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# Local curves, wild character varieties, and degenerations

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Conjectural results for cohomological invariants of wild character varieties are obtained by counting curves in degenerate Calabi-Yau threefolds. A conjectural formula for $E$-polynomials is derived from the Gromov-Witten theory of local Calabi-Yau threefolds with normal crossing singularities. A refinement is also conjectured, generalizing existing results of Hausel, Mereb and Wong as well as recent joint work of Donagi, Pantev and the author for weighted Poincaré polynomials of wild character varieties.



## 1. Introduction

The cohomology of character varieties and Higgs bundle moduli spaces is an important problem in geometry, topology and mathematical physics. Several approaches have been developed so far employing different methods. Arithmetic methods for character varieties have been used in [24–26, 33] while similar methods for Higgs bundles have been employed in [18, 32, 46, 55]. The motive of the moduli space of irregular connections as well as irregular Higgs bundles over arbitrary fields has been determined in [19]. Moreover, a different approach based on wallcrossing for linear chains has been used in [20, 21] for motivic and Hirzebruch genus computations. At the same the topology of wild character varieties has been related to ploynomial invariants of legendrian knots in [56, 57]. Finally, a string theoretic framework for this problem has been developed in [10–12] based on an identification of perverse Betti numbers of Higgs moduli spaces with degeneracies of spinning BPS states in M-theory. In particular the conjectural formulas derived by Hausel and Rodriquez-Villegas [26], Hausel, Letellier and Rogriguez-Villegas [24] are identified with Gopakumar-Vafa expansions in the refined stable pair theory of certain Calabi-Yau threefolds. Furthermore, this framework also provides important evidence for the $P = W$ conjecture formulated by de Cataldo, Hausel and Migliorini [13].

The string theoretic approach has been recently generalized to wild character varieties in [14], providing a physical derivation and a generalization for the results of Hausel, Mereb and Wong [25]. While the geometric framework and the spectral construction of [14] are general, explicit formulas are obtained only for wild character varieties with one singular point on the projective line. The goal of the present paper is to further extend these results to wild character varieties with multiple singular points on higher genus curves.

### 1.1. Wild character varieties

Wild character varieties are moduli spaces of Stokes data of irregular connections on curves with fixed irregular type. Such moduli spaces played a central role in Witten's work on geometric Langlands correspondence [58] and were rigorously constructed by Boalch in [5, 7]. The setup consists of a smooth projective curve $C$ with $m$ pairwise distinct marked points $p_a$, $1 \leq a \leq m$, and a complex reductive algebraic group $G$, which in this paper will be the general linear group, $GL(r, \mathbb{C})$, $r \geq 1$.

Let $\mathbf{T}_r \subset GL(r, \mathbb{C})$ be the standard maximal torus, and $\mathbf{t}_r \subset \mathfrak{g}l(r, \mathbb{C})$ the corresponding Cartan subalgebra. As in [7], a $\mathbf{t}_r$-valued irregular type at a point $p \in C$ is an equivalence class

$$\mathsf{Q} \in \mathbf{t}_r(\widehat{\mathcal{K}}_p)/\mathbf{t}_r(\widehat{\mathcal{O}}_p),$$

where $\widehat{\mathcal{O}}_p$ is the natural completion of the local ring at $p$ and $\widehat{\mathcal{K}}_p$ is its field of fractions. Given a local coordinate $z$ centered at $p$, any irregular type $\mathsf{Q}$

admits a representative of the form

$$Q = \sum_{i=1}^{n-1} \frac{A_i}{z^i}$$

for some integer $n \geq 2$ and some diagonal matrices $A_1, \ldots, A_{n-1} \in \mathbf{t}_r$. The common cetralizer of these Cartan elements depends only on the equivalence class $\mathsf{Q}$ and will be denoted by $H_{\mathsf{Q}}$. Since $A_1, \ldots, A_{n-1}$ are diagonal, $H_{\mathsf{Q}}$ is equivalent under conjugation with a canonical subgroup $\times_{i=1}^{\ell} GL(m_i, \mathbb{C}) \subset GL(r, \mathbb{C})$ for an ordered partition $r = m_1 + \cdots + m_\ell$.

Following [5, 7], a wild curve is determined by the data $(C, p_a, \mathsf{Q}_a)$, $1 \leq a \leq m$, where $\mathsf{Q}_a$ is a $\mathbf{t}_r$-valued irregular type at $p_a$. For concreteness let $z_a$ be an affine coordinate at each $p_a$, and let

$$Q_a = \sum_{i=1}^{n_a - 1} \frac{A_{a,i}}{z_a^i}$$

be a representative of $\mathsf{Q}_a$ for each $1 \leq a \leq m$. Let $\mathsf{Q} = (\mathsf{Q}_1, \ldots, \mathsf{Q}_m)$. The wild character variety associated to a wild curve is the moduli space of Stokes data for irregular flat connections on $C \setminus \{p_1, \ldots, p_m\}$ which are locally gauge equivalent to

$$dQ_a + \text{terms of order } \geq -1$$

in the infinitesimal neighhborhood of each marked point $p_a$. One of the main results of [7] proves that this is a smooth quasi-projective variety with a natural Poisson structure. Moreover, the symplectic leaves of the Poisson structure are obtained by fixing the conjugacy class $\tau_a$ in $H_{\mathsf{Q}_a}$ of the formal monodromy of the irregular flat connections at each $p_a$ for all $1 \leq a \leq m$. Note that the conjugacy class of the formal mondromy in $H_{\mathsf{Q}_a}$ coincides with the conjugacy class of the local monodromy around the puncture, which is determined by the residue of the irregular connection.

Assuming the marked curve fixed, let $\mathcal{S}_{\mathsf{Q},\tau}$ be the symplectic leaf determined by $\tau = (\tau_1, \ldots, \tau_m)$. Throughout this paper all irregular types will be chosen such that

$(W.1)$ $H_{\mathsf{Q}} = \times_{i=1}^{\ell} GL(m_i, \mathbb{C}) \subset GL(r, \mathbb{C})$ and $H_{\mathsf{Q}}$ coincides with the centralizer of the leading term in the Laurent expansion.

$(W.2)$ Each $\tau_a$, $1 \leq a \leq m$, is the conjugacy class of a diagonal matrix $\mathsf{T}_a$ with eigenvalues

$$(\underbrace{\tau_{a,1}, \ldots, \tau_{a,1}}_{m_{a,1}}, \ldots, \underbrace{\tau_{a,\ell_a}, \ldots, \tau_{a,\ell_a}}_{m_{a,\ell_a}}), \tag{1.1}$$

where $\tau_{a,1}, \ldots, \tau_{a,\ell_a}$ are pairwise distinct complex numbers.

For each $a$ let $\mu_a$ denote the partition of $r$ determined by $(m_{a,1}, \ldots, m_{a,\ell_a})$ and let $\underline{\mu} = (\mu_1, \ldots, \mu_m)$.

As shown in [7, Thm. 8.2], for fixed irregular types $\mathsf{Q} = (\mathsf{Q}_1, \ldots, \mathsf{Q}_m)$ and fixed conjugacy classes $\tau = (\tau_1, \ldots, \tau_m)$ the associated moduli space $\mathcal{S}_{\mathsf{Q},\tau}$ of Stokes data has an explicit presentation as an affine algebraic quotient. For sufficiently generic $\tau$ this moduli space is a smooth quasi-projective variety equipped with a holomorphic symplectic structure. Its dimension is given by

$$d(\underline{\mu}, \underline{n}, g) = 2r^2(g-1) + \sum_{a=1}^{m} n_a \left( r^2 - \sum_{i=1}^{\ell_a} \mu_{a,i}^2 \right) + 2. \tag{1.2}$$

where $\underline{n} = (n_1, \ldots, n_m)$.

### 1.2. Higgs bundles and wild non-abelian Hodge correspondence

Results proven by Biquard and Boalch [4] and Sabbah [54] show that moduli spaces of irregular filtered flat connections are related by hyper-Kähler rotations to moduli spaces of irregular parabolic Higgs bundles. This relation is commonly refered to as the wild non-abelian Hodge correspondence. A very clear and explicit statement can be found in [6] to which the reader is refered for more details.

For the purposes of the present paper, it suffices to note that given sufficiently generic parameters $(\mathsf{Q}, \tau)$ the resulting Higgs bundle moduli problems admit a specific formulation leading to a natural connection to Calabi-Yau enumerative geometry. This construction as well as its relation to the moduli spaces of [6] is explained in detail in Sections 2.1 and 2.3 of [14]. Very briefly, one employs Higgs bundles with a coefficient line bundle $M = K_C(D)$ where $D = \sum_{a=1}^m n_a p_a$ is the total polar divisor of the fixed irregular types. The parabolic structure consists of a locally-free filtration along each non-reduced divisor $D_a = n_a p_a$, $1 \leq a \leq m$. The numerical type of such a filtration is specified by a sequence of positive integers $(m_{a,1}, \ldots, m_{a,\ell_a})$, where $m_{a,i}$ is the length of the $i$-th successive quotient of the $a$-th filtration as an $\mathcal{O}_{D_a}$-module. Let $\underline{m} = (m_{a,i})$, $1 \le i \le \ell_a$, $1 \le a \le m$. The Higgs field is required to preserve all these filtrations, and moreover the associated graded $\mathrm{gr}\big(\Phi|_{D_a}\big)$ is required to act as a scalar on each succesive quotient. More precisely, for each $1 \le a \le m$ one requires

$$\mathrm{gr}\big(\Phi|_{D_a}\big) = \oplus_{i=1}^{\ell_a} \xi_{a,i} \otimes \mathbf{1}_i$$

where $\xi_{a,i} \in H^0(D_a, M|_{D_a})$, $1 \le i \le \ell_a$ are some fixed sections of $M$ over $D_a = n_a p_a$. Let $\underline{\xi}_a = (\xi_{a,i})$ with $1 \le i \le \ell_a$ and let also $\underline{\xi} = (\xi_{a,i})$, $1 \le i \le \ell_a$, $1 \le a \le m$. Throughout this paper the sections $\underline{\xi}_a$ will be assumed generic for all $1 \le a \le m$, meaning that their values $\xi_{a,i}(p_a)$ at the reduced marked point are pairwise distinct and nonzero.

In addition one has to specify real parabolic weights $\underline{\alpha} = (\alpha_{a,i})$, $1 \le i \le \ell_a$, $1 \le a \le m$ and impose the standard parabolic stability conditions on such objects as in [41]. This results in a moduli stack $\mathfrak{H}^{ss}_{\underline{\xi}}(C, D; \underline{\alpha}, \underline{m}, d)$ of semistable objects, where $d$ is the degree of the Higgs bundles. Moreover, strictly semistable objects are absent for sufficiently generic parabolic weights $\underline{\alpha}$. In such cases the moduli stack is a $\mathbb{C}^\times$-gerbe over a smooth quasi-projective moduli space $\mathcal{H}^s_{\underline{\xi}}(C, D; \underline{\alpha}, \underline{m}, d)$ as usual.

To conclude, note that the wild non-abelian Hodge correspondence leads to a wild variant of the $P = W$ conjecture of de Cataldo, Hausel and Migliorini [13]. This correspondence identifies the weight filtration on the cohomology of a character variety and the perverse Leray filtration on the cohomology of the associated Higgs bundle moduli space.

### 1.3. Spectral correspondence and Calabi-Yau threefolds

As shown in detail in [14, Sect. 3.1], the string theoretic construction is based on a spectral correspondence relating the irregular Higgs bundles to dimension one sheaves on a holomorphic symplectic surface $S_{\underline{\xi}}$. Employing the construction of [31], this surface is obtained by blowing up the images of the sections $\xi_{a,i}$, taking a minimal resolution of singularities, and removing an anti-canonical divisor.

The compact curve classes on $S_{\underline{\xi}}$ are in one-to-one correspondence with collections non-negative integers $\underline{m} = (m_{a,i})$, $1 \le i \le \ell_a$, $1 \le a \le m$ such that the sum $\sum_{i=1}^{\ell_a} m_{a,i}$ takes the same value, $r \ge 1$, for each $1 \le a \le m$. Any compact curve in such a class is a finite $r:1$ cover of $C$. The curve class corresponding to $\underline{m} = (m_{a,i})$ will be denoted by $\beta(\underline{m})$.

As shown in [14, Sect.3.2-3.4], for sufficiently generic $\underline{\xi}$ the moduli stack $\mathfrak{H}^{ss}_{\underline{\xi}}(C, D; \underline{\alpha}, \underline{m}, d)$ is isomorphic to a moduli stack of Bridgeland semistable pure dimension one sheaves $F$ on $S_{\underline{\xi}}$ with compact support. The topological invariants of these sheaves are

$$\mathrm{ch}_1(F) = \beta(\underline{m}), \qquad \chi(F) = d - r(g-1)$$

while the Bridgeland stability condition is determined by the parabolic weights $\underline{\alpha}$.

This correspondence can be lifted one step further. Namely let $Y_{\underline{\xi}}$ be the total space of the canonical bundle $K_{S_{\underline{\xi}}}$, which is isomorphic to $S_{\underline{\xi}} \times \mathbb{A}^1$. Then any moduli stack of compactly supported Bridgeland stable pure dimension one sheaves on $Y_{\underline{\xi}}$ is isomorphic to a product of the form $\mathfrak{H}^{s}_{\underline{\xi}}(C, D; \underline{\alpha}, \underline{m}, d) \times \mathbb{A}^1$. The threefold $Y_{\underline{\xi}}$ will be called a wild local curve throughout this paper. Occasionally it will be also referred to as a spectral threefold for ease of exposition.

Applying the mathematical framework of [27, 45], in the present context the degeneracies of spinning BPS particles in M-theory are given by the perverse Betti numbers of the Higgs bundle moduli spaces $\mathfrak{H}^{s}_{\underline{\xi}}(C, D; \underline{\alpha}, \underline{m}, d)$. Therefore, using the $P = W$ conjecture and the refined Gopakumar-Vafa expansion [17, 22, 28, 30], it follows that the weighted Betti numbers of wild character varieties are completely determined by the refined stable pair theory of $Y_{\underline{\xi}}$.

### 1.4. Curve counting invariants and degenerations

Computing the refined stable pair theory of the threefolds $Y_{\underline{\xi}}$ constructed above is a difficult and challenging problem. Conjectural formulas were derived in [14] for genus zero curves with a single marked point. The derivation in loc. cit. uses refined virtual localization as in [47] and the refined colored variant [15] of the Oblomkov-Shende-Rasmussen conjectures [48, 49] to reduce the problem to refined torus link invariants. The latter are computed in turn using refined Chern-Simons theory [2] and large $N$ duality.

This approach cannot be implemented directly for a genus $g$ curve with multiple marked points since there is no torus action on the associated spectral surface $S_{\underline{\xi}}$. In fact, given the absence of a torus action, even the computation of unrefined invariants poses significant problems. The present work develops an alternative strategy employing nodal degenerations of the curve $C$ to construct a normal crossing spectral surface equipped with a suitable torus action. This torus action is then used to find an explicit formula for the unrefined stable pair theory of the resulting degenerate threefolds, providing conjectural formulas for $E$-polynomials of wild character varieties. The resulting formulas admit a further conjectural refinement which leads to explicit conjectures for weighted Poincaré polynomials. In more detail, the main steps of this approach are summarized below.

$(i)$ As defined by Pandharipande and Thomas [53], a stable pair $(F, s)$ on the smooth spectral threefold $Y_{\underline{\xi}}$ is a pure dimension one sheaf $F$ equipped with a generically surjective section $s : \mathcal{O}_{Y_{\underline{\xi}}} \to F$. The support of $F$ is required to be compact and one fixes the invariants $\mathrm{ch}_1(F) = \beta(\underline{m}) \in H_2(Y_{\underline{\xi}}, \mathbb{Z})$ and $c = \chi(F)$. The resulting moduli space of pairs $\mathcal{P}(Y_{\underline{\xi}}, \underline{m}, c)$ has a perfect obstruction theory and a virtual cycle of degree zero. However, since it is non-compact and there is no torus action with compact fixed locus, hence stable pair invariants cannot be defined by virtual integration. One has to employ instead Behrend's constructible function approach [3], definining the unrefined invariants as weighted Euler characteristics of moduli spaces,

$$PT(Y_{\underline{\xi}}, \underline{m}, c) = \chi(\mathcal{P}(Y_{\underline{\xi}}, \underline{m}, c), \nu^B). \tag{1.3}$$

The weights are encoded in the constructible function $\nu^B : \mathcal{P}(Y_{\underline{\xi}}, \underline{m}, c) \to \mathbb{Z}$ constructed in [3]. As opposed to virtual integration, this definition is not known to be deformation invariant in general. However, in the present context, it will be assumed that the constructible invariants are in fact invariant under deformations of $Y_{\underline{\xi}}$ induced by deformations of the base curve $C$ and the marked points.

$(ii)$ A nodal degeneration $\overline{C}$ of the curve $C$ is constructed in Section 2.1 consisting of a smooth central component $C_0$ and $m$ pairwise disjoint rational components $C_1, \ldots, C_m$. Each projective line $C_a$ contains exactly one marked point $p_a$ and intersects $C_0$ transversely at another point $\nu_a$. For fixed data $\underline{\xi}$, this degeneration yields normal crossing degenerations of the spectral surface $S_{\underline{\xi}}$ as well as the threefold $Y_{\underline{\xi}}$. Moreover, as shown in Section 2.2, for a suitable choice of data, the resulting normal crossing threefold admits a torus action preserving the normal crossing Calabi-Yau structure. Using the work of J. Li and B. Wu [37], for any pair $(\underline{m}, c)$ there is a moduli space of stable pairs on the normal crossing threefold $\overline{Y}_{\underline{\xi}}$ equipped with a perfect obstruction theory. Moreover, the fixed locus of the torus action is compact, hence in this limit one can construct equivariant residual stable pair invariants $PT(\overline{Y}_{\underline{\xi}}, \underline{m}, c)$. Then a stronger deformation invariance assumption will be made in this paper, claiming that

$$PT(\overline{Y}_{\underline{\xi}}, \underline{m}, c) = PT(Y_{\underline{\xi}}, \underline{m}, c) \tag{1.4}$$

for all $\underline{m}, c$.

(*iii*) The equivariant residual stable pair theory of degenerations is explicitely computed in Section 3 using GW/PT correspondence, which will be assumed without proof in the present context. In principle one should be able to give a proof by analogy with [44], but this would be beyond the scope of the present paper. Granting this correspondence, the residual Gromov-Witten theory of degenerations is computed in Section 3 using J. Li's theory of relative stable maps [34] and the degeneration formula [35], the relative virtual localization theorem of Graber and Vakil [23], as well as the local TQFT formalism of Bryan and Pandharipande [9]. The computation also uses the Marino-Vafa formula proven by C.-C. M. Liu, K. Liu and J. Zhou [38] respectively Okounkov and Pandharipande [50] as well as results of Bryan and Pandharipande [8], Okounkov and Pandharipande [51, 52] and C.-C. M. Liu, K. Liu and J. Zhou [39] on relative Gromov-Witten invariants of (local) curves. For completeness, a self-contained exposition of relative stable maps and relative virtual localization, including one the main examples used in the computation, is provided in Section 4. The final outcome is presented below.

### 1.5. The main formula

To summarize the current setup, in this section $\overline{C}$ will be a nodal curve consisting of a smooth genus $g$ central component $C_0$ and $m$ pairwise disjoint smooth genus zero components $C_1, \ldots, C_m$. Each component $C_a$ intersects $C_0$ transversely at one point $\nu_a$ and. One also specifies a nonreduced effective divisor $D = \sum_{a=1}^{m} n_a p_a$ on $\overline{C}$ such that $p_a \in C_a \setminus \{\nu_a\}$ for each $1 \leq a \leq m$. Each marked point is assigned in addition a collection $\underline{\xi}_a = (\underline{\xi}_{a,1}, \ldots, \underline{\xi}_{a,\ell_a})$ of $\ell_a \geq 1$ pairwise disjoint, nonzero, sections of $M = \omega_{\overline{C}}(D)$ over the thickening $D_a = n_a p_a$, where $\omega_{\overline{C}}$ is the dualizing sheaf of $\overline{C}$.

Using this data one constructs a normall crossing spectral surface $\overline{S}_{\underline{\xi}}$ following the algorithm of [31], as in [14, Sect. 3.1]. One has to blow up the images of the sections $\underline{\xi}$ in the total space of $M$, take a minimal resolution of singularities, and remove a certain divisor. The spectral threefold $\overline{Y}_{\underline{\xi}}$ is the total space of the dualizing sheaf of $\overline{S}_{\underline{\xi}}$, hence it has a normal-crossing Calabi-Yau structure. It consists of a central component, $Y_0$, which is isomorphic to the total space of a rank two bundle over $C_0$ and $m$ components $Y_a$, $1 \leq a \leq m$, each of them isomorphic to the total space of a line bundle over a surface $S_a$, $1 \leq a \leq m$. By construction, both $Y_a$ and $S_a$ have natural log Calabi-Yau structures for all $1 \leq a \leq m$. The polar divisor of the dualizing sheaf of $Y_a$, denoted by $\Delta_a$, $1 \leq a \leq m$, is isomorphic to the affine plane. Moreover, the threefolds $Y_a$ are glued to the central component $Y_0$ along the divisors $\Delta_a$, which are naturally identified with fibers of the projection $Y_0 \to C_0$. Each pair $(Y_a, \Delta_a)$ will be called a wild cap in the following.

