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Riemann Hypothesis for DAHA superpolynomials and plane curve singularities

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Dedicated with admiration to Yuri Ivanovich Manin on the occasion of his 80th birthday

Stable Khovanov-Rozansky polynomials of algebraic knots are expected to coincide with certain generating functions, superpolynomials, of nested Hilbert schemes and flagged Jacobian factors of the corresponding plane curve singularities. Also, these 3 families conjecturally match the DAHA superpolynomials. These superpolynomials can be considered as singular counterparts and generalizations of the Hasse-Weil zeta-functions. We conjecture that all a-coefficients of the DAHA superpolynomials upon the substitution $q \mapsto qt$ satisfy the Riemann Hypothesis for sufficiently small q for uncolored algebraic knots, presumably for $q \le 1/2$ as $a = 0$. This can be partially extended to algebraic links at least for $a = 0$. Colored links are also considered, though mostly for rectangle Young diagrams. Connections with Kapranov's motivic zeta and the Galkin-Stöhr zeta-functions are discussed.

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List of basic notations

$R=\{\alpha\}\subset\mathbb{R}^{n+1},\ \alpha_i=\epsilon_i-\epsilon_{i+1}$	root system of type A_n, simple roots
$W=\mathbb{S}_{n+1},\ P,Q,\ \Pi=P/Q$	the Weyl group, weight/root lattices
$\widetilde{R}=\{\widetilde{\alpha}=[\alpha,j]\},\ \ \alpha_0=[-\theta,1]$	affine root system, θ =maximal root
$\widetilde{W}=\langle s_{\widetilde{\alpha}}\rangle,\ \widehat{W}=W\ltimes P=\widetilde{W}\rtimes\Pi$	affine, extended Weyl groups: (2.3)
$\mathcal{H\!\!H}=\langle X_b,Y_b,T_i,\ q^{\pm\frac{1}{n+1}},t^{\pm\frac{1}{2}}\rangle$	DAHA: Definition 2.1, $Y_b\ (b\in P)$: (2.8)
$\tau_{\pm},\sigma=\tau_+\tau_-^{-1}\tau_+,\varphi:$ (2.12), $\star$	automorphisms & involutions: (2.9)
$P_b=$ Macdonald polynomials	$\mathcal{V}$ = polynomial module: Sect. 2.4.1
$\{H\}_{ev}=H(1)(t^{-\rho}),\ H\in\mathcal{H\!\!H}$	coinvariant, $H(1)=H\Downarrow\,\in\mathcal{V}$: (2.14)
$P_b^{\circ}\ =\ P_b(X)/(P_b(t^{-\rho}),b\in P_+$	spherical normalization of P_b: (2.22)
$J_\lambda=h_\lambda P_b,\ \ \lambda=\lambda(b),\ \ b\in P_+$	J-polynomials for diagrams λ: (2.24)
$h_\lambda=\prod_{\Box\in\lambda}(1-q^{arm(\Box)}t^{leg(\Box)+1})$	arm and leg numbers in : Sect. 2.5.1
$\vec{\mathsf{r}}=\{\mathsf{r}_1,\ldots,\mathsf{r}_\ell\},\ \vec{\mathsf{s}}=\{\mathsf{s}_1,\ldots,\mathsf{s}_\ell\}$	Newton's pairs, $gcd(\mathsf{r}_i,\mathsf{s}_i)=1$: (3.1)
$\mathsf{a}_1=\mathsf{s}_1,\ \ \mathsf{a}_i=\mathsf{a}_{i-1}\mathsf{r}_{i-1}\mathsf{r}_i+\mathsf{s}_i$	cable (topological) parameters: (3.2)
$\mathcal{L}\ =\ \mathcal{L}^{\Upsilon,(b^j)}_{(\vec{\mathsf{r}}^j,\vec{\mathsf{s}}^j)},\ \ \ {}'\mathcal{L}\ =\ {}'\mathcal{L}^{\,'\Upsilon,('b^j)}_{('\vec{\mathsf{r}}^j,'\vec{\mathsf{s}}^j)}$	pairs of labeled/colored graphs: (3.9)
$\widehat{JD}^{\,min}_{(\vec{\mathsf{r}}^j,\vec{\mathsf{s}}^j),('\vec{\mathsf{r}}^j,'\vec{\mathsf{s}}^j)}((b^j),('b^j);q,t)$	hat-DAHA-Jones polynomials: (3.14)
$\widehat{\mathcal{H}}^{min}_{\mathcal{L},'\mathcal{L}}(q,t,a)=\sum_{i=0}^{d}\mathcal{H}^i(q,t)a^i$	superpolynomials, $d=deg_a(\widehat{\mathcal{H}})$: (4.16)
$\widehat{H}^i(q,t)=\mathcal{H}^i(qt,t)\widehat{\ },\ \ \varsigma_i,\pi_i,S^i$	hat-normalized $\mathcal{H}^i(qt,t)$: Conject. 4.9
$\widehat{H}^i_{sym}=(H^i(q,t)+H^i(q,\frac{1}{qt}))\widehat{\ }$	hat-symmetrization of $H^i(q,t)$: (4.42)

$\widehat{\mathbb{H}}(q,t,a,u)=\sum_{m=0}^{\infty}\Big(\frac{u}{t}\Big)^{g(m)}\widehat{\mathcal{H}}_m$	family superpolynomials $(\tau_-^m\gamma_1)$: (4.21)
$\mathcal{H}_{mot}(q,t,a), \mathcal{H}^0_{mot}=\mathcal{H}_{mot}(a=0)$	flagged motivic s-polynomials: (4.34)
$\mathcal{Z}(q,t,a), \mathcal{L}(q,t,a),\ Z, L(q,t)$	flagged Galkin-Stöhr functions: (4.33)
given $\widehat{H}^i$, $\varpi_i = \min(\omega')$ s.t.	(weak) RH holds for $\omega=\frac{1}{q}>\omega'$: (4.44)
$\{\gamma[3,2]\gamma[2,1](P)\}=\{\widehat{\gamma}_{3,2}(\widehat{\gamma}_{2,1}(P)\Downarrow)\}$	coinvariants' abbreviations: Sect. 5.1.2

1. Introduction

The aim of the paper is to approach the Riemann Hypothesis, RH, for DAHA superpolynomials of algebraic links colored by Young diagrams upon the substitution $q \mapsto qt$. The parameter q, a counterpart of the cardinality of $\mathbb{F}$ in the Weil conjectures, is assumed sufficiently small, which is complementary to the classical theory. Then RH presumably holds for any a–coefficients of DAHA superpolynomials of *uncolored algebraic knots*; moreover, $q \leq 1/2$ seems sufficient when $a=0$. For links, stable (any q) *irregular* (non-RH) zeros appear. For instance, the number of their pairs is conjectured to coincide with the number of components of uncolored algebraic *links* minus 1 as $a=0$. We provide tools for finding such bounds for any Young diagrams and arbitrary a–coefficients; finding the exact RH–range of q is much more subtle, which is somewhat parallel to the theory of *spectral zeta-functions*.

Let us try to put this conjecture into perspective and explain the rationale behind it and its relations to the classical Weil conjectures.

1.1. Superpolynomials

1.1.1. Topological and geometric theories. The superpolynomials have several reincarnations in mathematics and physics; the origin is the theory of *stable Khovanov-Rozansky polynomials*, which are Poincaré polynomials of the HOMFLY-PT triply-graded link homology [Kh, KhR1]. They depend on 3 parameters q, t, a and are actually infinite series in the *unreduced* case. Generally, they are difficult to calculate and there are unsettled problems with the formulas for links, in the presence of colors and in the reduced case, though the *categorification theory* generally provides their definition for any colors (dominant weights). For uncolored unreduced algebraic links, they were conjectured to coincide with the *ORS superpolynomials*, certain generating series for *nested Hilbert schemes* of the corresponding *plane curve singularties* [ORS].

These two families in the reduced setting are conjecturally related to the *geometric superpolynomials* introduced in [ChP1, ChP2]. They were defined there for any algebraic knots colored by columns (wedge powers of the fundamental representation), developing [ChD1, GM, Gor]. Their construction is in terms of the *flagged Jacobian factors* of unibranch plane curve singularities. Jacobian factors are (indeed) factors of the corresponding *compactified Jacobians*; the definition is entirely local. In turn, Jacobian factors are almost directly related to the affine Springer fibers in type A (the nil-elliptic case), and therefore to the corresponding p–adic orbital integrals; see the end of [ChP1] for some references and discussion.

Inspired by [ORS, GORS], [ChP1, ChP2] and various prior works, especially [Kap, GSh], the *geometric superpolynomials* can be considered as "singular" analogs of the *Weil polynomials*, the numerators of the Hasse-Weil zeta-functions of smooth curves. To analyze this we switch to the 4_{th} type of superpolynomials, the DAHA ones from [Ch2, Ch3, GN] and further works; the most comprehensive paper on them by now is [ChD2].

1.1.2. DAHA superpolynomials. The DAHA superpolynomials deal with the combinatorial data of iterated torus links and allow any colors (Young diagrams). Importantly, they almost directly reflect the *topological* type of singularity, in contrast to the ORS construction and the geometric superpolynomials from [ChP1, ChP2]. We note that the latter are related to *restricted* nested Hilbert schemes of singularities, some subvarieties of those used in [ORS] (geometrically simpler). See Section 4.2.5. The DAHA superpolynomials are expected to be connected with physics superpolynomials [DGR, AS, GS, DMS, FGS], which is not discussed in this work.

They are defined so far for *iterated torus links*. All initial *intrinsic* conjectures from [Ch2] concerning torus knots were proved (but the *positivity* discussed below) and extended to any colored iterated torus links (not only algebraic). The theory of DAHA-Jones polynomials is very much uniform for any colors (dominant weights) and root systems; the DAHA superpolynomials are in type A.

In other approaches, the limitations and practical problems are more significant, especially with links and colors. The ORS polynomials are generally difficult to calculate since they are based on the *weight filtration* in cohomology of the corresponding nested Hilbert schemes. The *KhR polynomials* are known only for simple knots. For torus knots, they were recently calculated using Soergel bimodules [Mel2]; see also Corollary 3.4 there, which proves Conjecture 2.7 (*ii*) (uncolored) from [Ch2]; not all details are provided in [Mel2], but the proof seems essentially a direct identification of Gorsky's combinatorial formulas with those in [Ch2]. See also [EH, Hog] concerning $T(m\mathsf{r} \pm 1, \mathsf{r})$ and links $T(m\mathsf{r}, \mathsf{r})$. The geometric superpolynomials and *flagged Jacobian factors* from [ChP1, ChP2] are relatively simple to define, but this is done so far only for algebraic links colored by columns.

1.2. DAHA and Weil conjectures

Let us present the main features of the theory of DAHA superpolynomials as analogs of the Weil conjectures. This connection is mostly heuristic and our *RH* mostly serves the sector of $q < 1$, complementary to Weil's *RH*. The key point of the whole work is the identification of t in DAHA superpolynomials with T in the singular counterpart of the Weil zeta-function.

1.2.1. Polynomiality and super-duality. First of all, the DAHA theory and the geometric construction from [ChP1, ChP2] *directly* provide superpolynomials, counterparts of the Weil $P_1(T)$. This is in contrast to the classical theory, where $P_1(T)$ appears due to the rationality theorem:

$$\zeta(X,T) \stackrel{\mathbf{def}}{=\!=} \exp\left(\sum_{n=1}^{\infty} T^n\, |X(\mathbb{F}_{q^n})|/n\right) = \frac{P_1(T)}{(1-T)(1-qT)},$$

where X is a smooth projective curve over a finite field $\mathbb{F} = \mathbb{F}_q$ with q elements and P_1 is of degree $2g$ for the genus g of X (its smoothness is actually needed only for *RH*).

The coefficients of DAHA superpolynomials are presumably all positive for rectangle Young diagrams and algebraic knots (the positivity conjectures from [Ch2, ChD1]), which is not present in the Weil-Deligne theory [Del1, Del2]. Such a positivity hints at a possible geometric interpretation and "categorification" of these polynomials. However, they can be positive for "very" non-algebraic knots. For links and non-rectangle Young diagrams, the positivity holds only upon the division by some powers of $(1-t), (1-q)$. For instance, $(1-t)^{\kappa-1}$ is presumably sufficient for uncolored links with κ components.

To avoid misunderstanding, let us emphasize that the positivity of DAHA superpolynomials for knots (and for links upon the division above) is neither necessary nor sufficient for the validity of the corresponding *RH*. However when such positivity holds, one may expect a geometric interpretation of

DAHA superpolynomials as in [ChP1, ChP2], which actually implies *RH* for sufficiently small q.

Thus, a counterpart of the existence of $P_1(T)$ is the polynomiality for *DAHA-Jones polynomials* from Theorem 1.2,[Ch3] (for torus knots, any colors, and root systems) and its generalizations to iterated torus links. The passage to the superpolynomials, Theorem 1.3 there, was announced in [Ch2] (based on [SV]); its complete proof was provided in [GN]. See [ChD2] for the most general version. We provide here many examples of algebraic links and the corresponding DAHA procedures, significantly developing and extending those from [ChD1, ChD2].

Accordingly, the *super-duality* of DAHA superpolynomials matches *the Weil's functional equation*: $\zeta(X, q^{-1}T^{-1}) = q^{1-g}T^{2-2g}\zeta(X,T)$. It was conjectured in [Ch2] (let us mention prior [GS] in the context of physics superpolynomials) and proved in [GN] on the basis of the $q \leftrightarrow t$–duality of the modified Macdonald polynomials. An alternative approach to the proof via roots of unity and the generalized *level-rank duality* was presented in [Ch3]; it can work for classical root systems and some other families. The proof of the duality from Proposition 3 from [ORS] is parallel to that of the motivic functional equation [Kap]; see also Section 6 in [Gal], Section 3 in [Sto] and formula (4.29) below concerning the functional equation for the Galkin-Stöhr functions.

We note that our parameter a and adding colors (numerically, we mostly consider rectangle Young diagrams) do not have direct origins in the theory of Hasse-Weil zeta-functions. The parameter a is associated with *flagged* Jacobian factors in [ChP1, ChP2]; we also define flagged Galkin-Stöhr functions by considering *standardizable* flags of ideals.

1.2.2. Riemann Hypothesis. For large q, corresponding to the cardinality $|\mathbb{F}|$, *RH* for our superpolynomials and the zeta-functions from [Gal, Sto] generally fails. However the inequality $q \leq 1/2$ is (surprisingly) sufficient for uncolored *algebraic* knots at $a=0$ we considered; $\mathbb{F}[[z^4, z^6 + z^{7+2m}]]$ presumably make $1/2$ sharp here as $m \to \infty$. For uncolored algebraic *links*, the number of stable pairs of irregular zeros is conjectured to be the number of components minus 1 as $a = 0$. Adding colors to knots and links is more subtle, though rectangle Young diagrams satisfy *RH* for sufficiently small q upon the symmetrization at least for $a = 0$ in all examples we considered. We note that *RH* can hold for non-algebraic links too, but algebraic links, more generally *positively iterated torus links*, are a major class for *RH* (for small q).

Deviations from the classical theory. Focusing on the sector $0<q<1$ is a significant deviation. Even for q close to 0, *irregular* zeros generally appear. For instance, one pair (always real) of such zeros is conjectured to occur for uncolored algebraic links with 2 components.

The main change is of course that q is *arbitrary* real in the DAHA approach. It is a counterpart of $|\mathbb{F}|$ in the Weil conjectures, but this is just a parameter for us. Accordingly, we calculate minimal ϖ such that *RH* holds for all $\omega \stackrel{\mathbf{def}}{=} 1/q > \varpi$. Also, we add colors and one more parameter a due to the flagged Jacobian factors (or Hilbert schemes), which is a clear extension of the Weil conjectures.

The field with one element is of importance. It corresponds to $q=t$ in the DAHA parameters (which becomes $q = 1$ after the substitution $q \mapsto qt$), and describes the HOMFLY-PT polynomials in topology. However, the bound ϖ is generally beyond 1; *RH* is not expected to hold for $q = t$ for sufficiently general non-torus knots. We note that $t = 1$ can be also interpreted as the case of "field with one element"; see [ChP1, ChP2].

1.2.3. On the structure of the paper. The motivic Conjecture 4.5 is the best we have to clarify the meaning of the substitution $q \mapsto qt$. Finding π_i and polynomials S^i from Conjecture 4.9 is an entirely algebraic procedure and for any Young diagrams; these invariants are expected to be meaningful topologically.

Conjecture 4.10, the "qualitative *RH*", for small q is actually a corollary of Conjecture 4.9. Conjecture 4.11, the "quantitative *RH*", gives a bound for q in the case of uncolored algebraic knots.

If the motivic superpolynomials are known and coincide with the DAHA superpolynomials (which is conjectured), then the validity of *RH* for sufficiently small q (sufficiently large ω), i.e. Conjectures 4.9,4.10, can be checked by an entirely algebraic and relatively straightforward procedure. For uncolored algebraic knots, the geometric superpolynomials were defined in [ChP1] and they do satisfy Conjecture 4.9, which almost directly follows from their definition in terms of the corresponding flagged Jacobian factors.

The motivic superpolynomials at $a = 0$ are quite close to the Hasse-Weil zeta-function for singular curves and the Galkin-Stöhr (local) zeta-functions. The exact relation is Conjecture 4.5; Conjecture 4.7 is its generalization to any a. Adding colors is more subtle; the motivic superpolynomials of algebraic links are known by now for (any) columns. The DAHA superpolynomials are well developed for any Young diagrams.

1.3. Motivic approach

Let us discuss some arithmetic-geometric details. Connecting DAHA superpolynomials with the numerators of the Hasse-Weil zeta-functions seems *a priori* some stretch, but we think that the following chain of steps provides a sufficiently solid link.

1.3.1. Kapranov's zeta. The first step is the Kapranov zeta-function of a smooth algebraic curve C of genus g over a field k. It is defined via the classes of $[C^{[n]}]$ of the n-fold symmetric products of C in the Grothendieck ring $K_0(Var/k)$ of varieties over k. The motivic zeta-function of C from [Kap] is then a formal series $\zeta(C,u) \stackrel{\mathbf{def}}{=\!=} \sum_{n\ge 0} u^n[C^{[n]}]$. Here one can replace $[C^{[n]}]$ by $\mu([C^{[n]}])$ for any *motivic measure* μ. If $k=\mathbb{F}=\mathbb{F}_q$ and $\mu(X)$ is the number of $\mathbb{F}$–points of X, then this is the classical presentation of the *Hasse-Weil zeta-function* for $u=T$.

Theorem 1.1.9 from [Kap] establishes the first two Weil conjectures (the rationality and the functional equation) in the motivic setting. The justification is close to the Artin's proof in the case of $\mathbb{F}$.

One can then extend the definition of motivic zeta to reduced *singular* curves C, replacing $C^{[n]}$ by the corresponding Hilbert schemes of n points on C, subschemes of length n to be exact. Let us assume now (and later) that C is a *rational* planar projective reduced curve of arithmetic genus δ. Then Conjecture 17 from [GSh] states that in $K_0(Var/\mathbb{C})$:

$$\sum_{n\ge 0} u^{n+1-\delta}[C^{[n]}] = \sum_{0\le i\le\delta} \mathcal{N}_c(i)\left(\frac{u}{(1-u)(1-u[\mathbb{A}^1])}\right)^{i+1-\delta}, \tag{1.1}$$

where $\mathcal{N}_C^{(i)}\in\mathbb{Z}_+[\mathbb{A}^1]$. To be exact, this is stated for any reduced curve, not only rational, the total arithmetic genus g must be then used in the left-hand side instead of δ and the resulting expression must be divided by the same series for the normalization of C (calculated by Macdonald for smooth C). In the right-hand side, we must set $i+1-\delta\mapsto i-\delta$. This substitution is due to the Macdonald formula for $\mathbb{P}^1$.

1.3.2. Nested Hilbert schemes. When the classes $[X]$ are replaced by their (topological) Euler numbers $e(X)$, we arrive at

$$\sum_{n\ge 0} u^{n+1-\delta}e(C^{[n]}) = \sum_{0\le i\le\delta} n_C(i)\left(\frac{u}{(1-u)^2}\right)^{i+1-\delta}. \tag{1.2}$$

The rationality here was motivated by Gopakumar and Vafa (via the BPS invariants) and justified in [PaT]. The positivity of $n_C^{(i)}$ was deduced from the approach based on versal deformations: [FGvS] for $i=\delta$ and then (for arbitrary i) in [Sh].

The OS-conjecture [ObS], (extended and) proved in [Ma], is a geometric interpretation and an a-generalization of (1.2) for rational planar curves C and their *nested* Hilbert schemes $C^{[n\le n+m]}$. It is actually a local formula and one can switch from a rational curve C to its *germ* $\mathcal{C}$ at the singular point under consideration. Then $\mathcal{C}^{[n\le n+m]}=\{I_{n+m}\subset I_n \mid \mathfrak{m}I_n\subset I_{n+m}\}$, where $I_n\in \mathcal{C}^{[n]}$ and $\mathfrak{m}$ is the maximal ideal in the (local) ring of $\mathcal{C}$.

1.3.3. ORS polynomials. Let $\mathcal{C}$ be an arbitrary plane curve singularity of arithmetic genus δ (its Serre number); the Hilbert schemes are defined correspondingly. Considering the construction above for $K_0(Var/\mathbb{F})$ and then applying the motivic integration from Example 1.3.2b from [Kap] is essentially what was suggested in [ORS] (for nested Hilbert schemes of punctual pairs). The reduced ORS polynomial is

$$\mathscr{P}_{\text{alg}}=\Big(\frac{q_{st}}{a_{st}}\Big)^{\mu}\,\frac{1-q_{st}^2}{1+a_{st}^2t_{st}}\sum_{n,m\ge 0} q_{st}^{2n}\,a_{st}^{2m}\,t_{st}^{m^2}\,\mathfrak{w}(\mathcal{C}^{[n\le n+m]}). \tag{1.3}$$

Here μ is the Milnor number ($\mu=2\delta$ in the unibranch case) and $\mathfrak{w}$ is the *weight filtration* in the compactly supported cohomology of the corresponding scheme. See the Overview and Section 4 in [ORS]; Proposition 3 there contains the functional equation. And also see Section 9.1 from [GORS]. The parameter t_{st} is associated with this filtration, we put q_{st},t_{st},a_{st} here to distinguish these parameters from the DAHA parameters (below). They are really *standard* in quite a few topological-geometric papers; see (1.4) below. For $t_{st}=1$, $u=q_{st}$ and at the minimal possible degree of a_{st}, the sum in (1.3) essentially reduces to the right-hand side of (1.2).

Conjecture 2 of [ORS] states that $\mathscr{P}_{\text{alg}}$ coincides with the *reduced* stable Khovanov-Rozansky polynomial of the corresponding link, the Poincarè polynomials of the triply-graded HOMFLY-PT homology. Accordingly, its unreduced version $\overline{\mathscr{P}}_{\text{alg}}$ is related to the unreduced stable KhR polynomial. The problem with the identification of the ORS and KhR polynomials is that the number of examples is very limited in both theories. Though see [Mel2] concerning the stable KhR polynomials for torus knots via Soergel bimodules.

1.4. DAHA approach

The following two families of superpolynomials are much more explicit and calculatable. *DAHA superpolynomials* are the key for us; their full definition will be provided in this paper. They were defined in [Ch2] for torus knots, triggered by [AS], and extended to any iterated torus links in further papers.

1.4.1. Geometric superpolynomials. The connection of the DAHA superpolynomials to geometry of plane curve singularities goes through the *geometric superpolynomials*, a class of superpolynomials introduced in [ChP1, ChP2], which generalizes [ChD1], [GM] (for $a = 0$ and torus knots) and Gorsky's approach from [Gor] (a combinatorial theory for torus knots and any powers of a). *Flagged Jacobian factors* were used in [ChP1] instead of the nested Hilbert schemes in [ORS]; though see Section 4.2.5 where *restricted* nested Hilbert schemes emerge (the pairs of ideals that become trivial upon tensoring with the normalization ring).

The geometric superpolynomials do not require root systems at and the *a–stabilization* of the *DAHA-Jones-WRT polynomials*. They are uniformly defined for any root systems, but their a–stabilization naturally requires classical series A, B, C, D. However see [ChE] for some superpolynomials for the Deligne-Gross exceptional family [DG]. Presumably they exist for any root systems and their construction is geometric, in terms of the corresponding *spectral curve*.

The relation between the DAHA superpolynomials and geometric superpolynomials for algebraic knots from [ChP1, ChP2] is confirmed in many examples and is verifiable theoretically. One can connect the standard (monoidal) transformations of the plane curve singularities with the corresponding recurrence relations in the DAHA theory.

1.4.2. ORS polynomials vs. geometric ones. Our uncolored geometric superpolynomials are parallel to $\mathscr{P}_{\text{alg}}$ from [ORS]. Upon the division by $(1-t)$ there is a conjectural connection with those. See Sections 4.2.4 and 4.2.5 below. They can be expressed is terms of the weight filtration too via a theorem due to N. Katz; see the end of [ChP1]. The DAHA parameter q (or t^{-1} due to the super-duality) is a counterpart of cardinality of $\mathbb{F}$ there. Accordingly, $q = 1$ is the case of the "field with one element", which leads to the DAHA theory at critical center charge for $q = 1$. When t^{-1} is used, the case $t = 1$ is the so-called "free theory". This is different from the usage of the weight filtration in [ORS], where $|\mathbb{F}|$ corresponds to $q_{st}^2 = q/t$. Generally, the DAHA parameters a, q, t are connected with those in (1.3) as follows:

$$
\begin{aligned}
&t = q_{st}^2,\ q = (q_{st}t_{st})^2,\ a = a_{st}^2 t_{st},\\
&q_{st}^2 = t,\ t_{st} = \sqrt{\frac{q}{t}},\ a_{st}^2 = a\sqrt{\frac{t}{q}}.
\end{aligned}
\tag{1.4}
$$

In the ORS polynomials, the motivic measure and finite fields vanish at $t_{st} = 1$, i.e. at $q = t$ in the DAHA parameters, which is quite different from $q = 1$ in geometric superpolynomials. Also, the normalization of singularities is heavily used for the *flagged Jacobian factors* and our construction does not require ideals and Hilbert schemes, though the Galkin-Stöhr zeta-functions are directly related to the latter.

Let us mention here that colors and arbitrary root systems (available in the DAHA approach) are a challenge from the geometric-motivic perspective. However the *OS conjecture* (the case $t = q$ in DAHA parameters) was established and proved in [Ma] for *any* colors (Young diagrams) and algebraic links using *non-reduced* singularities.

1.4.3. Galkin-Stöhr zeta. We will consider here only the case $a = 0$. The motivic measure will be the count of points over a finite field $\mathbb{F}$ of cardinality q. With some simplifications, the zeta-function from [Gal, Sto] is defined as the sum $Z_{\mathcal{R}}(t)$ of $t^{dim_{\mathbb{F}}(\mathcal{R}/\mathfrak{a})}$ over all ideals $\mathfrak{a} \subset \mathcal{R}$ for the local ring $\mathcal{R}$ of a curve singularity. This ring is assumed Gorenstein to ensure the functional equation for $(1-t)Z_{\mathcal{R}}(t)$, which is for the substitution $t \mapsto 1/(tq)$. The function $Z_{\mathcal{R}}(t)$ is a (local) version of the Weil zeta for singularities. It is quite natural from number theoretical viewpoint, but the corresponding *RH* generally fails. However if we treat q as a variable (which is q/t in the DAHA parameters), then $q \le 1/2$ is presumably sufficient for *RH* in the unibranch case at $a=0$.

The connection with our geometric superpolynomials can be stated as some "combinatorial" identity, which seems not straightforward to check; for instance, it generally holds only for *planar* singularities (not arbitrary Gorenstein rings). Also, the positivity of the coefficients of $(1-t)Z_{\mathcal{R}}(t)$, which follows from this connection, generally requires plane curve singularities. The formula from [Sto] has many positive and negative terms canceling each other in a non-trivial way. We provide examples of *non-plane* curve singularities where the connection with our formula can be fixed, but some non-trivial adjustments are needed for this. Importantly, the functional equation for $(1-t)Z_{\mathcal{R}}(t)$ is not difficult to established (see [Sto]) and we do not see any *direct* proof of super-duality for our geometric (motivic) superpolynomials, without the passage to ideals.

1.4.4. Modular periods. The construction of DAHA superpolynomials is actually parallel to that for modular periods, a starting point for p–adic measures (Mazur, Manin, Katz, eigenvarieties), and for Manin's zeta-polynomials [OnRS]. Namely, the DAHA-Jones polynomials of torus knots $T(\mathsf{r},\mathsf{s})$ colored by Young diagrams λ are essentially obtained by applying $\gamma_{\mathsf{r},\mathsf{s}} = \begin{pmatrix} \mathsf{r} & * \\ \mathsf{s} & * \end{pmatrix} \in PSL(2,\mathbb{Z})$ to the Macdonald polynomials P_λ followed by taking the DAHA coinvariant. The superpolynomials are due to the a–stabilization for the root systems A_n; this is important, since the super-duality holds only upon the a–stabilization.

This procedure is analogous to taking the integral of a cusp form $\Phi(z)$ for $z \in \mathbb{H}$ multiplied by z^k for certain integers k over the paths $\gamma[0, \imath\infty]$. Here z^k can be seen (with some stretch) as counterparts of P_λ, the integration $\int\{\cdot\}\Phi(y)dy$ plays the role of the *coinvariant*.

The latter obviously has nothing to do with modular forms. However, it is the simplest level-one coinvariant among all DAHA coinvariants of arbitrary levels $\ell > 0$ from [ChM]. They are in one-to-one correspondence with elliptic functions of level ℓ (Looijenga functions for any root systems). Using them makes these constructions closer to each other. Vice versa, a challenge is to find modular counterparts of the DAHA-Jones polynomials and superpolynomials for iterated non-torus algebraic knots. It is not impossible that they can be related to Manin's iterated Shimura integrals [Man].

This is connected with the following (heuristic) interpretation of the Dirichlet L–functions of conductor r via the *families* $T(\mathsf{r},*)$. The sums of DAHA superpolynomials over the knots in such families are supposed to be considered. In a more conceptual way, one applies $\sum_{m=1}^{\infty} \chi(m) \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$ inside the coinvariant for the corresponding Dirichlet character χ. The q–analogs of zeta and Dirichlet functions from [Ch4] are of this kind, but for a different action of $PSL(2,\mathbb{Z})$. They are the integrals of $\sum_{m=1}^{\infty} \chi(m) q^{mx^2/2}$ for the Gaussian $q^{x^2/2}$ with respect to the Macdonald measure for A_1. See also Section 3 in [ChD2].

2. Double Hecke algebras

2.1. Affine root systems

Let us adjust the standard DAHA definitions to the case of the root systems A_n, which is $R = \{\alpha = \varepsilon_i - \varepsilon_j, i \neq j\}$ for the basis $\{\varepsilon_i, 1 \leq i \leq n+1\} \in \mathbb{R}^{n+1}$, orthonormal with respect to the usual euclidean form $(\cdot,\cdot)$. The Weyl group is $W = \mathbb{S}_{n+1}$; it is generated by the reflections (transpositions) s_α for the set of positive roots $R_+ = \{\varepsilon_i - \varepsilon_j, i < j\}$; $R_- = -R_+$. The simple roots are

$\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. The weight lattice is $P = \oplus_{i=1}^n \mathbb{Z}\omega_i$, where $\{\omega_i\}$ are fundamental weights: $(\omega_i, \alpha_j) = \delta_{ij}$. Explicitly,

$$\omega_i = \varepsilon_1 + \cdots + \varepsilon_i - \frac{i}{n+1}(\varepsilon_1 + \cdots + \varepsilon_{n+1}) \quad \text{for} \quad i = 1, \ldots, n, \tag{2.1}$$
$$\rho = \omega_1 + \cdots + \omega_n = \frac{1}{2}\big((n-1)\varepsilon_1 + (n-3)\varepsilon_2 + \cdots + (1-n)\varepsilon_n\big).$$

The root lattice is denoted by $Q = \oplus_{i=1}^n \mathbb{Z}\alpha_i$. Replacing $\mathbb{Z}$ by $\mathbb{Z}_\pm = \{m \in \mathbb{Z}, \pm m \geq 0\}$, we obtain $P_\pm, Q_\pm$. See e.g., [Bo] or [Ch1].

The vectors $\widetilde{\alpha} = [\alpha, j] \in \mathbb{R}^{n+2}$ for $\alpha \in R, j \in \mathbb{Z}$ form the *affine root system* $\widetilde{R} \supset R$, where $\alpha \in R$ are identified with $[\alpha, 0]$. We add $\alpha_0 \stackrel{\mathbf{def}}{=\!=} [-\theta, 1]$ to the simple roots for the *maximal root* $\theta = \varepsilon_1 - \varepsilon_{n+1}$. The corresponding set $\widetilde{R}_+$ of positive roots is $R_+ \cup \{[\alpha, j],\ \alpha \in R,\ j > 0\}$.

2.1.1. Affine Weyl group. Given $\widetilde{\alpha} = [\alpha, j] \in \widetilde{R},\ b \in P$, let

$$s_{\widetilde{\alpha}}(\widetilde{z}) \ = \ \widetilde{z} - (z, \alpha^\vee)\widetilde{\alpha}, \quad b'(\widetilde{z}) \ = \ [z, \zeta - (z, b)] \tag{2.2}$$

for $\widetilde{z} = [z, \zeta] \in \mathbb{R}^{n+2}$. The *affine Weyl group* $\widetilde{W} = \langle s_{\widetilde{\alpha}}, \widetilde{\alpha} \in \widetilde{R}_+ \rangle$ is the semidirect product $W \ltimes Q$ of its subgroups $W = \langle s_\alpha, \alpha \in R_+ \rangle$ and Q, where α is identified with

$$s_\alpha s_{[\alpha,1]} = s_{[-\alpha,1]} s_\alpha \quad \text{for} \quad \alpha \in R.$$

The *extended Weyl group* $\widehat{W}$ is $W \ltimes P$, where the action is

$$(wb)([z, \zeta]) \ = \ [w(z), \zeta - (z, b)] \quad \text{for} \quad w \in W, b \in P. \tag{2.3}$$

It is isomorphic to $\widetilde{W} \rtimes \Pi$ for $\Pi \stackrel{\mathbf{def}}{=\!=} P/Q$. The latter group consists of $\pi_0 = \mathrm{id}$ and the images π_r of ω_r in P/Q. Note that π_r^{-1} is π_{r^ι}, where ι is the standard involution of the *non-affine* Dynkin diagram of R, induced by $\alpha_i \mapsto \alpha_{n+1-i}$. Generally, we set $\iota(b) = -w_0(b) = b^\iota$, where w_0 is the longest element in W sending $\{1, 2, \ldots, n+1\}$ to $\{n+1, \ldots, 2, 1\}$.

