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Differential zeros of period integrals and generalized hypergeometric functions

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In this paper, we study the zero loci of locally constant sheaves of the form $\delta\Pi$, where Π is the period sheaf of the universal family of CY hypersurfaces in a suitable ambient space X, and δ is a given differential operator on the space of sections $V^\vee = \Gamma(X, K_X^{-1})$. Using earlier results of three of the authors and their collaborators, we give several different descriptions of the zero locus of $\delta\Pi$. As applications, we prove that the locus is algebraic and in some cases, non-empty. We also give an explicit way to compute the polynomial defining equations of the locus in some cases. This description gives rise to a natural stratification to the zero locus.

1. Introduction

Zeros of special functions have been of interests to many authors since the times of Riemann. He of course famously conjectured that the zeros of the Riemann zeta functions occur only on a certain critical line. Inspired by works of Stieltjes, Hilbert and Klein, Hurwitz [Hu1][Hu2] and Van Vleck [V] determined the number of zeros of the Gauss hypergeometric function ${}_2F_1(a, b, c; z)$ for real a, b, c. Subsequently, many authors generalized their results to confluent hypergeometric functions. Runckel [R] gave a simpler proof of the results of Hurwitz and Van Vleck using the argument principle. Eichler and Zagier [EZ] gave a complete description of the zeros of the Weierstrass $\wp$ function in terms of a classical Eisenstein series. Duke and Imamoḡlu [DI] later used it to prove transcendence of values of certain classical generalized hypergeometric functions at algebraic arguments. More recently following Hille [H], Ki and Kim [KiK] studied the zeros of generalized hypergeometric functions of the form ${}_pF_p$. For real parameters for such a function, they showed that it can only have finitely many zeros, and that they are all real.

Since all (except the Riemann zeta function) of those special functions are solutions to ordinary differential equations, it is natural to consider the higher dimensional analogues of these functions and their zeros. It is well known that the theory of Gel'fand-Kapranov-Zelevinsky (GKZ) hypergeometric functions [GKZ] generalize classical *special functions*, including the Euler-Gauss, Appell, Clausen-Thomae, Lauricella hypergeometric functions, and their multivariable generalizations. Therefore, GKZ hypergeometric functions can be viewed as generalized special functions. Since the theory of tautological systems generalizes the GKZ theory [LY], solutions to tautological systems and their derivatives can be thought of as further generalizations of special functions. The zero loci of their derivatives amount to zeros of these vast generalizations of those for classical special functions.

In this paper, we shall study the zeros of derivatives of GKZ hypergeometric functions and their generalizations in the context of Calabi-Yau geometry. It is well-known that period integrals of CY hypersurfaces in a toric variety are GKZ hypergeometric functions. Moreover, since these functions are local sections of locally constant sheaves, each admits a multivalued analytic continuation. Thus it is natural to consider zero loci that are monodromy invariant. Recall that the period sheaf Π of the universal family of smooth CY hypersurfaces in a suitable ambient space X form a locally constant sheaf, which is generated by pairings between a nonvanishing holomorphic top form and middle dimensional cycles on a CY hypersurface.

Since every such hypersurface has at least one nonzero period, the zero locus of the period sheaf is always empty. However, as it turns out, it is more natural to consider the zero locus of a locally constant sheaf of the form $\delta\Pi$, where δ is a differential operator on the affine space $V^\vee = \Gamma(X, K_X^{-1})$. This zero locus will be the main object of study in this paper.

We will follow closely the notations introduced in [HLZ][BHLSY]. Given a Lie algebra $\hat{\mathfrak{g}}$, a $\hat{\mathfrak{g}}$-module $V^\vee$, and a $\hat{\mathfrak{g}}$-invariant ideal I of the commutative algebra $\mathbb{C}[V]$, then a *tautological system* τ is a $D_{V^\vee}$-module of the form

$$\tau = D_{V^\vee}/(D_{V^\vee}\tilde{I} + D_{V^\vee}\hat{\mathfrak{g}})$$

where $\tilde{I} \subset D_{V^\vee}$ is the Fourier transform of I. In this paper, we consider the following special case of τ.

Let G be a connected complex algebraic group. Let X be a complex projective G-variety and let L be a very ample G-equivariant line bundle over X. This gives rise to a G-equivariant embedding

$$X \to \mathbb{P}(V),$$

where $V = \Gamma(X, L)^\vee$. We assume that the action of G on X is locally effective, i.e. ker $(G \to \mathrm{Aut}(X))$ is finite. Let $\hat{G} := G \times \mathbb{C}^\times$, whose Lie algebra is $\hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C}e$, where e acts on V by identity. We denote by $Z : \hat{G} \to \mathrm{GL}(V)$ the group action induced on V, and by $Z : \hat{\mathfrak{g}} \to \mathrm{End}(V)$ the corresponding Lie algebra representation. Note that under our assumption, $Z : \hat{\mathfrak{g}} \to \mathrm{End}(V)$ is injective.

Let $\hat{\iota} : \hat{X} \subset V$ be the cone of X, and $I(\hat{X})$ its defining ideal. Let $\beta : \hat{\mathfrak{g}} \to \mathbb{C}$ be a Lie algebra homomorphism. Then a *tautological system* as defined in [LSY][LY] is the cyclic D-module on $V^\vee$

$$\tau(X, L, G, \beta) = D_{V^\vee}/\big(D_{V^\vee}\tilde{I} + D_{V^\vee}(Z(x) + \beta(x), x \in \hat{\mathfrak{g}})\big),$$

where

$$\tilde{I} = \{\tilde{P} \mid P \in I(\hat{X})\}$$

is the ideal of the commutative subalgebra $\mathbb{C}[\partial] \subset D_{V^\vee}$ obtained by the Fourier transform of $I(\hat{X})$. Here $\tilde{P}$ denotes the Fourier transform of P.

Given a basis $\{a_1, \ldots, a_n\}$ of V, we have $Z(x) = \sum_{ij} x_{ij} a_i \frac{\partial}{\partial a_j}$, where (x_{ij}) is the matrix representing x in the basis. Since the a_i are also linear coordinates on $V^\vee$, we can view $Z(x) \in \mathrm{Der}\, \mathbb{C}[V^\vee] \subset D_{V^\vee}$. In particular, the identity operator $Z(e) \in \mathrm{End}\, V$ becomes the Euler vector field on $V^\vee$.

Let X be an m-dimensional compact complex manifold such that its anticanonical line bundle K_X^{-1} is very ample. Let $L := K_X^{-1}$. We shall regard the basis elements a_i of $V = \Gamma(X, L)^\vee$ as linear coordinates on $V^\vee$. Let $B := \Gamma(X, L)_{sm} \subset V^\vee$ be the space of smooth sections. Let $\pi : \mathcal{Y} \to B$ be the family of smooth CY hyperplane sections $Y_b \subset X$, and let $\mathbb{H}^{\text{top}}$ be the Hodge bundle over B whose fiber at $b \in B$ is the line $\Gamma(Y_b, \omega_{Y_b}) \subset H^{m-1}(Y_b)$. In [LY] the period integrals of this family are constructed by giving a canonical trivialization of $\mathbb{H}^{\text{top}}$. Let Π be the period sheaf of this family, i.e. the locally constant sheaf generated by the period integrals. Let G be a connected algebraic group acting on X.

Theorem 1.1 (See [LY]). *The period integrals of the family $\pi : \mathcal{Y} \to B$ are solutions to*

$$\tau \equiv \tau(X, K_X^{-1}, G, \beta_0)$$

where β_0 is the Lie algebra homomorphism with $\beta_0(\mathfrak{g}) = 0$ and $\beta_0(e) = 1$.

In [LSY] and [LY], it is shown that if G acts on X by finitely many orbits, then τ is regular holonomic. *We shall assume this holds throughout the paper.*

Let $R = \mathbb{C}[V]/I(\hat{X})$. Let $f = \sum a_i a_i^\vee$ be the universal section. Then the Lie algebra $\hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C}e$ acts on $R[V^\vee]e^f$ by the homomorphism $Z^\vee : \hat{\mathfrak{g}} \to \operatorname{End} V^\vee$ which is dual to Lie algebra action Z on V. Thus it takes the form

$$Z^\vee(x) = -\sum x_{ij} a_j^\vee \frac{\partial}{\partial a_i^\vee} - \beta(x), \qquad x \in \hat{\mathfrak{g}}.$$

Here $\{a_i\}$,$\{a_i^\vee\}$ are the bases of $V, V^\vee$ dual to each other. Note that since $I(\hat{X})$ is a $\hat{\mathfrak{g}}$-invariant ideal of $\mathbb{C}[V]$, there is an induced $\hat{\mathfrak{g}}$-action on R hence on $R[V^\vee]e^f = R[a]e^f$. Recall that the $D_{V^\vee}$-module structure on $R[V^\vee]e^f$ is that $a_i \in D_{V^\vee}$ acts by left multiplication, while $\partial_i \in D_{V^\vee}$ acts by the usual derivative $\frac{\partial}{\partial a_i}$. In particular, this action commutes with the $\hat{\mathfrak{g}}$-action given by $Z^\vee$, and with left multiplication by R.

Theorem 1.2. *[BHLSY],[HLZ] There is a canonical isomorphism of $D_{V^\vee}$-modules*

$$\begin{array}{ccc} \tau(X, L, G, \beta_0) & \stackrel{\Phi}{\longleftrightarrow} & R[V^\vee]e^f/\hat{\mathfrak{g}}(R[V^\vee]e^f) \\ 1 & \longleftrightarrow & e^f. \end{array}$$

Denote by $\mathrm{sol}(\tau)$ the sheaf of classical solutions to τ. We will prove in Section 2.

Theorem 1.3. *Let $\delta \in D_{V^\vee}$, and $b \in V^\vee$. The following statements are equivalent:*

1) $\delta\mathfrak{s}(b) = 0$ *for all* $\mathfrak{s} \in sol(\tau)_b$.
2) $\delta e^{f(b)} = 0$ *in* $Re^{f(b)}/\hat{\mathfrak{g}}Re^{f(b)}$, *i.e.* $\delta e^{f(b)} \in \hat{\mathfrak{g}}Re^{f(b)}$.

This theorem generalizes [CHL, Corollary 4.2].

For any $\delta \in D_{V^\vee}$, we introduce

$$\mathcal{N}(\delta) = \{b \in B \mid \delta\mathfrak{s}(b) = 0, \forall \mathfrak{s} \in \mathrm{sol}(\tau)_b\}. \tag{1.1}$$

This will be a main object of study in this paper. By Theorem 1.3, we have

$$\mathcal{N}(\delta) = \left\{b \in B \mid \delta e^{f(b)} \in \hat{\mathfrak{g}}(Re^{f(b)})\right\}.$$

In the special case $\delta = p(\partial) \in \mathbb{C}[\partial]$ has constant coefficients, we have

$$p(\partial)e^{f(b)} = p(a^\vee)e^{f(b)}$$

Thus making the identification $R \equiv \tilde{R}$ by Fourier transform $\tilde{}: R \to \tilde{R}$, $p(a^\vee) \mapsto p(\partial)$, we get

$$\mathcal{N}(p) \equiv \mathcal{N}(\tilde{p}) = \left\{b \in B \mid p(a^\vee)e^{f(b)} \in \hat{\mathfrak{g}}(Re^{f(b)})\right\}.$$

This recovers the definition of $\mathcal{N}(p)$ introduced in [CHL].

We will prove in Section 5.

Theorem 1.4. *If $\delta \in D_{V^\vee}$ is homogeneous under scaling by $\mathbb{C}^\times$, $\mathcal{N}(\delta)$ is algebraic.*

In Sections 6 and 8, we discuss the non-emptiness of $\mathcal{N}(\delta)$ in a number of cases. In Section 7, we give an explicit way to compute the polynomial equations defining $\mathcal{N}(\delta)$ in $\mathbb{P}V^\vee$ in the case $X = \mathbb{P}^m$. We also show that $\mathcal{N}(\delta)$ has a natural stratification in this case.

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2. A coinvariant description of differential zeros

Let $J = D_{V^\vee}\tilde{I} + D_{V^\vee}(Z(x) + \beta_0(x), x \in \hat{\mathfrak{g}})$ be the defining left ideal of a regular holonomic tautological system τ. Then $\tau = D_{V^\vee}/J$. Since τ is cyclic, $\mathrm{sol}(\tau)$ can be identified as a subsheaf of local analytic functions in $\mathcal{O}_{V^\vee} \equiv \mathcal{O}^{an}_{V^\vee}$ annihilated by the left ideal J. Then we have the canonical isomorphism of sheaves

$$\mathcal{H}om_{D_{V^\vee}}(\tau, \mathcal{O}_{V^\vee}) \to \mathrm{sol}(\tau), \qquad \varphi \mapsto \varphi(1).$$

Theorem 2.1. *Let $\delta \in D_{V^\vee} \equiv \mathbb{C}[a_i, \partial_{a_i}]$, and $b \equiv \sum b_i a_i^\vee \in V^\vee$. The following statements are equivalent:*

1) $\delta\mathfrak{s}(b) = 0$ *for all* $\mathfrak{s} \in sol(\tau)_b$,
2) $\delta e^{f(b)} = 0$ *in* $Re^{f(b)}/\hat{\mathfrak{g}}Re^{f(b)}$, *i.e.* $\delta e^{f(b)} \in \hat{\mathfrak{g}}Re^{f(b)}$.
3) $\delta \in \mathfrak{m}_b D_{V^\vee} + J$, *where* $\mathfrak{m}_b := \langle a_i - b_i \rangle$ *is the ideal sheaf of the point* b.

Proof. First we prove (1)⇔(2). Consider the evaluation map

$$e_b : D_{V^\vee,b} \to \oplus_\alpha \mathbb{C}\partial^\alpha \equiv \mathbb{C}[\partial], \qquad \sum g_\alpha \partial^\alpha \mapsto \sum g_\alpha(b)\partial^\alpha.$$

Let $i_b : b \to V^\vee$ be the inclusion and $\mathcal{O}_b \equiv \mathbb{C}$ be the constant sheaf over b.

Claim 2.2. *The morphism*

$$e'_b : i_b^* D_{V^\vee} = \mathcal{O}_b \otimes_{i_b^{-1}\mathcal{O}_{V^\vee}} D_{V^\vee,b} \xrightarrow{\simeq} e_b(D_{V^\vee,b}) = \mathbb{C}[\partial], \qquad 1 \otimes \delta \mapsto e_b(\delta)$$

is well-defined and it is an isomorphism.

Proof. It is clear that the map

$$e'_b : \mathcal{O}_b \otimes_{\mathbb{C}} D_{V^\vee,b} \to e_b(D_{V^\vee,b}), \qquad 1 \otimes \delta \mapsto e_b(\delta)$$

is well-defined. Let $f \in i_b^{-1}\mathcal{O}_{V^\vee} = \mathcal{O}_{V^\vee,b}$, then

$$1 \otimes f\delta - f(b) \otimes \delta \mapsto e_b(f\delta) - f(b)e_b(\delta) = 0.$$

Thus e'_b descends and it is well-defined on $i_b^* D_{V^\vee}$.

Surjectivity: For any $\delta_c := \sum_\alpha c_\alpha \partial^\alpha \in \mathbb{C}[\partial]$ where $c_\alpha \in \mathbb{C}$, we have $\delta_c \in D_{V^\vee}$ and $e_b(\delta_c) = \delta_c$. Thus $e'_b(1 \otimes \delta_c) = e_b(\delta_c) = \delta_c$.