As shown in Section 2.2, assuming that

$$n_1 = \cdots = n_m = n,$$

and making a suitable choice of local data $\underline{\xi}$, one constructs a torus action on $\overline{S}_{\underline{\xi}}$. This action lifts to an action on the spectral threefold $\overline{Y}_{\underline{\xi}}$ preserving the normal-crossing Calabi-Yau structure. Moreover for each $1 \leq a \leq m$ the cone of compact curve classes on $Y_a$ is freely generated by $\ell_a$ torus invariant sections of the projection map $Y_a \to C_a$. Therefore the compact curve classes on $Y_a$ are in one-to-one correspondence with collections $(m_{i,a}) \neq (0, \ldots, 0)$, $1 \leq i \leq \ell_a$ of non-negative integers. At the same time the cone of compact curve classes in $Y_0$ is freely generated by the zero section. Hence any such class is classified by the degree $r \geq 1$. In conclusion the compact curve classes on $\overline{Y}_{\underline{\xi}}$ in one-to-one correspondence with numerical data $(\underline{m}, r)$.

As shown in [42, 43, 53], in the context of GW/DT/PT correspondence one has to consider stable maps to $\overline{Y}_{\underline{\xi}}$ with disconnected domains such that no connected component is mapped to a point. Using the degeneration formula proven in [35] it follows that only pairs $(\underline{m}, r)$ satisfying

$$\sum_{i=1}^{\ell_a} m_{a,i} = r \geq 1 \tag{1.5}$$

for all $1 \leq a \leq m$ contribute to the Gromov-Witten theory of $\overline{Y}_{\underline{\xi}}$. In particular $r$ is determined by $\underline{m}$, hence the residual equivariant invariants will be denoted by $GW^{\bullet}_{\underline{m},h}$, where $h \in \mathbb{Z}$ is the arithmetic genus of the domain. By convention for any $h \in \mathbb{Z}$, and any integral vector $\underline{m}$ let $GW^{\bullet}_{\underline{m},h} = 0$ unless $\underline{m}$ has non-negative entries satisfying condition (1.5). Therefore the partition function takes the form

$$Z(\overline{Y}_{\underline{\xi}}; \mathsf{x}, g_s) = 1 + \sum_{\underline{m}} \sum_{h \in \mathbb{Z}} g_s^{2h-2} GW_{\underline{m},h} \prod_{a=1}^{m} \prod_{i=1}^{\ell_a} x_{a,i}^{m_{a,i}}. \tag{1.6}$$

An explicit formula for $Z(\overline{Y}_{\underline{\xi}}; \mathsf{x}, g_s)$ is derived in Section 3 by relative virtual localization. The formula is written as a sum over partitions

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{l(\lambda)}),$$

identified with Young diagrams consisting of $l(\lambda)$ left-aligned rows such that the $i$-th row contains $\lambda_i$ boxes. The following notation and definitions are used in the computation.

- The total number of boxes in $\lambda$ will be denoted by $|\lambda|$ and its transpose will be denoted by $\lambda^t$.
- The content of $\lambda$ is defined as

$$c(\lambda) = \sum_{\square\in\lambda} (a(\square) - l(\square))$$

  where $a(\square), l(\square)$ are the arm and leg length respectively.
- The Schur function corresponding to $\lambda$ will be denoted by $s_\lambda(\mathsf{x})$. The fusion coefficients for $\ell$ partitions $\nu_1, \ldots, \nu_\ell$ are defined by

$$\prod_{i=1}^{\ell} s_{\nu_i}(\mathsf{x}) = \sum_{|\lambda|=|\nu_1|+\cdots+|\mu_\ell|} c^{\lambda}_{\nu_1,\ldots,\nu_\ell} s_\lambda(\mathsf{x}).$$

Then the formula derived in Section 3 reads

$$Z_{GW}(\overline{Y}_{\underline{\xi}}; \mathsf{x}, g_s) = 1 + \sum_{|\lambda|>0} s_{\lambda^t}(\underline{q})^{2-2g-m} \prod_{a=1}^{m} F_{n-1,\ell_a,\lambda}(\mathsf{x}_a, q) \tag{1.7}$$

where $q = e^{ig_s}$, $\underline{q} = (q^{1/2}, q^{3/2}, \ldots)$, and

$$F_{k,\ell,\lambda}(\mathsf{x}, q) = q^{-kc(\lambda)} \sum_{|\nu_1|+\cdots+|\nu_\ell|=|\lambda|} c^{\lambda}_{\nu_1,\ldots,\nu_\ell} \prod_{i=1}^{\ell} x_i^{|\nu_i|} q^{kc(\nu_i)} s_{\nu_i^t}(\underline{q}). \tag{1.8}$$

for any integers $k, \ell \geq 1$. The factors $F_{n-1,\ell_a,\lambda}(\mathsf{x}_a, q)$ encode the relative Gromov-Witten invariants of the wild caps $(Y_a, \Delta_a)$, which are the main novelty in this computation.

Assuming GW/PT correspondence, the above partition function coincides with the generating function of residual stable pairs invariants $PT(\overline{Y}_{\underline{\xi}}, \underline{m}, c)$ of the threefolds $Y_{\underline{\xi}}$. Further assuming deformation invariance as in(1.4), the above formula provides an explicit expression for the generating function

$$Z_{PT}(Y_{\underline{\xi}}, \mathsf{x}, q) = 1 + \sum_{\underline{m}} \sum_{c\in\mathbb{Z}} PT(Y_{\underline{\xi}}, \underline{m}, c)(-q)^c \prod_{a=1}^{m} \prod_{i=1}^{\ell_a} x_{a,i}^{m_{a,i}}$$

associated to a smooth spectral threefold $Y_{\underline{\xi}}$.

The above formula is derived under the assumption that all integers $n_a$ used as input data in the construction of $Y_{\underline{\xi}}$ are equal, $n_1 = \cdots = n_m$. Moreover, one also requires the sections $\underline{\xi}_a$ to satisfy an equivariance condition for all $1 \leq a \leq m$ in order for a torus action to exist in the degenerate limit. At the same time, the invariants (1.3) are defined more generally for any values of $n_a$ and for generic sections $\underline{\xi}_a$, $1 \leq a \leq m$. The next section formulates a conjectural refined generalization of formula (1.7) allowing arbitrary values of $n_a$, as well as arbitrary generic sections $\underline{\xi}_a$, $1 \leq a \leq m$.

### 1.6. Conjectural refinement

The conjectural refinement of the partition function (1.7) is inferred from the structure of the wild factors (1.8) by comparison with previous refined formulas derived in [10, 12, 14]. The refinement will be written in terms of Macdonald polynomials $P_\lambda(s,t;\mathsf{x})$ and uses the following quantities:

- the fusion coefficients $N^\lambda_{\nu_1,\ldots,\nu_\ell}(s,t)$ defined by the product identity

$$\prod_{i=1}^{\ell} P_{\nu_i}(s,t;\mathsf{x}) = \sum_{|\lambda|=|\nu_1|+\cdots+|\nu_\ell|} N^\lambda_{\nu_1,\ldots,\nu_\ell}(s,t) P_\lambda(s,t;\mathsf{x})$$

- the specializations

$$R_\mu(q,y) = P_\mu(s,t;\underline{t})\big|_{s=qy,\ t=qy^{-1}}, \quad L_\mu(q,y) = P_\mu(t,s;\underline{s})|_{s=qy,\ t=qy^{-1}}$$
$$\widetilde{N}^\lambda_{\nu_1,\ldots,\nu_\ell}(q,y) = N^\lambda_{\nu_1,\ldots,\nu_\ell}(qy,qy^{-1})$$

  where

$$\underline{t} = (t^{1/2}, t^{3/2}, \ldots), \qquad \underline{s} = (s^{1/2}, s^{3/2}, \ldots),$$

  and

- the framing factors

$$g_\mu(q,y) = f_\mu(qy,qy^{-1}), \qquad f_\mu(s,t) = \prod_{\Box\in\mu} s^{a(\Box)} t^{-l(\Box)}.$$

One of the main building blocks of the refined formulas derived in [10, 12] is the polynomial function

$$T_{g,\lambda}(q,y) = \prod_{\Box\in\lambda} (qy)^{-(2a(\Box)+1)g} (1 - y^{a(\Box)-l(\Box)} q^{h(\Box)})^{2g}$$

encoding the genus dependence of the partition function. Furthermore, by comparison with the refined formula of [14], one is led to conjecture the following refinement of the wild factor (1.8).

$$
(1.9) \qquad G_{k,\ell,\lambda}(\mathsf{x},q,y) = \sum_{\substack{\mu_1,\ldots,\mu_\ell \\ |\mu_1|+\cdots+|\mu_\ell|=|\lambda|}} \widetilde{N}^{\lambda}_{\mu_1,\ldots,\mu_\ell}(q,y) g_\lambda(q,y)^{-k} \times \prod_{i=1}^{\ell} \left( x_i^{|\mu_i|} g^k_{\mu_i}(q,y) L_{\mu_i^t}(q,y) \right).
$$

Then the refined stable pair theory of a threefold $Y_{\underline{\xi}}$ with arbitrary data $(n_a, \ell_a, \underline{\xi}_a)$, $1 \leq a \leq m$, at the marked points is conjectured to be

$$
(1.10) \qquad Z^{ref}_{PT}(Y_{\underline{\xi}}; q, y, \mathsf{x}) = 1 + \sum_{|\lambda|>0} g_\lambda(q,y) R_\lambda(q,y) T_{g,\lambda}(q,y) L_{\lambda^t}(q,y)^{1-m} \times \prod_{a=1}^{m} G_{n_a-1,\ell_a,\lambda}(\mathsf{x}_a, q, y)
$$

As a consistency check, this formula reduces to the one conjectured in [12] in the absence of marked points, in which case the wild cap factors are absent. Moreover, it also reduces to the main formula conjectured in [14] for a genus zero curve with one marked point. Moreover hard numerical evidence for this conjecture is obtained by comparison with the main conjecture of Hausel, Mereb and Wong [25], as discussed in the next section.

### 1.7. Conjectures for wild character varieties

As explained in Section 1.2, for sufficiently generic data $\underline{\xi}$ the wild non-abelian Hodge correspondence relates moduli spaces of irregular parabolic Higgs bundles with singular type $\underline{\xi}$ to moduli spaces of filtered flat connections. A set of generic of local data $\underline{\xi}$ and a set of generic parabolic weights determine uniquely a collection $\Gamma(\underline{\xi})$ of irregular types as well as a collection $\tau$ of conjugacy classes associated to the marked points. Both $\Gamma(\underline{\xi})$ and $\tau$ satisfy conditions $(W.1)$ and $(W.2)$ in Section 1.1. This relation leads to a wild variant of the $P = W$ conjecture of [13] identifying the perverse Leray filtration on the moduli space of Higgs bundles with the weight filtration on the cohomology of wild character varieties. Using the refined Gopakumar-Vafa expansion, the refined stable pair partition function (1.10) yields explicit predictions for weighted Poincaré polynomials of wild character varieties. By analogy with [10, 12, 14], in the present context the refined

Gopakumar-Vafa formula is a conjectural expansion

$$(1.11)\qquad \ln Z_{ref}(Y_{\underline{\xi}}; \mathsf{x}, q, y) = -\sum_{k\geq 1} \sum_{|\mu_1|=\cdots=|\mu_m|=r\geq 1} \prod_{a=1}^{m} \frac{m_{\mu_a}(x_a^k)}{k} \frac{y^{-kr}(qy^{-1})^{kd(\underline{\mu},\underline{n},g)/2} P_{\underline{\mu},\underline{n}}((qy)^{-k}, y^k))}{(1-(qy)^{-k})(1-(qy^{-1})^k)}$$

where $P_{\underline{\mu},\underline{n}}(u, v)$ are polynomials with integer coefficients which count the degeneracies of spinning BPS states in M-theory. The coefficients $m_\mu(\mathsf{x}^k)$ are the monomial symmetric functions evaluated at $\mathsf{x}^k = (x_1^k, x_2^k, \ldots)$ for any $k \geq 1$. Note that $\mathsf{x}_a = (x_1, \ldots, x_{\ell_a}, 0, 0, \ldots)$ for each $1 \leq a \leq m$.

Using the mathematical theory of [27, 45], since $Y_{\underline{\xi}}$ is a spectral threefold, the polynomials $P_{\underline{\mu},\underline{n}}$ are identified with the perverse Poincaré polynomials of the Higgs bundle moduli spaces $\mathcal{H}^s_{\underline{\xi}}(C, D; \underline{\alpha}, \underline{m}, d)$ for sufficiently generic $\underline{\alpha}$. This yields the following conjectural statement for wild character varieties.

Let $\mathcal{S}_{\Gamma(\underline{\xi}),\tau}$ be the wild character variety corresponding to some generic sections $\underline{\xi}$ and generic conjugacy classes $\tau$ satisfying conditions $(W.1)$, $(W.2)$ in Section 1.1. Let

$$P(\mathcal{S}_{\Gamma(\underline{\xi}),\tau}; u, v) = \sum_{i,j} \dim \mathrm{Gr}_i^W H^j(\mathcal{S}_{\Gamma(\underline{\xi}),\tau}) u^{i/2}(-v)^j$$

be the weighted Poincaré polynomial of $\mathcal{S}_{\Gamma(\underline{\xi}),\tau}$ where $\mathrm{Gr}_i^W H^j(\mathcal{S}_{\Gamma(\underline{\xi}),\tau})$ are the successive quotients of the weight filtration on cohomology. Then

$$P(\mathcal{S}_{\Gamma(\underline{\xi}),\tau}; u, v) = P_{\underline{\mu},\underline{n}}(u, v).$$

In particular, under the current genericity assumptions the weighted Poincaré polynomial depends only on the discrete data $(\underline{\mu}, \underline{n})$.

The main supporting evidence for this conjecture is obtained from the comparison with the conjecture of Hausel, Mereb and Wong [25] which yields explicit predictions for partitions of the form $\underline{\mu} = ((1^r), \ldots, (1^r))$. Their partition function is defined as

$$Z_{HMW}(z, w) = 1 + \sum_{|\lambda|>0} \Omega_\lambda^{g,n}(z, w) \prod_{a=1}^{m} \widetilde{H}_\lambda(\mathsf{x}_a; z^2, w^2)$$

where:

- the sum in the right hand side is over all Young diagrams $\lambda$ with $|\lambda| > 0$,

- for each $\lambda$

$$\Omega_\lambda^{g,n}(z,w) = \prod_{\Box\in\lambda} \frac{(-z^{2a(\Box)}w^{2l(\Box)})^{n-m}(z^{2a(\Box)+1} - w^{2l(\Box)+1})^{2g}}{(z^{2a(\Box)+2} - w^{2l(\Box)})(z^{2a(\Box)} - w^{2l(\Box)+2})},$$

- $\mathsf{x}_a = (x_{a,1}, x_{a,2}, \ldots)$ is an infinite set of formal variables for each $1 \leq a \leq m$, and $\widetilde{H}_\lambda(\mathsf{x}_a; z^2, w^2)$ are the modified Macdonald polynomials.

Let $\mathbb{H}_{\mu,n}(z,w)$ be defined by

(1.12)
$$\ln Z_{HMW}(z,w) = \sum_{k\geq 1}\sum_{\mu} \frac{(-1)^{(n-m)|\mu|}w^{kd(\mu,n,g)}\,\mathbb{H}_{\underline{\mu},n}(z^k,w^k)}{(1-z^{2k})(w^{2k}-1)} \prod_{a=1}^{m} m_{\mu_a}(x_a^k)$$

where the sum is again over all Young diagrams, $m_\mu(\mathsf{x})$ are the monomial symmetric functions and $\mathsf{x}^k = (x_1^k, x_2^k, \ldots)$. Then for $\mu_a = (1^r)$, $1 \leq a \leq r$, one has the following conjectural formula

$$WP(\mathcal{S}_{\mathsf{Q},\mathsf{T}}; u, v) = \mathbb{H}_{(1^r),n}(u^{1/2}, u^{-1/2}v^{-1}) \tag{1.13}$$

where $n = \sum_{a=1}^m n_a$. Note that the $v = 1$ specialization of this conjecture is proven in [25] using arithmetic methods.

Although the two conjectural formulas are quite different, direct numerical comparison shows that they yield the same predictions for

- $0 \leq g \leq 2$, $m = 2$, $\mu_1 = \mu_2 = (1^3)$, $2 \leq \deg(D) = n_1 + n_2 \leq 7$
- $0 \leq g \leq 1$, $m = 3$, $\mu_1 = \mu_2 = \mu_3 = (1^3)$, $3 \leq \deg(D) = n_1 + n_2 + n_3 \leq 9$.

It should be noted that these are highly nontrivial tests. For example for $g = 2$, $\deg(D) = 7$, $\mu_1 = \mu_2 = (1^3)$, as well as $g = 1$, $\deg(D) = 9$, $\mu_1 = \mu_2 = \mu_3 = (1^3)$ the polynomials $P_{\underline{\mu},n}(u,v)$ have bidegree $(56, 56)$, hence a total of 3248 terms. Moreover, as a further consistency check, explicit computations for rank $r = 3$ and $r = 4$ character varieties confirm that the partition function (1.10) has indeed a BPS expansion of the form (1.11) for arbitrary partitions $\mu_a$, including $\mu_a \neq (1^r)$. For illustration some numerical results for the polynomials $P_{\underline{\mu},\underline{n}}$ are listed in Appendix A.

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## 2. Wild degenerate local curves

### 2.1. Degenerations

In this section $\overline{C}$ will be a nodal curve consisting of $m$ rational components $C_1, \ldots, C_m$ and a smooth projective genus $g$ curve $C_0$. Each rational component $C_a$, $1 \leq a \leq m$, intersects $C_0$ transversely at a node $\nu_a \in C_0$ and is at the same time disjoint from all other rational components $C_b$, $b \neq a$. Moreover, each component $C_a$ contains a marked point $p_a \neq \nu_a$, and each marked point is assigned the pair of positive integers $(n_a, \ell_a)$. This data will be denoted by $\underline{n} = (n_a)$, $\underline{\ell} = (\ell_a)$ with $1 \leq a \leq m$. Furthermore let $D_a = n_a p_a$, $1 \leq a \leq m$, and $D = \sum_{a=1}^{m} D_a$.

A degenerate wild curve is then specified by the collection $(\overline{C}, p_1, \ldots, p_m)$ and a collection $\underline{\xi}_a = (\xi_{a,i})$, $1 \leq i \leq \ell_a$ of generic sections of $M|_{D_a}$ for each $1 \leq a \leq m$. Recall that the genericity condition requires the values $\xi_{a,i}(p_a)$ to be pairwise distinct and nonzero for each $1 \leq a \leq m$. Let $\underline{\xi} = (\underline{\xi}_{a,i})$, $1 \leq i \leq \ell_a$, $1 \leq a \leq m$.

Let $M = \omega_{\overline{C}}(D)$, where $\omega_{\overline{C}}$ is the dualizing line bundle of $\overline{C}$. In analogy with [14, 31], one constructs a surface $\overline{S}_{\underline{\xi}}$ by blowing-up the total space of $M$ along the images of the sections $\underline{\xi}_{a,i}$, taking a minimal resolution of singularities, and removing an anticanonical divisor. This process can be alternatively viewed as a sequence of successive blowups of M at closed points lying recursively on the strict transforms of the sections $\xi_{a,i}$. The resulting surface is reducible, with normal crossing singularities, and there is a natural projection map $\eta : \overline{S}_{\underline{\xi}} \to \overline{C}$. The smooth irreducible components of $\overline{S}_{\underline{\xi}}$ are $S_a = \eta^{-1}(C_a)$ and $S_0 = \eta^{-1}(C_0)$. The central component $S_0 = \pi^{-1}(C_0)$ is isomorphic to the total space of

$$M|_{C_0} \simeq K_{C_0}(\nu_1 + \cdots + \nu_m)$$

and intersects each $S_a$ transversely along the fiber $F_a = \eta^{-1}(\nu_a)$. For all $1 \leq a \leq m$ the surface $S_a$ is obtained by succesively blowing up the total space of

$$M|_{C_a} \simeq K_{C_a}(n_a p_a + \nu_a)$$

and removing a certain divisor, as shown in detail in Section 2.2. The divisor in question is chosen such that the pair $(S_a, F_a)$ is a log Calabi-Yau surface, $K_{S_a} \simeq \mathcal{O}_{S_a}(-F_a)$, for all $1 \leq a \leq m$.