The group Π is naturally identified with the subgroup of $\widehat{W}$ of the elements of the length zero; the *length* is defined as follows:

$$l(\widehat{w}) = |\Lambda(\widehat{w})| \quad \text{for} \quad \Lambda(\widehat{w}) \stackrel{\mathbf{def}}{=\!=} \widetilde{R}_+ \cap \widehat{w}^{-1}(-\widetilde{R}_+).$$

One has $\omega_r = \pi_r u_r$ for $1 \leq r \leq n$, where u_r is the element $u \in W$ of *minimal* length such that $u(\omega_r) \in P_-$, equivalently, $w = w_0 u$ is of *maximal* length such that $w(\omega_r) \in P_+$. The elements u_r are very explicit. Let w_0^r be the longest element in the subgroup $W_0^r \subset W$ of the elements preserving ω_r; this subgroup is generated by simple reflections. One has:

$$u_r = w_0 w_0^r \text{ and } (u_r)^{-1} = w_0^r w_0 = u_{r^\iota} \text{ for } 1 \le r \le n. \tag{2.4}$$

Setting $\widehat{w} = \pi_r \widetilde{w} \in \widehat{W}$ for $\pi_r \in \Pi$, $\widetilde{w} \in \widetilde{W}$, $l(\widehat{w})$ coincides with the length of any reduced decomposition of $\widetilde{w}$ in terms of the simple reflections s_i, $0 \le i \le n$. Thus, indeed, Π is a subgroup of $\widehat{W}$ of the elements of length 0.

2.2. Definition of DAHA

We follow [Ch3, Ch2, Ch1]. Let $\mathsf{m}=n+1$; generally it is the least natural number such that $(P,P) = (1/\mathsf{m})\mathbb{Z}$. The *double affine Hecke algebra, DAHA*, of type A_n depends on the parameters q,t and will be defined over the ring $\mathbb{Z}_{q,t} \stackrel{\mathbf{def}}{=} \mathbb{Z}[q^{\pm 1/\mathsf{m}}, t^{\pm 1/2}]$ formed by polynomials in terms of $q^{\pm 1/\mathsf{m}}$ and $t^{\pm 1/2}$. Note that the coefficients of the Macdonald polynomials will belong to $\mathbb{Q}(q,t)$.

It is convenient to use the following notation:

$$t = q^k, \quad \rho_k \stackrel{\mathbf{def}}{=} \frac{k}{2}\sum_{\alpha>0}\alpha = k\sum_{i=1}^{n}\omega_i.$$

We set $m_{i\,i+1} = 3 = m_{n0}$ for $0 \le i \le n$ and $m_{ij} = 2$ otherwise, generally, $\{2,3,4,6\}$ when the number of links between α_i, α_j in the affine Dynkin diagram is $\{0,1,2,3\}$.

For pairwise commutative $X_1, \ldots, X_n$,

$$X_{\widetilde{b}} \stackrel{\mathbf{def}}{=} \prod_{i=1}^{n} X_i^{l_i} q^j \text{ if } \widetilde{b} = [b,j],\ \widehat{w}(X_{\widetilde{b}}) = X_{\widehat{w}(\widetilde{b})}, \tag{2.5}$$
$$\text{where } b = \sum_{i=1}^{n} l_i\omega_i \in P,\ j \in \frac{1}{\mathsf{m}}\mathbb{Z},\ \widehat{w} \in \widehat{W}.$$

For instance, $X_0 \stackrel{\mathbf{def}}{=} X_{\alpha_0} = qX_\theta^{-1}$.

Definition 2.1. The double affine Hecke algebra $\mathcal{H\!\!H}$ is generated over $\mathbb{Z}_{q,t}$ by the elements $\{T_i,\ 0 \le i \le n\}$, pairwise commutative $\{X_b,\ b \in P\}$ satisfying (2.5) and the group Π, where the following relations are imposed:

(o) $(T_i - t^{1/2})(T_i + t^{-1/2}) = 0,\ 0 \le i \le n;$

(i) $T_iT_jT_i... \;=\; T_jT_iT_j...,\ m_{ij}$ factors on each side;
(ii) $\pi_rT_i\pi_r^{-1} \;=\; T_j$ if $\pi_r(\alpha_i)=\alpha_j$;
(iii) $T_iX_b \;=\; X_bX_{\alpha_i}^{-1}T_i^{-1}$ if $(b,\alpha_i)=1,\ 0\le i\le n$;
(iv) $T_iX_b \;=\; X_bT_i$ when $(b,\alpha_i)=0$ for $0\le i\le n$;
(v) $\pi_rX_b\pi_r^{-1}=X_{\pi_r(b)} \;=\; X_{u_r^{-1}(b)}q^{(\omega_{\iota(r)},b)},\ 1\le r\le n.$

Given $\widetilde{w}\in\widetilde{W}, 1\le r\le n,$ the product

$$T_{\pi_r\widetilde{w}} \stackrel{\mathbf{def}}{=\!=} \pi_rT_{i_l}\cdots T_{i_1},\quad\text{where }\ \widetilde{w}=s_{i_l}\cdots s_{i_1}\ \text{ for }\ l=l(\widetilde{w}), \tag{2.6}$$

does not depend on the choice of the reduced decomposition of $\widetilde{w}$. Moreover,

$$T_{\widehat{v}}T_{\widehat{w}} \;=\; T_{\widehat{v}\widehat{w}}\ \text{ whenever }\ l(\widehat{v}\widehat{w})=l(\widehat{v})+l(\widehat{w})\ \text{ for }\ \widehat{v},\widehat{w}\in\widehat{W}. \tag{2.7}$$

In particular, we arrive at the pairwise commutative elements

$$Y_b \stackrel{\mathbf{def}}{=\!=} \prod_{i=1}^n Y_i^{l_i}\ \text{ if }\ b=\sum_{i=1}^n l_i\omega_i\in P,\ Y_i \stackrel{\mathbf{def}}{=\!=} T_{\omega_i}, b\in P. \tag{2.8}$$

When acting in the polynomial representation $\mathcal{V}$ (see below), they are called *difference Dunkl operators.*

2.3. The automorphisms

The following maps can be (uniquely) extended to automorphisms of $\mathcal{HH}$, where $q^{1/(2\mathsf{m})}$ must be added to $\mathbb{Z}_{q,t}$ (see [Ch1], (3.2.10)–(3.2.15)):

$$\tau_+:\ X_b\mapsto X_b,\ T_i\mapsto T_i\,(i>0),\quad Y_r\mapsto X_rY_rq^{-\frac{(\omega_r,\omega_r)}{2}}, \tag{2.9}$$
$$\tau_+:\ T_0\mapsto q^{-1}X_\theta T_0^{-1},\ \pi_r\mapsto q^{-\frac{(\omega_r,\omega_r)}{2}}X_r\pi_r\ (1\le r\le n),$$
$$\tau_-:\ Y_b\mapsto Y_b,\ T_i\mapsto T_i\,(i\ge 0),\quad X_r\mapsto Y_rX_rq^{\frac{(\omega_r,\omega_r)}{2}}, \tag{2.10}$$
$$\tau_-(X_\theta)=qT_0X_\theta^{-1}T_{s_\theta}^{-1};\quad \sigma \stackrel{\mathbf{def}}{=\!=} \tau_+\tau_-^{-1}\tau_+ \;=\; \tau_-^{-1}\tau_+\tau_-^{-1},$$
$$\sigma(X_b)=Y_b^{-1},\ \sigma(Y_b)=T_{w_0}^{-1}X_{b^\iota}^{-1}T_{w_0},\ \sigma(T_i)=T_i(i>0). \tag{2.11}$$

These automorphisms fix $t,\ q$ and their fractional powers, as well as the following *anti-involution*:

$$\varphi:\ X_b\mapsto Y_b^{-1}, Y_b\mapsto X_b^{-1},\ T_i\mapsto T_i\ (1\le i\le n). \tag{2.12}$$

The following anti-involution results directly from the group nature of the DAHA relations:

$$H^\star = H^{-1} \;\text{ for }\; H\in\{T_{\widehat{w}}, X_b, Y_b, q, t\}. \tag{2.13}$$

To be exact, it is naturally extended to the fractional powers of q, t:

$$\star:\; t^{\frac{1}{2}} \mapsto t^{-\frac{1}{2}},\; q^{\frac{1}{2m}} \mapsto q^{-\frac{1}{2m}}.$$

This anti-involution serves the inner product in the theory of the DAHA polynomial representation.

Let us list the matrices corresponding to the automorphisms and anti-automorphisms above upon the natural projection onto $SL_2(\mathbb{Z})$, corresponding to $t^{\frac{1}{2}} = 1 = q^{\frac{1}{2m}}$. The matrix $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ will then represent the map $X_b \mapsto X_b^\alpha Y_b^\gamma, Y_b \mapsto X_b^\beta Y_b^\delta$ for $b\in P$. One has:

$$\tau_+ \rightsquigarrow \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix},\; \tau_- \rightsquigarrow \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix},\; \sigma \rightsquigarrow \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\; \varphi \rightsquigarrow \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}.$$

The *projective* $PSL_2(\mathbb{Z})$ (due to Steinberg) is the group generated by $\tau_\pm$ subject to the relation $\tau_+\tau_-^{-1}\tau_+ = \tau_-^{-1}\tau_+\tau_-^{-1}$. The notation will be $PSL_2^\wedge(\mathbb{Z})$; it is isomorphic to the braid group B_3.

2.3.1. The coinvariant. The projective $PSL_2(\mathbb{Z})$ and the *coinvariant*, to be defined now, are the main ingredients of our approach.

Any $H\in\mathcal{H\!H}$ can be uniquely represented in the form

$$H = \sum_{a,w,b} c_{a,w,b}\, X_a T_w Y_b \;\text{ for }\; w\in W, a, b\in P$$

(the DAHA-PBW theorem, see [Ch1]). Using this presentation, the *coinvariant* is a functional $\mathcal{H\!H}\to\mathbb{C}$ defined as follows:

$$\{\,\}_{ev}:\; X_a \;\mapsto\; q^{-(\rho_k,a)},\quad Y_b \;\mapsto\; q^{(\rho_k,b)},\quad T_i \;\mapsto\; t^{1/2}. \tag{2.14}$$

The main symmetry of the coinvariant is

$$\{\,\varphi(H)\,\}_{ev} = \{\,H\,\}_{ev} \;\text{ for }\; H\in\mathcal{H\!H}. \tag{2.15}$$

Also, $\{\,\iota(H)\,\}_{ev} = \{\,H\,\}_{ev}$, where we extend ι to $\mathcal{H\!H}$ as follows:

$$\iota(X_b){=}X_{\iota(b)},\quad \iota(Y_b){=}Y_{\iota(b)},\quad T_i^\iota{=}T_{\iota(i)},\quad 1\le i\le n. \tag{2.16}$$

The following interpretation of the coinvariant is important. For any $H \in \mathcal{HH}$, one has: $\{HT_wYb\}_{ev} = \{H\}_{ev}\,\chi(T_wY_b)$, where χ is the standard character (one-dimensional representation) of the affine Hecke algebra $\mathcal{H}_Y$, generated by T_w, Y_b for $w \in W, b \in P$; χ sends $Y_b \mapsto q^{(\rho_k, b)}$ and $T_i \mapsto t^{1/2}$. Therefore $\{\ldots\}_{ev}$ acts via the projection $H \mapsto H\Downarrow \stackrel{\textbf{def}}{=\!=} H(1)$ of $\mathcal{HH}$ onto the *polynomial representation* $\mathcal{V}$, which is the $\mathcal{HH}$–module induced from χ; see [Ch1, Ch2, Ch3] and the next section.

2.4. Macdonald polynomials

2.4.1. Polynomial representation. It is isomorphic to $\mathbb{Z}_{q,t}[X_b]$ as a vector space with the action of $T_i (0 \le i \le n)$ given by the *Demazure-Lusztig operators*:

$$T_i = t^{1/2}s_i \; + \; (t^{1/2} - t^{-1/2})(X_{\alpha_i} - 1)^{-1}(s_i - 1), \quad 0 \le i \le n. \tag{2.17}$$

The elements X_b become the multiplication operators and $\pi_r (1 \le r \le n)$ act via the general formula $\widehat{w}(X_b) = X_{\widehat{w}(b)}$ for $\widehat{w} \in \widehat{W}$. Note that τ_- naturally acts in the polynomial representation. See formula (1.37) from [Ch3], which is based on the identity

$$\tau_-(H\Downarrow) \;=\; \tau_-(H)\Downarrow \;=\; \Big(\tau_-\big(H\big)\Big)(1) \;\text{ for }\; H \in \mathcal{HH}. \tag{2.18}$$

Symmetric Macdonald polynomials. The standard notation is $P_b(X)$ for $b \in P_+$; see [Mac, Ch1] (they are due to Kadell for the classical root systems and due to Rogers for A_1). The usual definition is as follows. Let c_+ be such that $c_+ \in W(c) \cap P_+$ (it is unique); recall that $Q_+ = \oplus_{i=1}^n \mathbb{Z}_+\alpha_i$. For $b \in P_+$, the following are the defining relations:

$$P_b - \sum_{a \in W(b)} X_a \in \oplus_{c_+ \in b - Q_+}^{c_+ \neq b}\, \mathbb{Q}(q,t)X_c \;\text{ and }\; \langle\, P_b\, X_{c^\iota}\, \mu(X;q,t)\,\rangle = 0 \;\text{ for}$$

$$\text{all } c \text{ in } \oplus \text{ above; } \;\mu(X;q,t) \stackrel{\textbf{def}}{=\!=} \prod_{\alpha \in R_+} \prod_{j=0}^{\infty} \frac{(1-X_\alpha q^j)(1-X_\alpha^{-1}q^{j+1})}{(1-X_\alpha tq^j)(1-X_\alpha^{-1}tq^{j+1})}\,.$$

Here and further $\langle f\rangle$ is the *constant term* of a Laurent series or polynomial $f(X)$; μ is considered a Laurent series of X_b with the coefficients expanded in terms of positive powers of q. The coefficients of P_b belong to the field

$\mathbb{Q}(q,t)$. One has (see (3.3.23) from [Ch1]):

$$P_b(X^{-1}) \,=\, P_{b^\iota}(X) \,=\, P_b^\star(X),\ \ P_b(q^{-\rho_k}) = P_b(q^{\rho_k}) \tag{2.19}$$

$$= (P_b(q^{-\rho_k}))^\star = q^{-(\rho_k,b)} \prod_{\alpha>0} \prod_{j=0}^{(\alpha,b)-1} \Bigl(\frac{1-q^j\, t\, X_\alpha(q^{\rho_k})}{1-q^j X_\alpha(q^{\rho_k})}\Bigr). \tag{2.20}$$

Recall that $\iota(b) = b^\iota = -w_0(b)$ for $b \in P$.

DAHA provides an important alternative (operator) approach to the P–polynomials; namely, they satisfy the (defining) relations

$$L_f(P_b) = f(q^{-\rho_k - b})P_b,\ L_f \stackrel{\mathbf{def}}{=\!=} f(X_a \mapsto Y_a) \tag{2.21}$$

for any symmetric (W–invariant) polynomial $f \in \mathbb{C}[X_a, a \in P]^W$. Here $b \in P_+$ and the coefficient of X_b in P_b is assumed 1.

Spherical normalization. We call $P_b^\circ \stackrel{\mathbf{def}}{=\!=} P_b/P_b(q^{-\rho_k})$ *spherical Macdonald polynomials* for $b \in P_+$. One has (the evaluation theorem):

$$P_b(q^{-\rho_k}) = q^{-(\rho_k,b)} \prod_{\alpha>0}^{(\alpha,b)>0} \prod_{j=0}^{(\alpha,b)-1} \Bigl(\frac{1-q^j t X_\alpha(q^{\rho_k})}{1-q^j X_\alpha(q^{\rho_k})}\Bigr). \tag{2.22}$$

2.5. *J*-polynomials

They are necessary below for managing algebraic links (spherical polynomials P_b° are sufficient for knots) and are important for the justification of the *super-duality*.

For $b = \sum_{i=1}^n b_i\omega_i \in P_+$, i.e. for a *dominant* weight with $b_i \ge 0$ for all i, the corresponding *Young diagram* is as follows:

$$\begin{aligned} &\lambda = \lambda(b) = \{\lambda_1 = b_1 + \cdots + b_n, \lambda_2 = b_2 + \cdots + b_n, \ldots, \lambda_n = b_n\}, \\ &b = \sum_{i=1}^n \lambda_i\varepsilon_i - \frac{|\lambda|}{n+1}\bigl(\varepsilon_1 + \cdots + \varepsilon_{n+1}\bigr) \ \text{ for } \ |\lambda| \stackrel{\mathbf{def}}{=\!=} \sum_{i=1}^n \lambda_i. \end{aligned} \tag{2.23}$$

One has: $(b, \varepsilon_i - \varepsilon_j) = b_i + \cdots + b_{j-1} = \lambda_i - \lambda_j$, $(b,\rho) = (|\lambda| - \lambda_1)/2$. Also, $b^2 \stackrel{\mathbf{def}}{=\!=} (b,b) = \sum_{i=1}^n \lambda_i^2 - |\lambda|^2/(n+1)$.

Let us calculate the set of all $[\alpha, j]$ in the product from (2.22); it is

$$\{\,[\alpha,j],\ \alpha = \varepsilon_l - \varepsilon_m \in R_+,\ j > 0 \ \mid\ b_l + \cdots + b_{m-1} > j > 0\}.$$

2.5.1. Their definition. The *J–polynomials* are as follows:

$$J_\lambda \stackrel{\textbf{def}}{=\!=} h_\lambda P_b \ \text{ for } \ \lambda=\lambda(b),\ h_\lambda=\prod_{\square\in\lambda}(1-q^{arm(\square)}t^{leg(\square)+1}); \tag{2.24}$$

they are q,t–integral.

Here $arm(\square)$ is the *arm number*, which is the number of boxes in the same row as $\square$ strictly after it; $leg(\square)$ is the *leg number*, which is the number of boxes in the column of $\square$ strictly below it. This is for the standard presentation of λ: $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_{n-1}\geq\lambda_n$ are the numbers of boxes in the corresponding rows and the $i_{\rm th}$ row is above the $(i+1)_{\rm th}$.

Equivalently:

$$J_\lambda=t^{-(\rho,b)}\prod_{p=1}^{n}\prod_{j=0}^{\lambda_{p^*}-1}\Bigl(1-q^jt^{p+1}\Bigr)P_b^\circ,\ \ p^*=n-p+1,\ b\in P_+. \tag{2.25}$$

See, for instance, Theorem 2.1 from [GN]. Note that the *arms and legs* do not appear in the latter presentation (in terms of P_b°). In this approach, counterparts of J–polynomials exist for any root systems, but there are some deviations. See [ChD2], Section 2.6.

2.5.2. Stabilization. The following formula is important:

$$J_\lambda(t^{-\rho})=(a^2)^{-\frac{\lambda_1}{4}}\,t^{\frac{|\lambda|}{2}}\prod_{p=1}^{n}\prod_{j=0}^{\lambda_p-1}\Bigl(1+q^j\,a\,t^{-p+1}\Bigr)\ \text{ for }\ a=-t^{n+1}. \tag{2.26}$$

More generally, we have the following a–stabilization; see [ChD2]. We note that the geometric/motivic superpolynomials are defined without any A_n (and further stabilization), but so far they are known only for columns $(b=m\omega_1)$.

Proposition 2.2. *Given two Young diagrams λ and μ, the values $P_\lambda(q^{\mu+\rho_k})$ are a–stable, which means that there is a universal expression in terms of q,t,a such that its value at $a=-t^{n+1}$ coincides with $P_b(q^{c+\rho_k})$ for $\lambda=\lambda(b),\mu=\lambda(c)$, $b,c\in P_+$ for A_n with n no smaller than the number of rows in λ and in μ. Up to powers of $a^{1/2}$ and $t^{1/2}$, they are rational function in q,t,a. Also, $P^\circ_\lambda(q^{\mu+\rho_k})$, $\langle P_\lambda,P_\lambda\rangle$ and $\langle P^\circ_\lambda,P^\circ_\lambda\rangle$ are a–stable (in the same sense).*

3. DAHA-Jones theory

3.1. Iterated torus knots

We will first introduce the data necessary in the construction of DAHA-Jones polynomials and DAHA superpolynomials of algebraic iterated torus *knots*.

3.1.1. Newton pairs. The (algebraic) torus knots $T(\mathsf{r},\mathsf{s})$ are defined for any positive integers assuming that $\gcd(\mathsf{r},\mathsf{s})=1$. One has the symmetry $T(\mathsf{r},\mathsf{s})=T(\mathsf{s},\mathsf{r})$, where we use "=" for the ambient isotopy equivalence. Also $T(\mathsf{r},\mathsf{s})=\bigcirc$ if $\mathsf{r}=1$ or $\mathsf{s}=1$ for the *unknot* $\bigcirc$. Here and below see e.g., [RJ, EN, ChD1] for details and/or Knot Atlas for the corresponding invariants.

Following [ChD1], the $[\mathsf{r},\mathsf{s}]$-*presentation* of an *iterated torus knots* (defined below) will be $\mathcal{T}(\vec{\mathsf{r}},\vec{\mathsf{s}})$ for any two sequences of positive integers:

$$\vec{\mathsf{r}}=\{\mathsf{r}_1,\ldots,\mathsf{r}_\ell\},\ \vec{\mathsf{s}}=\{\mathsf{s}_1,\ldots,\mathsf{s}_\ell\} \text{ such that } \gcd(\mathsf{r}_i,\mathsf{s}_i)=1; \tag{3.1}$$

ℓ will be called the *length* of $\vec{\mathsf{r}},\vec{\mathsf{s}}$. The pairs $[\mathsf{r}_i,\mathsf{s}_i]$ can be interpreted as *characteristic* or *Newton pairs* in terms of plane curve singularities. The necessary and sufficient condition for being algebraic is $\mathsf{r}_i,\mathsf{s}_i>0$, which will be imposed in this paper unless stated otherwise.

3.1.2. Cabling parameters. The above parameters are the ones needed in the DAHA approach. However they are not optimal for establishing the symmetries of our polynomials and the justification that our construction depends only on the corresponding knot/link. We need the *cable presentation* for this. It requires one more sequence of integers:

$$\mathsf{a}_1=\mathsf{s}_1,\quad \mathsf{a}_i=\mathsf{a}_{i-1}\mathsf{r}_{i-1}\mathsf{r}_i+\mathsf{s}_i\quad (i=2,\ldots,m). \tag{3.2}$$

See e.g., [EN]. In terms of the *cabling* discussed below, the corresponding knots are as follows. First, $T(\mathsf{r},\mathsf{s})=\mathit{Cab}(\mathsf{s},\mathsf{r})(\bigcirc)$ (note that we transpose r,s here), and then we set:

$$\mathcal{T}(\vec{\mathsf{r}},\vec{\mathsf{s}})\rightsquigarrow \mathit{Cab}(\vec{\mathsf{a}},\vec{\mathsf{r}})(\bigcirc)=\big(\mathit{Cab}(\mathsf{a}_\ell,\mathsf{r}_\ell)\cdots \mathit{Cab}(\mathsf{a}_2,\mathsf{r}_2)\big)\big(T(\mathsf{r}_1,\mathsf{s}_1)\big). \tag{3.3}$$

Knots and links will be considered up to *ambient isotopy*; we use "=" for it. The *cabling* $\mathit{Cab}(\mathsf{a},\mathsf{b})(K)$ of any oriented knot K in (oriented) $\mathbb{S}^3$ is defined as follows; see e.g., [Mo, EN] and references therein. We consider a small 2–dimensional torus around K and put there the torus knot $T(\mathsf{a},\mathsf{b})$

in the direction of K, which is $Cab(\mathsf{a},\mathsf{b})(K)$ (up to ambient isotopy); we set $Cab(\vec{\mathsf{a}},\vec{\mathsf{r}}) \stackrel{\mathbf{def}}{=\!=} Cab(\vec{\mathsf{a}},\vec{\mathsf{r}})(\bigcirc)$.

This procedure depends on the order of a,b and orientation of K. We choose them in the standard way: the parameter a gives the number of turns around K. This construction also depends on the *framing* of the cable knots; we take the natural one, associated with the parallel copy of the torus where a given cable knot sits (its parallel copy has zero linking number with this knot).

3.2. From knots to links

Switching to links, we need to add *colors* to the cables above, which are dominant weights b. For knots, there is always one color, so it gives an extra (external) piece of information on top of the topological data above. Now adding colors becomes an internal part of the definition.

3.2.1. Graphs and labels. The [r,s]-presentation of a *iterated torus link* will be a union of κ *colored knots*

$$\mathcal{L}^{\Upsilon,\,(b^j)}_{(\vec{\mathsf{r}}^j,\vec{\mathsf{s}}^j)} = \Bigl(\{\mathcal{T}(\vec{\mathsf{r}}^j,\vec{\mathsf{s}}^j),\, b^j \in P_+\},\, j=1,\ldots,\kappa\Bigr) \text{ together with} \tag{3.4}$$
the incidence matrix $\Upsilon = (\upsilon_{j,k})$, where $0 \le \upsilon_{j,k} = \upsilon_{k,j} \le \min\{\ell^j,\ell^k\}$ implies that $[\mathsf{r}^j_i,\mathsf{s}^j_i]=[\mathsf{r}^k_i,\mathsf{s}^k_i]$ for all $1 \le i \le \upsilon_{j,k}$ and any $1 \le j,k \le \kappa$.

Here ℓ^j is the length of $\vec{\mathsf{r}}^j = \{\mathsf{r}^j_i\}$ and $\vec{\mathsf{s}}^j = \{\mathsf{s}^j_i\}$ for $1 \le j \le \kappa$; we naturally set $\upsilon_{j,j} = \ell^j$.

Thus Υ determines a graph $\mathcal{L}$ with the vertices $\{i,j\}$ identified as in (3.4). The *paths* are sequences of increasing consecutive i–vertices with fixed j; their orientation is from i to $i+1$. The vertices for neighboring i in the same path j are naturally connected by the *edges*. This graph is a disjoint union of trees. Any its subtree contains a unique *initial vertex* $i=i_0$ such that the i–indices are the distances in this subtree from i_0 one plus i_0. Every subtree has at least one *base path*, the one that intersects all other paths in this subtree.

For $i \le \ell^j$, the pairs $[\mathsf{r}^j_i,\mathsf{s}^j_i]$ become labels, called [r,s]-*labels* of the *vertices* i,j of $\mathcal{L}$; the square brackets will be used for them.

Additionally, we put *arrowheads* at the end of *every* path (which is at the vertex with $i=\ell^j$). The paths with coinciding vertices but different arrowheads will be treated as different paths. So j are the indices of all different maximal paths in $\mathcal{L}$ including the arrowheads at their ends. The colors b^j

will be assigned to the arrowheads. Topologically, the j_{th} path corresponds to the knot $\mathcal{T}(\vec{\mathsf{r}}^j,\vec{\mathsf{s}}^j)$ colored by $b^j \in P_+$ (later, by the corresponding Young diagrams λ^j). If a graph contains no vertices; then it is a collection of paths that are pure arrowheads (a set of colors).

3.2.2. Topological parameters. The a–parameters above must be now calculated along the paths exactly as we did for the knots (i.e. starting from $i=1, \mathsf{a}_1=\mathsf{s}_1$); then a^j_i depends only on the corresponding vertex. The pairs $\{\mathsf{a}^j_i, \mathsf{r}^j_i\}$ will be called the *cab-labels* of the vertices. Actually only the $[\mathsf{r},\mathsf{s}]$-labels will be needed in the DAHA constructions; we will call them simply *labels* (and use $[\, ,\,]$ only for them). However the cab-labels are necessary for understanding the topological symmetries.

The torus knot colored by $b \in P_+$ (or by the corresponding λ) is denoted by $T^b_{\mathsf{r},\mathsf{s}}$; $Cab^b_{\mathsf{a},\mathsf{r}}(\mathbf{L})$, equivalently $Cab^b_{0,1}Cab_{\mathsf{a},\mathsf{r}}(\mathbf{L})$, is the cable $Cab(\mathsf{a},\mathsf{r})(\mathbf{L})$ of a link $\mathbf{L}$ colored by b. The color is attached to the last Cab in the sequence of cables. In the absence of vertices, the notation is $\bigcirc^b$ (the unknot colored by $b \in P_+$) or $Cab(0,1)^b$. *We will use the same notation $\mathcal{L}$ for the graph and the corresponding link* $\mathbf{L}$.

The passage from the $[\mathsf{r},\mathsf{s}]$-*presentation* to the *cab-presentation* is

$$\mathcal{L}\big(\vec{\mathsf{r}}^j,\vec{\mathsf{s}}^j,\, 1\le j\le \kappa\big) \rightsquigarrow \left(\coprod_{j=1}^{\kappa} Cab(\vec{\mathsf{a}}^j,\vec{\mathsf{r}}^j)\right)(\bigcirc), \tag{3.5}$$

where the composition and coproduct of cables is with respect to the graph structure and $Cab(\vec{\mathsf{a}}^j,\vec{\mathsf{r}}^j) = \cdots Cab(\mathsf{a}^j_2,\mathsf{r}^j_2)T(\mathsf{r}^j_1,\mathsf{s}^j_1)$ is as in (3.3). In this work, the coproduct symbol $\coprod$, which stands for the union of cables, will be simply replaced by comma; we set $\big(Cab(\mathsf{a},\mathsf{r}), Cab(\mathsf{a}',\mathsf{r}')\big)$ instead of $Cab(\mathsf{a},\mathsf{r})\coprod Cab(\mathsf{a}',\mathsf{r}')$. The a–parameters are calculated as above along the corresponding paths. See [ChD2] for details and (many) examples. Actually, we do not need much the topological aspects in this work; the graphs are sufficient for the DAHA construction.

3.3. Pairs of graphs

This extension is necessary to incorporate *all* iterated torus links (in this work, all *algebraic* links); see [ChD2] for details. Let $\{\mathcal{L}, {}'\mathcal{L}\}$ be a pair of labeled graphs defined above.

3.3.1. Twisted union. The cabling construction provides a canonical embedding of the iterated torus links into the solid torus. Their *twisted* *union* for the pair $\{\mathcal{L}, {}'\!\mathcal{L}\}$ is as follows: we put the links for $\mathcal{L}$ and ${}'\!\mathcal{L}$ into the horizontal solid torus and the complementary vertical one.

Since we consider only *algebraic* links, we will always change here the natural orientation of the second component by the opposite one. Without this switch, the resulting link is never algebraic. For instance, $\big\{\mathcal{L} = \{\circ \to \Box\}, {}'\!\mathcal{L} = \{\circ \to \Box\}\big\}$ represents uncolored Hopf 2–links with the linking number $lk = +1$, which is $\{\circ \to \Box, \circ \to \Box^\vee\}$ in [ChD2].

Thus the pairs $\{\mathcal{L}, {}'\!\mathcal{L}\}$ *in this work are actually* $\{\mathcal{L}, {}'\!\mathcal{L}^\vee\}$ *in the notation from [ChD2]*; we consider only such pairs (with $\vee$) here. We note that $\{\mathcal{L}, {}'\!\mathcal{L}\}$ and $\{{}'\!\mathcal{L}, \mathcal{L}\}$ result in isotopic links; this corresponds to formula (4.24) from [ChD2] in the DAHA setting.

3.3.2. Positivity conditions. Arbitrary algebraic links can be obtained using this construction including the *twisted union* for the pairs of positive algebraic graphs subject to the inequality ${}'\mathsf{s}_1\mathsf{s}_1 > {}'\mathsf{r}_1\mathsf{r}_1$ for any pairs of the first vertices of these two graphs. Then $\{\mathcal{L}, {}'\!\mathcal{L}\}$ is called a *positive pair*. See e.g. [New] [ChD2]. These inequalities are imposed upon the full usage of the symmetries of the corresponding *splice diagram*. In the absence of vertices or if there is only one vertex with $\mathsf{r}_1 = 1$, the pair $\{\mathsf{r}_1, \mathsf{s}_1\}$ is technically treated as $\{0, 1\}$ for $\mathcal{L}$ (or for ${}'\!\mathcal{L}$, with primes), so the inequality above holds tautologically. Recall that $Cab(0,1)L = L$ for any link L (it is a path along the link). Note that the transposition of r_1 and s_1 (only in the first pair!) does not change the isotopy type of the corresponding component, but this may influence the resulting twisted union. Let us comment on it.

Let ${}'\!\mathcal{L}$ be a pure arrow (no vertices) colored by λ and $\mathcal{L}$ is a positive graph. Then the pair $\{\mathcal{L},{}'\!\mathcal{L}\}$ is positive and it corresponds to adding "the meridian" colored by λ to $\mathcal{L}$. The meridian and its linking number with $\mathcal{L}$ (always positive) obviously may change when $\mathsf{r}_1, \mathsf{s}_1$ are transposed. In the DAHA construction, this pair will be treated as taking the coinvariant of $P_\lambda(X_b \mapsto Y_b)$ applied to the *pre-polynomial* associated to $\mathcal{L}$. If the color here is $\lambda = \Box$ (common in our numerical examples), then $P_\Box$ is the $\mathbb{S}_{n+1}$–orbit sum of X_{ω_1}.

See [EN, New] and [ChD2] for details. The theory in [EN] is without colors, as well as that in [ObS, ORS]. Attaching colors to the branches can be incorporated topologically using framed links, but this requires more involved combinatorial definitions (and some usage of *the skein*). Colors are natural in the DAHA construction, but colored DAHA superpolynomials are of course more complicated than the uncolored ones. See also [Ma, ChP2].

3.3.3. Algebraic links. We provide here only basic facts; see [EN] for details and references, especially Theorem 9.4 there. Generally, one begins with a polynomial equation $f(x,y)=0$ considered in a neighborhood of an isolated singularity $0=(x=0,y=0)$. Its intersection with a small 3-dimensional sphere in $\mathbb{C}^2$ around 0 is called an *algebraic link*. Assuming that $\mathsf{r}_i^j,\mathsf{s}_i^j>0$, any *labeled graph* $\mathcal{L}=\mathcal{L}^{\Upsilon}_{(\vec{\mathsf{r}}^j,\vec{\mathsf{s}}^j)}$ (in the [r,s]-presentation, without colors) corresponds to a germ of *plane curve singularity* at 0. If these inequalities hold, the graph is called *positive*. Using positive pairs $\{\mathcal{L},{}'\mathcal{L}\}$ provide all of them.

