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Wrońskian factorizations and Broadhurst–Mellit determinant formulae

Yajun Zhou

Drawing on Vanhove's contributions to mixed Hodge structures for Feynman integrals in two-dimensional quantum field theory, we compute two families of determinants whose entries are Bessel moments. Via explicit factorizations of certain Wrońskian determinants, we verify two recent conjectures proposed by Broadhurst and Mellit, concerning determinants of arbitrary sizes. With some extensions to our methods, we also relate two more determinants of Broadhurst–Mellit to the logarithmic Mahler measures of certain polynomials.

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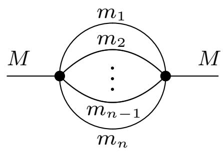
1. Introduction

In perturbative expansions for two-dimensional quantum field theory, we often need to evaluate Feynman diagrams such as [33, §8]

(1.1)



$$= 2^{n-1} \int_0^\infty I_0(Mx) \left[\prod_{i=1}^n K_0(m_i x) \right] x \,\mathrm{d}\, x,$$

where $I_0(t) = \frac{1}{\pi} \int_0^\pi e^{t \cos\theta} \,\mathrm{d}\,\theta$ and $K_0(t) = \int_0^\infty e^{-t\cosh u} \,\mathrm{d}\, u$ are modified Bessel functions of zeroth order. When all the external legs and all the internal lines bear the same parameters (say, $M = m_1 = \cdots = m_n = 1$ in the diagram above), we are left with the single-scale Bessel moments [5, 16, 21, 23]

$$\mathbf{IKM}(a, b; n) := \int_0^\infty [I_0(t)]^a [K_0(t)]^b t^n \,\mathrm{d}\, t \tag{1.2}$$

for certain non-negative integers $a, b, n \in \mathbb{Z}_{\geq 0}$.

In addition to their important rôles in the computation of anomalous magnetic dipole moment [24, 25, 27] in quantum electrodynamics, these single-scale Bessel moments are also intimately related to motivic integrations in algebraic geometry [7] and modular forms in number theory [31], thus having stimulated intensive mathematical research. For example, various linear relations among Bessel moments, such as $\pi^2 \mathbf{IKM}(3, 5; 1) = \mathbf{IKM}(1, 7; 1)$ [conjectured in 16, (148)] and $9\pi^2 \mathbf{IKM}(4, 4; 1) = 14 \times \mathbf{IKM}(2, 6; 1)$ [conjectured in 16, (147)] had been discovered by numerical experiments, before their formal proofs [35, 36] were constructed by algebraic and analytic methods.

Recently, based on a collaboration with Anton Mellit [21], David Broadhurst has laid out several dazzling conjectures about non-linear algebraic relations among $\mathbf{IKM}(a, b; n)$ with fixed $a + b$ and varying n [16]. They revolve around certain determinants whose entries are Bessel moments, two of which are recapitulated below.

Conjecture 1.1 (Broadhurst–Mellit [16, Conjecture 4]). *If* $\mathbf{M}_k$ *is a* $k \times k$ *matrix with elements*

$$(\mathbf{M}_k)_{a,b} := \int_0^\infty [I_0(t)]^a [K_0(t)]^{2k+1-a} t^{2b-1} \,\mathrm{d}\, t, \tag{1.3}$$

then its determinant evaluates to

$$\det \mathbf{M}_k = \prod_{j=1}^{k} \frac{(2j)^{k-j}\pi^j}{\sqrt{(2j+1)^{2j+1}}}. \tag{1.4}$$

Conjecture 1.2 (Broadhurst–Mellit [16, Conjecture 7]). *If* $\mathbf{N}_k$ *is a* $k \times k$ *matrix with elements*

$$(\mathbf{N}_k)_{a,b} := \int_0^\infty [I_0(t)]^a [K_0(t)]^{2k+2-a} t^{2b-1}\,\mathrm{d}\,t, \tag{1.5}$$

then its determinant evaluates to

$$\det \mathbf{N}_k = \frac{2\pi^{(k+1)^2/2}}{\Gamma((k+1)/2)} \prod_{j=1}^{k+1} \frac{(2j-1)^{k+1-j}}{(2j)^j}, \tag{1.6}$$

an expression that involves Euler's gamma function $\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}\,t$ *for* $x > 0$.

In our previous work [36, §3], we established the determinant formula

$$\det \mathbf{M}_2 = \det \begin{pmatrix} \mathbf{IKM}(1,4;1) & \mathbf{IKM}(1,4;3) \\ \mathbf{IKM}(2,3;1) & \mathbf{IKM}(2,3;3) \end{pmatrix} = \frac{2\pi^3}{\sqrt{3^3 5^5}} \tag{1.7}$$

by evaluating all the four entries of $\mathbf{M}_2$ in closed form. These analytic evaluations were made possible by integrations of some special modular forms. It appears uneconomical, if not utterly infeasible, to probe into the remaining scenarios in Conjectures 1.1 and 1.2 through analytic expressions for all the individual elements in these matrices. Indeed, only a limited number of individual Bessel moments $\mathbf{IKM}(a,b;n)$ for $a+b \geq 5$ are currently known in closed form (say, as special L-values attached to certain automorphic forms) [16, 36].

In this work, we verify Conjectures 1.1–1.2, in their entirety, via Vanhove's studies of mixed Hodge structures for Feynman integrals [33], and factorizations of certain Wrońskian determinants. This approach allows us to find a recursive mechanism underlying the Broadhurst–Mellit determinant formulae, without going through the ordeals of evaluating individual matrix elements by brute force. The same method can be extended to certain determinants whose entries involve the vacuum diagrams $V_n := \mathbf{IKM}(0,n;1) = \int_0^\infty [K_0(t)]^n t\,\mathrm{d}\,t$ for $n \in \{5,6\}$. These extensions allow us to evaluate two other determinants that were studied numerically by Broadhurst–Mellit [16,

(101) and (114)], in terms of logarithmic Mahler measures, which are defined as

$$m(P):=\int_0^1\mathrm{d}\,t_1\cdots\int_0^1\mathrm{d}\,t_n\,\log|P(e^{2\pi it_1},\dots,e^{2\pi it_n})| \tag{1.8}$$

for all non-zero Laurent polynomials $P\in\mathbb{C}[x_1^{\pm1},\dots,x_n^{\pm1}]\smallsetminus\{0\}$.

This article runs as follows. In §2, we write a new proof for $\det\mathbf{M}_2=\frac{2\pi^3}{\sqrt{3^35^5}}$, using algebraic manipulations of determinants, rather than automorphic representations of individual matrix entries. We carry on these algebraic arguments in §3 to produce a proof of $\det\mathbf{N}_2=\frac{\pi^4}{2^63^2}$, before devoting §4 to the treatments of $\det\mathbf{M}_k$ and $\det\mathbf{N}_k$ that come in arbitrary sizes ($k\in\mathbb{Z}_{\geq2}$). In §5, we open with an overview of current understandings for the relations between vacuum diagrams and Mahler measures, before presenting a proof of the results stated below.

Theorem 1.3 (Broadhurst–Mellit determinants and Mahler measures). *We have the following determinant evaluations, in terms of the logarithmic Mahler measures defined in* (1.8)*:*

$$\begin{aligned}\det\check{\mathbf{M}}_2:={}&\det\begin{pmatrix}\mathbf{IKM}(0,5;1) & \mathbf{IKM}(0,5;3)\\ \mathbf{IKM}(2,3;1) & \mathbf{IKM}(2,3;3)\end{pmatrix}\\ ={}&\frac{2\pi^3}{15\sqrt{15}}m(1+x_1+x_2+x_3+x_4),\end{aligned} \tag{1.9}$$

$$\begin{aligned}\det\check{\mathbf{N}}_2:={}&\det\begin{pmatrix}\mathbf{IKM}(0,6;1) & \mathbf{IKM}(0,6;3)\\ \mathbf{IKM}(2,4;1) & \mathbf{IKM}(2,4;3)\end{pmatrix}\\ ={}&\frac{\pi^4}{96}m(1+x_1+x_2+x_3+x_4+x_5).\end{aligned} \tag{1.10}$$

2. An algebraic evaluation of $\det\mathbf{M}_2$

As announced in the introduction, we now calculate $\det\mathbf{M}_2$ without evaluating each element in the matrix $\mathbf{M}_2$. In §2.1, using variations on the single-scale Bessel moments, we construct a 3×3 Wrońskian determinant as a function $\Omega_3(u)$ of a parameter $u\in(0,4)$, and characterize $\Omega_3(u),u\in(0,4)$ up to an overall multiplicative constant. In §2.2, we determine the aforementioned multiplicative constant by the asymptotic behavior $\Omega_3(u),u\to0^+$, and compute $\det\mathbf{M}_2$ via the special value $\Omega_3(1)$.

2.1. A 3 × 3 Wrońskian determinant

To simplify notations, we introduce a few abbreviations involving Bessel moments and their analogs.

Definition 2.1. We write $\widetilde{\mathbf{I}}\mathbf{KM}$ (resp. $\mathbf{I}\widetilde{\mathbf{K}}\mathbf{M}$) for two-scale Bessel moments with a rescaled argument in one I_0 (resp. K_0) factor. Concretely speaking, we have

$$\widetilde{\mathbf{I}}\mathbf{KM}(a+1,b;n|u) := \int_0^\infty I_0(\sqrt{u}t)[I_0(t)]^a[K_0(t)]^b t^n \,\mathrm{d}\,t, \tag{2.1}$$

$$\mathbf{I}\widetilde{\mathbf{K}}\mathbf{M}(a,b+1;n|u) := \int_0^\infty K_0(\sqrt{u}t)[I_0(t)]^a[K_0(t)]^b t^n \,\mathrm{d}\,t, \tag{2.2}$$

for certain non-negative integers $a, b, n \in \mathbb{Z}_{\geq 0}$ that make these integral expressions absolutely convergent for a given scaling parameter $u > 0$. Differentiations in the variable u will be denoted by short-hands like $D^m f(u) := \mathrm{d}^m f(u)/\mathrm{d}\,u^m$, where $m \in \mathbb{Z}_{\geq 0}$. It is understood that $D^0 f(u) = f(u)$. For each $N \in \mathbb{Z}_{>1}$, the Wrońskian determinant $W[f_1(u), \dots, f_N(u)]$ refers to $\det(D^{i-1}f_j(u))_{1\leq i,j\leq N}$.

Here, for the convergence test of the two-scale Bessel moments, it would suffice to remind our readers of the asymptotic expansions for the modified Bessel functions:

$$I_0(t) = \frac{e^t}{\sqrt{2\pi t}}\left[1 + O\left(\frac{1}{t}\right)\right], \quad K_0(t) = \sqrt{\frac{\pi}{2t}}e^{-t}\left[1 + O\left(\frac{1}{t}\right)\right], \tag{2.3}$$

as $t \to \infty$. In the $t \to 0^+$ regime, the bounded term $I_0(t) = 1 + O(t^2)$ and the mild singularity $K_0(t) = O(\log t)$ do not contribute to the convergence test of single-scale Bessel moments $\mathbf{IKM}$ and their two-scale analogs $\widetilde{\mathbf{I}}\mathbf{KM}, \mathbf{I}\widetilde{\mathbf{K}}\mathbf{M}$. Later in this section, we will also find the following facts

$$\sup_{t>0} \sqrt{t} I_0(t)K_0(t) < \infty, \quad \sup_{t>0} t^3 \left|[I_0(t)K_0(t)]^2 - \frac{1}{4t^2}\right| < \infty \tag{2.4}$$

and

$$\sup_{t>0} \frac{K_0(t)}{1 + |\log t|} < \infty \tag{2.5}$$

useful in bound estimates for $\mathbf{I}\widetilde{\mathbf{K}}\mathbf{M}(a, b+1; n|u)$, as $u \to 0^+$.

Setting

$$
\begin{cases}\mu_{2,1}^{\ell}(u)=\frac{\widetilde{\mathbf{I}}\mathbf{KM}(1,4;2\ell-1|u)+4\,\mathbf{I}\widetilde{\mathbf{K}}\mathbf{M}(1,4;2\ell-1|u)}{5},\\ \mu_{2,2}^{\ell}(u)=\widetilde{\mathbf{I}}\mathbf{KM}(2,3;2\ell-1|u),\\ \mu_{2,3}^{\ell}(u)=\mathbf{I}\widetilde{\mathbf{K}}\mathbf{M}(2,3;2\ell-1|u),\end{cases}\tag{2.6}
$$

we study the Wrońskian determinant

$$
\begin{aligned}\varOmega_3(u)&:=W[\mu_{2,1}^{1}(u),\mu_{2,2}^{1}(u),\mu_{2,3}^{1}(u)]\\ &=\det\begin{pmatrix}D^0\mu_{2,1}^1(u) & D^0\mu_{2,2}^1(u) & D^0\mu_{2,3}^1(u)\\ D^1\mu_{2,1}^1(u) & D^1\mu_{2,2}^1(u) & D^1\mu_{2,3}^1(u)\\ D^2\mu_{2,1}^1(u) & D^2\mu_{2,2}^1(u) & D^2\mu_{2,3}^1(u)\end{pmatrix}\end{aligned}\tag{2.7}
$$

in the next lemma.

Lemma 2.2 (Vanhove differential equation for $\varOmega_3(u)$). *For $0<u<4$, the Wrońskian determinant $\varOmega_3(u):=W[\mu_{2,1}^{1}(u),\mu_{2,2}^{1}(u),\mu_{2,3}^{1}(u)]$ satisfies the following differential equation:*

$$
D^1\varOmega_3(u)=\frac{3\varOmega_3(u)}{2}D^1\log\frac{1}{u^2(4-u)(16-u)}.\tag{2.8}
$$

Proof. Using integration by parts in the variable t, one can verify that the following holonomic differential operator [33, Table 1, $n=4$]

$$
\begin{aligned}\widetilde{L}_3:={}&u^2(u-4)(u-16)D^3+6u(u^2-15u+32)D^2\\ &+(7u^2-68u+64)D^1+(u-4)D^0\end{aligned}\tag{2.9}
$$

annihilates every member of the set $\{\mu_{2,1}^{1}(u),\mu_{2,2}^{1}(u),\mu_{2,3}^{1}(u)\}$, for $u\in(0,4)$.

With the Kronecker delta

$$
\delta_{i,j}=\begin{cases}0, & \text{if } i\neq j,\\ 1, & \text{if } i=j,\end{cases}\tag{2.10}
$$

we can show that

$$\begin{aligned}(2.11)\qquad & D^1W[\mu^1_{2,1}(u),\mu^1_{2,2}(u),\mu^1_{2,3}(u)]\\ &=\sum_{k=1}^{3}\det(D^{i+\delta_{i,k}-1}\mu^1_{2,j}(u))_{1\le i,j\le 3}\\ &=\det(D^{i+\delta_{i,3}-1}\mu^1_{2,j}(u))_{1\le i,j\le 3}\\ &=\det\begin{pmatrix} D^0\mu^1_{2,1}(u) & D^0\mu^1_{2,2}(u) & D^0\mu^1_{2,3}(u)\\ D^1\mu^1_{2,1}(u) & D^1\mu^1_{2,2}(u) & D^1\mu^1_{2,3}(u)\\ D^3\mu^1_{2,1}(u) & D^3\mu^1_{2,2}(u) & D^3\mu^1_{2,3}(u)\end{pmatrix}\\ &=-\frac{6u(u^2-15u+32)}{u^2(u-4)(u-16)}\det\begin{pmatrix} D^0\mu^1_{2,1}(u) & D^0\mu^1_{2,2}(u) & D^0\mu^1_{2,3}(u)\\ D^1\mu^1_{2,1}(u) & D^1\mu^1_{2,2}(u) & D^1\mu^1_{2,3}(u)\\ D^2\mu^1_{2,1}(u) & D^2\mu^1_{2,2}(u) & D^2\mu^1_{2,3}(u)\end{pmatrix}.\end{aligned}$$

Here, in the last step, we have subtracted linear combinations of the first two rows from the last row in the penultimate determinant, while appealing to the homogeneous differential equations $\widetilde{L}_3\mu^1_{2,k}(u)=0$ for $k\in\{1,2,3\}, u\in(0,4)$. Clearly, the differential equation in (2.11) is equivalent to (3.3). □

Remark After carefully collecting boundary contributions to the Newton–Leibniz formula (see Lemma 4.2 for technical details), one can show that $\widetilde{L}_3\,\widetilde{\mathbf{I}}\mathbf{KM}(1,4;1|u)=-3$ holds for $u\in(0,16)$ and $\widetilde{L}_3\,\mathbf{I}\widetilde{\mathbf{K}}\mathbf{M}(1,4;1|u)=\frac{3}{4}$ holds for $u\in(0,\infty)$. This justifies our choice of the particular linear combination in $\mu^1_{2,1}(u)=\frac{1}{5}\widetilde{\mathbf{I}}\mathbf{KM}(1,4;1|u)+\frac{4}{5}\mathbf{I}\widetilde{\mathbf{K}}\mathbf{M}(1,4;1|u)$. The homogeneous differential equation $\widetilde{L}_3\mu^1_{2,1}(u)=0$ was also crucially important in a previous study [36, §5] of the single-scale 6-loop sunrise diagram in two-dimensional quantum field theory.