Injectivity: Let $\mathfrak{m}_b := \langle a_i - b_i \rangle$ be the ideal sheaf of the point b. Then $e'_b(1 \otimes \sum_\alpha g_\alpha \partial^\alpha) = \sum g_\alpha(b)\partial^\alpha = 0$ implies that $g_\alpha \in \mathfrak{m}_b$ for all α. Thus $1 \otimes \sum_\alpha g_\alpha \partial^\alpha = \sum_\alpha g_\alpha(b) \otimes \partial^\alpha = 0$, which means that $\ker e'_b = 0$. □

Since $\mathcal{O}_b \otimes_{i_b^{-1}\mathcal{O}_{V^\vee}} J_b = \{1 \otimes \delta \mid \delta \in J_b\}$, similarly we can show that

$$\mathcal{O}_b \otimes_{i_b^{-1}\mathcal{O}_{V^\vee}} J_b \simeq e_b(J_b).$$

Next we claim that e_b induces a map $e_b : \tau_b \to i_b^*\tau$. Since $i_b^{-1}\tau = \tau_b$, we have

$$i_b^*\tau := \mathcal{O}_b \otimes_{i_b^{-1}\mathcal{O}_{V^\vee}} i_b^{-1}\tau = \mathcal{O}_b \otimes_{i_b^{-1}\mathcal{O}_{V^\vee}} \tau_b.$$

Consider the exact sequence

$$0 \to J_b \xrightarrow{\iota} D_{V^\vee,b} \xrightarrow{p} \tau_b = D_{V^\vee,b}/J_b \to 0.$$

Since tensoring over any ring is right exact, we have

$$\mathcal{O}_b \otimes_{i_b^{-1}\mathcal{O}_{V^\vee}} J_b \xrightarrow{\mathcal{O}_b\otimes\iota} \mathcal{O}_b \otimes_{i_b^{-1}\mathcal{O}_{V^\vee}} D_{V^\vee,b} \xrightarrow{\mathcal{O}_b\otimes p} \mathcal{O}_b \otimes_{i_b^{-1}\mathcal{O}_{V^\vee}} (D_{V^\vee,b}/J_b) \to 0.$$

Thus

$$\ker \mathcal{O}_b \otimes p = \operatorname{Im} \mathcal{O}_b \otimes \iota = \mathcal{O}_b \otimes_{i_b^{-1}\mathcal{O}_{V^\vee}} J_b,$$

hence

$$i_b^*\tau = \mathcal{O}_b \otimes_{i_b^{-1}\mathcal{O}_{V^\vee}} (D_{V^\vee,b}/J_b) \simeq (\mathcal{O}_b \otimes_{i_b^{-1}\mathcal{O}_{V^\vee}} D_{V^\vee,b})/(\mathcal{O}_b \otimes_{i_b^{-1}\mathcal{O}_{V^\vee}} J_b).$$

Therefore by Claim 2.2 we have

$$i_b^*\tau \simeq (\mathcal{O}_b \otimes_{i_b^{-1}\mathcal{O}_{V^\vee}} D_{V^\vee,b})/(\mathcal{O}_b \otimes_{i_b^{-1}\mathcal{O}_{V^\vee}} J_b) \simeq e_b(D_{V^\vee,b})/e_b(J_b).$$

Now we have a surjective map

$$e_b : \tau_b \to e_b(D_{V^\vee,b})/e_b(J_b) \simeq i_b^*\tau.$$

Consider the pairing

$$\tau \otimes_{\mathbb{C}} \mathcal{H}om_{D_{V^\vee}}(\tau, \mathcal{O}_{V^\vee}) \to \mathcal{O}_{V^\vee}, \quad \delta \otimes \varphi \mapsto \delta(\varphi). \tag{2.1}$$

And note that evaluation is $\mathcal{O}_{V^\vee}$-bilinear. Taking a b-germ of (2.1) yields

$$\tau_b \otimes_{\mathbb{C}} \mathcal{H}om_{D_{V^\vee}}(\tau, \mathcal{O}_{V^\vee})_b \to \mathcal{O}_{V^\vee,b}.$$

Applying $\mathcal{O}_b \otimes_{i_b^{-1}\mathcal{O}_{V^\vee}} -$ to both sides, we get

$$\alpha : \mathcal{O}_b \otimes_{i_b^{-1}\mathcal{O}_{V^\vee}} \tau_b \otimes_{\mathbb{C}} \mathcal{H}om_{D_{V^\vee}}(\tau, \mathcal{O}_{V^\vee})_b \to \mathcal{O}_b \otimes_{i_b^{-1}\mathcal{O}_{V^\vee}} \mathcal{O}_{V^\vee,b}. \tag{2.2}$$

The morphism is given by $\alpha(1 \otimes \delta \otimes \varphi) = 1 \otimes \varphi(\delta) = e_b(\varphi(\delta))$. Here

$$\mathcal{O}_b \otimes_{i_b^{-1}\mathcal{O}_{V^\vee}} \mathcal{O}_{V^\vee,b} = i_b^*(\mathcal{O}_{V^\vee}) = \mathcal{O}_b.$$

Since τ is regular holonomic, it follows that

$$\mathcal{H}om_{D_{V^\vee}}(\tau, \mathcal{O}_{V^\vee})_b \xrightarrow{\simeq} \mathcal{H}om_{\mathbb{C}}(i_b^*\tau, \mathcal{O}_b),$$

where $\varphi \mapsto \bar{\varphi}$ and $\bar{\varphi}(e_b(\delta)) := e_b(\varphi(\delta))$.

Next, consider the canonical non-degenerate pairing

$$\beta : i_b^*\tau \otimes \mathcal{H}om_{\mathbb{C}}(i_b^*\tau, \mathcal{O}_b) \to \mathcal{O}_b \equiv \mathbb{C}, \tag{2.3}$$

together with pairing (2.2) we have a diagram

$$\begin{array}{ccc} \mathcal{O}_b \otimes_{i_b^{-1}\mathcal{O}_{V^\vee}} \tau_b \otimes_{\mathbb{C}} \mathcal{H}om_{D_{V^\vee}}(\tau, \mathcal{O}_{V^\vee})_b & \xrightarrow{\alpha} & \mathcal{O}_b \otimes_{i_b^{-1}\mathcal{O}_{V^\vee}} \mathcal{O}_{V^\vee,b} \\ \simeq \downarrow \gamma & & \Vert \\ i_b^*\tau \otimes \mathcal{H}om_{\mathbb{C}}(i_b^*\tau, \mathcal{O}_b) & \xrightarrow{\beta} & \mathcal{O}_b. \end{array}$$

Since

$$\beta \circ \gamma((1 \otimes \delta) \otimes \varphi) = \beta(e_b(\delta) \otimes \bar{\varphi}) = \bar{\varphi}(e_b(\delta)) = e_b(\varphi(\delta)) = \alpha(1 \otimes \delta \otimes \varphi),$$

the above diagram commutes.

Since

$$\mathcal{H}om_{D_{V^\vee}}(\tau, \mathcal{O}_{V^\vee}) \xrightarrow{\simeq} \mathrm{sol}(\tau), \quad \varphi \mapsto \varphi(1),$$

then condition (1): $\delta\mathfrak{s}(b) = 0$ for all $\mathfrak{s} \in \mathrm{sol}(\tau)_b$ is equivalent to

$$(\delta\varphi(1))(b) = (\varphi(\delta \cdot 1))(b) = e_b(\varphi(\delta)) = \beta(e_b(\delta) \otimes \bar{\varphi}) = 0$$

for all $\varphi \in \mathcal{H}om_{D_{V^\vee}}(\tau, \mathcal{O}_{V^\vee})$. By the non-degeneracy of pairing (2.3), this is equivalent to $e_b(\delta) = 0$ in $i_b^*\tau$.

On the other hand, by the isomorphism

$$\tau \xrightarrow{\simeq} (R[V^\vee]e^f/\hat{\mathfrak{g}}R[V^\vee]e^f), \qquad \delta \mapsto \delta e^f,$$

we have

$$\begin{aligned} i_b^*\tau \xrightarrow{\simeq} i_b^*(R[V^\vee]e^f/\hat{\mathfrak{g}}R[V^\vee]e^f) &\simeq i_b^*R[V^\vee]e^f/\hat{\mathfrak{g}}i_b^*R[V^\vee]e^f \\ &\simeq Re^{f(b)}/\hat{\mathfrak{g}}Re^{f(b)}. \end{aligned}$$

Thus $e_b(\delta) = 0$ in $i_b^*\tau$ is equivalent to $(\delta e^f)(b) = 0$ in $Re^{f(b)}/\hat{\mathfrak{g}}Re^{f(b)}$, which is the condition (2). This completes the proof of (1)⇔(2).

Next, we prove (2)⇔(3).

Claim 2.3. *The following diagram commutes:*

$$\begin{array}{ccc} \tau & \xrightarrow{\quad e_b \quad} & i_b^*\tau \\ \Phi \downarrow & & \downarrow \mathcal{O}_b\otimes\Phi \\ R[V^\vee]e^f/\hat{\mathfrak{g}}R[V^\vee]e^f & \xrightarrow{e_b} & Re^{f(b)}/\hat{\mathfrak{g}}Re^{f(b)} \end{array}$$

where the e_b are evaluation maps, and

$$\begin{aligned} \Phi\left(\sum g_\alpha \partial^\alpha\right) &= \left(\sum g_\alpha \partial^\alpha\right)\cdot e^f = \sum g_\alpha (a^\vee)^\alpha e^f, \\ (\mathcal{O}_b \otimes \Phi)\left(\sum g(b)_\alpha \partial^\alpha\right) &= \left(\left(\sum g(b)_\alpha \partial^\alpha\right)\cdot e^f\right)(b) \\ &= \left(\sum g(b)_\alpha (a^\vee)^\alpha e^f\right)(b) \\ &= \sum g(b)_\alpha (a^\vee)^\alpha e^{f(b)}. \end{aligned}$$

Define the map

$$\Theta_b : D_{V^\vee} \to Re^{f(b)}/\hat{\mathfrak{g}}Re^{f(b)}, \quad \delta \mapsto \delta e^{f(b)}.$$

Let

$$\bar{\Theta}_b : \tau = D_{V^\vee}/J \to Re^{f(b)}/\hat{\mathfrak{g}}Re^{f(b)}, \quad \delta \mapsto \delta e^{f(b)}.$$

Let

$$\Theta_b' : \tau \to \tau/\mathfrak{m}_b\tau \xrightarrow{\simeq} i_b^*\tau, \quad \delta \mapsto \delta + \mathfrak{m}_b\tau \mapsto \delta(b),$$

which is an $\mathcal{O}_{V^\vee}$-module morphism. Then we have a diagram:

$$\begin{array}{ccc} \tau & \xrightarrow{\bar{\Theta}_b} & Re^{f(b)}/\hat{\mathfrak{g}}Re^{f(b)} \\ \Big\| & & \simeq \Big\uparrow \mathcal{O}_b\otimes\Phi\equiv\Phi' \\ \tau & \xrightarrow{\Theta'_b} & \tau/\mathfrak{m}_b\tau \simeq i_b^*\tau. \end{array}$$

Claim 2.4. *$\bar{\Theta}_b = \Phi' \circ \Theta'_b$, i.e. the above diagram commutes.*

Proof. Let $\delta = \sum g_\alpha \partial^\alpha$, $\bar{\Theta}_b(\delta) = (\delta e^f)(b) = \sum g(b)_\alpha (\partial^\alpha e^f)(b)$. On the other hand, $\Phi'(\delta(b)) = (\delta(b) e^f)(b) = \sum (g(b)_\alpha \partial^\alpha e^f)(b) = \bar{\Theta}_b(\delta)$. Thus $\bar{\Theta}_b = \Phi' \circ \Theta'_b$. □

Let pr : $D_{V^\vee} \to \tau = D_{V^\vee}/J$ be the projection.

Proposition 2.5. *For all $b \in V^\vee$,*

$$\textit{ker}\ \Theta_b = \mathfrak{m}_b D_{V^\vee} + J.$$

(Note that $\mathfrak{m}_b D_{V^\vee}$ is a right ideal and J is a left ideal.)

Proof. By the previous claim $\Theta_b = \bar{\Theta}_b \circ pr = \Phi' \circ \Theta'_b \circ \mathrm{pr}$. Since Φ' is an isomorphism,

$$\ker\ \Theta_b = \ker\ \Theta'_b \circ \mathrm{pr}\,.$$

$\ker\ \Theta'_b \circ \mathrm{pr} = \ker\ (D_{V^\vee} \to \tau = D_{V^\vee}/J \to \tau/\mathfrak{m}_b\tau) = \mathfrak{m}_b D_{V^\vee} + J.$ □

Therefore given $\delta \in D_{V^\vee}$, then $\delta e^{f(b)} = 0$ in $Re^{f(b)}/\hat{\mathfrak{g}}Re^{f(b)}$ iff $\Theta_b(\delta) = 0$ iff $\delta \in \ker\ \Theta_b = \mathfrak{m}_b D_{V^\vee} + J$, i.e. (2)⇔(3). This completes the proof of Theorem 2.1. □

The theorem shows that for each $b \in V^\vee$, the membership condition $\delta e^{f(b)} \in \hat{\mathfrak{g}}Re^{f(b)}$ determines exactly if b is a zero of the sheaf $\delta\mathrm{sol}(\tau)$ of analytic functions. Thus describing the vector subspace $\hat{\mathfrak{g}}Re^{f(b)} \subset Re^{f(b)}$ is crucial in understanding differential zeros of the solutions to τ in general, and of generalized hypergeometric functions in particular. In Appendix A, we give an explicit basis for $\hat{\mathfrak{g}}Re^{f(b)}$ for a number of interesting examples.

3. Analyticity along singularity

In this section, we shall consider the zero locus of certain sheaf of analytic functions on a complex manifold B.[1]

Definition 3.1. Let B be a complex manifold. A locally constant sheaf $\mathcal{S}$ of finite dimensional vector spaces on B is called analytic (ALCS) if it is equipped with an embedding $\mathcal{S} \hookrightarrow \mathcal{O}_B$ of sheaves. We shall identify an ALCS $\mathcal{S}$ with its image in $\mathcal{O}_B$ via the given embedding, and treat $\mathcal{S}$ as a subsheaf of $\mathcal{O}_B$.

The classical solution sheaf $\mathrm{sol}(\tau)$ of a holonomic D-module τ on B is an ALCS. For a given ALCS $\mathcal{S}$ and for any $\delta \in D_{V^\vee}$, let $\delta\mathcal{S}$ be the sheaf such that $(\delta\mathcal{S})_b = \{\delta\mathfrak{s} \mid \mathfrak{s} \in \mathcal{S}_b\}$, then it is also an ALCS. An ALCS of the form $\delta\mathrm{sol}(\tau)$ for a tautological system τ will be our primary focus here.

Definition 3.2. Let $\overline{B}$ be a smooth partial compactification of B such that $D = \overline{B}\backslash B$ is a normal crossing divisor in $\overline{B}$. We say that an ALCS $\mathcal{S}$ on B has regular singularity along D, if for each $b_0 \in D$, there exists local coordinates $z = (z_1, \ldots, z_n)$ on $\overline{B}$ in some polydisk U centered at b_0 such that $U \cap D = U \cap (\bigcup_{i=1}^r \{z_i = 0\})$ for some $1 \le r \le n$ and every $\mathfrak{s} \in \mathcal{S}(U\backslash D)$ has the form

$$\mathfrak{s} = \sum_{\alpha\in\Lambda} \sum_{I\in\Theta} g_{\alpha,I}(z)[z]_r^\alpha [\log z]_r^I \tag{3.1}$$

on $U\backslash D$, where Λ is a finite subset of $\mathbb{C}^r$, $[z]_r^\alpha = z_1^{\alpha^1} \cdots z_r^{\alpha^r}$; Θ is a finite subset of $\mathbb{Z}^r_{\ge 0}$, $[\log z]_r^I = (\log z_1)^{I^1} \cdots (\log z_r)^{I^r}$, and $g_{\alpha,I}$ are meromorphic functions with poles along D.

Note that if $\mathcal{S}$ is the solution sheaf of a regular holonomic D-module with singular hypersurface being a normal crossing divisor D, then $\mathcal{S}$ is an ALCS with regular singularity along D (cf. [KK, p.862], [SST, p.83]).

The typical situation we shall consider is when $\mathcal{S} = \delta\mathrm{sol}(\tau)$, where τ is a regular holonomic tautological system defined on $V^\vee$ as before and $\delta \in D_{V^\vee}$. Since B is a Zariski open subset of $V^\vee$, $V^\vee$ can be viewed as a smooth partial compactification of B. However, it may be the case that the divisor $D = V^\vee\backslash B$ fails to be normal crossing. In that case we can remedy

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[1]We thank Professor M. Kashiwara for his helpful insights which provide the basis for the analytic argument in this section.

this by blowing up $V^\vee$ along D to achieve normal crossing, which we will talk about in the next section.

Our main goal here is to show that the differential zero locus $\mathcal{N}(\delta)$ of $\delta\mathrm{sol}(\tau)$ has an analytic closure in $V^\vee$ if D is a normal crossing divisor. Then we can use the proper mapping theorem to conclude for the general case.

For the rest of this section, $\mathcal{S}$ is assumed to be an ALCS on B with regular singularity along $D = \overline{B}\backslash B$.

3.1. Regular singularities

For fixed $I \in \Theta$, we can combine terms in (3.1) with log component being $[\log z]_r^I$. Then we have a finite sum of the form $(\sum_{\alpha\in\Lambda} g_{\alpha,I}(z)[z]_r^\alpha)[\log z]_r^I$. Let $\alpha_{I1}, \ldots, \alpha_{I\Lambda_I}$ denote all the α's that appear in this sum, and let $g_{Ik}(z) := g_{\alpha_{Ik},I}(z)$. Then we can rewrite (3.1) as

$$\mathfrak{s} = \sum_{I\in\Theta}\left(\sum_{k=1}^{\Lambda_I} g_{Ik}(z)[z]_r^{\alpha_{Ik}}\right)[\log z]_r^I. \tag{3.2}$$

For fixed I, if there exist k, k' such that $\alpha_{Ik} - \alpha_{Ik'} = n_I \in \mathbb{Z}^r$, then

$$g_{Ik}(z)[z]_r^{\alpha_{Ik}} + g_{Ik'}(z)[z]_r^{\alpha_{Ik'}} = (g_{Ik}(z) + g_{Ik'}(z)[z]_r^{n_I})[z]_r^{\alpha_{Ik}}$$

and $g_{Ik}(z) + g_{Ik'}(z)[z]_r^{n_I}$ is a meromorphic function with poles along $\bigcup_{i=1}^r\{z_i = 0\}$. So without loss of generality we can assume further in the expression (3.2) that for each I,

$$\forall 1 \le k \le \Lambda_I,\ \mathrm{Re}\,\alpha_{Ik} \in [0,1)^r \text{ and } \forall\, 1 \le k \ne k' \le \Lambda_I,\ \alpha_{Ik} \ne \alpha_{Ik'}. \tag{3.3}$$

We say that $\mathfrak{s}$ is of *reduced form* if (3.3) holds.