The corresponding (germs of) singularities are unions of *unibranch* components corresponding for the paths of Υ (numbered by j), which are given as follows:

$$y=c_1^j\,x^{\mathsf{s}_1^j/\mathsf{r}_1^j}\Big(1+c_2^j\,x^{\mathsf{s}_2^j/(\mathsf{r}_1^j\mathsf{r}_2^j)}\Big(1+c_3^j\,x^{\mathsf{s}_3^j/(\mathsf{r}_1^j\mathsf{r}_2^j\mathsf{r}_3^j)}(\ldots)\Big)\Big)\ \text{ at }\ 0. \tag{3.6}$$

The numbers r,s are obtained from the corresponding labels; the parameters $c_i^j\in\mathbb{C}$ must be sufficiently general here. The simplest example is the equation $y^{\mathsf{r}\kappa}=x^{\mathsf{s}\kappa}$ under $\gcd(\mathsf{r},\mathsf{s})=1$, which corresponds to the torus link $T(\mathsf{r}\kappa,\mathsf{s}\kappa)$ with κ knot components isotopic to $T(\mathsf{r},\mathsf{s})$. The pairwise linking numbers here are all equal to $\mathsf{r}\,\mathsf{s}$ in this case.

The unibranch components and their (pairwise) linking numbers uniquely determine the corresponding germ due to the Reeve theorem; see e.g. [EN]. All linking numbers must be strictly positive for algebraic links. The DAHA constructions works for any (not only positive) labels. The above discussion is of course in the absence of colors.

3.4. DAHA-Jones polynomials

They can be uniformly defined for any (reduced, irreducible) root systems R; for its twisted affinization $\widetilde{R}$, to be exact. We need only $R=A_n$ in this work. The P,J–polynomials and the necessary DAHA tools are from the previous sections.

3.4.1. Data and ingredients. The combinatorial data will be the [r,s]-labeled graphs $\mathcal{L}^{\Upsilon,(b^j)}_{(\vec{\mathsf{r}}^j,\vec{\mathsf{s}}^j)}$ from (3.4) and their pairs. Recall that

$$1\le j\le\kappa,\quad \vec{\mathsf{r}}^j=\{\mathsf{r}_i^j\},\quad \vec{\mathsf{s}}^j=\{\mathsf{s}_i^j\},\quad 1\le i\le\ell^j,$$

and Υ is the *incidence graph/matrix*; the *arrowheads* (at the ends of all *paths*) are colored by $b^j\in P_+$. The incidence graph is not supposed to be connected here and the paths can contain no vertices; see (3.4). The construction below will be for *two* arbitrary such graphs $\mathcal{L}$, ${}'\mathcal{L}$ (the second can be empty).

In the case of algebraic *knots*, spherical polynomials P°_λ are sufficient; one obviously has $J^\circ_\lambda \stackrel{\mathbf{def}}{=\!=} J_\lambda/J_\lambda(t^\rho) = P^\circ_\lambda$. Generally (for links), we need the J–polynomials; see (2.24). For the latter reference, let $\lambda^j = \lambda(b^j)$ for dominant b^j. We set:

$$(3.7)\qquad (b^1,\ldots,b^m)^J_{ev} = (\lambda^1,\ldots,\lambda^m)^J_{ev} = LCM\bigl(J_{\lambda^1}(t^\rho),\ldots,J_{\lambda^m}(t^\rho)\bigr),$$

where LCM is normalized by the condition that it is a q,t–polynomial with the constant term 1.

One has the following combinatorially transparent formula:

$$(3.8)\qquad (\lambda^1,\ldots,\lambda^m)^J_{ev} \;=\; (\lambda^1\vee\cdots\vee\lambda^m)^J_{ev}\,, \quad \text{where}$$
$$\lambda^1\vee\cdots\vee\lambda^m \;\text{ is the union of diagrams }\; \{\lambda^j\}.$$

The latter union is by definition the smallest Young diagram containing all diagrams $\lambda^1,\ldots,\lambda^m$.

Note that the J–polynomials in the A_n–case are not *minimal* integral (without q,t–denominators) even for $t=q$. They are important for the a–stabilization (including HOMFLY-PT polynomials for *links*) and for the super-duality. However, the switch from P to J does not influence the DAHA-construction for iterated knots, though their role is still important even for knots; see [GN],[ChD2].

Let us now go to the DAHA construction. Recall that $H\Downarrow \stackrel{\mathbf{def}}{=\!=} H(1)$, where the action of $H\in\mathcal{H\!H}$ in $\mathcal{V}$ is used. We represent torus knots $T(\mathsf{r},\mathsf{s})$ by the matrices $\gamma[\mathsf{r},\mathsf{s}]=\gamma_{\mathsf{r},\mathsf{s}}\in PSL_2(\mathbb{Z})$ with the first column $(\mathsf{r},\mathsf{s})^{tr}$ (tr is the transposition) for $\mathsf{r},\mathsf{s}\in\mathbb{N}$ provided $\gcd(\mathsf{r},\mathsf{s})=1$. Let $\widehat{\gamma}_{\mathsf{r},\mathsf{s}}\in PSL^{\wedge}_2(\mathbb{Z})$ be *any* pullback of $\gamma_{\mathsf{r},\mathsf{s}}$ to the projective $PSL_2(\mathbb{Z})$.

3.4.2. Pre-polynomials. The definition is for any pair $\{\mathcal{L},'\mathcal{L}\}$ from (3.4) (the *positivity* of $\mathcal{L},'\mathcal{L}$ and the pair is actually not needed). Let

$$(3.9)\qquad \mathcal{L}=\mathcal{L}^{\Upsilon,(b^j)}_{(\vec{\mathsf{r}}^j,\vec{\mathsf{s}}^j)},\quad '\mathcal{L}={}'\mathcal{L}^{'\Upsilon,('b^j)}_{('\vec{\mathsf{r}}^j,'\vec{\mathsf{s}}^j)}\quad \text{where }\; b^j,\,'b^j\in P_+,$$
$$1\le j\le\kappa,\,'\kappa \;\text{ for }\; \mathcal{L},\,'\mathcal{L},\;\vec{\mathsf{r}}^j=(\mathsf{r}^j_i\mid 1\le i\le \ell^j),\;'\vec{\mathsf{r}}^j=('\mathsf{r}^j_i\mid 1\le i\le {}'\ell^j).$$

First of all, we lift $(\mathsf{r}^j_i,\mathsf{s}^j_i)^{tr}$, $('\mathsf{r}^j_i,'\mathsf{s}^j_i)^{tr}$ to $\widehat{\gamma}^j_i,\,'\widehat{\gamma}^j_i\in PSL^{\wedge}_2(\mathbb{Z})$ as above. Then for any vertex of $\mathcal{L}$, i.e. a pair $\{i,j\}$, we begin the (inductive) definition with:

$$\mathbf{P}^j_{\ell^j+1}\stackrel{\mathbf{def}}{=\!=} J_{b^j},\;\gamma^j_{\ell^j+1}\stackrel{\mathbf{def}}{=\!=}\text{id}\;\text{ for }\;1\le j\le\kappa.$$

Recall that $\ell^j=0$ when the j_{th} path contains only an arrowhead and $v_{j,k}=0$ if the corresponding paths do not intersect.

For a given *path* with the index j, we define the *pre-polynomials* $\mathbf{P}^j_i$ by induction with respect to i, starting with $i=\ell^j$ down to $i=0$:

$$\mathbf{P}^j_i = \prod_{1\le k\le \kappa}^{\upsilon(k,j)=i} \bigl(\widehat{\gamma}^k_{i+1}(\mathbf{P}^k_{i+1})\Downarrow\bigr). \tag{3.10}$$

I.e. the last product is over all paths k that enter (intersect) the path for j exactly at the index $i\ge 0$, including $k=j$ when $i=\ell^j$. Note that $\mathbf{P}^j_{\ell^j}=\prod_{1\le k\le\kappa}^{\upsilon(k,j)=\ell^j} J_{b^k}$ for a *base path* j, where this product is over all κ *arrowheads* from (originated at) the vertex $\{i=\ell^j, j\}$.

The polynomial $\mathbf{P}^j_0$ actually depends only on the corresponding *subtree* for *any* path j there. If $\mathcal{L}$ is a union of subtrees, then $\mathbf{P}^j_0$ is the product of the corresponding polynomials $\bigl(\widehat{\gamma}^k_1(\mathbf{P}^k_1)\Downarrow\bigr)$ over all these subtrees. The *pre-polynomial* is defined then as $\mathbf{P}^{tot}_0=\mathbf{P}^j_0$ (the end of the above inductive procedure). The pre-polynomial $'\mathbf{P}^{tot}_0$ for $'\mathcal{L}$ is defined in the same way.

3.4.3. Finale. Using the notations $\mathbf{b}=(b^j)$, $'\mathbf{b}=('b^j)$, the *DAHA-Jones polynomial* for the J–polynomials J_{b^j}, $J_{'b^j}$ and a fixed index $1\le j_o\le\kappa$ (which determines the normalization) is as follows:

$$\begin{aligned} JD^{R,j_o,\Upsilon,'\Upsilon}_{(\vec{r}^j,\vec{s}^j),('\vec{r}^j,'\vec{s}^j)}((b^j),('b^j)\,;q,t) &= JD^{j_o,\Upsilon,'\Upsilon}_{(\vec{r}^j,\vec{s}^j),('\vec{r}^j,'\vec{s}^j)}(\mathbf{b},'\mathbf{b}\,;q,t)\\ = JD^{j_o}_{\mathcal{L},'\mathcal{L}} &\stackrel{\mathbf{def}}{=\!=} \Bigl\{\,'\mathbf{P}^{tot}_0(Y)\,\mathbf{P}^{tot}_0/J_{b^{j_o}}(q^{-\rho_k})\,\Bigr\}_{ev}. \end{aligned} \tag{3.11}$$

Here j_o can be $\emptyset$, which means that there is no divisions by the evaluations at $q^{-\rho_k}=t^{-\rho}$; R is the root system, which is A_n.

In the case of iterated torus *knots* (when there is only one path) and in the absence of $'\mathcal{L}$, we arrive at formula (2.12) from [ChD1]:

$$JD_{\vec{r},\vec{s}}(b;q,t)=\Bigl\{\widehat{\gamma}_1\Bigl(\cdots\Bigl(\widehat{\gamma}_{\ell-1}\Bigl(\bigl(\widehat{\gamma}_\ell(P_b)/P_b(q^{-\rho_k})\bigr)\Downarrow\Bigr)\Downarrow\Bigr)\cdots\Bigr)\Bigr\}_{ev}. \tag{3.12}$$

It includes only one $b\in P_+$ and therefore one can use P_b instead of J_λ.

The simplest link is for the union of any number of arrowheads colored by $b^1,\ldots,b^\kappa$. It is represented by the graph $\overset{\rightarrow}{\underset{\rightarrow}{\vdots}}$. A single $\rightarrow$ is associated with $P_b/P_b(q^{-\rho_k})$; generally we arrive at the product:

$$\mathbf{P}^1_\ell = J^1_{\ell+1}\cdots J^\kappa_{\ell+1}/J_{b^{j_o}}(q^{-\rho_k}) = J_{b^1}\cdots J_{b^\kappa}/J_{b^{j_o}}(q^{-\rho_k}), \tag{3.13}$$

where $j_\circ$ can be $\emptyset$. It will be later allowed to divide by the *LCM* of the evaluations for all J_{b^j} here, for the *minimal normalization* from (3.14).

3.5. Polynomiality etc

The following theorem and other statements in this and the next sections are from [ChD1] and [ChD2].

Theorem 3.1. *For any choice of the normalization index* $1 \le j_o \le \kappa$ *(it can be* $\emptyset$*), the DAHA-Jones polynomial* $JD^{j_o}_{\mathcal{L},{}'\mathcal{L}}$ *is indeed a polynomial in terms of* q,t *up to a factor* $q^{\bullet}t^{\bullet}$*, where the powers* $\bullet$ *can be rational. Modulo such factors, it does not depend on the particular choice of the lifts* $\gamma^j_i \in PSL_2(\mathbb{Z})$ *and* $\widehat{\gamma}^j_i \in PSL^{\wedge}_2(\mathbb{Z})$ *for* $1 \le i \le \ell^j$*.*

Up to the $q^{\bullet}t^{\bullet}$*–equivalence and* $\mathbb{Q}$*–proportionality we pick the* hat-normalization*, denoted* $\widehat{JD}^{j_o}_{\mathcal{L},{}'\mathcal{L}}$*, as follows. It is a* q,t*–polynomial not divisible by* q *and by* t*, with integral coefficients of all* q,t*–monomials such that their* GCD *is* 1 *and, finally, the coefficient of the minimal pure power of* t *is assumed positive.*

3.5.1. Minimal normalization. The q,t–integrality and other claims from Theorem 3.1 hold for the following modifications of DAHA-Jones polynomials (which is sharper and does not require picking j_o). Make $j_o = \emptyset$ (no normalization in $\widehat{\mathbf{P}}$), we set:

$$\widehat{JD}^{\,min}_{(\vec{\mathsf{r}}^j,\vec{\mathsf{s}}^j),({}'\vec{\mathsf{r}}^j,{}'\vec{\mathsf{s}}^j)}(\mathbf{b},{}'\mathbf{b}\,;\,q,t) = \widehat{JD}^{\,min}_{\mathcal{L},{}'\mathcal{L}} \stackrel{\mathbf{def}}{=\!=} \left\{ \frac{\varphi\circ\iota\,(\,{}'\widehat{\mathbf{P}}^{tot}_0)\,\widehat{\mathbf{P}}^{tot}_0}{(\mathbf{b},{}'\mathbf{b})^J_{ev}} \right\}_{ev}, \tag{3.14}$$

in the notation from (3.8), (2.16). We put $j_o = min$ if the minimal normalization (division by the corresponding LCM) is taken.

Some modification is needed for $t=1$ to ensure the connection with the HOMFLY-PT polynomials. We take $P^{(k=1)}_b$ instead of J_λ for $\lambda=\lambda(b)$ and $t=q$, which do not depend on q at all; they simply coincide with the corresponding Schur functions. The $LCM\ (b^1,\ldots,b^m)_{ev}$ then must be understood correspondingly. Recall that the polynomials J_λ generally have extra factors vs. the Schur functions in this case.

3.5.2. Topological symmetries. The polynomial $\widehat{JD}^{j_o}_{\mathcal{L},{}'\mathcal{L}}$ defined in Theorem 3.1 and $\widehat{JD}^{\,min}_{\mathcal{L},{}'\mathcal{L}}$ introduced above actually depend only on the topological link corresponding to the pair of graphs $\{\mathcal{L},{}'\mathcal{L}\}$. For instance, the reduction of the vertices with $\mathsf{r}=1$ can be applied in $\mathcal{L}$ or in ${}'\mathcal{L}$. Also, the transposition $[\mathsf{r}^j_1,\mathsf{s}^j_1] \mapsto [\mathsf{s}^j_1,\mathsf{r}^j_1]$ (only for $i=1$) does not influence $\widehat{JD}^{j_o}_{\mathcal{L}}$ or $\widehat{JD}^{\,min}_{\mathcal{L}}$ provided that ${}'\mathcal{L}=\emptyset$, and the pairs $\{\mathcal{L},{}'\mathcal{L}\}$ and $\{{}'\mathcal{L},\mathcal{L}\}$ result in coinciding polynomials.

The justification of this and other symmetries is essentially parallel to Theorem 1.2 from [Ch3]. Let us discuss torus knots. Essentially, one needs to check here that $T(m\mathsf{r}+\mathsf{s},\mathsf{r})$ results in the same DAHA-Jones polynomial as the "2-cable" corresponding to the tree $[m,1]\to[\mathsf{r},\mathsf{s}]$. Topologically, $T(m\mathsf{r}+\mathsf{s},\mathsf{r})$ is isotopic to $Cab(m\mathsf{r}+\mathsf{s},\mathsf{r})T(m,1)$, since $T(m,1)$ is unknot. The corresponding relation for the *JD*–polynomials readily follows from the commutativity τ_-^m with $\Downarrow$, which simply means that τ_- acts in $\mathcal{V}$.

3.5.3. Specialization $q=1$. We now make $q=1$, assuming that t is generic and using the notation $(b^1,\ldots,b^m)^J_{ev}$ from (3.7). Then

$$\frac{(b^1,\ldots,b^\kappa,\,'b^1,\ldots,'b'^\kappa)^J_{ev}}{(b^1)^J_{ev}\cdots(b^\kappa)^J_{ev}\,('b^1)^J_{ev}\cdots('b'^\kappa)^J_{ev}}(q=1)\ \widehat{JD}^{min}_{\mathcal{L},'\mathcal{L}}(q=1)$$
$$=\prod_{j=1}^{\kappa} JD_{\vec{\mathsf{r}}^j,\vec{\mathsf{s}}^j}\bigl(b^j\,;\,q=1,\,t\bigr)\prod_{j=1}^{'\kappa} JD_{'\vec{\mathsf{r}}^j,'\vec{\mathsf{s}}^j}\bigl('b^j\,;\,q=1,\,t\bigr), \tag{3.15}$$
$$\text{where } JD_{\vec{\mathsf{r}},\vec{\mathsf{s}}}\bigl(b;\,q=1,t\bigr)=\textstyle\prod_{p=1}^n JD_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(\omega_p;\,q=1,t)^{b_p},$$

for $b=\sum_{p=1}^n b_p\omega_p\in P_+$, where the *JD*–polynomials from (3.12) are used (for knots). See formula (2.18) in [ChD1].

Notice the factor in the left-hand side. It would be 1 if the polynomials P°_b were taken in this construction (without any further division) instead of J_λ under the minimal normalization. This can result in the q,t–denominators of the resulting coinvariants, but there will be no correction factor in the left-hand side.

4. DAHA-superpolynomials

4.1. Existence and duality

4.1.1. Stabilization. Following [Ch2, GN, Ch3, ChD1, ChD2], the construction from Theorem 3.1 and other statements above can be extended to the *DAHA- superpolynomials*, the result of the stabilization of $\widehat{JD}^{A_n,j_o}_{\mathcal{L},'\mathcal{L}}$ (including $j_o=min$, the minimal normalization).

The a–stabilization for torus knots was announced in [Ch2]; its proof was published in [GN]. Both approaches use [SV]. The super-duality conjecture was proposed in [Ch2] (let us also mention [GS]) and proven in [GN] for torus knots; see also [Ch3] for an alternative approach based on the generalized level-rank duality. The justifications of the a–stabilization and the super-duality was extended to arbitrary iterated torus knots in [ChD1].

The main change for links vs. knots is that the polynomiality of the superpolynomials for links requires the usage of J_λ–polynomials. Actually $\{J_\lambda\}$ were already employed in [GN] for the stabilization and duality, but the construction of (reduced) *JD*–polynomials and superpolynomials for *knots* requires only spherical P°_λ. For *links* (vs. knots), the role of polynomials $\{J_\lambda\}$ is the key; without the usage of *J*–polynomials the superpolynomials have non-trivial *t*–denominators.

The sequences $\vec{\mathsf{r}}^j$, $\vec{\mathsf{s}}^j$ of length ℓ^j for the graph $\mathcal{L}$ and $'\vec{\mathsf{r}}^j$,$'\vec{\mathsf{s}}^j$ of length $'\ell^j$ for the graph $'\mathcal{L}$ will be from the previous sections. We always use the DAHA-Jones polynomials under the hat-normalization, i.e. $\widehat{JD}^{j_o}$, $\widehat{JD}^{min}$. For $t=q$, the Schur functions $P^{(k=1)}_\lambda$ will be employed when discussing the connection with the HOMFLY-PT polynomials. Recall that $\lambda=\lambda(b)$ is the Young diagram representing $b\in P_+$.

We consider now $P_+\ni b=\sum_{i=1}^n b_i\omega_i$ for A_n as a (dominant) weight for *any* A_m (for sl_{m+1}) for $m\geq n-1$; we set $\omega_n=0$ upon its restriction to A_{n-1}. See [Ch2, GN, Ch3, ChD1] concerning the versions of the following theorem for torus knots and iterated torus *knots*.

Theorem 4.1. *[ChD2, Theorem 2.3] Given a pair $\{\mathcal{L}, '\mathcal{L}\}$ colored by $\mathbf{b}=(b^j)$, $'\mathbf{b}=('b^j)$ and the normalization index $1\leq j_o\leq\kappa$ or $j_o=min$, there exists a unique polynomial from $\mathbb{Z}[q,t^{\pm1},a]$*

$$\widehat{\mathcal{H}}^{j_o}_{\mathcal{L},'\mathcal{L}} \;=\; \widehat{\mathcal{H}}^{\Upsilon,j_o}_{(\vec{\mathsf{r}}^j,\vec{\mathsf{s}}^j),('\vec{\mathsf{r}}^j,'\vec{\mathsf{s}}^j)}(\mathbf{b},\,'\mathbf{b};\,q,t,a) \tag{4.16}$$

such that for any $m\geq n-1$ and proper powers of q,t (possibly rational):

$$\widehat{\mathcal{H}}^{j_o}_{\mathcal{L},'\mathcal{L}}(q,t,a=-t^{m+1}) \;=\; \pm\, q^\bullet t^\bullet\, \widehat{JD}^{A_m,j_o}_{\mathcal{L},'\mathcal{L}}(q,t). \tag{4.17}$$

Let us pick $\widehat{\mathcal{H}}$ such that $\widehat{\mathcal{H}}(a=0)$ is under the hat-normalization from Theorem 3.1. Then relations (4.17) will automatically hold for sufficiently large m without any correction factors $\pm q^\bullet t^\bullet$ (and one sufficiently large m is actually sufficient to fix $\widehat{\mathcal{H}}$ uniquely).

4.1.2. Symmetries. The polynomials $\widehat{\mathcal{H}}$ depend only on the isotopy class of the corresponding iterated torus links. All symmetries from the previous section hold for $\widehat{\mathcal{H}}=\widehat{\mathcal{H}}^{j_o}$, including $j_o=min$. For instance, this includes the *specialization* relation from (3.15) at $q=1$ for $\widehat{\mathcal{H}}$. The exact product formula there holds when spherical polynomials $P^\circ_b=P_b/P_b(t^\rho)$ are used in the formulas for $\widehat{JD},\widehat{\mathcal{H}}$ instead of *J*–polynomials. We note that *mirroring* of torus iterated links results in changing $\mathsf{a}_i\mapsto-\mathsf{a}_i$ for all i and in $q\mapsto q^{-1},t\mapsto$

$t^{-1}, a \mapsto a^{-1}$ in the superpolynomials (followed by the hat-normalization). Also, algebraically Y must be replaced by Y^{-1} in the case of two graphs (the change or their relative orientation); see [ChD2].

The key new feature of the stable theory is *super-duality*. We switch from $\mathbf{b}$, $'\mathbf{b}$ to the corresponding sets of Young diagrams $\{\lambda^j\}$, $\{'\lambda^j\}$; their transpositions will be denoted by $\{\cdot\}^{tr}$. Up to powers of q and t denoted here and below by $q^\bullet t^\bullet$, one has:

$$\widehat{\mathcal{H}}_{\mathcal{L},'\mathcal{L}}(\{\lambda^j\},\{'\lambda^j\};q,t,a) = q^\bullet t^\bullet \widehat{\mathcal{H}}_{\mathcal{L},'\mathcal{L}}(\{\lambda^j\}^{tr},\{'\lambda^j\}^{tr};t^{-1},q^{-1},a). \tag{4.18}$$

Let us discuss the a–degree of $\widehat{\mathcal{H}}$–polynomials. We conjectured in [ChD2] (for algebraic links only) that

$$\begin{aligned} \deg_a \widehat{\mathcal{H}}^{min}_{\mathcal{L},'\mathcal{L}} = {} & \textstyle\sum_{j=1}^{\kappa} \min\{\mathsf{r}_1^j, \mathsf{s}_1^j\}\, \mathsf{r}_2^j \cdots \mathsf{r}_{\ell^j}^j\, |\lambda^j| \\ & + \textstyle\sum_{j=1}^{'\kappa} \min\{'\mathsf{r}_1^j, '\mathsf{s}_1^j\}\, '\mathsf{r}_2^j \cdots '\mathsf{r}_{'\ell^j}^j\, |'\lambda^j| - \Delta \\ & \text{for } \Delta = |\lambda^1 \vee \ldots \vee \lambda^\kappa \vee {}'\lambda^1 \vee \ldots \vee {}'\lambda^{'\kappa}| \text{ from (3.8)}, \end{aligned} \tag{4.19}$$

where $|\lambda|$ is the number of boxes in λ. A somewhat weaker statement can be justified. This is connected with the product formula at $q = 1$, which gives that the a–degree is no smaller than that in (4.19). The right-hand side of this formula is actually the multiplicity of the corresponding singularity generalized to the colored case, e.g. $|\lambda|(\mathsf{s}-1)$ for $T(\mathsf{r},\mathsf{s})$ colored by λ, where $\mathsf{s} < \mathsf{r}$.

From now on we will always impose the *minimal normalization* of $\widehat{\mathcal{H}}$ unless stated otherwise.

4.1.3. Family superpolynomials. There are important reasons to consider superpolynomials in the "families" of links. Namely, (a) the corresponding generating functions have better algebraic structure than individual superpolynomials, (b) DAHA provides uniform tools for calculations within families (see below), (c) the families are natural to match the classical theory of zeta and [Ch4], (d) considering families is similar to Iwasawa theory in number theory (see [Mor] and below).

Generally, the definition is as follows. One replaces one (or several) γ_i in the formula for DAHA superpolynomials by $\sum v^m \tau_\pm^m \gamma_i$ for $m \in \mathbb{Z}_+$ and a new variable/parameter v. When $i = 1$ in the case of uncolored torus iterated *knots* described by formula (3.12), we set

$$\mathbb{JD}_{\vec{r},\vec{s}}(b;q,t,u) = \sum_{m=0}^{\infty} \Big(\frac{u}{t}\Big)^{g(m)} JD_m \tag{4.20}$$

$$= \sum_{m=0}^{\infty} \Big(\frac{u}{t}\Big)^{g(m)} \Big\{\widehat{\tau_-^m\gamma_1}\Big(\cdots \Big(\widehat{\gamma}_{\ell-1}\Big(\big(\widehat{\gamma}_{\ell}(P_b)/P_b(q^{-\rho_k})\big)\Downarrow\Big)\Downarrow\Big)\cdots\Big)\Big\}_{ev},$$

where $g(m)$ is the arithmetic genus of the corresponding singularity; this is δ from Section 4.2. Combinatorially, the pair $\mathsf{r}_1,\mathsf{s}_1$ is replaced by $\mathsf{r}_1, m\mathsf{r}_1+\mathsf{s}_1$. The corresponding rings are naturally embedded for $i=1$. The division by $t^{g(m)}$ provides the exact super-duality $q\leftrightarrow 1/t$ (not only up to proportionality). The corresponding *family superpolynomial* is defined accordingly. For $\widehat{\mathcal{H}}_m$ corresponding to JD_m,

$$\widehat{\mathbb{H}}(q,t,a,u) \stackrel{\mathbf{def}}{=\!=} \sum_{m=0}^{\infty} \Big(\frac{u}{t}\Big)^{g(m)} \widehat{\mathcal{H}}_m(q,t,a). \tag{4.21}$$

The simplest example is for the family $T(2,2m+1)$, i.e. for the summation over $\tau_-^m\tau_+\tau_-$ with $m\in\mathbb{Z}_+$. The formula is as follows

$$\widehat{\mathbb{H}}_{2,1+2\mathbb{Z}_+}(q,t,a,u) = \frac{1+auq/t}{(1-u/t)(1-uq)}, \tag{4.22}$$

which is simple to establish; the superpolynomials for $T(2,2m+1)$ are well known. From the DAHA perspective, this is the simplest application of the formula for the *pre-polynomial* $\mathbf{P}_{2,1}=(\tau_+\tau_-(P_{\omega_1}))(1)$. Importantly the polynomials P_b in (4.20) can be replaced by the corresponding *nonsymmetric* polynomials E_b for $b\in P_+$. They are eigenfunctions of $Y-$operators: $Y_a(E_b)=q^{-(a,b)}t^{-(\rho,b)}E_b$ (for dominant b); the normalization is $E_b=X_b$ modulo lower terms. We use that

$$\frac{P_b}{P_b(t^{-\rho})} = \mathscr{P}\Big(\frac{E_b}{E_b(t^{-\rho})}\Big)$$

for $\mathscr{P}=\frac{\sum_{w\in W} t^{l(w)/2}T_w}{\sum_{w\in W} t^{l(w)}}$; $\mathscr{P}$ commutes with taking the coinvariant. See Section 2.4.1 and [Ch1, Ch2, Ch3]. Also we can switch here to GL_n instead of A_n. Thus, we need $\mathbf{E}_{2,1}=(\tau_+\tau_-(E_{\omega_1}))(1)$, which is proportional to $X_{\omega_1}=X_1$. Using the standard GL_n variables X_i for $1\le i\le n$, $X_{\omega_i}=X_1X_2\cdots X_i$

and

$$E_{2\omega_1}=X_1^2\frac{1-q}{1-qt}+\frac{(1-t)q}{1-qt}(X_1+\cdots+X_n)X_1,\quad E_{2\omega_1}(t^{-\rho})=\frac{1-qt^n}{t^{n-1}(1-qt)}.$$

Applying $\mathscr{P}$ here, we arrive at some DAHA-Rosso-Jones relation.

Proposition 4.2. *Let* $\gamma_1=\gamma_{\mathsf{r}_1,\mathsf{s}_1}$ *be a matrix from* $PSL(2,\mathbb{Z})$, $\gamma_2=\tau_+\tau_-=\gamma_{2,1}$, $\widehat{\mathcal{H}}^{\square}_{\gamma_1,\gamma_2}$ *the corresponding (hat-normalized) uncolored superpolynomial, corresponding to the cable* $Cab(2\mathsf{r}_1\mathsf{s}_1+1,2)T(\mathsf{r}_1,\mathsf{s}_1)$. *Then*

$$\widehat{\mathcal{H}}^{\square}_{\gamma_1,\gamma_2}=\frac{1+aq}{1-q}\widehat{\mathcal{H}}^{\square\square}_{\mathsf{r}_1,\mathsf{s}_1}-\frac{q}{1-q}\widehat{\mathcal{H}}^{\square}_{2\mathsf{r}_1,2\mathsf{s}_1}\qquad for\ \ \mathsf{r},\mathsf{s}\ge 0, \tag{4.23}$$

where $\widehat{\mathcal{H}}^{\square\square}_{\mathsf{r},\mathsf{s}}$ *is the superpolynomial for* $\gamma=\gamma_{\mathsf{r},\mathsf{s}}$ *colored by* $2\omega_1$, $\widehat{\mathcal{H}}^{\square}_{2\mathsf{r},2\mathsf{s}}$ *is the uncolored superpolynomial for the link* $2T(\mathsf{r},\mathsf{s})$.

When $\gamma_1=\tau_-^m=\gamma_{1,m}$ for $m\in\mathbb{Z}_+$, we use (4.26) below and obtain:

$$\begin{aligned}\widehat{\mathcal{H}}^{\square}_{2,2m+1}&=\frac{1+aq}{1-q}-\frac{q}{1-q}\widehat{\mathcal{H}}^{\square}_{2,2m};\ \text{ e.g. for } m=1:\\ 1+qt+aq&=\frac{1+aq}{1-q}-\frac{q}{1-q}\big(1+aq-t(1-q)\big).\end{aligned} \tag{4.24}$$

Generally, this approach gives that $\mathbb{H}(q,t,a,u)$ are rational functions in terms of q,t,u,a (the details will be published elsewhere).

Corollary 4.2 is a typical recurrence relation in the DAHA theory of superpolynomials; topological justification of such DAHA formulas within the Khovanov -Rozansky theory is an obvious challenge. The calculation of $\widehat{\mathcal{H}}^{\square}_{2\mathsf{r}_2,2\mathsf{s}_2}$ can be generally performed using the Pieri rules from (1.4.50),[Ch1]. See also Proposition 4.3 below.

Let us provide the formula for $\mathbb{H}(q,t,a,u)$ in the case of the family $T(3,3n+1)$, where $g(n)=3n$ and the superpolynomials are sufficiently well known; see e.g. Conjecture 7 from [GORS] and [Ch2]. One has:

$$\begin{aligned}\widehat{\mathbb{H}}_{3,1+3\mathbb{Z}_+}(q,t,a,u)&=\frac{1}{(1-(u/t)^3)(1-(uq)^3)(1-u^3q^2/t^2)}\\ &\times\Bigg(1+u^3\left(\frac{q}{t^2}+\frac{q^2}{t}\right)+a^2u^3\frac{q^3}{t^3}\left(1+u^3\left(\frac{q}{t^2}+\frac{q^2}{t}\right)\right)\\ &+au^3\frac{q}{t}\left(\frac{1}{t^2}+\frac{q}{t}+q^2+\frac{q}{t^2}+\frac{q^2}{t}+u^3\frac{q^3}{t^3}\right)\Bigg).\end{aligned} \tag{4.25}$$

Family Cab(13+2m,2)T(3,2). These cables can be obtained for $\gamma_1 = \tau_+\tau_-^2$ and $\gamma_2 = \tau_-^m\tau_+\tau_-$, i.e. the procedure $\gamma \mapsto \tau_-^m\gamma$ is used her for γ_2. We use the above formula for $E_{2\omega_1}$ rewriting it as follows:

$$X_1^2 = E_{2\omega_1} - \frac{(1-t)q}{1-qt}(X_2+\cdots+X_n)X_1.$$

Formula (2.18) defines the action of τ_- in $\mathcal{V}$. We used in [ChD2] the notation $\dot{\tau}_-$ to avoid confusion with the action of τ_- in $\mathcal{H}$. Formula (1.37) there states that $\tau_-(E_b) = q^{-b_+^2/2}t^{-(\rho,b_+)}$, where b_+ is the unique weight in $P_+ \cap W(b)$ for $b \in P$. We need here the relations

$$\tau_-(E_{2\omega_1}) = q^{-2}t^{1-n}E_{2\omega_1},\ \tau_-(X_iX_j) = q^{-1/2}t^{(1-n)/2}, 1 \le i < j \le n. \tag{4.26}$$

One has: $\tau_-^m(X_1^2) =$

$$\frac{(q^{-2}t^{1-n})^m}{(1-qt)}\left(\frac{E_{2\omega_1}}{E_{2\omega_1}(t^{-\rho})}\frac{(1-qt^n)}{t^{n-1}} - (1-t)q(qt)\,X_1(X_2+\cdots+X_n)\right).$$

Proposition 4.3. *For any $m \in \mathbb{Z}_+$, $\mathsf{r}_1, \mathsf{s}_1 \ge 0$, let $\gamma_1 = \gamma_{\mathsf{r}_1,\mathsf{s}_1} \in PSL(2,\mathbb{Z})$, $\gamma_2(m) = \tau_-^m\tau_+\tau_- = \gamma_{2,2m+1}$, $\widehat{\mathcal{H}}^{\square}_{\gamma_1,\gamma_2(m)}$ be the corresponding superpolynomial for the cable Cab$(2\mathsf{r}_1\mathsf{s}_1+2m+1,2)T(\mathsf{r}_1,\mathsf{s}_1)$. Then*

$$\begin{aligned}\widehat{\mathcal{H}}^{\square}_{\gamma_1,\gamma_2(m)} &= \frac{1+aq}{1-qt}\left(1+(qt)^m\frac{q(1-t)}{1-q}\right)\widehat{\mathcal{H}}^{\square\square}_{\mathsf{r}_1,\mathsf{s}_1} - (qt)^m\frac{q}{1-q}\widehat{\mathcal{H}}^{\square}_{2\mathsf{r}_1,2\mathsf{s}_1}\\ &= \frac{1-(qt)^m}{1-qt}(1+aq)\widehat{\mathcal{H}}^{\square\square}_{\mathsf{r}_1,\mathsf{s}_1} + (qt)^m\widehat{\mathcal{H}}^{\square}_{\gamma_1,\gamma_2(0)},\end{aligned} \tag{4.27}$$

where $\widehat{\mathcal{H}}^{\square\square}_{\mathsf{r},\mathsf{s}}$, $\widehat{\mathcal{H}}^{\square}_{2\mathsf{r},2\mathsf{s}}$ are as in the previous proposition (which is used to obtain the second formula from the first). The formula for $\gamma_2(m)$ in terms of $\gamma_2(0)$ actually works for any $m \in \mathbb{Z}$.