2.2. Reduction to $\det \mathbf{M_2}$

We recall that the modified Bessel functions of first order are related to derivatives of their counterparts of zeroth order:

$$(2.12)\qquad I_1(t)=\frac{\mathrm{d}\,I_0(t)}{\mathrm{d}\,t},\quad K_1(t)=-\frac{\mathrm{d}\,K_0(t)}{\mathrm{d}\,t},$$

and we have a bound

$$(2.13)\qquad \sup_{t>0}\frac{|tK_1(t)-1|}{t(1+|\log t|)}<\infty.$$

Reserving the symbol D^1 for partial derivatives in the variable u, we have

$$D^1 I_0(\sqrt{u}t)=\frac{tI_1(\sqrt{u}t)}{2\sqrt{u}},\quad D^1 K_0(\sqrt{u}t)=-\frac{tK_1(\sqrt{u}t)}{2\sqrt{u}}. \tag{2.14}$$

This motivates us to introduce additional short-hand notations, to accommodate for derivatives of two-scale Bessel moments $\widetilde{\mathbf{I}}\mathbf{KM}$ and $\mathbf{I}\widetilde{\mathbf{K}}\mathbf{M}$ with respect to u.

Definition 2.3. We write $\acute{\mathbf{I}}\mathbf{KM}$ (resp. $\mathbf{I}\acute{\mathbf{K}}\mathbf{M}$) for the replacement of one $I_0(t)$ (resp. $K_0(t)$) factor in the single-scale Bessel moments by one $I_1(\sqrt{u}t)$ (resp. $-K_1(\sqrt{u}t)$) factor. Concretely speaking, we define

$$\acute{\mathbf{I}}\mathbf{KM}(a+1,b;n|u):=+\int_0^\infty I_1(\sqrt{u}t)[I_0(t)]^a[K_0(t)]^b t^{n+1}\,\mathrm{d}\,t, \tag{2.15}$$

$$\mathbf{I}\acute{\mathbf{K}}\mathbf{M}(a,b+1;n|u):=-\int_0^\infty K_1(\sqrt{u}t)[I_0(t)]^a[K_0(t)]^b t^{n+1}\,\mathrm{d}\,t, \tag{2.16}$$

for certain non-negative integers $a,b,n\in\mathbb{Z}_{\geq 0}$ that guarantee convergence of these integrals for a given parameter $u>0$.

With the understanding that $D^m f(1)=\mathrm{d}^m f(u)/\mathrm{d}\,u^m|_{u=1}$, we now investigate

$$\Omega_3(1)=\det\begin{pmatrix}D^0\mu^1_{2,1}(1) & D^0\mu^1_{2,2}(1) & D^0\mu^1_{2,3}(1)\\ D^1\mu^1_{2,1}(1) & D^1\mu^1_{2,2}(1) & D^1\mu^1_{2,3}(1)\\ D^2\mu^1_{2,1}(1) & D^2\mu^1_{2,2}(1) & D^2\mu^1_{2,3}(1)\end{pmatrix}. \tag{2.17}$$

To save space for matrix entries, we also define

$$\begin{cases}\acute{\mu}^\ell_{2,1}(u)=\frac{\acute{\mathbf{I}}\mathbf{KM}(1,4;2\ell-1|u)+4\,\mathbf{I}\acute{\mathbf{K}}\mathbf{M}(1,4;2\ell-1|u)}{5},\\ \acute{\mu}^\ell_{2,2}(u)=\acute{\mathbf{I}}\mathbf{KM}(2,3;2\ell-1|u),\\ \acute{\mu}^\ell_{2,3}(u)=\mathbf{I}\acute{\mathbf{K}}\mathbf{M}(2,3;2\ell-1|u).\end{cases} \tag{2.18}$$

Proposition 2.4 (Factorization of $\Omega_3(1)$). *We have the following identity:*

$$\Omega_3(1)=\frac{\mathbf{IKM}(1,2;1)}{2^3}\det\mathbf{M}_2. \tag{2.19}$$

Proof. With the Bessel differential equations $(uD^2+D^1)I_0(\sqrt{u}t)=\frac{t^2}{4}I_0(\sqrt{u}t)$ and $(uD^2+D^1)K_0(\sqrt{u}t)=\frac{t^2}{4}K_0(\sqrt{u}t)$, we can verify

$$2^3u^{3/2}\Omega_3(u)=\det\begin{pmatrix}\mu^1_{2,1}(u) & \mu^1_{2,2}(u) & \mu^1_{2,3}(u)\\ \acute{\mu}^1_{2,1}(u) & \acute{\mu}^1_{2,2}(u) & \acute{\mu}^1_{2,3}(u)\\ \mu^2_{2,1}(u) & \mu^2_{2,2}(u) & \mu^2_{2,3}(u)\end{pmatrix}\tag{2.20}$$

for all $u\in(0,4)$, upon using elementary row operations. In particular, we may identify $2^3\Omega_3(1)$ with

$$\det\begin{pmatrix}\mathbf{IKM}(1,4;1) & \mathbf{IKM}(2,3;1) & \mathbf{IKM}(2,3;1)\\ \acute{\mu}^1_{2,1}(1) & \acute{\mu}^1_{2,2}(1) & \acute{\mu}^1_{2,3}(1)\\ \mathbf{IKM}(1,4;3) & \mathbf{IKM}(2,3;3) & \mathbf{IKM}(2,3;3)\end{pmatrix}.\tag{2.21}$$

Now, subtracting the second column from the last column in the determinant above, while keeping in mind that $I_0(t)K_1(t)+I_1(t)K_0(t)=\frac{1}{t}$ leads to $\acute{\mu}^1_{2,3}(1)-\acute{\mu}^1_{2,2}(1)=-\mathbf{IKM}(1,2;1)$, we may equate $2^3\Omega_3(1)$ with

$$\det\begin{pmatrix}\mathbf{IKM}(1,4;1) & \mathbf{IKM}(2,3;1) & 0\\ \acute{\mu}^1_{2,1}(1) & \acute{\mu}^1_{2,2}(1) & -\mathbf{IKM}(1,2;1)\\ \mathbf{IKM}(1,4;3) & \mathbf{IKM}(2,3;3) & 0\end{pmatrix},\tag{2.22}$$

thereby establishing our claim in (2.19). □

In the next proposition, we examine the Wrońskian determinant in the $u\to 0^+$ limit.

Proposition 2.5 (Factorization of $\Omega_3(0^+)$). *The limit*

$$\lim_{u\to 0^+}u^3\Omega_3(u)=\frac{[\mathbf{IKM}(1,3;1)]^2}{2^35}\tag{2.23}$$

entails

$$\Omega_3(u)=\frac{\pi^4}{2^25[u^2(4-u)(16-u)]^{3/2}},\quad \forall u\in(0,4).\tag{2.24}$$

In particular, this implies the evaluation $\det\mathbf{M}_2=\frac{2\pi^3}{\sqrt{3^35^5}}$.

Proof. From (3.3), we know that $[u^2(4-u)(16-u)]^{3/2}\mathit{\Omega}_3(u)$ remains constant for $u\in(0,4)$. We will determine this constant by computing

$$2^9\lim_{u\to0^+}u^3\mathit{\Omega}_3(u)\tag{2.25}$$

from

$$2^3u^3\mathit{\Omega}_3(u)=\det\begin{pmatrix}\mu^1_{2,1}(u) & \mu^1_{2,2}(u) & \mu^1_{2,3}(u)\\ \sqrt{u}\acute{\mu}^1_{2,1}(u) & \sqrt{u}\acute{\mu}^1_{2,2}(u) & \sqrt{u}\acute{\mu}^1_{2,3}(u)\\ u\mu^2_{2,1}(u) & u\mu^2_{2,2}(u) & u\mu^2_{2,3}(u)\end{pmatrix}.\tag{2.26}$$

In the $u\to0^+$ regime, we have [cf. (2.4) and (2.13)]

$$\begin{aligned}\mu^1_{2,3}(u)={}&\int_0^\infty K_0(\sqrt{u}t)[I_0(t)K_0(t)]^2t\,\mathrm{d}\,t\\ ={}&O\left(\int_0^\infty K_0(\sqrt{u}t)\,\mathrm{d}\,t\right)=O\left(\frac{1}{\sqrt{u}}\right),\end{aligned}\tag{2.27}$$

$$\begin{aligned}-\sqrt{u}\,\mathbf{I\acute{K}M}(1,4;1|u)={}&\int_0^\infty I_0(t)[K_0(t)]^3t\,\mathrm{d}\,t\\ &+\int_0^\infty[\sqrt{u}tK_1(\sqrt{u}t)-1]I_0(t)[K_0(t)]^3t\,\mathrm{d}\,t\\ ={}&\mathbf{IKM}(1,3;1)+O(\sqrt{u}\log u),\end{aligned}\tag{2.28}$$

along with several other asymptotic expansions, so $2^3u^3\mathit{\Omega}_3(u)$ becomes

$$\det\begin{pmatrix}O(\log u) & \mathbf{IKM}(1,3;1)+O(u) & O(1/\sqrt{u})\\ -\frac{4\,\mathbf{IKM}(1,3;1)}{5}+O(\sqrt{u}\log u) & O(u) & \sqrt{u}\acute{\mu}^1_{2,3}(u)\\ O(u\log u) & O(u) & u\mu^2_{2,3}(u)\end{pmatrix}.\tag{2.29}$$

Noting that [cf. (2.4)]

$$\begin{aligned}-\sqrt{u}\acute{\mu}^1_{2,3}(u)={}&\int_0^\infty\sqrt{u}K_1(\sqrt{u}t)[I_0(t)K_0(t)]^2t^2\,\mathrm{d}\,t\\ ={}&O\left(\int_0^\infty\sqrt{u}K_1(\sqrt{u}t)t\,\mathrm{d}\,t\right)=O\left(\frac{1}{\sqrt{u}}\right),\end{aligned}\tag{2.30}$$

and [cf. (2.4)]

$$\begin{aligned} u\mu_{2,3}^2(u) &= \frac{u}{4}\int_0^\infty K_0(\sqrt{u}t)t\,\mathrm{d}\,t \\ &\quad + u\int_0^\infty K_0(\sqrt{u}t)\left\{[I_0(t)K_0(t)]^2-\frac{1}{4t^2}\right\}t^3\,\mathrm{d}\,t \\ &= \frac{1}{4}\int_0^\infty K_0(t)t\,\mathrm{d}\,t + O\left(u\int_0^\infty K_0(\sqrt{u}t)\,\mathrm{d}\,t\right) \\ &= \frac{1}{4}+O(\sqrt{u}), \end{aligned}\tag{2.31}$$

we find

(2.32)

$$\begin{aligned} &2^3u^3\Omega_3(u) \\ &= \det\begin{pmatrix} O(\log u) & \mathbf{IKM}(1,3;1)+O(u) & O(1/\sqrt{u}) \\ -\frac{4\,\mathbf{IKM}(1,3;1)}{5}+O(\sqrt{u}\log u) & O(u) & O(1/\sqrt{u}) \\ O(u\log u) & O(u) & \frac{1}{4}+O(\sqrt{u}) \end{pmatrix} \\ &= \frac{[\mathbf{IKM}(1,3;1)]^2}{5}+O(\sqrt{u}\log u). \end{aligned}$$

As we have $\mathbf{IKM}(1,3;1)=\frac{\pi^2}{2^4}$ [5, (55)], we see that the limit in (2.25) must be equal to $\frac{\pi^4}{2^2 5}$.

Recalling the well-known evaluation $\mathbf{IKM}(1,2;1)=\frac{\pi}{3\sqrt{3}}$ from [5, (23)], we can compute $\det\mathbf{M}_2=\frac{2\pi^3}{\sqrt{3^3 5^5}}$ with the aid of (2.19) and (2.24). □

3. An algebraic evaluation of $\det\mathbf{N}_2$

In §2, we built $\det\mathbf{M}_2$ on the knowledge of (the retroactively defined 1×1 "determinants") $\det\mathbf{M}_1=\mathbf{IKM}(1,2;1)$ and $\det\mathbf{N}_1=\mathbf{IKM}(1,3;1)$. Our task in this section is to compute $\det\mathbf{N}_2$ from $\det\mathbf{M}_2$ and $\det\mathbf{N}_1$.

3.1. A 4×4 Wrońskian determinant

Setting

$$\begin{cases} \nu_{2,1}^\ell(u)=\frac{\widetilde{\mathbf{I}}\mathbf{KM}(1,5;2\ell-1|u)+5\,\mathbf{I}\widetilde{\mathbf{K}}\mathbf{M}(1,5;2\ell-1|u)}{6}, \\ \nu_{2,2}^\ell(u)=\widetilde{\mathbf{I}}\mathbf{KM}(2,4;2\ell-1|u), \\ \nu_{2,3}^\ell(u)=\widetilde{\mathbf{I}}\mathbf{KM}(3,3;2\ell-1|u), \\ \nu_{2,4}^\ell(u)=\mathbf{I}\widetilde{\mathbf{K}}\mathbf{M}(2,4;2\ell-1|u), \end{cases}\tag{3.1}$$

and

$$
\begin{cases}\acute{\nu}_{2,1}^{\ell}(u)=\frac{\mathbf{\acute{I}KM}(1,5;2\ell-1|u)+5\,\mathbf{I\acute{K}M}(1,5;2\ell-1|u)}{6},\\ \acute{\nu}_{2,2}^{\ell}(u)=\mathbf{\acute{I}KM}(2,4;2\ell-1|u),\\ \acute{\nu}_{2,3}^{\ell}(u)=\mathbf{\acute{I}KM}(3,3;2\ell-1|u),\\ \acute{\nu}_{2,4}^{\ell}(u)=\mathbf{I\acute{K}M}(2,4;2\ell-1|u),\end{cases}\tag{3.2}
$$

we begin our study of the Wrońskian determinant $\omega_4(u):=W[\nu_{2,1}^1(u),\nu_{2,2}^1(u),\nu_{2,3}^1(u),\nu_{2,4}^1(u)]$ from the next lemma.

Lemma 3.1 (Vanhove differential equation for $\omega_4(u)$). *For $0<u<1$, the Wrońskian determinant $\omega_4(u):=W[\nu_{2,1}^1(u),\nu_{2,2}^1(u),\nu_{2,3}^1(u),\nu_{2,4}^1(u)]$ satisfies the following differential equation:*

$$
D^1\omega_4(u)=2\omega_4(u)D^1\log\frac{1}{u^2(1-u)(9-u)(25-u)}.\tag{3.3}
$$

Proof. Using integration by parts in the variable t, one can verify that the following holonomic differential operator [33, Table 1, $n=5$]

$$
\begin{aligned}\widetilde{L}_4:={}&u^2(u-25)(u-9)(u-1)D^4\\&+2u(5u^3-140u^2+777u-450)D^3\\&+(25u^3-518u^2+1839u-450)D^2\\&+(3u-5)(5u-57)D^1+(u-5)D^0\end{aligned}\tag{3.4}
$$

annihilates every member in the set $\{\nu_{2,1}^1(u),\nu_{2,2}^1(u),\nu_{2,3}^1(u),\nu_{2,4}^1(u)\}$. One may then proceed as in Lemma 2.2. □

Remark We have $\widetilde{L}_4\,\widetilde{\mathbf{I}}\mathbf{KM}(1,5;1|u)=-\frac{15}{2}$ for $u\in(0,25)$ and $\widetilde{L}_4\,\mathbf{I}\widetilde{\mathbf{K}}\mathbf{M}(1,5;1|u)=\frac{3}{2}$ for $u\in(0,\infty)$. Such computations will be put into a broader context in Lemma 4.2.

3.2. Reduction to $\det\mathbf{N}_2$

We now describe an analog of Proposition 2.4.

Proposition 3.2 (Factorization of $\omega_4(1^-)$). *We have the following identity:*

$$
\lim_{u\to1^-}(1-u)^2\omega_4(u)=-\frac{\mathbf{IKM}(1,3;1)}{2^7}\det\mathbf{N}_2.\tag{3.5}
$$

Proof. Through row operations and the Bessel differential equations for I_0 and K_0, we find

$$\text{(3.6)}\qquad 2^6u^3\omega_4(u)=\det\begin{pmatrix}\nu^1_{2,1}(u) & \nu^1_{2,2}(u) & \nu^1_{2,3}(u) & \nu^1_{2,4}(u)\\ \acute{\nu}^1_{2,1}(u) & \acute{\nu}^1_{2,2}(u) & \acute{\nu}^1_{2,3}(u) & \acute{\nu}^1_{2,4}(u)\\ \nu^2_{2,1}(u) & \nu^2_{2,2}(u) & \nu^2_{2,3}(u) & \nu^2_{2,4}(u)\\ \acute{\nu}^2_{2,1}(u) & \acute{\nu}^2_{2,2}(u) & \acute{\nu}^2_{2,3}(u) & \acute{\nu}^2_{2,4}(u)\end{pmatrix}$$

for all $u\in(0,1)$. In particular, as $u\to1^-$, we have

$$\text{(3.7)}\qquad 2^6u^3\omega_4(u)\\ =\det\begin{pmatrix}\mathbf{IKM}(1,5;1)+\circ & \nu^1_{2,2}(1)+\circ & \nu^1_{2,3}(1)+\circ & \nu^1_{2,4}(1)+\circ\\ \sharp & \acute{\nu}^1_{2,2}(1)+\circ & \acute{\nu}^1_{2,3}(u) & \acute{\nu}^1_{2,4}(1)+\circ\\ \mathbf{IKM}(1,5;3)+\circ & \nu^2_{2,2}(1)+\circ & \nu^2_{2,3}(u) & \nu^2_{2,4}(1)+\circ\\ \sharp & \sharp & \acute{\nu}^2_{2,3}(u) & \sharp\end{pmatrix}$$

where a hash (resp. circle) denotes a bounded (resp. infinitesimal) quantity. Here, it is also worth pointing out that $\nu^1_{2,2}(1)=\nu^1_{2,4}(1)=\mathbf{IKM}(2,4;1)$ and $\nu^2_{2,2}(1)=\nu^2_{2,4}(1)=\mathbf{IKM}(2,4;3)$.