Proposition 3.3. *Assume $\mathcal{S}$ on B has regular singularity along D. For $b_0 \in D$, let U be a polydisk centered at $b_0 \in U \cap D = U \cap (\bigcup_{i=1}^r\{z_i = 0\})$ such that for every $\mathfrak{s} \in \mathcal{S}(U\backslash D)$,*

$$\mathfrak{s} = \sum_{I\in\Theta^{(\mathfrak{s})}}\left(\sum_{k=1}^{\Lambda_I^{(\mathfrak{s})}} g_{Ik}^{(\mathfrak{s})}(z)[z]_r^{\alpha_{Ik}^{(\mathfrak{s})}}\right)[\log z]_r^I$$

on $U\backslash D$ and is of reduced form. Then $\mathfrak{s}(b) = 0$ for all $\mathfrak{s} \in \mathcal{S}_b$ if and only if $g_{Ik}^{(\mathfrak{s})}(z(b)) = 0$ for all $g_{Ik}^{(\mathfrak{s})}$ on $U\backslash D$.

We are going to prove this proposition for $r = 1$ and $r = 2$. Then by a straightforward induction the proposition holds for general cases.

3.2. Case $r = 1$

Consider $\mathfrak{s} \in \mathcal{S}(U\backslash D)$,

$$\mathfrak{s} = \sum_{j=0}^{d} \left(\sum_{k=1}^{\Lambda_j} g_{jk}(z) z_1^{\alpha_{jk}} \right) (\log z_1)^j \tag{3.4}$$

where $g_{jk}(z)$ are meromorphic functions with poles along $\{z_1 = 0\}$. Then $\mathfrak{s}$ is of reduced form if it satisfies further that

$$\operatorname{Re} \alpha_{jk} \in [0, 1) \text{ and when } k \neq k',\ \alpha_{jk} \neq \alpha_{jk'}. \tag{3.5}$$

Suppose for some $b \in U\backslash D$, $\mathfrak{s}(b) = 0$ for all $\mathfrak{s} \in \mathcal{S}_b$. Then the zero locus is monodromy invariant. Let $z(b)$ denote the coordinate of b in U, then $z_1(b) \neq 0$. Fix $z_i = z_i(b)$ for $2 \leq i \leq n$ in $\mathfrak{s}$, the analytic continuation of $\mathfrak{s}$ around $z_1 = 0$ also vanishes at b. Let $\log z_1(b) = w + 2\pi i m$, $m \in \mathbb{Z}$ for some $w \in \mathbb{C}$, then

$$0 = \mathfrak{s}(m) = \sum_{j=0}^{d} \left(\sum_{k=1}^{\Lambda_j} c_{jk} e^{2\pi i m \alpha_{jk}} \right) (w + 2\pi i m)^j, \quad \forall m \in \mathbb{Z}$$

where $c_{jk} = g_{jk}(z(b)) e^{\alpha_{jk} w} \in \mathbb{C}$.

Claim 3.4. *$c_{jk} = 0$ for all $0 \leq j \leq d, 1 \leq k \leq \Lambda_j$.*

Proof. Let $\{\alpha_1, \ldots, \alpha_s\} := \{\alpha_{jk}\}_{j,k}$ where $\alpha_1, \ldots, \alpha_s$ are pairwise distinct. Then we can write

$$\mathfrak{s}(m) = \sum_{l=1}^{s} e^{2\pi i m \alpha_l} \left(\sum_{\{j,k | \alpha_{jk} = \alpha_l\}} c_{jk} (w + 2\pi i m)^j \right).$$

Let $P_l'(m) := \sum_{\{j,k|\alpha_{jk}=\alpha_l\}} c_{jk}(w + 2\pi i m)^j$. Since (3.5) holds, the j's appearing in the summands are pairwise distinct. We have

$$0 = \mathfrak{s}(m) = \sum_{l=1}^{s} e^{2\pi i m \alpha_l} P_l'(m), \quad \forall m \in \mathbb{Z}. \tag{3.6}$$

Let $\beta := \min_{1\le l\le s}\{\operatorname{Im}\alpha_l\}$ and let p be the number of α_l's that reaches this minimum. Without loss of generality we can assume $\operatorname{Im}\alpha_1 = \cdots = \operatorname{Im}\alpha_p = \beta$. Consider

$$\begin{aligned} 0 &= \left(\sum_{l=1}^{s} e^{2\pi i m\alpha_l} P_l'(m)\right) / e^{2\pi i m(i\beta)} \\ &= e^{2\pi i m \operatorname{Re}\alpha_1} P_1'(m) + \cdots + e^{2\pi i m \operatorname{Re}\alpha_p} P_p'(m) \\ &\quad + \sum_{l=p+1}^{s} e^{2\pi m(\beta - \operatorname{Im}\alpha_l)} e^{2\pi i m \operatorname{Re}\alpha_l} P_l'(m). \end{aligned}$$

Since $\beta - \operatorname{Im}\alpha_l < 0$ for $l > p$, let $m \to \infty$,

$$\begin{aligned} &\lim_{m\to\infty} |e^{2\pi m(\beta - \operatorname{Im}\alpha_l)} e^{2\pi i m \operatorname{Re}\alpha_l} P_l'(m)| \\ &= \lim_{m\to\infty} |e^{2\pi m(\beta - \operatorname{Im}\alpha_l)} P_l'(m)| = 0 \qquad \text{for } l > p. \end{aligned}$$

Thus

$$\lim_{m\to\infty} e^{2\pi i m \operatorname{Re}\alpha_1} P_1'(m) + \cdots + e^{2\pi i m \operatorname{Re}\alpha_p} P_p'(m) = 0. \tag{3.7}$$

We have

$$\begin{aligned} &\sum_{l=1}^{p} e^{2\pi i m \operatorname{Re}\alpha_l} P_l'(m) \\ &= \sum_{l=1}^{p} e^{2\pi i m \operatorname{Re}\alpha_l} \left(\sum_{\{j,k \mid \alpha_{jk}=\alpha_l\}} c_{jk}(w + 2\pi i m)^j \right) \\ &= \sum_{j=0}^{d} (w + 2\pi i m)^j \left(\sum_{l=1}^{p} \sum_{\{1\le k\le \Lambda_j \mid \alpha_{jk}=\alpha_l\}} c_{jk} e^{2\pi i m \operatorname{Re}\alpha_{jk}} \right). \end{aligned}$$

Since for every $0 \le j \le d$, $\sum_{l=1}^{p} \sum_{\{k \mid \alpha_{jk}=\alpha_l\}} c_{jk} e^{2\pi i m \operatorname{Re}\alpha_{jk}}$ is bounded for all m, then (3.7) implies

$$\lim_{m\to\infty} \sum_{l=1}^{p} \sum_{\{1\le k\le \Lambda_j \mid \alpha_{jk}=\alpha_l\}} c_{jk} e^{2\pi i m \operatorname{Re}\alpha_{jk}} = 0 \qquad \text{for } 0 \le j \le d. \tag{3.8}$$

Note that (3.5) implies that for fixed j and l, there is at most one k such that $\alpha_{jk} = \alpha_l$. Thus for fixed j, α_{jk} appearing in (3.8) are pairwise distinct.

By our assumption their imaginary parts all equal β, then $\operatorname{Re}\alpha_{jk} \in [0,1)$ and are pairwise distinct in the summands of (3.8).

Lemma 3.5. *Given $\alpha_l \in \mathbb{R}$, $a_l \in \mathbb{C}$, $1 \le l \le p$. If $\alpha_i - \alpha_j \notin \mathbb{Z}$ when $i \neq j$, then*

$$\lim_{m\to\infty} e^{2\pi i m\alpha_1} a_1 + \cdots + e^{2\pi i m\alpha_p} a_p = 0$$

implies that $a_l = 0$ for all $1 \le l \le p$.

Proof. When $p = 1$, we have

$$\lim_{m\to\infty} e^{2\pi i m\alpha_1} a_1 = 0.$$

Then

$$\lim_{m\to\infty} |a_1| = 0$$

and thus $a_1 = 0$. Assume that lemma holds for $p = n$. Now we consider

$$\lim_{m\to\infty} e^{2\pi i m\alpha_1} a_1 + \cdots + e^{2\pi i m\alpha_{n+1}} a_{n+1} = 0. \tag{3.9}$$

The difference of replacing m by $m+1$ in (3.9) and multiplying (3.9) by $e^{2\pi i\alpha_{n+1}}$ becomes

$$\lim_{m\to\infty} e^{2\pi i m\alpha_1}(e^{2\pi i\alpha_1} - e^{2\pi i\alpha_{n+1}})a_1 + \cdots + e^{2\pi i m\alpha_n}(e^{2\pi i\alpha_n} - e^{2\pi i\alpha_{n+1}})a_n = 0.$$

Then by our inductive hypothesis we can conclude that

$$(e^{2\pi i\alpha_l} - e^{2\pi i\alpha_{n+1}})a_l = 0$$

for $1 \le l \le n$. Since by our assumption $e^{2\pi i\alpha_l} - e^{2\pi i\alpha_{n+1}} \neq 0$ for $1 \le l \le n$, then $a_1 = \cdots = a_n = 0$. Thus

$$\lim_{m\to\infty} e^{2\pi i m\alpha_{n+1}} a_{n+1} = 0$$

and therefore $a_{n+1} = 0$. By induction the lemma holds for all p. $\square$

Hence by Lemma 3.5 we can conclude that $c_{jk} = 0$ for all $\{j,k \mid \alpha_{jk} = \alpha_l, l = 1,\ldots,p\}$.

Now our original summation (3.6) reduces to

$$\sum_{l=p+1}^{s} e^{2\pi i m\alpha_l} P_l'(m) = 0.$$

We can repeat our strategy of considering terms that reach minimum imaginary part in this sum, then eventually we have $c_{jk} = 0$ for all j, k. $\square$

Since $c_{jk} = g_{jk}(z(b))e^{\alpha_{jk}w}$, it implies $g_{jk}(z(b)) = 0$ for all j, k.

We just showed that if $\mathfrak{s}(b) = 0$ for all $\mathfrak{s} \in \mathcal{S}_b$, then $g_{jk}^{(\mathfrak{s})}(z(b)) = 0$ for all $g_{jk}^{(\mathfrak{s})}$. On the other hand, it is clear that if $g_{jk}^{(\mathfrak{s})}(z(b)) = 0$, then $\mathfrak{s}(b) = 0$. Therefore Proposition 3.3 holds if $r = 1$.

3.3. Case $r = 2$

Consider $\mathfrak{s} \in \mathcal{S}(U\backslash D)$,

$$\mathfrak{s} = \sum_{i,j}\left(\sum_{k} g_{ijk}(z) z_1^{\alpha_{ijk}} z_2^{\beta_{ijk}}\right)(\log z_1)^i (\log z_2)^j$$

where $g_{ijk}(z)$ are meromorphic functions with poles along $\{z_1 = 0\} \cup \{z_2 = 0\}$. Then $\mathfrak{s}$ is of reduced form if

$$\begin{aligned} &\operatorname{Re}\alpha_{ijk} \in [0,1),\ \operatorname{Re}\beta_{ijk} \in [0,1);\\ &\text{when } k \neq k',\ \text{either } \alpha_{ijk} \neq \alpha_{ijk'} \text{ or } \beta_{ijk} \neq \beta_{ijk'}. \end{aligned} \tag{3.10}$$

For each i, let $\{\alpha_{ijk}\}_{j,k} = \{\alpha_{i1}, \ldots, \alpha_{is_i}\}$ where $\alpha_{i1}, \ldots, \alpha_{is_i}$ are pairwise distinct. We can rewrite $\mathfrak{s}$ as

$$\mathfrak{s} = \sum_{i} (\log z_1)^i \left(\sum_{l_i=1}^{s_i} z_1^{\alpha_{il_i}} \left(\sum_{\{j,k|\alpha_{ijk}=\alpha_{il_i}\}} g_{ijk}(z) z_2^{\beta_{ijk}} (\log z_2)^j\right)\right).$$

Suppose for some $b \in U\backslash D$, $\mathfrak{s}(b) = 0$ for all $\mathfrak{s} \in \mathcal{S}_b$. Then $z_1(b)z_2(b) \neq 0$. First we fix $z_i = z_i(b)$ for $2 \leq i \leq n$ and consider the analytic continuation around $z_1 = 0$. Then

$$\mathfrak{s} = \sum_i (\log z_1)^i \left(\sum_{l_i=1}^{s_i} z_1^{\alpha_{il_i}} \left(\sum_{\{j,k|\alpha_{ijk}=\alpha_{il_i}\}} g_{ijk}(z_1, z_2(b), \ldots, z_n(b)) \right.\right.$$
$$\left.\left. \times\, z_2(b)^{\beta_{ijk}} (\log z_2(b))^j \right)\right).$$

Let $\mathfrak{s}_{i,l_i}(z) := \sum_{\{j,k|\alpha_{ijk}=\alpha_{il_i}\}} g_{ijk}(z) z_2^{\beta_{ijk}} (\log z_2)^j$, then

$$\mathfrak{s} = \sum_i (\log z_1)^i \left(\sum_{l_i=1}^{s_i} z_1^{\alpha_{il_i}} \mathfrak{s}_{i,l_i}(z_1, z_2(b), \ldots, z_n(b)) \right) \tag{3.11}$$

and $\mathfrak{s}_{i,l_i}(z_1, z_2(b), \ldots, z_n(b))$ is a meromorphic function in z_1 with poles along $\{z_1 = 0\}$. Then (3.11) satisfies (3.5) and by case $r = 1$ of Proposition 3.3 we have

$$\mathfrak{s}_{i,l_i}(z(b)) = \sum_{\{j,k|\alpha_{ijk}=\alpha_{il_i}\}} g_{ijk}(z(b)) z_2(b)^{\beta_{ijk}} (\log z_2(b))^j = 0.$$

for all i, l_i.

Fix i, l_i. Note that if $k \neq k'$ and $\alpha_{ijk} = \alpha_{ijk'} = \alpha_{il_i}$, (3.10) implies $\beta_{ijk} \neq \beta_{ijk'}$. Now in $\mathfrak{s}_{i,l_i}$ we fix $z_i = z_i(b)$ for $i \neq 2, 1 \leq i \leq n$ and do analytic continuation around $z_2 = 0$, then by case $r = 1$ of Proposition 3.3 again $\mathfrak{s}_{i,l_i}(z(b)) = 0$ implies $g_{ijk}(z(b)) = 0$ for all j, k such that $\alpha_{ijk} = \alpha_{il_i}$.

Hence if $\mathfrak{s}(b) = 0$ for all $\mathfrak{s} \in \mathcal{S}_b$, then $g_{ijk}^{(\mathfrak{s})}(z(b)) = 0$ for all $g_{ijk}^{(\mathfrak{s})}$. Therefore Proposition 3.3 holds for $r = 2$.

3.4. Analyticity of the zero locus

Let $\mathcal{N} := \{b \in B \mid \mathfrak{s}(b) = 0, \forall \mathfrak{s} \in \mathcal{S}_b\}$. Let $\overline{\mathcal{N}}$ denote its analytic closure in $\overline{B}$.

Proposition 3.6. *If an ALCS $\mathcal{S}$ on B has regular singularity along D, then $\overline{\mathcal{N}}$ is analytic.*

Proof. $\mathfrak{s}$ is locally holomorphic away from D, thus $\mathcal{N}$ is an analytic subvariety of B. In particular, $\mathcal{N}$ is a closed subset of B.