In the case of $\gamma_1 = \gamma_{3,2}$ these formulas are an important addition to the (mostly numerical) analysis of this family in [ChD1]. For instance, for

$m=-5$ we obtain the superpolynomial (**c**) from Section 4.1 there:

(**c**) $\vec{\mathsf{r}}=\{3,2\}$, $\vec{\mathsf{s}}=\{2,-9\}$ for $Cab(3,2)T(3,2)$;

$$\mathcal{H}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(\square;q,t,a)=1-q^2+qt+q^2t-q^3t+q^2t^2+q^3t^3+a^3\left(-\frac{q^4}{t^2}-\frac{q^5}{t}\right)$$
$$+a^2\left(q^3-q^4-q^5-\frac{q^3}{t^2}-\frac{q^3}{t}-\frac{2q^4}{t}\right)$$
$$+a\left(q+q^2-2q^3-q^4-\frac{q^2}{t}-\frac{q^3}{t}+q^2t+q^3t-q^4t+q^3t^2\right).$$

Negative m result in non-algebraic knots, however the formula is uniform for any m. Negative powers of q,t cancel each other for $m<0$. A geometric interpretation of non-algebraic torus iterated knots/links is an interesting problem. See Section 4.2.3 below. We note that Proposition 4.3 and its generalizations provide important tools for counting points of Jacobian factors over finite fields in *families*, which is parallel to Iwasawa theory in number theory. See the end of Section 4.4.4.

Conjecture 4.7 is expected to follow from such recurrence relations (in full generality), to be considered elsewhere. We also hope that these relations will connect DAHA superpolynomials with the stable KhR–polynomials (via the approach based on Soergel modules).

There are no actual problems with applying this formula to any γ_1; practically, its complexity is generally comparable with that of the starting $\widehat{\mathcal{H}}_{\gamma_1,\gamma_2(0)}$. The simplest example is for $\gamma_1=\gamma_{3,2}$ and the range $m\geq 0$. Here the genus is $g(m)=8+m$ and we have:

$$\left(\frac{t}{u}\right)^8\widehat{\mathbb{H}}_{\{3,2\},\{2,3+2\mathbb{Z}_+\}}=\sum_{m=0}^{\infty}\left(\frac{u}{t}\right)^m\widehat{\mathcal{H}}_{\{3,2\},\{2,3+2m\}},\tag{4.28}$$

$$(1-u/t)(1-uq)\left(\frac{t}{u}\right)^8\widehat{\mathbb{H}}_{\{3,2\},\{2,3+2\mathbb{Z}_+\}}$$
$$=1+qt+q^2t+q^3t+q^2t^2+q^3t^2+2q^4t^2+q^3t^3+q^4t^3+2q^5t^3$$
$$+q^4t^4+q^5t^4+2q^6t^4+q^5t^5+q^6t^5+q^7t^5+q^6t^6+q^7t^6$$
$$+q^7t^7+q^8t^8+\big(-q-q^2t-q^3t-q^4t-q^3t^2-q^4t^2-2q^5t^2$$
$$-q^4t^3-q^5t^3-2q^6t^3-q^5t^4-q^6t^4-q^7t^4-q^6t^5-q^7t^5$$
$$-q^7t^6-q^8t^7\big)\mathbf{u}+\mathbf{a}^3\Big(q^6+q^7t+q^8t^2+\big(-q^7-q^8t\big)\mathbf{u}\Big)$$

$$
\begin{aligned}
&+\mathbf{a}^2\Big(q^3+q^4+q^5+q^4t+2q^5t+2q^6t+q^5t^2+2q^6t^2+2q^7t^2\\
&+q^6t^3+2q^7t^3+q^8t^3+q^7t^4+q^8t^4+q^8t^5+\big(-q^4-q^5-q^6\\
&-q^5t-2q^6t-2q^7t-q^6t^2-2q^7t^2-q^8t^2-q^7t^3-q^8t^3-q^8t^4\big)\mathbf{u}\Big)\\
&+\mathbf{a}\Big(q+q^2+q^3+q^2t+2q^3t+3q^4t+q^5t+q^3t^2+2q^4t^2+4q^5t^2\\
&+q^6t^2+q^4t^3+2q^5t^3+4q^6t^3+q^7t^3+q^5t^4+2q^6t^4+3q^7t^4+q^6t^5\\
&+2q^7t^5+q^8t^5+q^7t^6+q^8t^6+q^8t^7+\big(-q^2-q^3-q^4-q^3t-2q^4t\\
&-3q^5t-q^6t-q^4t^2-2q^5t^2-4q^6t^2-q^7t^2-q^5t^3-2q^6t^3-3q^7t^3\\
&-q^6t^4-2q^7t^4-q^8t^4-q^7t^5-q^8t^5-q^8t^6\big)\mathbf{u}\Big).
\end{aligned}
$$

We note that when $a=0$, the coefficient of u^1 considered upon the substitution $q\mapsto qt$ satisfies RH (its t–roots ζ are complex such that $|\xi|=q^{1/2}$) provided that $q<0.7562688464467736\ldots$. This is better than the bound $q\le 1/2$ in Conjecture 4.11 for (individual) m.

4.2. Motivic superpolynomials

In the unibranch uncolored case, let us compare the *motivic superpolynomials* from [ChP1] and the *Galkin-Stöhr zeta-functions*.

4.2.1. Standard modules. Let $\mathcal{R}\subset O \overset{\textbf{def}}{=\!=} \mathbb{F}[[z]]$ be a Gorenstein ring over a finite field $\mathbb{F}=\mathbb{F}_q$ of cardinality q, $\delta=dim_{\mathbb{F}}(\mathcal{O}/\mathcal{R})$ (the arithmetic genus), $\Gamma_{\mathcal{R}}=val_z(\mathcal{R})$ for the usual z–valuation. It is a semigroup and $\delta=|\mathbb{Z}_+\setminus\Gamma|$. For the later, we assume that $\mathcal{R}=\widetilde{\mathcal{R}}_{\mathbb{F}_q}\overset{\textbf{def}}{=\!=}\widetilde{\mathcal{R}}\otimes_{\mathbb{Z}}\mathbb{F}$, where $\widetilde{\mathcal{R}}\subset\mathbb{Z}[[z]]$, and $\Gamma_{\mathbb{C}}$ for $\widetilde{\mathcal{R}}\otimes_{\mathbb{Z}}\mathbb{C}$ coincides with Γ over $\mathbb{F}$.

Given an $\mathcal{R}$–module $M\subset\mathcal{O}$, $\Delta=\Delta_M\overset{\textbf{def}}{=\!=}val_z(M)$ is Γ–module, i.e. $\Gamma+\Delta=\Delta$. One has: $dev(M)\overset{\textbf{def}}{=\!=}\delta-dim_{\mathbb{F}}(\mathcal{O}/M)=\delta-|\mathbb{Z}_+\setminus\Delta|\overset{\textbf{def}}{=\!=}dev(\Delta)$. If $0\in\Delta$, then Δ and the corresponding M are called *standard*; equivalently, $M\cdot\mathcal{O}=\mathcal{O}$. For any $\mathcal{R}$–module M, let $M_{st}\overset{\textbf{def}}{=\!=}z^{-m}M$ for $m=\min\Delta_M$, which is a standard module corresponding to $\Delta_{st}=\Delta-\min\{\Delta\}$. Also, $M^\star\overset{\textbf{def}}{=\!=}\{x\in\mathbb{F}((z))\mid xM\subset\mathcal{R}\}$ corresponds to $\Delta^\star\overset{\textbf{def}}{=\!=}\{n\in\mathbb{Z}_+\mid n+\Delta\subset\Gamma\}$ for any modules M,Δ.

Definition 4.4. For any $\mathbb{F}$–subalgebra $\mathcal{R}\subset\mathcal{O}$ with $\mathbb{F}((z))$ as its field of fractions, let $J=J_{\mathcal{R}}(\mathbb{F})\overset{\textbf{def}}{=\!=}\{M=M_{st}\}$ be the Jacobian factor, $J_\Delta\overset{\textbf{def}}{=\!=}\{M=M_{st},\Delta(M)=\Delta\}$, $\mathcal{H}^0_{mot}(q,t)=\sum_{M\in J}t^{dim_{\mathbb{F}}(\mathcal{O}/M)}$. For a standard Δ, we set

$\mathcal{H}^0_{mot}(\Delta,q,t)=\sum_{M\in J_\Delta} t^{dim_{\mathbb{F}}(\mathcal{O}/M)}$. Following [Sto] for Gorenstein $\mathcal{R}$, $M\subset \mathcal{O}$, and standard Δ:

$$Z(\Delta,q,t)=\sum_{M\supset\mathcal{R}}^{M_{st}\in J_\Delta} t^{dim_{\mathbb{F}}(M/\mathcal{R})}=\sum_{M\subset\mathcal{R}}^{(M^\star)_{st}\in J_\Delta} t^{dim_{\mathbb{F}}(\mathcal{R}/M)}.$$

Accordingly, $Z(q,t)=\sum_{\Delta=\Delta_{st}} Z(\Delta,q,t)$. Also we set

$$L(\Delta,q,t)\stackrel{\mathbf{def}}{=\!=}(1-t)\,Z(\Delta,q,t),\ \ L(q,t)\stackrel{\mathbf{def}}{=\!=}(1-t)\,Z(q,t).$$

4.2.2. Stöhr's formula. We begin with the *functional equation*:

$$(qt^2)^\delta\, L\left(\Delta,q,\frac{1}{qt}\right)=L((\Delta^\star)_{st},q,t)\ \text{ for standard }\Delta. \tag{4.29}$$

The natural setting here is when the summation over M from Definition 4.4 is reduced to (any) $\mathcal{O}^*$–orbits; $\mathcal{O}^*=\{x\in\mathcal{O}\mid x^{-1}\in\mathcal{O}\}$ acts in J_Δ by multiplication. See [Sto],(3.10). We are going to state a version of Theorem 3.1 there, which almost directly provides (4.29). Let

$$g_\Delta(n)=\bigl|\{\mathbb{N}\ni m\not\in\Delta\ \mid\ m<n\}\bigr|\ \text{ for }\ n\in\mathbb{Z}_+,\ \text{so}\ g_\Delta(0)\!=\!0\!=\!g_\Delta(1).$$

Using the relations $dev(M\cap(z^n\mathcal{O}))=dev(M)-n+g_\Delta(n)$, one has:

$$Z(\Delta,q,t)=(qt)^{dev(\Delta)}q^{-\delta}\,\bigl|J_\Delta\bigr|\sum_{n\in\Delta} t^n\, q^{g_\Delta(n)}\ \text{ for }\ \Delta=\Delta_{st},$$

$$L(\Delta,q,t)=(qt)^{dev(\Delta)}q^{-\delta}\,\bigl|J_\Delta\bigr| \times\left(\sum_{n\in\Delta\not\ni n-1} t^n\, q^{g_\Delta(n)}-\sum_{n\not\in\Delta\ni n-1} t^n\, q^{g_\Delta(n-1)}\right). \tag{4.30}$$

Note that the second formula readily gives that $L(\Delta,q,t=1)=\bigl|J_\Delta\bigr|$.

Conjecture 4.5. *Let $\mathcal{R}_{\mathbb{C}}\subset\mathbb{C}[[z]]$ be the ring of plane curve singularity (with 2 generators and the same field of fractions). Within its topological type, there exists $\mathcal{R}_{\mathbb{Z}}\subset\mathbb{Z}[[z]]$ such that $\mathcal{R}_{\mathbb{C}}$ and $\mathcal{R}_{\mathbb{F}}$ have coinciding Γ. Let $\widehat{\mathcal{H}}(q,t,a)$ be the corresponding uncolored DAHA superpolynomial for the link of $\mathcal{R}_{\mathbb{C}}$. Then $\mathcal{H}^0_{mot}(q,t)=\widehat{\mathcal{H}}^0\stackrel{\mathbf{def}}{=\!=}\widehat{\mathcal{H}}(q,t,a\!=\!0)$, which is the case $a=0$ of Conjecture 2.3,(iii) from [ChP1]. Then one has:*

$$\mathcal{H}^0_{mot}(q\mapsto qt,t)\ =\ L(q,t)\ =\ \widehat{\mathcal{H}}(q\mapsto qt,t,a\!=\!0), \tag{4.31}$$

$$L(\Gamma,q,t)=\widehat{\mathcal{H}}(q,t,a\mapsto -(t/q))\big|_{q\mapsto qt},\ \ \bigl|J_\Gamma\bigr|=q^\delta. \tag{4.32}$$

This conjecture (and its a–generalization, see below), clarifies the substitution $q\mapsto qt$; the DAHA super-duality then becomes the functional equation. The DAHA super-duality (with a and for any colors) is not very difficult to justify, but some theory is needed; see [GN] and [Ch3] for a sketch of the proof via roots of unity. (which can work beyond A_n). The positivity of $L(q,t)$ seems new for the Galkin-Stöhr zeta (beyond some special values/ coefficients). The corresponding cancelations in (4.30) generally hold only for plane curve singularities.

Corollary 4.6. *Provided (4.31),* $\widehat{\mathcal{H}}(t,t,a\!=\!0)\!=\!\sum_{\Delta=\Delta_{st}} t^{dev(\Delta)}\text{æ}(\Delta)$*, where* $\text{æ}(\Delta)$ *is the image of* J_Δ *considered as an abstract (projective) variety in the quotient of the Grothendieck ring* $K_0(Var)$ *by* $1-\mathbb{L}$ *for* $\mathbb{L}=[\mathbb{A}^1]$*, e.g.* $\text{æ}(\Delta)=1$ *if* J_Δ *is an affine space* $\mathbb{A}^N$*.*

4.2.3. A non-planar example. Relation (4.31) was checked numerically for quite a few examples of plane curve singularities, including many torus knots, $\mathcal{R}=\mathbb{F}[[z^4,z^6+z^{7+2m}]]$ for many $m\geq 0$, $\mathbb{F}[[z^6,z^8+z^{9,\ldots,15}]]$, and $\mathcal{R}=\mathbb{F}[[z^6,z^9+z^{10}]]$.

Interestingly, it fails for *non-planar* Gorenstein singularities. Let $\mathcal{R}=\mathbb{F}[[z^4,z^6,z^9]]$. Here $\delta=6$ and it corresponds in some sense (which we omit here) to the non-algebraic knot $\mathit{Cab}(9,2)T(3,2)$, called *pseudo-algebraic* in [ChD1]. See also Corollary 1.4 from [Hed]. The positivity of the DAHA superpolynomial and its algebraic similarity to that for $\mathit{Cab}(13,2)T(3,2)$ were the defining features in [ChD1]. One has:

$$\begin{aligned}\widehat{\mathcal{H}} &= 1+a^3q^6+qt+q^2t+q^3t+q^2t^2+q^3t^2+2q^4t^2+q^3t^3+q^4t^3+q^5t^3\\ &+q^4t^4+q^5t^4+q^5t^5+q^6t^6+a^2(q^3+q^4+q^5+q^4t+2q^5t+q^6t+q^5t^2\\ &+q^6t^2+q^6t^3)+a(q+q^2+q^3+q^2t+2q^3t+3q^4t+q^5t+q^3t^2+2q^4t^2\\ &+3q^5t^2+q^4t^3+2q^5t^3+q^6t^3+q^5t^4+q^6t^4+q^6t^5).\end{aligned}$$

The smallest positive $\widehat{\mathcal{H}}$ is for $\mathit{Cab}(7,2)T(3,2)$ associated with $\mathbb{F}[[z^4,z^6,z^7]]$, but its a–degree drops to 2.

The motivic interpretation of $\widehat{\mathcal{H}}$ above at $a=0$ is as follows. We define ${}'\mathcal{H}^0_{mot}$, ${}'L$ by restricting the summation in $\mathcal{H}^0_{mot}, Z$ to Δ such that $\Delta_{1,2}\setminus\Gamma\neq\{2,11\},\{2,7,11\}$. We note that $\Delta_{1,2}$ satisfy $(\Delta^\star)_{st}=\Delta$. Then ${}'\mathcal{H}^0_{mot}(q\mapsto qt,t)={}'L(q,t)$ (and with a, see (4.34) below), but $\mathcal{H}^0_{mot}(q\mapsto qt)\neq L$. We note that (4.32) holds without prime.

Namely, $\mathcal{H}^0_{mot}-{}'\mathcal{H}^0_{mot}=q^6t^4+q^5t^3$; the latter monomials are contributions of Δ_1,Δ_2. One has: $L(q,t)-{}'L(q,t)=q^2t^2-q^2t^4+q^3t^4-q^3t^6+q^4t^6-q^4t^8+q^5t^8+q^6t^{10}$; by the way, $L(q,t)$ has positive coefficients. Here

$L(q,t))/t^6$ and $'L(q,t)/t^6$ satisfy the functional equation $t\mapsto 1/(qt)$. However, $\mathcal{H}^0_{mot}/t^6$ is not self-super-dual for $q\leftrightarrow t^{-1}$. This failure is typical for motivic pseudo-algebraic superpolynomials.

Observation. For planar $\mathcal{R}$, let $\ddot{\mathcal{R}}$ be the corresponding *quasi-homogeneous* ring, which is $\ddot{\mathcal{R}} \stackrel{\mathbf{def}}{=\!=} \mathbb{F}[z^v, v\in\Gamma]$, and $\ddot{\mathcal{H}}_{mot}(q,t,a)$ as in (4.34). The following holds for $\mathbb{F}[[z^4, z^6+z^{7+2m}]]$ we checked and for $\mathcal{R}=\mathbb{F}[[z^6,z^8+z^9]]$: *$\widehat{\mathcal{H}}$ is $\ddot{\mathcal{H}}^{sym}_{mot}$, which is $\ddot{\mathcal{H}}_{mot}$ minus the sum of positive terms in $\ddot{\mathcal{H}}_{mot}(q,t,a)-$ $(qt)^\delta\ddot{\mathcal{H}}_{mot}(\frac{1}{t},\frac{1}{q},a)$.* The same tendency holds for (non-planar) $\mathcal{R}$ associated with *pseudo-algebraic* knots.

The smallest counterexample we found at $a=0$ is $\mathbb{F}[[z^6,z^9+z^{10}]]$ ($\delta=21$), where $\ddot{\mathcal{H}}^{sym}_{mot}-\widehat{\mathcal{H}}\big|_{a=0} = q^{13}t^5+q^{14}t^5+2q^{14}t^6+2q^{15}t^6+2q^{15}t^7+q^{16}t^7+q^{16}t^8$. The number of Δ is 447 for the corresponding $\ddot{\mathcal{R}}$. By the way, $J_{\ddot{R}}$ has a component of $dim=\delta+2$ and there exist 9 non-affine cells $\ddot{J}_\Delta$, but they do not contribute to $\ddot{\mathcal{H}}^{sym}_{mot}-\widehat{\mathcal{H}}\big|_{a=0}$ (all J_Δ are affine).

The rationale here is that $i-j=dim(J_\Delta)+dev(\Delta)-\delta$ is high for Δ (with affine J_Δ) and the corresponding q^it^j in $\widehat{\mathcal{H}}$ if $dim(\ddot{J}_\Delta)>dim(J_\Delta)$ ($dim\,\emptyset=-\infty$). In the difference above, $i-j=8,9$, when $max\{i-j\}=9$ in $\widehat{\mathcal{H}}^0$, and only very few Δ there are with $i-j\geq 8$.

For instance, let $\mathcal{R}=\mathbb{F}[[z^4,z^6,z^{11}]]$. Then the reduction above and the relations $'\mathcal{H}_{mot}(q\mapsto qt,t,a) ={}'\mathcal{L}(q,t,a)=\widehat{\mathcal{H}}(q\mapsto qt,t,a)$ hold for non-admissible $\Delta\setminus\Gamma=\{2,13\},\{2,9,13\}$. For z^7 instead of $z^{9,11}$ (not pseudo-algebraic according to [ChD1]): $\widehat{\mathcal{H}}={}'\mathcal{H}_{mot}$ if $\{2,9\},\{1,5,9\}$ are excluded, but then $\widehat{\mathcal{H}}-{}'\mathcal{L}=q^2t^4(1-t)(1-qt)(1+aq+aqt)$, which requires further adjustments. Also, the observation above holds only for $a=0$. We note that $\mathcal{H}^0_{mot}-\widehat{\mathcal{H}}(a=0)$ for $11,9,7,5,3$ is uniform: $(qt)^\delta(t^{-2}+t^{-3}q^{-1})$, where $\delta=7,6,5,4,3$.

4.2.4. Flagged zeta-functions. Following [ChP1], the *standard ℓ–flag* of Γ–modules is a sequence $\vec{\Delta}=\{\Delta_0\subset\Delta_1=\Delta_0\cup\{g_1\}\subset\cdots\subset\Delta_i=\Delta_{i-1}\cup\{g_i\}\subset\Delta_\ell\}$ of standard Γ–modules Δ_i such that $0\neq g_i\in\mathbb{Z}_+\setminus\Delta_{i-1}$ and $g_{i-1}<g_i$ for $1\leq\ell$. Thus $dev(D_i)=dev(D_{i-1})+1$. We set $dev(\vec{\Delta})\stackrel{\mathbf{def}}{=\!=} dev(\Delta_m)$ and define the *standardizable* flag of $\mathcal{R}$–modules of length ℓ as the sequences of $\mathcal{R}$–modules $\overrightarrow{M}=\{M_0\subset M_1\cdots\subset M_\ell\}$ such that $\Delta(\overrightarrow{M})\stackrel{\mathbf{def}}{=\!=}\{\Delta(M_i)\}$ becomes a standard flag as above upon the subtraction of $m=min(\Delta_\ell)$, i.e. for $\overrightarrow{M}_{st}\stackrel{\mathbf{def}}{=\!=} z^{-m}\overrightarrow{M}$. Accordingly,

$$\mathcal{Z}(q,t,a)\stackrel{\mathbf{def}}{=\!=}\sum_{\overrightarrow{M}\subset\mathcal{R}} a^\ell\, t^{dim_{\mathbb{F}}(\mathcal{R}/M_\ell)},\ \ \mathcal{L}(q,t,a)\stackrel{\mathbf{def}}{=\!=}(1-t)\,\mathcal{Z}(q,t,a), \tag{4.33}$$

where the summation is over *standardizable* flags. The (full) *motivic superpolynomial* from [ChP1] is as follows:

$$\mathcal{H}_{mot}(q,t,a) = \sum_{\overrightarrow{M}\in \vec{J}} a^{\ell}\, t^{dim_{\mathbb{F}}(\mathcal{O}/M_{\ell})}, \tag{4.34}$$

where $\vec{J} = \vec{J}_{\mathcal{R}}$ is a scheme of all *standard* flags of submodules in $\mathcal{O}$.

Conjecture 4.7. *In the case of plane curve singularity,*

$$\widehat{\mathcal{H}}(q,t,a) = \mathcal{H}_{mot}(q,t,a) = \mathcal{L}\left(\frac{q}{t},t,a\right), \quad \mathcal{H}_{mot}\left(qt,t,a=-\frac{1}{q}\right) = L(\Gamma,q,t),$$

where the latter possibly holds for any Gorenstein rings $\mathcal{R}$.

It was checked in quite a few case; as for $\mathcal{H}_{mot}(q,t,a)=\mathcal{L}(\frac{q}{t},t,a)$, including $\mathbb{F}[[z^6,z^8+z^9]]$ for $\ell=0,1$ and $\mathbb{F}[[z^6,z^9+z^{10}]]$ for $\ell=0$. We will omit here flag generalizations of formulas from (4.30), which result in a relatively straightforward proof of the flagged functional equation for $t^{-\delta}\mathcal{L}(q,t,a)$ upon the (same) transformation $t\mapsto 1/(qt)$. The generalization to algebraic (uncolored) links is also known, as well as some steps of the justification based on the analysis of the monoidal transformations of singularities. (at least for some families).

4.2.5. Nested Hilbert schemes. Our flags of $\mathcal{R}$–modules can be interpreted via *restricted* nested Hilbert schemes $Hilb_{res}^{m,m-\ell}$, and nested Jacobian factors $J_{res}^{m,m-\ell}$:

$$Hilb_{res}^{m,m+\ell} = \{M\supset M'\supset \mathfrak{m}M \mid M\subset \mathcal{R},\ \dim(\mathcal{R}/M)=m,\ \dim(M/M')=\ell,\ M\otimes_{\mathcal{R}}\mathcal{O}=M'\otimes_{\mathcal{R}}\mathcal{O}\}, \tag{4.35}$$

$$J_{res}^{m,m+\ell} = \{M\supset M'\supset \mathfrak{m}M \mid M\subset \mathcal{O},\ \dim(\mathcal{O}/M)=m,\ \dim(M/M')=\ell,\ M'=M'_{st}\}. \tag{4.36}$$

Here $\mathfrak{m}$ is the maximal ideal of $\mathcal{R}$. If the condition $M\otimes_{\mathcal{R}}\mathcal{O}=M'\otimes_{\mathcal{R}}\mathcal{O}$ or $M'=M'_{st}$ is omitted, one obtains nested $Hilb$ from [ORS] or its variant for J. As above, the pairs satisfying this condition are called *standardizable*; also, $M'=M'_{st}$ obviously implies $M=M_{st}$.

Using Proposition 2.3 from [ChP1], we obtain that $\mathfrak{m}M_{\ell}\subset M_0$ for any standardizable flag $\overrightarrow{M}$. Moreover, given any standardizable pair $\{M=M_{\ell}\supset M'=M_0\}$, the number of the corresponding standardizable ℓ–flags $\overrightarrow{M}$ is $q^{\ell(\ell-1)/2}$. Furthermore, we have the following proposition.

Proposition 4.8. *For an $\mathcal{R}$–module M, let $\Delta=\Delta(M)=\Delta(\mathfrak{m}M)\cup\{d_1< d_2<,\ldots,<d_r\}$, where $d_1=\min\{\Delta(M)\}$ and $r=dim_{\mathcal{R}/\mathfrak{m}}(M/\mathfrak{m}M)$. And let $\Delta'=\Delta(M)\setminus\{g_1<g_2<,\ldots,<g_\ell\}$ for g_j taken from the set $\{d_i,\, i=2,\ldots,r\}$; it is a Γ–module (containing d_1). Setting $\{d_i\}\setminus\{g_i\}=\{d_1=g_1^\circ<g_2^\circ<,\ldots,< g_{r-\ell}^\circ\}$, the number of standardizable ℓ–flags $\overrightarrow{M}=\{M_i\}$ with $M_\ell=M$ and $\Delta(M_0)=\Delta'$ equals q^N for $N=N(M,\Delta')=$*

$$|\{g_i>g_1^\circ\}|+|\{g_i>g_2^\circ\}|+\cdots+|\{g_i>g_{r-\ell}^\circ\}|+\frac{\ell(\ell-1)}{2},\ \ 1\le i\le \ell<r.$$

Geometrically, the flags with fixed M_ℓ and $\Delta_0=\Delta(M_0)$ form an affine space $\mathbb{A}^{N(M_\ell,\Delta_0)}$. We use here the Nakayama Lemma; cf. Section 2.1 from [ORS] and Section 9.1 from [GORS]. Thus, assuming that the stratification of J_{Δ_ℓ} with respect to the Nakayama rank $r(M_\ell)$ is known, the calculation of $\mathcal{H}_{mot}(q,t,a)$ and $\mathcal{L}(q,t,a)$ becomes in terms of Δ_l,Δ_0. Using that $r(M_\ell)=const$ within J_Γ–orbits (they are affine spaces), we obtain the functional equation for $\mathcal{L}(q,t,a)$ (following Stöhr). This proposition coupled with our conjectures provides far-reaching generalizations of the so-called Shuffle Conjecture; see [CaM].

Let us relax the definition of standardizable flags by allowing g_1 to be $m=\min(D_0)$. Such flags can be called (partial) full *gap-increasing* due to $g_{i-1}<g_i$. We actually add to the standardizable ℓ–flags the standardizable $(\ell-1)$–flags from M_0 extended by $M_{-1}=M_0\cap z^{m+1}\mathcal{O}$. This gives the following connection with the usual nested Hilbert schemes:

$$(4.37)\qquad (1+a)\mathcal{Z}(q,t,a)\;=\;\sum_{m,\ell=0}^{\infty} q^{(\ell-1)\ell/2}\,a^\ell t^m\,\big|Hilb^{m+\ell,m}(\mathbb{F}_q)\big|,$$

where $|\cdot|$ is the number of points, which is directly related to the ORS conjecture. Namely, one replaces $\mathfrak{w}$ in (1.3) by the count of $\mathbb{F}_q$–points (we will not comment on that) and substitutes $q_{st}^2\mapsto t$, $a_{st}^2 t_{st}\mapsto a$, $t_{st}^2\mapsto q$; recall that q in this section is q/t via the DAHA parameters q,t.

Using Proposition 4.8. In fact, we gave a different definition of $\mathcal{H}_{mot}$ in [ChP1]: $\mathcal{H}_{mot}^{\checkmark}\stackrel{\mathbf{def}}{=\!=}t^\delta\sum_{\overrightarrow{M}\in\overrightarrow{J}}a^\ell\,q^{dev(M_\ell)}$ for the field $\mathbb{F}_{1/t}$. Accordingly, if $J_{\vec{\Delta}}$ is $\mathbb{A}^N$, its contribution to $\mathcal{H}_{mot}^{\checkmark}$ is $a^\ell q^{dev(M_\ell)}t^{\delta-N}$. It coincides with $\mathcal{H}_{mot}$ assuming its self-duality, but can be more convenient.

Let $\mathcal{R}=\mathbb{F}[[z^4, z^6+z^7]]$. For $\ell=3=deg_a(\widehat{\mathcal{H}})$, we use [ChP1],(4.1):

$$
\begin{aligned}
(4.38)\qquad & D_0 = [9,11,15],\ \vec{g} = (2,5,7),\ \dim = 8 && \rightsquigarrow q^6t^0a^3,\\
& D_0 = [7,9,11,15],\ \vec{g} = (2,3,5),\ \dim = 7 && \rightsquigarrow q^7t^1a^3,\\
& D_0 = [5,7,9,11,15],\ \vec{g} = (1,2,3),\ \dim = 6 && \rightsquigarrow q^8t^2a^3,
\end{aligned}
$$

where $D_i=\Delta_i\backslash\Gamma$, $\vec{g}$ are the gaps (consecutively) added to D_0, and we show the contributions of the corresponding cells to $\mathcal{H}^{\checkmark}_{mot}$. One has:

$$D_3 = [2,5,7,9,11,15],\ \ [2,3,5,7,9,11,15],\ \ [1,2,3,5,7,9,11,15].$$

The corresponding $M_3/\mathfrak{m}M_3$ are all of rank $r = 4$ for any M_3 for these D_3 with an important reservation. In the case of $D_3 = [2,5,7,9,11,15]$, the rank is $r = 3$ for generic M_3, and an affine subspace of codimension 1 in J_{Δ_3} must be taken to ensure $r = 4$; otherwise $\Delta[\mathfrak{m}M_3] = [7,9,11,15]$. Thus the proposition gives that the dimensions in (4.38) are (indeed) $\ell(\ell-1)/2+\ell =$ 6 plus $\dim J_{\Delta_3} = 3,1,0$ minus 1 for the first D_3. See Table 1,[ChP1]; all cells J_Δ are affine for $\mathcal{R}=\mathbb{F}[[z^4, z^6+z^7]]$.

The case $\boldsymbol{\ell=2}$, $\boldsymbol{D_2}$=[2,5,7,9,11,15]. There are now 3 possibilities for D_0 here: $D_0 = [2,9,11,15], D_0' = [7,9,11,15], D_0'' = [5,9,11,15]$; see now Table 3. The dimension of J_{Δ_2}, which is 3, must be diminished by 1 for D_0 and D_0'' due to the absence of 7 there (similar to the example above; the same subspace serves D_0 and D_0''). Then $r = 4$ and $\{d_i\} = \{0,2,5,7\}$. The summation in Proposition 4.8 becomes: $\frac{\ell(\ell-1)}{2} + \{4,2,3\} = \{7,6,6\}$ for $\{D_0, D_0', D_0''\}$. The corresponding contribution to $\mathcal{H}^{\checkmark}_{mot}$ is $2q^6t^2 + q^6t$. This matches Table 3 in [ChP1].

Z(q,t,a) for trefoil. This proposition can be equally used for ideals in $\mathcal{R}$, though for $\mathcal{R} = \mathbb{F}[[z^2, z^3]]$ this is simple. The corresponding standardizable ideals $M = \langle\cdot\rangle_d \subset \mathcal{R}$ with $d = \dim\mathcal{R}/M$ are as follows:

$$\langle 1, z^2, z^3\rangle_0,\ \langle z^2, z^3\rangle_1,\ \langle z^2+\lambda z^3\rangle_2,\ \langle z^3, z^4\rangle_2,\ \langle z^3+\lambda z^4\rangle_3,\ \langle z^4, z^5\rangle_3, \ldots,$$

where $\lambda \in \mathbb{F}$. The standardizable pairs of $\ell=1$ are $\langle z^i, z^{i+1}\rangle_{i-1} \supset \langle z^i+\lambda z^{i+1}\rangle_i$ for $i \geq 2$. Thus $\mathcal{Z} = (1+qt^2+aqt)/(1-t)$.