From a bound

$$\text{(3.8)}\qquad \sup_{t>0}t^{2s}\left|[I_0(t)K_0(t)]^2-\frac{1}{4t^2}\right|<\infty,\quad s\in\{1,2\}$$

and generalized Weber–Schafheitlin integrals [cf. 34, §13.45] for $u\in(0,1)$:

$$\text{(3.9)}\qquad \int_0^\infty I_0(\sqrt{u}t)K_0(t)t\,\mathrm{d}\,t=\frac{1}{1-u},$$

$$\text{(3.10)}\qquad \int_0^\infty I_1(\sqrt{u}t)K_0(t)\,\mathrm{d}\,t=-\frac{\log(1-u)}{2\sqrt{u}},$$

$$\text{(3.11)}\qquad \int_0^\infty I_1(\sqrt{u}t)K_0(t)t^2\,\mathrm{d}\,t=\frac{2\sqrt{u}}{(1-u)^2},$$

we may deduce the following asymptotic formulae in the $u\to1^-$ regime:

$$\text{(3.12)}\qquad \begin{aligned}\acute{\nu}^1_{2,3}(u)&=\frac14\int_0^\infty I_1(\sqrt{u}t)K_0(t)\,\mathrm{d}\,t\\ &\quad+\int_0^\infty I_1(\sqrt{u}t)K_0(t)\left\{[I_0(t)K_0(t)]^2-\frac{1}{4t^2}\right\}t^2\,\mathrm{d}\,t\\ &=O(\log(1-u)),\end{aligned}$$

$$(1-u)\nu_{2,3}^2(u) \\ = \frac{(1-u)}{4}\int_0^\infty I_0(\sqrt{u}t)K_0(t)t\,\mathrm{d}\,t \\ + (1-u)\int_0^\infty I_0(\sqrt{u}t)K_0(t)\left\{[I_0(t)K_0(t)]^2-\frac{1}{4t^2}\right\}t^3\,\mathrm{d}\,t \\ = O(1), \tag{3.13}$$

$$(1-u)^2\acute{\nu}_{2,3}^2(u) \\ = \frac{(1-u)^2}{4}\int_0^\infty I_1(\sqrt{u}t)K_0(t)t^2\,\mathrm{d}\,t \\ + (1-u)^2\int_0^\infty I_1(\sqrt{u}t)K_0(t)\left\{[I_0(t)K_0(t)]^2-\frac{1}{4t^2}\right\}t^4\,\mathrm{d}\,t \\ = \frac{\sqrt{u}}{2}+O((1-u)^2\log(1-u)). \tag{3.14}$$

Therefore, we have

$$2^6u^2(1-u)^2\omega_4(u) \\ = \det\begin{pmatrix}\mathbf{IKM}(1,5;1)+\circ & \nu_{2,2}^1(1)+\circ & \circ & \nu_{2,4}^1(1)+\circ \\ \sharp & \acute{\nu}_{2,2}^1(1)+\circ & \circ & \acute{\nu}_{2,4}^1(1)+\circ \\ \mathbf{IKM}(1,5;3)+\circ & \nu_{2,2}^2(1)+\circ & \circ & \nu_{2,4}^1(1)+\circ \\ \sharp & \sharp & \frac{1}{2}+\circ & \sharp\end{pmatrix} \\ = -\frac{1}{2}\det\begin{pmatrix}\mathbf{IKM}(1,5;1)+\circ & \mathbf{IKM}(2,4;1)+\circ & \mathbf{IKM}(2,4;1)+\circ \\ \sharp & \acute{\nu}_{2,2}^1(1)+\circ & \acute{\nu}_{2,4}^1(1)+\circ \\ \mathbf{IKM}(1,5;3)+\circ & \mathbf{IKM}(2,4;3)+\circ & \mathbf{IKM}(2,4;3)+\circ\end{pmatrix} \\ + o(1) \tag{3.15}$$

by cofactor expansion, as $u\to1^-$. After eliminating the second column from the last column in the last 3×3 determinant, and employing $\acute{\nu}_{2,4}^1(1)-\acute{\nu}_{2,2}^1(1)=-\mathbf{IKM}(1,3;1)$, in a similar fashion as (2.22), we arrive at the factorization formula in (3.5). □

Next, we consider an extension of Proposition 2.5.

Proposition 3.3 (Factorization of $\omega_4(0^+)$). *The limit*

$$\lim_{u\to0^+}u^4\omega_4(u)=-\frac{5(\det\mathbf{M}_2)^2}{2^73} \tag{3.16}$$

entails

$$\omega_4(u)=-\frac{\pi^6}{2^5[u^2(1-u)(9-u)(25-u)]^2},\quad\forall u\in(0,1).\tag{3.17}$$

In particular, this implies the evaluation $\det\mathbf{N}_2=\frac{\pi^4}{2^6 3^2}$.

Proof. We will evaluate $\lim_{u\to0^+}u^4\omega_4(u)$, starting from the expansion

(3.18)
$$\begin{aligned}2^6u^4\omega_4(u)&=\det\begin{pmatrix}\nu^1_{2,1}(u)&\nu^1_{2,2}(u)&\nu^1_{2,3}(u)&\nu^1_{2,4}(u)\\\sqrt{u}\acute{\nu}^1_{2,1}(u)&\sqrt{u}\acute{\nu}^1_{2,2}(u)&\sqrt{u}\acute{\nu}^1_{2,3}(u)&\sqrt{u}\acute{\nu}^1_{2,4}(u)\\\nu^2_{2,1}(u)&\nu^2_{2,2}(u)&\nu^2_{2,3}(u)&\nu^2_{2,4}(u)\\\sqrt{u}\acute{\nu}^2_{2,1}(u)&\sqrt{u}\acute{\nu}^2_{2,2}(u)&\sqrt{u}\acute{\nu}^2_{2,3}(u)&\sqrt{u}\acute{\nu}^2_{2,4}(u)\end{pmatrix}\\
&=\det\begin{pmatrix}O(\log u)&\mu^1_{2,1}(1)+O(u)&\mu^1_{2,2}(1)+O(u)&O(\log u)\\\sqrt{u}\acute{\nu}^1_{2,1}(u)&O(u)&O(u)&\sqrt{u}\acute{\nu}^1_{2,4}(u)\\O(\log u)&\mu^2_{2,1}(1)+O(u)&\mu^2_{2,2}(1)+O(u)&O(\log u)\\\sqrt{u}\acute{\nu}^2_{2,1}(u)&O(u)&O(u)&\sqrt{u}\acute{\nu}^2_{2,4}(u)\end{pmatrix}\\
&=-\det\begin{pmatrix}\mathbf{IKM}(1,4;1)&\mathbf{IKM}(2,3;1)\\\mathbf{IKM}(1,4;3)&\mathbf{IKM}(2,3;3)\end{pmatrix}\det\begin{pmatrix}\sqrt{u}\acute{\nu}^1_{2,1}(u)&\sqrt{u}\acute{\nu}^1_{2,4}(u)\\\sqrt{u}\acute{\nu}^2_{2,1}(u)&\sqrt{u}\acute{\nu}^2_{2,4}(u)\end{pmatrix}\\
&\quad+O(u^2\log^2u),\end{aligned}$$

where $\mu^\ell_{2,1}(1)=\mathbf{IKM}(1,4;2\ell-1)$ and $\mu^\ell_{2,2}(1)=\mathbf{IKM}(2,3;2\ell-1)$. Arguing in a similar vein as (2.28), we find

(3.19)
$$\begin{aligned}&\begin{pmatrix}\sqrt{u}\acute{\nu}^1_{2,1}(u)&\sqrt{u}\acute{\nu}^1_{2,4}(u)\\\sqrt{u}\acute{\nu}^2_{2,1}(u)&\sqrt{u}\acute{\nu}^2_{2,4}(u)\end{pmatrix}\\
&=\begin{pmatrix}-\frac{5}{6}\mathbf{IKM}(1,4;1)+o(1)&-\mathbf{IKM}(2,3;1)+o(1)\\-\frac{5}{6}\mathbf{IKM}(1,4;3)+o(1)&-\mathbf{IKM}(2,3;3)+o(1)\end{pmatrix}\end{aligned}$$

as $u\to0^+$. Therefore, our goal is achieved. □

4. Broadhurst–Mellit formulae for $\det\mathbf{M}_k$ and $\det\mathbf{N}_k$

The major goal of this section is to generalize the algebraic manipulations in §§2–3 to the following recursions of Broadhurst–Mellit determinants for all $k\in\mathbb{Z}_{\geq 2}$:

$$\det\mathbf{M}_{k-1}\det\mathbf{M}_k=\frac{k[\Gamma(k/2)]^2(\det\mathbf{N}_{k-1})^2}{2(2k+1)}\prod_{j=1}^{k}\left[\frac{(2j)^2}{(2j)^2-1}\right]^{k-\frac12},\tag{4.1}$$

$$\det\mathbf{N}_{k-1}\det\mathbf{N}_k=\frac{2k+1}{k+1}\frac{(\det\mathbf{M}_k)^2}{(k-1)!}\prod_{j=2}^{k+1}\left[\frac{(2j-1)^2}{(2j-1)^2-1}\right]^{k}.\tag{4.2}$$

Once these recursions are established, we can verify Conjectures 1.1 and 1.2 by induction.

4.1. Wrońskians for two-scale Bessel moments

The analysis in §§2–3 motivates us to introduce the following notations for matrix elements.

Definition 4.1. For each $k\in\mathbb{Z}_{\geq 2}$, we set

$$\begin{cases}\mu_{k,1}^{\ell}(u)=\frac{\widetilde{\mathbf{I}}\mathbf{KM}(1,2k;2\ell-1|u)+2k\,\mathbf{I}\widetilde{\mathbf{K}}\mathbf{M}(1,2k;2\ell-1|u)}{2k+1},\\ \mu_{k,j}^{\ell}(u)=\widetilde{\mathbf{I}}\mathbf{KM}(j,2k+1-j;2\ell-1|u),\forall j\in\mathbb{Z}\cap[2,k],\\ \mu_{k,j}^{\ell}(u)=\mathbf{I}\widetilde{\mathbf{K}}\mathbf{M}(j-k+1,3k-j;2\ell-1|u),\forall j\in\mathbb{Z}\cap[k+1,2k-1],\end{cases}\tag{4.3}$$

and

$$\begin{cases}\nu_{k,1}^{\ell}(u)=\frac{\widetilde{\mathbf{I}}\mathbf{KM}(1,2k+1;2\ell-1|u)+(2k+1)\,\mathbf{I}\widetilde{\mathbf{K}}\mathbf{M}(1,2k+1;2\ell-1|u)}{2(k+1)},\\ \nu_{k,j}^{\ell}(u)=\widetilde{\mathbf{I}}\mathbf{KM}(j,2k+2-j;2\ell-1|u),\forall j\in\mathbb{Z}\cap[2,k+1],\\ \nu_{k,j}^{\ell}(u)=\mathbf{I}\widetilde{\mathbf{K}}\mathbf{M}(j-k,3k+2-j;2\ell-1|u),\forall j\in\mathbb{Z}\cap[k+2,2k].\end{cases}\tag{4.4}$$

For $a,b\in\mathbb{Z}\cap[1,k]$, we also write $\mu_{k,a}^{b}=\mu_{k,a}^{b}(1)$ and $\nu_{k,a}^{b}=\nu_{k,a}^{b}(1)$, as the abbreviations for the entries in the Broadhurst–Mellit matrices:

$$\mu_{k,a}^{b}=(\mathbf{M}_k)_{a,b}:=\int_0^\infty[I_0(t)]^a[K_0(t)]^{2k+1-a}t^{2b-1}\,\mathrm{d}\,t,\tag{4.5}$$

$$\nu_{k,a}^{b}=(\mathbf{N}_k)_{a,b}:=\int_0^\infty[I_0(t)]^a[K_0(t)]^{2k+2-a}t^{2b-1}\,\mathrm{d}\,t.\tag{4.6}$$

For each $k\in\mathbb{Z}_{\geq 2}$, we will be concerned with

$$\varOmega_{2k-1}(u):=W[\mu_{k,1}^{1}(u),\dots,\mu_{k,2k-1}^{1}(u)],\tag{4.7}$$

$$\omega_{2k}(u):=W[\nu_{k,1}^{1}(u),\dots,\nu_{k,2k}^{1}(u)],\tag{4.8}$$

the Wrońskian determinants for two-scale Bessel moments.

If we further define

(4.9)
$$\begin{cases}\acute{\mu}^{\ell}_{k,1}(u)=\frac{\mathbf{\acute{I}KM}(1,2k;2\ell-1|u)+2k\,\mathbf{I\acute{K}M}(1,2k;2\ell-1|u)}{2k+1},\\ \acute{\mu}^{\ell}_{k,j}(u)=\mathbf{\acute{I}KM}(j,2k+1-j;2\ell-1|u),\forall j\in\mathbb{Z}\cap[2,k],\\ \acute{\mu}^{\ell}_{k,j}(u)=\mathbf{I\acute{K}M}(j-k+1,3k-j;2\ell-1|u),\forall j\in\mathbb{Z}\cap[k+1,2k-1],\end{cases}$$

and

(4.10)
$$\begin{cases}\acute{\nu}^{\ell}_{k,1}(u)=\frac{\mathbf{\acute{I}KM}(1,2k+1;2\ell-1|u)+(2k+1)\,\mathbf{I\acute{K}M}(1,2k+1;2\ell-1|u)}{2(k+1)},\\ \acute{\nu}^{\ell}_{k,j}(u)=\mathbf{\acute{I}KM}(j,2k+2-j;2\ell-1|u),\forall j\in\mathbb{Z}\cap[2,k+1],\\ \acute{\nu}^{\ell}_{k,j}(u)=\mathbf{I\acute{K}M}(j-k,3k+2-j;2\ell-1|u),\forall j\in\mathbb{Z}\cap[k+2,2k],\end{cases}$$

then we can verify

$$(2\sqrt{u})^{(k-1)(2k-1)}\varOmega_{2k-1}(u)=\det\begin{pmatrix}\mu^1_{k,1}(u) & \cdots & \mu^1_{k,2k-1}(u)\\ \acute{\mu}^1_{k,1}(u) & \cdots & \acute{\mu}^1_{k,2k-1}(u)\\ \cdots & \cdots & \cdots\\ \mu^k_{k,1}(u) & \cdots & \mu^k_{k,2k-1}(u)\end{pmatrix}\tag{4.11}$$

for $u\in(0,4)$, and

$$(2\sqrt{u})^{(2k-1)k}\omega_{2k}(u)=\det\begin{pmatrix}\nu^1_{k,1}(u) & \cdots & \nu^1_{k,2k-1}(u)\\ \acute{\nu}^1_{k,1}(u) & \cdots & \acute{\nu}^1_{k,2k-1}(u)\\ \cdots & \cdots & \cdots\\ \nu^k_{k,1}(u) & \cdots & \nu^k_{k,2k-1}(u)\\ \acute{\nu}^k_{k,1}(u) & \cdots & \acute{\nu}^k_{k,2k-1}(u)\end{pmatrix}\tag{4.12}$$

for $u\in(0,1)$, through iterated applications of the Bessel differential equations

$$(uD^2+D^1)I_0(\sqrt{u}t)=\frac{t^2}{4}I_0(\sqrt{u}t),$$
$$(uD^2+D^1)K_0(\sqrt{u}t)=\frac{t^2}{4}K_0(\sqrt{u}t).$$

Lemma 4.2 (Vanhove differential equations for $\varOmega_{2k-1}(u)$ and $\omega_{2k}(u)$).