Finally, let $\overline{Y}_{\underline{\xi}}$ be the total space of the dualizing line bundle of the normal crossing surface $\overline{S}_{\underline{\xi}}$. Note that there is a natural projection $\pi : \overline{Y}_{\underline{\xi}} \to$

$\overline{C}$ and $\overline{Y}_{\underline{\xi}}$ has normal crossing singularities along the divisors $\Delta_a = \pi^{-1}(\nu_a)$, $1 \leq a \leq m$. The irreducible components of $\overline{Y}_{\underline{\xi}}$ are $Y_a = \pi^{-1}(C_a)$, $1 \leq a \leq m$, and $Y_0 = \pi^{-1}(C_0)$. Each $Y_a$ is isomorphic to the total space of the line bundle $K_{S_a}(F_a) \simeq \mathcal{O}_{S_a}$ over $S_a$, hence $K_{Y_a} \simeq \mathcal{O}_{Y_a}(-\Delta_a)$ for all $1 \leq a \leq m$. The central component $Y_0$ is isomorphic to the total space of the rank two bundle

$$\mathcal{O}_{C_0} \oplus M|_{C_0} \simeq \mathcal{O}_{C_0} \oplus K_{C_0}(\nu_1 + \cdots + \nu_m). \tag{2.1}$$

Hence $K_{Y_0} \simeq \mathcal{O}_{Y_a}(-\Delta_1 - \cdots - \Delta_m)$ as expected. The log Calabi-Yau threefolds $(Y_a, \Delta_a)$ will be called wild caps in the following.

### 2.2. Torus action and invariant curves

A torus action on the degenerate threefold $\overline{Y}_{\underline{\xi}}$ will be constructed in this section satisfying the following criteria:

- The restriction to $Y_a$, $1 \leq a \leq m$, is a lift of the natural action on the rational curve $C_a$ which preserves the log Calabi-Yau structure.
- The restriction to $Y_0$ is an anti-diagonal torus action on the total space of the rank two bundle (2.1).

The torus action will be obtained by gluing separate torus actions on each irreducible component of $\overline{Y}_{\underline{\xi}}$. To fix notation, let $M_a$ denote the total space of $M|_{C_a}$ and let $(z_a, w_a)$ be standard affine coordinates on the rational component $C_a$ centered at $p_a, \nu_a$ respectively. Let $k_a = \deg(M|_{C_a}) = n_a - 1$, $1 \leq a \leq m$. The inverse image of the standard open cover of $C_a$ determines an affine open cover $(U_a, V_a)$ of $M_a$ with coordinates $(z_a, u_a), (w_a, v_a)$ related by the transition function

$$w_a = z_a^{-1}, \qquad v_a = z_a^{k_a} u_a$$

on the overlap. Then there is a one dimensional torus action $\mathbf{T} \times M_a \to M_a$,

$$t \times (z_a, u_a) \mapsto (t^{-1} z_a, u_a) \quad \text{and} \quad t \times (w_a, v_a) \mapsto (t w_a, t^{k_a} v_a), \tag{2.2}$$

on $M_a$ leaving the fiber over $p_a$ pointwise fixed.

In order for this action to lift to the blow-up, the local sections $\xi_{a,i} : D_a \to M_{D_a}$ must be chosen to be equivariant. This means the image of each section $\xi_{a,i} : D_a \to M_a$ will be a non-reduced zero dimensional subscheme

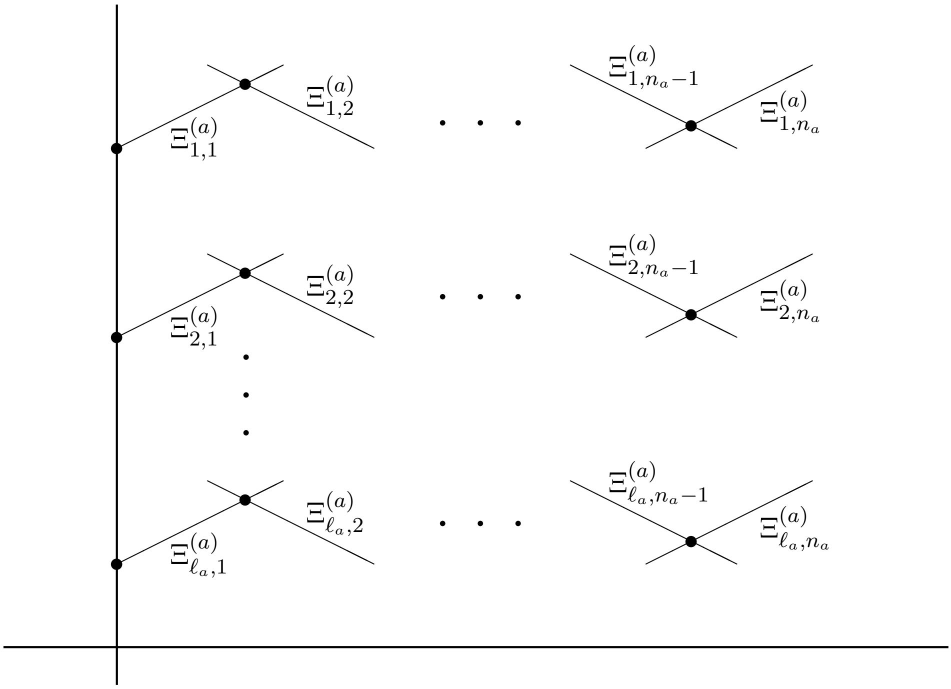


Figure 1: Chains of exceptional curves on the blow-up.

$\delta_{a,i} \subset M_a$ given by

$$z_a^{n_a} = 0, \qquad u_a = \lambda_{a,i},$$

with $\lambda_{a,i} \in \mathbb{C}$, $1 \leq i \leq \ell_a$. By the genericity assumption, $(\lambda_{a,i})$, $1 \leq i \leq \ell_a$ must be pairwise distinct and nonzero for each $1 \leq a \leq m$. Then there is a canonical lift of the action (2.2) to the blow-up along the union $\cup_{i=1}^{\ell_a} \delta_i$.

In more detail, the surface obtained by blowing up $M_a$ contains $\ell_a$ linear chains of exceptional divisors $\Xi^{(a)}_{i,j}$, $1 \leq i \leq \ell_a$, $1 \leq j \leq n_a$ as shown in Figure 1. The surface $S_a$ is the complement of the union

$$\cup_{i=1}^{\ell_a} \cup_{j=1}^{n_a-1} \Xi^{(a)}_{i,j}.$$

Note in particular that the last component $\Xi^{(a)}_{i,n_a}$, $1 \leq i \leq \ell_a$ of each chain is not removed, but it restricts to a non-compact curve on $S_a$. Moreover, the torus action (2.2) restricts to a torus action on $S_a$.

By construction there is an affine open subset of $S_a$ which is canonically identified with affine chart $V_a \subset M_a$ by the blow-up map. Therefore one can use $(w_a, v_a)$ as local coordinates on $S_a$ as well. The threefold $Y_a$ is the total space of the line bundle $K_{S_a}(F_a) \simeq \mathcal{O}_{S_a}$, where $F_a$ is the divisor $w_a = 0$. In particular, $Y_a$ is a log Calabi-Yau threefold with $K_{Y_a} \simeq \mathcal{O}_{Y_a}(-\Delta_a)$, where $\Delta_a$ is the inverse image of $F_a$. The torus action on $S_a$ lifts to a torus action on $Y_a$ preserving the log Calabi-Yau structure. In order to write a local formula, note that the inverse image of $V_a$ in $Y_a$ is an affine coordinate chart on $Y_a$ with coordinates $(w_a, y_a, v_a)$ where $y_a$ is a natural vertical affine coordinate. In this chart the torus action reads

$$t \times (w_a, y_a, v_a) = (tw_a, t^{-k_a} y_a, t^{k_a} v_a). \tag{2.3}$$

It is important to note that there is a finite collection of smooth rational torus invariant curves $X_{a,i}$ in $Y_a$ locally given by

$$y_a = 0, \qquad v_a = \lambda_{a,i} w_a^{k_a}, \qquad 1 \leq i \leq \ell_a. \tag{2.4}$$

This is sketched in Figure 2. Each of these curves is a $(0, -1)$ curve on $Y_a$ intersecting the exceptional divisor $\Xi^{(a)}_{i,n_a}$ transversely at a torus fixed point $p_{a,i}$. In fact for each $1 \leq i \leq \ell$ the blow-up construction yields an affine open coordinate chart on $Y_a$ with coordinates $(z_{i,a}, x_{i,a}, u_{i,a})$ centered at $p_{a,i}$ which are related to $(w_a, y_a, v_a)$ by the transition functions

$$w_a = z_{i,a}^{-1}, \qquad y_a = x_{i,a}, \qquad v_a = z_{i,a} u_{i,a}.$$

In this coordinate chart the curve $X_{i,a}$ is cut by $u_{i,a} = x_{i,a} = 0$. Furthermore, the local form of the torus action reads

$$t \times (z_{i,a}, x_{i,a}, u_{i,a}) \mapsto (t^{-1} z_{i,a}, t^{-k_a} x_{i,a}, t^{k_a+1} u_{i,a}). \tag{2.5}$$

In conclusion, if the local sections $\underline{\xi}_a$ are equivariant, each irreducible component $Y_a$, $1 \leq a \leq m$ has a torus action preserving the log Calabi-Yau structure. According to equation (2.3), the restriction of this action to the divisor $\Delta_a \simeq \mathbb{A}^2$ has weights $(k_a, -k_a)$ on the tangent space at the origin, which is spanned by $(\partial/\partial y_a,\ \partial/\partial v_a)$. In order to construct a torus action on the normal crossing threefold $\overline{Y}_{\underline{\xi}}$, the weights $(k_a, -k_a)$ must be identical for all values of $a$, that is $k_a = k$ for all $1 \leq a \leq m$. If this condition is satisfied, there is a fiberwise torus with weights $(k, -k)$ on the central component $Y_0$ which agrees with the torus actions on the components $Y_a$ along the gluing divisor $\Delta_a$ for $1 \leq a \leq m$. This will be assumed to be the case in the next

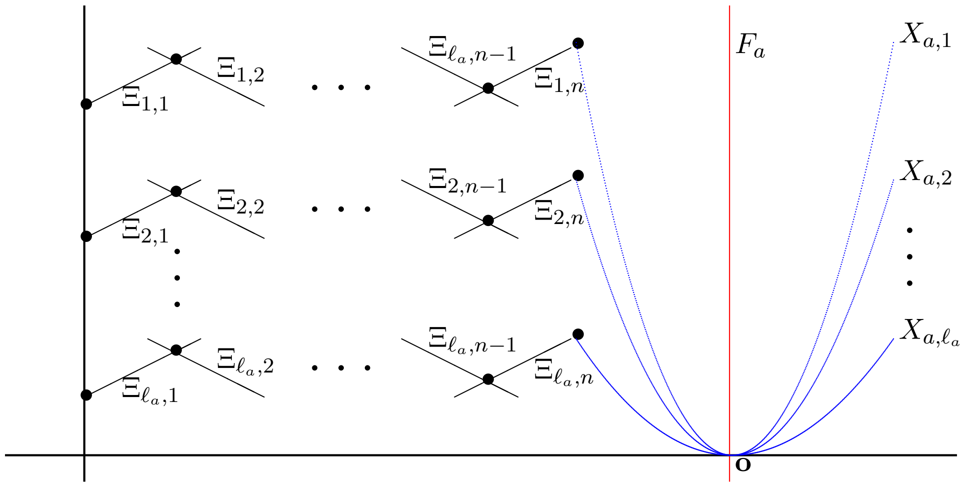


Figure 2: Torus invariant curves in $S_a$.

section. The log Calabi-Yau threefolds $(Y_a, \Delta_a)$, $1 \leq a \leq m$ equipped with the above torus action will be called degree $k$ wild caps in the following.

## 3. Wild degenerate Gromov-Witten theory

The goal of this section is to derive an explicit formula for the Gromov-Witten theory of a degenerate wild local curve. This will be carried out using J. Li's theory of relative stable morphisms [34] and degeneration formula [35], the relative virtual localization theorem of Graber and Vakil [23], as well as the local TQFT formalism of Bryan and Pandharipande [9]. Several results proven in [8, 38, 39, 50–52] will be also needed in the process.

The main building blocks will be residual relative Gromov-Witten invariants of log Calabi-Yau threefolds $(Y, \Delta)$ equipped with a torus action, where $\Delta \subset Y$ is a smooth divisor. For completeness a review of relative stable maps is provided in Section 4. As explained there, simple relative stable maps to the pair $(Y, \Delta)$ are stable maps $f : \Sigma \to Y$ with prescribed contact conditions along $\Delta$ specified by a Young diagram $\lambda = (\lambda_1, \ldots, \lambda_{l_\lambda})$. More precisely, one requires that $f^{-1}(\Delta) = \sum_{j=1}^{l_\lambda} \lambda_j \sigma_j$ for some pairwise distinct smooth points $\sigma_1, \ldots, \sigma_{l_\lambda} \in \Sigma$. The topological invariants of a relative stable map are the genus $h \in \mathbb{Z}$ and the homology class $\beta = f_*[\Sigma] \in H_2(Y, \mathbb{Z})$. As in [8, 9, 51, 52], disconnected domains are allowed as long as no connected component is mapped to a point. Moreover, following loc. cit., the points $\sigma_1, \ldots, \sigma_{l_\lambda}$ in the domain will be unmarked, as opposed to [34, 35].

### 3.1. The wild cap

A wild cap is a log Calabi-Yau threefold $(Y_a, \Delta_a)$, as constructed in the previous section, equipped with a torus action as in Section 2.2. Since the relative target will be fixed throughout this subsection, it will be simply denoted by $(Y, \Delta)$.

The first observation is that the cone of compact curve classes on $Y$ is freely generated by the classes $[X_i]$, $1 \leq i \leq \ell$, of torus invariant curves determined by the equations (2.4). Again the subscript $a$ will be dropped. Given any collection of non-negative integers $\underline{m} = (m_1, \ldots, m_\ell)$, let $\beta(\underline{m}) = \sum_{i=1}^{\ell} m_i[X_i]$ be the corresponding curve class, and let $\overline{M}^{\bullet}_{h,\underline{m}}(\lambda)$ be the moduli stack of genus $h$ relative stable maps to $(Y, \Delta)$ with homology class $\beta(\underline{m})$ and relative conditions specified by $\lambda$. The torus fixed locus $\overline{M}^{\bullet}_{h,\underline{m}}(\lambda)^{\mathbf{T}}$ is compact and the residual relative invariants are defined by virtual integration

$$GW^{\bullet}(h, \underline{m}, \lambda) = \int_{[\overline{M}^{\bullet}_{h,\underline{m}}(\lambda)^{\mathbf{T}}]^{vir}} \frac{1}{e_{\mathbf{T}}(N^{vir})} \tag{3.1}$$

where $N^{vir}$ denotes the virtual normal bundle to the fixed locus. The wild cap is the generating function

$$W_\lambda(\mathsf{x}, g_s) = \sum_{h\in\mathbb{Z}} \sum_{(m_1,\ldots,m_\ell)} g_s^{2h-2+l(\mu)} GW^{\bullet}(h, \underline{m}, \lambda) \prod_{i=1}^{\ell} x^{m_i} \tag{3.2}$$

The computation of the residual invariants (3.1) proceeds by analogy with the $(0, -1)$ cap reviewed in Section 4.3. Using the relative virtual localization theorem [23], the partition function of the wild cap is a convolution

$$W_\lambda(\mathsf{x}, g_s) = S_\lambda(\mathsf{x}, g_s) + \sum_{|\rho|=|\lambda|} (ik\mathsf{u})^{2l(\rho)} \zeta(\rho) S_\rho(\mathsf{x}, g_s) R_{\rho,\lambda}(g_s) \tag{3.3}$$

where

$$S_\rho(\mathsf{x}, g_s) = \sum_{h\in\mathbb{Z}} \sum_{(m_1,\ldots,m_\ell)} g_s^{2h-2+l(\rho)} GW^{\bullet}_s(h, \underline{m}, \rho) \prod_{i=1}^{\ell} x^{m_i}$$

is the generating function of equivariant simple residues and $R_{\rho,\lambda}(g_s)$ is the generating function of rubber integrals. In the above formula $\mathsf{u} \in H^2(B\mathbf{T})$ is the equivariant parameter and the combinatorial factor $\zeta(\rho)$ is defined by

$$\zeta(\rho) = \prod_{j\geq 1} k_j! j^{k_j}$$

for a partition $\rho = (1^{k_1} 2^{k_2} \cdots)$.

An important observation is that the rubber target occuring in the expanded degenerations of the wild cap is $\mathbb{P}^1 \times \mathbb{A}^2$ with antidiagonal torus action of type $(k, -k)$ on $\mathbb{A}^2$. Therefore using the results of [39, 51, 52] collected in Section 4.5, the rubber generating function is

$$R_{\rho,\lambda}(g_s) = (ik\mathsf{u})^{-(l(\rho)+l(\lambda))} \sum_{|\nu|=d} \left(e^{-ikc(\nu)g_s} - 1\right) \frac{\chi^\nu(\rho)}{\zeta(\rho)} \frac{\chi^\nu(\lambda)}{\zeta(\lambda)}. \tag{3.4}$$

for any Young diagrams $(\rho, \lambda)$ with $|\rho| = |\lambda| = d$. Note that $\chi^\nu(\lambda)$ denotes the character of the irreducible symmetric group representation corresponding to $\nu$ evaluated on the conjugacy class corresponding to $\lambda$.

The generating function $S_\rho(\mathsf{x}, g_s)$ for the residues of the simple fixed loci is computed by localization arguments generalizing those used in Section 4.3 for the $(0, -1)$ Calabi-Yau cap. The domain of a simple fixed map in the current setup is union

$$\Sigma = \left( \cup_{i=1}^{\ell} \Sigma_i \right) \cup \left( \cup_{i=1}^{\ell} \cup_{j=1}^{k_i} \Lambda_{i,j} \right)$$

where

- each $\Sigma_i$, $1 \leq i \leq \ell$ is a possibly disconnected genus $h_i$ component mapped to the fixed point $p_i$ in $Y$,
- for each $1 \leq i \leq \ell$, the components $\Lambda_{i,1}, \ldots, \Lambda_{i,k_i}$ are projective lines attached to $\Sigma_i$ mapped in a torus invariant fashion to the curve $X_i$ in the target with some degrees $d_{i,1}, \ldots, d_{i,k_i} \geq 1$.

Note that

$$h = \sum_{i=1}^{\ell} h_i, \qquad m_i = \sum_{j=1}^{k_i} d_{i,j}$$

for all $1 \leq i \leq \ell$. Moreover, the infinitesimal neighborhood of each target curve $X_i \subset Y$ is isomorphic to the infinitesimal neighborhood of the zero section in the $(0,-1)$ cap geometry discussed in detail in Section 4.3. In conclusion a simple fixed locus is isomorphic to a product

$$\times_{i=1}^{\ell}\Gamma_s(h_i,\rho_i)$$

where $\rho_i$ be the partition of $m_i$ determined by $(d_{i,j})$, $1\leq j\leq k_i$, and each factor $\Gamma_s(h_i,\rho_i)$ is a fixed locus of type $(h_i,\rho_i)$ in the $(0,-1)$ cap geometry. Furthermore, the Young diagram $\rho$ encoding the contact conditions along $\Delta$ is given by

$$\rho=\cup_{i=1}^{\ell}\rho_i.$$

This yields

$$S_\lambda(\mathsf{x},g_s)=\sum_{\rho_1\cup\cdots\cup\rho_\ell=\lambda}\prod_{i=1}^{\ell}\Big(x_i^{|\rho_i|}S_{\rho_i}(g_s)\Big) \tag{3.5}$$

where $S_\rho(g_s)$ is the contribution of simple fixed loci with relative conditions $\rho$ for the $(0,-1)$ cap geometry. As explained in Section 4.4, the latter is given by the Marino-Vafa formula [40] proven in [38, 50]. This reads

$$S_\rho(Y,\Delta;g_s)=(ik\mathsf{u})^{-l(\rho)}\sum_{|\nu|=|\rho|}\frac{\chi^\nu(\rho)}{\zeta(\rho)}V_\nu^{(k+1)}(q),$$

where $q=e^{ig_s}$ and

$$V_\nu^{(k+1)}(q)=q^{(k+1)c(\nu)}s_\nu(\underline{q})=q^{kc(\nu)}s_{\nu^t}(\underline{q}),\qquad \underline{q}=(q^{1/2},q^{3/2},\ldots).$$

is the one leg topological vertex [1].