The pair $\mathcal{R} \supset \mathfrak{m}$ is non-standardizable of $\ell=1$; the other such pairs are $\langle z^i, z^{i+1}\rangle_{i-1} \supset \langle z^{i+1}, z^{i+2}\rangle_i$ and $\langle z^i+\lambda z^{i+1}\rangle_i \supset \langle z^{i+2}, z^{i+3}\rangle_{i+1}$. Also, there are pairs $\langle z^i, z^{i+1}\rangle_{i-1} \supset \langle z^{i+2}, z^{i+3}\rangle_{i+1}$ with $\ell=2$. Formula (4.37) gives then $(1+a)(1+qt^2+aqt)/(1-t)$, which matches (1.3).

4.3. On connection conjectures

4.3.1. HOMFLY-PT polynomials. Given a link colored by a set of Young diagrams, let $\mathrm{HOM}(q_{st}, a_{st})$ be the corresponding unreduced HOMFLY-PT polynomial. They can be defined via Quantum Groups (in type A) or using the corresponding *skein relations* and Hecke algebras. See e.g. [QS] and references there; we provide here only a sketchy discussion.

Recall that iterated torus links are determined by the pairs of graphs $\{\mathcal{L}, '\mathcal{L}\}$ colored by arbitrary sequences $\{\lambda^j\}, \{'\lambda^j\}$ of Young diagrams. We need to switch to the *reduced* HOMFLY-PT polynomials with respect to one of its components, say $j_\circ$, and then perform the hat-normalization; the notation will be $\widehat{\mathrm{HOM}}^{j_o}_{\mathcal{L},'\mathcal{L}}(q_{st}, a_{st})$. Let us mention that the q–polynomiality of the *unreduced* $\mathrm{HOM}(q_{st}, a_{st})$ generally does *not* hold for links.

We put $\widehat{\mathcal{H}}^{j_\circ}_{\mathcal{L},'\mathcal{L}}(q,t,a)_{st}$ for $\widehat{\mathcal{H}}^{j_\circ}_{\mathcal{L},'\mathcal{L}}(q,t,a)$ from Theorem 4.1 expressed in terms of the *standard topological parameters* (see [Ch2], Section 1 in [ORS], and (1.4) above):

$$
\begin{aligned}
&t = q_{st}^2,\ q = (q_{st}t_{st})^2,\ a = a_{st}^2 t_{st},\\
&q_{st}^2 = t,\ t_{st}^2 = q/t,\ a_{st}^2 = a\sqrt{t/q}.
\end{aligned}
\tag{4.39}
$$

I.e. we use the substitutions from the first line here in $\widehat{\mathcal{H}}^{j_\circ}_{\mathcal{L},'\mathcal{L}}(q,t,a)$. Taking $j_o = min$ here (the main setting of the paper) is generally "non-topological", though very reasonable algebraically.

The case $t = q$ results in HOMFLY-PT polynomials, i.e. we set $k=1$ in $t = q^k$. Recall that J–polynomials must be replaced in the definition of $\widehat{\mathcal{H}}$ by $P_\lambda^{(k=1)}$ in this case, which are Schur functions. Finally:

$$
\widehat{\mathcal{H}}^{j_o}_{\mathcal{L},'\mathcal{L}}(q, t\mapsto q, a\mapsto -a)_{st} = \widehat{\mathrm{HOM}}^{j_o}_{\mathcal{L},'\mathcal{L}}(q_{st}, a_{st}),
\tag{4.40}
$$

A combination of [ChD2] with Section 7.1 from [MS] (the case of iterated knots) proves (4.40) for any iterated *links*. Another way to justify this coincidence is via a relatively straightforward generalization of Proposition 2.3 from [Ch2] (for torus knots), where we used [Ste]; see also [ChD1]. This approach is based on the DAHA shift operators and Verlinde algebras. Also, instead of using [MS] or the knot operators from *CFT* (and the Verlinde algebras), one can directly apply here the *Rosso-Jones cabling formula* [RJ, Mo, ChE] upon its relatively straightforward adjustment to iterated torus links.

4.3.2. Khovanov-Rozansky theory. Let us restrict ourselves to $\lambda^i = \Box = '\lambda^j$ for all i, j (the uncolored case). Then $\{J_\Box\}_{ev} = t^{1/2}(1+a)/(a^2)^{1/4}$

and we conjecture that for the hat-normalization of the stable KhR polynomials:

$$\left(\frac{\widehat{\mathcal{H}}^{\emptyset}_{\mathcal{L},{}'\mathcal{L}}}{(1-t)^{\kappa+{}'\kappa}}\right)_{st} = \left(\frac{(1+a)\,\widehat{\mathcal{H}}^{min}_{\mathcal{L},{}'\mathcal{L}}}{(1-t)^{\kappa+{}'\kappa}}\right)_{st} = \widehat{KhR}^{\,\mathrm{stab}}_{\mathcal{L},{}'\mathcal{L}}(q_{st},t_{st},a_{st}). \tag{4.41}$$

The topological setting is *unreduced* here; recall that $j_o=\emptyset$ in the first term means that we do not divide by the evaluations of Macdonald polynomials at t^ρ. Unreduced $KhR^{\,\mathrm{stab}}$ are polynomials in terms of a_{st}, q_{st}, but their coefficients are generally infinite t_{st}*–series*.

The stable *Khovanov-Rozansky homology* is the sl_N homology from [KhR1, KhR2, Rou] in the range of N where the isomorphism in Theorem 1 from [Ras1] holds (see also [Kh]). They can be obtained for any N from the *triply-graded HOMFLY-PT homology*, assuming that the corresponding *differentials* are known, which are generally involved.

Let us also mention the relation to the *Heegaard-Floer homology*: $N=0$. Also, the *Alexander polynomial* of the corresponding singularity is $\widehat{\mathcal{H}}^{min}_{\mathcal{L}}(q,q,a=-1)/(1-q)^{\kappa-\delta_{\kappa,1}}$ in the case of one *uncolored* graph $\mathcal{L}$ with κ paths (the number of connected components in the corresponding link). This is the *zeta-monodromy* from [DGPS] upon $t\mapsto q$ (unless for the unknot). The DAHA parameters are used here.

The ORS conjecture, namely Conjecture 2 from [ORS], states (in the unreduced setting) that $KhR^{\,\mathrm{stab}}_{\mathcal{L},{}'\mathcal{L}} = \overline{\mathscr{P}}_{\mathrm{alg}}$, where the latter series is defined there for (the germ of) the corresponding plane curve singularity $\mathcal{C}$ from (3.6) in terms of the *weight filtration* in the cohomology of its nested Hilbert scheme. This (conjecturally) connects the DAHA superpolynomial upon the division from (4.41) with $\overline{\mathscr{P}}_{\mathrm{alg}}$ under the hat-normalization. See Section 4.2.5 above.

There are also other conjectures connecting KhR polynomials with rational DAHA, Gorsky's combinatorial polynomials (for torus knots), Hilbert scheme of $\mathbb{C}^2$ and physics superpolynomials (the name "superpolynomials" came from [DGR]). We will not discuss these and other related directions in the present paper.

4.4. Riemann Hypothesis

We are now ready to state *RH* for DAHA superpolynomials. The notation is from the previous section. We will use Theorem 4.1; see also formula (4.19) for $\deg_a\widehat{\mathcal{H}}$.

4.4.1. The RH-substitution. For any positive pair of graphs $\{\mathcal{L}, {}'\mathcal{L}\}$, let

$$\widehat{\mathcal{H}}(q,t,a) = \widehat{\mathcal{H}}^{min}_{\mathcal{L},'\mathcal{L}}(q,t,a) = \sum_{i=0}^{d} \mathcal{H}^i(q,t)a^i \quad \text{for} \quad d = \deg_a \widehat{\mathcal{H}}(q,t,a),$$
$$H(q,t;a) = \widehat{\mathcal{H}}(q \mapsto qt, t, a), \quad H^i(q,t) = \mathcal{H}^i(q \mapsto qt, t), \ 0 \le i \le d.$$

We also set $\widehat{H}^i(q,t) \stackrel{\mathbf{def}}{=\!=} q^{-m}t^{-n}H^i(q,t)$ for the minimal degrees m,n of q,t in H^i. Switching from $\mathcal{H}^i$ to their *hat-normalizations* $\widehat{\mathcal{H}}^i$ with the constant term 1 as above, one has $\widehat{H}^i(q,t) = \widehat{\mathcal{H}}^i(qt,t)$. They are considered as polynomials in terms of t, and we will almost entirely switch below to $\omega \stackrel{\mathbf{def}}{=\!=} 1/q$ from q. The super-duality for $H(q,t,a)$ is now for the map $q \mapsto q, t \mapsto 1/(qt)$.

For any uncolored algebraic links, the t–degree of $\widehat{H}^0(q,t)$ is conjecturally the sum of δ–invariants of its components plus $(\kappa-1)$ for the number κ of the components, which follows from Conjecture 2.4 of [ChD1]; for any a, see Conjecture 2.3 from [ChP1]. Indeed, the top term in $\mathcal{H}^i$ is "diagonal" for uncolored algebraic knots, i.e. of the form $(qt)^{m_i}$ for any i due to the DAHA super-duality; the passage to links results from (3.15).

We mostly stick to rectangle Young diagrams in this work; the square diagrams are especially valuable for us because they are transposition-invariant. The (expected) geometric interpretation of DAHA superpolynomials is directly related to RH below; this is known for $\square$ and any columns [ChP1, ChP2]. For non-square rectangles, $l \times m$ (corresponding to $m\omega_l$), one can do the following *symmetrization*:

$$H^i \mapsto H^i_{sym} \stackrel{\mathbf{def}}{=\!=} H^i(q,t) + H^i(q, 1/(qt)) \quad \text{for} \quad 0 \le i \le \deg_a \widehat{\mathcal{H}}. \tag{4.42}$$

Then we employ the hat-normalization: $\widehat{H}^i_{sym} \stackrel{\mathbf{def}}{=\!=} \widehat{H^i_{sym}} = q^{-m}t^{-n}H^i_{sym}$ for the minimal degrees m,n of q,t in H^i_{sym}. We *always* switch to $\widehat{H}^i_{sym}$ below if the transposition of all Young diagrams involved changes the isomorphism class of the diagram/link $\mathcal{L}$ or the pair $\{\mathcal{L},'\mathcal{L}\}$.

4.4.2. Algebraic RH. We assume that $\omega = 1/q$ is real positive. Recall that $\widehat{\mathcal{H}} = \widehat{\mathcal{H}}^{min}_{\mathcal{L},'\mathcal{L}}(\boldsymbol{\lambda}, {}'\boldsymbol{\lambda}; q,t,a) = \sum_i \mathcal{H}^i a^i$ for $0 \le i \le \deg_a \widehat{\mathcal{H}}$. Also $\widehat{H}^i$ is the hat-normalization of $\mathcal{H}^i$ upon the substitution $q \mapsto qt$; they are polynomial in terms of q,t with the constant term 1. For the sake of definiteness, we assume that $\deg_q \le \deg_t$ in $\widehat{\mathcal{H}}(a=0)$, employing the super-duality if necessary. For instance, $l \ge m$ will be always imposed for *knots* colored by rectangles $l \times m$ (columns instead of rows).

Furthermore, let $\mathcal{H}^i_\bullet = \mathcal{H}^i$, $\widehat{H}^i_\bullet = \widehat{H}^i$ if the corresponding colored link is self-dual with respect to the transposition of the diagrams in $\boldsymbol{\lambda}$, $'\boldsymbol{\lambda}$ (and the equivalence of graphs). Otherwise $\mathcal{H}^i_\bullet = \mathcal{H}^i_{sym}$, $\widehat{H}^i_\bullet = \widehat{H}^i_{sym}$.

Conjecture 4.9. *(i) For any pair of iterated links $\mathcal{L}, '\mathcal{L}$ (possibly non-algebraic), any colors (Young diagrams) and arbitrary $0 \le i \le deg_a \widehat{\mathcal{H}}$, the following limits exist and coincide:*

$$\text{(4.43)} \qquad \widehat{H}^i_\dagger = \varsigma_i t^{\pi_i}\, S^i(t) \overset{\mathbf{def}}{=\!=} \lim_{q\to 0} \widehat{H}^i_\bullet\left(q, t\mapsto \frac{t}{q^{\frac{1}{2}}}\right) = t^{-\pi_i} \lim_{q\to 0} \mathcal{H}^i_\bullet\left(q, t\mapsto \frac{t^2}{q}\right),$$

where $\varsigma_i = \pm$, $\pi_i \ge 0$, $S^i(t{=}0) = 1$. Moreover, the relation $2(\pi_i+\sigma_i) = deg_t \widehat{H}^i$ always holds, where we set $2\sigma_i \overset{\mathbf{def}}{=\!=} deg_t S^i$.

(ii) All zeros of the polynomials S^i are roots of unity if $\mathcal{L}, '\mathcal{L}$ are colored only by rectangle diagrams. They are simple for any iterated torus links if either $i = 0$ or they are uncolored. Furthermore, $S^i = (1 - t^{2\sigma_i+2})/(1 - t^2)$ for uncolored algebraic knots and $S^i = (1 + t^{2\sigma_i})$ for those colored by columns upon the symmetrization (including $\square$); i are arbitrary and $\pi_i = 0$ in such cases. For uncolored algebraic links, π_0 coincides with the number of components minus 1.

The passage to t^2/q in (i) is not difficult to justify; thus S^i are actually polynomials in terms of t^2. This is somewhat parallel to [OnRS] and a recent work [GORZ] of Griffin, Ono, Rolen, and Zagier. We expect that at least in the uncolored case, the invariants π_i and $S^i(t)$ have topological meaning beyond iterated torus links. For instance, if the stable, reduced and hat-normalized KhR polynomials at $a = 0$ are used instead of $\widehat{\mathcal{H}}$, then $(\pi_i + \sigma_i)$ is presumably the *4-genus* for any *positively iterated torus knot* (not only algebraic). See [Ras2] for the connection to the Khovanov polynomial and other invariants and also [Hed]. For instance, assuming that $i \le 2j$ in $\widehat{KhR}^{\text{stab}}_{\text{red}}(a_{st}{=}0) = \sum_{i,j} c_{i,j} q_{st}^{2i} t_{st}^{2j}$ (possibly upon some adjustment of i, j), π_0 and $S^0(t)$ can exists for the sum of borderline terms $\sum_i c_{2i,i} q_{st}^{4i} t_{st}^{2i}$ not only for algebraic knots.

For our *geometric superpolynomials* of algebraic knots, $\pi_i + \sigma_i = \delta$, e.g. $= (\mathsf{r}-1)(\mathsf{s}-1)/2$ for $T(\mathsf{r}, \mathsf{s})$. This becomes significantly more involved for non-rectangle Young diagrams. It seems that *non-cyclotomic factors of S^i appear only for non-rectangle diagrams* for any knots.

The diagram $\boxplus$ deserves a special comment; no symmetrization is necessary here. In this case, there are no multiple zeros of S^i for algebraic knots in the examples we reached (though not many) for any $i \ge 0$. Also,

$\pi_i = 0$ for even i and $\pi_i = 1$ for odd i, when $\widehat{H}^i$ always has two *trivial* irregular zeros $\{-1, -\omega\}$. Correspondingly (conjecturally) for even and odd i: $\varsigma_i t^{\pi_i} S^i = \frac{1-t^{2\sigma_i+4}}{1-t^4}$ and $\varsigma_i t^{\pi_i} S^i = t\frac{1-t^{2\sigma_i+2}}{1-t^4}$.

Part (i) is expected to hold under *antisymmetrization* instead of symmetrization, which is $H^i_{asym} \stackrel{\mathbf{def}}{=\!=} H^i(q,t) - H^i(q, 1/(qt))$ followed by $H^i_{asym} \mapsto \widehat{H}^i_{asym}$. As above, we assume that $\deg_q \leq \deg_t$ in $\widehat{\mathcal{H}}(a=0)$; in particular, $l > m$ for non-square diagrams $l \times m$. Then the properties of π_i, S^i generally become somewhat "better" vs. the symmetrization. Extra *trivial* zeros of $\widehat{H}^i$ now can emerge, which are $t = \pm\sqrt{\omega}$. Presumably $S^i = t^{2\sigma_i} - 1$ in this case for any algebraic knots colored by columns and for any i.

4.4.3. Analytic aspects. Conjecture 4.9 gives that the number of non-*RH* (irregular) zeros of $\widehat{\mathcal{H}}^i$ in the vicinity of $q = 1/\omega = 0$ is no greater that $2\pi_i$ plus the number of non-unimodular zeros of S^i and multiplicities of its multiple zeros. It can be smaller when some multiple zeros of S^i are unimodular (not always). We arrive at the following.

Conjecture 4.10. *(i)* Strong RH . *For an arbitrary uncolored algebraic knot and any given $0 \leq i \leq deg_a \widehat{\mathcal{H}}$, there exists $\omega' = \omega'_i > 0$ such that for all $\omega > \omega'_i$, the t–zeros ξ (if any) of $\widehat{H}^i$ are all complex, simple and satisfying the RH*–equality*: $|\xi| = \sqrt{\omega}$.*

(ii) The same holds for algebraic knots colored by $\boxplus$ if the trivial zeros *$\xi = -1, -\omega$ are omitted, which occur in $\widehat{H}^i$ if and only if i is odd. Also, $\widehat{H}^i_{sym}$ satisfies (i) for columns (any $i \geq 0$) and non-square rectangles $l \times m$ with $l > m$ for $i = 0$.*

(iii) Weak RH . *Given an uncolored algebraic link, there exists ω'_0 such that the number of pairs of* stable irregular zeros *of $\widehat{H}^{i=0}$ (satisfying $|\xi| \neq \sqrt{\omega}$) for $\omega > \omega'_0$ equals the number of components minus* 1.

A connection is expected with the *spectral zeta-functions*, especially in the case of Schottky uniformization of Riemann surfaces; see e.g. [CM]. The following conjecture is of this nature.

Conjecture 4.11. *For any uncolored algebraic* knot, *$\varpi_i \stackrel{\mathbf{def}}{=\!=} \inf \omega'_i$ for $i = 0$ is smaller than 2. Moreover, $\lim_{m\to\infty} \varpi_0 = 2$ for $\mathcal{R} = \mathbb{C}[[z^4, z^6 + z^{7+2m}]]$ corresponding to the cables $Cab(13+2m, 2)T(3,2)$, which sequence of ϖ_0 is actually increasing when m* mod 4 *is fixed. Also,* $\sup_m \varpi_1 = 2.2132458\ldots$, $\sup_m \varpi_2 = 1$.

This is actually the only "family" with ϖ_0 near 2 we found. Let us provide ϖ_i for $i = 0, 1, 2$ in the following 3 cases:

$$Cab(13\ ,2)T(3,2) : \varpi_i = 1.495583269,\ 2.176487419,\ 0.9430445115,$$
$$Cab(113,2)T(3,2) : \varpi_i = 1.993679388,\ 2.134669951,\ 0.9955504853,$$
$$Cab(313,2)T(3,2) : \varpi_i = 1.997705951,\ 2.210868584,\ 0.9992171315.$$

Using formula (4.28) for $m = 110, 210, 310, 510, 1010, 1510$, we have:

$$\varpi_0 = 1.996921, 1.998340, 1.998863, 1.999303, 1.999645, 1.99976261$$

and the corresponding

$$q = \frac{1}{\varpi_0} = 0.500771, 0.500415, 0.500284, 0.500174, 0.500088, 0.50005935.$$

The convergence is the best for $m \bmod 4 = 1, 2$. Such limits can be generally calculated using the conjectural coincidence of $\widehat{\mathcal{H}}$ with the geometric superpolynomials from [ChP1].

This family also has the largest ϖ_i for $i = 1, 2$ among uncolored algebraic knots we considered. Here $\deg_a \widehat{\mathcal{H}}=3$, and $\widehat{H}^3=1$; $\varpi_i(i > 0)$ exceed 2 only for this family in our "database".

The greatest ϖ_0 we found so far among uncolored algebraic *links* is 2.062433590332 for $(Cab(5,3), Cab(4,3))T(1,1)$ corresponding to the coinvariant $\{\gamma[1,1](\gamma[3,1]P(1)\gamma[3,2]P(1))\}$ in the notation from the table. Here Weak *RH* holds for all i. The corresponding singularity is $(x^5-y^3)(x^3-y^4)=0$ with $Z = 1 + q^7 + q^8 + q^{14} + q^{15} + q^{16} + q^{22} + q^{23} + q^{30}$ and $lk = 9$ (in this table, $\varpi_0 < 2$ for uncolored algebraic links).

4.4.4. Comments. For algebraic knots colored by columns, the statements of Conjecture 4.9 can be verified if one switches from DAHA superpolynomials $\widehat{\mathcal{H}}$ to geometric (motivic) ones. The proof goes as follows. Only J_Δ with dimensions no smaller than $dev(\Delta)$ contribute to the limit of $\mathcal{H}_{mot}$ from Definition 4.4. Given $dev(\Delta)$, such Δ occurs only for a *single* Δ, namely, the one obtained by adding *consecutive* gaps to Γ starting with the top one, which is $\max\{\mathbb{Z}_+ \setminus \Gamma\}$.

The generalization to any powers of a and columns follows from [ChP1, ChP2]. Part (i) of Conjecture 4.9 for (at least) algebraic knots upon $a = 0$ and in the case of columns can be managed within the DAHA theory. We use that only positive powers of $\tau_\pm$ appear in the formulas and that X_{ω_i} are *nonsymmetric* Macdonald polynomials.

An extension of Part (*ii*) there to *rectangle* Young diagrams larger that $\boxplus$ is likely, but there is a lack of (numerical) examples. Also, the restriction $i=0$ in our conjectures can presumably be replaced by "for any even i", but we will stick to $i=0$ in this paper.

Torus knots. The polynomials $\widehat{H}^i$ at $\omega=1$ are products of cyclotomic polynomials for *uncolored torus knots* and any $i \geq 0$. They can be calculated explicitly; for $a = 0$, the formula can be deduced from the *shuffle conjecture* (proven in [CaM]). In the absence of multiple zeros at $\omega=1$ (which is always true for $i=0$), this implies RH in some neighborhood of $\omega=1$. However this is not generally the case for $i > 0$ (for torus knots); multiple zeros do occur. We note that ϖ_0 can be beyond 1 for uncolored torus knots for sufficiently general torus knots. Then RH fails somewhere between $\omega = 1$ and ϖ_0.

We mention that there are sufficiently explicit known/conjectured formulas for uncolored torus knots $T(m, km+1)$. See [GM, Mel1, ChP1] and also [ORS, DMS, FGS]; the proof of the formula for DAHA superpolynomials colored by any rows for $T(2m+1, 2)$ is in [Ch3]. Even when explicit formulas are known/conjectured, they are generally involved for explicit finding ϖ_i, but can be helpful (at least) to examine the point $\omega = 1$ for torus knots and obtain *family superpolynomials* (actually rational functions) as was done in (4.22,4.25,4.28).

Note that $H^i(\omega=1)$ upon $t \mapsto q_{st}^2$ is the a_{st}^{2i}–coefficient of the corresponding HOMFLY-PT polynomial up to $\pm q_{st}^{\bullet}$. See (4.40). This coincidence for torus knots was justified in [Ch2] using [Ste]. See also [MS, ChD2] for any iterated torus knots and links.

Asymptotic class numbers. A natural application of Conjecture 4.11 is to the growth estimates for $|J(\mathbb{F}_q)|$ for the Jacobian factors $J = J_{\mathcal{R}}$ from Definition 4.4. This is a classical track (for any curves). The conjecture that RH holds for $\omega = 1/q \leq 2$ provides some estimates for $|J(\mathbb{F}_q)|$ with any q. Here we obtain "pure singular" contributions; the smooth case is covered by the Weil RH.

The switch to the *families* and the family polynomials $\widehat{\mathbb{H}}$ from (4.21) seems the most relevant here. This can potentially clarify the parallelism between the Alexander knot polynomials and *Iwasawa theory* observed by B.Mazur; see e.g. [Mor] and around (6.46) below. The Iwasawa polynomials give exact formulas for the growth of ideal class groups inΓ-extensions; we do the towers of Puiseux field extensions in the *families* for $i = 1$. This is connected with the so-called *Drinfeld-Vladut bound*, but superpolynomials provide *exact* formulas (as in the Iwasawa theory) ; see e.g. [GaS].

4.5. Non-classical features

Similar to the classical Riemann zeta, all $t^\bullet \widehat{H}^i$ are real-valued at $U_{\sqrt{\omega}} \stackrel{\mathbf{def}}{=\!=} \{z\in\mathbb{C} \mid |z|=\sqrt{\omega}\}$ for proper powers of t due to the super-symmetry (our functional equation) and the reality of $\widehat{H}^i$. Also, if ξ is a zero of $\widehat{H}^i$, then $\sqrt{\omega}/\,\overline{\xi}$ is its zero with the same angle, where $\overline{\xi}$ is the complex conjugation of ξ.

4.5.1. The range of $\boldsymbol{\omega}$. There are new opportunities here vs. the classical theory, since ω is arbitrary for us (a free parameter).

Lemma 4.12. *Assuming that ω_i' from Conjecture 4.10,(iii) exists for some $i \geq 0$, let ϖ_i be (as above) the lowest such ω_i'. Then ϖ_i is a (real) root of the* reduced t–discriminant *D^i of $\widehat{H}^i$, which is the product of all simple factors in the actual discriminant of $\widehat{H}^i$. For instance, ϖ_i is an algebraic number; it coincides with the greatest real zero ω_i^{top} of D^i if we add the simplicity of zeros ξ for $\omega > \omega_i'$ to the definition of ω_i'.*

Also, assuming that $\widehat{H}^i$ satisfies Weak RH in an interval beyond (greater than) ω_i^{top}, Weak RH holds then for all $\omega \geq \omega_i^{top}$. The same is true if Weak RH holds in the interval $\omega_i^{top} - \epsilon < \omega < \omega_i^{top}$ for some $\epsilon > 0$ and $\widehat{H}^i$ has no multiple roots at $\omega = \omega_i^{top}$ of norm $\sqrt{\omega}$.

Proof. Here we use that the zeros apart from $U_{\sqrt{\omega}}$ vanish or emerge only at (real) zeros of D^i. Indeed, they appear in pairs $\{z, z'\}$ with coinciding angles and therefore create multiple zeros of $\widehat{H}^i$ when approaching $U_{\sqrt{\omega}}$.

In particular, if Weak *RH* holds for ω in an interval greater than ω^{top}, then the formation of a non-*RH* pair (i.e. that apart from $U_{\sqrt{\omega}}$) at some $\omega > \omega_i^{top}$ results in a multiple zero of $\widehat{H}^i$ beyond ω_i^{top}, which is impossible. Similarly, if Weak *RH* holds for $\omega_i^{top} - \epsilon < \omega < \omega_i^{top}$, then the multiple zeros at $\omega = \omega_i^{top}$ can emerge only from some pairs of zeros of norm $\sqrt{\omega}$. $\square$

4.5.2. Non-RH zeros. By *RH* for links, we will always mean Weak *RH* from Conjecture 4.10, (*iii*), allowing multiple zeros and $\kappa - 1$ super-dual pairs of (stable) irregular zeros for κ branches. Accordingly,

$$\varpi_i \stackrel{\mathbf{def}}{=\!=} \inf\{\omega' \mid \text{Weak RH holds for } \widehat{H}^i \text{ for } \omega \geq \omega'\}; \tag{4.44}$$

this is a real zero of D^i from the lemma.

Thus, ϖ_i conjecturally exist for uncolored algebraic *knots* for any $i \geq 0$. Also, they exist for algebraic knots colored by (a) $\boxplus$, where *trivial zeros*

$-1, -\omega$ are excluded, (b) columns ω_l upon the symmetrization, (c) any rectangles upon the symmetrization and for $i = 0$.

Concerning (4.44), it is possible that $\varpi_i < \omega_i^{top}$; then multiple zeros appear after ϖ_i; this occurs only twice in the table. Also, there are no reasons for "nice" formulas for ϖ_i, ω_i^{top}, but at least they are algebraic numbers (zeros of D^i) and can be calculated as exactly as necessary.

If $\varpi_i = 0$, then Weak RH holds for any $\omega > 0$. This happens for any uncolored $T(2m+1, 2)$ and for iterated Hopf-type links $2T(m, 1)$. Here $\deg_a = 2$ and $i = 0, 1$. For the Hopf 2-link, $\varsigma_0 t^{\pi_0} S^0 = t\frac{1-t^{2m}}{1-t^2}$. It corresponds to the 2–branch singularity $(x^m - y)(x^m + y) = 0$ with the linking number $lk = m$. One pair of irregular (real) zeros of H^0 approaching $\{1, \omega\}$ as $\omega \to \infty$ occurs here after $\omega_0^{top} = (m+1)^2/m^2$; the other zeros for any $\omega > 0$ are of norm $\sqrt{\omega}$. Thus $\omega_0^{top} > \varpi_0$ in this case. There is only one instance of non-Hopf link in the table when this happens: entry 35 ($N_z^0 = 26$).

5. RH numerically

We calculated quite a few examples in the range $\deg_a \leq 8$ (and sometimes beyond). The most instructional ones are collected in the table below, though we provide many examples beyond it.

5.1. Table organization

It is based on the $\widehat{\mathcal{H}}$–polynomials above with the following deviation: we consider $\widehat{H}^i$ *not* only for algebraic links/knots. The DAHA construction is fully applicable without any positivity conditions for r, s, though the connection with plane singularities will be lost then. Numerical experiments clearly indicate that the class of iterated links with $\widehat{\mathcal{H}}$–polynomials satisfying RH is wider than algebraic links only (combinatorially, *positive* pairs of graphs).

5.1.1. Main notations. *By RH, we will mean below Weak RH* from (*iii*) of Conjecture 4.10; multiple zeros will be allowed. The number of pairs $\{\xi, \omega/\xi\}$ of *irregular* zeros of $\widehat{H}^i$ is shown in the last column of the table below after "$-$". Since $|\xi| > \sqrt{\omega}$ (assuming the simplicity) for one of the zeros of a non-RH pair, they significantly influence expected counterparts of Weil-style estimates (those in his proof of RH).

Status. For knots and links we simply put "knot" or "link" in the corresponding entry; "alg" or "~~alg~~" are naturally stand for algebraic and non-algebraic knots/links. The first column provides the status of Weak RH with the following 3 options:

"*HOLDS*", if it holds for all a^i, "*FAILS*", when it fails at $a{=}0$,
"*OK* $a{=}0$", if it holds for $a{=}0$ but fails for some other a^i $(i{>}0)$.

In the "*OK*"-case, we give the range of a^i when RH holds; the total number of polynomials $\widehat{H}^i$ is also provided, which is $(\deg_a + 1)$. We calculate only until the first failure of RH in this table; then we stop. We numerate the examples and provide he corresponding numbers N_z^0 of all zeros of $\widehat{H}^0$ (only for $i{=}0$) after "No". See the first column in the table below. Note that $N_z^0 = 2\delta$ for algebraic knots.

The last two columns contain ϖ_0 and the maximum, denoted by $\varpi_{1\ldots}^{max}$, of $\varpi_1, \varpi_2, \ldots$ calculated till the first failure of RH. They can be zero when all $\omega > 0$ satisfy RH. We put "$-$" for ϖ_0 if RH fails at $a = 0$, and make $\varpi_{1\ldots}^{max} =$"$-$" if RH fails for $a = 0$ or fails for a^1.

5.1.2. DAHA formulas. The 4th column contains the DAHA formula used to calculate the corresponding $\widehat{\mathcal{H}}$, where $\{\cdots\}$ means the coinvariant and $\gamma[\mathsf{r},\mathsf{s}]$ is understood as the lift of $\gamma_{\mathsf{r},\mathsf{s}} \in PSL_2(\mathbb{Z})$ to $\widehat{\gamma}_{\mathsf{r},\mathsf{s}} \in PSL_2^{\wedge}(\mathbb{Z})$. We somewhat abuse the notations by omitting $H\Downarrow = H(1)$; the a–stabilization is also assumed. For instance, $\{\gamma[3,2]\gamma[2,1](P)\}$ must be understood as $\{\widehat{\gamma}_{3,2}(\widehat{\gamma}_{2,1}(P_\Box)\Downarrow)\}$, *not* as a product of two γ inside the coinvariant, and upon the further a–stabilization.

Also, we actually use the J–polynomials (not P–polynomials) for links and always do division by the LCM of the evaluations at t^ρ; see the definition of the min-normalization of $\widehat{\mathcal{H}}$. In the table, we simply put P assuming the rest. There are the following abbreviations: P stands for $P_\Box = P_{\omega_1}$, $P(1+1), P(2+0), P(2+1), P(2+2)$, and $P(3+3)$ stand for $2\omega_1 = \Box\!\Box$ and $\omega_2 = \begin{smallmatrix}\Box\\\Box\end{smallmatrix}$, $\omega_1+\omega_2 = \begin{smallmatrix}\Box\Box\\\Box\end{smallmatrix}$ (the hook), $2\omega_2 = \begin{smallmatrix}\Box\Box\\\Box\Box\end{smallmatrix}$, $2\omega_3 = \begin{smallmatrix}\Box\Box\\\Box\Box\\\Box\Box\end{smallmatrix}$. Recall that if some of r,s are negative then the corresponding link/knot is non-algebraic.

5.1.3. Basic cables. The cab-presentations in the table are partial: we omit the *arrows* and do not show at which vertex the corresponding polynomials are inserted. This can be seen from the DAHA presentations. For instance, entry 46 with the number of zeros $N_z^0 = 72$ is represented by the graph $\circ\!\!\stackrel{\to}{\searrow}\!\circ\!\!\to$ with both arrows colored by $\Box$ (omitted) and the $[\mathsf{r},\mathsf{s}]$–labels $[5,2]$ (for the left vertex) and $[2,1]$. Also, applying $P(Y)$ in the DAHA-formula means that we consider a pair of graphs: the one shown in the table and $'\mathcal{L}$ that is a pure arrow $\to$ colored by $\Box$ (without vertices).

To give another example, $\big(Cab(11,3), Cab(11,3)\big)T(3,1)$ from entry No$=$ 45 (with 94 zeros) means the tree $\circ\!\rightrightarrows\!\begin{smallmatrix}\circ\\\circ\end{smallmatrix}\!\rightrightarrows$, where the first vertex is labeled by $[3,1]$ and the other two by (coinciding) labels $[3,2]$ with the arrows colored by $\Box$. RH holds here for all $\widehat{H}^i$ (from $i = 0$ through $i = \deg_a = 5$). This link is algebraic, $\varpi_0 = 1.39031$ (we calculate them with much greater accuracy), $\varpi^{max}_{1,\dots} = 1.55923$, which is the maximum of ϖ_i from $i\!=\!1$ to $i\!=\!6$ in this case. Generally, this maximum is taken from $\widehat{H}^1$ till the last $\widehat{H}^i$ where RH holds.