(a) *For each $n\in\mathbb{Z}_{\geq1}$, there exists a holonomic differential operator $\widetilde{L}_n$ whose leading term is $f_n(u)D^n$, such that $f_n(u)$ is a monic polynomial and*

$$\begin{cases}\widetilde{L}_n\,\widetilde{\mathbf{I}}\mathbf{KM}(1,n+1,1|u)=-\frac{(n+1)!}{2^n},\\ \widetilde{L}_n\,\mathbf{I}\widetilde{\mathbf{K}}\mathbf{M}(1,n+1,1|u)=\frac{n!}{2^n},\\ \widetilde{L}_n\,\widetilde{\mathbf{I}}\mathbf{KM}(j,n+2-j,1|u)=0, & \forall j\in\mathbb{Z}\cap[2,\frac{n}{2}+1],\\ \widetilde{L}_n\,\mathbf{I}\widetilde{\mathbf{K}}\mathbf{M}(j,n+2-j,1|u)=0, & \forall j\in\mathbb{Z}\cap[2,\frac{n+1}{2}].\end{cases}\tag{4.13}$$

(b) *For $u\in(0,4)$, we have*

$$D^1\varOmega_{2k-1}(u)=\frac{2k-1}{2}\varOmega_{2k-1}(u)D^1\log\frac{1}{u^k\prod_{j=1}^{k}[(2j)^2-u]};\tag{4.14}$$

for $u\in(0,1)$, we have

$$D^1\omega_{2k}(u)=k\omega_{2k}(u)D^1\log\frac{1}{u^k\prod_{j=1}^{k+1}[(2j-1)^2-u]}.\tag{4.15}$$

Proof. (a) With the notations $\eth^0 f(t)=f(t)$ and $\eth^{n+1}f(t)=t\frac{\mathrm{d}}{\mathrm{d}\,t}\eth^n f(t)$ for all $n\in\mathbb{Z}_{\geq0}$, we have the Bessel differential equations $\eth^2 I_0(t)=t^2\eth^0 I_0(t)$ and $\eth^2 K_0(t)=t^2\eth^0 K_0(t)$. The Borwein–Salvy operator L_{n+1} [8, Lemma 3.3], being the n-th symmetric power of the Bessel differential operator $\eth^2-t^2\eth^0$, annihilates each member in the set $\{[I_0(t)]^j[K_0(t)]^{n-j}|j\in\mathbb{Z}\cap[0,n]\}$. The Borwein–Salvy operator $L_{n+1}=\mathscr{L}_{n+1,n+1}$ can be constructed by the Bronstein–Mulders–Weil algorithm [22, Theorem 1]:

$$\begin{cases}\mathscr{L}_{n+1,0}=\eth^0,\mathscr{L}_{n+1,1}=\eth^1,\\ \mathscr{L}_{n+1,k+1}=\eth^1\mathscr{L}_{n+1,k}-k(n+1-k)t^2\mathscr{L}_{n+1,k-1}, & \forall k\in\mathbb{Z}\cap[1,n].\end{cases}\tag{4.16}$$

For each fixed $j\in\mathbb{Z}\cap[0,n]$, one can use the aforementioned recursion for the operators $\mathscr{L}_{n+1,k}$, the Leibniz rule for derivatives, and the Bessel differential equation, to prove a formula [cf. 22, Theorem 1]

$$\begin{aligned}&\mathscr{L}_{n+1,k}\{[I_0(t)]^j[K_0(t)]^{n-j}\}\\ &=\sum_{\ell=0}^{k}\frac{k!}{\ell!(k-\ell)!}\frac{j!}{(j-\ell)!}\frac{(n-j)!}{(n-j-k+\ell)!}\\ &\quad\times[\eth^1 I_0(t)]^\ell[I_0(t)]^{j-\ell}[\eth^1 K_0(t)]^{k-\ell}[K_0(t)]^{n-j-k+\ell}\end{aligned}\tag{4.17}$$

by induction on $k \in \mathbb{Z} \cap [0, n]$. (Here, we need the convention $1/(-m)! = 0$ for all positive integers m.) In particular, we have the following identities for $k \in \mathbb{Z} \cap [0, n]$ [cf. 8, Lemma 3.1]

$$\begin{aligned} &\mathscr{L}_{n+1,k}\{[K_0(t)]^n\} \\ &= \frac{n!}{(n-k)!}[K_0(t)]^{n-k}[\eth^1 K_0(t)]^k, \end{aligned} \tag{4.18}$$

$$\begin{aligned} &\mathscr{L}_{n+1,k}\{I_0(t)[K_0(t)]^{n-1}\} \\ &= \frac{(n-1)!k}{(n-k)!}[\eth^1 I_0(t)][K_0(t)]^{n-k}[\eth^1 K_0(t)]^{k-1} \\ &\quad + \frac{(n-1)!}{(n-k-1)!} I_0(t)[K_0(t)]^{n-k-1}[\eth^1 K_0(t)]^k. \end{aligned} \tag{4.19}$$

Once we have obtained

$$L_{n+1} = \sum_{k=0}^{n+1} \lambda_{n+1,k}(t) \frac{\partial^k}{\partial t^k} \tag{4.20}$$

from the Bronstein–Mulders–Weil algorithm described above [with the understanding that $\frac{\partial^0}{\partial t^0} g(t, u) = g(t, u)$], we can define the action of its formal adjoint L^*_{n+1} on a bivariate function $g(t, u)$ as follows:

$$L^*_{n+1} g(t, u) = \sum_{k=0}^{n+1} (-1)^k \frac{\partial^k}{\partial t^k}[\lambda_{n+1,k}(t) g(t, u)]. \tag{4.21}$$

The design of Vanhove's operators $\widetilde{L}_n, n \in \mathbb{Z}_{\geq 1}$ in [33, §9] ensures that

$$\begin{cases} t\widetilde{L}_n I_0(\sqrt{u}t) = \frac{(-1)^n}{2^n} L^*_{n+2} \frac{I_0(\sqrt{u}t)}{t}, \\ t\widetilde{L}_n K_0(\sqrt{u}t) = \frac{(-1)^n}{2^n} L^*_{n+2} \frac{K_0(\sqrt{u}t)}{t}. \end{cases} \tag{4.22}$$

Starting from the vanishing identity

$$0 = \int_0^\infty \frac{I_0(\sqrt{u}t)}{t} L_{n+1}\{[K_0(t)]^n\} \,\mathrm{d}\, t, \tag{4.23}$$

we may perform successive integrations by parts, while carefully treating boundary contributions from the $t \to 0^+$ regime. We recall the recursion $L_{n+1} = \mathscr{L}_{n+1,n+1} = \eth^1 \mathscr{L}_{n+1,n} - nt^2 \mathscr{L}_{n+1,n-1}$ from (4.16) and the closed-form formula for $\mathscr{L}_{n+1,k}\{[K_0(t)]^n\}$ from (4.18). These identities enable us to rewrite (4.23) as

$$\begin{aligned}0={}&\int_0^\infty I_0(\sqrt{u}t)\frac{\partial}{\partial t}\mathscr{L}_{n+1,n}\{[K_0(t)]^n\}\,\mathrm{d}\,t\\&-n\int_0^\infty tI_0(\sqrt{u}t)\mathscr{L}_{n+1,n-1}\{[K_0(t)]^n\}\,\mathrm{d}\,t\\={}&-(-1)^n n!-\int_0^\infty \mathscr{L}_{n+1,n}\{[K_0(t)]^n\}\frac{\partial I_0(\sqrt{u}t)}{\partial t}\,\mathrm{d}\,t\\&-n\int_0^\infty tI_0(\sqrt{u}t)\mathscr{L}_{n+1,n-1}\{[K_0(t)]^n\}\,\mathrm{d}\,t,\end{aligned}\tag{4.24}$$

where the boundary contribution comes from

$$\lim_{t\to0^+}\mathscr{L}_{n+1,n}\{[K_0(t)]^n\}=n!\lim_{t\to0^+}[-tK_1(t)]^n=(-1)^n n!.$$

None of the subsequent integrations by parts will incur any non-vanishing boundary contributions, because we have $\lim_{t\to0^+}t^\ell\log^m t=0$ for all $\ell,m\in\mathbb{Z}_{>0}$. Thus, we can recast (4.24) into

$$\begin{aligned}0={}&-(-1)^n n!+\int_0^\infty[K_0(t)]^n L^*_{n+1}\frac{I_0(\sqrt{u}t)}{t}\,\mathrm{d}\,t\\={}&-(-1)^n n!+(-1)^{n-1}2^{n-1}\widetilde{L}_{n-1}\,\mathbf{IKM}(1,n,1|u),\end{aligned}\tag{4.25}$$

which proves the first identity in (4.13).

In a similar vein, we may integrate by parts with the help from (4.16) and (4.19):

$$\begin{aligned}0={}&\int_0^\infty\frac{K_0(\sqrt{u}t)}{t}L_{n+1}\{I_0(t)[K_0(t)]^{n-1}\}\,\mathrm{d}\,t\\={}&-\int_0^\infty\mathscr{L}_{n+1,n}\{I_0(t)[K_0(t)]^{n-1}\}\frac{\partial K_0(\sqrt{u}t)}{\partial t}\,\mathrm{d}\,t\\&-n\int_0^\infty tK_0(\sqrt{u}t)\mathscr{L}_{n+1,n-1}\{I_0(t)[K_0(t)]^{n-1}\}\,\mathrm{d}\,t\\={}&\lim_{t\to0^+}\left(t\frac{\partial K_0(\sqrt{u}t)}{\partial t}\mathscr{L}_{n+1,n-1}\{I_0(t)[K_0(t)]^{n-1}\}\right)\\&+(-1)^{n-1}2^{n-1}\widetilde{L}_{n-1}\,\mathbf{I\widetilde{K}M}(1,n,1|u)\\={}&(-1)^n(n-1)!+(-1)^{n-1}2^{n-1}\widetilde{L}_{n-1}\,\mathbf{I\widetilde{K}M}(1,n,1|u),\end{aligned}\tag{4.26}$$

which proves the second identity in (4.13).

All the remaining cases in (4.13) can be proved by examining the asymptotic behavior of (4.17) in the $t \to 0^+$ regime.

(b) From (4.13), we know that for each $k \in \mathbb{Z}_{\geq 2}$, Vanhove's operator $\widetilde{L}_{2k-1}$ (resp. $\widetilde{L}_{2k}$) annihilates every member in the set $\{\mu^1_{k,j}(u)|j \in \mathbb{Z} \cap [1, 2k - 1]\}$ (resp. $\{\nu^1_{k,j}(u)|j \in \mathbb{Z} \cap [1, 2k]\}$).

For $k \in \mathbb{Z}_{\geq 2}$, Vanhove's operators $\widetilde{L}_{2k-1}$ and $\widetilde{L}_{2k}$ take the following forms [33, (9.11)–(9.12)]:

$$\widetilde{L}_{2k-1} = \mathfrak{m}_{2k-1}(u)D^{2k-1} + \frac{2k-1}{2}\frac{\mathrm{d}\,\mathfrak{m}_{2k-1}(u)}{\mathrm{d}\,u}D^{2k-2} + L.O.T., \tag{4.27}$$

$$\widetilde{L}_{2k} = \mathfrak{n}_{2k}(u)D^{2k} + k\frac{\mathrm{d}\,\mathfrak{n}_{2k}(u)}{\mathrm{d}\,u}D^{2k-1} + L.O.T., \tag{4.28}$$

where

$$\mathfrak{m}_{2k-1}(u) = u^k \prod_{j=1}^{k}[u - (2j)^2], \quad \mathfrak{n}_{2k}(u) = u^k \prod_{j=1}^{k+1}[u - (2j-1)^2], \tag{4.29}$$

and "$L.O.T.$" stands for "lower order terms". Therefore, the corresponding Wrońskians must evolve according to (4.14) and (4.15). □

Remark Prior to the work of Vanhove [33], various authors [1, 26, 29] have considered the operator $\widetilde{L}_2$. Although Vanhove formulated his theory in [33, §9] only for "sunrise diagrams" $\widetilde{\mathbf{IKM}}(1, n; 1|u)$, his ideas generalize well to Feynman graphs with other topologies, as indicated in the proof above. For an extension of Vanhove's differential equations to quantum field theory in arbitrary dimensions, see Müller-Stach–Weinzierl–Zayadeh [30].

Remark For $n \in \mathbb{Z}_{>1}$, Kluyver's function $p_n(x) = \int_0^\infty J_0(xt)[J_0(t)]^n xt\,\mathrm{d}\,t$ represents the probability density for the distance traveled by a random walker in the Euclidean plane after n consecutive unit steps aiming at random directions. Here, $J_0(t) := \frac{2}{\pi}\int_0^{\pi/2}\cos(t\cos\varphi)\,\mathrm{d}\,\varphi$ is the Bessel function of the first kind. It has been shown by Borwein–Straub–Wan–Zudilin that $p_n(x)$ is holonomic, whose annihilator has the form $g_n(x)\frac{\mathrm{d}^{n-1}}{\mathrm{d}\,x^{n-1}} + L.O.T.$ where [10, (2.8)]

$$g_n(x) = x^{n-1} \prod_{\substack{m \in \mathbb{Z}\cap[1,n] \\ m \equiv n \pmod 2}} (x^2 - m^2). \tag{4.30}$$

The resemblance between (4.29) and (4.30) is not accidental. We refer our readers to [37] for the connection between Kluyver's probability density function and two-scale Bessel moments.

4.2. Reduction of $\det \mathbf{M}_k$ to $\det \mathbf{M}_{k-1}$ and $\det \mathbf{N}_{k-1}$

Now we factorize Ω_{2k-1} in a similar spirit as Propositions 2.4 and 2.5.

Proposition 4.3 (Factorization of $\Omega_{2k-1}(1)$). *For each $k \in \mathbb{Z}_{\geq 2}$, we have*

$$\Omega_{2k-1}(1) = (-1)^{\frac{(k-1)(k-2)}{2}} \frac{\det \mathbf{M}_{k-1}}{2^{(k-1)(2k-1)}} \det \mathbf{M}_k. \tag{4.31}$$

Proof. In the formula

$$\begin{aligned} & 2^{(k-1)(2k-1)}\Omega_{2k-1}(1) \\ &= \det\begin{pmatrix} \mu^1_{k,1}(1) & \cdots & \mu^1_{k,k}(1) & \mu^1_{k,k+1}(1) & \cdots & \mu^1_{k,2k-1}(1) \\ \acute{\mu}^1_{k,1}(1) & \cdots & \acute{\mu}^1_{k,k}(1) & \acute{\mu}^1_{k,k+1}(1) & \cdots & \acute{\mu}^1_{k,2k-1}(1) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mu^k_{k,1}(1) & \cdots & \mu^k_{k,k}(1) & \mu^k_{k,k+1}(1) & \cdots & \mu^k_{k,2k-1}(1) \end{pmatrix}, \end{aligned} \tag{4.32}$$

we observe that

$$\begin{cases} \mu^\ell_{k,j}(1) = \mu^\ell_{k,k+j-1}(1) = \mu^\ell_{k,j}, \\ \acute{\mu}^\ell_{k,k+j-1}(1) - \acute{\mu}^\ell_{k,j}(1) = -\mu^\ell_{k-1,j-1} \end{cases} \tag{4.33}$$

for all $j \in \mathbb{Z} \cap [2, k]$. Thus, we obtain, after column eliminations and row bubble sorts,

$$\begin{aligned} & 2^{(k-1)(2k-1)}\Omega_{2k-1}(1) \\ &= \det\begin{pmatrix} \mu^1_{k,1} & \cdots & \mu^1_{k,k} & 0 & \cdots & 0 \\ \acute{\mu}^1_{k,1}(1) & \cdots & \acute{\mu}^1_{k,k}(1) & -\mu^1_{k-1,1} & \cdots & -\mu^1_{k-1,k-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mu^k_{k,1} & \cdots & \mu^k_{k,k} & 0 & \cdots & 0 \end{pmatrix} \end{aligned} \tag{4.34}$$

$$= (-1)^{\frac{k(k-1)}{2}} \det \left(\begin{array}{ccc|c} & \mathbf{M}_k^{\mathrm{T}} & & \mathbf{O} \\ \hline \acute{\mu}_{k,1}^{1}(1) & \cdots & \acute{\mu}_{k,k}^{1}(1) & \\ \cdots & \cdots & \cdots & -\mathbf{M}_{k-1}^{\mathrm{T}} \\ \acute{\mu}_{k,1}^{k-1}(1) & \cdots & \acute{\mu}_{k,k}^{k-1}(1) & \end{array} \right),$$

which factorizes as claimed. □

Proposition 4.4 (Factorization of $\Omega_{2k-1}(0^+)$). *The limit*

$$\lim_{u\to 0^+} u^{k(2k-1)/2}\Omega_{2k-1}(u) = (-1)^{\frac{(k-1)(k-2)}{2}} \frac{k[\Gamma(k/2)]^2}{(2k+1)} \frac{(\det \mathbf{N}_{k-1})^2}{2^{(k-1)(2k-1)+1}} \tag{4.35}$$

entails

$$\begin{aligned} \Omega_{2k-1}(u) = {} & \frac{(-1)^{\frac{(k-1)(k-2)}{2}} k[\Gamma(k/2)]^2}{u^{k(2k-1)/2}(2k+1)} \frac{(\det \mathbf{N}_{k-1})^2}{2^{(k-1)(2k-1)+1}} \\ & \times \prod_{j=1}^{k} \left[\frac{(2j)^2}{(2j)^2 - u} \right]^{k-\frac{1}{2}}, \quad \forall u \in (0,4). \end{aligned} \tag{4.36}$$

Proof. As we compare the representation

$$\begin{aligned} & 2^{(k-1)(2k-1)} u^{k(2k-1)/2} \Omega_{2k-1}(u) \\ & = \det \left(\begin{array}{ccc|c} \mu_{k,1}^{1}(u) & \cdots & \mu_{k,2k-2}^{1}(u) & \mu_{k,2k-1}^{1}(u) \\ \sqrt{u}\acute{\mu}_{k,1}^{1}(u) & \cdots & \sqrt{u}\acute{\mu}_{k,2k-2}^{1}(u) & \sqrt{u}\acute{\mu}_{k,2k-1}^{1}(u) \\ \cdots & \cdots & \cdots & \cdots \\ \hline u^{k/2}\mu_{k,1}^{k}(u) & \cdots & u^{k/2}\mu_{k,2k-2}^{k}(u) & u^{k/2}\mu_{k,2k-1}^{k}(u) \end{array} \right) \end{aligned} \tag{4.37}$$

with (4.11), we see that each row involving $\acute{\mu}_{k,j}^{\ell}$ now bears an additional pre-factor of $\sqrt{u}$; the first $(k-1)$ rows involving $\mu_{k,j}^{\ell}$ are left intact, but the bottom row in (4.11) is multiplied by a factor of $u^{k/2}$. Clearly, this setting hearkens back to (2.26).

