 and 11. The special points in both cases are $\infty, 0, -1, 1$ but with indices $2, 3, 2, 2$ in example 9 and $2, 2, 2, 3$ in example 11. Testing again the finite number of possible transformations with the sigma invariants gives $\lambda_2 = (1 + \lambda_1)/(1 - \lambda_1)$. It is proved in [3, Subsection 8.8] that this again makes the two base fibrations isogenous. Thus, we do not need to consider examples 8 and 11, being isogenous to 6 and 9 respectively.

No. 6 - For $V_6/(w_2, w_3)$ it turns out to be slightly better to work with the coordinate $\zeta = 1/(1-t)$ so that the elliptic points are at $\zeta = 0, 1$ and ∞ with indices 6, 2 and 4 respectively. To get the required ellipticity behavior for the λ-line, with elliptic points at $\lambda = 4, 1, \infty, 0$ with indices $2, 2, 2, 4$, one may consider a degree 3 map having ramification type $(2, 1)$ over ∞, producing indices 2 and 4, ramification 3 above 0, producing an additional index 2 and ramification type $(1, 2)$ above 1, producing one additional index 2 and an additional non-elliptic point. This can be arranged by a map of the form $\zeta = c\lambda^{-1}((\lambda-1)^3$ for the appropriate c for which this ramifies above 1. So c is the value for which one of the roots of the derivative $(c(\lambda-1)^3 - \lambda)' = 3c(\lambda-1)^2 - 1$ is mapped to $\zeta = 1$. We have for that root

$$1 = c\frac{(\lambda-1)^3}{\lambda} = \frac{\lambda-1}{3\lambda}$$

hence $\lambda = -1/2$ and $c = 4/27$. Consider the equation for λ to map to $\zeta = 1$. As an equation on $\lambda - 1$ the sum of the 3 roots should be 0, hence the third root should be 3, so that the additional preimage of 1 is 4. Thus, the cover we found matches perfectly with the λ-line. Summarizing, we have

$$t = 1 - \frac{1}{\zeta} = 1 - \frac{27\lambda}{4(\lambda-1)^3}.$$

This is confirmed by the sigma invariants.

No. 9 - Here the elliptic points are at $1, -1, \infty$ and 0 with indices $2, 2, 2$ and 3. In trying to relate them with the elliptic points for ζ it is very easy to guess the relation $\zeta = \lambda^2$, and this is confirmed by the σ-invariants. Thus, the λ-line is a double cover of V_6^* ramified above the elliptic points of order 4 and 6. This was used in [3, Subsection 8.2] to prove that the λ-line is $V_6/\langle w_6\rangle$.

No. 7 - The elliptic points are $-25/3, 0, \infty, 1/5$ and with indices 2 at the first 3 points and 4 at the last point. Here we guess that the base for the family of twists is isomorphic to the quotient $V_{6,5}/\langle w_2, w_3, w_5\rangle$. Elkies finds a coordinate x on $V_{6,5}/\langle w_2, w_3\rangle$ for which the action of w_5 is given by [5, (37)] $w_5(x) = (42 - 55x)/(55 + 300x)$. The two fixed points of this action are $7/30$ and $-3/5$. Thus, a coordinate on the quotient is provided by

$$y = ((x + 3/5)/(x - 7/30))^2. \tag{8.1}$$

The map from $X_0^*(5)$ to $V_6/(w_2, w_3)$ is given by [5, Equation 36] by

$$t = (1 + 3x + 6x^2)^2(1 - 6x + 15x^2) = 1 + 27x^4(5 + 12x + 20x^2).$$

The relation between t and ζ is

$$\zeta = 1/(1 - t) = \frac{-1}{27x^4(5 + 12x + 20x^2)}. \tag{8.2}$$

The ramification above $\zeta = 0$ is of order 6 at infinity. The ramification over $\zeta = \infty$ is of order 4 at $x = 0$ and order 1 at each of the roots of $5 + 12x + 20x^2$. The ramification over $\zeta = 1$, or $t = 0$, is of order 1 at each of the roots of $1 - 6x + 15x^2$ and of order 2 at each of the roots of $1 + 3x + 6x^2$. Thus the elliptic points of the cover are of order 4 at the roots of $5 + 12x + 20x^2$ and of order 2 at each of the roots of $1 - 6x + 15x^2$. These two pairs of points are interchanged by w_5. The elliptic points of order 4 are mapped to $y = -9/16$ while those of order 2 are mapped to $y = -24$. In addition we get elliptic points at the ramification points of the covering at $y = 0$ and $y = \infty$, both of order 2. Now, if we guessed correctly, there would be a Möbius transformation sending the 4 elliptic points to the 4 singular points of the elliptic surface, sending $y = -9/16$ to $\lambda = 1/5$. It is easy to check that the unique transformation of this type is $\lambda = -25y/(3(24 + y))$. Composing with (8.1) we get

$$\lambda = \frac{-(3 + 5x)^2}{5(1 - 6x + 15x^2)}.$$

This, as usual, is confirmed by pulling back the σ-invariants. Our guess can be proved rigorously [3, Subsection 8.6].