Let $b_0 \in D \cap \overline{\mathcal{N}}$. Then by Proposition 3.3 there exists a polydisk U centered at b_0 such that

$$\mathcal{N} \cap (U \backslash D) = \left\{ b \in B \mid g_{Ik}^{(\mathfrak{s})}(z(b)) = 0,\ \forall \mathfrak{s} \in \mathcal{S}_b,\ \forall I, k \right\} \cap (U \backslash D)$$

where g_{Ik} are meromorphic functions with poles along $\bigcup_{i=1}^{r} \{z_i = 0\}$. Let $\chi_{Ik}^{i} \in \mathbb{Z}^r$ be the order of poles of $g_{Ik}(z)$ corresponding to z_i respectively. Then $[z]_r^{\chi_{Ik}} g_{Ik}(z)$ is holomorphic on the neighborhood U. Then

$$\overline{\mathcal{N}} \cap U = \left\{ b \in \overline{B} \mid [z(b)]_r^{\chi_{Ik}^{(\mathfrak{s})}} g_{Ik}^{(\mathfrak{s})}(z(b)) = 0,\ \forall \mathfrak{s} \in \mathcal{S}_b,\ \forall I, k \right\} \cap U,$$

i.e. $\overline{\mathcal{N}}$ is analytic. $\square$

4. Algebraicity of $\mathcal{N}(\delta)$

As before, let τ be a regular holonomic tautological system on $V^\vee$, B be a Zariski dense open subset of $V^\vee$, and $D = V^\vee \backslash B$.

By Hironaka's Theorem [Hi] there exists a proper analytic morphism (blow-up) f:

$$\begin{array}{ccc} \tilde{V}^\vee & \xrightarrow{\quad f \quad} & V^\vee \\ \cup \big\uparrow & & \big\uparrow \cup \\ \tilde{B} = \tilde{V}^\vee \backslash \tilde{D} & \xrightarrow{\ \simeq\ } & B = V^\vee \backslash D \end{array}$$

such that $\tilde{D} := f^{-1}(D)$ is a normal crossing divisor in $\tilde{V}^\vee$. We can then consider the D-module $\tilde{\tau} = f^*\tau$ on $\tilde{V}^\vee$ and its solution sheaf. Since τ is regular holonomic, $\tilde{\tau}$ is also regular holonomic. Note that $f|_{\tilde{B}}$ induces an isomorphism from $\delta\mathrm{sol}(\tilde{\tau})$ on $\tilde{B}$ to $\delta\mathrm{sol}(\tau)$ on B. Let $\tilde{\mathcal{N}}(\delta) := \{b \in \tilde{B} \mid \tilde{\mathfrak{s}}(b) = 0,\ \forall \tilde{\mathfrak{s}} \in \delta\mathrm{sol}(\tilde{\tau})_b\}$.

Claim 4.1. *The closure in analytic topology $\overline{\tilde{\mathcal{N}}(\delta)}$ is analytic in $\tilde{V}^\vee$.*

Proof. Since $\tilde{D}$ is a normal crossing divisor and $\tilde{\tau}$ is regular holonomic, $\mathrm{sol}(\tilde{\tau})$ has regular singularity along $\tilde{D}$. Then it is clear that $\delta\mathrm{sol}(\tilde{\tau})$ also has regular singularity along $\tilde{D}$. Then by Proposition 3.6, $\overline{\tilde{\mathcal{N}}(\delta)}$ is analytic in $\tilde{V}^\vee$. $\square$

Proposition 4.2. *The closure in analytic topology $\overline{\mathcal{N}(\delta)}$ is analytic in $V^\vee$.*

Proof. First we claim two properties.

$f|_{\overline{\tilde{\mathcal{N}}(\delta)}}$ *is proper:* Given a compact subset $C \subset V^\vee$,

$$(f|_{\overline{\tilde{\mathcal{N}}(\delta)}})^{-1}(C) = \overline{\tilde{\mathcal{N}}(\delta)} \cap f^{-1}(C).$$

Since f is proper, $f^{-1}(C)$ is compact. Since $\overline{\tilde{\mathcal{N}}(\delta)}$ is closed, $\overline{\tilde{\mathcal{N}}(\delta)} \cap f^{-1}(C)$ is compact in $\overline{\tilde{\mathcal{N}}(\delta)}$.

$f|_{\overline{\tilde{\mathcal{N}}(\delta)}}$ *is holomorphic:* The restriction of a holomorphic map to an analytic space is holomorphic.

Then by Proper Mapping Theorem (cf. [GR, p.162]) $f(\overline{\tilde{\mathcal{N}}(\delta)})$ is analytic.

Since f is continuous, $f(\overline{\tilde{\mathcal{N}}(\delta)}) \subset \overline{f(\tilde{\mathcal{N}}(\delta))}$. On the other hand, given any sequence $\tilde{x}_k \in \tilde{\mathcal{N}}(\delta)$ such that $\lim_{k\to\infty} f(\tilde{x}_k) = y \in D$. We can take a compact neighborhood $C \subset V^\vee$ of y. Then for $k >> 0$, $f(\tilde{x}_k) \in C$, i.e. $\tilde{x}_k \in f^{-1}(C) \subset \tilde{V}^\vee$. Since f is proper, $f^{-1}(C)$ is compact. Thus there exists a convergent subsequence $x_{k'}$ such that $\lim_{k'\to\infty} x_{k'}$ exists. Therefore by continuity of f we have

$$f\left(\lim_{k'\to\infty} x_{k'}\right) = \lim_{k'\to\infty} f(x_{k'}) = y$$

which means $y \in f(\overline{\tilde{\mathcal{N}}(\delta)})$. Thus

$$f(\overline{\tilde{\mathcal{N}}(\delta)}) = \overline{f(\tilde{\mathcal{N}}(\delta))} = \overline{\mathcal{N}(\delta)}$$

and therefore $\overline{\mathcal{N}(\delta)}$ is analytic. $\square$

Proposition 4.3. *If $\delta \in D_{V^\vee}$ is homogeneous under scaling by $\mathbb{C}^\times$, $\overline{\mathcal{N}(\delta)}$ is algebraic.*

Proof. By Proposition 4.2, $\overline{\mathcal{N}(\delta)} \subset V^\vee = \mathbb{C}^n$ is closed analytic. Suppose δ is homogeneous of degree d under scaling by $\mathbb{C}^\times$. Given $\lambda \in \mathbb{C}^\times$, for $\mathfrak{s} \in \mathrm{sol}(\tau)_b$,

$$(\delta\mathfrak{s})(\lambda b) = \lambda^{d-\beta(e)}(\delta\mathfrak{s})(b).$$

Thus $\lambda b \in \mathcal{N}(\delta)$ if $b \in \mathcal{N}(\delta)$, i.e. the $\mathbb{C}^\times$-action by scaling on $V^\vee$ leaves $\mathcal{N}(\delta)$ invariant. Hence $\mathbb{C}^\times$ also leaves $\overline{\mathcal{N}(\delta)}$ invariant.

Let $p : \mathbb{C}^n\backslash\{0\} \to \mathbb{P}^{n-1}$ be the projection. Then $p(\overline{\mathcal{N}(\delta)}\backslash\{0\})$ is a closed analytic subspace of $\mathbb{P}^{n-1}$, by Chow's theorem it is an algebraic subvariety. Thus its cone $\overline{\mathcal{N}(\delta)}$ is an algebraic variety. $\square$

Since $\mathcal{N}(\delta)$ is a closed subset of B, $\mathcal{N}(\delta) = \overline{\mathcal{N}(\delta)} \cap B$.

Theorem 4.4. *If $\delta \in D_{V^\vee}$ is homogeneous under scaling by $\mathbb{C}^\times$, $\mathcal{N}(\delta)$ is algebraic.*

5. Non-emptiness of $\mathcal{N}(\delta)$: $\mathbb{P}^1$ case

We now consider the problem of non-emptiness of $\mathcal{N}(\delta)$, starting with the simplest nontrivial case when $X = \mathbb{P}^1$, $G = SL_2$. In this case $R \equiv \mathbb{C}[x_1^2, x_2^2, x_1x_2]$, $f = a_0x_1x_2 + a_1x_1^2 + a_2x_2^2$. Recall that

$$\begin{aligned} Z(h) &= -2a_1\partial_1 + 2a_2\partial_2 \\ Z(x) &= -2a_2\partial_0 - a_0\partial_1 \\ Z(y) &= -2a_1\partial_0 - a_0\partial_2. \end{aligned}$$

for

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Proposition 5.1. *If $X = \mathbb{P}^1, G = SL_2$, given a positive integer d, then $\mathcal{N}(\delta) \neq \emptyset$ for every $\delta \in \mathbb{C}[\partial]_d$.*

Proof. Step 1: $\mathfrak{sl}_2$ acts on $\mathbb{C}[\partial]_d$ by commutator $[Z(\xi), \delta]$ for $\xi \in \mathfrak{sl}_2$, $\delta \in \mathbb{C}[\partial]_d$. Since $\mathfrak{sl}_2$ is a semisimple Lie algebra and $\mathbb{C}[\partial]_d$ is a finite dimensional $\mathfrak{sl}_2$-module, $\mathbb{C}[\partial]_d$ is a semisimple $\mathfrak{sl}_2$-module.

Step 2: Let $\Delta := a_0^2 - 4a_1a_2$. When $X = \mathbb{P}^1$, up to scalar the solution of τ is $\Delta^{-\frac{1}{2}}$. Define

$$Ann_d := Ann_{\mathbb{C}[\partial]_d}(\Delta^{-\frac{1}{2}}) := \left\{ \alpha \in \mathbb{C}[\partial]_d \mid \alpha(\Delta^{-\frac{1}{2}}) = 0 \right\}.$$

Given $\alpha \in Ann_d$, then $\alpha(\Delta^{-\frac{1}{2}}) = 0$, thus $[Z(\xi), \alpha](\Delta^{-\frac{1}{2}}) = 0$ and $[Z(\xi), \alpha] \in Ann_d$. Therefore Ann_d is an $\mathfrak{sl}_2$-submodule of $\mathbb{C}[\partial]_d$.

Step 3: By Step 1, $\mathbb{C}[\partial]_d$ is a semisimple $\mathfrak{sl}_2$-module, then there exists an $\mathfrak{sl}_2$-submodule S_d such that $\mathbb{C}[\partial]_d = Ann_d \oplus S_d$ as $\mathfrak{sl}_2$-modules.

Step 4: It is well known that $\mathfrak{sl}_2$-invariant ring is

$$\{\alpha \in \mathbb{C}[\partial]_d \mid [Z(\xi), \alpha] = 0 \; \forall \xi \in \mathfrak{sl}_2\} = \mathbb{C}[\partial]_d^{\mathfrak{sl}_2} = \mathbb{C}[\partial_0^2 - \partial_1\partial_2]_d$$

where $\mathbb{C}[\partial_0^2 - \partial_1\partial_2]$ denotes the polynomial ring generated by a single element $\partial_0^2 - \partial_1\partial_2$. It is clear that $\mathbb{C}[\partial]_d^{\mathfrak{sl}_2} \subset Ann_d$ for $d > 0$.

Step 5: Let $\delta \in \mathbb{C}[\partial]_d$, we claim that $\mathcal{N}(\delta) = \emptyset$ if and only if $\delta(\Delta^{-\frac{1}{2}}) \in \mathbb{C}^\times \Delta^{-\frac{d+1}{2}}$. If $\delta(\Delta^{-\frac{1}{2}}) \in \mathbb{C}^\times \Delta^{-\frac{d+1}{2}}$, then $\delta(\Delta^{-\frac{1}{2}})$ is nowhere vanishing. Thus

$\mathcal{N}(\delta) = \emptyset$. For the other direction, we first observe that

$$\delta(\Delta^{-\frac{1}{2}}) = \Delta^{-\frac{1}{2}-d} P_d(a_0, a_1, a_2)$$

where P_d is a homogeneous polynomial of degree d. Suppose P_d factors into $P_d = \Delta^k q_{d-2k}$ where $\gcd(\Delta, q_{d-2k}) = 1$. Then $\delta(\Delta^{-\frac{1}{2}}) = \Delta^{-\frac{1}{2}-d+k} q_{d-2k}$. $\mathcal{N}(\delta) = \emptyset$ implies that

$$\left\{\Delta^{-\frac{1}{2}-d+k} q_{d-2k} = 0\right\} \cap \{\Delta \neq 0\} = \emptyset$$

and thus $\{q_{d-2k} = 0\} \subset \{\Delta = 0\}$. But q_{d-2k} and Δ are coprime, it implies that $q_{d-2k} \subset \mathbb{C}^\times$. Thus $d = 2k$ and $\delta(\Delta^{-\frac{1}{2}}) \in \mathbb{C}^\times \Delta^{-\frac{d+1}{2}}$.

Step 6: Suppose $\mathcal{N}(\delta) = \emptyset$, then by Step 5 we have $\delta(\Delta^{-\frac{1}{2}}) \in \mathbb{C}^\times \Delta^{-\frac{d+1}{2}}$. Since $Z(\mathfrak{sl}_2)(\Delta^{-\frac{d+1}{2}}) = 0$, it implies that $[Z(\mathfrak{sl}_2), \delta] \subset Ann_d$. Step 3 tells us that $\delta = \delta' + \delta''$ where $\delta' \in Ann_d$, $\delta'' \in S_d$ and

$$[Z(\xi), \delta] = [Z(\xi), \delta'] + [Z(\xi), \delta''].$$

Since $[Z(\xi), \delta] \subset Ann_d$ and $[Z(\xi), \delta'] \subset Ann_d$, the direct sum forces $[Z(\xi), \delta''] = 0$ for all $\xi \in \mathfrak{sl}_2$. This implies $\delta'' \in \mathbb{C}[\partial]_d^{\mathfrak{sl}_2} \subset Ann_d$ when $d > 0$. Then the direct sum further forces that $\delta'' = 0$. Thus $\delta = \delta' \in Ann_d$. Therefore $\delta(\Delta^{-\frac{1}{2}}) = 0$, contradicts $\delta(\Delta^{-\frac{1}{2}}) \in \mathbb{C}^\times \Delta^{-\frac{d+1}{2}}$.

Therefore given a positive integer d, $\mathcal{N}(\delta) \neq \emptyset$ for every $\delta \in \mathbb{C}[\partial]_d$. □

6. A degree bound

In this section we consider $X = \mathbb{P}^m$, $G = SL_{m+1}$. In this case, we will view $R = \mathbb{C}[a^\vee]$ as the subring of $\mathbb{C}[x_0, \ldots, x_m]$ generated by the degree $m+1$ monomials in the x_i. This degree however will *not* be used below. The *degree* deg below shall refer to the degree in the variables $a_i^\vee$ which can be identified with a monomial basis of $V^\vee$. We now prove an important degree bound and use a rank approach to give another proof of $\mathcal{N}(\delta)$ being algebraic.

Lemma 6.1 (Degree bound lemma). *Take $X = \mathbb{P}^m$, $\hat{\mathfrak{g}} = \mathfrak{sl}_{m+1} \oplus \mathbb{C}$. Let $Z_i := Z^\vee(x_i)$ where x_i is a basis of $\hat{\mathfrak{g}}$. Suppose $f(b)$ is nonsingular. For $h \in R$, $he^{f(b)} \equiv 0$ in $H_0(\hat{\mathfrak{g}}, Re^{f(b)})$ iff*

$$he^{f(b)} = \sum Z_i(r_i e^{f(b)})$$

for some $r_i \in R$, and $\deg r_i \leq \deg h - 1$, $\forall i$.

Proof. The 'if' direction is obvious. For the 'only if' direction, consider the homogeneous ideal $I := \langle x_u \partial_v f(b) | 0 \le u, v \le m \rangle$ of R. Let B_k denote a $\mathbb{C}$-basis for the degree k part of R/I. First, since $f(b)$ is homogeneous of degree 1, the degree 0 part of R/I is nonzero, and is spanned by 1. For any $h \in R$, consider expanding the highest degree component of h, which we denote by h_0, in degree $= \deg h$ part of R/I in terms of the chosen basis: i.e. by definition, there exist elements $s_i \in R$, such that $h_0 - \sum s_i Z_i(f(b))$ can be written as a linear combination of the chosen basis elements in degree $= \deg h$. Obviously, we can require that $\deg s_i \le \deg h - 1$ for each i by dropping all higher degree components of each of these r_i, if there are any. Working degree by degree, it is clear that we can choose $r_i \in R$ with $\deg r_i \le \deg h - 1, \forall i$, such that $he^{f(b)} = \sum Z_i(r_i e^{f(b)}) + \sum c_k B_k$, where $\sum c_k B_k$ denote a linear combination of elements of the B_k with all $k \le \deg h$. Therefore, $H_0(\hat{\mathfrak{g}}, Re^{f(b)})$ is spanned by B_k.