Akin to what we had in Proposition 2.5 when u is a positive infinitesimal, we can establish the following asymptotic behavior of the first $(2k-2)$

columns in (4.37):

$$\mu^{\ell}_{k,j}(u)=\begin{cases}O(\log u), & j\in\{1\}\cup(\mathbb{Z}\cap[k+1,2k-2])\\ \nu^{\ell}_{k-1,j-1}+O(u), & j\in\mathbb{Z}\cap[2,k]\end{cases}\tag{4.38}$$

for $\ell\in\mathbb{Z}\cap[1,k]$, and

$$\sqrt{u}\acute{\mu}^{\ell}_{k,j}(u)=\begin{cases}-\frac{2k}{2k+1}\nu^{\ell}_{k-1,1}+o(1), & j=1\\ O(u), & j\in\mathbb{Z}\cap[2,k]\\ -\nu^{\ell}_{k-1,j-k-1}+o(1), & j\in\mathbb{Z}\cap[k+1,2k-2]\end{cases}\tag{4.39}$$

for $\ell\in\mathbb{Z}\cap[1,k-1]$. Here, it is understood that when $k=2$, the closed interval $[k+1,2k-2]=[3,2]=\varnothing$ is the empty set, so $\{1\}\cup(\mathbb{Z}\cap[k+1,2k-2])$ degenerates to $\{1\}$ in this scenario. We also bear in mind that the bottom row in (4.37) carries an additional factor of $u^{k/2}$, so the estimate in (4.38) tells us that the bottom-left section of the partitioned matrix in (4.37) contains only infinitesimal elements, with order at most $O(u^{k/2}\log u)$.

Meanwhile, we point out that the top-right block in (4.37) contains elements of order $O(1/\sqrt{u})$, according to the rationale in (2.27) and (2.30). The bottom-right element behaves like

$$\begin{aligned}&u^{k/2}\mu^{k}_{k,2k-1}(u)\\ &=\frac{u^{k/2}}{2^k}\int_0^\infty K_0(\sqrt{u}t)t^{k-1}\,\mathrm{d}\,t\\ &\quad+u^{k/2}\int_0^\infty K_0(\sqrt{u}t)\left\{[I_0(t)K_0(t)]^k-\frac{1}{(2t)^k}\right\}t^{2k-1}\,\mathrm{d}\,t\\ &=\frac{1}{2^k}\int_0^\infty K_0(t)t^{k-1}\,\mathrm{d}\,t+O\left(u^{k/2}\int_0^\infty K_0(\sqrt{u}t)t^{k-2}\,\mathrm{d}\,t\right)\\ &=\frac{[\Gamma(k/2)]^2}{4}+O(\sqrt{u}),\end{aligned}\tag{4.40}$$

where we have quoted the evaluation of $\int_0^\infty K_0(t)t^{k-1}\,\mathrm{d}\,t$ from Heaviside's integral formula [34, §13.21(8)].

After taking care of the sign changes due to row and column permutations, we conclude that

$$\begin{aligned}&2^{(k-1)(2k-1)}\lim_{u\to0^+}u^{k(2k-1)/2}\varOmega_{2k-1}(u)\\ &=(-1)^{\frac{(k-1)(k-2)}{2}}\frac{k[\Gamma(k/2)]^2}{2(2k+1)}(\det\mathbf{N}_{k-1})^2\end{aligned}\tag{4.41}$$

as claimed. □

Therefore, we obtain the recursion relation in (4.1), after comparing (4.31) with (4.36).

4.3. Reduction of $\det \mathbf{N}_k$ to $\det \mathbf{M}_k$ and $\det \mathbf{N}_{k-1}$

Before factorizing ω_{2k} (as generalizations of Propositions 3.2 and 3.3), we need to build some asymptotic formulae on hypergeometric techniques.

Lemma 4.5 (Euler–Gauß–Schafheitlin–Weber). *We have*

(4.42)
$$\int_0^\infty I_0(\sqrt{u}t)K_0(t)t^n \,\mathrm{d}\, t = \begin{cases} -\dfrac{\log(1-u)}{2} + O(1), & n = 0, \\ \dfrac{2^{n-1}(n-1)!}{(1-u)^n} + o\left(\dfrac{1}{(1-u)^n}\right), & n \in \mathbb{Z}_{>0}, \end{cases}$$

and

(4.43)
$$\int_0^\infty I_1(\sqrt{u}t)K_0(t)t^n \,\mathrm{d}\, t = \begin{cases} -\dfrac{\log(1-u)}{2} + O(1), & n = 0, \\ \dfrac{2^{n-1}(n-1)!}{(1-u)^n} + o\left(\dfrac{1}{(1-u)^n}\right), & n \in \mathbb{Z}_{>0}, \end{cases}$$

as $u \to 1^-$.

Proof. According to the modified Weber–Schafheitlin integral formula [34, §13.45], we have

(4.44)
$$\int_0^\infty I_0(\sqrt{u}t)K_0(t)t^n \,\mathrm{d}\, t = 2^{n-1}\left[\Gamma\left(\frac{n+1}{2}\right)\right]^2 {}_2F_1\left(\left. \begin{matrix} \frac{n+1}{2}, \frac{n+1}{2} \\ 1 \end{matrix} \right| u \right),$$

(4.45)
$$\int_0^\infty I_1(\sqrt{u}t)K_0(t)t^n \,\mathrm{d}\, t = 2^{n-1}\sqrt{u}\left[\Gamma\left(\frac{n+2}{2}\right)\right]^2 {}_2F_1\left(\left. \begin{matrix} \frac{n+2}{2}, \frac{n+2}{2} \\ 2 \end{matrix} \right| u \right),$$

where the ${}_2F_1$'s are hypergeometric functions. When $n = 0$, the asymptotic behavior $-\frac{\log(1-u)}{2} + O(1)$ can be found directly in both cases above; to prove (4.42) [resp. (4.43)] when $n \in \mathbb{Z}_{>0}$, we need to specialize the Gauß summation [2, Theorem 2.2.2]:

$$ {}_2F_1\left(\begin{array}{c} a,b \\ c \end{array}\middle| 1\right)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\quad \text{for }\operatorname{Re}(c-a-b)>0 \tag{4.46}$$

and the Euler transformation [2, Theorem 2.2.5]:

$$ {}_2F_1\left(\begin{array}{c} a,b \\ c \end{array}\middle| u\right)=(1-u)^{c-a-b}{}_2F_1\left(\begin{array}{c} c-a,c-b \\ c \end{array}\middle| u\right) \tag{4.47}$$

to $a=\frac{1-n}{2},b=\frac{1-n}{2},c=1$ (resp. $a=\frac{2-n}{2},b=\frac{2-n}{2},c=2$). $\square$

Proposition 4.6 (Factorization of $\omega_{2k}(1^-)$). *We have the following identity:*

$$\lim_{u\to1^-}(1-u)^k\omega_{2k}(u)=(-1)^{\frac{k(k-1)}{2}}\frac{(k-1)!}{2^{(2k-1)k+1}}\det\mathbf{N}_{k-1}\det\mathbf{N}_k. \tag{4.48}$$

Proof. We will use the representation of $(2\sqrt{u})^{(2k-1)k}\omega_{2k}(u)$ in (4.12).

From the exponential decays (for large t) in the respective integrands, it is clear that the following limits exist as finite real numbers, so long as $j\in[1,k]\cup[k+2,2k]$ and $\ell\in\mathbb{Z}\cap[1,k]$:

$$\lim_{u\to1^-}\nu_{k,j}^{\ell}(u)=\nu_{k,j}^{\ell}(1), \tag{4.49}$$

$$\lim_{u\to1^-}\acute{\nu}_{k,j}^{\ell}(u)=\acute{\nu}_{k,j}^{\ell}(1). \tag{4.50}$$

So we need to examine the behavior of $(1-u)^k\nu_{k,k+1}^{\ell}(u)$ and $(1-u)^k\acute{\nu}_{k,k+1}^{\ell}(u)$, as u approaches 1 from below.

First, we consider

$$\nu_{k,k+1}^{\ell}(u)=\int_0^\infty I_0(\sqrt{u}t)[I_0(t)]^k[K_0(t)]^{k+1}t^{2\ell-1}\,\mathrm{d}\,t. \tag{4.51}$$

When $2\ell-k-1<0$, the integral $\nu_{k,k+1}^{\ell}(1)$ is finite (thanks to power law decay of the integrand for large t), and is equal to $\lim_{u\to1^-}\nu_{k,k+1}^{\ell}(u)$. Using the fact that

$$\sup_{t>0}t^k\left|[I_0(t)K_0(t)]^k-\frac{1}{(2t)^k}\right|<\infty, \tag{4.52}$$

we may deduce

$$\begin{aligned}(4.53)\quad \nu^{\ell}_{k,k+1}(u) &= \frac{1}{2^k}\int_0^\infty I_0(\sqrt{u}t)K_0(t)t^{2\ell-1-k}\,\mathrm{d}\,t\\ &\quad+\int_0^\infty I_0(\sqrt{u}t)K_0(t)\left\{[I_0(t)K_0(t)]^k-\frac{1}{(2t)^k}\right\}t^{2\ell-1}\,\mathrm{d}\,t\\ &= O\left(\int_0^\infty I_0(\sqrt{u}t)K_0(t)t^{2\ell-1-k}\,\mathrm{d}\,t\right)\\ &= \begin{cases}O((1-u)^{k+1-2\ell}), & 2\ell>k+1\\ O(\log(1-u)), & 2\ell=k+1\end{cases}\end{aligned}$$

when $2\ell-1-k\in\mathbb{Z}_{\geq 0}$, and (4.42) is applicable.

Then, we consider

$$(4.54)\qquad \acute{\nu}^{\ell}_{k,k+1}(u)=\int_0^\infty I_1(\sqrt{u}t)[I_0(t)]^k[K_0(t)]^{k+1}t^{2\ell}\,\mathrm{d}\,t.$$

When $2\ell-k<0$, the integral $\acute{\nu}^{\ell}_{k,k+1}(1)$ is finite (thanks to power law decay of the integrand for large t), and is equal to $\lim_{u\to1^-}\acute{\nu}^{\ell}_{k,k+1}(u)$. Using (4.52) and (4.43), we may deduce

$$\begin{aligned}(4.55)\quad \acute{\nu}^{\ell}_{k,k+1}(u) &= \frac{1}{2^k}\int_0^\infty I_1(\sqrt{u}t)K_0(t)t^{2\ell-k}\,\mathrm{d}\,t\\ &\quad+\int_0^\infty I_1(\sqrt{u}t)K_0(t)\left\{[I_0(t)K_0(t)]^k-\frac{1}{(2t)^k}\right\}t^{2\ell}\,\mathrm{d}\,t\\ &= O\left(\int_0^\infty I_1(\sqrt{u}t)K_0(t)t^{2\ell-k}\,\mathrm{d}\,t\right)\\ &= \begin{cases}O((1-u)^{k-2\ell}), & 2\ell>k\\ O(\log(1-u)), & 2\ell=k\end{cases}\end{aligned}$$

when $2\ell-k\in\mathbb{Z}_{\geq 0}$.

Summarizing the efforts in the last two paragraphs, we see that only the term $(1-u)^k\acute{\nu}^{k}_{k,k+1}(u)$ will play a consequential rôle in the $u\to1^-$ regime. Applying the bound

$$(4.56)\qquad \sup_{t>0}t^{k+1}\left|[I_0(t)K_0(t)]^k-\frac{1}{(2t)^k}\right|<\infty$$

to

$$\acute{\nu}^k_{k,k+1}(u)=\frac{1}{2^k}\int_0^\infty I_1(\sqrt{u}t)K_0(t)t^k\,\mathrm{d}\,t+\int_0^\infty I_1(\sqrt{u}t)K_0(t)\left\{[I_0(t)K_0(t)]^k-\frac{1}{(2t)^k}\right\}t^{2k}\,\mathrm{d}\,t,\tag{4.57}$$

we have

$$\lim_{u\to1^-}(1-u)^k\acute{\nu}^k_{k,k+1}(u)=\lim_{u\to1^-}\frac{(1-u)^k}{2^k}\int_0^\infty I_1(\sqrt{u}t)K_0(t)t^k\,\mathrm{d}\,t=\frac{(k-1)!}{2}\tag{4.58}$$

according to (4.43).

As we perform cofactor expansion with respect to the matrix element $\lim_{u\to1^-}(1-u)^k\acute{\nu}^k_{k,k+1}(u)$, manipulate columns according to

$$\begin{cases}\nu^\ell_{k,j}(1)=\nu^\ell_{k,k+j}(1)=\nu^\ell_{k,j},\\ \acute{\nu}^\ell_{k,k+j}(1)-\acute{\nu}^\ell_{k,j}(1)=-\nu^\ell_{k-1,j-1}\end{cases}\tag{4.59}$$

for all $j\in\mathbb{Z}\cap[2,k]$, and permute rows for a total of $\sum_{j=1}^k[(k+j)-2j]=\frac{k(k-1)}{2}$ times (according to bubble sort), we can identify $2^{(2k-1)k}\lim_{u\to1^-}(1-u)^k\omega_{2k}(u)$ with

$$(-1)^{k+1+\frac{k(k-1)}{2}}\frac{(k-1)!}{2}\det\left(\begin{array}{ccc|c}\multicolumn{3}{c|}{\mathbf{N}_k^{\mathrm{T}}} & \mathbf{O}\\ \hline \acute{\nu}^1_{k,1}(1) & \cdots & \acute{\nu}^1_{k,k}(1) & \\ \multicolumn{3}{c|}{\cdots\cdots\cdots\cdots} & -\mathbf{N}^{\mathrm{T}}_{k-1}\\ \acute{\nu}^{k-1}_{k,1}(1) & \cdots & \acute{\nu}^{k-1}_{k,k}(1) & \end{array}\right)$$
$$=(-1)^{\frac{k(k-1)}{2}}\frac{(k-1)!}{2}\det\mathbf{N}_{k-1}\det\mathbf{N}_k,\tag{4.60}$$

as expected. □

Proposition 4.7 (Factorization of $\omega_{2k}(0^+)$). *The limit*

$$\lim_{u\to0^+}u^{k^2}\omega_{2k}(u)=(-1)^{\frac{k(k-1)}{2}}\frac{(2k+1)(\det\mathbf{M}_k)^2}{2^{(2k-1)k+1}(k+1)}\tag{4.61}$$

entails

(4.62)
$$\omega_{2k}(u)=(-1)^{\frac{k(k-1)}{2}}\frac{(2k+1)(\det\mathbf{M}_k)^2}{2^{(2k-1)k+1}u^{k^2}(k+1)}\prod_{j=1}^{k+1}\left[\frac{(2j-1)^2}{(2j-1)^2-u}\right]^k,\quad\forall u\in(0,1).$$

Proof. In the formula

$$2^{(2k-1)k}u^{k^2}\omega_{2k}(u)=\det\begin{pmatrix}\nu^1_{k,1}(u) & \cdots & \nu^1_{k,2k-1}(u)\\ \sqrt{u}\acute{\nu}^1_{k,1}(u) & \cdots & \sqrt{u}\acute{\nu}^1_{k,2k-1}(u)\\ \cdots & \cdots & \cdots\\ \nu^k_{k,1}(u) & \cdots & \nu^k_{k,2k-1}(u)\\ \sqrt{u}\acute{\nu}^k_{k,1}(u) & \cdots & \sqrt{u}\acute{\nu}^k_{k,2k-1}(u)\end{pmatrix},\tag{4.63}$$

we observe that

$$\nu^\ell_{k,j}(u)=\begin{cases}O(\log u), & j\in\{1\}\cup(\mathbb{Z}\cap[k+2,2k])\\ \mu^\ell_{k-1,j-1}+O(u), & j\in\mathbb{Z}\cap[2,k]\end{cases}\tag{4.64}$$

and

$$\sqrt{u}\acute{\nu}^\ell_{k,j}(u)=\begin{cases}-\frac{2k+1}{2(k+1)}\mu^\ell_{k-1,1}+o(1), & j=1\\ O(u), & j\in\mathbb{Z}\cap[2,k]\\ -\mu^\ell_{k-1,j-k-1}+o(1), & j\in\mathbb{Z}\cap[k+2,2k]\end{cases}\tag{4.65}$$

apply to all $\ell\in\mathbb{Z}\cap[1,k]$, in the $u\to0^+$ limit. The factorization procedure is thus a straightforward generalization of Proposition 3.3. □

Comparing (4.48) with (4.62), we arrive at (4.2), thereby completing the proof of Broadhurst–Mellit determinant formulae (Conjectures 1.1 and 1.2).