The computation of the wild cap then proceeds by substituting equations (3.5) and (3.4) in (3.3). As shown in detail below, the final expression is

$$\begin{aligned} W_\lambda(\mathsf{x},g_s)&=(ik\mathsf{u})^{-l(\lambda)}\zeta(\lambda)^{-1}\\ &\times\sum_{|\mu|=|\lambda|}\sum_{\substack{\nu_1,\ldots,\nu_\ell\\ |\nu_1|+\cdots+|\nu_\ell|=|\lambda|}}q^{-kc(\mu)}c^{\mu}_{\nu_1,\ldots,\nu_\ell}\chi^\mu(\lambda)\prod_{i=1}^{\ell}\Big(x_i^{|\nu_i|}q^{kc(\nu_i)}s_{\nu_i^t}(\underline{q})\Big)\end{aligned} \tag{3.6}$$

where $c^{\mu}_{\nu_1,\ldots,\nu_\ell}$ are the fusion coefficients for Schur functions i.e.

$$\prod_{i=1}^{\ell}s_{\nu_i}(\mathsf{x})=\sum_{|\mu|=|\nu_1|+\cdots+|\nu_\ell|}c^{\mu}_{\nu_1,\ldots,\nu_\ell}s_\mu(\mathsf{x}). \tag{3.7}$$

The proof of formula (3.6) will proceed in a few steps.

**Step 1.** Using the orthogonality relation

$$\sum_{|\mu|=|\lambda|=|\rho|} \chi^{\mu}(\rho)\chi^{\mu}(\lambda) = \zeta(\lambda)\delta_{\rho,\lambda},$$

the generating function $S_\lambda(\mathsf{x}, g_s)$ in (3.5) will be rewritten as follows

$$\begin{aligned}
S_\lambda(\mathsf{x}, g_s) &= (iku)^{-l(\lambda)} \sum_{\rho_1\cup\cdots\cup\rho_\ell=\lambda} \sum_{\substack{\nu_1,\ldots,\nu_\ell \\ |\nu_i|=|\rho_i|,\ 1\leq i\leq \ell}} \prod_{i=1}^{\ell} \left( {x_i}^{|\rho_i|}\zeta(\rho_i)^{-1}\chi^{\nu_i}(\rho_i)V^{(k+1)}_{\nu_i}(q)\right) \\
&= (iku)^{-l(\lambda)}\zeta(\lambda)^{-1} \sum_{|\rho|=|\lambda|} \sum_{\rho_1\cup\cdots\cup\rho_\ell=\rho} \zeta(\lambda)\delta_{\rho,\lambda} \\
&\quad \times \sum_{\substack{\nu_1,\ldots,\nu_\ell \\ |\nu_i|=|\rho_i|,\ 1\leq i\leq \ell}} \prod_{i=1}^{\ell} \left( {x_i}^{|\rho_i|}\zeta(\rho_i)^{-1}\chi^{\nu_i}(\rho_i)V^{(k+1)}_{\nu_i}(q)\right) \\
&= (iku)^{-l(\lambda)}\zeta(\lambda)^{-1} \sum_{|\mu|=|\lambda|} \sum_{|\rho|=|\lambda|} \sum_{\rho_1\cup\cdots\cup\rho_\ell=\rho} \chi^{\mu}(\rho)\chi^{\mu}(\lambda) \\
&\quad \times \sum_{\substack{\nu_1,\ldots,\nu_\ell \\ |\nu_i|=|\rho_i|,\ 1\leq i\leq \ell}} \prod_{i=1}^{\ell} \left( {x_i}^{|\rho_i|}\zeta(\rho_i)^{-1}\chi^{\nu_i}(\rho_i)V^{(k+1)}_{\nu_i}(q)\right) \\
&= (iku)^{-l(\lambda)}\zeta(\lambda)^{-1} \sum_{\substack{\nu_1,\ldots,\nu_\ell \\ |\nu_1|+\cdots+|\nu_\ell|=|\lambda|}} \sum_{|\mu|=|\lambda|} \chi^{\mu}(\lambda) \prod_{i=1}^{\ell} \left( x_i^{|\nu_i|}V^{(k+1)}_{\nu_i}(q)\right) \\
&\quad \times \sum_{\substack{\rho_1,\ldots,\rho_\ell \\ |\rho_i|=|\nu_i|,\ 1\leq i\leq \ell}} \chi^{\mu}(\rho_1\cup\cdots\cup\rho_\ell) \prod_{i=1}^{\ell} \left(\zeta(\rho_i)^{-1}\chi^{\nu_i}(\rho_i)\right)
\end{aligned}$$

Now note the following combinatorial formula

$$c^{\mu}_{\nu_1,\ldots,\nu_\ell} = \sum_{\substack{\rho_1,\ldots,\rho_\ell \\ |\rho_i|=|\nu_i|,\ 1\leq i\leq \ell}} \chi^{\mu}(\rho_1\cup\cdots\cup\rho_\ell) \prod_{i=1}^{\ell} \left(\zeta(\rho_i)^{-1}\chi^{\nu_i}(\rho_i)\right) \tag{3.8}$$

for the fusion coefficients, which can be easily proven by writing the Schur functions in terms of power sum symmetric functions. This yields by substitution:

$$(3.9)\quad S_\lambda(\mathsf{x}, g_s) = (iku)^{-l(\lambda)}\zeta(\lambda)^{-1} \times \sum_{\substack{\nu_1,\ldots,\nu_\ell \\ |\nu_1|+\cdots+|\nu_\ell|=|\lambda|}} \sum_{|\mu|=|\lambda|} c^{\mu}_{\nu_1,\ldots,\nu_\ell}\chi^\mu(\lambda)\prod_{i=1}^{\ell}\left(x_i^{|\nu_i|}V^{(k+1)}_{\nu_i}(q)\right).$$

for the generating function of simple residues.

**Step 2**. The second term in the convolution formula (3.3) is computed using equation (3.5) and identity (3.8)

$$(3.10)\quad \begin{aligned} &\sum_{|\rho|=|\lambda|} (ik\mathsf{u})^{2l(\rho)}\zeta(\rho)S_\rho(\mathsf{x}, g_s)R_{\rho,\lambda}(g_s) \\ &= (ik\mathsf{u})^{-l_\lambda}\zeta(\lambda)^{-1}\sum_{|\rho|=|\lambda|}\sum_{|\mu|=|\lambda|}\left(e^{-ikc(\mu)g_s}-1\right)\chi^\mu(\rho)\chi^\mu(\lambda) \\ &\times \sum_{\rho_1\cup\cdots\cup\rho_\ell=\rho}\ \sum_{\substack{\nu_1,\ldots,\nu_\ell \\ |\nu_i|=|\rho_i|,\ 1\leq i\leq \ell}} \prod_{i=1}^{\ell}\left(x_i^{|\rho_i|}\zeta(\rho_i)^{-1}\chi^{\nu_i}(\rho_i)V^{(k+1)}_{\nu_i}(q)\right) \\ &= (ik\mathsf{u})^{-l_\lambda}\zeta(\lambda)^{-1}\sum_{|\mu|=|\lambda|}\left(e^{-ikc(\mu)g_s}-1\right)\chi^\mu(\lambda) \\ &\times \sum_{|\nu_1|+\cdots+|\nu_\ell|=|\lambda|} c^\mu_{\nu_1,\ldots,\nu_\ell}\prod_{i=1}^{\ell}\left(x_i^{|\nu_i|}V^{(k+1)}_{\nu_i}(q)\right) \end{aligned}$$

Finally, substituting (3.9) and (3.10) in (3.3) one obtains formula (3.6).

### 3.2. The central component

The central component of the degenerate threefold $\overline{Y}_{\underline{\xi}}$ is isomorphic to the total space of the rank two bundle

$$\mathcal{O}_{C_0} \oplus K_{C_0}(\nu_1 + \cdots + \nu_m)$$

over the smooth genus $g$ curve $C_0$. Under the current assumption there is a fiberwise torus action with weights $(k, -k)$ on $Y_0$. Recall that $Y_0$ is glued to the wild caps $Y_1, \ldots, Y_m$ along the fibers $\Delta_1, \ldots, \Delta_m$ over the points $\nu_1, \ldots, \nu_{2m}$. The goal of this section is to determine the generating function of relative Gromov-Witten invariants for the data $(Y_0, \Delta_1, \ldots, \Delta_m)$ with contact conditions $\lambda_1, \ldots, \lambda_m$. An explicit formula for this generating function follows from the two dimensional TQFT formalism for local curves constructed in [9]. The main building blocks are determined in the proof of Theorems 7.3 in loc. cit. as explained below, mainly using the same notation.

For the antidiagonal torus action, the TQFT constructed in [9] takes values in the category of $R$-modules, where $R = \mathbb{Q}[i](\mathsf{t})((g_s))$ is the ring of Laurent series in the genus counting variable $g_s$ over the algebraic extension $\mathbb{Q}[i]$. Given a fixed degree $d \geq 1$, the $R$-module $\mathbf{GW}_d(S^1)$ assigned to the circle is freely generated by elements $e_\alpha$ in one-to-one correspondence with Young diagrams $|\alpha| = d$. Let $e^\alpha$ be the dual generators given by

$$e^\alpha(e_\beta) = (-\mathsf{t}^2)^{l(\alpha)}\zeta(\alpha)\delta^\alpha_\beta.$$

Note that the coefficients $(-\mathsf{t}^2)^{l(\alpha)}\zeta(\alpha)$ are the gluing factors in the degeneration formula for local relative Gromov-Witten invariants [9, Thm. 3.2]. The multiplication operator is determined by the residual relative invariants of a local genus zero $(0,0)$ curve with contact conditions along three fibers

$$\mathsf{M} = \sum_{|\alpha|=|\beta|=|\gamma|=d} \mathsf{GW}(0|0,0)^\gamma_{\alpha,\beta} e^\alpha \otimes e^\beta \otimes e_\gamma. \tag{3.11}$$

The genus operator is defined as

$$\mathsf{G} = \sum_{|\alpha|=|\beta|=d} \mathsf{GW}(1|0,0)^\beta_\alpha e^\alpha \otimes e_\beta \tag{3.12}$$

where the coefficients $\mathsf{GW}(1|0,0)^\beta_\alpha$ are the residual relative invariants of a local genus one $(0,0)$ curve one $(0,0)$ curve with contact conditions along two fibers. An important observation is that the multiplication operator takes the canonical form

$$\mathsf{M}^\mu_{\lambda,\rho} = \delta^\mu_\lambda \delta^\mu_\rho. \tag{3.13}$$

with respect to the idempotent basis,

$$v_\lambda = \zeta(\lambda)^{-1} \sum_{|\alpha|=d} (i\mathsf{t})^{l(\alpha)-d}\chi^\lambda(\alpha) e_\alpha. \tag{3.14}$$

The remaining TQFT elements needed in this section are:

- the $(0,1)$ annulus in the $v$-basis,

$$\mathsf{A}_\lambda^\rho = (i\mathsf{t})^{-d}\zeta(\lambda)^{-1}s_{\lambda^t}(\underline{q})^{-1}\delta_{\rho,\lambda}, \tag{3.15}$$

- the genus operator in the $v$-basis,

$$\mathsf{G}_{\lambda,\rho} = (i\mathsf{t})^{2d}\zeta(\lambda)^2\delta_{\lambda,\rho}, \tag{3.16}$$

- and the counit in the $v$-basis, which is determined by the $(0,0)$ cap,

$$\mathsf{C}_\lambda = (i\mathsf{t})^{-2d}\zeta(\lambda)^{-2}. \tag{3.17}$$

In this formalism, a genus $g$ local curve of type $(p,q)$ with marked points $\underline{\nu} = (\nu_1,\ldots,\nu_m)$ determines an $R$-module morphism $\mathbf{GW}_d(C,\underline{\nu}|p,q) : \mathbf{GW}_d(S^1)^{\otimes m} \to R$ given by

$$\mathbf{GW}_d(C,\underline{\nu}|p,q) = \sum_{|\alpha_1|=\cdots=|\alpha_m|=d} \mathsf{GW}(g|p,q)_{\alpha_1,\ldots,\alpha_s} e^{\alpha_1}\otimes\cdots\otimes e^{\alpha_s}.$$

Here $\mathsf{GW}(g|p,q)_{\alpha_1,\ldots,\alpha_s}$ are residual relative Gromov-Witten invariants of the local $(p,q)$ curve $C$ with contact conditions $(\alpha_1,\ldots,\alpha_s)$ along the $m$ fibers $\Delta_1,\ldots,\Delta_m$. In the present case, recall that $(p,q)=(0,2g-2+m)$ and $\mathsf{t} = k\mathsf{u}$. In order to write down an explicit formula, let $\mathsf{P}^m : \mathbf{GW}_d(S^1)^{\otimes m} \to \mathbf{GW}_d(S^1)$ be the operator

$$\mathsf{P}^m(v_1,\ldots,v_m) = \mathsf{M}(\mathsf{M}(\cdots(\mathsf{M}(v_1,v_2),v_3),\ldots),v_m).$$

Then note that

$$\mathbf{GW}_d(C,\underline{\nu}|0,2g-2+m) = \mathsf{C}\mathsf{A}^{2g-2+m}\mathsf{G}\mathsf{P}^m. \tag{3.18}$$

Using equations (3.13), (3.15), (3.16) and (3.17), the coefficients of this operator in the $v$-basis are

$$Z_{\lambda_1,\ldots,\lambda_m} = (ik\mathsf{u})^{-(2g-2+m)d}\sum_{|\rho|=d} s_{\rho^t}(\underline{q})^{-(2g-2+m)}\prod_{a=1}^m \delta_{\lambda_a}^\rho. \tag{3.19}$$

### 3.3. The wild curve formula

The residual Gromov-Witten of a wild degenerate local curve is obtained from equations (3.6) and (3.18), (3.19) using the degeneration formula. For fixed degree $r \geq 1$ each wild cap yields an element

$$w = \sum_{|\alpha|=r} (ik\mathsf{u})^{2l(\alpha)} \zeta(\alpha) W_\alpha(\mathsf{x}, g_s) e_\alpha$$

in $\mathsf{GW}_d(S^1)$. The degree $r$ degenerate wild curve partition function is then given by

$$Z_r = \mathbf{GW}_r(C, \underline{\nu}|0, 2g-2+m)(w_1 \otimes \cdots \otimes w_m). \tag{3.20}$$

In order to derive an explicit formula, one needs the coefficients of the wild cap in the $v$-basis. Using the inverse change of basis

$$e_\alpha = (it)^{r-l(\alpha)} \zeta(\alpha)^{-1} \sum_{|\lambda|=r} \zeta(\lambda) \chi^\lambda(\alpha) v_\lambda. \tag{3.21}$$

one obtains

$$w^\lambda = \zeta(\lambda) \sum_{|\alpha|=r} (ik\mathsf{u})^{r+l(\alpha)} \chi^\lambda(\alpha) W_\alpha(\mathsf{x}, g_s).$$

Using equation (3.6), this yields

$$w^\lambda = (ik\mathsf{u})^{|\lambda|} \zeta(\lambda) q^{-kc(\lambda)} \sum_{|\nu_1|+\cdots+|\nu_\ell|=|\lambda|} c^\lambda_{\nu_1,\ldots,\nu_\ell} \prod_{i=1}^{\ell} x_i^{|\nu_i|} q^{kc(\nu_i)} s_{\nu_i^t}(\underline{q}). \tag{3.22}$$

Substituting (3.19) and (3.22) in (3.20), it follows that

$$Z_r = \sum_{|\lambda|=r} s_{\lambda^t}(\underline{q})^{2-2g-m} \prod_{a=1}^{m} F_{k,\ell_a,\lambda}(\mathsf{x}_a, q)$$

where

$$F_{k,\ell,\lambda}(\mathsf{x}, q) = q^{-kc(\lambda)} \sum_{|\nu_1|+\cdots+|\nu_\ell|=|\lambda|} c^\lambda_{\nu_1,\ldots,\nu_\ell} \prod_{i=1}^{\ell} x_i^{|\nu_i|} q^{kc(\nu_i)} s_{\nu_i^t}(\underline{q}).$$

This proves formula (1.7).

## 4. Relative stable maps

This section is a self-contained review of relative stable maps following [34, 35], and some of their applications [8, 9, 23, 38, 39, 50–52]. No results presented here are new, the goal being to collect all results needed in

Section 3.1 in a self-contained manner in order to streamline the derivation of the wild cap formula. This review is primarily aimed at the non-expert reader.

### 4.1. Background

Suppose $Y$ is a smooth complex projective variety and $\Delta \subset Y$ is a smooth connected divisor. A simple relative stable map to the pair $(Y, \Delta)$ consists of the data $(\Sigma, f, \sigma_1, \ldots, \sigma_l)$ where $\Sigma$ is a connected nodal curve $f : \Sigma \to Y$ a map to $Y$, and $\sigma_1, \ldots, \sigma_l$ are smooth pairwise distinct points on $\Sigma$ such that:

- $f^{-1}(\Delta) = \sum_{i=1}^{l} m_i \sigma_i$ for some positive integers $m_1, \ldots, m_l$, and
- the automorphism group of the data $(\Sigma, f, \sigma_1, \ldots, \sigma_l)$ is finite.

The topological invariants of simple relative stable maps are the arithmetic genus $g \geq 0$ of the domain and the homology class $\beta = f_*[\Sigma] \in H_2(X, \mathbb{Z})$.

The moduli stack of simple relative stable maps with fixed invariants $(g, l, \beta)$, $(m_1, \ldots, m_l)$ is not proper. A compactification is constructed in [34] by allowing the target to degenerate in a controlled way. The allowed degenerations of the target are normal crossing varieties constructed by gluing $Y$ and an arbitrary number of copies of the projective bundle

$$P = \mathbb{P}_\Delta(N_{\Delta/Y} \oplus \mathcal{O}_\Delta),$$

where $N_{\Delta/Y}$ is the normal bundle of $\Delta$ in $Y$. This projective bundle has two canonical sections $\Delta_0, \Delta_\infty$ with normal bundles $N_{\Delta/Y}^{-1}$, $N_{\Delta/Y}$ respectively. For any integer $n \geq 1$ let $P_n$ be the normal crossing variety obtained by gluing $n$ copies of $P$ such that the section $\Delta_\infty$ of the $i$-th copy is identified with section $\Delta_0$ of the $(i+1)$-th copy. Note that no gluing occurs along $\Delta_0$ in the first copy and $\Delta_\infty$ in the last. Abusing notation, these two divisors on $P_n$ will be denoted by $\Delta_0, \Delta_n$ respectively. The singular locus of $P_n$ consists of the union $\Delta_1 \cup \cdots \cup \Delta_{n-1}$ of copies of $\Delta$ where $\Delta_i$ is the intersection between the $i$-th and the $(i+1)$-th copy of $P$. For further reference let $\mathrm{Aut}_\Delta(P_n) \simeq (\mathbb{C}^\times)^n$ denote the group of automorphisms of $P_n$ acting trivially on all sections $\Delta_0, \ldots, \Delta_n$ over $\Delta$. The degenerate targets $Y_n$ are constructed by gluing $Y$ to $P_n$ so that $\Delta \subset Y$ is identified with $\Delta_0 \subset P_n$. Therefore one obtains a relative pair $(Y_n, \Delta_n)$ such that the singular set of $Y_n$ is the union of the $n$ copies $\Delta_0 \cup \cdots \cup \Delta_{n-1}$ of copies of $\Delta$. Note that there is an algebraic stack $\mathfrak{E}$ of expanded degenerations parameterizing all degenerate targets which occur in this construction, and a universal family $\mathcal{Y} \to \mathfrak{E}$.

Compactification of the moduli stack of relative stable maps is achieved by including relative stable morphisms to degenerate pairs $(Y_n, \Delta_n)$. Such relative maps consist of data $(\Sigma, f, \sigma_1, \ldots, \sigma_l)$ satisfying the following conditions

- $\Sigma$ is a connected nodal curve and $f : \Sigma \to Y_n$ is a predeformable map to $Y_n$.
- $\sigma_1, \ldots, \sigma_l$ are smooth points of $\Sigma$ such that

$$f^{-1}(\Delta_n) = \sum_{i=1}^{l} m_i \sigma_i.$$

Predeformable morphisms $f : \Sigma \to Y_n$ are defined by the following conditions:

- The set theoretic inverse image of any gluing divisor $\Delta_i \subset Y_n^{sing}$ is a set of nodes of $\Sigma$.
- The two branches of $\Sigma$ crossing at each such node are mapped to different irreducible components of $Y_n$ meeting along $\Delta_i$ such the contact orders along $\Delta_i$ are equal.

The topological invariants of a relative map to an expanded target are the arithmetic genus $g$ of the domain, and the homology class $\beta = \pi_{n*} f_*[\Sigma] \in H_2(Y, \mathbb{Z})$ where $\pi_n : Y_n \to Y$ is the natural projection.

Two relative maps $(\Sigma, f)$, $(\Sigma', f')$ to $Y_n$ are isomorphic if there is an isomorphism $\psi : \Sigma \to \Sigma'$ and an automorphism $\varphi : Y_n \to Y_n$ fixing $Y$ and the divisors $\Delta_0, \ldots, \Delta_n$ pointwise, such that $f' \circ \psi = \varphi \circ f$. Note that $\varphi|_{P_n} \in \mathrm{Aut}_\Delta(P_n)$ acts by fiberwise scalar multiplication on each copy of $P$ used in the construction of $P_n$. Stability is defined by requiring the relative maps to have a finite automorphism group according to this notion of isomorphism.