The last column gives the corresponding number of pairs of *irregular* zeros, where N^{max}_{irz} is calculated in the same manner; the first N_{irz} is calculated right after ϖ and the second one (after "$-$") is the stable number of such pairs (for any large ω).

5.2. On Conjecture 4.9

The presence of *non-unimodular* roots of polynomials $S^i(t)$ from Conjecture 4.9 is interesting. The only examples we found so far are only for non-rectangle diagrams, which includes links (possibly non-algebraic). Many examples were considered beyond the table and for all i, not only those till the first failure of RH, which are presented in the table below.

5.2.1. The case of 3-hook. We will give here all links from the table where S^i are not products of cyclotomic polynomials, also providing the corresponding π_0, S^0, even if S^0 are such products. Actually all S^0 are products of cyclotomic polynomials within the table (so only $S^{i>0}$ can be non-cyclotomic), but is not always the case. We will provide below an example when S^0 is non-cyclotomic. Also, *In all examples we considered* $\varsigma_i t^{\pi_i} = -t^3$, so we will omit it. This factor contributes 3 to the number of stable irregular (non-RH) pairs of zeros of $\widehat{H}^i$. Thus, let us focus on S^i.

(1) $T(5,2), \{\gamma[5,2](P(2+1))\}$:

$(i=0)$ $\varsigma_0 t^{\pi_0} S^0 = -t^3\big(\frac{1-t^6}{1-t^2}\big)\frac{1-t^{12}}{1-t^2}$, where the quantity $\big(\dots\big)$ is the multiple part of S^0. So there are potentially 4 pairs of non-RH zeros due to S^0 and 3 pairs due to t^3; this matches entry $No = 57$ with $N^0_z = 20$ from the table.

$(i=1)$ $\varsigma_1 t^{\pi_1} S^1 = -t^3(1+t^2)(1+2t^2+2t^4+3t^6+2t^8+2t^{10}+t^{12})$, where the latter factor is irreducible and *non-cyclotomic*. It has 4 non-unimodular zeros, which matches the total number of 6 non-RH pairs for $\widehat{H}^1$ (this is not in the table).

(2) $T(4,3), \{\gamma[4,3](P(2+1))\}$:

$(i=0)$ $S^0 = \frac{1-t^{20}}{1-t^2}\frac{1-t^8}{1-t^2}$, where the multiple zeros come from $\frac{1-t^4}{1-t^2}$. The expected number of non-RH pairs is therefore 5, which matches entry 59 with $N^0_z = 30$ in the table.

$(i=1)$ $S^1 = \frac{1-t^9}{1-t^2}(1+2t^2+2t^4+t^6+t^8+t^{10}+2t^{12}+2t^{14}+t^{16})$, where the latter factor is irreducible and *non-cyclotomic*.

(3) $T(5,3), \{\gamma[5,3](P(2+1))\}$:

$(i=0)$ $S^0 = \frac{1-t^{28}}{1-t^2}\frac{1-t^{10}}{1-t^2}$, where there are no multiple roots of S^0; this matches the number of non-RH pairs, which is 3, in entry 60 with $N_z^0 = 40$.

$(i=1)$ $S^1 = \frac{1-t^4}{1-t^2}\frac{1-t^6}{1-t^2}\frac{1-t^8}{1-t^4}\frac{1-t^{10}}{1-t^2}(1-t^4+t^6+t^{10}-t^{12}+t^{16})$, where there are no multiple roots and the last factor is irreducible *non-cyclotomic*.

$(i=3)$ $S^3 = (1+t^2)^2\frac{1-t^6}{1-t^2}(1+2t^2+2t^4+2t^6+t^8+t^{10}+t^{12}+2t^{14}+2t^{16}+2t^{18}+t^{20})$, where the latter factor is *non-cyclotomic*; we have totally 2 multiple roots and 4 non-unimodular zeros, so the expected number of non-RH zeros is $4+4+6=14$. This corresponds to the actual number for $\omega > 3.402358077$ (not in the table).

(4) $T(7,3), \{\gamma[7,3](P(2+1))\}$:

$(i=0)$ $S^0 = \frac{1-t^{44}}{1-t^2}\frac{1-t^{14}}{1-t^2}$, which has no multiple nonzero zeros, and the number of pairs of non-RH zeros in entry 61 with $N_z^0 = 60$ is 3 indeed.

$(i=1)$ $S^1 = (1+t^2)(1+2t^2+3t^4+4t^6+5t^8+6t^{10}+6t^{12}+7t^{14}+6t^{16}+7t^{18}+6t^{20}+7t^{22}+6t^{24}+7t^{26}+6t^{28}+7t^{30}+6t^{32}+7t^{34}+6t^{36}+7t^{38}+6t^{40}+6t^{42}+5t^{44}+4t^{46}+3t^{48}+2t^{50}+t^{52})$, where the last factor is irreducible *non-cyclotomic*.

$(i=3)$ $S^3 = (1+t^2)^2\frac{1-t^{14}}{1-t^2}\frac{1+t^{10}}{1+t^2}(1+t^2+t^4+t^6-t^{12}+t^{18}+t^{20}+t^{22}+t^{24})$, with *non-cyclotomic* last factor, 4 non-unimodular zeros and 2 multiple zeros $\pm\imath$ due to $(1+t^2)^2$. The total number of non-RH pairs is expected $2+(4/2)+3=7$, which is actually greater than the actual number 5; these 5 pairs occur after a huge $\varpi_3 = 159557.4798$. Thus, the q–deformations of $\pm\imath$ are RH–zeros for any $\omega >> 0$ in this case.

5.2.2. Non-cyclotomic $S^{i=0}$. The last example we will provide is for the 3–hook $\boxplus$ where S^0 is not a product of cyclotomic polynomials (even for $i=0$). This is not in the table.

$T(7,2), \{\gamma[7,2](P(2+1))\}$:

$(i=0)$ $\varsigma_0 t^{\pi_0} S^0 = -t^3\frac{1-t^{10}}{1-t^2}(1+t^2+2t^4+t^6+t^8+t^{10}+2t^{12}+t^{14}+t^{16})$, where the last factor has 4 pairs of non-unimodular zeros, which results in the total of 7 pairs of non-RH pairs for $\widehat{H}^0$. These 7 pairs occur after $\varpi_0 = 6522.513197$. The total number of zeros here is $N_z^0 = 30$.

This is actually unexpected because there is quite a regular behavior of S^0 for the family $T(3m\pm1,3)$. Namely, the following formula for $\boxplus$ is likely to hold for this family:

$$\varsigma_0 t^{\pi_0} S^0 = -t^3\, \frac{1-t^{4(2n-3)}}{1-t^2}\, \frac{1-t^{2n}}{1-t^2} \quad \text{for} \quad T(n,3),\ n = 3m \pm 1 > 2.$$

It was checked in the examples above and for $T(8,3)$. It obviously collapses for $T(3,2)$, where the actually one is $S^0 = \frac{1-t^6}{1-t^2}$.

Surprisingly, a similar formula for the 3–hook is more involved for $T(2m+1,2)$. Namely, non-cyclotomic factors occur in S^0 for $m \geq 3$. For instance for $T(11,2)$: $\varsigma_0 t^{\pi_0} S^0 = -t^3\, \frac{1-t^{10}}{1-t^2}\, (1+t^2+2t^4+2t^6+3t^8+3t^{10}+3t^{12}+3t^{14}+4t^{16}+3t^{18}+4t^{20}+3t^{22}+3t^{24}+3t^{26}+3t^{28}+2t^{30}+2t^{32}+t^{34}+t^{36})$. There is some pattern here, but not too simple.

5.2.3. Multiple zeros. Let us provide $\varsigma_i t^{\pi_i} S^i$ for all links from the table colored by rectangles where at least one S^i has multiple zeros. We give them for all $0 \leq i \leq \deg_a \widehat{\mathcal{H}}$ (not only when multiple roots occur).

(1) $\{\gamma[3,2](P(3+3)\}$, entry $58(N_z^0 = 60)$:

$$\varsigma_0 t^{\pi_0} S^0 = \frac{1-t^{120}}{1-t^{60}}, \quad \varsigma_1 t^{\pi_1} S^1 = t\frac{1-t^{109}}{1-t^{54}}, \quad \varsigma_2 t^{\pi_2} S^2 = (1+t^2)(1+t^{48}),$$
$$\varsigma_3 t^{\pi_3} S^3 = t\frac{1-t^{84}}{1-t^{42}}, \quad \varsigma_4 t^{\pi_4} S^4 = (1+t^2)(1+t^{34}),$$
$$\varsigma_5 t^{\pi_5} S^5 = t\frac{1-t^{52}}{1-t^{26}}, \quad \varsigma_6 t^{\pi_6} S^6 = (1-t^{36})(1-t^{18}).$$

The failure of Strong RH at $i=1$ is due to the factor t. The multiple factor is $(1+t^2)$ at $i=4$, but it does not result in non-RH zeros.

(2) $\{\gamma[3,2](P(2+0)P(1+1)\}$, entry $63(N_z^0 = 36)$:

$$\varsigma_0 t^{\pi_0} S^0 = -t^2(1+t^2)(1+t^{30}), \quad \varsigma_1 t^{\pi_1} S^1 = -t^2(1+t^2)(1+t^{28}),$$
$$\varsigma_2 t^{\pi_2} S^2 = -t^2\frac{1-t^{56}}{1-t^{28}}, \quad \varsigma_3 t^{\pi_3} S^3 = -t^2\frac{1-t^{44}}{1-t^{22}},$$
$$\varsigma_4 t^{\pi_4} S^4 = -t^2\frac{1-t^{28}}{1-t^{14}}, \quad \varsigma_5 t^{\pi_5} S^5 = -t^2\frac{1-t^{8}}{1-t^{4}}.$$

The expected number of pairs of non-RH zeros at $i=0$ is 4: 2 because of t^2 plus 2 due to the multiple $(1+t^2)$, which matches the table. Here the q–deformations of multiple $\pm\imath$ are *not* of the $RH-$ type.

(3) $\{\gamma[1,1](\gamma[2,1](P)(1)\gamma[3,2](P)(1))\}$, entry $37(N_z^0=22)$:

$$\varsigma_0 t^{\pi_0} S^0 = -t\frac{1-t^{22}}{1-t^2}, \quad \varsigma_1 t^{\pi_1} S^1 = -t\frac{1-t^{20}}{1-t^2}, \quad \varsigma_2 t^{\pi_2} S^2 = -t\frac{1-t^{16}}{1-t^2},$$
$$\varsigma_3 t^{\pi_3} S^3 = -t(1+t^2)\frac{1-t^8}{1-t^2}, \qquad \varsigma_4 t^{\pi_4} S^4 = 1+t^2.$$

Here the multiple factor is $(1+t^2)$ for $i=3$; however Weak RH holds (with one pair of real non-RH zeros due to $t=0$). The q–deformations of zeros of $1+t^2=0$ are (remain) of the RH–type.

(4) $\{\gamma[1,1](\gamma[3,2](P)(1)\gamma[3,2](P)(1))\}$, entry $64(N_z^0=34)$:

$$\varsigma_0 t^{\pi_0} S^0 = -t\frac{1-t^{34}}{1-t^2}, \quad \varsigma_1 t^{\pi_1} S^1 = -t\frac{1-t^{32}}{1-t^2}, \quad \varsigma_2 t^{\pi_2} S^2 = -t\frac{1-t^{28}}{1-t^2},$$
$$\varsigma_3 t^{\pi_3} S^3 = -t\frac{1-t^{22}}{1-t^2}, \quad \varsigma_4 t^{\pi_4} S^4 = -t(1+t^2)^2(1+t^4)^2, \quad \varsigma_5 t^{\pi_5} S^5 = (1+t^2)^2,$$

Here Weak RH fails at $i=4$ ($N_z^4=14$) with 5 pairs of nonzero non-RH zeros; the expected number is 7, i.e. *all* of them (including the contribution of $t=0$). However the q–deformations of the roots of $(1+t^2)=0$ remain of RH–type. Similarly, Weak RH holds for $i=5$.

5.3. The table

It is focused on the validity of Weak RH for all $0\le i\le \deg_a$, especially in the case $i=0$. See below some review of our calculations presented in the table (and beyond).

No (N_z^0) RH-type	GOOD deg_a+1	ALG type	DAHA-formula CABLE(basic)	ϖ_0 $\varpi_{1,\ldots}^{max}$	N_{irz}^0 N_{irz}^{max}
1 (12) HOLDS	$0\le i\le 2$ all= 3	alg knot	{γ[7,3](P)} T(7,3)	0.95272 1.	0-0 0-0
2 (14) HOLDS	$0\le i\le 2$ all= 3	alg knot	{γ[8,3](P)} T(8,3)	0.96465 0.962706	0-0 0-0
3 (18) HOLDS	$0\le i\le 2$ all= 3	alg knot	{γ[10,3](P)} T(10,3)	0.980586 0.95272	0-0 0-0
4 (20) HOLDS	$0\le i\le 2$ all= 3	alg knot	{γ[11,3](P)} T(11,3)	0.984635 1.	0-0 0-0
5 (360) HOLDS	$0\le i\le 2$ all= 3	alg knot	{γ[181,3](P)} T(181,3)	0.999995 1.	0-0 0-0
6 (366) HOLDS	$0\le i\le 2$ all= 3	alg knot	{γ[184,3](P)} T(184,3)	0.999995 0.999995	0-0 0-0

7 (16) HOLDS	$0 \le i \le 3$ all= 4	alg knot	$\{\gamma[3,2]\gamma[2,1](P)\}$ Cab(13,2)T(3,2)	1.49558 2.17649	0-0 0-0
8 (12) HOLDS	$0 \le i \le 3$ all= 4	alg knot	$\{\gamma[5,4](P)\}$ T(5,4)	0.945441 1.	0-0 0-0
9 (24) HOLDS	$0 \le i \le 3$ all= 4	alg knot	$\{\gamma[9,4](P)\}$ T(9,4)	0.987206 0.978578	0-0 0-0
10 (32) HOLDS	$0 \le i \le 4$ all= 5	alg knot	$\{\gamma[5,2](P(2{+}2))\}$ T(5,2)	2.17837 3.72664	0-0 0-0
11 (36) HOLDS	$0 \le i \le 4$ all= 5	alg knot	$\{\gamma[4,3](P(2{+}0))\}$ T(4,3)	1.69034 1.79669	0-0 0-0
12 (20) HOLDS	$0 \le i \le 4$ all= 5	alg knot	$\{\gamma[6,5](P)\}$ T(6,5)	0.98107 1.	0-0 0-0
13 (32) HOLDS	$0 \le i \le 4$ all= 5	alg knot	$\{\gamma[9,5](P)\}$ T(9,5)	0.993845 1.	0-0 0-0
14 (74) HOLDS	$0 \le i \le 5$ all= 6	alg knot	$\{\gamma[5,2]\gamma[3,2](P)\}$ Cab(32,3)T(5,2)	1.36037 1.57212	0-0 0-0
15 (36) HOLDS	$0 \le i \le 5$ all= 6	alg knot	$\{\gamma[4,3]\gamma[2,1](P)\}$ Cab(25,2)T(4,3)	1.42228 1.6464	0-0 0-0
16 (30) HOLDS	$0 \le i \le 5$ all= 6	alg knot	$\{\gamma[7,6](P)\}$ T(7,6)	0.993335 1.	0-0 0-0
17 (50) HOLDS	$0 \le i \le 5$ all= 6	alg knot	$\{\gamma[11,6](P)\}$ T(11,6)	1.14474 1.13811	0-0 0-0
18 (96) HOLDS	$0 \le i \le 6$ all= 7	alg knot	$\{\gamma[3,2]\gamma[2,1](P(2{+}0))\}$ Cab(13,2)T(3,2)	1.49797 1.65491	0-0 0-0
19 (108) HOLDS	$0 \le i \le 6$ all= 7	alg knot	$\{\gamma[3,2]\gamma[2,3](P(2{+}0))\}$ Cab(15,2)T(3,2)	1.45667 1.56317	0-0 0-0
20 (66) HOLDS	$0 \le i \le 6$ all= 7	alg knot	$\{\gamma[12,7](P)\}$ T(12,7)	1.119 1.11544	0-0 0-0
21 (72) HOLDS	$0 \le i \le 6$ all= 7	alg knot	$\{\gamma[13,7](P)\}$ T(13,7)	1.16157 1.17036	0-0 0-0
22 (84) HOLDS	$0 \le i \le 7$ all= 8	alg knot	$\{\gamma[3,2]\gamma[2,1]\gamma[2,1](P)\}$ Cab(53,2)Cab(13,2)T(3,2)	1.4672 1.56196	0-0 0-0
23 (80) HOLDS	$0 \le i \le 7$ all= 8	alg knot	$\{\gamma[3,2]\gamma[4,1](P)\}$ Cab(25,4)T(3,2)	1.37538 1.46552	0-0 0-0
24 (66) HOLDS	$0 \le i \le 7$ all= 8	alg knot	$\{\gamma[5,4]\gamma[2,3](P)\}$ Cab(43,2)T(5,4)	1.51732 1.58227	0-0 0-0
25 (48) HOLDS	$0 \le i \le 8$ all= 9	alg knot	$\{\gamma[4,3](P(2{+}2))\}$ T(4,3)	2.73447 6.01964	0-0 0-0
26 (90) HOLDS	$0 \le i \le 8$ all= 9	alg knot	$\{\gamma[4,3]\gamma[3,1](P)\}$ Cab(37,3)T(4,3)	1.31174 1.42245	0-0 0-0
27 (64) HOLDS	$0 \le i \le 8$ all= 9	alg knot	$\{\gamma[5,3](P(2{+}2))\}$ T(5,3)	2.1449 21.1022	0-0 0-0
28 (116) HOLDS	$0 \le i \le 8$ all= 9	alg knot	$\{\gamma[5,3]\gamma[3,2](P)\}$ Cab(47,3)T(5,3)	1.31641 1.57212	0-0 0-0
29 (72) HOLDS	$0 \le i \le 8$ all= 9	alg knot	$\{\gamma[10,9](P)\}$ T(10,9)	1.19316 1.21884	0-0 0-0
30 (80)	$0 \le i \le 8$	alg	$\{\gamma[11,9](P)\}$	1.20386	0-0

HOLDS	all= 9	knot	T(11,9)	1.26173	0-0
31 (96)	$0\leq i\leq 8$	alg	{γ[13,9](P)}	1.18157	0-0
HOLDS	all= 9	knot	T(13,9)	1.20367	0-0
32 (6)	$0\leq i\leq 1$	alg	{γ[3,1](PP)}	0	0-1
HOLDS	all= 2	link	T(3,1)	0	0-1
33 (24)	$0\leq i\leq 3$	alg	{γ[1,1](γ[1,1](PPPP)(1))}	1.87601	3-3
HOLDS	all= 4	link	(4Cab(1,1))(T(1,1))	1.89877	3-3
34 (28)	$0\leq i\leq 3$	alg	{γ[5,2](PP)}	1.60724	1-1
HOLDS	all= 4	link	T(5,2)	1.94384	1-1
35 (26)	$0\leq i\leq 3$	alg	{γ[2,1](P γ[3,2](P)(1))}	1.21155	0-1
HOLDS	all= 4	link	T(2,1)	1.54031	0-1
36 (52)	$0\leq i\leq 4$	alg	{γ[3,2](P(2+0)P)}	1.61361	1-1
HOLDS	all= 5	link	T(3,2)	1.81723	1-1
37 (22)	$0\leq i\leq 4$	alg	{γ[1,1](γ[2,1](P)(1)γ[3,2](P)(1))}	1.91393	1-1
HOLDS	all= 5	link	(Cab(5,3),Cab(3,2))(T(1,1))	2.00929	1-1
38 (42)	$0\leq i\leq 4$	alg	{γ[2,1](γ[2,1](P)(1)γ[3,2](P)(1))}	1.37608	1-1
HOLDS	all= 5	link	(Cab(7,3),Cab(5,2))(T(2,1))	1.91393	1-1
39 (24)	$0\leq i\leq 4$	alg	{P(Y)(γ[2,3](PP)(1))}	0.736757	2-2
HOLDS	all= 5	link	T(2,3)	1.2963	2-2
40 (46)	$0\leq i\leq 5$	alg	{γ[1,1]γ[3,2](PP)}	1.61257	1-1
HOLDS	all= 6	link	Cab(5,3)T(1,1)	1.94384	1-1
41 (76)	$0\leq i\leq 5$	alg	{γ[2,1]γ[3,2](PP)}	1.4269	1-1
HOLDS	all= 6	link	Cab(8,3)T(2,1)	1.61257	1-1
42 (42)	$0\leq i\leq 5$	alg	{γ[3,2](PPP)}	1.59651	2-2
HOLDS	all= 6	link	3T(3,2)	1.99525	2-2
43 (36)	$0\leq i\leq 5$	alg	{γ[4,3](PP)}	1.69177	1-1
HOLDS	all= 6	link	T(4,3)	1.72808	1-1
44 (64)	$0\leq i\leq 5$	alg	{γ[2,1](γ[3,2](P)(1)γ[3,2](P)(1))}	1.55923	1-1
HOLDS	all= 6	link	(Cab(8,3),Cab(8,3))(T(2,1))	2.18422	1-1
45 (94)	$0\leq i\leq 5$	alg	{γ[3,1](γ[3,2](P)(1)γ[3,2](P)(1))}	1.39031	1-1
HOLDS	all= 6	link	(Cab(11,3),Cab(11,3))(T(3,1))	1.55923	1-1
46 (72)	$0\leq i\leq 5$	alg	{γ[5,2](P γ[2,1](P))}	1.39868	1-1
HOLDS	all= 6	link	Cab(21,2)T(5,2)	1.60752	1-1
47 (100)	$0\leq i\leq 6$	alg	{P(Y)(γ[2,1]γ[3,2](PP)(1))}	1.35321	2-2
HOLDS	all= 7	link	Cab(8,3)T(2,1)	1.52945	2-2
48 (80)	$0\leq i\leq 7$	alg	{γ[3,2](PPPP)}	1.59986	3-3
HOLDS	all= 8	link	4T(3,2)	1.912	3-3
49 (84)	$0\leq i\leq 7$	alg	{γ[3,2]γ[2,1](PP)}	1.47591	1-1
HOLDS	all= 8	link	Cab(13,2)T(3,2)	1.60724	1-1
50 (62)	$0\leq i\leq 7$	alg	{γ[1,1](γ[5,2](P)(1)γ[3,2](P)(1))}	1.59801	1-1
HOLDS	all= 8	link	(Cab(7,5),Cab(5,3))(T(1,1))	2.06334	1-1
51 (82)	$0\leq i\leq 7$	alg	{γ[3,2](P γ[3,2](P))}	1.35745	1-1
HOLDS	all= 8	link	Cab(20,3)T(3,2)	1.59033	1-1
52 (90)	$0\leq i\leq 8$	alg	{γ[4,3](PPP)}	1.55728	2-2
HOLDS	all= 9	link	3T(4,3)	1.6737	2-2
53 (10)	$0\leq i\leq 2$	alg	{γ[3,2]γ[-2,5](P)}	1.8969	0-0
HOLDS	all= 3	knot	Cab(7,2)T(3,2)	0.75	0-0
54 (8)	$0\leq i\leq 3$	alg	{γ[3,2]γ[-2,7](P)}	0.88617	0-0

HOLDS	all= 4	knot	Cab(5,2)T(3,2)	0.5	0-0
55 (14)	0≤ i ≤3	~~a~~lg	{γ[3,2]γ[-2,1](P)}	1.42745	0-0
HOLDS	all= 4	knot	Cab(11,2)T(3,2)	1.88749	0-0
56 (26)	0≤ i ≤5	~~a~~lg	{γ[4,3]γ[2,-9](P)}	1.6649	0-0
HOLDS	all= 6	knot	Cab(15,2)T(4,3)	1.61525	0-0
57 (20)	0≤ i ≤-1	alg	{γ[5,2](P(2+1))}	–	–5
FAILS	all= 4	knot	T(5,2)	–	—
58 (60)	0≤ i ≤0	alg	{γ[3,2](P(3+3))}	2.15141	0-0
OK a=0	all= 7	knot	T(3,2)	–	—
59 (30)	0≤ i ≤-1	alg	{γ[4,3](P(2+1))}	–	–5
FAILS	all= 7	knot	T(4,3)	–	—
60 (40)	0≤ i ≤-1	alg	{γ[5,3](P(2+1))}	–	–3
FAILS	all= 7	knot	T(5,3)	–	—
61 (60)	0≤ i ≤-1	alg	{γ[7,3](P(2+1))}	–	–3
FAILS	all= 7	knot	T(7,3)	–	—
62 (8)	0≤ i ≤-1	alg	{γ[2,1](P(2+2)P)}	–	–2
FAILS	all= 2	link	T(2,1)	–	—
63 (36)	0≤ i ≤-1	alg	{γ[3,2](P(2+0)P(1+1))}	–	–4
FAILS	all= 6	link	T(3,2)	–	—
64 (34)	0≤ i ≤3	alg	{γ[1,1](γ[3,2](P)(1)γ[3,2](P)(1))}	2.18422	1-1
OK a=0	all= 6	link	(Cab(5,3),Cab(5,3))(T(1,1))	2.61867	1-1
65 (42)	0≤ i ≤-1	alg	{γ[3,2](P(2+2)P)}	–	–5
FAILS	all= 7	link	T(3,2)	–	—
66 (6)	0≤ i ≤-1	~~a~~lg	{γ[3,2]γ[-2,13](P)}	–	–2
FAILS	all= 4	knot	Cab(1,-2)T(3,2)	–	—
67 (8)	0≤ i ≤-1	~~a~~lg	{γ[3,2]γ[-2,15](P)}	–	–4
FAILS	all= 4	knot	Cab(3,-2)T(3,2)	–	—
68 (10)	0≤ i ≤-1	~~a~~lg	{γ[3,2]γ[-2,17](P)}	–	–3
FAILS	all= 4	knot	Cab(5,-2)T(3,2)	–	—
69 (12)	0≤ i ≤-1	~~a~~lg	{γ[3,2]γ[-2,19](P)}	–	–6
FAILS	all= 4	knot	Cab(7,-2)T(3,2)	–	—
70 (4)	0≤ i ≤0	~~a~~lg	{γ[3,2]γ[-2,11](P)}	1.33333	0-0
OK a=0	all= 4	knot	Cab(1,2)T(3,2)	–	—
71 (22)	0≤ i ≤-1	~~a~~lg	{γ[4,3]γ[-2,25](P)}	–	–6
FAILS	all= 6	knot	Cab(-1,2)T(4,3)	–	—
72 (20)	0≤ i ≤-1	~~a~~lg	{γ[4,3]γ[-2,23](P)}	–	–4
FAILS	all= 6	knot	Cab(1,2)T(4,3)	–	—
73 (18)	0≤ i ≤-1	~~a~~lg	{γ[4,3]γ[-2,21](P)}	–	–2
FAILS	all= 6	knot	Cab(3,2)T(4,3)	–	—
74 (16)	0≤ i ≤0	~~a~~lg	{γ[4,3]γ[-2,19](P)}	1.29137	0-0
OK a=0	all= 6	knot	Cab(5,2)T(4,3)	–	—
75 (18)	0≤ i ≤1	~~a~~lg	{γ[4,3]γ[-2,17](P)}	1.06963	0-0
OK a=0	all= 6	knot	Cab(7,2)T(4,3)	1.38291	0-0
76 (22)	0≤ i ≤3	~~a~~lg	{γ[4,3]γ[-2,13](P)}	1.69298	0-0
OK a=0	all= 6	knot	Cab(11,2)T(4,3)	1.51094	0-0

5.4. Brief analysis

The attachment to this paper contains the formulas for quite a few (not all) superpolynomials used in this table ; the link is: `http://intlpress.com/site/pub/files/_supp/CNTP-2017-v12n3-cherednik-s1.zip`

We will focus here on the most instructional cases and features. Only a few formulas for DAHA superpolynomials are provided in this work; the files are available and see our prior works, which contain many.

5.4.1. Non-algebraic links. The validity of *RH* is certainly most likely for *algebraic* knots/links. However, the table and calculations we performed show that it holds significantly beyond this class. For instance, the whole series of cables $Cab(2m+1,2)T(3,2)$ from $Cab(13,2)T(3,2)$ (which is the smallest algebraic) down to $Cab(3,2)T(3,2)$ satisfies *RH* for all $i \geq 0$. If $a=0$ and one continues to diminish m, then *RH* fails at $Cab(-1,2)T(3,2)$ (and further on).

We note that the DAHA superpolynomials remain positive in this series till $Cab(7,2)T(3,2)$ (all their q,t,a–coefficients are positive). We called them *pseudo-algebraic* in [ChD1], where this series was considered in detail. They do resemble algebraic knots (especially for $2m+1 = 11, 9$). Interestingly, the positivity of the superpolynomial recovers starting with $Cab(-7,2)T(3,2)$, but with a different pattern of superpolynomials (not like those for $m \geq 4$). However, generally, *RH* has a clear tendency to fail when "approaching" non-algebraic links.

Let us consider entry No$=64$ with 34 zeros (at $i=0$), already discussed above. This is an *algebraic link* where *RH* is valid only partially (till a^3, where $\deg_a = 5$). The outer γ is $\gamma[1,1]$ here, i.e. minimal possible to make it algebraic. Using $\gamma[2,1]$ and $\gamma[3,1]$ here instead of $\gamma[1,1]$ results in *RH* for *all* a^i (the entries are $44, 45$ with $N_z^0 = 64, 94$). This can be informally considered as "moving away" from non-algebraic ones. By the way, uncolored $Cab(13,2)T(3,2)$, the smallest non-torus cable, has $\varpi_3 > 2$, the only uncolored algebraic knot in the table with $\varpi_i > 2$ for $i > 0$; we conjecture that it is below 2 for $i = 0$.

5.4.2. Non-square diagrams. Strong *RH* always fails for $\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}$ in the examples we calculated. This diagram is self-dual with respect to transposition, which is valuable to us. See entries $57, 59, 60, 61$ (with $N_z^0 = 20, 30, 40, 60$) and Section 5.2.2.

For the entry 60, there are 3 pair of irregular zeros of $\widehat{H}^0$ (of t–degree 40) that tend to $\{\omega^{1/3}, \omega^{2/3}\}$ for the 3 different values of the cube root as

$\omega \to \infty$. This is directly related to the expansion $\widehat{H}^0 = \omega + 2t^2 - t^3 + O(\omega^2)$, so a variant of Weak *RH* holds in this case.

In the remaining 3 cases, there are two more complex (conjugated to each other) irregular pairs. See the last column of the table, which provides the (actual) number of pairs of irregular zeros. Recall that we provide this number only for $i = 0$ for entries with "*FAILS*". These additional pairs $\{\xi', \omega/\xi'\}$ stay in the vicinity of $U_{\sqrt{\omega}}$. More exactly, $\xi' = \pm(\sqrt{\omega} + C)\imath + (C + o(1))/\sqrt{\omega}$, where $C \in \mathbb{R}$ tends to some limit as $\omega \to \infty$. Similar ξ' occur for entries $62, 63, 65$ (with $N_z^0 = 8, 36, 42$).

For *non-square* rectangles, *RH* for $\widehat{H}^i_{sym}$ can hold beyond $i = 0$, where this was conjectured. For instance, it holds for any i for entries $18, 19$ ($N_z^0 = 96, 108$): $\{\widehat{\gamma}_{3,2}(\widehat{\gamma}_{2,1}(P(\boxminus)))\}$, $\{\widehat{\gamma}_{3,2}(\widehat{\gamma}_{2,3}(P(\boxminus)))\}$.

For $\{\widehat{\gamma}_{3,2}(P(\begin{smallmatrix}\square\square\\ \square\square\\ \square\square\end{smallmatrix}))\}$, only one *non-RH* pair appears for entry 58 ($N_z^0 = 60$, $\deg_a = 6$) when $i = 1$; it is real and quickly approaches $\{-1, -\omega\}$ as $\omega \to \infty$. See Section 5.2.3. *RH* also fails here for $i = 1, 3, 5$ upon the *antisymmetrization*; the corresponding $\varsigma_i t^{\pi_i} S^i$ are $-t(1 + t^{54,42,26})$.

For the two 2–rectangle and $2T(3,2)$, i.e. for $\{\widehat{\gamma}_{3,2}(P(\boxminus)^2)\}$, there are 2 real pairs of such zeros when $i = 0$ (from 48 pairs) approaching $\{1, \omega\}$ and $\{\pm\sqrt{\omega}\}$. One has: $\varsigma_0 t^{\pi_0} S^0 = -t\frac{1-t^{92}}{1-t^4}(1-t^2)^2(1+t^2)$ in this case. The multiple roots ± 1 and those due to t give the total number of pairs of non-*RH* zeros 3 (all are real). Upon the antisymmetrization here, $\varsigma_0 t^{\pi_0} S^0$ becomes $t(1-t^2)(1+t^{92})$; Weak *RH* is satisfied now.

Also, *RH* holds for $\{\widehat{\gamma}_{3,2}(P(\boxminus)P(\square)\}$ upon the symmetrization for all $0 \le i \le 4$. One respectively has $\varsigma_i t^{\pi_i} S^i = -t(1 + t^{50,46,40,32,22})$.

5.4.3. Large-small $\boldsymbol{\varpi}$. When *RH* holds, which is a *qualitative* property, the "actual" *RH* is in finding ϖ *quantitatively*. In the following example of 2×2–diagram, *RH* "almost" fails. The ϖ–number becomes much larger than "usual" $1 \sim 2$ due to the color and $i > 0$. This is entry No=27 with 64 zeros and $\deg_a = 8$ for $T(5,3)$ colored by $\boxplus$. The corresponding ϖ_4 is 21.1022; it is ω_4^{top} for D^4 (and can be calculated as exactly as necessary). By contrast, $\varpi_0 = 2.1449$, not too large.

This is similar to entry 25 (with 48 zeros). The latter is for $T(4,3)$ and $\boxplus$. Its superpolynomial is significantly simpler to calculate than that for $T(5,3)$ (which took about 4 days). One has $\varpi_4 = 6.01964$ for $T(4,3)$, which is large but not too much.

The inequality $\boldsymbol{\varpi < 2}$. This seems a counterpart of the bounds for the spectrum of Laplace-Dirac operators in the theory of *spectral zeta-functions.* We conjecture that $\varpi_0 < 2$ upon $a = 0$ for *uncolored* algebraic knots. For

$i > 0$, entry 7 (with 16 zeros) is an example when $\varpi_1 > 2$. This is for $Cab(13,2)T(3,2)$. Concerning algebraic uncolored *links* for $i = 0$, the greatest ϖ_0 in the table is 1.919393 for the entry 37 (with $N_z^0 = 22$), but it can go beyond 2.