On the other hand, observed that $R/I = (\mathbb{C}[x_0, \ldots, x_m]/J)^{\mu_{m+1}}$, where $J := \langle \partial_i f(b) | 0 \le i \le m \rangle$ is the Jacobian ideal of the nonsingular hypersurface $f(b)$, and μ_{m+1} is the group of $(m+1)$-th root of unity. By [AS][G], $\dim_{\mathbb{C}}(\mathbb{C}[x_0, \ldots, x_m]/J)^{\mu_{m+1}} = h^m(X - V(f(b)))$. Combining the algebraic and geometric rank formula for τ, we have in this case, $h^m(X - V(f(b))) = \dim H_0(\hat{\mathfrak{g}}, Re^{f(b)})$. Therefore, the collection of B_k consists of linearly independent elements, and $he^{f(b)} = 0$ in $H_0(\hat{\mathfrak{g}}, Re^{f(b)})$ iff all coefficients $c_k = 0$. □

We now apply the lemma to derive explicit polynomial equations for the variety $\mathcal{N}(\delta)$. Fix $\delta \in \mathbb{C}[\partial]$ be of degree d. Then $he^f = \delta e^f$ where $h \in R$ such that $\tilde{h} = \delta$. By the lemma, each $b \in \mathcal{N}(\delta)$ lies in the locus of an equation of the form

$$he^f = \sum Z_i(r_i e^f) = \sum_i (Z_i(r_i)e^f + Z_i(f) r_i e^f) \tag{6.1}$$

for some $r_i = \sum_{j \in J} \lambda^i_j e_j$, $\lambda^i_j \in \mathbb{C}$, where $\{e_j\}_{j \in \mathcal{J}_{d-1}}$ being a basis of the subspace of R of degree $\le d-1$. Thus $\deg Z_i(r_i) \le d-1$ and $Z_i(f) r_i$ is linear in the variables a_i, but is of degree $\le d$ in the variables $a_i^\vee$.

In the basis $\{e_j\}_{j \in \mathcal{J}_d}$, h can be viewed as a vector $\Theta \in \mathbb{C}^{\mathcal{J}_d}$. Let Λ be the column vector with entries λ^i_j, $\forall i, j$. Then comparing coefficients of the expansion of (6.1) gives us a matrix $M_d(a)$ (depending only on d but not on δ itself) whose entries lie in $\mathbb{C} + \sum_i \mathbb{C} a_i$ such that the following inhomogeneous linear system holds:

$$M_d(b)\Lambda = \Theta. \tag{6.2}$$

But in turn this is equivalent to the rank condition

$$\boxed{\operatorname{rk} M_d(b) = \operatorname{rk}[M_d(b)|\Theta].} \tag{6.3}$$

To summarize, let's fix $\delta \in \mathbb{C}[\partial]$ (hence fix h and Θ) of degree d. For a given $b \in B$, Lemma 6.1 says that $b \in \mathcal{N}(\delta)$ iff there exists $r_i \in R$ with $\deg r_i \leq d-1$ such that

$$he^{f(b)} = \sum Z_i(r_i e^{f(b)}). \tag{6.4}$$

This is equivalent to saying that (6.3) holds. Thus we can conclude:

Theorem 6.2. *For a given $\delta \in \mathbb{C}[\partial]$ of degree d*

$$\mathcal{N}(\delta) = \{b \in B \mid \operatorname{rk} M_d(b) = \operatorname{rk}[M_d(b)|\Theta]\}.$$

Therefore $\mathcal{N}(\delta)$ is an algebraic variety defined by the rank condition (6.3). In particular, $\mathcal{N}(\delta)$ has a natural stratification given by $\operatorname{rk} M_d(b)$.

7. Periods of elliptic curves

7.1. Some preparation

In this section we consider the case $X = \mathbb{P}^2$. $G = SL_3$. Then $\pi : \mathcal{Y} \to B$ is the family of smooth elliptic curves in X. We write the basis of V as $\{a_I \mid I = (ijk),\ i+j+k=3,\ i,j,k \geq 0\}$, which is dual to the monomial basis $x_1^i x_2^j x_3^k$ of sections in $V^\vee$. Let S, T be Aronhold invariants of a ternary cubic, then $\mathbb{C}[V^\vee]^{SL_3} = \mathbb{C}[S,T]$. Let $\Delta = 64S^3 - T^2$ be the discriminant.

Lemma 7.1. *There is a natural action of G on B. $B/G = \operatorname{Spec}\mathbb{C}[S,T,\Delta^{-1}]$. In particular, S, T give a global coordinate system on the two dimensional nonsingular variety B/G.*

Proof. We have $B = \operatorname{Spec}\mathbb{C}[a_I, \Delta^{-1}]$, the stable locus of the G-action on B (see [N, Theorem 1.6]). Thus every G-orbit in B is closed, and we have

$$B/G = \operatorname{Spec}\mathbb{C}[a_I,\Delta^{-1}]^G = \operatorname{Spec}\mathbb{C}[S,T,\Delta^{-1}],$$

where it is well known that S, T are algebraically independent. □

Lemma 7.2. *Let δ be a first order differential operator with constant coefficient (i.e. $\delta = \sum_I \lambda_I \frac{\partial}{\partial a_I}$, for constants λ_I). Let $h := \delta f$ where $f := \sum_I a_I a_I^\vee$*

is the universal section. Then given any point $b \in B$, the following are equivalent:

1) $b \in \mathcal{N}(\delta)$,
2) $h = Z_x f(b)$ *for some* $x \in \mathfrak{sl}_3$,
3) $(\delta S)(b) = 0$ *and* $(\delta T)(b) = 0$.

Proof. Since δ is of degree 1, by Lemma 6.1, $b \in \mathcal{N}(\delta)$ iff

$$he^{f(b)} = \sum_{i=1}^{8} Z_i(c_i e^{f(b)}) + (E+1)(c_E e^{f(b)}) \tag{7.1}$$

for some complex numbers c_i and c_E, where Z_i is a basis of $\mathfrak{sl}_3$, E is the Euler operator. $h = \delta f$ implies that h is a homogeneous polynomial in $a_I^\vee$ of degree 1 with constant coefficients, i.e. an element in the section space $V^\vee$. So (7.1) holds iff

$$he^{f(b)} = \sum_{i=1}^{8} Z_i(c_i e^{f(b)}) \tag{7.2}$$

for some complex numbers c_i. This is equivalent to

$$h = Z_x f(b) \tag{7.3}$$

for some $x \in \mathfrak{sl}_3$.

Next, we identify $V^\vee$ with its tangent space at b, where $a_I^\vee$ is identified with $\frac{\partial}{\partial a_I}$.

Note that the identification is compatible with the action of $\mathfrak{sl}_3$. Under this identification, it is clear that h is identified with δ. ($\delta = \sum_I \lambda_I \frac{\partial}{\partial a_I}$ is identified with $\sum_I \lambda_I a_I^\vee = h$.) We consider the projection map $p : B \mapsto B/G$. We denote the tangent map at b by dp_b. At b, $dp_b(\delta) = 0$ iff $\delta = h$ lies in the tangent space of the G-orbit $G \cdot b$, i.e. iff (7.3) holds.

S and T are global coordinates of $B/G \subset \operatorname{Spec}(V^\vee)^G = \operatorname{Spec} \mathbb{C}[S,T]$. Thus, we have

$$dp_b(\delta) = \frac{\partial}{\partial S}(\delta S)\Big|_b + \frac{\partial}{\partial T}(\delta T)\Big|_b . \tag{7.4}$$

So $dp_b(\delta) = 0$ iff $(\delta S)(b) = 0$ and $(\delta T)(b) = 0$. $\square$

This shows that

$$\mathcal{N}(\delta) = \mathcal{M} \cap B, \qquad \mathcal{M} := \{b \in B \mid \delta S(b) = \delta T(b) = 0\} \subset V^\vee.$$

In particular, this implies that $\mathcal{N}(\delta) = \emptyset$ iff $\mathcal{M} \subset \{\Delta = 64S^3 - T^2 = 0\}$. By Nullstellensatz, this is equivalent to

$$\Delta \in \sqrt{\langle \delta S, \delta T \rangle}$$

the radical of the ideal $\langle \delta S, \delta T \rangle$. In other words

$$\Delta^m \in \langle \delta S, \delta T \rangle \tag{7.5}$$

for some integer $m > 0$.

Remark 7.3. If we do not require δ to be constant coefficients, then $\mathcal{N}(\delta)$ can be empty. E.g. Take δ to be Euler and $\beta \neq 1$. Then $\mathcal{N}(\delta)$ is the set b where $\delta\mathfrak{s}(b) = \mathfrak{s}(b) = 0$ for all periods $\mathfrak{s}$, hence empty because there is no point $b \in B$ where all periods vanish.

7.2. Main theorem

Theorem 7.4. *Let $\delta = \sum_I \lambda_I \partial_I$, where $(\lambda_I) \in \mathbb{Z}[i]^{10}$ and*

$$\gcd(\lambda_I)_I = 1,\ (1+i)|\lambda_{111} \text{ in } \mathbb{Z}[i] \text{ and } \{\lambda_{300}, \lambda_{030}, \lambda_{003}\} \neq \{1, 0, 0\} \mod (1+i). \tag{7.6}$$

Then $\mathcal{N}(\delta) \neq \emptyset$.

We prove this by a series of lemmas.

Recall that

$$\mathbb{C}[V^\vee]^{SL_3} = \mathbb{C}[S, T]$$

where S, T are the Aronhold invariants, which are respectively polynomials of degree 4 and 6 with integer coefficients in 10 variables. In order to use their explicit expressions given in [S] which we include in Appendix B, we must multiply each variable $a_I = a_{ijk}$ appearing in our universal cubic section $f = \sum_I a_I x^I$ by the factor $\frac{i!j!k!}{3!}$. *All use of the a_I in this proof will be the a_I defined in [S].*

Recall that the discriminant polynomial for the cubic plane curves is $\Delta = 64S^3 - T^2 \in \mathbb{Z}[a]_{12}$. Evaluating Δ at the point $a_{111} = 1$ and $a_I = 0$ for $I \neq (111)$ yields $\Delta = 0$, since this point defines the singular curve $x_1x_2x_3 = 0$. Thus the monomial a_{111}^{12} does not appear in the polynomial Δ.

Suppose $\mathcal{N}(\delta) = \emptyset$. Then by Nullstellensatz there exists a positive integer m and

$$(h_S, h_T) \in W := \mathbb{C}[a]_{12m-3} \oplus \mathbb{C}[a]_{12m-5}$$

such that (7.5) becomes

$$\Delta^m = (64S^3 - T^2)^m = h_S \delta S + h_T \delta T. \tag{7.7}$$

Fix an ordering of the monomial basis in the a_I for each $\mathbb{Q}(i)[a]_k$, and so that we can now represent the polynomial Δ^m by the column $\mathbb{Z}$-vector θ given by the polynomial's coefficients, and (h_S, h_T) by a column $\mathbb{C}$-vector h. Then (7.7) is equivalent to a matrix equation of the form

$$Mh = \theta \tag{7.8}$$

where M is a matrix over $\mathbb{Z}[i]$ defined by the expressions of δS and δT. This equation has a solution h iff

$$\mathrm{rk}_{\mathbb{C}}(M) = \mathrm{rk}_{\mathbb{C}}[M|\theta]. \tag{7.9}$$

But since M and $[M|\theta]$ are defined over $\mathbb{Q}(i)$ this equation is equivalent to (because rank of a matrix remains the same under field extensions)

$$\mathrm{rk}_{\mathbb{Q}(i)}(M) = \mathrm{rk}_{\mathbb{Q}(i)}[M|\theta]. \tag{7.10}$$

Therefore, we can assume that

$$(h_S, h_T) \in W_{\mathbb{Q}(i)} := \mathbb{Q}(i)[a]_{12m-3} \oplus \mathbb{Q}(i)[a]_{12m-5}.$$

Write $h_S = \frac{1}{pd} r_S$, $h_T = \frac{1}{qd} r_T$, where the coefficients of each of the polynomials $r_S, r_T \in \mathbb{Z}[i][a]$ have gcd 1, and $p, q, d \in \mathbb{Z}[i]$, with $\gcd(p, q) = 1$ in $\mathbb{Z}[i]$. Then we get

Lemma 7.5. *There exist $r_S, r_T \in \mathbb{Z}[i][a]$ each having coefficients with* gcd *1, and $p, q, d \in \mathbb{Z}[i]$, with $\gcd(p, q) = 1$ in $\mathbb{Z}[i]$, and $m \in \mathbb{Z}_{>0}$ such that*

$$pqd(64S^3 - T^2)^m = q r_S \delta S + p r_T \delta T. \tag{7.11}$$

Notations. e_I denotes the standard unit vector in $\mathbb{Z}^{10}$ corresponding to I. For $\mu = (\mu_I) \in (\mathbb{Z}_{\geq 0})^{10}$, write $a^\mu = \prod_I a_I^{\mu_I}$ and $|\mu| = \sum_I \mu_I$. Note that a^μ is invariant under the diagonal maximal torus of $SL_3(\mathbb{C})$ iff the *index sum* $\sum_I \mu_I I$ of μ is equal to $|\mu|(1, 1, 1)$.

Lemma 7.6. *Let $\mu, \mu' \in (\mathbb{Z}_{\geq 0})^{10}$ such that $|\mu| = |\mu'| = \ell$ and $a^\mu, a^{\mu'}$ are torus invariant. Then for any $I \neq I'$, if $\partial_I a^\mu$ and $\partial_{I'} a^{\mu'}$ are nonzero, then they are not proportional (over any field).*

Proof. This is because the index sums of the monomials in $\partial_I a^\mu$ and $\partial_{I'} a^{\mu'}$ are different:

$$(\ell - i, \ell - j, \ell - k) \neq (\ell - i', \ell - j', \ell - k')$$

for $I \neq I'$. □

We will apply this to the cases $\ell = 4, 6$. For homogeneous polynomial $P \in \mathbb{C}[a]$, denote by $c_\mu(P)$ the coefficient of the monomial a^μ in P, so that

$$P = \sum_\mu c_\mu(P) a^\mu.$$

Lemma 7.7. *Consider the degree 4 invariant polynomial S. For each index I, there exists μ such that $c_\mu(S) = \pm 1$ and $\mu_I = 1$. For each such pair (I, μ)*

$$c_{\mu - e_I}(\delta S) = \pm \lambda_I.$$

Therefore the $\gcd(c_\nu(\delta S))_\nu = 1$. *More generally, without assuming* $\gcd(\lambda_I)_I = 1$, *we also have*

$$\gcd(c_\nu(\delta S))_\nu \,|\, \gcd(\lambda_I)_I.$$

Proof. The explicit expression of S shows the first statement holds. For the second statement, consider

$$\delta S = \sum_{I', \mu'} c_{\mu'}(S) \lambda_{I'} \partial_{I'} a^{\mu'} = \sum_{I', \mu'} c_{\mu'}(S) \lambda_{I'} \mu'_{I'} a^{\mu' - e_{I'}}.$$

For the given pair (I, μ), the summands on the right that are proportional to the monomial term $c_\mu(S) \lambda_I \partial_I a^\mu = \pm \lambda_I a^{\mu - e_I}$ must be those with $\mu' - e_{I'} = \mu - e_I$. But Lemma 7.6 forces $I = I'$, hence $\mu = \mu'$. This proves the second statement. Finally, the last statement follows from the second and the assumption (7.6). □

Now consider the explicit expression $T \in \mathbb{Z}[a]$. There are exactly 6 monomial terms with odd coefficients, namely

$$T_0 := a_{300}^2 a_{030}^2 a_{003}^2 - 3a_{300}^2 a_{021}^2 a_{012}^2 - 3a_{030}^2 a_{201}^2 a_{102}^2 - 3a_{003}^2 a_{210}^2 a_{120}^2 - 27a_{201}^2 a_{120}^2 a_{012}^2 - 27a_{210}^2 a_{102}^2 a_{021}^2 \tag{7.12}$$

so that T has the form $T = T_0 + 2T_1$ with $T_1 \in \mathbb{Z}[a]$.

(1) Note that $c_\nu(\delta T_0) \neq 0$ implies that ν satisfies the index sum condition that

$$\sum_I \nu_I I = (6,6,6) - (i,j,k), \qquad \text{for some } i,j,k \geq 0 \text{ with } i+j+k=3.$$

If $\nu = \mu - e_I$ then this condition uniquely determines μ and I, with $\sum_I \mu_I I = (6,6,6)$.

(2) Each $a_I \neq a_{111}$ appears in some monomial a^μ in T_0 and with exponent 2.

(3) Since $T = T_0 + 2T_1$, by Lemma 7.6, for $a_I \neq a_{111}$

$$c_{\mu - e_I}(\delta T_0) = c_\mu(T_0)\lambda_I \mu_I \tag{7.13}$$

where μ, I are uniquely determined by a given $\nu = \mu - e_I$.

(4) By the same lemma

$$c_{\mu-e_I}(\delta T) = \begin{cases} c_\mu(T_0)\lambda_I\mu_I \ \text{ or } 2c_\mu(T_1)\lambda_I\mu_I & I \neq (1,1,1) \\ 2c_\mu(T_1)\lambda_I\mu_I & I = (1,1,1) \end{cases}$$

where μ, I are uniquely determined by a given $\nu = \mu - e_I$. Note that if $c_{\mu-e_I}(\delta T) = c_\mu(T_0)\lambda_I\mu_I \neq 0$ then $\mu_I = 2$.