The main result of [34] proves that for fixed data $(g, l, \beta)$, $\underline{m} = (m_1, \ldots, m_l)$ there is a proper Deligne-Mumford moduli stack $\overline{M}_{g,\beta}(Y, \Delta, \underline{m})$ of relative stable morphisms equipped with a perfect obstruction theory and a universal relative morphism to the universal family $\mathcal{Y}$ of expanded degenerations.

A slightly different flavor of relative theory will be used throughout this paper, as in [8, 9, 51, 52]. Namely, as opposed to [34], the points $(\sigma_1, \ldots, \sigma_l)$ in the inverse image of the divisor $D$ are unmarked, and one also allows disconnected domains as long as no connected component is contracted to a point. In particular the numerical invariants $(m_1, \ldots, m_l)$ are unordered, hence they are encoded in a Young diagram $\mu$ with $l$ rows. The moduli stack of stable relative maps with fixed invariants $(g, \beta)$, $\mu$ will be denoted by $\overline{M}_{g,\beta}(Y, \Delta; \mu)$ for connected domains, respectively $\overline{M}^{\bullet}_{g,\beta}(Y, \Delta; \mu)$ for disconnected domains. Note that in the second case $g$ is allowed to be negative. In both cases the moduli stack is equipped with a virtual cycle and a universal morphism to the stack of expanded degenerations $\mathcal{Y}$.

Finally, given a torus action on $Y$ which preserves $\Delta$, there is a virtual localization formula for moduli stacks of relative stable maps, which has been extensively used in the literature. The foundations have been proven in [23]. In particular, this allows one to define residual relative invariants for non-compact pairs $(Y, \Delta)$ provided that there is a torus action on $Y$ which preserves $\Delta$ and has compact fixed locus.

### 4.2. Rubber target

The classification of torus fixed loci in moduli stacks of relative maps leads naturally to a variant of the above construction employing non-rigid targets. Namely one constructs a moduli stack of predeformable stable maps to degenerate targets $P_n$ imposing fixed contact conditions along the sections $\Delta_0, \Delta_\infty$ specified by two partitions $\rho, \mu$. Moreover two such maps $(\Sigma, f)$, $(\Sigma', f')$ are isomorphic if there is an isomorphism $\psi : \Sigma \to \Sigma'$ and an automorphism $\varphi \in \mathrm{Aut}_\Delta(P_n)$ such that $f' \circ \psi = \varphi \circ f$. This construction yields a Deligne-Mumford moduli stack $\overline{M}^{\bullet}_{h,\beta}(P, \rho, \mu)_\sim$ equipped with a perfect obstruction theory, where $\beta \in H_2(P, \mathbb{Z})$. Again, disconnected domains are allowed.

In this case there is again a stack of expanded degenerations $\mathfrak{C}$, a universal family $\mathcal{P} \to \mathfrak{C}$, and a universal map to $\mathcal{P}$. Moreover, $\mathfrak{C}$ is naturally isomorphic to an open substack of the algebraic moduli stack $\mathfrak{M}_{0,2}$ of genus zero curves with two marked points $0, \infty$. In particular one obtains a cohomology class $\psi_0$ on the rubber moduli stack by pulling back the $\psi$-class associated to the marked point 0 on $\mathfrak{M}_{0,2}$. Abusing the language, the class $\psi_0$ will be referred to as the rubber $\psi$-class at 0.

### 4.3. Local (0, −1) rational curves

As an example, this section presents the computation of relative Gromov-Witten theory by localization for a local $(-1, 0)$ rational curve. Although this computation has already been carried out for example in [8, 38, 50], a self-contained review will be helpful since many details are needed in Section 3.1.

Let $C = \mathbb{P}^1$ and let $(U, z)$, $(V, w)$ denote the standard affine coordinate charts on $C$. Let $\mathbf{T} \times C \to C$ be the torus action on $C$ given by

$$(t, z) \mapsto t^{-1} z, \qquad (t, w) \mapsto tw$$

in the two charts respectively. Let also $p \in U$ and $\delta \in V$ be the points $z = 0$, $w = 0$ respectively. Let $Y$ be the total space of $\mathcal{O}_C \oplus \mathcal{O}_C(-p)$ and let $\Delta$ be the fiber of $Y$ over $\delta$. The open subsets $Y_U, Y_V$ are affine coordinate charts on $Y$ with coordinates $(z, u, x)$, $(w, v, y)$ related by the transition functions

$$w = z^{-1}, \qquad x = y, \qquad v = zu.$$

Let $\mathbf{T} \times Y \to Y$ denote the lift of the torus action to $Y$ such that

$$t \times (z, u, x) \mapsto (t^{-1} z, t^{-k} x, t^{k+1} u), \qquad t \times (w, v, y) \mapsto (tw, t^{-k} y, t^k v).$$

Note that in this case any expanded degeneration $Y_n$, $n \geq$, is naturally isomorphic to the total space of the rank two bundle $\mathcal{O}_{C_n} \oplus \mathcal{O}_{C_n}(-p)$ where $C_n$ is the $n$ step degeneration of the target curve. As explained in Section 4.1, $C_n$ is constructed by gluing $C$ to a linear chain of $n$ rational curves, $(\mathbb{P}^1)_n$, where each curve in chain contains two marked points $\delta_0, \delta_\infty$. Similarly, any rubber target $P_n$ is isomorphic to $(\mathbb{P}^1)_n \times \mathbb{A}^2$ and the two divisors $\Delta_0, \Delta_n$ are the fibers of the projection to $(\mathbb{P}^1)_n$ over the marked points $\delta_0, \delta_n$. The above torus action lifts to a torus action $\mathbf{T} \times Y_n \to Y_n$ on any expanded degeneration of the target which scales the fibers over $C_n$ with weights $(k, -k)$ leaving the zero section pointwise fixed. This holds for $P_n$ as well.

The residual invariants of the pair $(Y, \Delta)$ are defined by evaluating an equivariant obstruction class on the virtual cycle of the moduli space of relative stable maps to $(C, \infty)$. Clearly, the degree of any such relative map is given by $d = \sum_{i=1}^{l} \mu_i = |\mu|$, where $\mu = (\mu_1, \ldots, \mu_l)$ is the partition specifying the relative conditions at $\delta_\infty$. Therefore the moduli stack of relative stable maps with disconnected domains can be denoted by $\overline{M}^\bullet_h(C, \delta; \mu)$, where $h \in \mathbb{Z}$ is the arithmetic genus of the domain.

Let $\phi : \mathcal{S} \to \mathcal{C}$ denote the universal relative stable map to $(C, \delta_\infty)$ and $\pi : \mathcal{S} \to \overline{M}^\bullet_h(C, \delta; \mu)$ denote the natural projection. Note that there is a line bundle $\mathcal{O}_{\mathcal{C}}(-\wp)$ which restricts to $\mathcal{O}_{C_n}(-p)$ on each closed point. Then the obstruction is the equivariant $K$-theory class

$$Ob = -T^{-k} R\pi_* \phi^* \mathcal{O}_{\mathcal{C}}(-\wp) - T^k R\pi_* \phi^* \mathcal{O}_C$$

where $T$ denotes the canonical representation of the torus and $\mathcal{O}_{\mathcal{C}}(-\wp)$, $\mathcal{O}_{\mathcal{C}}$ are equipped with their natural equivariant structures. The relative local invariants are defined by

$$GW^\bullet(h,\mu) = \int_{[\overline{M}^\bullet_h(C,\delta;\mu)]^{vir}_{\mathbf{T}}} e_{\mathbf{T}}(Ob), \tag{4.1}$$

where $[\overline{M}^\bullet_h(C,\delta;\mu)]^{vir}_{\mathbf{T}}$ is the equivariant virtual cycle. They can be computed by relative virtual localization as explained below.

The torus fixed loci in moduli stacks of relative stable maps have been analyzed in detail for example in [8, 38, 39, 50]. The main observation is that the domain of a generic torus invariant relative map $f : \Sigma \to C_n$ splits naturally as a union of two curves $\Sigma = \Sigma_s \cup \Sigma_\infty$ whose intersection is a finite set of nodes of $\Sigma$. The restriction $f_s = f|_{\Sigma_s}$ maps $\Sigma_s$ to $C$ while $f_\infty = f|_{\Sigma_\infty}$ maps $\Sigma_\infty$ to $(\mathbb{P}^1)_n$. Using the predeformability condition, it follows that $f_s : \Sigma_s \to C$ must be a simple relative stable map to $(C,\delta)$ with relative invariants specified by a Young diagram $\rho$ such that $|\rho| = |\mu|$. At the same time $f_\infty : \Sigma_\infty \to (\mathbb{P}^1)_n$ is a relative stable map to the triple $((\mathbb{P}^1)_n, \delta_0, \delta_n)$ with relative invariants $(\rho,\mu)$ respectively. Using the definition of isomorphisms of relative maps, it follows that the data $(\Sigma_\infty, f|_{\Sigma_\infty})$ determines a point in the rubber moduli stack of stable relative maps to $\mathbb{P}^1$.

At the same time, as explained in [8, Appendix A], identifying the ramification divisors on the two components $\Sigma_s$, $\Sigma_\infty$ over $\delta = \delta_0$ requires ordering the ramification points in the two components, which are unmarked in the present construction. This implies that a generic fixed locus $\Gamma(h,\mu)$ in the moduli stack $\overline{M}^\bullet_h(C,\delta,\mu)$ is isomorphic to a finite etale cover of the product $\Gamma_s(h_1,\rho) \times \overline{M}^\bullet_{h_2}(\mathbb{P}^1, \delta_0, \delta_\infty; \rho, \mu)$ where $h = h_1 + h_2$ while $\rho$ is a Young diagram with $|\rho| = |\mu|$ and $\Gamma_s(h_1,\rho)$ is a simple fixed locus in the moduli stack $\overline{M}^\bullet_{h_1}(C,\delta,\rho)$.

Using the relative virtual localization theorem [23], the equivariant residue of such a fixed locus factors as

$$Z_{\Gamma(h,\mu)} = \zeta(\rho) e_{\mathbf{T}}(T_\delta \Delta_0) Z_{\Gamma_s(h_1,\rho)} \int_{[\overline{M}^\bullet_{h_2}(\mathbb{P}^1,\delta_0,\delta_\infty;\rho,\mu)]^{vir}} \frac{e_{\mathbf{T}}(Ob_\infty)}{-\mathsf{u} - \psi_0} \tag{4.2}$$

where

- $Z_{\Gamma_s(h_1,\rho)}$ is the equivariant residue of the simple fixed locus $\Gamma(h_1,\rho)$,
- $Ob_\infty$ is an equivariant obstruction class on the rubber moduli space,
- $\psi_0$ is the rubber $\psi$-class at 0 defined as in Section 4.2,
- $\mathsf{u} \in H^2_{\mathbf{T}}(\text{point})$ is the natural generator of the equivariant cohomology ring of the point, and

- given a partition $\rho = (1^{k_1}2^{k_2}\cdots)$, the factor $\zeta(\rho)$ is defined by $\zeta(\rho) = \prod_{j\geq 1} k_j! j^{k_j}$.

Note that the factor $1/(-\mathsf{u}-\psi_0)$ corresponds to normal infinitesimal deformations in the ambient moduli stack $\overline{M}^\bullet_h(C,\delta;\mu)$ induced by deformations of the degenerate target. As shown in [8, Eqn. 14], the combinatorial factor $\zeta(\rho)$ encodes the degree of the finite etale cover involved in the presentation of the fixed locus as a direct product.

The rubber obstruction class is given by

$$Ob_\infty = \Big(-T^k[R\pi_*\phi^*\mathcal{O}_{\mathcal{P}^1}] - T^{-k}[R\pi_*\phi^*\mathcal{O}_{\mathcal{P}^1}]\Big)\,|_{\Gamma_\infty(h,\rho,\mu)}$$

where $\phi : \mathcal{S} \to \mathcal{P}^1$ is the universal relative morphism to the rubber target and $\pi : \mathcal{S} \to \overline{M}^\bullet_{h_2}(\mathbb{P}^1, \delta_0, \delta_\infty; \rho, \mu)$ the natural projection. Therefore

$$e_{\mathbf{T}}(Ob_\infty) = (k\mathsf{u})^{-1}(-k\mathsf{u})^{-1} e_{\mathbf{T}}(\mathbb{E}^\vee(k\mathsf{u}))e_{\mathbf{T}}(\mathbb{E}^\vee(-k\mathsf{u}))$$

where $\mathbb{E}$ is the Hodge bundle. Using Mumford's relation $c(\mathbb{E}^\vee)c(\mathbb{E}) = 1$, one further obtains

$$e_{\mathbf{T}}(Ob_\infty) = (-1)^{h-1}(k\mathsf{u})^{2h-2}. \tag{4.3}$$

Therefore the rubber factors in (4.2) reduce to

$$GW^\bullet(h,\rho,\mu)_\sim = (-1)^{h-1}(k\mathsf{u})^{2h-2}\int_{[\overline{M}^\bullet_{h_2}(\mathbb{P}^1,\delta_0,\delta_\infty;\rho,\mu)]^{vir}} \frac{1}{-\mathsf{u}-\psi_0}. \tag{4.4}$$

Next note that the equation (4.2) yields a gluing formula for the partition function

$$Z_\mu(Y,\Delta;g_s) = \sum_{h\in\mathbb{Z}} g_s^{2h-2+l(\mu)} GW^\bullet(h,\mu) \tag{4.5}$$

of residual local invariants with relative conditions $\mu$. Let

$$S_\mu(g_s) = \sum_{h\in\mathbb{Z}} g_s^{2h-2+l(\mu)} GW^\bullet_s(h,\mu)$$

be the generating function of simple residual invariants with relative conditions and

$$R_{\rho,\mu}(g_s) = \sum_{h\in\mathbb{Z}} g_s^{2h-2+l(\mu)+l(\rho)} GW^\bullet(h,\rho,\mu)_\sim$$

be the generating function of rubber relative invariants with relative conditions $(\rho, \mu)$. Then equation (4.5) yields

$$Z_\mu(Y, \Delta; g_s) = S_\mu(g_s) + \sum_{|\rho|=|\lambda|} (ik\mathsf{u})^{2l(\rho)}\zeta(\rho)S_\rho(g_s)R_{\rho,\mu}(g_s). \tag{4.6}$$

### 4.4. Simple fixed loci

The domain of a simple fixed map is a union

$$\Sigma = \Sigma_0 \cup \left( \cup_{j=1}^{l} \Lambda_i \right)$$

where

- $\Sigma_0$ is a possibly disconnected genus $h$ component mapped to the fixed point $p \in C$, and
- the components $\Lambda_1, \ldots, \Lambda_l$ are projective lines attached to $\Sigma_0$, which are mapped in a torus invariant fashion to $C$ with some degrees $d_i, \ldots, d_l \geq 1$.

The partition $\mu$ encoding the contact conditions at $\delta_\infty$ is determined by $(d_1, \ldots, d_l)$.

The equivariant residues of simple fixed loci are evaluated by standard localization computations in terms of Hodge integrals on moduli spaces of curves with marked points. Such computations have been done in detail for example in [8, 38, 39, 50], the results being in agreement with the open string Gromov-Witten invariants computed in [29, 36]. Omitting the details the end result is a closed form expression for the generating function $S^\bullet_\alpha(g_s)$ provided by the Marino-Vafa formula [40], proven in [38, 50]. To write this formula explicitly, for any Young diagram $\nu$ let $s_\nu(x_1, x_2, \ldots)$ are the corresponding Schur function and let $c(\nu) = \sum_{\Box\in\nu}(a(\Box) - l(\Box))/2$ be the content of $\nu$. As above, if $\nu = (1^{k_1}2^{k_2}\cdots)$, let $\zeta(\nu) = \prod_{j\geq 1} k_j! j^{k_j}$. Moreover let $\chi^\nu$ denote the character of the irreducible representation of the symmetric group $\mathcal{S}_{|\nu|}$ determined by $\nu$. For any Young diagram $\mu$ with $|\mu| = |\nu|$ let $\chi^\nu(\mu)$ denote the value of $\chi^\nu$ on the conjugacy class determined by $\mu$. Then the Marino-Vafa formula reads

$$S_\mu(Y, \Delta; g_s) = (ik\mathsf{u})^{-l(\mu)} \sum_{|\nu|=|\mu|} \frac{\chi^\nu(\mu)}{\zeta(\mu)} q^{(k+1)c(\nu)} s_\nu(\underline{q}). \tag{4.7}$$

where $q = e^{ig_s}$ and $\underline{q} = (q^{1/2}, q^{3/2}, \ldots)$.

### 4.5. Rubber integrals and Hurwitz numbers

In order to finish the computation one has to evaluate the rubber integrals in equation (4.4), which has been done in [39, 51, 52]. First note that

$$GW^\bullet(h,\rho,\mu)_\sim = (-1)^h(k\mathsf{u})^{2h-2}\sum_{n\geq 0}(-1)^n\mathsf{u}^{-n-1}\langle\rho,n|\mu\rangle_h^\sim$$

where

$$\langle\rho,n|\mu\rangle_h^\sim = \int_{[\overline{M}_h^\bullet(\mathbb{P}^1,\delta_0,\delta_\infty;\rho,\mu)_\sim]^{vir}} \psi_0^n.$$

These rubber correlators have been evaluated by rigidification in [39, 51, 52], the final formulas being expressed in terms of Hurwitz numbers,

$$\langle\rho,n|\mu\rangle_h^\sim = \frac{H_h^\bullet(\rho,\mu)}{(n+1)!} \tag{4.8}$$

if $n+1 = 2h-2+l(\rho)+l(\mu)$, respectively $\langle\rho,n|\mu\rangle_h^\sim = 0$ for all other values of $n$.

For completeness, recall that the double Hurwitz number $H_h^\bullet(\rho,\mu)$ is the weighted number of genus $h$ disconnected $d:1$ covers of $\mathbb{P}^1$ with fixed branch locus of type

$$\big(\mu,\nu,\underbrace{(2,1^{d-1}),\ldots,(2,1^{d-1})}_{n(h,\rho,\mu)}\big)$$

where $d = |\rho| = |\mu|$, and

$$n(h,\rho,\mu) = 2h-2+l(\rho)+l(\mu).$$

Each cover is weighted by the inverse of the order of its automorphism group. The double Hurwitz numbers are given by the following combinatorial formula [16]

$$H_h^\bullet(\rho,\mu) = \sum_{|\nu|=d} c(\nu)^{n(h,\rho,\mu)}\frac{\chi^\nu(\rho)}{\zeta(\rho)}\frac{\chi^\nu(\mu)}{\zeta(\mu)} \tag{4.9}$$

where $c(\nu) = \sum_{\square\in\nu}(a(\square)-l(\square))$ is the content of $\nu$.

Using equation (4.8), the generating series of rubber invariants is written as

$$\begin{aligned} R_{\rho,\mu}(g_s) &= (ik\mathsf{u})^{-(l(\rho)+l(\mu))} \\ &\quad\times \sum_{\substack{h\in\mathbb{Z}\\ 2h-2+l(\rho)+l(\mu)\geq 1}} (-ikg_s)^{2h-2+l(\rho)+l(\mu)} \frac{H^\bullet_h(\rho,\mu)}{(2h-2+l(\rho)+l(\mu))!} \\ &= (ik\mathsf{u})^{-(l(\rho)+l(\mu))} \sum_{n\geq 1} (-ikg_s)^n \frac{H_{h(n,\rho,\mu)}}{n!} \end{aligned}$$

where

$$h(n,\rho,\mu) = \begin{cases} (n-l(\rho)-l(\mu)+2)/2, & \text{if } n-l(\rho)-l(\mu) \text{ even} \\ 0, & \text{otherwise.} \end{cases}$$

Next note that formula (4.9) yields the following identity

$$\sum_{n\geq 1} \frac{t^n}{n!} H^\bullet_{h(n,\rho,\mu)}(\rho,\mu) = \sum_{\nu} \left(e^{tc(\nu)} - 1\right) \frac{\chi^\nu(\rho)}{\zeta(\rho)} \frac{\chi^\nu(\mu)}{\zeta(\mu)}. \tag{4.10}$$

for any variable $t$. Therefore the final formula for the rubber generating series for two partitions $(\rho,\mu)$ with $|\rho| = |\mu| = d$ is

$$R_{\rho,\mu}(g_s) = (ik\mathsf{u})^{-(l(\rho)+l(\mu))} \sum_{|\nu|=d} \left(e^{-ikc(\nu)g_s} - 1\right) \frac{\chi^\nu(\rho)}{\zeta(\rho)} \frac{\chi^\nu(\mu)}{\zeta(\mu)}. \tag{4.11}$$

The computation of the $(0,-1)$ cap is concluded by substituting formulas (4.7), (4.11) in equation (4.6). Using the orthogonality relations

$$\sum_{|\rho|=|\mu|=|\nu|} \zeta(\rho)^{-1}\chi^\mu(\rho)\chi^\nu(\rho) = \delta_{\mu,\nu},$$

it follows that

$$Z_\mu(Y,\Delta;g_s) = (ik\mathsf{u})^{-l(\mu)} \sum_{|\nu|=|\mu|} \frac{\chi^\nu(\mu)}{\zeta(\mu)} q^{c(\nu)} s_\nu(\underline{q}). \tag{4.12}$$

Using the identity,

$$q^{c(\nu)} s_\nu(\underline{q}) = s_{\nu^t}(\underline{q})$$

this formula agrees with the antidiagonal $(0,-1)$ cap computed in [9] for torus weight $\mathsf{t} = k\mathsf{u}$. Indeed, applying the TQFT formalism of [9], reviewed in Section 3.2, the element of $\mathbf{GW}_d(S^1)$ determined by the above formula is

$$\sum_{|\mu|=d} (ik\mathsf{u})^{2l(\mu)}\zeta(\mu)Z_\mu(Y,\Delta;g_s)e_\mu = \sum_{|\mu|=d}\sum_{|\nu|=d} (ik\mathsf{u})^{l(\mu)}\chi^\nu(\mu)q^{c(\nu)}s_\nu(\underline{q})e_\mu.$$

Using the change of basis formula (3.21), one finds

$$\sum_{|\rho|=d} (ik\mathsf{u})^d\zeta(\rho)s_{\rho^t}(\underline{q})v_\rho$$

in the indempotent basis (3.14). This is an agreement with the formula for the coefficients $\overline{\eta}_\rho$ derived on page 38 of [9].