5. Vacuum diagrams and Mahler measures

So far, each Wrońskian in our derivations concerns a set of functions that all reside in the kernel space $\ker\widetilde{L}_n$ of a certain Vanhove operator $\widetilde{L}_n$. The proofs of both Conjectures 1.1 and 1.2 were built on homogeneous evolution equations for the corresponding Wrońskian determinants, namely, (4.14) and (4.15). In this section, we will treat a pair of two-scale vacuum diagrams that are not annihilated by Vanhove's operators, along with the corresponding "vacuum analogs" $\check{\Omega}_3(u)$ and $\check{\omega}_4(u)$ of the Wrońskian determinants $\Omega_3(u)$ and $\omega_4(u)$ factorized in §§2–3. The inhomogeneous evolution equations for these new Wrońskians $\check{\Omega}_3(u)$ and $\check{\omega}_4(u)$ eventually enable us to verify Theorem 1.3, through factorizations of determinants.

5.1. Conjectures of Broadhurst–Mellit and Rodríguez-Villegas

For each positive integer n, the following integral

$$V_n := \mathbf{IKM}(0,n;1) = \int_0^\infty [K_0(t)]^n t \,\mathrm{d}\, t, \tag{5.1}$$

is known as the $(n-1)$-loop vacuum diagram [5, (1)] in two-dimensional quantum field theory. An integral representation $K_0(t) := \int_0^\infty e^{-t\cosh u}\,\mathrm{d}\, u$, $t > 0$ connects V_n to its avatar in statistical mechanics:

$$V_n = \int_0^\infty \mathrm{d}\, x_1 \cdots \int_0^\infty \mathrm{d}\, x_n \frac{1}{(\cosh x_1 + \cdots + \cosh x_n)^2}, \tag{5.2}$$

which is called the nth integral of Ising class [3, 4]. It has been shown that [5, 28]

(5.3)

$$V_1 = 1,\ V_2 = \frac{1}{2},\ V_3 = \frac{3}{4}\sum_{n=0}^{\infty}\left[\frac{1}{(3n+1)^2} - \frac{1}{(3n+2)^2}\right],\ V_4 = \sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}$$

and [4, Theorem 2]

$$\lim_{n\to\infty} \frac{2^n V_n}{n!} = 2e^{-2\gamma_0}, \tag{5.4}$$

where $\gamma_0 := \lim_{n\to\infty}\left(-\log n + \sum_{k=1}^n \frac{1}{k}\right)$ is the Euler–Mascheroni constant. The intermediate regime (namely, vacuum diagrams V_n for $n \in \mathbb{Z}_{>4}$) appears to be an uncharted territory.

In 2013, Broadhurst wrote that "we know nothing about the number theory of V_5" [15, §8.6], which stood in stark contrast with other physically relevant Bessel moments $\mathbf{IKM}(a,b;2k+1)$ involving $a+b=5$ Bessel factors, where k is a non-negative integer. In particular, conjectures on the closed-form expressions of $\mathbf{IKM}(1,4;2k+1)$ and $\mathbf{IKM}(2,3;2k+1)$ for $k \in \mathbb{Z}_{\geq 0}$ have been supported by numerical experiments [5] and confirmed by theoretical analyses [5, 7, 31, 36].

Rising to the challenge of understanding $V_5 = \mathbf{IKM}(0,5;1)$ and $V_6 = \mathbf{IKM}(0,6;1)$ arithmetically, Broadhurst and Mellit [16, 21] have proposed a possible link between Bessel moments and special L-values attached to two special modular forms

$$f_{3,15}(z) = [\eta(3z)\eta(5z)]^3 + [\eta(z)\eta(15z)]^3, \tag{5.5}$$

$$f_{4,6}(z) = [\eta(z)\eta(2z)\eta(3z)\eta(6z)]^2, \tag{5.6}$$

with $\eta(z) := e^{\pi iz/12}\prod_{n=1}^{\infty}(1-e^{2\pi inz})$ being the Dedekind eta function defined for complex numbers z in the upper half-plane $\mathfrak{H} := \{w \in \mathbb{C} | \operatorname{Im} w > 0\}$. Here, $f_{k,N}$ represents a modular form of weight k and level N.

We recapitulate their conjectures (see [21, (4.3), (5.8)] or [16, (101), (114)]) below.

Conjecture 5.1 (Broadhurst–Mellit). *We have the following evaluation of two 2×2 determinants filled with Bessel moments:*

$$\det \check{\mathbf{M}}_2 := \det\begin{pmatrix}\mathbf{IKM}(0,5;1) & \mathbf{IKM}(0,5;3)\\ \mathbf{IKM}(2,3;1) & \mathbf{IKM}(2,3;3)\end{pmatrix} \overset{?}{=} \frac{45}{8\pi^2}L(f_{3,15},4), \tag{5.7}$$

$$\det \check{\mathbf{N}}_2 := \det\begin{pmatrix}\mathbf{IKM}(0,6;1) & \mathbf{IKM}(0,6;3)\\ \mathbf{IKM}(2,4;1) & \mathbf{IKM}(2,4;3)\end{pmatrix} \overset{?}{=} \frac{27}{4\pi^2}L(f_{4,6},5), \tag{5.8}$$

where

$$L(f_{k,N},s) := \frac{(2\pi)^s}{\Gamma(s)}\int_0^{\infty} f_{k,N}(iy)y^{s-1}\,\mathrm{d}\,y. \tag{5.9}$$

In his seminal work [16, §7.4], Broadhurst has observed intricate connections between vacuum diagrams and logarithmic Mahler measures $m(P)$ of Laurent polynomials $P \in \mathbb{C}[x_1^{\pm1},\dots,x_n^{\pm1}]$ [cf. (1.8)]. Proven results in vacuum diagrams [5, 28] and Mahler measures [12] bring us the following identities [16, (118) and (119)]:

$$V_3 = \frac{\pi}{\sqrt{3}}m(1+x_1+x_2), \quad V_4 = \frac{\pi^2}{4}m(1+x_1+x_2+x_3). \tag{5.10}$$

Intriguingly, the special values $L(f_{3,15},4)$ and $L(f_{4,6},5)$ defined in (5.7) and (5.8) also show up in the conjectural evaluations of two logarithmic Mahler measures, due to Fernando Rodríguez-Villegas (see [11, §8], [10, (6.11), (6.12)] and [16, (120), (121)]).

Conjecture 5.2 (Rodríguez-Villegas). *We have*

$$m(1+x_1+x_2+x_3+x_4)\overset{?}{=}6\left(\frac{\sqrt{15}}{2\pi}\right)^5L(f_{3,15},4),\tag{5.11}$$

$$m(1+x_1+x_2+x_3+x_4+x_5)\overset{?}{=}3\left(\frac{\sqrt{6}}{\pi}\right)^6L(f_{4,6},5).\tag{5.12}$$

It appears that neither Conjecture 5.1 nor 5.2 would yield to the algebraic methods developed in this paper. In a recent review [32], Straub and Zudilin have stated that Conjecture 5.2 remains unproven, as of January 2018. Nevertheless, we can still achieve a modest goal of demonstrating the equivalence between Conjectures 5.1 and 5.2, as stated in Theorem 1.3.

As we will witness in the rest of §5, the bridge that connects Bessel moments to Mahler measures is Broadhurst's key formula (see [14, (9)], [10, last formula on p. 978 and penultimate formula on p. 981], as well as [16, (122)]):

(5.13)
$$m(1+x_1+\dots+x_{n-1})=-\gamma_0+\log 2-n\int_0^\infty J_1(t)[J_0(t)]^{n-1}\log t\,\mathrm{d}\,t,$$

which is provable by differentiating the "ramble integral" (see [10, §6] and [9, (2–2)])

$$\begin{aligned}W_n(s)&:=\int_0^1\mathrm{d}\,t_1\cdots\int_0^1\mathrm{d}\,t_n\left|\sum_{k=1}^n e^{2\pi i t_k}\right|^s\\&=-2^s\frac{\Gamma\left(1+\frac{s}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)}\int_0^\infty x^{-s}\frac{\mathrm{d}}{\mathrm{d}\,x}[J_0(x)]^n\,\mathrm{d}\,x,\quad\forall s\in(-1,2)\end{aligned}\tag{5.14}$$

at $s=0$. Here, we remind our readers that $J_0(x):=\frac{2}{\pi}\int_0^{\pi/2}\cos(x\cos\varphi)\,\mathrm{d}\,\varphi$ is the Bessel function of the first kind and zeroth order, whose derivative gives $\mathrm{d}\,J_0(x)/\,\mathrm{d}\,x=-J_1(x)$.

5.2. Relation between $\det\check{\mathbf{M}}_2$ and $m(1+x_1+x_2+x_3+x_4)$

If we assign a different parameter to one of the internal lines in the diagram V_5, then we obtain a family of two-scale vacuum diagrams

$$\int_0^\infty K_0(\sqrt{u}t)[K_0(t)]^4t\,\mathrm{d}\,t\tag{5.15}$$

parametrized by $u > 0$. To study this family of two-scale diagrams, we need a modest extension to Lemma 4.2, as given below.

Proposition 5.3 (Differential equation for two-scale 4-loop vacuums). *We have*

$$\widetilde{L}_3\,\widetilde{\mathbf{IKM}}(0,5;1|u) = \frac{3}{2}\log u, \quad \forall u \in (0,\infty), \tag{5.16}$$

where $\widetilde{L}_3$ is the third-order Vanhove operator defined in (2.9).

Proof. We first note that

$$\begin{aligned}\widetilde{L}_3 K_0(\sqrt{u}t) ={}& \frac{(2u^2-25u+32)t^2+2(u-4)}{2}K_0(\sqrt{u}t)\\ &-\frac{[(u-16)(u-4)t^2+12(u-6)]\sqrt{u}t}{8}K_1(\sqrt{u}t),\end{aligned} \tag{5.17}$$

where $K_1(x) = -\,\mathrm{d}\,K_0(x)/\,\mathrm{d}\,x$, which specializes to

$$\begin{aligned}&\widetilde{L}_3\,\widetilde{\mathbf{IKM}}(0,5;1|1)\\ &=\frac{3}{8}\int_0^\infty [4(3t^2-2)K_0(t)+5t(4-3t^2)K_1(t)][K_0(t)]^4 t\,\mathrm{d}\,t = 0.\end{aligned} \tag{5.18}$$

Here, we have canceled out integrals in the last step, thanks to the following formula for $n \in \mathbb{Z}_{>0}$:

$$\int_0^\infty K_1(t)[K_0(t)]^4 t^{2n}\,\mathrm{d}\,t = \frac{2n}{5}\,\mathbf{IKM}(0,5;2n-1), \tag{5.19}$$

which is a consequence of integration by parts.

We have

$$t\widetilde{L}_3 K_0(\sqrt{u}t) = -\frac{1}{2^3}L_5^*\frac{K_0(\sqrt{u}t)}{t}, \tag{5.20}$$

where

$$\begin{aligned}L_5^* :={}& -t^5\frac{\partial^5}{\partial t^5} - 15t^4\frac{\partial^4}{\partial t^4} + 5t^3(4t^2-13)\frac{\partial^3}{\partial t^3} + 90t^2(2t^2-1)\frac{\partial^2}{\partial t^2}\\ &- t(64t^4-392t^2+31)\frac{\partial}{\partial t} - (192t^4-184t^2+1).\end{aligned} \tag{5.21}$$

Here, the differential operator L_5^* is (formally) adjoint to the Borwein–Salvy operator [8, Example 4.1]

$$L_5 := \eth^5 - 20t^2\eth^3 - 60t^2\eth^2 + 8t^2(8t^2-9)t\eth^1 + 32t^2(4t^2-1)\eth^0 \quad \left[\text{where } \eth^n := \left(t\frac{\partial}{\partial t}\right)^n\right], \tag{5.22}$$

an annihilator of every member in the set $\{[I_0(t)]^j[K_0(t)]^{4-j}|j\in[0,4]\}$.

Using the fact that $L_5\{[K_0(t)]^4\}=0$, the recursive construction of $L_5 = \mathscr{L}_{5,5}$ via the the Bronstein–Mulders–Weil algorithm [22, Theorem 1]:

$$\begin{cases}\mathscr{L}_{5,0}=\eth^0, \mathscr{L}_{5,1}=\eth^1,\\ \mathscr{L}_{5,k+1}=\eth^1\mathscr{L}_{5,k}-k(5-k)t^2\mathscr{L}_{5,k-1}, \quad \forall k\in\mathbb{Z}\cap[1,4],\end{cases} \tag{5.23}$$

along with the identities $\mathscr{L}_{5,k}\{[K_0(t)]^4\} = \frac{4!}{(4-k)!}[K_0(t)]^{4-k}[\eth^1K_0(t)]^k, \forall k \in \mathbb{Z}\cap[1,4]$ [8, Lemma 3.1], we can integrate by parts as follows:

$$\begin{aligned}0 &= \int_0^\infty \frac{K_0(\sqrt{u}t)-K_0(t)}{t}L_5\{[K_0(t)]^4\}\,\mathrm{d}\,t\\ &= \int_0^\infty [K_0(\sqrt{u}t)-K_0(t)]\frac{\partial}{\partial t}\mathscr{L}_{5,4}\{[K_0(t)]^4\}\,\mathrm{d}\,t\\ &\quad -4\int_0^\infty t[K_0(\sqrt{u}t)-K_0(t)]\mathscr{L}_{5,3}\{[K_0(t)]^4\}\,\mathrm{d}\,t\\ &= 24\log\sqrt{u} - \int_0^\infty \mathscr{L}_{5,4}\{[K_0(t)]^4\}\frac{\partial[K_0(\sqrt{u}t)-K_0(t)]}{\partial t}\,\mathrm{d}\,t\\ &\quad -4\int_0^\infty t[K_0(\sqrt{u}t)-K_0(t)]\mathscr{L}_{5,3}\{[K_0(t)]^4\}\,\mathrm{d}\,t\\ &= 12\log u + \int_0^\infty [K_0(t)]^4 L_5^*\frac{K_0(\sqrt{u}t)-K_0(t)}{t}\,\mathrm{d}\,t.\end{aligned} \tag{5.24}$$

Here, in the first step of integration by parts, the boundary contribution arises from the asymptotic behavior

$$K_0(\sqrt{u}t)-K_0(t) = -\log\sqrt{u}+O(t^2\log t), \quad t\to0^+;$$

all the subsequent transfers of derivatives involve no boundary terms at all. Recalling (5.18) and (5.20), we see that (5.24) brings us (5.16). □

Remark As we specialize the relation

$$D^1\int_0^\infty [K_0(t)]^4 t\widetilde{L}_3 K_0(\sqrt{u}t)\,\mathrm{d}\,t=\frac{3}{2}D^1\log u \tag{5.25}$$

to $u=1$, we obtain

$$\mathbf{IKM}(0,5;5)=\frac{76}{15}\,\mathbf{IKM}(0,5;3)-\frac{16}{45}\,\mathbf{IKM}(0,5;1)+\frac{8}{15}, \tag{5.26}$$

a relation that was previously conjectured in [5, (120)].

We will be interested in a 3×3 determinant

$$\check{\Omega}_3(u):=W[\mathbf{I\widetilde{K}M}(0,5;1|u),\widetilde{\mathbf{I}}\mathbf{KM}(2,3;1|u),\mathbf{I\widetilde{K}M}(2,3;1|u)], \tag{5.27}$$

which is a "vacuum analog" of another Wrońskian studied in §2:

$$\Omega_3(u):=W\left[\frac{\widetilde{\mathbf{I}}\mathbf{KM}(1,4;1|u)+4\,\mathbf{I\widetilde{K}M}(1,4;1|u)}{5},\right.\\ \left.\widetilde{\mathbf{I}}\mathbf{KM}(2,3;1|u),\mathbf{I\widetilde{K}M}(2,3;1|u)\right]. \tag{5.28}$$

Lemma 5.4 (Differential equation for $\check{\Omega}_3(u)$). *For $u\in(0,4)$, we have*

$$D^1\check{\Omega}_3(u)=\frac{3\check{\Omega}_3(u)}{2}D^1\log\frac{1}{u^2(4-u)(16-u)}\\ +\frac{3}{2}\frac{\log u}{u^2(4-u)(16-u)}\det\begin{pmatrix}D^0\mu^1_{2,2}(u) & D^0\mu^1_{2,3}(u)\\ D^1\mu^1_{2,2}(u) & D^1\mu^1_{2,3}(u)\end{pmatrix}, \tag{5.29}$$

where $\mu^1_{2,2}(u)=\widetilde{\mathbf{I}}\mathbf{KM}(2,3;1|u)$ and $\mu^1_{2,3}(u)=\mathbf{I\widetilde{K}M}(2,3;1|u)$.