Lemma 7.8. *The prime $1+i$ appears in prime factorization of* $\gcd(c_\nu(\delta T))_\nu$ *in $\mathbb{Z}[i]$ with exponent exactly 2.*

Proof. By assumption (7.6), in $\mathbb{Z}[i]$

$$(1+i) \nmid \gcd(\lambda_I)_{I \neq (1,1,1)}.$$

Trivially in $\mathbb{Z}[i]$

$$2 = (1+i)^2(-i), \qquad (1+i) \nmid n, \quad n \in 2\mathbb{Z}+1.$$

Note that by (4), for $\nu = \mu - e_I$, $(1+i)^2 | c_\nu(\delta T)$ for all ν. So, it remains to show that $(1+i)^3 \nmid c_\nu(\delta T)$ for some ν. Pick $I \neq (1,1,1)$ such that $(1+i) \nmid$

λ_I. By (2) and (4), we can find μ such that

$$c_{\mu-e_I}(\delta T) = 2c_\mu(T_0)\lambda_I \neq 0.$$

Since $c_\mu(T_0)$ is odd, this number is not divisible by $(1+i)^3$. $\square$

Lemma 7.9. *p is a unit in $\mathbb{Z}[i]$. Therefore there exists $r_S, r_T \in \mathbb{Z}[i][a]$ each r_S, r_T having coefficients with* gcd *1, such that*

$$\boxed{qd(64S^3 - T^2)^m = qr_S\delta S + r_T\delta T.}$$

Proof. Consider (7.11) in Lemma 7.5. Since $\gcd(p,q) = 1$, if p_0 is a prime factor of p in $\mathbb{Z}[i]$, then the right side $\mod p_0$ is $r_S\delta S \mod p_0$, which is nonzero since the coefficients of r_S have gcd 1 by assumption, and likewise for δS by Lemma 7.7. But the left side of (7.11) is zero $\mod p_0$, a contradiction. This shows that p is a unit in $\mathbb{Z}[i]$ which we can assume it is 1, by absorbing p^{-1} into r_S, r_T. $\square$

Lemma 7.10. $(1+i)^3 \nmid q$.

Proof. For otherwise Lemma 7.9 implies that $(1+i)^3 | r_T\delta T$. Since $(1+i)^2|\delta T$ but $(1+i)^3 \nmid \delta T$ by Lemma 7.8, it follows that $(1+i)|r_T$, contradicting that $r_S \in \mathbb{Z}[i][a]$ has coefficients with gcd 1. $\square$

Lemma 7.11. $(1+i)^2 \nmid q$.

Proof. Suppose $2i = (1+i)^2$ divides q. Then we can write $q = 2q_1$ with $q_1 \in \mathbb{Z}[i]$ with $\gcd(q_1, 1+i) = 1$.

Since $\frac{1}{2}\delta T \in \mathbb{Z}[i][a]$, we can divide equation in Lemma 7.9 by 2 and get

$$q_1 d\Delta^m = q_1 r_S\delta S + r_T\frac{\delta T}{2} \in \mathbb{Z}[i][a]. \tag{7.14}$$

First we consider this equation in the residue field $\mod (1+i)$, which is $\mathbb{Z}/2\mathbb{Z}[a] = \mathbb{F}_2[a]$. Since $q_1 \equiv 1$, $64 \equiv 0$, $T \equiv H^2$, we get

$$dH^{4m} \equiv r_S\delta S + r_T\frac{\delta T}{2} \tag{7.15}$$

where

$$\begin{aligned} H = {} & a_{300}a_{030}a_{003} + a_{300}a_{021}a_{012} + a_{030}a_{201}a_{102} \\ & + a_{003}a_{210}a_{120} + a_{201}a_{120}a_{012} + a_{210}a_{102}a_{021}. \end{aligned}$$

This follows from direct observation that

$$\boxed{\Delta = T^2 = (\partial_{111} S)^2 = H^4 \mod 2.}$$

Claim 7.12. *H is irreducible in $\mathbb{F}_2[a]$.*

Proof. For otherwise, we can factorize

$$H = (P_1 a_{300} + P_2) P_3$$

with the $P_i \in \mathbb{F}_2[a]$ independent of a_{300} but P_3 is nonconstant, and $P_1 P_3 = a_{030}a_{003} + a_{021}a_{012}$. Check easily that this implies that P_1 is constant, so that we can set $P_1 = 1$. Therefore P_2 has degree 1 and

$$P_2(a_{030}a_{003} + a_{021}a_{012}) = a_{030}a_{201}a_{102} + a_{003}a_{210}a_{120} + a_{201}a_{120}a_{012} + a_{210}a_{102}a_{021}$$

which is clearly impossible. □

Now mod 2 (hence also mod $(1+i)$), the explicit expression of S gives

$$S \equiv H a_{111} + a_{111}^4 + P$$

for some $P \in \mathbb{F}_2[a]$ independent of a_{111} and has $\deg P = 4$. Setting $Q = \lambda_{111} H + \delta P$, we get

Claim 7.13. *There exists $Q \in \mathbb{F}_2[a]$ independent of a_{111} such that*

$$\delta S \equiv (\delta H) a_{111} + Q \mod (1+i).$$

Next, recall from (7.12) that $T = T_0 + 2T_1$. The explicit expression of T_1 has the form $T_1 = T_2 + T_3$ where $T_2 \in \mathbb{Z}[a]$ depends on a_{111} and $2|T_2$, and $T_3 \in \mathbb{Z}[a]$ is independent of a_{111}. Then

$$\frac{1}{2}\delta T = \frac{1}{2}\delta T_0 + \delta T_2 + \delta T_3.$$

Note that $\frac{1}{2}\delta T_0 \in \mathbb{Z}[a]$ is also independent of a_{111}. Taking $\mod (1+i)$, δT_2 drops out since $2|T_2$. This shows

Claim 7.14. $\frac{1}{2}\delta T \mod (1+i) \in \mathbb{F}_2[a]_5$ *is independent of a_{111}, and it is nonzero by Lemma 7.8.*

Suppose $(1+i)|d$. Let $u := \gcd(\delta S, \frac{\delta T}{2}) \in \mathbb{F}[i][a]$. By Lemma 7.7

$$\gcd(c_\nu(\delta S))_\nu = 1,$$

thus $u \neq 0 \mod (1+i)$. Then we have $P_S, P_T \in \mathbb{Z}[i][a]$ such that $\delta S = uP_S$, $\frac{\delta T}{2} = uP_T$. Then in $\mathbb{F}_2[a]$ equation (7.14) becomes

$$0 \equiv q_1 r_S \delta S + r_T \frac{\delta T}{2} \equiv u(q_1 r_S P_S + r_T P_T).$$

Thus

$$0 \equiv q_1 r_S P_S + r_T P_T \equiv r_S P_S + r_T P_T.$$

Since $(P_S, P_T) = 1 \in \mathbb{Z}[i][a]$, there exists $\alpha, \beta \in \mathbb{Z}[i][a]$ such that $\alpha P_S + \beta P_T = 1$. Thus $\alpha P_S + \beta P_T \equiv 1 \mod (1+i)$, i.e. $(P_S, P_T) = 1 \in \mathbb{F}_2[a]$. Therefore there exists some $h \in \mathbb{F}_2[a]$ such that $r_S \equiv hP_T$ and $r_T \equiv -hP_S \equiv -q_1 h P_S$.

Now we take $h_0 \in \mathbb{Z}[i][a]$ to be any lift of h in $\mathbb{Z}[i][a]$. Then

$$r_S = h_0 P_T + r'_S \text{ and } r_T = -q_1 h_0 P_S + r'_T$$

for some $r'_S,\ r'_T \in \mathbb{Z}[i][a]$ and $(i+1)|r'_S,\ (i+1)|r'_T$. Then equation (7.14) becomes

$$\begin{aligned} q_1 d \Delta^m &= q_1(h_0 P_T + r'_S)\delta S + (-q_1 h_0 P_S + r'_T)\frac{\delta T}{2} \\ &= q_1 r'_S \delta S + r'_T \frac{\delta T}{2} \in \mathbb{Z}[i][a]. \end{aligned}$$

Now we can see that $(i+1)$ is a common factor on both sides, so we have

$$q_1 \frac{d}{1+i}\Delta^m = q_1 \frac{r'_S}{1+i}\delta S + \frac{r'_T}{1+i}\frac{\delta T}{2} \in \mathbb{Z}[i][a].$$

Since $d \neq 0$, there exists a largest positive integer k such that

$$d_1 = \frac{d}{(1+i)^k} \in \mathbb{Z}[i], \quad (1+i) \nmid d_1.$$

Then we can repeat the process and get

$$q_1 d_1 \Delta^m = q_1 r''_S \delta S + r''_T \frac{\delta T}{2} \in \mathbb{Z}[i][a]. \tag{7.16}$$

for some $r''_S, r''_T \in \mathbb{Z}[i][a]$. Note that r''_S, r''_T not necessarily have the property that their coefficients have gcd 1.

Claim 7.13 shows that

$$\delta S \equiv (\delta H)a_{111} + Q \mod (1+i)$$

with $Q = \lambda_{111}H + \delta P$. By (7.6), $\lambda_{111} \equiv 0 \mod (1+i)$, then

$$\delta S \equiv (\delta H)a_{111} + \delta P \mod (1+i).$$

By looking at the explicit expression of T, we observe that $T \equiv H^2 \mod 4$. It implies that $\frac{\delta T}{2} \equiv H\delta H \mod 2$ and hence

$$\frac{\delta T}{2} \equiv H\delta H \mod (1+i).$$

Then (7.16) becomes

$$H^{4m} \equiv r''_S((\delta H)a_{111} + \delta P) + r''_T(H\delta H) \mod (1+i).$$

Now we evaluate both sides at $a_{111} = a_{120} = a_{102} = a_{210} = a_{012} = a_{021} = a_{201} = 0$. Let $Q|$ denotes the evaluation of Q under this condition. We observe that every term in P contains at least two of these a_I's, so $(\delta P)| = 0$. Then we get

$$(a_{300}a_{030}a_{003})^{4m-1} \equiv r''_T|(\delta H)|. \tag{7.17}$$

We observe that

$$(\delta H)| = \lambda_{300}a_{030}a_{003} + \lambda_{030}a_{300}a_{003} + \lambda_{003}a_{300}a_{030}.$$

Under condition (7.6), there are three types remaining:

1) $\{\lambda_{300}, \lambda_{030}, \lambda_{003}\} \equiv \{0,0,0\} \mod (1+i)$. Then $(\delta H)| \equiv 0 \mod (1+i)$ and it contradicts (7.17).
2) $\{\lambda_{300}, \lambda_{030}, \lambda_{003}\} \equiv \{1,1,1\} \mod (1+i)$. Then

$$(\delta H)| = a_{030}a_{003} + a_{300}a_{003} + a_{300}a_{030}.$$

It is irreducible in $\mathbb{F}_2[a_{300}, a_{030}, a_{003}]$ and it does not divide

$$(a_{300}a_{030}a_{003})^{4m-1},$$

it contradicts the fact that $\mathbb{F}_2[a_{300}, a_{030}, a_{003}]$ is an UFD.

3) $\{\lambda_{300}, \lambda_{030}, \lambda_{003}\} \equiv \{1, 1, 0\} \mod (1+i)$. If $\lambda_{300} = 1, \lambda_{030} = 1, \lambda_{003} = 0$,

$$(\delta H)| = a_{030}a_{003} + a_{300}a_{003} = (a_{030} + a_{300})a_{003}.$$

But $a_{030} + a_{300}$ doesn't divide $(a_{300}a_{030}a_{003})^{4m-1}$, it contradicts the fact that the fact that $\mathbb{F}_2[a_{300}, a_{030}, a_{003}]$ is an UFD.

This shows that our initial supposition that $(1+i)^2|q$ is false, hence proving Lemma 7.11. □

Lemma 7.15. $(1+i)|q$.

Proof. Suppose not.

Claim 7.16. $\delta S = H \mod (1+i)$.

Proof. We have

$$qdH^{4m} = qr_S\delta S + r_T\delta T. \tag{7.18}$$

Therefore in $\mathbb{F}_2[a]$, we have

$$dH^{4m} = r_S\delta S$$

since $2|\delta T$. Since the coefficients of r_S have gcd 1 by assumption, and same for δS by Lemma 7.7, the right side is nonzero, hence d is coprime to $(1+i)$, and hence $H = \delta S$ in $\mathbb{F}_2[a]$ (because the only unit in $\mathbb{F}_2$ is 1). □

Claim 7.17. *Without assuming* $\gcd(\delta) := \delta(\lambda_I)_I = 1$, *if* $\delta S = 0 \mod (1+i)$ *then* $\delta = 0 \mod (1+i)$.

Proof. We have $\delta S = 0 \mod (1+i)$ iff $(1+i)|\gcd(\delta S)$. Thus $(1+i)|\gcd(\lambda_I)$, by Lemma 7.7, hence $\delta = 0 \mod (1+i)$. □

Claim 7.18. $\delta = \partial_{111} \mod (1+i)$.

Proof. The explicit expression of S yields $\partial_{111}S = H \mod (1+i)$. So by Claim 7.16

$$(\delta - \partial_{111})S = H - H = 0 \mod (1+i).$$

Now letting $\delta - \partial_{111}$ play the role of δ in Claim 7.17, implies the claim. □

To finish the proof of Lemma 7.15, observe that Claim 7.18 contradicts (7.6). This shows that the supposition that $(1+i) \nmid q$ is false. □

Lemma 7.19. *q does not exists, hence Theorem 7.4 is proved.*

Proof. Lemmas 7.15 and 7.11 imply $(1+i)|q$ and $(1+i)^2 \nmid q$. Lemma 7.5 gives

$$\frac{q}{1+i} d\Delta^{4m} = \frac{q}{1+i} r_S \delta S + r_T \frac{\delta T}{1+i}.$$

Since $(1+i)|\frac{\delta T}{1+i}$, taking $\mod (1+i)$ yields

$$dH^{4m} = r_S \delta S \mod (1+i).$$

Again, since $\gcd(r_S) = 1$ and $\delta S \neq 0 \mod (1+i)$, $d \neq 0$ hence

$$H = \delta S \mod (1+i)$$

as in Claim 7.16, hence

$$\delta = \partial_{111} \mod (1+i)$$

as in Claim 7.18, which contradicts (7.6) again. □

Proposition 7.20. *The set of δ where $N(\delta)$ is nonempty, is dense in $V^\vee$, in analytic topology. Hence it is dense in Zariski topology.*

Proof. Let

$$S := \{(\lambda_I) \in \mathbb{Z}[i]^{10} \mid \gcd(\lambda_I)_I = 1,\ (1+i)|\lambda_{111} \text{ in } \mathbb{Z}[i] \\ \text{and } \{\lambda_{300}, \lambda_{030}, \lambda_{003}\} \neq \{1, 0, 0\} \mod (1+i)\}.$$

Then we have shown in Theorem 7.4 that $\mathcal{N}(\delta) \neq \emptyset$ for all $\lambda \in S$. We are going to show that given any point $\delta_{\bar{\lambda}} := \sum_I \bar{\lambda}_I \partial_I \in V^\vee$, we can find a sequence $\lambda^k \in \mathbb{Q}(i)^{10}$ such that $\lim_{k\to\infty} \lambda^k = \bar{\lambda}$ and $\mathcal{N}(\delta_{\lambda^k}) \neq \emptyset$ for all k.

We consider a subset of $S_0 \subset S$:

$$S_0 := \big\{(\lambda_I) \in \mathbb{Z}[i]^{10} \mid \gcd(\lambda_I)_I = 1, \lambda_{300} \equiv \lambda_{030} \equiv \lambda_{003} \equiv 1 \mod (1+i) \\ \text{and } \lambda_I \equiv 0 \mod (1+i) \text{ for } I \neq 300, 030, 003\big\}.$$

Since $\mathbb{Q}(i)^{10}$ is dense in $\mathbb{C}$, we can find a sequence $x^k \in \mathbb{Q}(i)^{10}$ such that $\lim_{k\to\infty} x^k = \bar{\lambda}$. For each k, we choose some $q^k \in \mathbb{Z}$ such that $q^k x^k \in \mathbb{Z}[i]^{10}$ and $\lim_{k\to\infty} q^k = \infty$.