## Appendix A. Examples

**Example 1.** $g=1$, $m=2$, $\mu_1=\mu_2=(2,1)$, $n_1=3$, $n_2=4$.

$$\begin{aligned}
P_{\underline{m},\underline{n}}(u,v) = {} & u^{30}v^{30} + 2\,u^{29}v^{30} - 2\,u^{29}v^{29} + 4\,u^{28}v^{30} - 6\,u^{28}v^{29} + 5\,u^{27}v^{30} \\
& + 2\,u^{28}v^{28} - 14\,u^{27}v^{29} + 7\,u^{26}v^{30} + 9\,u^{27}v^{28} - 22\,u^{26}v^{29} \\
& + 8\,u^{25}v^{30} - 2\,u^{27}v^{27} + 23\,u^{26}v^{28} - 32\,u^{25}v^{29} + 10\,u^{24}v^{30} \\
& - 10\,u^{26}v^{27} + 44\,u^{25}v^{28} - 40\,u^{24}v^{29} + 11\,u^{23}v^{30} + 2\,u^{26}v^{26} \\
& - 28\,u^{25}v^{27} + 68\,u^{24}v^{28} - 50\,u^{23}v^{29} + 11\,u^{22}v^{30} + 10\,u^{25}v^{26} \\
& - 60\,u^{24}v^{27} + 94\,u^{23}v^{28} - 58\,u^{22}v^{29} + 10\,u^{21}v^{30} - 2\,u^{25}v^{25} \\
& + 31\,u^{24}v^{26} - 100\,u^{23}v^{27} + 119\,u^{22}v^{28} - 58\,u^{21}v^{29} + 9\,u^{20}v^{30} \\
& - 10\,u^{24}v^{25} + 69\,u^{23}v^{26} - 148\,u^{22}v^{27} + 144\,u^{21}v^{28} - 54\,u^{20}v^{29} \\
& + 6\,u^{19}v^{30} + 2\,u^{24}v^{24} - 32\,u^{23}v^{25} + 124\,u^{22}v^{26} - 194\,u^{21}v^{27} \\
& + 144\,u^{20}v^{28} - 48\,u^{19}v^{29} + 4\,u^{18}v^{30} + 10\,u^{23}v^{24} - 74\,u^{22}v^{25} \\
& + 190\,u^{21}v^{26} - 240\,u^{20}v^{27} + 136\,u^{19}v^{28} - 34\,u^{18}v^{29} + u^{17}v^{30} \\
& - 2\,u^{23}v^{23} + 32\,u^{22}v^{24} - 140\,u^{21}v^{25} + 261\,u^{20}v^{26} - 240\,u^{19}v^{27} \\
& + 118\,u^{18}v^{28} - 20\,u^{17}v^{29} - 10\,u^{22}v^{23} + 77\,u^{21}v^{24} - 222\,u^{20}v^{25} \\
& + 324\,u^{19}v^{26} - 226\,u^{18}v^{27} + 86\,u^{17}v^{28} - 4\,u^{16}v^{29} + 2\,u^{22}v^{22} \\
& - 32\,u^{21}v^{23} + 149\,u^{20}v^{24} - 316\,u^{19}v^{25} + 323\,u^{18}v^{26} - 190\,u^{17}v^{27} \\
& + 44\,u^{16}v^{28} + 10\,u^{21}v^{22} - 78\,u^{20}v^{23} + 246\,u^{19}v^{24} - 392\,u^{18}v^{25} \\
& + 297\,u^{17}v^{26} - 134\,u^{16}v^{27} + 6\,u^{15}v^{28} - 2\,u^{21}v^{21} + 32\,u^{20}v^{22} \\
& - 154\,u^{19}v^{23} + 357\,u^{18}v^{24} - 384\,u^{17}v^{25} + 241\,u^{16}v^{26} - 56\,u^{15}v^{27}
\end{aligned}$$

$$
\begin{aligned}
&- 10\,u^{20}v^{21} + 78\,u^{19}v^{22} - 262\,u^{18}v^{23} + 443\,u^{17}v^{24} - 340\,u^{16}v^{25} \\
&+ 152\,u^{15}v^{26} - 4\,u^{14}v^{27} + 2\,u^{20}v^{20} - 32\,u^{19}v^{21} + 157\,u^{18}v^{22} \\
&- 386\,u^{17}v^{23} + 418\,u^{16}v^{24} - 260\,u^{15}v^{25} + 44\,u^{14}v^{26} + 10\,u^{19}v^{20} \\
&- 78\,u^{18}v^{21} + 271\,u^{17}v^{22} - 474\,u^{16}v^{23} + 354\,u^{15}v^{24} - 134\,u^{14}v^{25} \\
&+ u^{13}v^{26} - 2\,u^{19}v^{19} + 32\,u^{18}v^{20} - 158\,u^{17}v^{21} + 405\,u^{16}v^{22} \\
&- 428\,u^{15}v^{23} + 241\,u^{14}v^{24} - 20\,u^{13}v^{25} - 10\,u^{18}v^{19} + 78\,u^{17}v^{20} \\
&- 276\,u^{16}v^{21} + 484\,u^{15}v^{22} - 340\,u^{14}v^{23} + 86\,u^{13}v^{24} + 2\,u^{18}v^{18} \\
&- 32\,u^{17}v^{19} + 158\,u^{16}v^{20} - 412\,u^{15}v^{21} + 418\,u^{14}v^{22} - 190\,u^{13}v^{23} \\
&+ 4\,u^{12}v^{24} + 10\,u^{17}v^{18} - 78\,u^{16}v^{19} + 278\,u^{15}v^{20} - 474\,u^{14}v^{21} \\
&+ 297\,u^{13}v^{22} - 34\,u^{12}v^{23} - 2\,u^{17}v^{17} + 32\,u^{16}v^{18} - 158\,u^{15}v^{19} \\
&+ 405\,u^{14}v^{20} - 384\,u^{13}v^{21} + 118\,u^{12}v^{22} - 10\,u^{16}v^{17} + 78\,u^{15}v^{18} \\
&- 276\,u^{14}v^{19} + 443\,u^{13}v^{20} - 226\,u^{12}v^{21} + 6\,u^{11}v^{22} + 2\,u^{16}v^{16} \\
&- 32\,u^{15}v^{17} + 158\,u^{14}v^{18} - 386\,u^{13}v^{19} + 323\,u^{12}v^{20} - 48\,u^{11}v^{21} \\
&+ 10\,u^{15}v^{16} - 78\,u^{14}v^{17} + 271\,u^{13}v^{18} - 392\,u^{12}v^{19} + 136\,u^{11}v^{20} \\
&- 2\,u^{15}v^{15} + 32\,u^{14}v^{16} - 158\,u^{13}v^{17} + 357\,u^{12}v^{18} - 240\,u^{11}v^{19} \\
&+ 9\,u^{10}v^{20} - 10\,u^{14}v^{15} + 78\,u^{13}v^{16} - 262\,u^{12}v^{17} + 324\,u^{11}v^{18} \\
&- 54\,u^{10}v^{19} + 2\,u^{14}v^{14} - 32\,u^{13}v^{15} + 157\,u^{12}v^{16} - 316\,u^{11}v^{17} \\
&+ 144\,u^{10}v^{18} + 10\,u^{13}v^{14} - 78\,u^{12}v^{15} + 246\,u^{11}v^{16} - 240\,u^{10}v^{17} \\
&+ 10\,u^{9}v^{18} - 2\,u^{13}v^{13} + 32\,u^{12}v^{14} - 154\,u^{11}v^{15} + 261\,u^{10}v^{16} \\
&- 58\,u^{9}v^{17} - 10\,u^{12}v^{13} + 78\,u^{11}v^{14} - 222\,u^{10}v^{15} + 144\,u^{9}v^{16} \\
&+ 2\,u^{12}v^{12} - 32\,u^{11}v^{13} + 149\,u^{10}v^{14} - 194\,u^{9}v^{15} + 11\,u^{8}v^{16} \\
&+ 10\,u^{11}v^{12} - 78\,u^{10}v^{13} + 190\,u^{9}v^{14} - 58\,u^{8}v^{15} - 2\,u^{11}v^{11} \\
&+ 32\,u^{10}v^{12} - 140\,u^{9}v^{13} + 119\,u^{8}v^{14} - 10\,u^{10}v^{11} + 77\,u^{9}v^{12} \\
&- 148\,u^{8}v^{13} + 11\,u^{7}v^{14} + 2\,u^{10}v^{10} - 32\,u^{9}v^{11} + 124\,u^{8}v^{12} \\
&- 50\,u^{7}v^{13} + 10\,u^{9}v^{10} - 74\,u^{8}v^{11} + 94\,u^{7}v^{12} - 2\,u^{9}v^{9} + 32\,u^{8}v^{10} \\
&- 100\,u^{7}v^{11} + 10\,u^{6}v^{12} - 10\,u^{8}v^{9} + 69\,u^{7}v^{10} - 40\,u^{6}v^{11} + 2\,u^{8}v^{8} \\
&- 32\,u^{7}v^{9} + 68\,u^{6}v^{10} + 10\,u^{7}v^{8} - 60\,u^{6}v^{9} + 8\,u^{5}v^{10} - 2\,u^{7}v^{7} \\
&+ 31\,u^{6}v^{8} - 32\,u^{5}v^{9} - 10\,u^{6}v^{7} + 44\,u^{5}v^{8} + 2\,u^{6}v^{6} - 28\,u^{5}v^{7} \\
&+ 7\,u^{4}v^{8} + 10\,u^{5}v^{6} - 22\,u^{4}v^{7} - 2\,u^{5}v^{5} + 23\,u^{4}v^{6} - 10\,u^{4}v^{5} \\
&+ 5\,u^{3}v^{6} + 2\,u^{4}v^{4} - 14\,u^{3}v^{5} + 9\,u^{3}v^{4} - 2\,u^{3}v^{3} + 4\,u^{2}v^{4} - 6\,u^{2}v^{3} \\
&+ 2\,u^{2}v^{2} + 2\,uv^{2} - 2\,uv + 1.
\end{aligned}
$$

**Example 2.** $g=1$, $m=2$, $\mu_1=(2,1)$, $\mu_2=(1,1,1)$, $n_1=3$, $n_2=4$.

$$
\begin{aligned}
P_{\underline{\mu},\underline{n}}(u,v) = {} & u^{38}v^{38}+3\,u^{37}v^{38}-2\,u^{37}v^{37}+6\,u^{36}v^{38}-8\,u^{36}v^{37}+9\,u^{35}v^{38}\\
&+2\,u^{36}v^{36}-20\,u^{35}v^{37}+12\,u^{34}v^{38}+11\,u^{35}v^{36}-36\,u^{34}v^{37}\\
&+15\,u^{33}v^{38}-2\,u^{35}v^{35}+32\,u^{34}v^{36}-54\,u^{33}v^{37}+18\,u^{32}v^{38}\\
&-12\,u^{34}v^{35}+67\,u^{33}v^{36}-72\,u^{32}v^{37}+21\,u^{31}v^{38}+2\,u^{34}v^{34}\\
&-38\,u^{33}v^{35}+112\,u^{32}v^{36}-90\,u^{31}v^{37}+24\,u^{30}v^{38}+12\,u^{33}v^{34}\\
&-88\,u^{32}v^{35}+162\,u^{31}v^{36}-108\,u^{30}v^{37}+27\,u^{29}v^{38}-2\,u^{33}v^{33}\\
&+41\,u^{32}v^{34}-160\,u^{31}v^{35}+213\,u^{30}v^{36}-126\,u^{29}v^{37}+30\,u^{28}v^{38}\\
&-12\,u^{32}v^{33}+100\,u^{31}v^{34}-248\,u^{30}v^{35}+264\,u^{29}v^{36}-144\,u^{28}v^{37}\\
&+30\,u^{27}v^{38}+2\,u^{32}v^{32}-42\,u^{31}v^{33}+193\,u^{30}v^{34}-342\,u^{29}v^{35}\\
&+315\,u^{28}v^{36}-162\,u^{27}v^{37}+27\,u^{26}v^{38}+12\,u^{31}v^{32}-106\,u^{30}v^{33}\\
&+314\,u^{29}v^{34}-438\,u^{28}v^{35}+366\,u^{27}v^{36}-168\,u^{26}v^{37}+24\,u^{25}v^{38}\\
&-2\,u^{31}v^{31}+42\,u^{30}v^{32}-214\,u^{29}v^{33}+451\,u^{28}v^{34}-534\,u^{27}v^{35}\\
&+414\,u^{26}v^{36}-156\,u^{25}v^{37}+18\,u^{24}v^{38}-12\,u^{30}v^{31}+109\,u^{29}v^{32}\\
&-362\,u^{28}v^{33}+594\,u^{27}v^{34}-630\,u^{26}v^{35}+438\,u^{25}v^{36}-138\,u^{24}v^{37}\\
&+12\,u^{23}v^{38}+2\,u^{30}v^{30}-42\,u^{29}v^{31}+226\,u^{28}v^{32}-538\,u^{27}v^{33}\\
&+738\,u^{26}v^{34}-714\,u^{25}v^{35}+417\,u^{24}v^{36}-108\,u^{23}v^{37}+6\,u^{22}v^{38}\\
&+12\,u^{29}v^{30}-110\,u^{28}v^{31}+395\,u^{27}v^{32}-726\,u^{26}v^{33}+879\,u^{25}v^{34}\\
&-756\,u^{24}v^{35}+366\,u^{23}v^{36}-72\,u^{22}v^{37}-2\,u^{29}v^{29}+42\,u^{28}v^{30}\\
&-232\,u^{27}v^{31}+604\,u^{26}v^{32}-918\,u^{25}v^{33}+996\,u^{24}v^{34}-726\,u^{23}v^{35}\\
&+291\,u^{22}v^{36}-36\,u^{21}v^{37}-12\,u^{28}v^{29}+110\,u^{27}v^{30}-416\,u^{26}v^{31}\\
&+835\,u^{25}v^{32}-1098\,u^{24}v^{33}+1044\,u^{23}v^{34}-630\,u^{22}v^{35}+192\,u^{21}v^{36}\\
&+2\,u^{28}v^{28}-42\,u^{27}v^{29}+235\,u^{26}v^{30}-652\,u^{25}v^{31}+1073\,u^{24}v^{32}\\
&-1236\,u^{23}v^{33}+990\,u^{22}v^{34}-492\,u^{21}v^{35}+90\,u^{20}v^{36}+12\,u^{27}v^{28}\\
&-110\,u^{26}v^{29}+428\,u^{25}v^{30}-922\,u^{24}v^{31}+1287\,u^{23}v^{32}-1278\,u^{22}v^{33}\\
&+843\,u^{21}v^{34}-312\,u^{20}v^{35}-2\,u^{27}v^{27}+42\,u^{26}v^{28}-236\,u^{25}v^{29}\\
&+685\,u^{24}v^{30}-1202\,u^{23}v^{31}+1431\,u^{22}v^{32}-1182\,u^{21}v^{33}+627\,u^{20}v^{34}\\
&-120\,u^{19}v^{35}-12\,u^{26}v^{27}+110\,u^{25}v^{28}-434\,u^{24}v^{29}+988\,u^{23}v^{30}\\
&-1440\,u^{22}v^{31}+1449\,u^{21}v^{32}-972\,u^{20}v^{33}+360\,u^{19}v^{34}+2\,u^{26}v^{26}\\
&-42\,u^{25}v^{27}+236\,u^{24}v^{28}-706\,u^{23}v^{29}+1305\,u^{22}v^{30}-1578\,u^{21}v^{31}\\
&+1302\,u^{20}v^{32}-672\,u^{19}v^{33}+90\,u^{18}v^{34}+12\,u^{25}v^{26}-110\,u^{24}v^{27}\\
&+437\,u^{23}v^{28}-1036\,u^{22}v^{29}+1551\,u^{21}v^{30}-1554\,u^{20}v^{31}+1014\,u^{19}v^{32}
\end{aligned}
$$

$$
\begin{aligned}
&- 312\, u^{18}v^{33} - 2\, u^{25}v^{25} + 42\, u^{24}v^{26} - 236\, u^{23}v^{27} + 718\, u^{22}v^{28} \\
&- 1380\, u^{21}v^{29} + 1665\, u^{20}v^{30} - 1344\, u^{19}v^{31} + 627\, u^{18}v^{32} \\
&- 36\, u^{17}v^{33} - 12\, u^{24}v^{25} + 110\, u^{23}v^{26} - 438\, u^{22}v^{27} + 1068\, u^{21}v^{28} \\
&- 1620\, u^{20}v^{29} + 1590\, u^{19}v^{30} - 972\, u^{18}v^{31} + 192\, u^{17}v^{32} + 2\, u^{24}v^{24} \\
&- 42\, u^{23}v^{25} + 236\, u^{22}v^{26} - 724\, u^{21}v^{27} + 1425\, u^{20}v^{28} - 1692\, u^{19}v^{29} \\
&+ 1302\, u^{18}v^{30} - 492\, u^{17}v^{31} + 6\, u^{16}v^{32} + 12\, u^{23}v^{24} - 110\, u^{22}v^{25} \\
&+ 438\, u^{21}v^{26} - 1086\, u^{20}v^{27} + 1644\, u^{19}v^{28} - 1554\, u^{18}v^{29} + 843\, u^{17}v^{30} \\
&- 72\, u^{16}v^{31} - 2\, u^{23}v^{23} + 42\, u^{22}v^{24} - 236\, u^{21}v^{25} + 727\, u^{20}v^{26} \\
&- 1440\, u^{19}v^{27} + 1665\, u^{18}v^{28} - 1182\, u^{17}v^{29} + 291\, u^{16}v^{30} - 12\, u^{22}v^{23} \\
&+ 110\, u^{21}v^{24} - 438\, u^{20}v^{25} + 1092\, u^{19}v^{26} - 1620\, u^{18}v^{27} + 1449\, u^{17}v^{28} \\
&- 630\, u^{16}v^{29} + 12\, u^{15}v^{30} + 2\, u^{22}v^{22} - 42\, u^{21}v^{23} + 236\, u^{20}v^{24} \\
&- 728\, u^{19}v^{25} + 1425\, u^{18}v^{26} - 1578\, u^{17}v^{27} + 990\, u^{16}v^{28} - 108\, u^{15}v^{29} \\
&+ 12\, u^{21}v^{22} - 110\, u^{20}v^{23} + 438\, u^{19}v^{24} - 1086\, u^{18}v^{25} + 1551\, u^{17}v^{26} \\
&- 1278\, u^{16}v^{27} + 366\, u^{15}v^{28} - 2\, u^{21}v^{21} + 42\, u^{20}v^{22} - 236\, u^{19}v^{23} \\
&+ 727\, u^{18}v^{24} - 1380\, u^{17}v^{25} + 1431\, u^{16}v^{26} - 726\, u^{15}v^{27} + 18\, u^{14}v^{28} \\
&- 12\, u^{20}v^{21} + 110\, u^{19}v^{22} - 438\, u^{18}v^{23} + 1068\, u^{17}v^{24} - 1440\, u^{16}v^{25} \\
&+ 1044\, u^{15}v^{26} - 138\, u^{14}v^{27} + 2\, u^{20}v^{20} - 42\, u^{19}v^{21} + 236\, u^{18}v^{22} \\
&- 724\, u^{17}v^{23} + 1305\, u^{16}v^{24} - 1236\, u^{15}v^{25} + 417\, u^{14}v^{26} + 12\, u^{19}v^{20} \\
&- 110\, u^{18}v^{21} + 438\, u^{17}v^{22} - 1036\, u^{16}v^{23} + 1287\, u^{15}v^{24} - 756\, u^{14}v^{25} \\
&+ 24\, u^{13}v^{26} - 2\, u^{19}v^{19} + 42\, u^{18}v^{20} - 236\, u^{17}v^{21} + 718\, u^{16}v^{22} \\
&- 1202\, u^{15}v^{23} + 996\, u^{14}v^{24} - 156\, u^{13}v^{25} - 12\, u^{18}v^{19} + 110\, u^{17}v^{20} \\
&- 438\, u^{16}v^{21} + 988\, u^{15}v^{22} - 1098\, u^{14}v^{23} + 438\, u^{13}v^{24} + 2\, u^{18}v^{18} \\
&- 42\, u^{17}v^{19} + 236\, u^{16}v^{20} - 706\, u^{15}v^{21} + 1073\, u^{14}v^{22} - 714\, u^{13}v^{23} \\
&+ 27\, u^{12}v^{24} + 12\, u^{17}v^{18} - 110\, u^{16}v^{19} + 437\, u^{15}v^{20} - 922\, u^{14}v^{21} \\
&+ 879\, u^{13}v^{22} - 168\, u^{12}v^{23} - 2\, u^{17}v^{17} + 42\, u^{16}v^{18} - 236\, u^{15}v^{19} \\
&+ 685\, u^{14}v^{20} - 918\, u^{13}v^{21} + 414\, u^{12}v^{22} - 12\, u^{16}v^{17} + 110\, u^{15}v^{18} \\
&- 434\, u^{14}v^{19} + 835\, u^{13}v^{20} - 630\, u^{12}v^{21} + 30\, u^{11}v^{22} + 2\, u^{16}v^{16} \\
&- 42\, u^{15}v^{17} + 236\, u^{14}v^{18} - 652\, u^{13}v^{19} + 738\, u^{12}v^{20} - 162\, u^{11}v^{21} \\
&+ 12\, u^{15}v^{16} - 110\, u^{14}v^{17} + 428\, u^{13}v^{18} - 726\, u^{12}v^{19} + 366\, u^{11}v^{20} \\
&- 2\, u^{15}v^{15} + 42\, u^{14}v^{16} - 236\, u^{13}v^{17} + 604\, u^{12}v^{18} - 534\, u^{11}v^{19} \\
&+ 30\, u^{10}v^{20} - 12\, u^{14}v^{15} + 110\, u^{13}v^{16} - 416\, u^{12}v^{17} + 594\, u^{11}v^{18} \\
&- 144\, u^{10}v^{19} + 2\, u^{14}v^{14} - 42\, u^{13}v^{15} + 235\, u^{12}v^{16} - 538\, u^{11}v^{17}
\end{aligned}
$$