No. 2 - The elliptic points are at $-9/4, -8/9, \infty, 0$ with indices $2, 2, 2, 6$ respectively.

We try to guess that this family corresponds to $V_{6,7}/\langle w_2, w_3, w_7\rangle$. Elkies writes a coordinate x on $V_{6,7}/\langle w_2, w_3\rangle$ for which the action of w_7 is given by [5, (40)] $w_7(x) = (116 - 9x)/(9 + 20x)$. The two fixed points of this action are $-29/10$ and 2. Thus, a coordinate on the quotient is provided by

$$y = \left(\frac{x + 29/10}{x - 2}\right)^2. \tag{8.3}$$

The map from $V_{6,7}/\langle w_2, w_3\rangle$ to V_6^* is given by [5, (39)] by

$$t = \frac{-\left(25 + 4\,x + 4\,x^2\right)\,\left(2 - 12\,x + 3\,x^2 - 2\,x^3\right)^2}{108\,\left(37 - 8\,x + 7\,x^2\right)}$$
$$= 1 - \frac{(2x^2 - x + 8)^4}{108(7x^2 - 8x + 37)}.$$

This looks nicer with ζ

$$\zeta = 1/(1-t) = \frac{108\,\left(37 - 8\,x + 7\,x^2\right)}{\left(8 - x + 2\,x^2\right)^4}. \tag{8.4}$$

The preimage of $\zeta = 0$ is 6 times ∞ plus the two roots of $7x^2 - 8x + 37$. The preimage of $\zeta = \infty$ is 4 times each of the roots of $2x^2 - x + 8$. The preimage of $\zeta = 1$, or $t = 0$, is 2 times each of the roots of $2 - 12x + 3x^2 - 2x^3$ plus each of the roots of $4x^2 + 4x + 25$. Thus the elliptic points of $V_{6,7}/\langle w_2, w_3\rangle$ are the two roots of $7x^2 - 8x + 37$ with index 6, and the roots of $4x^2 + 4x + 25$ with index 2. Two pairs of points are interchanged by w_7. The elliptic points of order 6 are mapped to $y = -243/100$ while those of order 2 are mapped to $y = -24/25$. In addition we get elliptic points at the ramification points of the covering at $y = 0$ and $y = \infty$, both of order 2. Now, if we guessed correctly, there would be a Möbius transformation sending the 4 elliptic points to the 4 singular points of the elliptic surface, sending $y = -243/100$ to $\lambda = 0$. It is easy to check that the unique transformation of this type is

$$\lambda = \frac{486 + 200y}{-216 - 225y}.$$

Composing with (8.3) we get

$$\lambda = \frac{-8\,\left(37 - 8\,x + 7\,x^2\right)}{9\,\left(25 + 4\,x + 4\,x^2\right)}.$$

This is confirmed by the σ-invariants. Our guess can be proved rigorously [3, Subsection 8.5].

9. Software

All the relevant computations for this work can be downloaded from `http://www.math.bgu.ac.il/~bessera/picard-fuchs/`. They are in the form of a MATHEMATICA notebook. The notebook is self explanatory. It loads the following files:

- pf.m - main file contains all the algorithms
- data.m - file contains the equations for the elliptic fibrations on Herfurtner's list
- elkiesdata.m - contains the differential equations obtained by Elkies.

The relevant functions contained in the file pf.m

- picfucs function - computes the Picard-Fuchs equation for an elliptic surface. This is just a translation into Mathematica of the Maple script by Beukers, which may be found at `http://www.staff.science.uu.nl/~beuke106/picfuchs.maple`
- Twistpf function - computes the Picard-Fuchs equation for the family of twists given the equation for the original elliptic fibration.
- SigChVar function - makes a change of variable for the sigma invariant.
- DRes function - computes the residue of a quadratic differential.

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