Then we look at each entry, say $I = 111$. If $q^k x^k_{111} \equiv 0 \mod (1+i)$, let $\lambda^k_{111} = x^k$; if $q^k x^k_{111} \equiv 1 \mod (1+i)$, let $\lambda^k_{111} = \dfrac{q^k x^k_{111} + 1}{q^k}$. We repeat this process to make each entry of $q^k\lambda^k$ satisfy the $\mod (1+i)$ condition in S_0. Then it is clear that $\lim_{k\to\infty} |x^k - \lambda^k| = 0$ and thus

$$\lim_{k\to\infty} \lambda^k = \lim_{k\to\infty} x^k = \bar{\lambda}.$$

Note that $q^k\lambda^k$ is not necessarily in S_0 since it may not satisfy the gcd condition. Let $d^k = \gcd(q^k\lambda^k_I)_I \in \mathbb{Z}[i]$. Then by our construction $d^k \equiv 1 \mod (1+i)$. Then we consider $\frac{q^k\lambda^k}{d^k} \in \mathbb{Z}[i]^{10}$. It is clear that $\frac{q^k\lambda^k}{d^k} \in S_0$ and Theorem 7.4 implies that $\mathcal{N}(\delta_{\frac{q^k\lambda^k}{d^k}}) \neq \emptyset$. Since $\delta_{\frac{q^k\lambda^k}{d^k}}$ is homogeneous, $\mathcal{N}(\delta_{\lambda^k}) = \mathcal{N}(\delta_{\frac{q^k\lambda^k}{d^k}}) \neq \emptyset$, as desired. □

Corollary 7.21. *There exists a nonempty Zariski open subset $U_0 \subset V^\vee$, such that for each $\delta \in U_0$, $N(\delta) \neq \emptyset$.*

Proof. Consider the projection morphism of schemes of finite type over $\mathbb{C}$:

$$f : \operatorname{Spec} \mathbb{C}[\lambda_I, a_I, \Delta^{-1}]/\left\langle \sum_I \lambda_I \partial_I S, \sum_I \lambda_I \partial_I T \right\rangle \to \operatorname{Spec} \mathbb{C}[\lambda_I].$$

$Im(f)$ contains a dense subset of $V^\vee$ in analytic topology and therefore also in Zariski topology, so f is dominant, which implies that $Im(f)$ contains a non-empty Zariski open subset U_0, and consequently the corollary holds. □

7.3. Another proof

For the case $X = \mathbb{P}^2$, there is another simple proof for $N(\delta) \neq \emptyset$ where δ is a first order homogeneous constant coefficient differential operator. However, the proof cannot be generalized to higher dimension.

Proposition 7.22. *For $h = \delta$ (homogeneous, 1st order, constant coefficient) GIT-stable (i.e. in this case, smooth), and for each smooth section $f(b)$, we have $N(\delta) \cap G \cdot f(b) \neq \emptyset$ where $G \cdot f(b)$ denotes the G-orbit of $f(b)$. So in particular, $N(\delta) \neq \emptyset$.*

Proof. Since h is GIT-stable, h has finite stabilizer in $\mathbb{P}V^\vee$, under the action of $G = SL_3$. Therefore, the G-orbit of h in $\mathbb{P}V^\vee$ is a closed subvariety of dimension 8. For the same reason, the G-orbit of $f(b)$ in $\mathbb{P}V^\vee$ has dimension 8, so $f(b)$ is not killed by any nonzero Lie algebra element in $\mathfrak{sl}_3$, (otherwise the exponential map would give rise to a one-parameter subgroup infinite stabilizer of $f(b)$, under the action of G) and therefore the $\mathbb{C}$-vector space $W_b := \{Z_x f(b) | x \in \mathfrak{sl}_3\}$ has dimension 8. Therefore the projectivization $\mathbb{P}W_b$ is a closed subvariety of dimension 7 in $\mathbb{P}V^\vee$, which then must intersect with the G-orbit of $f(b)$ in $\mathbb{P}V^\vee$ by dimension reason. Therefore, there exists $g \in SL_3$, $\lambda \in \mathbb{C}, \lambda \neq 0$, and $x \in \mathfrak{sl}_3$ such that

$$gh = \lambda Z_x f(b). \tag{7.19}$$

Therefore, since $g^{-1} f(b) = f_{g^{-1}b}$, we have

$$h = g^{-1} Z_{\frac{1}{\lambda}x} g f_{g^{-1}b}. \tag{7.20}$$

Since $g^{-1} Z_{\frac{1}{\lambda}x} g = Z_{x'}$ for some $x' \in \mathfrak{sl}_3$, Lemma 7.2 implies that $g^{-1}b \in \mathcal{N}(\delta)$. Since $f_{g^{-1}b} \in G \cdot f(b)$, we have $\mathcal{N}(\delta) \cap G \cdot f(b) \neq \emptyset$. Hence the lemma follows. □

8. An application to classical invariant theory

Let $X = \mathbb{P}^{n-1}$ with $n \geq 3$, $V^\vee = \Gamma(X, K_X^{-1})$, and $G = SL_n$ as before. In this section, we prove the following

Theorem 8.1. *Let $\langle S_1, \ldots, S_w \rangle$ be a system of homogeneous polynomials that generate $\mathbb{C}[V^\vee]^G$, then there exists an S_k among these generators, such that $deg(S_k) \equiv 1 (\mod n)$.*

We first prove the following lemma for any $X = G/P$:

Lemma 8.2. *Let δ be a first order constant coefficient homogeneous differential operator, and $h = \delta f$ as before. Let $b \in B$. Then the following conditions are equivalent:*

1) $b \in \mathcal{N}(\delta)$.
2) $h = Z_x f(b)$ *for some* $x \in \mathfrak{g}$.
3) $(\delta P)(b) = 0$ *for any* $P \in \mathbb{C}[V^\vee]^G$.

Proof. We already proved that (1) and (2) are equivalent in Lemma 7.2. We now prove that (2) and (3) are equivalent. Again consider the projection morphism $p : B \to B/G$. By GIT theory, the function ring of B/G is identified with $\mathbb{C}[V^\vee, \Delta^{-1}]^G$: i.e. elements in $\mathbb{C}[V^\vee]^G$ divided by powers of Δ.

Assuming (3), take any regular function $\phi : B/G \to \mathbb{C}$, then $\phi \cdot p \in C[V^\vee, \Delta^{-1}]^G$, and therefore $(\delta(\phi \cdot p))(b) = 0$. (Note that $(\delta(\Delta))(b) = 0$ as

$\Delta \in \mathbb{C}[V^\vee]^G$.) So $(dp_b(\delta)\phi)(p(b)) = (\delta(\phi \cdot p))(b) = 0$, where again dp_b denotes the tangent map induced by p at b. Therefore $dp_b(\delta) = 0$, which is equivalent to (2) as we already know.

Assuming (2), for any $P \in \mathbb{C}[V^\vee]^G$, by abuse of notation we still denote P restricting to B by P. Then P is a regular function on B invariant under G, therefore $P = \phi_0 \cdot p$ for a regular function ϕ_0 on B/G. So (2) implies $dp_b(\delta) = 0$, which in turn implies $(dp_b(\delta)\phi_0)(p(b)) = 0$. i.e. $(\delta P)(b) = 0$. □

Now we prove Theorem 8.1:

Proof. Let $X = \mathbb{P}^{n-1}$ and let $\delta = \partial_{a_{1\ldots1}}$, so $h = x_1 \ldots x_n$. Let $b = x_1^n + \cdots + x_n^n$ be the Fermat point, which lies in B. As $n \geq 3$, it is clear that h does not satisfy condition (2) in lemma 8.2. Therefore Lemma 8.2 implies that there exists a homogeneous element $S \in \mathbb{C}[V^\vee]^G$, such that $(\delta S)(b) \neq 0$. i.e. $(\partial_{a_{1\ldots1}} S)(b) \neq 0$. This implies that S contains a monomial term that is linear in $a_{1\ldots1}$, which is a product of $a_{1\ldots1}$ with powers of $a_{n0\ldots0}, \ldots, a_{0\ldots0n}$. Since any monomial in S is invariant under the maximal torus action, for any monomial that appears in S, the sum of indexes at each position has to be equal. Therefore, this monomial is a nonzero multiple of $a_{1\ldots1}(a_{n0\ldots0} \cdots a_{0\ldots0n})^k$ for some nonzero $k \in \mathbb{N}$ (as it is clear that there is no invariant polynomial in degree 1).

Now, take S' to be an element in $\mathbb{C}[V^\vee]^G$ such that it contains a monomial term that is a nonzero multiple of $a_{1\ldots1}(a_{n0\ldots0} \cdots a_{0\ldots0n})^k$ with minimal k. Then it is clear that S' can not be written as a polynomial of invariant polynomials which do not contain monomial terms of this form. The theorem is therefore proved. □

Remark 8.3. It is possible to elaborate on this argument to extract further information about the invariant ring $\mathbb{C}[V^\vee]^G$, and to establish further relations between $\mathcal{N}(\delta)$ and the invariant ring. Indeed, theorem 8.1 does not hold for $n = 2$, precisely because in that case, the Fermat point does lie in $\mathcal{N}(\delta)$ for the δ in the above proof.

Remark 8.4. There are indications that our study of $\mathcal{N}(\delta)$ for 1st order derivatives is also related with the local Torelli theorem, as the vanishing loci of such derivatives of periods correspond to degenerations of the period map. It would also be interesting to investigate the invariant theoretic or geometric meaning of $\mathcal{N}(\delta)$ for higher order δ. We plan to study these questions in a future paper.

Appendix A. Some examples for $\mathbb{P}^m$

Making use of methods in [BHLSY], we can compute a basis of $\hat{\mathfrak{g}}Re^b$ explicitly at the large complex structure limit (LCSL) b_∞ and the Fermat point b_F for $\mathbb{P}^1$ and $\mathbb{P}^2$. By Theorem 2.1, this allows us to find explicit differential relations, i.e. linear relations for constant coefficient differential operators that kill periods at these points.

If $X = \mathbb{P}^m$, $G = SL_{m+1}$, then we can identify R with the subring of $\mathbb{C}[x_0, \ldots, x_m]$ generated by degree $m+1$ monomials.

Lemma A.1. *[BHLSY, Lemma 2.12] We have*

$$\hat{\mathfrak{g}} \cdot (Re^{f(b)}) = Re^{f(b)} \cap \sum_i \frac{\partial}{\partial x_i}(\mathbb{C}[x]e^{f(b)})$$

for all $b \in B$.

Computation for $\mathbb{P}^1$ at LCSL. For $X = \mathbb{P}^1$, $G = SL_2$, $R \equiv \mathbb{C}[x_1^2, x_2^2, x_1x_2]$, $f = a_0x_1x_2 + a_1x_1^2 + a_2x_2^2$ and $b_\infty = x_1x_2$.

Claim A.2. *For integers* $\alpha, \beta \geq 0$, $\alpha \neq \beta$, $\alpha + \beta > 0$ *and* $2|(\alpha+\beta)$,

$$x_1^\alpha x_2^\beta e^{x_1x_2} \in \hat{\mathfrak{g}} \cdot (Re^{x_1x_2}).$$

Proof. Without loss of generality we assume $\alpha > \beta \geq 0$. For $m, n \geq 0$, we observe

$$\frac{\partial}{\partial x_2}(x_1^m x_2^{n+1} e^{x_1x_2}) = x_1^{m+1}x_2^{n+1}e^{x_1x_2} + (n+1)x_1^m x_2^n e^{x_1x_2}. \tag{A.1}$$

Since $x_1^{\alpha-\beta}e^{x_1x_2} = \frac{\partial}{\partial x_2}x_1^{\alpha-\beta-1}e^{x_1x_2}$, then by (A.1),

$$x_1^\alpha x_2^\beta e^{x_1x_2} \in \sum_i \frac{\partial}{\partial x_i}(\mathbb{C}[x]e^{x_1x_2}).$$

Thus by Lemma A.1, $x_1^\alpha x_2^\beta e^{x_1x_2} \in \hat{\mathfrak{g}} \cdot (Re^{x_1x_2})$ if we further require $2|(\alpha+\beta)$. □

Consider the Euler operator, for $k \geq 0$,

$$\begin{aligned}(E+1)(x_1x_2)^k e^{x_1x_2} &= \left(\frac{1}{2}\sum_i x_i \frac{\partial}{\partial x_i} + 1\right)(x_1x_2)^k e^{x_1x_2} \\ &= \left((x_1x_2)^{k+1} + (k+1)(x_1x_2)^k\right) e^{x_1x_2} \in \hat{\mathfrak{g}} \cdot (Re^{x_1x_2}).\end{aligned}$$

By induction we have

$$\big((x_1x_2)^k + (-1)^{k+1}k!\big)e^{x_1x_2} \in \hat{\mathfrak{g}} \cdot (Re^{x_1x_2}).$$

Claim A.3. *For $X = \mathbb{P}^1$, at the LCSL we have a basis description*

$$\begin{aligned}\hat{\mathfrak{g}} \cdot (Re^{x_1x_2}) =& (\oplus_{k=1}^{\infty}\mathbb{C}\big((x_1x_2)^k + (-1)^{k+1}k!\big)e^{x_1x_2}) \\ & \oplus (\oplus_{\alpha+\beta>0,\alpha\neq\beta,2|(\alpha+\beta)}\mathbb{C}x_1^{\alpha}x_2^{\beta}e^{x_1x_2}) =: A.\end{aligned}$$

Proof. We already showed $\hat{\mathfrak{g}} \cdot (Re^{x_1x_2}) \supset A$. It is clear that $e^{x_1x_2} \notin A$ and $A \oplus \mathbb{C}e^{x_1x_2} = Re^{x_1x_2}$, thus $\dim_{\mathbb{C}} Re^{x_1x_2}/A = 1$. Since

$$(Re^{f(b)}/\hat{\mathfrak{g}} \cdot (Re^{f(b)}))^* \simeq \mathcal{H}om_{D^\vee}(\tau, \mathcal{O})_b \simeq \mathrm{sol}(\tau)_b$$

and we know in this case $\dim_{\mathbb{C}} \mathrm{sol}(\tau)_{b_\infty} = 1$, then $\dim_{\mathbb{C}} Re^{x_1x_2}/\hat{\mathfrak{g}} \cdot (Re^{x_1x_2}) = 1$. Therefore $\hat{\mathfrak{g}} \cdot (Re^{x_1x_2}) = A$. $\square$

Now consider

$$\begin{aligned}\delta e^f(b_\infty) &= \left(\sum c_\alpha \left(\frac{\partial}{\partial a_0}\right)^{\alpha_0}\left(\frac{\partial}{\partial a_1}\right)^{\alpha_1}\left(\frac{\partial}{\partial a_2}\right)^{\alpha_2}\right) e^f(b_\infty) \\ &= \left(\sum c_\alpha x_1^{\alpha_0+2\alpha_1}x_2^{\alpha_0+2\alpha_2}\right) e^{x_1x_2}.\end{aligned}$$

Let $\delta e^f(b_\infty) \in \hat{\mathfrak{g}} \cdot (Re^{x_1x_2})$. By Claim A.3, when $\alpha_1 \neq \alpha_2$, there is no restriction on c_α; when $\alpha_1 = \alpha_2$, let $d := \alpha_0 + \alpha_1 + \alpha_2$, it forces

$$\left(\sum c_\alpha (x_1x_2)^d\right) e^{x_1x_2} = \left(\sum_{d\geq 1} c_\alpha\big((x_1x_2)^d + (-1)^{d+1}(d)!\big)\right) e^{x_1x_2}.$$

Thus

$$c_{0,0,0} = \sum_{d\geq 1} c_{\alpha_0,\alpha_1,\alpha_1}(-1)^{d+1}(d)!.$$

We can rewrite this as

$$\sum_{\alpha_1=\alpha_2,|\alpha|=d} c_\alpha(-1)^d d! = 0.$$

Thus as a direct consequence of Theorem 2.1, we have:

Claim A.4. *When $X=\mathbb{P}^1$, if the coefficients of*

$$\delta = \sum c_\alpha \left(\frac{\partial}{\partial a_0}\right)^{\alpha_0} \left(\frac{\partial}{\partial a_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial a_2}\right)^{\alpha_2}$$

satisfy the linear relation

$$\sum_{\alpha_1=\alpha_2,\,|\alpha|=d} c_\alpha(-1)^d d! = 0,$$

then $\delta\mathfrak{s}(b_\infty)=0$ for all $\mathfrak{s}\in sol(\tau)_b$.

Computation for $\mathbb{P}^2$ at LCSL. For $X=\mathbb{P}^2$, $G=SL_3$, $R\equiv\mathbb{C}[x_\alpha,|\alpha|=3]$, $f=a_0x_1x_2x_3+a_1x_1^3+a_2x_1^2x_2+a_3x_1x_2^2+a_4x_2^3+a_5x_2^2x_3+a_6x_2x_3^2+a_7x_3^3+a_8x_1x_3^2+a_9x_1^2x_3$, $b_\infty=x_1x_2x_3$.

Similar to the $\mathbb{P}^1$ case, we can show

Claim A.5. *For $X=\mathbb{P}^2$, at the LCSL we have a basis description*

$$\begin{aligned}\hat{\mathfrak{g}}\cdot(Re^{x_1x_2x_3}) = (&\oplus_{k=1}^{\infty}\mathbb{C}\big((x_1x_2x_3)^k+(-1)^{k+1}k!\big)e^{x_1x_2x_3})\\ &\oplus(\oplus_{\iota_1+\iota_2+\iota_3>0,\iota_1,\iota_2,\iota_3 \text{ not all equal, } 3|(\iota_1+\iota_2+\iota_3)}\mathbb{C}x_1^{\iota_1}x_2^{\iota_2}x_3^{\iota_3}e^{x_1x_2x_3}).\end{aligned}$$

In this case we know $\dim_{\mathbb{C}} \mathrm{sol}(\tau)_{b_\infty}=1$, so $\hat{\mathfrak{g}}\cdot(Re^{x_1x_2x_3})$ is of codimension 1.