Cyclotomic polynomials at $\omega = 1$. Let us discuss $T(11,6)$; its superpolynomial was posted in the online version of [Ch2]. See entry 17 ($N_z^0 = 50$); strong *RH* for $i{=}0$ holds for $\varpi_0 = 1.1447417735112874\ldots$ and for $0.9985190700739621 < \omega < 1.0021178996517260$. We expect that $\varpi_0 = 1 + O(1/\mathsf{r})$ for *knots* $T(\mathsf{r}\to\infty, \mathsf{s}>2)$, which is based on numerical evidence for $\mathsf{s} = 3, 6, 7, 9$ and the considerations of *geometric superpolynomials* [ChP1] as $\mathsf{r}\to\infty$. One has: $\widehat{H}^0(\omega{=}1) = \prod_{j=1}^{\mathsf{s}-1} \frac{1-t^{\mathsf{r}+\mathsf{s}-j}}{1-t^{1+j}}$, which follows from the Shuffle Conjecture proved in [CaM].

This can be deduced from Corollary 4.6; here $\widehat{H}^0(\omega{=}1)$ is the *rational slope t–Catalan number*. The following formula can be checked using the *geometric superpolynomials* from [ChP1], their (conjectured) connection with DAHA and the a–version of Corollary 4.6. For $\mathsf{r} > \mathsf{s}$, $0 \le i \le \mathsf{s}-1$,

$$\widehat{H}^i(\omega = 1) = \prod_{j=1}^{\mathsf{s}-1} \frac{(1 - t^{\mathsf{r}+\mathsf{s}-j})}{(1 - t^{1+\mathsf{s}-j})}$$
$$\times \frac{\big((1-t^{\mathsf{r}-1})(1-t^{\mathsf{r}-2})\cdots(1-t^{\mathsf{r}-i})\big)\big((1-t^{\mathsf{s}-1})(1-t^{\mathsf{s}-2})\cdots(1-t^{\mathsf{s}-i})\big)}{\big((1-t)(1-t^2)\cdots(1-t^i)\big)\ \big((1-t^{\mathsf{r}+\mathsf{s}-1})(1-t^{\mathsf{r}+\mathsf{s}-2})\cdots(1-t^{\mathsf{r}+\mathsf{s}-i})\big)}.$$

The second line here is trivial when $i = 0$. Similar formulas exist for (at least) the family $Cab(13+2m,2)T(3,2)$ with $m \in \mathbb{Z}_+$, where all Piontkowski cells J_Δ are affine spaces.

We note that $\widehat{H}^i(\omega{=}1)$ for $i > 0$ generally have multiple zeros. For instance, in the case of $T(11,6)$: $\widehat{H}^1(\omega{=}1) = \Phi_5^2\Phi_7\Phi_{10}\Phi_{12}\Phi_{13}\Phi_{14}\Phi_{15}$ for cyclotomic polynomials Φ_m. It has 4 pairs of irregular (non-RH) zeros in any punctured neighborhood of $\omega{=}1$. Strong *RH* begins only after $\varpi_1 = 1.1381148969721394\ldots = \omega_1^{top}$.

5.4.4. Irregular zeros. We provide the number of super-dual-invariant pairs of *irregular zeros* in the last column; the upper two numbers are for $a = 0$, the lower ones are for the maximum among $i > 0$. The first gives the number of pairs right after ϖ; the second, after "−", is for the stable number for $\omega >> 0$. Mostly they coincide in this table.

Trivial zeros, $-1, -\omega$, are not counted in the table. For $\omega > \omega_i^{top}$ such zeros appear for $\boxplus$ for odd i. They are likely to reflect some symmetries of the DAHA construction, and presumably can be interpreted geometrically.

We note that more general squares result in huge superpolynomials, which we do not have by now.

If the number of pairs of *irregular zeros* is $\{\cdot\}-0$ in the last column, then Strong RH holds for $\omega >> 0$; multiple $RH-$zeros appear after ϖ in this case. If it is $0-0$, then RH holds starting with ϖ from the table. Thus Strong RH always holds in the table for uncolored algebraic knots and those colored by $\boxplus$ (where we disregard trivial zeros).

Recall that Weak RH allows $\kappa-1$ pairs of irregular zeros for *uncolored* algebraic links, where κ is the number of components of a link. Practically, the number of allowed irregular pairs (disregarding trivial ones) equals the total number of symbols P in the DAHA presentation of a link minus 1. Mostly this difference is 0 or 1 in the table. It is 1 for 2–links and the corresponding irregular pair is automatically real (the number of irregular *complex* pairs is always even). For entry 33 (with $N_z^0=24$), there are four P in its DAHA presentation and, indeed, $\widehat{H}^i$ have 3 irregular pairs of zeros. Similarly, RH holds for $\{\widehat{\gamma}(3,2)(P^3)\}$ (entry 52 with $N_z^0=90$), with 2 pairs of (automatically complex) irregular roots. We provide one algebraic *link* (entry 64 with $N_z^0=34$) where Weak RH fails for $i=4$ (only for such i). See below.

5.5. Composite theory, etc

We do not discuss in this work other root systems. The a–stabilization was conjectured for classical series in [Ch2]; the corresponding polynomials are called DAHA *hyperpolynomials.* The *hyper-duality* is expected to hold too; one can expect the corresponding RH for algebraic knots/links. However, the corresponding DAHA hyperpolynomials are known so far only for small knots. We think it makes some sense to provide at least one example from the *composite theory* for the exceptional series from [DG], which is topologically for the annulus multiplied by $\mathbb{R}^1$ instead of $\mathbb{S}^3$. Algebraically, this is the case of a–stabilization when *two* Young diagrams are placed at the opposite ends of the (nonaffine) Dynkin diagram of type A and the stabilization is with respect to the distance between these diagrams.

We will consider only the case of uncolored $T(4,3)$ from [ChE]. Then RH is "OK" for the corresponding *composite superpolynomial*. Namely, it holds for $i=0,1$ but fails for $i=2$ ($\deg_a=5$ in this case). One has: $\varpi_0=0.84405$, $\varpi_1=0.6874328$. The first coincides with ω_0^{top}, the second is among the toots $\omega=1/q$ of D^1 (which is always true for ϖ), but smaller than $\omega_1^{top}=2$. The factor of D^1 corresponding to ϖ_1 is the square of:

$$-9248+12492q-14345q^2+844q^3+6308q^4-1608q^5+112q^6.$$

Some recalculation is necessary from the setting of [ChE] to make the super-duality exactly as in the present work. So we will provide the corresponding $\widehat{\mathcal{H}}(q,t,a)$:

$$1 + 2qt + 2q^2t + 3q^2t^2 + 2q^3t^2 + q^4t^2 + 4q^3t^3 + 2q^4t^3 + 3q^4t^4 + 2q^5t^4 + 2q^5t^5 + q^6t^6 + a^5\big(-q^5 + q^6 - q^4t + 2q^5t - q^6t + q^4t^2 - q^5t^2\big) + a^4\big(-q^3 + 2q^4 - q^5 + q^2t - 4q^4t + 4q^5t - q^6t + q^3t^2 + q^4t^2 - 4q^5t^2 + 2q^6t^2 + q^4t^3 - q^6t^3 + q^5t^4\big) + a^3\big(q - q^2 - 3q^3 + 3q^4 + q^5 - q^6 + qt + 3q^2t - q^3t - 8q^4t + 4q^5t + q^6t + 2q^2t^2 + 5q^3t^2 - 2q^4t^2 - 8q^5t^2 + 3q^6t^2 + 3q^3t^3 + 5q^4t^3 - q^5t^3 - 3q^6t^3 + 2q^4t^4 + 3q^5t^4 - q^6t^4 + q^5t^5 + q^6t^5\big) + a^2\big(1 + 2q - 2q^2 - 3q^3 + q^5 + q^6 + 4qt + 5q^2t - 2q^3t - 8q^4t + q^6t + 7q^2t^2 + 9q^3t^2 - 2q^4t^2 - 8q^5t^2 + 8q^3t^3 + 9q^4t^3 - 2q^5t^3 - 3q^6t^3 + 7q^4t^4 + 5q^5t^4 - 2q^6t^4 + 4q^5t^5 + 2q^6t^5 + q^6t^6\big) + a\big(2 + q - q^2 - q^3 - q^4 + 5qt + 5q^2t - q^3t - 3q^4t - 2q^5t + 8q^2t^2 + 7q^3t^2 + q^4t^2 - 3q^5t^2 - q^6t^2 + 9q^3t^3 + 7q^4t^3 - q^5t^3 - q^6t^3 + 8q^4t^4 + 5q^5t^4 - q^6t^4 + 5q^5t^5 + q^6t^5 + 2q^6t^6\big).$$

Let us also mention here that *Heegaard-Floer homology* is the specialization of Khovanov-Rozansky link homology at for the differential at $a=-1$ (in DAHA parameters). This specialization preserves the super-duality, in contrast to the differentials at $a=-t^{n+1}$ (in DAHA parameters) to the A_n–theories. Practically everything we conjecture for $\widehat{H}^i$ is applicable when $a=-1$. In this paper, we do not discuss polynomials/series that are sums of $\mathcal{H}^i$ over all i, which is a clear potential of the theory. The specialization $a \mapsto -(t/q)$ from (4.31) is of obvious importance too. Also, employing the connection conjectures, one has an opportunity to interpret *RH* for the topological and other superpolynomials, including the HOMFLY-PT polynomials, KhR polynomials, physical ones and those associated with the rational DAHA and the Hilbert scheme of $\mathbb{C}^2$.

6. Some formulas, conclusion

6.1. A non-RH link for $i > 0$

The only counterexample to the total (all a^i) Weak *RH* among algebraic knots/links in the table is for the entry 64 ($N_z^0 = 34$), where it fails for a^4 (but holds for all other a^i). See also Section 5.2.3, (4) and Section 6.3.3 below. Let us discuss it.

The corresponding cable is $\mathcal{L}_1 = (\mathit{Cab}(5,3), \mathit{Cab}(5,3))T(1,1)$, which is the link of the singularity $\mathcal{C}_1 = \{(x^5 - y^3)(x^3 - y^5) = 0\}$ at $x\!=\!0\!=\!y$. The linking number is $lk = 9$ for its two components, which are $T(5,3)$. The zeta-monodromy from [DGPS] upon $t \mapsto q$, essentially the Alexander polynomial, is $Z_1 = q^{32} + 2q^{24} + 3q^{16} + 2q^8 + 1$. According to Section 5.4 from [ChD2], the following connection with our superpolynomials is expected (unless for the unknot):

$$Z = \widehat{\mathcal{H}}^{min}_{\mathcal{L}}(q, q, a = -1)/(1-q)^{\kappa - \delta_{\kappa,1}}. \tag{6.45}$$

This is for *uncolored* graph $\mathcal{L}$ (without $'\mathcal{L}$) with κ paths (the number of connected components in the corresponding cable). The linking number is then $Z(q\!=\!1)$. Recall that we always impose the minimal normalization $\widehat{\mathcal{H}}^{min}$ in the present paper.

Let us mention that the link $2\,T(5,3) = (\mathit{Cab}(1,1), \mathit{Cab}(1,1))T(5,3)$, corresponding to $\mathcal{C}_1' = \{(x^5\!-\!y^3)(x^5\!+\!y^3) = 0\}$ with the linking number $Z_1'(1) = 15$, satisfies RH. One has: $Z_1' = q^{44} + q^{38} + q^{34} + q^{32} + q^{28} + q^{26} + q^{24} + q^{22} + q^{20} + q^{18} + q^{16} + q^{12} + q^{10} + q^6 + 1$ in this case. This is actually entry No= 40 (with $N^0_z = 46$), because the cables $\mathit{Cab}(1,1)T(3,2)$ and $T(5,3)$ are isotopic. Note that the degree of Z is $N^0_z - 2$.

Also, RH holds for all i for the following direct modifications of $\mathcal{L}_1$: $\mathcal{L}_2 = (\mathit{Cab}(8,3), \mathit{Cab}(8,3))T(2,1)$, $\mathcal{L}_3 = (\mathit{Cab}(11,3), \mathit{Cab}(11,3))T(3,1)$, which are entries $44, 45$ ($N^0_z = 64, 94$). They are correspondingly unions of two copies of $T(8,3)$ and $T(11,3)$ with linking numbers 18 and 27.

Finding the equations of the corresponding plane curve singularities is more involved in these examples. Say for $\mathcal{C}_2$, we begin with $(x^8 - y^3)(x^3 - y^5) = 0$ and replace $x^3 = y^5$ by "its double", which is $x^3 = y^8$, provided that the corresponding link is in the vicinity of $T(5,3)$ (then the resulting linking number becomes 18). This is the meaning of cabling in this case. The equations and Z–polynomials are as follows:

$$\begin{aligned}
&\mathcal{C}_2 = \{(x^8\!-\!y^3)((y\!+\!x^2)^3\!+\!x^8)\!=\!0\}, \quad \mathcal{C}_3 = \{(x^{11}\!-\!y^3)((y\!+\!x^3)^3\!+\!x^{11})\!=\!0\},\\
&Z_2 = 1\!+\!q^6\!+\!2q^{14}\!+\!2q^{20}\!+\!3q^{28}\!+\!3q^{34}\!+\!2q^{42}\!+\!2q^{48}\!+\!q^{56}\!+\!q^{62},\\
&Z_3 = 1\!+\!q^6 + q^{12}\!+\!2q^{20}\!+\!2q^{26}\!+\!2q^{32}\!+\!3q^{40}\!+\!3q^{46}\!+\!3q^{52}\!+\!2q^{60}\!+\!2q^{66}\\
&\qquad +2q^{72}\!+\!q^{80}\!+\!q^{86}\!+\!q^{92}.
\end{aligned}$$

6.1.1. Non-RH superpolynomial. The failure of Weak RH for the link of $\mathcal{C}_1 = \{(x^5 - y^3)(x^3 - y^5) = 0\}$ above is at a^4; let us provide $\widehat{\mathcal{H}}^{min}_{\mathcal{L}}$:

$$
\begin{aligned}
&1 - t + qt + q^2t + q^3t + q^4t + q^5t - qt^2 + q^4t^2 + 2q^5t^2 + 4q^6t^2 + q^7t^2 + q^8t^2\\
&- q^2t^3 - q^4t^3 - q^5t^3 + 5q^7t^3 + 4q^8t^3 + 4q^9t^3 - q^3t^4 - q^5t^4 - 2q^6t^4 - 2q^7t^4\\
&+ 3q^8t^4 + 5q^9t^4 + 7q^{10}t^4 + q^{11}t^4 - q^4t^5 - q^6t^5 - 2q^7t^5 - 4q^8t^5 + q^9t^5\\
&+ 5q^{10}t^5 + 8q^{11}t^5 + 2q^{12}t^5 - q^5t^6 - q^7t^6 - 2q^8t^6 - 5q^9t^6 + 5q^{11}t^6 + 8q^{12}t^6\\
&+ q^{13}t^6 - q^6t^7 - q^8t^7 - 2q^9t^7 - 6q^{10}t^7 + 5q^{12}t^7 + 7q^{13}t^7 - q^7t^8 - q^9t^8\\
&- 2q^{10}t^8 - 5q^{11}t^8 + q^{12}t^8 + 5q^{13}t^8 + 4q^{14}t^8 - q^8t^9 - q^{10}t^9 - 2q^{11}t^9 - 4q^{12}t^9\\
&+ 3q^{13}t^9 + 4q^{14}t^9 + q^{15}t^9 - q^9t^{10} - q^{11}t^{10} - 2q^{12}t^{10} - 2q^{13}t^{10} + 5q^{14}t^{10}\\
&+ q^{15}t^{10} - q^{10}t^{11} - q^{12}t^{11} - 2q^{13}t^{11} + 4q^{15}t^{11} - q^{11}t^{12} - q^{13}t^{12} - q^{14}t^{12}\\
&+ 2q^{15}t^{12} + q^{16}t^{12} - q^{12}t^{13} - q^{14}t^{13} + q^{15}t^{13} + q^{16}t^{13} - q^{13}t^{14} + q^{16}t^{14}\\
&- q^{14}t^{15} + q^{16}t^{15} - q^{15}t^{16} + q^{16}t^{16} - q^{16}t^{17} + q^{17}t^{17} + a^5\big(q^{15} + 2q^{16}t + q^{17}t^2\big)\\
&+ \mathbf{a^4}\big(\mathbf{q^{10}} + \mathbf{q^{11}} + q^{12} + q^{13} + q^{14} - q^{10}t + q^{11}t + 3q^{12}t + 3q^{13}t + 3q^{14}t\\
&+ 3q^{15}t - 2q^{11}t^2 - q^{12}t^2 + 3q^{13}t^2 + 4q^{14}t^2 + 4q^{15}t^2 + 3q^{16}t^2 - 3q^{12}t^3 - q^{13}t^3\\
&+ 2q^{14}t^3 + 4q^{15}t^3 + 3q^{16}t^3 + q^{17}t^3 - 4q^{13}t^4 - q^{14}t^4 + 3q^{15}t^4 + 3q^{16}t^4 + q^{17}t^4\\
&- 3q^{14}t^5 - q^{15}t^5 + 3q^{16}t^5 + q^{17}t^5 - 2q^{15}t^6 + q^{16}t^6 + q^{17}t^6 - q^{16}t^7 + \mathbf{q^{17}t^7}\big)\\
&+ a^3\big(q^6 + q^7 + 2q^8 + 2q^9 + 2q^{10} + q^{11} + q^{12} - q^6t + q^8t + 4q^9t + 6q^{10}t\\
&+ 8q^{11}t + 5q^{12}t + 4q^{13}t + q^{14}t - q^7t^2 - 2q^8t^2 - 3q^9t^2 + q^{10}t^2 + 7q^{11}t^2\\
&+ 14q^{12}t^2 + 10q^{13}t^2 + 7q^{14}t^2 + 2q^{15}t^2 - q^8t^3 - 2q^9t^3 - 6q^{10}t^3 - 4q^{11}t^3\\
&+ 5q^{12}t^3 + 16q^{13}t^3 + 12q^{14}t^3 + 7q^{15}t^3 + q^{16}t^3 - q^9t^4 - 2q^{10}t^4 - 8q^{11}t^4\\
&- 8q^{12}t^4 + 5q^{13}t^4 + 16q^{14}t^4 + 10q^{15}t^4 + 4q^{16}t^4 - q^{10}t^5 - 2q^{11}t^5 - 8q^{12}t^5\\
&- 8q^{13}t^5 + 5q^{14}t^5 + 14q^{15}t^5 + 5q^{16}t^5 + q^{17}t^5 - q^{11}t^6 - 2q^{12}t^6 - 8q^{13}t^6\\
&- 4q^{14}t^6 + 7q^{15}t^6 + 8q^{16}t^6 + q^{17}t^6 - q^{12}t^7 - 2q^{13}t^7 - 6q^{14}t^7 + q^{15}t^7\\
&+ 6q^{16}t^7 + 2q^{17}t^7 - q^{13}t^8 - 2q^{14}t^8 - 3q^{15}t^8 + 4q^{16}t^8 + 2q^{17}t^8 - q^{14}t^9\\
&- 2q^{15}t^9 + q^{16}t^9 + 2q^{17}t^9 - q^{15}t^{10} + q^{17}t^{10} - q^{16}t^{11} + q^{17}t^{11}\big)\\
&+ a^2\big(q^3 + q^4 + 2q^5 + 2q^6 + 2q^7 + q^8 + q^9 - q^3t + 3q^6t + 6q^7t + 8q^8t + 7q^9t\\
&+ 6q^{10}t + 2q^{11}t + q^{12}t - q^4t^2 - q^5t^2 - 3q^6t^2 - 2q^7t^2 + 3q^8t^2 + 12q^9t^2\\
&+ 16q^{10}t^2 + 15q^{11}t^2 + 7q^{12}t^2 + 3q^{13}t^2 - q^5t^3 - q^6t^3 - 4q^7t^3 - 6q^8t^3 - 6q^9t^3\\
&+ 8q^{10}t^3 + 20q^{11}t^3 + 23q^{12}t^3 + 11q^{13}t^3 + 4q^{14}t^3 - q^6t^4 - q^7t^4 - 4q^8t^4\\
&- 8q^9t^4 - 13q^{10}t^4 + 2q^{11}t^4 + 21q^{12}t^4 + 26q^{13}t^4 + 11q^{14}t^4 + 3q^{15}t^4 - q^7t^5\\
&- q^8t^5 - 4q^9t^5 - 9q^{10}t^5 - 16q^{11}t^5 - q^{12}t^5 + 21q^{13}t^5 + 23q^{14}t^5 + 7q^{15}t^5\\
&+ q^{16}t^5 - q^8t^6 - q^9t^6 - 4q^{10}t^6 - 10q^{11}t^6 - 16q^{12}t^6 + 2q^{13}t^6 + 20q^{14}t^6\\
&+ 15q^{15}t^6 + 2q^{16}t^6 - q^9t^7 - q^{10}t^7 - 4q^{11}t^7 - 9q^{12}t^7 - 13q^{13}t^7 + 8q^{14}t^7\\
&+ 16q^{15}t^7 + 6q^{16}t^7 - q^{10}t^8 - q^{11}t^8 - 4q^{12}t^8 - 8q^{13}t^8 - 6q^{14}t^8 + 12q^{15}t^8
\end{aligned}
$$

$$+7q^{16}t^8+q^{17}t^8-q^{11}t^9-q^{12}t^9-4q^{13}t^9-6q^{14}t^9+3q^{15}t^9+8q^{16}t^9+q^{17}t^9$$
$$-q^{12}t^{10}-q^{13}t^{10}-4q^{14}t^{10}-2q^{15}t^{10}+6q^{16}t^{10}+2q^{17}t^{10}-q^{13}t^{11}-q^{14}t^{11}$$
$$-3q^{15}t^{11}+3q^{16}t^{11}+2q^{17}t^{11}-q^{14}t^{12}-q^{15}t^{12}+2q^{17}t^{12}-q^{15}t^{13}+q^{17}t^{13}$$
$$-q^{16}t^{14}+q^{17}t^{14}\big)+a\big(q+q^2+q^3+q^4+q^5-qt+q^3t+2q^4t+4q^5t+6q^6t$$
$$+3q^7t+2q^8t+q^9t-q^2t^2-q^3t^2-q^4t^2-q^5t^2+2q^6t^2+10q^7t^2+10q^8t^2$$
$$+9q^9t^2+4q^{10}t^2+q^{11}t^2-q^3t^3-q^4t^3-2q^5t^3-4q^6t^3-4q^7t^3+5q^8t^3$$
$$+13q^9t^3+18q^{10}t^3+9q^{11}t^3+3q^{12}t^3-q^4t^4-q^5t^4-2q^6t^4-5q^7t^4-8q^8t^4$$
$$-2q^9t^4+11q^{10}t^4+22q^{11}t^4+14q^{12}t^4+4q^{13}t^4-q^5t^5-q^6t^5-2q^7t^5-5q^8t^5$$
$$-11q^9t^5-7q^{10}t^5+9q^{11}t^5+23q^{12}t^5+14q^{13}t^5+3q^{14}t^5-q^6t^6-q^7t^6-2q^8t^6$$
$$-5q^9t^6-13q^{10}t^6-8q^{11}t^6+9q^{12}t^6+22q^{13}t^6+9q^{14}t^6+q^{15}t^6-q^7t^7-q^8t^7$$
$$-2q^9t^7-5q^{10}t^7-13q^{11}t^7-7q^{12}t^7+11q^{13}t^7+18q^{14}t^7+4q^{15}t^7-q^8t^8-q^9t^8$$
$$-2q^{10}t^8-5q^{11}t^8-11q^{12}t^8-2q^{13}t^8+13q^{14}t^8+9q^{15}t^8+q^{16}t^8-q^9t^9-q^{10}t^9$$
$$-2q^{11}t^9-5q^{12}t^9-8q^{13}t^9+5q^{14}t^9+10q^{15}t^9+2q^{16}t^9-q^{10}t^{10}-q^{11}t^{10}$$
$$-2q^{12}t^{10}-5q^{13}t^{10}-4q^{14}t^{10}+10q^{15}t^{10}+3q^{16}t^{10}-q^{11}t^{11}-q^{12}t^{11}-2q^{13}t^{11}$$
$$-4q^{14}t^{11}+2q^{15}t^{11}+6q^{16}t^{11}-q^{12}t^{12}-q^{13}t^{12}-2q^{14}t^{12}-q^{15}t^{12}+4q^{16}t^{12}$$
$$+q^{17}t^{12}-q^{13}t^{13}-q^{14}t^{13}-q^{15}t^{13}+2q^{16}t^{13}+q^{17}t^{13}-q^{14}t^{14}-q^{15}t^{14}$$
$$+q^{16}t^{14}+q^{17}t^{14}-q^{15}t^{15}+q^{17}t^{15}-q^{16}t^{16}+q^{17}t^{16}\big).$$

It makes sense to provide $\widehat{H}^4$, responsible for the failure of RH. It is $\mathcal{H}^4$ upon the substitution $q\mapsto qt$ and under the hat-normalization: $\widehat{H}^4(1+qt^2)^{-2}=1-t+qt-qt^2+q^2t^2+q^2t^3+q^3t^3+q^3t^4+q^4t^4-2q^2t^5+q^4t^5+q^4t^6+q^5t^6+q^4t^7+q^5t^7-q^4t^8+q^5t^8-q^4t^9+q^5t^9+q^5t^{10}$. We note that $\varsigma_4 t^{\pi_4}S^4=-t\big((1+t^2)(1+t^4)\big)^2$. In the case of *Heegaard-Floer substitution*, which is $a=-1$ in $\widehat{\mathcal{H}}_{\mathcal{L}}$, Weak RH holds for this link.

6.2. An example of 3-link

The appearance of $\kappa-1$ (super-dual) pairs of zeros certainly deserves a comment. Non-real non-RH roots can occur only when the number of branches is $\kappa>2$ and they do appear. Let

$$\mathcal{L}=3\,T(4,3)=(Cab(1,1),Cab(1,1),Cab(1,1))T(4,3)$$

be the link of $(x^4-y^3)(x^4+y^3)(x^4+2y^3)=0$, which has 3 components $T(4,3)$ and the pairwise linking numbers 12. The corresponding zeta-monodromy from (6.45) is $Z=1+q^9+q^{12}+q^{18}+q^{21}+q^{24}+q^{27}+q^{30}+q^{33}-$

$q^{36}+q^{39}+q^{42}-q^{45}-q^{48}+q^{51}-q^{54}-q^{57}-q^{60}-q^{63}-q^{66}-q^{69}-q^{75}-q^{78}-q^{87}$.

This example is entry No= 52 with $N_z^0=90$ from the table. The corresponding $\widehat{\mathcal{H}}$ is large (4308 different a,q,t–monomials). The polynomial $\widehat{H}^0$ is of degree 90 with respect to t (and 45 with respect to q). Weak RH holds after $\varpi_0=1.55727521033844259502499...$, coinciding with the top real zero ω_0^{top} of the reduced discriminant D^0. Let us make $\omega=2$. Then there is only one non-RH pair of zeros up to the complex conjugation:

$$\xi \;=\; 1.99999963844688175553480+0.00001272650573633190499\,\imath,$$
$$\frac{\omega}{\xi}=1.00000018073610079338078-6.363255168562838442\times 10^{-6}\imath.$$

Their product is 2. The first quickly approaches the corresponding ω as $\omega\to\infty$ (the difference is approximately some power of $1/\omega$). These zeros cannot become real for any $\omega>\omega_0^{top}$ because this would result in multiple roots after ω_0^{top}. Weak RH holds for $\omega=2$ and any $\widehat{H}^i$, $0\le i\le 8=\deg_a$ with 2 pairs of complex irregular zeros. The number of zeros ξ is correspondingly $90,88,84,78,70,60,48,34,18$ for $i=0,\ldots,8$.

6.2.1. Superpolynomial at $a=0$. Let us provide $\mathcal{H}^0=\widehat{\mathcal{H}}(a=0)$ (not $\widehat{H}^0$, i.e. without the substitution $q\mapsto qt$) for this 3–link:

$$\begin{aligned}
&1-2t+qt+q^2t+q^3t+q^4t+q^5t+q^6t+q^7t+q^8t+t^2-2qt^2-q^2t^2-q^3t^2\\
&+q^6t^2+q^7t^2+2q^8t^2+4q^9t^2+4q^{10}t^2+2q^{11}t^2+2q^{12}t^2+q^{13}t^2+q^{14}t^2\\
&+qt^3-q^2t^3-2q^4t^3-q^5t^3-2q^6t^3-q^7t^3-2q^8t^3-q^9t^3+5q^{11}t^3+5q^{12}t^3\\
&+6q^{13}t^3+5q^{14}t^3+6q^{15}t^3+3q^{16}t^3+q^{17}t^3+q^{18}t^3+q^2t^4-q^3t^4+q^4t^4\\
&-q^5t^4-q^6t^4-2q^7t^4-q^8t^4-4q^9t^4-3q^{10}t^4-5q^{11}t^4-2q^{12}t^4+4q^{14}t^4\\
&+4q^{15}t^4+11q^{16}t^4+11q^{17}t^4+7q^{18}t^4+5q^{19}t^4+3q^{20}t^4+q^3t^5-q^4t^5+q^5t^5\\
&-2q^8t^5-3q^{10}t^5-3q^{11}t^5-6q^{12}t^5-5q^{13}t^5-7q^{14}t^5-2q^{15}t^5-5q^{16}t^5\\
&+3q^{17}t^5+10q^{18}t^5+14q^{19}t^5+12q^{20}t^5+11q^{21}t^5+5q^{22}t^5+q^{23}t^5+q^4t^6\\
&-q^5t^6+q^6t^6+q^8t^6-q^9t^6-2q^{11}t^6-q^{12}t^6-5q^{13}t^6-4q^{14}t^6-9q^{15}t^6\\
&-6q^{16}t^6-10q^{17}t^6-6q^{18}t^6-4q^{19}t^6+7q^{20}t^6+14q^{21}t^6+18q^{22}t^6+14q^{23}t^6\\
&+8q^{24}t^6+q^{25}t^6+q^5t^7-q^6t^7+q^7t^7+q^9t^7+q^{11}t^7-2q^{12}t^7-3q^{14}t^7\\
&-2q^{15}t^7-7q^{16}t^7-6q^{17}t^7-13q^{18}t^7-8q^{19}t^7-12q^{20}t^7-8q^{21}t^7+2q^{22}t^7
\end{aligned}$$

$$
\begin{aligned}
&+17q^{23}t^{7}+21q^{24}t^{7}+18q^{25}t^{7}+8q^{26}t^{7}+q^{27}t^{7}+q^{6}t^{8}-q^{7}t^{8}+q^{8}t^{8}+q^{10}t^{8}\\
&+2q^{12}t^{8}-q^{13}t^{8}-2q^{15}t^{8}-5q^{17}t^{8}-3q^{18}t^{8}-12q^{19}t^{8}-9q^{20}t^{8}-13q^{21}t^{8}\\
&-14q^{22}t^{8}-14q^{23}t^{8}+3q^{24}t^{8}+21q^{25}t^{8}+26q^{26}t^{8}+15q^{27}t^{8}+6q^{28}t^{8}+q^{29}t^{8}\\
&+q^{7}t^{9}-q^{8}t^{9}+q^{9}t^{9}+q^{11}t^{9}+2q^{13}t^{9}+q^{15}t^{9}-2q^{16}t^{9}+q^{17}t^{9}-3q^{18}t^{9}-q^{19}t^{9}\\
&-9q^{20}t^{9}-7q^{21}t^{9}-13q^{22}t^{9}-13q^{23}t^{9}-21q^{24}t^{9}-15q^{25}t^{9}+10q^{26}t^{9}+31q^{27}t^{9}\\
&+23q^{28}t^{9}+12q^{29}t^{9}+3q^{30}t^{9}+q^{8}t^{10}-q^{9}t^{10}+q^{10}t^{10}+q^{12}t^{10}+2q^{14}t^{10}\\
&+2q^{16}t^{10}-q^{17}t^{10}+q^{18}t^{10}-2q^{19}t^{10}+q^{20}t^{10}-7q^{21}t^{10}-4q^{22}t^{10}-11q^{23}t^{10}\\
&-12q^{24}t^{10}-20q^{25}t^{10}-25q^{26}t^{10}-5q^{27}t^{10}+25q^{28}t^{10}+30q^{29}t^{10}+18q^{30}t^{10}\\
&+5q^{31}t^{10}+q^{32}t^{10}+q^{9}t^{11}-q^{10}t^{11}+q^{11}t^{11}+q^{13}t^{11}+2q^{15}t^{11}+2q^{17}t^{11}\\
&+2q^{19}t^{11}-2q^{20}t^{11}+2q^{21}t^{11}-5q^{22}t^{11}-2q^{23}t^{11}-8q^{24}t^{11}-10q^{25}t^{11}\\
&-20q^{26}t^{11}-27q^{27}t^{11}-14q^{28}t^{11}+13q^{29}t^{11}+34q^{30}t^{11}+23q^{31}t^{11}+7q^{32}t^{11}\\
&+q^{33}t^{11}+q^{10}t^{12}-q^{11}t^{12}+q^{12}t^{12}+q^{14}t^{12}+2q^{16}t^{12}+2q^{18}t^{12}+3q^{20}t^{12}\\
&-q^{21}t^{12}+2q^{22}t^{12}-4q^{23}t^{12}-6q^{25}t^{12}-7q^{26}t^{12}-20q^{27}t^{12}-29q^{28}t^{12}\\
&-17q^{29}t^{12}+7q^{30}t^{12}+33q^{31}t^{12}+23q^{32}t^{12}+9q^{33}t^{12}+q^{34}t^{12}+q^{11}t^{13}\\
&-q^{12}t^{13}+q^{13}t^{13}+q^{15}t^{13}+2q^{17}t^{13}+2q^{19}t^{13}+3q^{21}t^{13}+3q^{23}t^{13}-4q^{24}t^{13}\\
&+q^{25}t^{13}-4q^{26}t^{13}-5q^{27}t^{13}-18q^{28}t^{13}-30q^{29}t^{13}-24q^{30}t^{13}+6q^{31}t^{13}\\
&+35q^{32}t^{13}+23q^{33}t^{13}+7q^{34}t^{13}+q^{35}t^{13}+q^{12}t^{14}-q^{13}t^{14}+q^{14}t^{14}+q^{16}t^{14}\\
&+2q^{18}t^{14}+2q^{20}t^{14}+3q^{22}t^{14}+4q^{24}t^{14}-3q^{25}t^{14}+q^{26}t^{14}-3q^{27}t^{14}-4q^{28}t^{14}\\
&-16q^{29}t^{14}-29q^{30}t^{14}-26q^{31}t^{14}+6q^{32}t^{14}+33q^{33}t^{14}+23q^{34}t^{14}+5q^{35}t^{14}\\
&+q^{13}t^{15}-q^{14}t^{15}+q^{15}t^{15}+q^{17}t^{15}+2q^{19}t^{15}+2q^{21}t^{15}+3q^{23}t^{15}+4q^{25}t^{15}\\
&-2q^{26}t^{15}+2q^{27}t^{15}-3q^{28}t^{15}-4q^{29}t^{15}-15q^{30}t^{15}-29q^{31}t^{15}-24q^{32}t^{15}\\
&+7q^{33}t^{15}+34q^{34}t^{15}+18q^{35}t^{15}+3q^{36}t^{15}+q^{14}t^{16}-q^{15}t^{16}+q^{16}t^{16}+q^{18}t^{16}\\
&+2q^{20}t^{16}+2q^{22}t^{16}+3q^{24}t^{16}+4q^{26}t^{16}-2q^{27}t^{16}+3q^{28}t^{16}-3q^{29}t^{16}\\
&-4q^{30}t^{16}-16q^{31}t^{16}-30q^{32}t^{16}-17q^{33}t^{16}+13q^{34}t^{16}+30q^{35}t^{16}+12q^{36}t^{16}\\
&+q^{37}t^{16}+q^{15}t^{17}-q^{16}t^{17}+q^{17}t^{17}+q^{19}t^{17}+2q^{21}t^{17}+2q^{23}t^{17}+3q^{25}t^{17}\\
&+4q^{27}t^{17}-2q^{28}t^{17}+3q^{29}t^{17}-3q^{30}t^{17}-4q^{31}t^{17}-18q^{32}t^{17}-29q^{33}t^{17}\\
&-14q^{34}t^{17}+25q^{35}t^{17}+23q^{36}t^{17}+6q^{37}t^{17}+q^{16}t^{18}-q^{17}t^{18}+q^{18}t^{18}+q^{20}t^{18}\\
&+2q^{22}t^{18}+2q^{24}t^{18}+3q^{26}t^{18}+4q^{28}t^{18}-2q^{29}t^{18}+2q^{30}t^{18}-3q^{31}t^{18}-5q^{32}t^{18}\\
&-20q^{33}t^{18}-27q^{34}t^{18}-5q^{35}t^{18}+31q^{36}t^{18}+15q^{37}t^{18}+q^{38}t^{18}+q^{17}t^{19}\\
&-q^{18}t^{19}+q^{19}t^{19}+q^{21}t^{19}+2q^{23}t^{19}+2q^{25}t^{19}+3q^{27}t^{19}+4q^{29}t^{19}-2q^{30}t^{19}\\
&+q^{31}t^{19}-4q^{32}t^{19}-7q^{33}t^{19}-20q^{34}t^{19}-25q^{35}t^{19}+10q^{36}t^{19}+26q^{37}t^{19}\\
&+8q^{38}t^{19}+q^{18}t^{20}-q^{19}t^{20}+q^{20}t^{20}+q^{22}t^{20}+2q^{24}t^{20}+2q^{26}t^{20}+3q^{28}t^{20}\\
&+4q^{30}t^{20}-3q^{31}t^{20}+q^{32}t^{20}-6q^{33}t^{20}-10q^{34}t^{20}-20q^{35}t^{20}-15q^{36}t^{20}
\end{aligned}
$$