$$
\begin{aligned}
&+ 315\,u^{10}v^{18} + 12\,u^{13}v^{14} - 110\,u^{12}v^{15} + 395\,u^{11}v^{16} - 438\,u^{10}v^{17} \\
&+ 27\,u^{9}v^{18} - 2\,u^{13}v^{13} + 42\,u^{12}v^{14} - 232\,u^{11}v^{15} + 451\,u^{10}v^{16} \\
&- 126\,u^{9}v^{17} - 12\,u^{12}v^{13} + 110\,u^{11}v^{14} - 362\,u^{10}v^{15} + 264\,u^{9}v^{16} \\
&+ 2\,u^{12}v^{12} - 42\,u^{11}v^{13} + 226\,u^{10}v^{14} - 342\,u^{9}v^{15} + 24\,u^{8}v^{16} \\
&+ 12\,u^{11}v^{12} - 110\,u^{10}v^{13} + 314\,u^{9}v^{14} - 108\,u^{8}v^{15} - 2\,u^{11}v^{11} \\
&+ 42\,u^{10}v^{12} - 214\,u^{9}v^{13} + 213\,u^{8}v^{14} - 12\,u^{10}v^{11} + 109\,u^{9}v^{12} \\
&- 248\,u^{8}v^{13} + 21\,u^{7}v^{14} + 2\,u^{10}v^{10} - 42\,u^{9}v^{11} + 193\,u^{8}v^{12} \\
&- 90\,u^{7}v^{13} + 12\,u^{9}v^{10} - 106\,u^{8}v^{11} + 162\,u^{7}v^{12} - 2\,u^{9}v^{9} \\
&+ 42\,u^{8}v^{10} - 160\,u^{7}v^{11} + 18\,u^{6}v^{12} - 12\,u^{8}v^{9} + 100\,u^{7}v^{10} \\
&- 72\,u^{6}v^{11} + 2\,u^{8}v^{8} - 42\,u^{7}v^{9} + 112\,u^{6}v^{10} + 12\,u^{7}v^{8} - 88\,u^{6}v^{9} \\
&+ 15\,u^{5}v^{10} - 2\,u^{7}v^{7} + 41\,u^{6}v^{8} - 54\,u^{5}v^{9} - 12\,u^{6}v^{7} + 67\,u^{5}v^{8} \\
&+ 2\,u^{6}v^{6} - 38\,u^{5}v^{7} + 12\,u^{4}v^{8} + 12\,u^{5}v^{6} - 36\,u^{4}v^{7} - 2\,u^{5}v^{5} \\
&+ 32\,u^{4}v^{6} - 12\,u^{4}v^{5} + 9\,u^{3}v^{6} + 2\,u^{4}v^{4} - 20\,u^{3}v^{5} + 11\,u^{3}v^{4} \\
&- 2\,u^{3}v^{3} + 6\,u^{2}v^{4} - 8\,u^{2}v^{3} + 2\,u^{2}v^{2} + 3uv^{2} - 2uv + 1.
\end{aligned}
$$

**Example 3.** $g = 1$, $m = 2$, $\mu_1 = \mu_2 = (2, 2)$, $n_1 = 3$, $n_2 = 4$.

$$
\begin{aligned}
P_{\underline{\mu},\underline{n}}(u, v) = {} & u^{58}v^{58} + 2\,u^{57}v^{58} - 2\,u^{57}v^{57} + 6\,u^{56}v^{58} - 6\,u^{56}v^{57} + 9\,u^{55}v^{58} \\
&+ 2\,u^{56}v^{56} - 18\,u^{55}v^{57} + 17\,u^{54}v^{58} + 9\,u^{55}v^{56} - 36\,u^{54}v^{57} \\
&+ 22\,u^{53}v^{58} - 2\,u^{55}v^{55} + 28\,u^{54}v^{56} - 68\,u^{53}v^{57} + 34\,u^{52}v^{58} \\
&- 10\,u^{54}v^{55} + 68\,u^{53}v^{56} - 108\,u^{52}v^{57} + 41\,u^{51}v^{58} + 2\,u^{54}v^{54} \\
&- 34\,u^{53}v^{55} + 136\,u^{52}v^{56} - 164\,u^{51}v^{57} + 57\,u^{50}v^{58} + 10\,u^{53}v^{54} \\
&- 90\,u^{52}v^{55} + 245\,u^{51}v^{56} - 228\,u^{50}v^{57} + 66\,u^{49}v^{58} - 2\,u^{53}v^{53} \\
&+ 37\,u^{52}v^{54} - 196\,u^{51}v^{55} + 391\,u^{50}v^{56} - 308\,u^{49}v^{57} + 86\,u^{48}v^{58} \\
&- 10\,u^{52}v^{53} + 102\,u^{51}v^{54} - 378\,u^{50}v^{55} + 591\,u^{49}v^{56} - 396\,u^{48}v^{57} \\
&+ 97\,u^{47}v^{58} + 2\,u^{52}v^{52} - 38\,u^{51}v^{53} + 238\,u^{50}v^{54} - 648\,u^{49}v^{55} \\
&+ 826\,u^{48}v^{56} - 500\,u^{47}v^{57} + 121\,u^{46}v^{58} + 10\,u^{51}v^{52} - 108\,u^{50}v^{53} \\
&+ 480\,u^{49}v^{54} - 1030\,u^{48}v^{55} + 1122\,u^{47}v^{56} - 612\,u^{46}v^{57} \\
&+ 134\,u^{45}v^{58} - 2\,u^{51}v^{51} + 38\,u^{50}v^{52} - 262\,u^{49}v^{53} + 873\,u^{48}v^{54} \\
&- 1520\,u^{47}v^{55} + 1447\,u^{46}v^{56} - 740\,u^{45}v^{57} + 162\,u^{44}v^{58} \\
&- 10\,u^{50}v^{51} + 111\,u^{49}v^{52} - 550\,u^{48}v^{53} + 1445\,u^{47}v^{54} - 2136\,u^{46}v^{55} \\
&+ 1839\,u^{45}v^{56} - 876\,u^{44}v^{57} + 169\,u^{43}v^{58} + 2\,u^{50}v^{50} - 38\,u^{49}v^{51}
\end{aligned}
$$

$$
\begin{aligned}
&+ 274\, u^{48}v^{52} - 1040\, u^{47}v^{53} + 2235\, u^{46}v^{54} - 2864\, u^{45}v^{55} \\
&+ 2254\, u^{44}v^{56} - 1028\, u^{43}v^{57} + 183\, u^{42}v^{58} + 10\, u^{49}v^{50} \\
&- 112\, u^{48}v^{51} + 594\, u^{47}v^{52} - 1790\, u^{46}v^{53} + 3242\, u^{45}v^{54} \\
&- 3720\, u^{44}v^{55} + 2742\, u^{43}v^{56} - 1150\, u^{42}v^{57} + 177\, u^{41}v^{58} \\
&- 2\, u^{49}v^{49} + 38\, u^{48}v^{50} - 280\, u^{47}v^{51} + 1152\, u^{46}v^{52} - 2864\, u^{45}v^{53} \\
&+ 4499\, u^{44}v^{54} - 4688\, u^{43}v^{55} + 3246\, u^{42}v^{56} - 1232\, u^{41}v^{57} \\
&+ 177\, u^{40}v^{58} - 10\, u^{48}v^{49} + 112\, u^{47}v^{50} - 618\, u^{46}v^{51} + 2049\, u^{45}v^{52} \\
&- 4294\, u^{44}v^{53} + 5977\, u^{43}v^{54} - 5784\, u^{42}v^{55} + 3738\, u^{41}v^{56} \\
&- 1256\, u^{40}v^{57} + 159\, u^{39}v^{58} + 2\, u^{48}v^{48} - 38\, u^{47}v^{49} + 283\, u^{46}v^{50} \\
&- 1224\, u^{45}v^{51} + 3367\, u^{44}v^{52} - 6118\, u^{43}v^{53} + 7717\, u^{42}v^{54} \\
&- 6988\, u^{41}v^{55} + 4062\, u^{40}v^{56} - 1232\, u^{39}v^{57} + 144\, u^{38}v^{58} \\
&+ 10\, u^{47}v^{48} - 112\, u^{46}v^{49} + 630\, u^{45}v^{50} - 2226\, u^{44}v^{51} + 5199\, u^{43}v^{52} \\
&- 8336\, u^{42}v^{53} + 9671\, u^{41}v^{54} - 8150\, u^{40}v^{55} + 4243\, u^{39}v^{56} \\
&- 1152\, u^{38}v^{57} + 112\, u^{37}v^{58} - 2\, u^{47}v^{47} + 38\, u^{46}v^{48} - 284\, u^{45}v^{49} \\
&+ 1268\, u^{44}v^{50} - 3746\, u^{43}v^{51} + 7582\, u^{42}v^{52} - 10968\, u^{41}v^{53} \\
&+ 11885\, u^{40}v^{54} - 9034\, u^{39}v^{55} + 4189\, u^{38}v^{56} - 1020\, u^{37}v^{57} \\
&+ 88\, u^{36}v^{58} - 10\, u^{46}v^{47} + 112\, u^{45}v^{48} - 636\, u^{44}v^{49} + 2340\, u^{43}v^{50} \\
&- 5916\, u^{42}v^{51} + 10577\, u^{41}v^{52} - 14000\, u^{40}v^{53} + 14025\, u^{39}v^{54} \\
&- 9528\, u^{38}v^{55} + 3966\, u^{37}v^{56} - 824\, u^{36}v^{57} + 56\, u^{35}v^{58} + 2\, u^{46}v^{46} \\
&- 38\, u^{45}v^{47} + 284\, u^{44}v^{48} - 1292\, u^{43}v^{49} + 4015\, u^{42}v^{50} - 8824\, u^{41}v^{51} \\
&+ 14169\, u^{40}v^{52} - 17426\, u^{39}v^{53} + 15749\, u^{38}v^{54} - 9528\, u^{37}v^{55} \\
&+ 3502\, u^{36}v^{56} - 624\, u^{35}v^{57} + 34\, u^{34}v^{58} + 10\, u^{45}v^{46} - 112\, u^{44}v^{47} \\
&+ 639\, u^{43}v^{48} - 2412\, u^{42}v^{49} + 6453\, u^{41}v^{50} - 12554\, u^{40}v^{51} \\
&+ 18404\, u^{39}v^{52} - 20824\, u^{38}v^{53} + 16718\, u^{37}v^{54} - 9020\, u^{36}v^{55} \\
&+ 2850\, u^{35}v^{56} - 416\, u^{34}v^{57} + 14\, u^{33}v^{58} - 2\, u^{45}v^{45} + 38\, u^{44}v^{46} \\
&- 284\, u^{43}v^{47} + 1304\, u^{42}v^{48} - 4194\, u^{41}v^{49} + 9817\, u^{40}v^{50} \\
&- 17134\, u^{39}v^{51} + 23193\, u^{38}v^{52} - 23538\, u^{37}v^{53} + 16823\, u^{36}v^{54} \\
&- 7964\, u^{35}v^{55} + 2128\, u^{34}v^{56} - 232\, u^{33}v^{57} + 6\, u^{32}v^{58} - 10\, u^{44}v^{45} \\
&+ 112\, u^{43}v^{46} - 640\, u^{42}v^{47} + 2456\, u^{41}v^{48} - 6842\, u^{40}v^{49} \\
&+ 14208\, u^{39}v^{50} - 22596\, u^{38}v^{51} + 28013\, u^{37}v^{52} - 25108\, u^{36}v^{53} \\
&+ 15875\, u^{35}v^{54} - 6420\, u^{34}v^{55} + 1422\, u^{33}v^{56} - 96\, u^{32}v^{57} + 2\, u^{31}v^{58} \\
&+ 2\, u^{44}v^{44} - 38\, u^{43}v^{45} + 284\, u^{42}v^{46} - 1310\, u^{41}v^{47} + 4308\, u^{40}v^{48}
\end{aligned}
$$

$$
\begin{aligned}
&- 10568\,u^{39}v^{49} + 19733\,u^{38}v^{50} - 28854\,u^{37}v^{51} + 31799\,u^{36}v^{52}\\
&- 25232\,u^{35}v^{53} + 13903\,u^{34}v^{54} - 4740\,u^{33}v^{55} + 762\,u^{32}v^{56}\\
&- 24\,u^{31}v^{57} + 10\,u^{43}v^{44} - 112\,u^{42}v^{45} + 640\,u^{41}v^{46} - 2480\,u^{40}v^{47}\\
&+ 7113\,u^{39}v^{48} - 15538\,u^{38}v^{49} + 26397\,u^{37}v^{50} - 35134\,u^{36}v^{51}\\
&+ 33951\,u^{35}v^{52} - 23668\,u^{34}v^{53} + 11017\,u^{33}v^{54} - 3080\,u^{32}v^{55}\\
&+ 294\,u^{31}v^{56} - 8\,u^{30}v^{57} - 2\,u^{43}v^{43} + 38\,u^{42}v^{44} - 284\,u^{41}v^{45}\\
&+ 1313\,u^{40}v^{46} - 4380\,u^{39}v^{47} + 11115\,u^{38}v^{48} - 21900\,u^{37}v^{49}\\
&+ 34131\,u^{36}v^{50} - 40020\,u^{35}v^{51} + 33933\,u^{34}v^{52} - 20372\,u^{33}v^{53}\\
&+ 7953\,u^{32}v^{54} - 1592\,u^{31}v^{55} + 42\,u^{30}v^{56} - 10\,u^{42}v^{43} + 112\,u^{41}v^{44}\\
&- 640\,u^{40}v^{45} + 2492\,u^{39}v^{46} - 7292\,u^{38}v^{47} + 16565\,u^{37}v^{48}\\
&- 29708\,u^{36}v^{49} + 41846\,u^{35}v^{50} - 42580\,u^{34}v^{51} + 31455\,u^{33}v^{52}\\
&- 15768\,u^{32}v^{53} + 4931\,u^{31}v^{54} - 536\,u^{30}v^{55} + 12\,u^{29}v^{56} + 2\,u^{42}v^{42}\\
&- 38\,u^{41}v^{43} + 284\,u^{40}v^{44} - 1314\,u^{39}v^{45} + 4424\,u^{38}v^{46}\\
&- 11506\,u^{37}v^{47} + 23641\,u^{36}v^{48} - 38818\,u^{35}v^{49} + 47736\,u^{34}v^{50}\\
&- 42196\,u^{33}v^{51} + 26439\,u^{32}v^{52} - 10952\,u^{31}v^{53} + 2372\,u^{30}v^{54}\\
&- 48\,u^{29}v^{55} + 10\,u^{41}v^{42} - 112\,u^{40}v^{43} + 640\,u^{39}v^{44} - 2498\,u^{38}v^{45}\\
&+ 7406\,u^{37}v^{46} - 17326\,u^{36}v^{47} + 32473\,u^{35}v^{48} - 47886\,u^{34}v^{49}\\
&+ 50467\,u^{33}v^{50} - 38400\,u^{32}v^{51} + 19799\,u^{31}v^{52} - 6372\,u^{30}v^{53}\\
&+ 648\,u^{29}v^{54} - 8\,u^{28}v^{55} - 2\,u^{41}v^{41} + 38\,u^{40}v^{42} - 284\,u^{39}v^{43}\\
&+ 1314\,u^{38}v^{44} - 4448\,u^{37}v^{45} + 11777\,u^{36}v^{46} - 25002\,u^{35}v^{47}\\
&+ 42804\,u^{34}v^{48} - 54550\,u^{33}v^{49} + 49370\,u^{32}v^{50} - 31348\,u^{31}v^{51}\\
&+ 13023\,u^{30}v^{52} - 2688\,u^{29}v^{53} + 42\,u^{28}v^{54} - 10\,u^{40}v^{41}\\
&+ 112\,u^{39}v^{42} - 640\,u^{38}v^{43} + 2501\,u^{37}v^{44} - 7478\,u^{36}v^{45}\\
&+ 17875\,u^{35}v^{46} - 34684\,u^{34}v^{47} + 53025\,u^{33}v^{48} - 57160\,u^{32}v^{49}\\
&+ 43870\,u^{31}v^{50} - 22448\,u^{30}v^{51} + 6918\,u^{29}v^{52} - 536\,u^{28}v^{53}\\
&+ 2\,u^{27}v^{54} + 2\,u^{40}v^{40} - 38\,u^{39}v^{41} + 284\,u^{38}v^{42} - 1314\,u^{37}v^{43}\\
&+ 4460\,u^{36}v^{44} - 11956\,u^{35}v^{45} + 26033\,u^{34}v^{46} - 46046\,u^{33}v^{47}\\
&+ 60161\,u^{32}v^{48} - 54904\,u^{31}v^{49} + 34540\,u^{30}v^{50} - 13752\,u^{29}v^{51}\\
&+ 2372\,u^{28}v^{52} - 24\,u^{27}v^{53} + 10\,u^{39}v^{40} - 112\,u^{38}v^{41} + 640\,u^{37}v^{42}\\
&- 2502\,u^{36}v^{43} + 7522\,u^{35}v^{44} - 18266\,u^{34}v^{45} + 36388\,u^{33}v^{46}\\
&- 57102\,u^{32}v^{47} + 62238\,u^{31}v^{48} - 47392\,u^{30}v^{49} + 23364\,u^{29}v^{50}\\
&- 6372\,u^{28}v^{51} + 294\,u^{27}v^{52} - 2\,u^{39}v^{39} + 38\,u^{38}v^{40} - 284\,u^{37}v^{41}
\end{aligned}
$$