Now consider

$$\begin{aligned}\delta e^f(b_\infty) &= \left(\sum c_\alpha\left(\frac{\partial}{\partial a_0}\right)^{\alpha_0}\cdots\left(\frac{\partial}{\partial a_9}\right)^{\alpha_9}\right)e^f(b_\infty)\\ &= \Bigg(\sum c_\alpha x_1^{\alpha_0+3\alpha_1+2\alpha_2+\alpha_3+\alpha_8+2\alpha_9}x_2^{\alpha_0+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6}\\ &\qquad\times x_3^{\alpha_0+\alpha_5+2\alpha_6+3\alpha_7+2\alpha_8+\alpha_9}\Bigg)e^{x_1x_2x_3}.\end{aligned}$$

Let

$$\text{(A.2)} \qquad \begin{cases} \beta_1 := \alpha_0 + 3\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_8 + 2\alpha_9, \\ \beta_2 := \alpha_0 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6, \\ \beta_3 := \alpha_0 + \alpha_5 + 2\alpha_6 + 3\alpha_7 + 2\alpha_8 + \alpha_9. \end{cases}$$

By Claim A.5 we can see that there is no restriction on the coefficient c_α unless

$$\beta_1 = \beta_2 = \beta_3 = |\alpha|.$$

Let $\delta e^f(b_\infty) \in \hat{\mathfrak{g}} \cdot (Re^{x_1 x_2 x_3})$, it forces

$$\sum_{\beta_1=\beta_2=\beta_3=d} c_\alpha (-1)^d (d!) = 0.$$

Thus by Theorem 2.1, we have:

Claim A.6. *When $X = \mathbb{P}^2$, if the coefficients of*

$$\delta = \sum c_\alpha \left(\frac{\partial}{\partial a_0}\right)^{\alpha_0} \cdots \left(\frac{\partial}{\partial a_9}\right)^{\alpha_9}$$

satisfy the linear relation

$$\sum_{\beta_1=\beta_2=\beta_3=d} c_\alpha (-1)^d (d!) = 0,$$

then $\delta\mathfrak{s}(b_\infty) = 0$ for all $\mathfrak{s} \in sol(\tau)_b$.

Computation for $\mathbb{P}^1$ at the Fermat point. $X = \mathbb{P}^1$, $G = SL_2$, $b_F = x_1^2 + x_2^2$.

Let $(-1)!! = 1$. By straightforward induction which we omit here, we can show

Claim A.7. *For $X = \mathbb{P}^1$, at the Fermat point we have a basis description*

$$\hat{\mathfrak{g}} \cdot (Re^{x_1^2+x_2^2}) = \left(\oplus_{k \equiv l \equiv 1 (mod\ 2)} \mathbb{C} x_1^k x_2^l e^{x_1^2+x_2^2}\right) \\ \oplus \left(\oplus_{k \equiv l \equiv 0 (mod\ 2), k+l \geq 2} \mathbb{C} \left(x_1^k x_2^l - (-1)^{\frac{k+l}{2}} \frac{(k-1)!!(l-1)!!}{2^{(k+l)/2}}\right) e^{x_1^2+x_2^2}\right).$$

In this case we know $\dim_{\mathbb{C}} \mathrm{sol}(\tau)_{b_F} = 1$ and $\hat{\mathfrak{g}} \cdot (Re^{x_1^2+x_2^2})$ is of codimension 1.

And by Theorem 2.1, we have:

Claim A.8. *When $X = \mathbb{P}^1$, if the coefficients of*

$$\delta = \sum c_\alpha \left(\frac{\partial}{\partial a_0}\right)^{\alpha_0} \left(\frac{\partial}{\partial a_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial a_2}\right)^{\alpha_2}$$

satisfy the linear relation

$$\sum_{\alpha_0 \equiv 0 \mod 2} c_{\alpha_0,\alpha_1,\alpha_2} (-1)^{\alpha_0+\alpha_1+\alpha_2} \frac{(\alpha_0 + 2\alpha_1 - 1)!!(\alpha_0 + 2\alpha_2 - 1)!!}{2^{(\alpha_0+\alpha_1+\alpha_2)}} = 0,$$

then $\delta\mathfrak{s}(b_F) = 0$ for all $\mathfrak{s} \in sol(\tau)_b$.

Computation for $\mathbb{P}^2$ at the Fermat point. $X = \mathbb{P}^2$, $G = SL_3$, $b_F = x_1^3 + x_2^3 + x_3^3$.

Let $(-1)!!! = (-2)!!! = 1$. By straightforward induction, we can show

Claim A.9. *Let $\iota_0 + \iota_1 + \iota_2 := c$. For $X = \mathbb{P}^2$, at the Fermat point we have a basis description*

$$\begin{aligned}
\hat{\mathfrak{g}} \cdot (Re^{x_0^3+x_1^3+x_2^3}) = &\left(\oplus_{one\ of\ \iota_i \equiv 2 (\mathrm{mod}\ 3)} \mathbb{C} x_0^{\iota_0} x_1^{\iota_1} x_2^{\iota_2} e^{x_0^3+x_1^3+x_2^3}\right) \\
\oplus &\Bigg(\oplus_{\iota_0 \equiv \iota_1 \equiv \iota_2 \equiv 0 (\mathrm{mod}\ 3), c \geq 3} \\
&\mathbb{C}\left(x_0^{\iota_0} x_1^{\iota_1} x_2^{\iota_2} - (-1)^{\frac{c}{3}} \frac{(\iota_0 - 2)!!!(\iota_1 - 2)!!!(\iota_2 - 2)!!!}{3^{\frac{c}{3}}}\right) e^{x_0^3+x_1^3+x_2^3}\Bigg) \\
\oplus &\Bigg(\oplus_{\iota_0 \equiv \iota_1 \equiv \iota_2 \equiv 1 (\mathrm{mod}\ 3), c \geq 6} \\
&\mathbb{C}\left(x_0^{\iota_0} x_1^{\iota_1} x_2^{\iota_2} + (-1)^{\frac{c}{3}} \frac{(\iota_0 - 2)!!!(\iota_1 - 2)!!!(\iota_2 - 2)!!!}{3^{\frac{c}{3}-1}} x_0 x_1 x_2\right) e^{x_0^3+x_1^3+x_2^3}\Bigg).
\end{aligned}$$

In this case we know $\dim_{\mathbb{C}} \mathrm{sol}(\tau)_{b_F} = 2$ and $\hat{\mathfrak{g}} \cdot (Re^{x_1^3+x_2^3+x_3^3})$ is of codimension 2.

Then by Theorem 2.1, we have:

Claim A.10. *When $X = \mathbb{P}^2$, if the coefficients of*

$$\delta = \sum c_\alpha \left(\frac{\partial}{\partial a_0}\right)^{\alpha_0} \cdots \left(\frac{\partial}{\partial a_9}\right)^{\alpha_9}$$

satisfy the linear relation

$$\begin{cases} \sum_{\beta_1\equiv\beta_2\equiv\beta_3\equiv 0 \mod 3} c_\alpha(-1)^{\frac{\beta_1+\beta_2+\beta_3}{3}} \dfrac{(\beta_1-2)!!!(\beta_2-2)!!!(\beta_3-2)!!!}{3^{\frac{\beta_1+\beta_2+\beta_3}{3}}} = 0, \\ \sum_{\beta_1\equiv\beta_2\equiv\beta_3\equiv 1 \mod 3} c_\alpha(-1)^{\frac{\beta_1+\beta_2+\beta_3}{3}-1} \dfrac{(\beta_1-2)!!!(\beta_2-2)!!!(\beta_3-2)!!!}{3^{\frac{\beta_1+\beta_2+\beta_3}{3}-1}} = 0 \end{cases}$$

where β_i *are defined in* (A.2), *then* $\delta\mathfrak{s}(b_F) = 0$ *for all* $\mathfrak{s} \in sol(\tau)_b$.

Appendix B. Expressions of S and T

The degree 4 invariant S of a ternary cubic equals (see [S, p.167])

$$\begin{aligned} S = &-a_{300}a_{012}^2a_{120} + a_{012}^2a_{210}^2 + a_{300}a_{012}a_{021}a_{111} - a_{012}a_{021}a_{201}a_{210} \\ &- a_{012}a_{102}a_{120}a_{210} + a_{030}a_{300}a_{012}a_{102} - 2a_{012}a_{111}^2a_{210} \\ &+ 3a_{012}a_{111}a_{120}a_{201} - a_{030}a_{012}a_{201}^2 - a_{300}a_{021}^2a_{102} + a_{021}^2a_{201}^2 \\ &+ 3a_{021}a_{102}a_{111}a_{210} - a_{021}a_{102}a_{120}a_{201} - 2a_{021}a_{111}^2a_{201} \\ &+ a_{003}a_{300}a_{021}a_{120} - a_{003}a_{021}a_{210}^2 + a_{102}^2a_{120}^2 - a_{030}a_{102}^2a_{210} \\ &- 2a_{102}a_{111}^2a_{120} + a_{030}a_{102}a_{111}a_{201} + a_{111}^4 + a_{003}a_{111}a_{120}a_{210} \\ &- a_{003}a_{030}a_{300}a_{111} - a_{003}a_{120}^2a_{201} + a_{003}a_{030}a_{201}a_{210}. \end{aligned}$$

The degree 6 invariant T of the ternary cubic equals (see [S, p.171])

$$\begin{aligned} T = &\, a_{003}^2a_{030}^2a_{300}^2 - 6a_{003}^2a_{030}a_{120}a_{210}a_{300} + 4a_{003}^2a_{030}a_{210}^3 + 4a_{003}^2a_{120}^3a_{300} \\ &- 6a_{003}a_{012}a_{021}a_{030}a_{300}^2 + 18a_{003}a_{012}a_{021}a_{120}a_{210}a_{300} \\ &- 12a_{003}a_{012}a_{021}a_{210}^3 + 12a_{003}a_{012}a_{030}a_{111}a_{210}a_{300} \\ &+ 6a_{003}a_{012}a_{030}a_{120}a_{201}a_{300} - 12a_{003}a_{012}a_{030}a_{201}a_{210}^2 \\ &- 24a_{003}a_{012}a_{111}a_{120}^2a_{300} + 12a_{003}a_{012}a_{111}a_{120}a_{210}^2 \\ &+ 6a_{003}a_{012}a_{120}^2a_{201}a_{210} - 24a_{003}a_{021}^2a_{111}a_{210}a_{300} \\ &- 12a_{003}a_{021}^2a_{120}a_{201}a_{300} + 24a_{003}a_{021}^2a_{201}a_{210}^2 \\ &+ 6a_{003}a_{021}a_{030}a_{102}a_{210}a_{300} + 12a_{003}a_{021}a_{030}a_{111}a_{201}a_{300} \\ &- 12a_{003}a_{021}a_{030}a_{201}^2a_{210} - 12a_{003}a_{021}a_{102}a_{120}^2a_{300} \\ &+ 6a_{003}a_{021}a_{102}a_{120}a_{210}^2 + 36a_{003}a_{021}a_{111}^2a_{120}a_{300} \\ &+ 12a_{003}a_{021}a_{111}^2a_{210}^2 - 60a_{003}a_{021}a_{111}a_{120}a_{201}a_{210} \\ &+ 24a_{003}a_{021}a_{120}^2a_{201}^2 - 6a_{003}a_{030}^2a_{102}a_{201}a_{300} + 4a_{003}a_{030}^2a_{201}^3 \end{aligned}$$

$$
\begin{aligned}
&+ 12a_{003}a_{030}a_{102}a_{111}a_{120}a_{300} - 24a_{003}a_{030}a_{102}a_{111}a_{210}^2\\
&+ 18a_{003}a_{030}a_{102}a_{120}a_{201}a_{210} - 20a_{003}a_{030}a_{111}^3a_{300}\\
&+ 36a_{003}a_{030}a_{111}^2a_{201}a_{210} - 24a_{003}a_{030}a_{111}a_{120}a_{201}^2\\
&+ 12a_{003}a_{102}a_{111}a_{120}^2a_{210} - 12a_{003}a_{102}a_{120}^3a_{201} - 12a_{003}a_{111}^3a_{120}a_{210}\\
&+ 12a_{003}a_{111}^2a_{120}^2a_{201} + 4a_{012}^3a_{030}a_{300}^2 - 12a_{012}^3a_{120}a_{210}a_{300}\\
&+ 8a_{012}^3a_{210}^3 - 3a_{012}^2a_{021}^2a_{300}^2 + 12a_{012}^2a_{021}a_{111}a_{210}a_{300}\\
&+ 6a_{012}^2a_{021}a_{120}a_{201}a_{300} - 12a_{012}^2a_{021}a_{201}a_{210}^2 - 12a_{012}^2a_{030}a_{102}a_{210}a_{300}\\
&- 24a_{012}^2a_{030}a_{111}a_{201}a_{300} + 24a_{012}^2a_{030}a_{201}^2a_{210} + 24a_{012}^2a_{102}a_{120}^2a_{300}\\
&+ 12a_{012}^2a_{111}^2a_{120}a_{300} - 24a_{012}^2a_{111}^2a_{210}^2 + 36a_{012}^2a_{111}a_{120}a_{201}a_{210}\\
&- 27a_{012}^2a_{120}^2a_{201}^2 + 6a_{012}a_{021}^2a_{102}a_{210}a_{300} + 12a_{012}a_{021}^2a_{111}a_{201}a_{300}\\
&- 12a_{012}a_{021}^2a_{201}^2a_{210} + 18a_{012}a_{021}a_{030}a_{102}a_{201}a_{300} - 12a_{012}a_{021}a_{030}a_{201}^3\\
&- 60a_{012}a_{021}a_{102}a_{111}a_{120}a_{300} + 36a_{012}a_{021}a_{102}a_{111}a_{210}^2\\
&- 6a_{012}a_{021}a_{102}a_{120}a_{201}a_{210} - 12a_{012}a_{021}a_{111}^3a_{300}\\
&- 12a_{012}a_{021}a_{111}^2a_{201}a_{210} + 36a_{012}a_{021}a_{111}a_{120}a_{201}^2\\
&- 12a_{012}a_{030}a_{102}^2a_{120}a_{300} + 24a_{012}a_{030}a_{102}^2a_{210}^2\\
&+ 36a_{012}a_{030}a_{102}a_{111}^2a_{300} - 60a_{012}a_{030}a_{102}a_{111}a_{201}a_{210}\\
&+ 6a_{012}a_{030}a_{102}a_{120}a_{201}^2 + 12a_{012}a_{030}a_{111}^2a_{201}^2 - 12a_{012}a_{102}^2a_{120}^2a_{210}\\
&+ 36a_{012}a_{102}a_{111}a_{120}^2a_{201} + 24a_{012}a_{111}^4a_{210} - 36a_{012}a_{111}^3a_{120}a_{201}\\
&+ 8a_{021}^3a_{201}^3 + 24a_{021}^2a_{102}^2a_{120}a_{300} - 27a_{021}^2a_{102}^2a_{210}^2 + 12a_{021}^2a_{102}a_{111}^2a_{300}\\
&+ 36a_{021}^2a_{102}a_{111}a_{201}a_{210} - 12a_{021}^2a_{102}a_{120}a_{201}^2 - 24a_{021}^2a_{111}^2a_{201}^2\\
&+ 6a_{021}a_{030}a_{102}^2a_{201}a_{210} + 12a_{021}a_{030}a_{102}a_{111}a_{201}^2\\
&+ 36a_{021}a_{102}^2a_{111}a_{120}a_{210} - 12a_{021}a_{102}^2a_{120}^2a_{201}\\
&- 36a_{021}a_{102}a_{111}^3a_{210} - 12a_{021}a_{102}a_{111}^2a_{120}a_{201} + 4a_{030}^2a_{102}^3a_{300}\\
&- 3a_{030}^2a_{102}^2a_{201}^2 - 12a_{030}a_{102}^3a_{120}a_{210} + 12a_{030}a_{102}^2a_{111}^2a_{210}\\
&+ 12a_{030}a_{102}^2a_{111}a_{120}a_{201} - 12a_{030}a_{102}a_{111}^3a_{201} + 8a_{102}^3a_{120}^3\\
&- 24a_{102}^2a_{111}^2a_{120}^2 - 3a_{003}^2a_{120}^2a_{210}^2 + 4a_{003}a_{021}^3a_{300}^2 - 12a_{012}^2a_{102}a_{120}a_{210}^2\\
&- 12a_{012}a_{102}a_{111}^2a_{120}a_{210} - 24a_{021}a_{030}a_{102}^2a_{111}a_{300} + 24a_{021}a_{111}^4a_{201}\\
&- 12a_{021}^3a_{102}a_{201}a_{300} + 24a_{102}a_{111}^4a_{120} - 8a_{111}^6.
\end{aligned}
$$

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