Proof. Differentiating each row of the Wrońskian determinant $\check{\Omega}_3(u)$, we obtain

$$D^1\check{\Omega}_3(u)=\det\begin{pmatrix}D^0\,\mathbf{I\widetilde{K}M}(0,5;1|u) & D^0\mu^1_{2,2}(u) & D^0\mu^1_{2,3}(u)\\ D^1\,\mathbf{I\widetilde{K}M}(0,5;1|u) & D^1\mu^1_{2,2}(u) & D^1\mu^1_{2,3}(u)\\ D^3\,\mathbf{I\widetilde{K}M}(0,5;1|u) & D^3\mu^1_{2,2}(u) & D^3\mu^1_{2,3}(u)\end{pmatrix}. \tag{5.30}$$

Using the differential equations in (5.16) to reduce the third-order derivatives to linear combinations of lower-order derivatives, we may convert the equation above into

$$D^1\check{\Omega}_3(u)=\frac{3\check{\Omega}_3(u)}{2}D^1\log\frac{1}{u^2(4-u)(16-u)}+\det\begin{pmatrix}D^0\,\widetilde{\mathbf{IKM}}(0,5;1|u) & D^0\mu^1_{2,2}(u) & D^0\mu^1_{2,3}(u)\\ D^1\,\widetilde{\mathbf{IKM}}(0,5;1|u) & D^1\mu^1_{2,2}(u) & D^1\mu^1_{2,3}(u)\\ \frac{3\log u}{2u^2(4-u)(16-u)} & 0 & 0\end{pmatrix}, \tag{5.31}$$

which is equivalent to the claimed identity. □

Proposition 5.5 (An integral representation for $\check{\Omega}_3(u)$). *The 2×2 determinant appearing in (5.29) has an integral representation for $u\in(0,4)$:*

$$\det\begin{pmatrix}D^0\mu^1_{2,2}(u) & D^0\mu^1_{2,3}(u)\\ D^1\mu^1_{2,2}(u) & D^1\mu^1_{2,3}(u)\end{pmatrix}=-\frac{\pi^4}{24}\frac{1}{\sqrt{u^2(4-u)(16-u)}}\int_0^\infty J_0(\sqrt{u}t)[J_0(t)]^4t\,\mathrm{d}\,t. \tag{5.32}$$

As a result, there exists a constant $\check{C}_3\in\mathbb{R}$ such that

$$[u^2(4-u)(16-u)]^{3/2}\check{\Omega}_3(u)=\check{C}_3-\frac{\pi^4\sqrt{u}\log u}{8}\int_0^\infty J_1(\sqrt{u}t)[J_0(t)]^4\,\mathrm{d}\,t+\frac{\pi^4}{4}\int_0^\infty\frac{1-J_0(\sqrt{u}t)}{t}[J_0(t)]^4\,\mathrm{d}\,t \tag{5.33}$$

for $u\in(0,4)$.

Proof. By direct computation, one can show that

$$\widetilde{L}_3\left[\sqrt{u^2(4-u)(16-u)}\det\begin{pmatrix}D^0f_1(u) & D^0f_2(u)\\ D^1f_1(u) & D^1f_2(u)\end{pmatrix}\right]=0 \tag{5.34}$$

holds for any two functions $f_1,f_2\in\ker\widetilde{L}_3$ that are annihilated by $\widetilde{L}_3$. Therefore, for $u\in(0,4)$,

$$\Psi_2(u):=\sqrt{u^2(4-u)(16-u)}\det\begin{pmatrix}D^0\mu^1_{2,2}(u) & D^0\mu^1_{2,3}(u)\\ D^1\mu^1_{2,2}(u) & D^1\mu^1_{2,3}(u)\end{pmatrix} \tag{5.35}$$

is a linear combination of

$$\frac{\widetilde{\mathbf{I}}\mathbf{KM}(1,4;1|u)+4\,\mathbf{I}\widetilde{\mathbf{K}}\mathbf{M}(1,4;1|u)}{5},\qquad(5.36)$$
$$\widetilde{\mathbf{I}}\mathbf{KM}(2,3;1|u),\quad\text{and}\quad\mathbf{I}\widetilde{\mathbf{K}}\mathbf{M}(2,3;1|u),$$

in view of §2. However, we can infer from [36, Propositions 3.1.2 and 5.1.4] that

$$\begin{aligned}(5.37)\qquad&\widetilde{\mathbf{I}}\mathbf{KM}(1,4;1|u)+4\,\mathbf{I}\widetilde{\mathbf{K}}\mathbf{M}(1,4;1|u)\\&=\frac{\pi^4}{6}\frac{p_4(\sqrt{u})}{\sqrt{u}}:=\frac{\pi^4}{6}\int_0^\infty J_0(\sqrt{u}t)[J_0(t)]^4t\,\mathrm{d}\,t\end{aligned}$$

holds for $u\in(0,4)$, so we have

$$\begin{aligned}(5.38)\qquad\Psi_2(u)=c_1\frac{p_4(\sqrt{u})}{\sqrt{u}}&+c_2\,\widetilde{\mathbf{I}}\mathbf{KM}(2,3;1|u)\\&+c_3\,\mathbf{I}\widetilde{\mathbf{K}}\mathbf{M}(2,3;1|u),\quad\forall u\in(0,4),\end{aligned}$$

where the constants c_1,c_2,c_3 will be determined from the asymptotic behavior of $\Psi_2(u)$ in the $u\to0^+$ limit and the special value $\Psi_2(1)$.

We note that in the decomposition

$$\begin{aligned}(5.39)\quad\mathbf{I}\widetilde{\mathbf{K}}\mathbf{M}(2,3;1|u)=&\frac{1}{2}\int_0^\infty K_0(\sqrt{u}t)I_0(t)K_0(t)\,\mathrm{d}\,t\\&+\int_0^\infty K_0(\sqrt{u}t)I_0(t)K_0(t)\left[I_0(t)K_0(t)-\frac{1}{2t}\right]t\,\mathrm{d}\,t,\end{aligned}$$

we have [cf. 6, (3.3)]

$$\begin{aligned}(5.40)\qquad&\int_0^\infty K_0(\sqrt{u}t)I_0(t)K_0(t)\,\mathrm{d}\,t\\&=\frac{\mathbf{K}\left(\sqrt{\frac{1}{2}\left(1+i\sqrt{\frac{4-u}{u}}\right)}\right)\mathbf{K}\left(\sqrt{\frac{1}{2}\left(1-i\sqrt{\frac{4-u}{u}}\right)}\right)}{\sqrt{u}}\\&=\frac{1}{2\sqrt{4-u}}\log^2\sqrt{\frac{4-u}{u}}+O\left(\log\frac{4-u}{u}\right),\quad\text{as }u\to0^+,\end{aligned}$$

with $\mathbf{K}(\sqrt{\lambda}) = \int_0^{\pi/2}(1-\lambda\sin^2\theta)^{-1/2}\,\mathrm{d}\,\theta$, and

$$\begin{aligned}(5.41)\qquad &\int_0^\infty K_0(\sqrt{u}t)I_0(t)K_0(t)\left[I_0(t)K_0(t)-\frac{1}{2t}\right]t\,\mathrm{d}\,t\\ &= O\left(\int_0^\infty K_0(\sqrt{u}t)I_0(t)K_0(t)\frac{\mathrm{d}\,t}{\sqrt{t}}\right)\\ &= O\left(\int_0^\infty [1+|\log(\sqrt{u}t)|]I_0(t)K_0(t)\frac{\mathrm{d}\,t}{\sqrt{t}}\right)\\ &= O(\log u)\end{aligned}$$

according to

$$(5.42)\qquad \sup_{t>0} t^{3/2}\left|I_0(t)K_0(t)-\frac{1}{2t}\right|<\infty \quad\text{and}\quad \sup_{t>0}\frac{K_0(t)}{1+|\log t|}<\infty.$$

Thus, we have

$$(5.43)\qquad \widetilde{\mathbf{IKM}}(2,3;1|u) = \frac{\log^2 u}{32}+O(\log u),\quad \text{as } u\to 0^+,$$

and similarly,

$$(5.44)\qquad uD^1\,\widetilde{\mathbf{IKM}}(2,3;1|u) = \frac{\log u}{16}+O(1),\quad \text{as } u\to 0^+.$$

Therefore, we obtain

$$\begin{aligned}(5.45)\qquad \varPsi_2(u) &= 8[D^0\,\widetilde{\mathbf{IKM}}(2,3;1|u)][uD^1\,\widetilde{\mathbf{IKM}}(2,3;1|u)]+O(1)\\ &= 8\,\mathbf{IKM}(1,3;1)\frac{\log u}{16}+O(1) = \frac{\pi^2}{32}\log u+O(1)\end{aligned}$$

in the regime $u\to 0^+$. Meanwhile, we recall that $\frac{p_4(\sqrt{u})}{\sqrt{u}} = -\frac{3\log u}{4\pi^2}+O(1)$ [10, Theorem 4.4] and $\widetilde{\mathbf{IKM}}(2,3;1|u)=O(1)$ as $u\to 0^+$, so we must have

$$(5.46)\qquad \varPsi_2(u) = -\frac{\pi^4}{24}\frac{p_4(\sqrt{u})}{\sqrt{u}}+c_2\,\widetilde{\mathbf{IKM}}(2,3;1|u),\quad u\in(0,4),$$

for a certain constant c_2.

Bearing in mind that

$$D^1\,\mathbf{I}\widetilde{\mathbf{K}}\mathbf{M}(2,3;1|1) - D^1\,\widetilde{\mathbf{I}}\mathbf{KM}(2,3;1|1) = -\frac{1}{2}\int_0^\infty [I_0(t)K_1(t)+I_1(t)K_0(t)]I_0(t)[K_0(t)]^2t^2\,\mathrm{d}\,t = -\frac{1}{2}\int_0^\infty I_0(t)[K_0(t)]^2t\,\mathrm{d}\,t = -\frac{\pi}{6\sqrt{3}}, \tag{5.47}$$

we compute

$$\varPsi_2(1) = 3\sqrt{5}\det\begin{pmatrix}\mathbf{IKM}(2,3;1) & 0\\ D^1\,\widetilde{\mathbf{I}}\mathbf{KM}(2,3;1|1) & -\frac{\pi}{6\sqrt{3}}\end{pmatrix} = -\frac{\pi\sqrt{5}}{2\sqrt{3}}\,\mathbf{IKM}(2,3;1) = -\frac{\pi^4}{24}p_4(1), \tag{5.48}$$

where the last equality can be inferred from [36, Proposition 3.1.2]. Therefore, we have $c_2=0$ in (5.46), which allows us to confirm (5.32).

A solution to (5.29), namely

$$D^1\check{\varOmega}_3(u) = \frac{3\check{\varOmega}_3(u)}{2}D^1\log\frac{1}{u^2(4-u)(16-u)} - \frac{\pi^4}{16}\frac{\log u}{[u^2(4-u)(16-u)]^{3/2}}\int_0^\infty J_0(\sqrt{u}t)[J_0(t)]^4t\,\mathrm{d}\,t, \tag{5.29$'$}$$

has the form

$$\check{\varOmega}_3(u) = \frac{1}{[u^2(4-u)(16-u)]^{3/2}} \times\left(\check{C}_3 - \frac{\pi^4}{16}\int_0^u\left\{\int_0^\infty J_0(\sqrt{v}t)[J_0(t)]^4t\,\mathrm{d}\,t\right\}\log v\,\mathrm{d}\,v\right), \tag{5.49}$$

where the constant of integration $\check{C}_3$ is equal to $2^9\lim_{u\to0^+}u^3\check{\varOmega}_3(u)$.

Here, noting that

$$J_0(\sqrt{v}t) = \frac{\partial}{\partial v}\frac{2\sqrt{v}J_1(\sqrt{v}t)}{t},\qquad \frac{tJ_1(\sqrt{v}t)}{2\sqrt{v}} = -\frac{\partial J_0(\sqrt{v}t)}{\partial v} \tag{5.50}$$

we may integrate by parts, as follows:

$$\begin{aligned}\tag{5.51} &\int_0^u\left\{\int_0^\infty J_0(\sqrt{v}t)[J_0(t)]^4t\,\mathrm{d}\,t\right\}\log v\,\mathrm{d}\,v\\ ={}&(2\sqrt{u}\log u)\int_0^\infty J_1(\sqrt{u}t)[J_0(t)]^4\,\mathrm{d}\,t\\ &-\int_0^u\left\{\int_0^\infty\frac{2J_1(\sqrt{v}t)}{\sqrt{v}}[J_0(t)]^4\,\mathrm{d}\,t\right\}\mathrm{d}\,v\\ ={}&(2\sqrt{u}\log u)\int_0^\infty J_1(\sqrt{u}t)[J_0(t)]^4\,\mathrm{d}\,t\\ &-4\int_0^\infty\frac{1-J_0(\sqrt{u}t)}{t}[J_0(t)]^4\,\mathrm{d}\,t.\end{aligned}$$

This completes the proof of (5.33). □

To facilitate computations of the Wrońskian matrix $\check{\Omega}_3(u)$, we recall the notations $\acute{\mathbf{I}}\mathbf{KM}$ and $\mathbf{I}\acute{\mathbf{K}}\mathbf{M}$ from Definition 2.3, before writing down the following analog of (2.20):

$$\begin{aligned}\tag{5.52} 2^3u^{3/2}\check{\Omega}_3(u)={}&\det\begin{pmatrix}\mathbf{I}\widetilde{\mathbf{K}}\mathbf{M}(0,5;1|u) & \widetilde{\mathbf{I}}\mathbf{KM}(2,3;1|u) & \mathbf{I}\widetilde{\mathbf{K}}\mathbf{M}(2,3;1|u)\\ \mathbf{I}\acute{\mathbf{K}}\mathbf{M}(0,5;1|u) & \acute{\mathbf{I}}\mathbf{KM}(2,3;1|u) & \mathbf{I}\acute{\mathbf{K}}\mathbf{M}(2,3;1|u)\\ \mathbf{I}\widetilde{\mathbf{K}}\mathbf{M}(0,5;3|u) & \widetilde{\mathbf{I}}\mathbf{KM}(2,3;3|u) & \mathbf{I}\widetilde{\mathbf{K}}\mathbf{M}(2,3;3|u)\end{pmatrix}\\ ={}&\det\begin{pmatrix}\mathbf{I}\widetilde{\mathbf{K}}\mathbf{M}(0,5;1|u) & \mu^1_{2,2}(u) & \mu^1_{2,3}(u)\\ \mathbf{I}\acute{\mathbf{K}}\mathbf{M}(0,5;1|u) & \acute{\mu}^1_{2,2}(u) & \acute{\mu}^1_{2,3}(u)\\ \mathbf{I}\widetilde{\mathbf{K}}\mathbf{M}(0,5;3|u) & \mu^2_{2,2}(u) & \mu^2_{2,3}(u)\end{pmatrix}.\end{aligned}$$

In the next proposition, we factorize the last determinant in the $u\to0^+$ regime.

Proposition 5.6 (Factorization of $\check{\Omega}_3(0^+)$). *We have*

$$\check{C}_3=2^9\lim_{u\to0^+}u^3\check{\Omega}_3(u)=\pi^2V_4.\tag{5.53}$$

Consequently, we have

$$135\sqrt{5}\check{\Omega}_3(1)=\pi^2V_4+\frac{\pi^4}{4}\int_0^\infty\frac{1-J_0(t)}{t}[J_0(t)]^4\,\mathrm{d}\,t.\tag{5.54}$$

Proof. Using methods in Proposition 2.5, we can show that

(5.55)

$$
\begin{aligned}
&2^3u^3\check{\Omega}_3(u)\\
&=\det\begin{pmatrix} \mathbf{I\widetilde{K}M}(0,5;1|u) & \mathbf{\widetilde{I}KM}(2,3;1|u) & \mathbf{I\widetilde{K}M}(2,3;1|u)\\ \sqrt{u}\,\mathbf{I\acute{K}M}(0,5;1|u) & \sqrt{u}\,\mathbf{\acute{I}KM}(2,3;1|u) & \sqrt{u}\,\mathbf{I\acute{K}M}(2,3;1|u)\\ u\,\mathbf{I\widetilde{K}M}(0,5;3|u) & u\,\mathbf{\widetilde{I}KM}(2,3;3|u) & u\,\mathbf{I\widetilde{K}M}(2,3;3|u)\end{pmatrix}\\
&=\det\begin{pmatrix} O(\log u) & \mathbf{IKM}(1,3;1)+O(u) & O(1/\sqrt{u})\\ -V_4+O(\sqrt{u}\log u) & O(u) & O(1/\sqrt{u})\\ O(u\log u) & O(u) & \frac14+O(\sqrt{u})\end{pmatrix}\\
&=\frac{\pi^2V_4}{2^6}+o(1),\quad \text{as } u\to0^+,
\end{aligned}
$$

thereby proving our claims. □

Proposition 5.7 (Factorization of $\check{\Omega}_3(1)$). *We have the following factorization*

$$\check{\Omega}_3(1)=\frac{\mathbf{IKM}(1,2;1)}{2^3}\det\check{\mathbf{M}}_2 \tag{5.56}$$

where

$$
\begin{aligned}
\det\check{\mathbf{M}}_2&:=\det\begin{pmatrix}\mathbf{IKM}(0,5;1) & \mathbf{IKM}(0,5;3)\\ \mathbf{IKM}(2,3;1) & \mathbf{IKM}(2,3;3)\end{pmatrix}\\
&=\frac{2\pi^3}{15\sqrt{15}}m(1+x_1+x_2+x_3+x_4).
\end{aligned} \tag{5.57}
$$