$$
\begin{aligned}
&+ 1314\, u^{36}v^{42} - 4466\, u^{35}v^{43} + 12070\, u^{34}v^{44} - 26784\, u^{33}v^{45} \\
&+ 48532\, u^{32}v^{46} - 64342\, u^{31}v^{47} + 58388\, u^{30}v^{48} - 35648\, u^{29}v^{49} \\
&+ 13023\, u^{28}v^{50} - 1592\, u^{27}v^{51} + 6\, u^{26}v^{52} - 10\, u^{38}v^{39} \\
&+ 112\, u^{37}v^{40} - 640\, u^{36}v^{41} + 2502\, u^{35}v^{42} - 7546\, u^{34}v^{43} \\
&+ 18536\, u^{33}v^{44} - 37648\, u^{32}v^{45} + 60049\, u^{31}v^{46} - 65386\, u^{30}v^{47} \\
&+ 48614\, u^{29}v^{48} - 22448\, u^{28}v^{49} + 4931\, u^{27}v^{50} - 96\, u^{26}v^{51} \\
&+ 2\, u^{38}v^{38} - 38\, u^{37}v^{39} + 284\, u^{36}v^{40} - 1314\, u^{35}v^{41} + 4469\, u^{34}v^{42} \\
&- 12142\, u^{33}v^{43} + 27309\, u^{32}v^{44} - 50272\, u^{31}v^{45} + 66922\, u^{30}v^{46} \\
&- 59576\, u^{29}v^{47} + 34540\, u^{28}v^{48} - 10952\, u^{27}v^{49} + 762\, u^{26}v^{50} \\
&+ 10\, u^{37}v^{38} - 112\, u^{36}v^{39} + 640\, u^{35}v^{40} - 2502\, u^{34}v^{41} + 7558\, u^{33}v^{42} \\
&- 18712\, u^{32}v^{43} + 38508\, u^{31}v^{44} - 61844\, u^{30}v^{45} + 66448\, u^{29}v^{46} \\
&- 47392\, u^{28}v^{47} + 19799\, u^{27}v^{48} - 3080\, u^{26}v^{49} + 14\, u^{25}v^{50} \\
&- 2\, u^{37}v^{37} + 38\, u^{36}v^{38} - 284\, u^{35}v^{39} + 1314\, u^{34}v^{40} - 4470\, u^{33}v^{41} \\
&+ 12186\, u^{32}v^{42} - 27656\, u^{31}v^{43} + 51295\, u^{30}v^{44} - 67792\, u^{29}v^{45} \\
&+ 58388\, u^{28}v^{46} - 31348\, u^{27}v^{47} + 7953\, u^{26}v^{48} - 232\, u^{25}v^{49} \\
&- 10\, u^{36}v^{37} + 112\, u^{35}v^{38} - 640\, u^{34}v^{39} + 2502\, u^{33}v^{40} - 7564\, u^{32}v^{41} \\
&+ 18820\, u^{31}v^{42} - 39000\, u^{30}v^{43} + 62450\, u^{29}v^{44} - 65386\, u^{28}v^{45} \\
&+ 43870\, u^{27}v^{46} - 15768\, u^{26}v^{47} + 1422\, u^{25}v^{48} + 2\, u^{36}v^{36} - 38\, u^{35}v^{37} \\
&+ 284\, u^{34}v^{38} - 1314\, u^{33}v^{39} + 4470\, u^{32}v^{40} - 12210\, u^{31}v^{41} \\
&+ 27855\, u^{30}v^{42} - 51632\, u^{29}v^{43} + 66922\, u^{28}v^{44} - 54904\, u^{27}v^{45} \\
&+ 26439\, u^{26}v^{46} - 4740\, u^{25}v^{47} + 34\, u^{24}v^{48} + 10\, u^{35}v^{36} - 112\, u^{34}v^{37} \\
&+ 640\, u^{33}v^{38} - 2502\, u^{32}v^{39} + 7567\, u^{31}v^{40} - 18880\, u^{30}v^{41} \\
&+ 39158\, u^{29}v^{42} - 61844\, u^{28}v^{43} + 62238\, u^{27}v^{44} - 38400\, u^{26}v^{45} \\
&+ 11017\, u^{25}v^{46} - 416\, u^{24}v^{47} - 2\, u^{35}v^{35} + 38\, u^{34}v^{36} - 284\, u^{33}v^{37} \\
&+ 1314\, u^{32}v^{38} - 4470\, u^{31}v^{39} + 12221\, u^{30}v^{40} - 27920\, u^{29}v^{41} \\
&+ 51295\, u^{28}v^{42} - 64342\, u^{27}v^{43} + 49370\, u^{26}v^{44} - 20372\, u^{25}v^{45} \\
&+ 2128\, u^{24}v^{46} - 10\, u^{34}v^{35} + 112\, u^{33}v^{36} - 640\, u^{32}v^{37} + 2502\, u^{31}v^{38} \\
&- 7568\, u^{30}v^{39} + 18900\, u^{29}v^{40} - 39000\, u^{28}v^{41} + 60049\, u^{27}v^{42} \\
&- 57160\, u^{26}v^{43} + 31455\, u^{25}v^{44} - 6420\, u^{24}v^{45} + 56\, u^{23}v^{46} + 2\, u^{34}v^{34} \\
&- 38\, u^{33}v^{35} + 284\, u^{32}v^{36} - 1314\, u^{31}v^{37} + 4470\, u^{30}v^{38} - 12224\, u^{29}v^{39} \\
&+ 27855\, u^{28}v^{40} - 50272\, u^{27}v^{41} + 60161\, u^{26}v^{42} - 42196\, u^{25}v^{43} \\
&+ 13903\, u^{24}v^{44} - 624\, u^{23}v^{45} + 10\, u^{33}v^{34} - 112\, u^{32}v^{35} + 640\, u^{31}v^{36}
\end{aligned}
$$

$$
\begin{aligned}
&- 2502\, u^{30}v^{37} + 7568\, u^{29}v^{38} - 18880\, u^{28}v^{39} + 38508\, u^{27}v^{40} \\
&- 57102\, u^{26}v^{41} + 50467\, u^{25}v^{42} - 23668\, u^{24}v^{43} + 2850\, u^{23}v^{44} \\
&- 2\, u^{33}v^{33} + 38\, u^{32}v^{34} - 284\, u^{31}v^{35} + 1314\, u^{30}v^{36} - 4470\, u^{29}v^{37} \\
&+ 12221\, u^{28}v^{38} - 27656\, u^{27}v^{39} + 48532\, u^{26}v^{40} - 54550\, u^{25}v^{41} \\
&+ 33933\, u^{24}v^{42} - 7964\, u^{23}v^{43} + 88\, u^{22}v^{44} - 10\, u^{32}v^{33} \\
&+ 112\, u^{31}v^{34} - 640\, u^{30}v^{35} + 2502\, u^{29}v^{36} - 7568\, u^{28}v^{37} \\
&+ 18820\, u^{27}v^{38} - 37648\, u^{26}v^{39} + 53025\, u^{25}v^{40} - 42580\, u^{24}v^{41} \\
&+ 15875\, u^{23}v^{42} - 824\, u^{22}v^{43} + 2\, u^{32}v^{32} - 38\, u^{31}v^{33} + 284\, u^{30}v^{34} \\
&- 1314\, u^{29}v^{35} + 4470\, u^{28}v^{36} - 12210\, u^{27}v^{37} + 27309\, u^{26}v^{38} \\
&- 46046\, u^{25}v^{39} + 47736\, u^{24}v^{40} - 25232\, u^{23}v^{41} + 3502\, u^{22}v^{42} \\
&+ 10\, u^{31}v^{32} - 112\, u^{30}v^{33} + 640\, u^{29}v^{34} - 2502\, u^{28}v^{35} + 7567\, u^{27}v^{36} \\
&- 18712\, u^{26}v^{37} + 36388\, u^{25}v^{38} - 47886\, u^{24}v^{39} + 33951\, u^{23}v^{40} \\
&- 9020\, u^{22}v^{41} + 112\, u^{21}v^{42} - 2\, u^{31}v^{31} + 38\, u^{30}v^{32} - 284\, u^{29}v^{33} \\
&+ 1314\, u^{28}v^{34} - 4470\, u^{27}v^{35} + 12186\, u^{26}v^{36} - 26784\, u^{25}v^{37} \\
&+ 42804\, u^{24}v^{38} - 40020\, u^{23}v^{39} + 16823\, u^{22}v^{40} - 1020\, u^{21}v^{41} \\
&- 10\, u^{30}v^{31} + 112\, u^{29}v^{32} - 640\, u^{28}v^{33} + 2502\, u^{27}v^{34} - 7564\, u^{26}v^{35} \\
&+ 18536\, u^{25}v^{36} - 34684\, u^{24}v^{37} + 41846\, u^{23}v^{38} - 25108\, u^{22}v^{39} \\
&+ 3966\, u^{21}v^{40} + 2\, u^{30}v^{30} - 38\, u^{29}v^{31} + 284\, u^{28}v^{32} - 1314\, u^{27}v^{33} \\
&+ 4470\, u^{26}v^{34} - 12142\, u^{25}v^{35} + 26033\, u^{24}v^{36} - 38818\, u^{23}v^{37} \\
&+ 31799\, u^{22}v^{38} - 9528\, u^{21}v^{39} + 144\, u^{20}v^{40} + 10\, u^{29}v^{30} - 112\, u^{28}v^{31} \\
&+ 640\, u^{27}v^{32} - 2502\, u^{26}v^{33} + 7558\, u^{25}v^{34} - 18266\, u^{24}v^{35} \\
&+ 32473\, u^{23}v^{36} - 35134\, u^{22}v^{37} + 16718\, u^{21}v^{38} - 1152\, u^{20}v^{39} \\
&- 2\, u^{29}v^{29} + 38\, u^{28}v^{30} - 284\, u^{27}v^{31} + 1314\, u^{26}v^{32} - 4470\, u^{25}v^{33} \\
&+ 12070\, u^{24}v^{34} - 25002\, u^{23}v^{35} + 34131\, u^{22}v^{36} - 23538\, u^{21}v^{37} \\
&+ 4189\, u^{20}v^{38} - 10\, u^{28}v^{29} + 112\, u^{27}v^{30} - 640\, u^{26}v^{31} + 2502\, u^{25}v^{32} \\
&- 7546\, u^{24}v^{33} + 17875\, u^{23}v^{34} - 29708\, u^{22}v^{35} + 28013\, u^{21}v^{36} \\
&- 9528\, u^{20}v^{37} + 159\, u^{19}v^{38} + 2\, u^{28}v^{28} - 38\, u^{27}v^{29} + 284\, u^{26}v^{30} \\
&- 1314\, u^{25}v^{31} + 4469\, u^{24}v^{32} - 11956\, u^{23}v^{33} + 23641\, u^{22}v^{34} \\
&- 28854\, u^{21}v^{35} + 15749\, u^{20}v^{36} - 1232\, u^{19}v^{37} + 10\, u^{27}v^{28} \\
&- 112\, u^{26}v^{29} + 640\, u^{25}v^{30} - 2502\, u^{24}v^{31} + 7522\, u^{23}v^{32} \\
&- 17326\, u^{22}v^{33} + 26397\, u^{21}v^{34} - 20824\, u^{20}v^{35} + 4243\, u^{19}v^{36} \\
&- 2\, u^{27}v^{27} + 38\, u^{26}v^{28} - 284\, u^{25}v^{29} + 1314\, u^{24}v^{30} - 4466\, u^{23}v^{31}
\end{aligned}
$$

$$
\begin{aligned}
&+ 11777\, u^{22}v^{32} - 21900\, u^{21}v^{33} + 23193\, u^{20}v^{34} - 9034\, u^{19}v^{35} \\
&+ 177\, u^{18}v^{36} - 10\, u^{26}v^{27} + 112\, u^{25}v^{28} - 640\, u^{24}v^{29} + 2502\, u^{23}v^{30} \\
&- 7478\, u^{22}v^{31} + 16565\, u^{21}v^{32} - 22596\, u^{20}v^{33} + 14025\, u^{19}v^{34} \\
&- 1256\, u^{18}v^{35} + 2\, u^{26}v^{26} - 38\, u^{25}v^{27} + 284\, u^{24}v^{28} - 1314\, u^{23}v^{29} \\
&+ 4460\, u^{22}v^{30} - 11506\, u^{21}v^{31} + 19733\, u^{20}v^{32} - 17426\, u^{19}v^{33} \\
&+ 4062\, u^{18}v^{34} + 10\, u^{25}v^{26} - 112\, u^{24}v^{27} + 640\, u^{23}v^{28} - 2502\, u^{22}v^{29} \\
&+ 7406\, u^{21}v^{30} - 15538\, u^{20}v^{31} + 18404\, u^{19}v^{32} - 8150\, u^{18}v^{33} \\
&+ 177\, u^{17}v^{34} - 2\, u^{25}v^{25} + 38\, u^{24}v^{26} - 284\, u^{23}v^{27} + 1314\, u^{22}v^{28} \\
&- 4448\, u^{21}v^{29} + 11115\, u^{20}v^{30} - 17134\, u^{19}v^{31} + 11885\, u^{18}v^{32} \\
&- 1232\, u^{17}v^{33} - 10\, u^{24}v^{25} + 112\, u^{23}v^{26} - 640\, u^{22}v^{27} + 2501\, u^{21}v^{28} \\
&- 7292\, u^{20}v^{29} + 14208\, u^{19}v^{30} - 14000\, u^{18}v^{31} + 3738\, u^{17}v^{32} \\
&+ 2\, u^{24}v^{24} - 38\, u^{23}v^{25} + 284\, u^{22}v^{26} - 1314\, u^{21}v^{27} + 4424\, u^{20}v^{28} \\
&- 10568\, u^{19}v^{29} + 14169\, u^{18}v^{30} - 6988\, u^{17}v^{31} + 183\, u^{16}v^{32} \\
&+ 10\, u^{23}v^{24} - 112\, u^{22}v^{25} + 640\, u^{21}v^{26} - 2498\, u^{20}v^{27} + 7113\, u^{19}v^{28} \\
&- 12554\, u^{18}v^{29} + 9671\, u^{17}v^{30} - 1150\, u^{16}v^{31} - 2\, u^{23}v^{23} + 38\, u^{22}v^{24} \\
&- 284\, u^{21}v^{25} + 1314\, u^{20}v^{26} - 4380\, u^{19}v^{27} + 9817\, u^{18}v^{28} \\
&- 10968\, u^{17}v^{29} + 3246\, u^{16}v^{30} - 10\, u^{22}v^{23} + 112\, u^{21}v^{24} - 640\, u^{20}v^{25} \\
&+ 2492\, u^{19}v^{26} - 6842\, u^{18}v^{27} + 10577\, u^{17}v^{28} - 5784\, u^{16}v^{29} \\
&+ 169\, u^{15}v^{30} + 2\, u^{22}v^{22} - 38\, u^{21}v^{23} + 284\, u^{20}v^{24} - 1314\, u^{19}v^{25} \\
&+ 4308\, u^{18}v^{26} - 8824\, u^{17}v^{27} + 7717\, u^{16}v^{28} - 1028\, u^{15}v^{29} + 10\, u^{21}v^{22} \\
&- 112\, u^{20}v^{23} + 640\, u^{19}v^{24} - 2480\, u^{18}v^{25} + 6453\, u^{17}v^{26} - 8336\, u^{16}v^{27} \\
&+ 2742\, u^{15}v^{28} - 2\, u^{21}v^{21} + 38\, u^{20}v^{22} - 284\, u^{19}v^{23} + 1313\, u^{18}v^{24} \\
&- 4194\, u^{17}v^{25} + 7582\, u^{16}v^{26} - 4688\, u^{15}v^{27} + 162\, u^{14}v^{28} - 10\, u^{20}v^{21} \\
&+ 112\, u^{19}v^{22} - 640\, u^{18}v^{23} + 2456\, u^{17}v^{24} - 5916\, u^{16}v^{25} + 5977\, u^{15}v^{26} \\
&- 876\, u^{14}v^{27} + 2\, u^{20}v^{20} - 38\, u^{19}v^{21} + 284\, u^{18}v^{22} - 1310\, u^{17}v^{23} \\
&+ 4015\, u^{16}v^{24} - 6118\, u^{15}v^{25} + 2254\, u^{14}v^{26} + 10\, u^{19}v^{20} - 112\, u^{18}v^{21} \\
&+ 640\, u^{17}v^{22} - 2412\, u^{16}v^{23} + 5199\, u^{15}v^{24} - 3720\, u^{14}v^{25} + 134\, u^{13}v^{26} \\
&- 2\, u^{19}v^{19} + 38\, u^{18}v^{20} - 284\, u^{17}v^{21} + 1304\, u^{16}v^{22} - 3746\, u^{15}v^{23} \\
&+ 4499\, u^{14}v^{24} - 740\, u^{13}v^{25} - 10\, u^{18}v^{19} + 112\, u^{17}v^{20} - 640\, u^{16}v^{21} \\
&+ 2340\, u^{15}v^{22} - 4294\, u^{14}v^{23} + 1839\, u^{13}v^{24} + 2\, u^{18}v^{18} - 38\, u^{17}v^{19} \\
&+ 284\, u^{16}v^{20} - 1292\, u^{15}v^{21} + 3367\, u^{14}v^{22} - 2864\, u^{13}v^{23} + 121\, u^{12}v^{24} \\
&+ 10\, u^{17}v^{18} - 112\, u^{16}v^{19} + 639\, u^{15}v^{20} - 2226\, u^{14}v^{21} + 3242\, u^{13}v^{22}
\end{aligned}
$$

$$
\begin{aligned}
&- 612\,u^{12}v^{23} - 2\,u^{17}v^{17} + 38\,u^{16}v^{18} - 284\,u^{15}v^{19} + 1268\,u^{14}v^{20}\\
&- 2864\,u^{13}v^{21} + 1447\,u^{12}v^{22} - 10\,u^{16}v^{17} + 112\,u^{15}v^{18} - 636\,u^{14}v^{19}\\
&+ 2049\,u^{13}v^{20} - 2136\,u^{12}v^{21} + 97\,u^{11}v^{22} + 2\,u^{16}v^{16} - 38\,u^{15}v^{17}\\
&+ 284\,u^{14}v^{18} - 1224\,u^{13}v^{19} + 2235\,u^{12}v^{20} - 500\,u^{11}v^{21} + 10\,u^{15}v^{16}\\
&- 112\,u^{14}v^{17} + 630\,u^{13}v^{18} - 1790\,u^{12}v^{19} + 1122\,u^{11}v^{20} - 2\,u^{15}v^{15}\\
&+ 38\,u^{14}v^{16} - 284\,u^{13}v^{17} + 1152\,u^{12}v^{18} - 1520\,u^{11}v^{19} + 86\,u^{10}v^{20}\\
&- 10\,u^{14}v^{15} + 112\,u^{13}v^{16} - 618\,u^{12}v^{17} + 1445\,u^{11}v^{18} - 396\,u^{10}v^{19}\\
&+ 2\,u^{14}v^{14} - 38\,u^{13}v^{15} + 283\,u^{12}v^{16} - 1040\,u^{11}v^{17} + 826\,u^{10}v^{18}\\
&+ 10\,u^{13}v^{14} - 112\,u^{12}v^{15} + 594\,u^{11}v^{16} - 1030\,u^{10}v^{17} + 66\,u^{9}v^{18}\\
&- 2\,u^{13}v^{13} + 38\,u^{12}v^{14} - 280\,u^{11}v^{15} + 873\,u^{10}v^{16} - 308\,u^{9}v^{17}\\
&- 10\,u^{12}v^{13} + 112\,u^{11}v^{14} - 550\,u^{10}v^{15} + 591\,u^{9}v^{16} + 2\,u^{12}v^{12}\\
&- 38\,u^{11}v^{13} + 274\,u^{10}v^{14} - 648\,u^{9}v^{15} + 57\,u^{8}v^{16} + 10\,u^{11}v^{12}\\
&- 112\,u^{10}v^{13} + 480\,u^{9}v^{14} - 228\,u^{8}v^{15} - 2\,u^{11}v^{11} + 38\,u^{10}v^{12}\\
&- 262\,u^{9}v^{13} + 391\,u^{8}v^{14} - 10\,u^{10}v^{11} + 111\,u^{9}v^{12} - 378\,u^{8}v^{13}\\
&+ 41\,u^{7}v^{14} + 2\,u^{10}v^{10} - 38\,u^{9}v^{11} + 238\,u^{8}v^{12} - 164\,u^{7}v^{13}\\
&+ 10\,u^{9}v^{10} - 108\,u^{8}v^{11} + 245\,u^{7}v^{12} - 2\,u^{9}v^{9} + 38\,u^{8}v^{10}\\
&- 196\,u^{7}v^{11} + 34\,u^{6}v^{12} - 10\,u^{8}v^{9} + 102\,u^{7}v^{10} - 108\,u^{6}v^{11}\\
&+ 2\,u^{8}v^{8} - 38\,u^{7}v^{9} + 136\,u^{6}v^{10} + 10\,u^{7}v^{8} - 90\,u^{6}v^{9} + 22\,u^{5}v^{10}\\
&- 2\,u^{7}v^{7} + 37\,u^{6}v^{8} - 68\,u^{5}v^{9} - 10\,u^{6}v^{7} + 68\,u^{5}v^{8} + 2\,u^{6}v^{6}\\
&- 34\,u^{5}v^{7} + 17\,u^{4}v^{8} + 10\,u^{5}v^{6} - 36\,u^{4}v^{7} - 2\,u^{5}v^{5} + 28\,u^{4}v^{6}\\
&- 10\,u^{4}v^{5} + 9\,u^{3}v^{6} + 2\,u^{4}v^{4} - 18\,u^{3}v^{5} + 9\,u^{3}v^{4} - 2\,u^{3}v^{3} + 6\,u^{2}v^{4}\\
&- 6\,u^{2}v^{3} + 2\,u^{2}v^{2} + 2\,uv^{2} - 2\,uv + 1
\end{aligned}
$$

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