$$
\begin{aligned}
&+21q^{37}t^{20}+18q^{38}t^{20}+q^{39}t^{20}+q^{19}t^{21}-q^{20}t^{21}+q^{21}t^{21}+q^{23}t^{21}+2q^{25}t^{21}\\
&+2q^{27}t^{21}+3q^{29}t^{21}+4q^{31}t^{21}-4q^{32}t^{21}-8q^{34}t^{21}-12q^{35}t^{21}-21q^{36}t^{21}\\
&+3q^{37}t^{21}+21q^{38}t^{21}+8q^{39}t^{21}+q^{20}t^{22}-q^{21}t^{22}+q^{22}t^{22}+q^{24}t^{22}+2q^{26}t^{22}\\
&+2q^{28}t^{22}+3q^{30}t^{22}+3q^{32}t^{22}-4q^{33}t^{22}-2q^{34}t^{22}-11q^{35}t^{22}-13q^{36}t^{22}\\
&-14q^{37}t^{22}+17q^{38}t^{22}+14q^{39}t^{22}+q^{40}t^{22}+q^{21}t^{23}-q^{22}t^{23}+q^{23}t^{23}+q^{25}t^{23}\\
&+2q^{27}t^{23}+2q^{29}t^{23}+3q^{31}t^{23}+2q^{33}t^{23}-5q^{34}t^{23}-4q^{35}t^{23}-13q^{36}t^{23}\\
&-14q^{37}t^{23}+2q^{38}t^{23}+18q^{39}t^{23}+5q^{40}t^{23}+q^{22}t^{24}-q^{23}t^{24}+q^{24}t^{24}+q^{26}t^{24}\\
&+2q^{28}t^{24}+2q^{30}t^{24}+3q^{32}t^{24}-q^{33}t^{24}+2q^{34}t^{24}-7q^{35}t^{24}-7q^{36}t^{24}-13q^{37}t^{24}\\
&-8q^{38}t^{24}+14q^{39}t^{24}+11q^{40}t^{24}+q^{23}t^{25}-q^{24}t^{25}+q^{25}t^{25}+q^{27}t^{25}+2q^{29}t^{25}\\
&+2q^{31}t^{25}+3q^{33}t^{25}-2q^{34}t^{25}+q^{35}t^{25}-9q^{36}t^{25}-9q^{37}t^{25}-12q^{38}t^{25}+7q^{39}t^{25}\\
&+12q^{40}t^{25}+3q^{41}t^{25}+q^{24}t^{26}-q^{25}t^{26}+q^{26}t^{26}+q^{28}t^{26}+2q^{30}t^{26}+2q^{32}t^{26}\\
&+2q^{34}t^{26}-2q^{35}t^{26}-q^{36}t^{26}-12q^{37}t^{26}-8q^{38}t^{26}-4q^{39}t^{26}+14q^{40}t^{26}\\
&+5q^{41}t^{26}+q^{25}t^{27}-q^{26}t^{27}+q^{27}t^{27}+q^{29}t^{27}+2q^{31}t^{27}+2q^{33}t^{27}+q^{35}t^{27}\\
&-3q^{36}t^{27}-3q^{37}t^{27}-13q^{38}t^{27}-6q^{39}t^{27}+10q^{40}t^{27}+7q^{41}t^{27}+q^{42}t^{27}+q^{26}t^{28}\\
&-q^{27}t^{28}+q^{28}t^{28}+q^{30}t^{28}+2q^{32}t^{28}+2q^{34}t^{28}-q^{35}t^{28}+q^{36}t^{28}-5q^{37}t^{28}\\
&-6q^{38}t^{28}-10q^{39}t^{28}+3q^{40}t^{28}+11q^{41}t^{28}+q^{42}t^{28}+q^{27}t^{29}-q^{28}t^{29}+q^{29}t^{29}\\
&+q^{31}t^{29}+2q^{33}t^{29}+2q^{35}t^{29}-2q^{36}t^{29}-7q^{38}t^{29}-6q^{39}t^{29}-5q^{40}t^{29}+11q^{41}t^{29}\\
&+3q^{42}t^{29}+q^{28}t^{30}-q^{29}t^{30}+q^{30}t^{30}+q^{32}t^{30}+2q^{34}t^{30}+q^{36}t^{30}-2q^{37}t^{30}\\
&-2q^{38}t^{30}-9q^{39}t^{30}-2q^{40}t^{30}+4q^{41}t^{30}+6q^{42}t^{30}+q^{29}t^{31}-q^{30}t^{31}+q^{31}t^{31}\\
&+q^{33}t^{31}+2q^{35}t^{31}-3q^{38}t^{31}-4q^{39}t^{31}-7q^{40}t^{31}+4q^{41}t^{31}+5q^{42}t^{31}+q^{43}t^{31}\\
&+q^{30}t^{32}-q^{31}t^{32}+q^{32}t^{32}+q^{34}t^{32}+2q^{36}t^{32}-q^{37}t^{32}-5q^{39}t^{32}-5q^{40}t^{32}\\
&+6q^{42}t^{32}+q^{43}t^{32}+q^{31}t^{33}-q^{32}t^{33}+q^{33}t^{33}+q^{35}t^{33}+2q^{37}t^{33}-2q^{38}t^{33}\\
&-q^{39}t^{33}-6q^{40}t^{33}-2q^{41}t^{33}+5q^{42}t^{33}+2q^{43}t^{33}+q^{32}t^{34}-q^{33}t^{34}+q^{34}t^{34}\\
&+q^{36}t^{34}+q^{38}t^{34}-2q^{39}t^{34}-3q^{40}t^{34}-5q^{41}t^{34}+5q^{42}t^{34}+2q^{43}t^{34}+q^{33}t^{35}\\
&-q^{34}t^{35}+q^{35}t^{35}+q^{37}t^{35}-3q^{40}t^{35}-3q^{41}t^{35}+4q^{43}t^{35}+q^{34}t^{36}-q^{35}t^{36}\\
&+q^{36}t^{36}+q^{38}t^{36}-q^{39}t^{36}-4q^{41}t^{36}-q^{42}t^{36}+4q^{43}t^{36}+q^{35}t^{37}-q^{36}t^{37}\\
&+q^{37}t^{37}+q^{39}t^{37}-2q^{40}t^{37}-q^{41}t^{37}-2q^{42}t^{37}+2q^{43}t^{37}+q^{44}t^{37}+q^{36}t^{38}\\
&-q^{37}t^{38}+q^{38}t^{38}-2q^{41}t^{38}-q^{42}t^{38}+q^{43}t^{38}+q^{44}t^{38}+q^{37}t^{39}-q^{38}t^{39}\\
&+q^{39}t^{39}-q^{41}t^{39}-2q^{42}t^{39}+q^{43}t^{39}+q^{44}t^{39}+q^{38}t^{40}-q^{39}t^{40}+q^{40}t^{40}\\
&-q^{41}t^{40}-q^{42}t^{40}+q^{44}t^{40}+q^{39}t^{41}-q^{40}t^{41}+q^{41}t^{41}-2q^{42}t^{41}+q^{44}t^{41}\\
&+q^{40}t^{42}-q^{41}t^{42}-q^{43}t^{42}+q^{44}t^{42}+q^{41}t^{43}-q^{42}t^{43}-q^{43}t^{43}+q^{44}t^{43}\\
&+q^{42}t^{44}-2q^{43}t^{44}+q^{44}t^{44}+q^{43}t^{45}-2q^{44}t^{45}+q^{45}t^{45}.
\end{aligned}
$$

The irregular zeros become more distant from ω for H^i with i close to $\deg_a$, but the tendency remains the same. The counterpart of irregular ξ above for $i = \deg_a = 8$ is $\xi = 1.973849767 + 0.055623630$.

Let us very briefly discuss entry No=48 ($N_z^0 = 80$) with the link $4\,T(3,2)$, corresponding to the 4–branch plane curve singularity

$$(x^3 - y^2)(x^3 + y^2)(x^3 - 2y^2)(x^3 + 2y^2) = 0.$$

It has 1 real and 2 complex pairs of irregular zeros. Up to the complex conjugation and $\xi \mapsto \omega/\xi$, they are for $\omega = 2$: 1.999451149, $2.000252243 + 0.000499389\imath$, and for $\omega = 20$: 19.999999999999995, $20.000000000000003 + 4\times 10^{-15}\imath$.

6.3. Some simple cases

Let us provide the simplest algebraic uncolored knots, links, and discuss the simplest non-algebraic cable where Weak RH fails, which are $Cab(-1-2m)T(3,2)$ for $m \geq 0$.

6.3.1. Trefoil, Hopf link. For the simplest unibranch plane curve singularities $\mathcal{C}_{32} = \{x^3 = y^2\}$ at $x=0=y$ and $\mathcal{C}_{52} = \{x^5 = y^2\}$:

$$\begin{aligned}
&\widehat{\mathcal{H}}_{32} = 1 + qt + aq, \quad \widehat{\mathcal{H}}_{52} = 1 + qt + q^2t^2 + a(q + q^2t),\\
&H_{32} = 1 + qt^2 + aqt, \quad H_{52} = 1 + qt^2 + q^2t^4 + a(qt + q^2t^3).
\end{aligned}$$

The corresponding $\widehat{H}^i$ obviously have only (complex) zeros satisfying RH; note that $\widehat{H}^1_{32} = 1$ and $\widehat{H}^1_{52} = 1 + qt^2$.

For 2–branch $\mathcal{C}_{22} = \{(x+y)(x-y) = 0\}$, $\mathcal{C}_{42} = \{(x^2+y)(x^2-y) = 0\}$:

$$\begin{aligned}
&\widehat{\mathcal{H}}_{22} = 1 - t + qt + aq, \quad \widehat{\mathcal{H}}_{42} = 1 - t + qt - qt^2 + q^2t^2 + a(q - qt + q^2t),\\
&H_{22} = 1 - t + qt^2 + aqt, \quad H_{42} = 1 - t + qt^2 - qt^3 + q^2t^4 + a(qt - qt^2 + q^2t^3).
\end{aligned}$$

The zeros are obviously real irregular if $\omega = 1/q > 4$ for $\widehat{H}^0_{22}$ and $\widehat{H}^1_{42}$. One (real) pair of *irregular zeros* occurs if $\omega > 2.25$ for $\widehat{H}^0_{42}$; this pair approaches $\{1, \omega\}$ as $\omega \to \infty$, which is obvious from the formula. Otherwise their norms are $\sqrt{\omega}$.

6.3.2. Adding Y, colors. Let us provide 2 examples in the case of the non-trivial pairs $\{\mathcal{L}, {}'\mathcal{L}\}$. In the notation from the table, they are

$$\{P(Y)(\gamma[2,3](P))\} \quad \text{and} \quad \{P(Y)(\gamma[3,2](P))\}.$$

The corresponding singularities are $\mathcal{C}_{1,23} = \{(x^3-y^2)x=0\}, \mathcal{C}_{1,32} = \{(x^3-y^2)y=0\}$; their links are $T(3,2)\cup\bigcirc$ with the linking numbers $2,3$. One has:

$$
\begin{aligned}
\widehat{\mathcal{H}}_{1,23} &= 1 - t + qt + q^2t - qt^2 + q^2t^2 - q^2t^3 + q^3t^3 + a^2q^3\\
&\quad + a(q + q^2 - qt + q^2t + q^3t - q^2t^2 + q^3t^2),\\
H_{1,23} &= 1 - t + qt^2 + q^2t^3 - qt^3 + q^2t^4 - q^2t^5 + q^3t^6 + a^2q^3t^3\\
&\quad + a(qt + q^2t^2 - qt^2 + q^2t^3 + q^3t^4 - q^2t^4 + q^3t^5),\\
\widehat{\mathcal{H}}_{1,32} &= 1 - t + qt + q^2t - qt^2 + q^3t^2 - q^2t^3 + q^3t^3 - q^3t^4 + q^4t^4\\
&\quad + a^2(q^3 - q^3t + q^4t)\\
&\quad + a(q + q^2 - qt + 2q^3t - q^2t^2 + q^4t^2 - q^3t^3 + q^4t^3),\\
H_{1,32} &= 1 - t + qt^2 + q^2t^3 - qt^3 + q^3t^5 - q^2t^5 + q^3t^6 - q^3t^7 + q^4t^8\\
&\quad + a^2(q^3t^3 - q^3t^4 + q^4t^5)\\
&\quad + a(qt + q^2t^2 - qt^2 + 2q^3t^4 - q^2t^4 + q^4t^6 - q^3t^6 + q^4t^7).
\end{aligned}
$$

They satisfy Weak *RH* with one pair of stable real irregular zeros, approaching $1, \omega$ for $\omega \to \infty$. For instance, $\widehat{H}^1_{1,23} = (1 - t + qt^2)(1 + qt + qt^2)$.

Using colors. The simplest colored superpolynomials is for $T(3,2)$ colored by ω_2: $\widehat{\mathcal{H}} = 1 + a^2\frac{q^2}{t} + qt + qt^2 + q^2t^4 + a(q + \frac{q}{t} + q^2t + q^2t^2)$. Accordingly, $\widehat{H}^0_{sym} = 1 + q^2t^3 + q^3t^4 + 2q^4t^6 + q^5t^8 + q^5t^9 + q^6t^{12}$, which has 12 *RH*−zeros for $\omega = 1/q > \varpi = 1.464541725162\ldots$.

Let us also provide $\mathcal{H}^0$ for $\{\widehat{\gamma}_{2,1}(P(\begin{smallmatrix}\square\\\square\end{smallmatrix})P(\square\square))\} = \{\widehat{\gamma}_{2,1}(P(\begin{smallmatrix}\square\square\\\square\square\end{smallmatrix})P(\square))\}$ (they coincide!). It is $1 - t^2 + q^2t^2 - q^2t^4 + q^4t^4$. The corresponding $\varsigma_0 t^{\pi_0} S^0$ is $-t^2(1+t^4)$, so Weak *RH* fails in this case with 2 pairs of non-*RH* zeros due to t^2.

6.3.3. A failure at $i > 0$. The superpolynomials $\widehat{\mathcal{H}}_{1,23}, \widehat{\mathcal{H}}_{1,32}$ can be also obtained as $\{(\gamma[2,3](P)\Downarrow)(Y)(P)\}$ and $\{(\gamma[3,2](P)\Downarrow)(Y)(P)\}$, i.e. using the pairs $\{\mathcal{L}, {}'\mathcal{L}\}$ with non-trivial ${}'\mathcal{L}$. We use here that the DAHA construction is isotopy-invariant. A similar one is $\{(\gamma[2,3](P)\Downarrow)(Y)(\gamma[3,2](P))\}$, corresponding to $\mathcal{C}_{32,32} = \{(x^3-y^2)(x^3+y^2) = 0\}$ with the link $2\,T(3,2)$ and $lk = 6$.

Transposing 3 and 2 in the second factor of the last equation, the singularity $\mathcal{C}_{32,23} = \{(x^3-y^2)(x^2-y^3) = 0\}$ with $Z = 1 + 2q^5 + q^{10}$ and $lk = 4$ provides a counterexample to Weak *RH* with $i > 0$ among uncolored algebraic *links*. Here $\deg_a = 3$ and the failure of *RH* is only at $i = 2$. This is a simplification of the counterexample from Section 6.1, where the failure is at $i\!=\!\deg_a\!-\!1(=\!4)$ too. The DAHA procedure in this case is $\{(\gamma[2,3](P)\Downarrow)(Y)(\gamma[2,3](P))\}$; the superpolynomial is $\widehat{\mathcal{H}}_{32,23}=$

$$a^3q^6 + a^2(q^3 + q^4 + q^5 - q^3t + q^4t + q^5t + q^6t - 2q^4t^2 + q^5t^2 + q^6t^2 - q^5t^3 + q^6t^3) + a(q + q^2 + q^3 - qt + 2q^3t + 2q^4t + q^5t - q^2t^2 - 2q^3t^2 + 2q^4t^2 + 2q^5t^2 - q^3t^3 - 2q^4t^3 + 2q^5t^3 + q^6t^3 - q^4t^4 + q^6t^4 - q^5t^5 + q^6t^5) + 1 - t + qt + q^2t + q^3t - qt^2 + q^3t^2 + q^4t^2 - q^2t^3 - q^3t^3 + q^4t^3 + q^5t^3 - q^3t^4 + q^5t^4 - q^4t^5 + q^5t^5 - q^5t^6 + q^6t^6.$$

Then $\widehat{H}^2_{32,23}\!=\!1\!-\!t\!+\!qt\!+\!qt^2\!+\!q^2t^2\!-\!2qt^3\!+\!q^2t^3\!+\!q^2t^4\!+\!q^3t^4\!-\!q^2t^5\!+\!q^3t^5\!+\!q^3t^6$ has 1 pair of real zeros approaching $\{1,\omega\}$ for $\omega >> 0$, and 2 conjugated pairs of complex zeros not satisfying RH (though staying in the vicinity of $U_{\sqrt{\omega}}$). One has: $\varsigma_1 t^{\pi_2}S^2\!=\!-t(1\!+\!t^2)^2$ in this case; some "irregular behavior" of the corresponding flagged Jacobian factor can be expected.

We note that unless for $(x^a - y^b)(x^b - y^a)$, the corresponding singularities satisfy Weak RH in the examples we calculated. For instance, RH holds for all i for $\mathcal{C}_{34,23}\!=\!\{(x^4\!-\!y^3)(x^2\!-\!y^3)\!=\!0\}$ with $Z = 1 + 2q^6 + 2q^{12} + q^{18}$ and the linking number $Z(1) = 6$. The DAHA procedure here is $\{(\gamma[2,3](P)\Downarrow)(Y)(\gamma[3,4](P))\}$.

Non-algebraic knots. The first failures of RH for $a = 0$ in the family $Cab(2m\!+\!1,2)T(3,2)$ are for $Cab(-1,2)T(3,2)$ and $Cab(-3,2)T(3,2)$. Let us provide the corresponding $\widehat{H}^0$ for the latter: $1 + 2qt^2 + qt^3 - qt^4 + 2q^2t^4 + q^2t^5 + 2q^3t^6 + q^4t^8$. Actually there are no zeros at all of norm $\sqrt{\omega}$ in this case for $\omega >> 0$. This remains equally chaotic for all $Cab(-3\!-\!2m,2)T(3,2)$ as $m \geq 0$. We note that $\widehat{\mathcal{H}}$ become positive starting with $Cab(-7,2)T(3,2)$; the corresponding $\widehat{H}^0$ for -7 is $1 + 2qt^2 + qt^3 + 2q^2t^4 + q^2t^5 + q^2t^6 + 2q^3t^6 + q^3t^7 + 2q^4t^8 + q^4t^9 + 2q^5t^{10} + q^6t^{12}$. In this case, $\varsigma_0 t^{\pi_0}S^0 = t^6$, all zeros are non-$RH$ (and quite random).

6.4. Concluding remarks

Let us begin with the computational aspects. Superpolynomials have many symmetries: super-duality, evaluation at $q = 1$, color exchange, $\deg_a$–formula and more of these. They are routinely checked by the programs that calculate superpolynomials, including extra evaluations $a = -t^{n+1}$, and it is very unlikely that there are mistakes with the formulas for $\widehat{\mathcal{H}}$. *The attachment to this paper contains the formulas for quite a few (not all) superpolynomials used in the table from Section 5.3; the link is:* `http://intlpress.com/site/pub/files/_supp/CNTP-2017-v12n3-cherednik-s1.zip`

Numerical finding the zeros of $\widehat{H}^i$ is a relatively simple (and fast) task. We mostly rely here on the standard software. The symmetry $\xi \mapsto \omega/\xi$ provides a good independent test of the correctness of this part of our programs. Then the program increases ω to reach the RH–range (if it exists) and then diminishes ω to find the lower bonds ϖ_i. Then it checks that they are (within the accuracy) roots of the reduced discriminants D^i. The (automated) comparison with the number of pairs of non-RH zeros resulting from π_i, S^i concludes the analysis.

6.4.1. Toward Riemann's zeta. The most optimistic expectations are that DAHA superpolynomials can be a move toward the Riemann zeta and Dirichlet L–functions (and Grand RH). However, quite a few steps are needed.

Families. First of all, $\widehat{\mathcal{H}}$ must be extended to the *families* of iterated torus links; the *family superpolynomials* $\widehat{\mathbb{H}}(q,t,a,u)$ from Section 4.1.3 are natural candidates (they are actually rational functions). Algebraic links emerge in the DAHA theory as sequences of matrices $\gamma \in PSL_2(\mathbb{Z})$. The match of this interpretation with the *splice diagrams* of [EN] is a surprising outcome of [ChD2]. The families are when we multiply one of these γ by $\tau_{\pm}^m$. For instance, $\{T(\mathsf{r}+m\mathsf{s},\mathsf{s}), m \in \mathbb{Z}_+\}$ and $\{Cab(13+2m,2)T(3,2)\}$ for $m \in \mathbb{Z}_+$ are families.

For algebraic *knots* and when $\gamma_1 \mapsto \tau_-^m\gamma_1$, there are natural embeddings of the corresponding rings $\mathcal{R}_m$. Geometrically, this means that we count submodules M from Section 4.2 with some weights in terms of its (full) ring of endomorphisms $\mathcal{R}$ from a given family. Algebraically, we sum the corresponding $\widehat{\mathcal{H}}$–polynomials over a given family with the weights $(u/t)^{\text{genus}}$ for uncolored algebraic knots, where u is an additional parameter.

We use that the *same* super-symmetry serves all rings $\mathcal{R}$. The corresponding $\widehat{\mathbb{H}}(q,t,a,u)$ are generally algebraically simpler than individual $\widehat{\mathcal{H}}_m(q,t,a)$ and the ϖ_i for their a–coefficients are generally better (smaller) than those for individual $\widehat{\mathcal{H}}_m$ as $m >> 0$. Cf. formulas (4.25),(4.28),(4.28).

Analytic DAHA superpolynomials. The key step could be a passage from algebraic superpolynomials to "analytic" ones, parallel to Section "Topological vertex" from [ChD2]. As it was observed there, the analytic counterparts of superpolynomials for Hopf links extend (by adding t) the Rogers-Ramanujan expansions. The latter are interpreted in [ChF, GOW] as expansions of powers/products of theta-functions in terms of q–Hermite and Hall-Littlewood polynomials. They are (closely related) limits $t \to 0, q \to 0$ of the Macdonald polynomials. Since the invariants S^i, π_i we define are when $q \to 0$ (though this is not a direct substitution), one can expect interesting connections here, which we will not discuss in the present paper.

To define analytic DAHA superpolynomials, we essentially replace the DAHA-Jones polynomials by some integrals of the products of the powers of the Gaussian $q^{x^2/2}$ and their images under the action of $\gamma \in PSL(2,\mathbb{Z})$. The sums of such integrals with proper weights with respect to the *families* above generalize the q–analogs of the Riemann zeta and Dirichlet L–functions from [Ch4]. For instance, the q–zeta there is the integral of $q^{x^2/2}/(1-q^{x^2/2})$ with respect to the Macdonald measure $\mu(X;q,t)$ in type A_1.

The theory of *analytic* DAHA superpolynomials is of clear independent interest regardless of zeta-functions. Actually its main objective is in obtaining the invariants of Seifert and lens spaces; fruitful algebraic applications are expected too. The details will be published elsewhere. In contrast to knot invariants (though these two theories are closely related), the invariants of Seifert spaces are given in terms of modular functions, Maass forms and Mock theta-functions.

An obvious problem with the passage to the analytic superpolynomials is as follows. They are calculated in terms of a proper completion of the polynomial representation $\mathcal{V}$ in contrast to the algebraic theory (we present here) based on the "adjoint representation", which is in $\mathrm{End}_{\mathbb{C}}\mathcal{V}$ via the conjugation (actually in $\mathcal{HH}$). However this sufficiently transparent relation does not guarantee any connection at the level of the *zeros* of the corresponding superpolynomials. Generally, approaches to the Riemann Hypothesis (Grand RH) via any theories of "zeta-polynomials" satisfying RH, including the Hasse-Weil zeta functions, have little support in the classical and modern mathematics.

6.4.2. Further perspectives. A connection of superpolynomials with the zeta-functions of Laplace/Dirac operators of Riemann surfaces and p–adic strings would be a fundamental development.

Spectral zeta-functions. The motivic zeta-functions are quite parallel to the so-called *spectral zeta-functions*. Namely, let us consider the Schottky uniformization of Riemann surfaces and the corresponding Dirac operators. The corresponding "pure" zeta-functions then depend only on the genus in the smooth case; see e.g. [CM]. This fact (but not the formula itself) matches Macdonald's formula, a starting point of the Kapranov zeta-function. Then we switch to plane curve singularities. The bound $q \le 1/2$ from Conjecture 4.11 resembles the inequalities in the theory of spectral zeta. Presumably we can arrive at the same superpolynomials of plane curve singularities within this approach. Importantly, the p–adic Schottky uniformization is closely related to the curve singularities (as the closed fibers), which can be potentially a tool for establishing the link with superpolynomials.

Let us mention here Witten's p–adic strings, which can be hopefully revisited and extended toward the superpolynomials of plain curve singularities. At least, the *matrix models* can be used for this; see e.g. [DMS], which is actually closely connected with the DAHA approach. The physics insight certainly can help here.

Adelic zeta-functions. Let us connect our considerations with the classical theory of zeta-functions of arithmetic varieties. The compactified Jacobians and flagged Jacobian factors can be naturally defined over $\mathbb{Z}$. Accordingly, one can consider their *adelic zeta-functions*, the products of local zeta-functions. The latter are given in terms of the q–coefficients of the motivic superpolynomials when $a=0, t=1$; see Section 4.2. If all J_Δ are affine spaces, these coefficients simply give the numbers of cells in each dimension and readily result in the formula for the adelic zeta. It will be the product of the corresponding powers of the zeta-functions of affine spaces.

Generally, J_Δ are not always affine. However the flagged Jacobian factors are conjecturally *strongly polynomial-count* due to the discussion at the end of [ChP1]. It is not impossible that they are even paved by affine spaces (no counterexamples are known). Thus their local zeta-functions (presumably) uniformly depend on $|\mathbb{F}|$, ignoring the points of bad reduction (which are not a problem within a given topological class of the singularity). Such adelic zeta-functions generalize those of projective spaces, flag and Schubert varieties; flagged Jacobian factors can be naturally seen as the next level of *Schubert calculus.*

On Iwasawa polynomials. A similarity between the Iwasawa polynomials and the Alexander polynomials observed by B.Mazur is basically as follows in our setting. We use that the DAHA superpolynomial $\widehat{\mathcal{H}}(q,t,a)$ (in the DAHA parameters) conjecturally coincide with $\widehat{\mathcal{H}}_{mot}(q,t,a)$. Due to (4.32), the corresponding Alexander polynomial up to a normalization is

$$(6.46)\quad \widehat{\mathcal{H}}\,(t,t,a=-1) = \widehat{\mathcal{H}}\,(q,t,a=-t/q)\big|_{q\mapsto t} = L(\Gamma,q/t,t)\big|_{q\mapsto t} = L(\Gamma,1,t) = Z(\Gamma,1,t)/(1-t).$$

Recall that $Z(\Gamma,q,t)=\sum_{M\subset\mathcal{R}}^{\Delta(M)=\Gamma} t^{dim_{\mathbb{F}}(\mathcal{R}/M)}$ (considered by Zúñiga-Galindo); i.e. the summation here is over *principal* ideals M. Considering only the *group* of principal ideals (the generalized Jacobian) matches the *group* of classes of ideals in the Iwasawa theory. Finding Iwasawa-type analogs of the whole $\widehat{\mathcal{H}}(q,t,a)$ (presumably coinciding with reduced stable KhR–polynomials of algebraic links) is a challenge.

According to what we discussed above, the passage to the *families* for $\gamma_1 \mapsto \tau_-^m \gamma_1$ is natural here. The corresponding Puiseux extensions play the role of Iwasawa towers. A u–counterpart of the Iwasawa polynomial is then a weighted sum of the corresponding zeta-functions for principal ideals. The limit of this construction at $q=1$ (the field with one element) becomes the corresponding weighted sum of Alexander polynomials, namely $\widehat{\mathbb{H}}(t,t,a=-1,u)$ for $\widehat{\mathbb{H}}$ in (4.21). The techniques used to calculate the latter allow to present $\widehat{\mathbb{H}}(t,t,a=-1,u)$ as finite sums of Alexander polynomials with sufficiently simple denominators. We omit the action of the Galois group and other related aspects here.

The deep connection of the Iwasawa polynomials with *p-adic analytic L–functions* is of obvious importance to us. See Section 7 of [Mor], especially formula (7.2) and its further discussion there. Using *flags* and *families* (parameters a,u) is beyond the approach there, and we have something else: a connection with q–zeta from [Ch4]. We note that motivic superpolynomials can be defined practically in the same way for local p–adic rings; we do not really need Jacobian factors to be algebraic varieties, but the count of modules becomes more involved. They may coincide with our ones (as in the Fundamental Lemma).

What DAHA can provide. The coincidence of the DAHA superpolynomials with the motivic superpolynomials and Galkin-Stöhr zeta-functions can be checked as follows. One uses the DAHA recurrence relations similar to those in Propositions 4.2, 4.3 and compare them with the transformations of geometric superpolynomials under the blowups. This was checked for some *families* and seems doable in general.

For instance, this gives that the Galkin-Stöhr zeta-functions depend only on the topological type of the singularity (i.e. on the corresponding link). For $L(\Gamma,q,t)$ from (4.32) and for any $L(\Delta,q,t)$ such that J_Δ is affine of the same dimension as over $\mathbb{C}$, this follows from Stöhr's formula. However the affineness of *all* J_Δ holds only for torus knots and some "small" non-torus families. The DAHA superpolynomials are topological invariants, which is a relatively simple theorem.

Generally, the connection of the DAHA superpolynomials to the Khovanov-Rozansky stable polynomials requires the recurrence relations for the latter of Rosso-Jones type. They are not known, though the approach via Soergel modules seems quite relevant. At $t=q$ in the DAHA parameters, the DAHA superpolynomials were identified with the HOMFLY-PT polynomials in full generality. The identification of the DAHA-Jones polynomials with the corresponding *WRT* invariants was also done in quite a few examples (including some cases of special root systems). The CFT approach, Rosso-Jones formulas, and the so-called Skein are used here.

The DAHA theory and Macdonald polynomials are also connected with affine flag varieties, Hilbert schemes of $\mathbb{C}P^2$ (and some similar surfaces), Nekrasov's instanton sums and the mixed Hodge polynomials of certain related character varieties. Linking these theories to (classical and motivic) zeta-functions is quite a challenge. Also, the geometric superpolynomials can be expected to be connected with the spectral zeta-functions of the plane curve singularities considered under the Schottky uniformization, but this is only in the beginning and we do not see *apriori* reasons for DAHA to occur here.

A clear potential of the DAHA superpolynomials is their connection with q–analogs of Riemann's zeta and L–functions from [Ch4]. Numerically, the zeros of these q–analogs are absent in the left/right half-spaces in terms of $k=s-\frac{1}{2}$ (for s from the zeta and $t=q^k$); if true, this would give the Grand RH. We also suggested there a "straight" q–Riemann hypothesis upon the symmetrization: Conjecture 6.3. The geometric applications of DAHA outlined above may be not very surprising due to their origin: they are deformations of Heisenberg and Weyl algebras. However, their link to the classical zeta theory is certainly a surprising and promising development.

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