Proof. Setting $u=1$ in (5.52), and referring back to (5.47), we may equate $2^3\check{\Omega}_3(1)$ with

$$
\begin{aligned}
&\det\begin{pmatrix}\mathbf{IKM}(0,5;1) & \mathbf{IKM}(2,3;1) & \mathbf{IKM}(2,3;1)\\ \mathbf{I\acute{K}M}(0,5;1) & \mathbf{\acute{I}KM}(2,3;1) & \mathbf{I\acute{K}M}(2,3;1)\\ \mathbf{IKM}(0,5;3) & \mathbf{IKM}(2,3;3) & \mathbf{IKM}(2,3;3)\end{pmatrix}\\
&=\det\begin{pmatrix}\mathbf{IKM}(0,5;1) & \mathbf{IKM}(2,3;1) & 0\\ \mathbf{I\acute{K}M}(0,5;1) & \mathbf{\acute{I}KM}(2,3;1) & -\mathbf{IKM}(1,2;1)\\ \mathbf{IKM}(0,5;3) & \mathbf{IKM}(2,3;3) & 0\end{pmatrix}\\
&=\mathbf{IKM}(1,2;1)\det\check{\mathbf{M}}_2=\frac{\pi\det\check{\mathbf{M}}_2}{3\sqrt{3}}.
\end{aligned} \tag{5.58}
$$

Substituting into the integral representation for $\check{\Omega}_3(1)$ in (5.54), we see that

$$\frac{45\sqrt{5}\pi\det\check{\mathbf{M}}_2}{2^3\sqrt{3}}=\pi^2V_4+\frac{\pi^4}{4}\int_0^\infty\frac{1-J_0(t)}{t}[J_0(t)]^4\,\mathrm{d}\,t. \tag{5.59}$$

Meanwhile, integrating by parts, we find

$$\begin{aligned}&\int_0^\infty\frac{1-J_0(t)}{t}[J_0(t)]^4\,\mathrm{d}\,t\\&=4\int_0^\infty J_1(t)[J_0(t)]^3\log t\,\mathrm{d}\,t-5\int_0^\infty J_1(t)[J_0(t)]^4\log t\,\mathrm{d}\,t\\&=m(1+x_1+x_2+x_3+x_4)-m(1+x_1+x_2+x_3),\end{aligned} \tag{5.60}$$

as a result of Broadhurst's integral representation for Mahler measures, given in (5.13). Combining the last two equations while recalling $m(1+x_1+x_2+x_3)=\frac{4V_4}{\pi^2}$ from (5.10), we achieve our goal. □

5.3. Relation between $\det\check{\mathbf{N}}_2$ and $m(1+x_1+x_2+x_3+x_4+x_5)$

As a variation on the Wrońskian determinant

$$\begin{aligned}\omega_4(u)=W\Bigg[&\frac{\widetilde{\mathbf{I}}\mathbf{KM}(1,5;1|u)+5\,\mathbf{I}\widetilde{\mathbf{K}}\mathbf{M}(1,5;1|u)}{6},\widetilde{\mathbf{I}}\mathbf{KM}(2,4;1|u),\\&\widetilde{\mathbf{I}}\mathbf{KM}(3,3;1|u),\mathbf{I}\widetilde{\mathbf{K}}\mathbf{M}(2,4;1|u)\Bigg]\end{aligned} \tag{5.61}$$

treated in §3, we consider its "vacuum analog"

$$\begin{aligned}\breve{\omega}_4(u)=W[&\mathbf{I}\widetilde{\mathbf{K}}\mathbf{M}(0,6;1|u),\widetilde{\mathbf{I}}\mathbf{KM}(2,4;1|u),\\&\widetilde{\mathbf{I}}\mathbf{KM}(3,3;1|u),\mathbf{I}\widetilde{\mathbf{K}}\mathbf{M}(2,4;1|u)].\end{aligned} \tag{5.62}$$

Lemma 5.8 (Differential equation for $\breve{\omega}_4(u)$). *For $u\in(0,1)$, we have*

$$\begin{aligned}D^1\omega_4(u)=&\,2\omega_4(u)D^1\log\frac{1}{u^2(1-u)(9-u)(25-u)}\\&+\frac{15\log u}{4u^2(1-u)(9-u)(25-u)}\\&\times\det\begin{pmatrix}D^0\nu_{2,2}^1(u)&D^0\nu_{2,3}^1(u)&D^0\nu_{2,4}^1(u)\\D^1\nu_{2,2}^1(u)&D^1\nu_{2,3}^1(u)&D^1\nu_{2,4}^1(u)\\D^2\nu_{2,2}^1(u)&D^2\nu_{2,3}^1(u)&D^2\nu_{2,4}^1(u)\end{pmatrix},\end{aligned} \tag{5.63}$$

where $\nu^1_{2,2}(u) = \widetilde{\mathbf{I}}\mathbf{KM}(2,4;1|u)$, $\nu^1_{2,3}(u) = \widetilde{\mathbf{I}}\mathbf{KM}(3,3;1|u)$ *and* $\nu^1_{2,4}(u) = \mathbf{I}\widetilde{\mathbf{K}}\mathbf{M}(2,4;1|u)$.

Proof. With the fourth-order Vanhove operator $\widetilde{L}_4$ defined in (3.4), we can establish (using methods similar to those in Lemma 5.3) the following differential equations:

$$\begin{cases} \widetilde{L}_4\,\mathbf{I}\widetilde{\mathbf{K}}\mathbf{M}(0,6;1|u) = \frac{15}{4}\log u, & \forall u\in(0,\infty);\\ \widetilde{L}_4\,\widetilde{\mathbf{I}}\mathbf{KM}(2,4;1|u) = 0, & \forall u\in(0,9);\\ \widetilde{L}_4\,\widetilde{\mathbf{I}}\mathbf{KM}(3,3;1|u) = 0, & \forall u\in(0,1);\\ \widetilde{L}_4\,\mathbf{I}\widetilde{\mathbf{K}}\mathbf{M}(2,4;1|u) = 0, & \forall u\in(0,\infty). \end{cases} \tag{5.64}$$

One can subsequently differentiate $\check{\omega}_4(u)$, with manipulations similar to those intended for $\check{\Omega}_3(u)$. □

Proposition 5.9. *For* $u\in(0,1)$, *we have*

$$\det\begin{pmatrix} D^0\nu^1_{2,2}(u) & D^0\nu^1_{2,3}(u) & D^0\nu^1_{2,4}(u)\\ D^1\nu^1_{2,2}(u) & D^1\nu^1_{2,3}(u) & D^1\nu^1_{2,4}(u)\\ D^2\nu^1_{2,2}(u) & D^2\nu^1_{2,3}(u) & D^2\nu^1_{2,4}(u) \end{pmatrix} = \frac{\pi^6}{80u^2(1-u)(9-u)(25-u)}\int_0^\infty J_0(\sqrt{u}t)[J_0(t)]^5 t\,\mathrm{d}\,t. \tag{5.65}$$

Consequently, there exists a constant $\check{c}_4\in\mathbb{R}$ *such that*

$$\begin{aligned}[] [u^2(1-u)(9-u)(25-u)]^2\check{\omega}_4(u) &= \check{c}_4 + \frac{3\pi^6\sqrt{u}\log u}{32}\int_0^\infty J_1(\sqrt{u}t)[J_0(t)]^5\,\mathrm{d}\,t\\ &\quad - \frac{3\pi^6}{16}\int_0^\infty \frac{1-J_0(\sqrt{u}t)}{t}[J_0(t)]^5\,\mathrm{d}\,t \end{aligned} \tag{5.66}$$

is valid for $u\in(0,1)$.

Proof. First, we point out that

$$\widetilde{L}_4\left[u^2(1-u)(9-u)(25-u)\det\begin{pmatrix} D^0f_1(u) & D^0f_2(u) & D^0f_3(u)\\ D^1f_1(u) & D^1f_2(u) & D^1f_3(u)\\ D^2f_1(u) & D^2f_2(u) & D^2f_3(u) \end{pmatrix}\right] = 0 \tag{5.67}$$

is true for any three functions $f_1, f_2, f_3 \in \ker \widetilde{L}_4$ residing in the null space of $\widetilde{L}_4$. So we may assert that there are constants C_1, C_2, C_3, C_4 satisfying

$$\begin{aligned}(5.68)\quad \psi_3(u) :=&\, u^2(1-u)(9-u)(25-u)\\ &\times \det\begin{pmatrix} D^0\nu^1_{2,2}(u) & D^0\nu^1_{2,3}(u) & D^0\nu^1_{2,4}(u)\\ D^1\nu^1_{2,2}(u) & D^1\nu^1_{2,3}(u) & D^1\nu^1_{2,4}(u)\\ D^2\nu^1_{2,2}(u) & D^2\nu^1_{2,3}(u) & D^2\nu^1_{2,4}(u)\end{pmatrix}\\ =&\, C_1\frac{\widetilde{\mathbf{I}}\mathbf{KM}(1,5;1|u)+5\,\mathbf{I}\widetilde{\mathbf{K}}\mathbf{M}(1,5;1|u)}{6}+C_2\,\widetilde{\mathbf{I}}\mathbf{KM}(2,4;1|u)\\ &+C_3\,\widetilde{\mathbf{I}}\mathbf{KM}(3,3;1|u)+C_4\,\mathbf{I}\widetilde{\mathbf{K}}\mathbf{M}(2,4;1|u),\end{aligned}$$

for $u \in (0,1)$.

Next, we point out that the following limits

$$(5.69)\qquad \begin{aligned}\lim_{u\to0^+}\psi_3(u) &= \frac{3\pi^2}{8}\,\mathbf{IKM}(1,4;1)\\ \lim_{u\to0^+}D^1\psi_3(u) &= \frac{3\pi^2}{32}\,\mathbf{IKM}(1,4;3).\end{aligned}$$

allow us to determine

$$(5.70)\qquad C_1 = 0,\quad C_2 = \frac{3\pi^2}{8},\quad C_3 = 0,\quad C_4 = 0.$$

Here, before evaluating $\lim_{u\to0^+}\psi_3(u)$, we put down

$$\begin{aligned}(5.71)\qquad &2^3u^2\det\begin{pmatrix} D^0\nu^1_{2,2}(u) & D^0\nu^1_{2,3}(u) & D^0\nu^1_{2,4}(u)\\ D^1\nu^1_{2,2}(u) & D^1\nu^1_{2,3}(u) & D^1\nu^1_{2,4}(u)\\ D^2\nu^1_{2,2}(u) & D^2\nu^1_{2,3}(u) & D^2\nu^1_{2,4}(u)\end{pmatrix}\\ &=\det\begin{pmatrix} \nu^1_{2,2}(u) & \nu^1_{2,3}(u) & \nu^1_{2,4}(u)\\ \sqrt{u}\acute{\nu}^1_{2,2}(u) & \sqrt{u}\acute{\nu}^1_{2,3}(u) & \sqrt{u}\acute{\nu}^1_{2,4}(u)\\ \nu^2_{2,2}(u) & \nu^2_{2,3}(u) & \nu^2_{2,4}(u)\end{pmatrix},\end{aligned}$$

where the last determinant is asymptotic to (cf. Propositions 3.3 and 2.5)

$$\det\begin{pmatrix}\mu^1_{2,1}+O(u) & \mu^1_{2,2}+O(u) & O(\log u)\\ O(u) & O(u) & -\mu^1_{2,2}+o(1)\\ \mu^2_{2,1}+O(u) & \mu^2_{2,2}+O(u) & O(\log u)\end{pmatrix} \tag{5.72}$$
$$=\mathbf{IKM}(2,3;1)\det\begin{pmatrix}\mu^1_{2,1} & \mu^1_{2,2}\\ \mu^2_{2,1} & \mu^2_{2,2}\end{pmatrix}+o(1)$$
$$=\frac{2\pi^3\,\mathbf{IKM}(2,3;1)}{\sqrt{3^3 5^5}}+o(1)$$

in the $u\to0^+$ limit. Here, we recall from [36, Theorem 2.2.2 and Proposition 3.1.2] that

$$\mathbf{IKM}(2,3;1)=\frac{\sqrt{15}}{2\pi}\,\mathbf{IKM}(1,4;1), \tag{5.73}$$

so the evaluation of $\lim_{u\to0^+}\psi_3(u)$ in (5.69) is now confirmed.

To compute $\lim_{u\to0^+}D^1\psi_3(u)$, we need the observations that

$$D^1[u^2D^2I_0(\sqrt{u}t)]=\frac{t^3\sqrt{u}}{8}I_1(\sqrt{u}t),$$
$$D^1[u^2D^2K_0(\sqrt{u}t)]=-\frac{t^3\sqrt{u}}{8}K_1(\sqrt{u}t),$$

which entail

$$D^1\psi_3(u)=\frac{(-3u^2+70u-259)\psi_3(u)}{(1-u)(9-u)(25-u)}+\frac{(1-u)(9-u)(25-u)}{16} \tag{5.74}$$
$$\times\det\begin{pmatrix}\nu^1_{2,2}(u) & \nu^1_{2,3}(u) & \nu^1_{2,4}(u)\\ \acute{\nu}^1_{2,2}(u) & \acute{\nu}^1_{2,3}(u) & \acute{\nu}^1_{2,4}(u)\\ \acute{\nu}^2_{2,2}(u) & \acute{\nu}^2_{2,3}(u) & \acute{\nu}^2_{2,4}(u)\end{pmatrix}.$$

Here, as $u\to0^+$, the last determinant is asymptotic to

$$\det\begin{pmatrix}\mu^1_{2,1}+O(u) & \mu^1_{2,2}+O(u) & O(\log u)\\ \frac{\sqrt{u}}{2}[\mu^2_{2,1}+O(u)] & \frac{\sqrt{u}}{2}[\mu^2_{2,2}+O(u)] & -\frac{\mu^1_{2,2}+o(1)}{\sqrt{u}}\\ \frac{\sqrt{u}}{2}[\mu^3_{2,1}+O(u)] & \frac{\sqrt{u}}{2}[\mu^3_{2,2}+O(u)] & -\frac{\mu^2_{2,2}+o(1)}{\sqrt{u}}\end{pmatrix}. \tag{5.75}$$

We recall the following closed-form formulae (conjectured in [5, (95)–(100)], proved in [36, §3])

(5.76)
$$\begin{cases}\frac{\mu^1_{2,1}}{\pi^2}=C, & \frac{\mu^2_{2,1}}{\pi^2}=\left(\frac{2}{15}\right)^2\left(13C-\frac{1}{10C}\right), & \frac{\mu^3_{2,1}}{\pi^2}=\left(\frac{4}{15}\right)^3\left(43C-\frac{19}{40C}\right)\\ \frac{2\mu^1_{2,2}}{\sqrt{15}\pi}=C, & \frac{2\mu^2_{2,2}}{\sqrt{15}\pi}=\left(\frac{2}{15}\right)^2\left(13C+\frac{1}{10C}\right), & \frac{2\mu^3_{2,2}}{\sqrt{15}\pi}=\left(\frac{4}{15}\right)^3\left(43C+\frac{19}{40C}\right)\end{cases}$$

where $C=\frac{1}{240\sqrt{5}\pi^2}\Gamma\left(\frac{1}{15}\right)\Gamma\left(\frac{2}{15}\right)\Gamma\left(\frac{4}{15}\right)\Gamma\left(\frac{8}{15}\right)$ is the "Bologna constant" attributed to Broadhurst [5, 13] and Laporta [24]. It is then clear that

$$\lim_{u\to0^+}D^1\psi_3(u)=-\frac{259\pi^4C}{600}+\frac{\pi^4(2720C^2-1)}{6000C}=\frac{3\pi^2}{32}\mathbf{IKM}(1,4;3),\tag{5.77}$$

as claimed in (5.69).

Now, to guarantee the finiteness of both

$$\lim_{u\to0^+}\psi_3(u)\quad\text{and}\quad\lim_{u\to0^+}D^1\psi_3(u),$$

we must have $C_1=C_4=0$. Fitting

$$\psi_3(u)=C_2\,\widetilde{\mathbf{IKM}}(2,4;1|u)+C_3\,\widetilde{\mathbf{IKM}}(3,3;1|u)$$

to (5.69), we obtain $C_2=\frac{3\pi^2}{8}, C_3=0$.

Last, but not the least, we recall from [37, Lemma 2.1] that

$$\begin{aligned}\frac{p_5(\sqrt{u})}{\sqrt{u}}:={}&\int_0^\infty J_0(\sqrt{u}t)[J_0(t)]^5t\,\mathrm{d}\,t\\={}&\frac{30\,\widetilde{\mathbf{IKM}}(2,4;1|u)}{\pi^4},\quad\forall u\in[0,1],\end{aligned}\tag{5.78}$$

which turns $\psi_3(u)=\frac{3\pi^2}{8}\widetilde{\mathbf{IKM}}(2,4;1|u)$ into $\psi_3(u)=\frac{\pi^6}{80}\frac{p_5(\sqrt{u})}{\sqrt{u}}$, just as stated in (5.65).

Following procedures similar to those in Proposition 5.5, we can deduce (5.66) from (5.65). □

In the next proposition, we study the determinant

(5.79)
$$
\begin{aligned}
&2^6 u^4 \breve{\omega}_4(u)\\
&= \det\begin{pmatrix} \mathbf{I}\widetilde{\mathbf{K}}\mathbf{M}(0,6;1|u) & \nu^1_{2,2}(u) & \nu^1_{2,3}(u) & \nu^1_{2,4}(u)\\ \sqrt{u}\,\mathbf{I}\acute{\mathbf{K}}\mathbf{M}(0,6;1|u) & \sqrt{u}\acute{\nu}^1_{2,2}(u) & \sqrt{u}\acute{\nu}^1_{2,3}(u) & \sqrt{u}\acute{\nu}^1_{2,4}(u)\\ \mathbf{I}\widetilde{\mathbf{K}}\mathbf{M}(0,6;3|u) & \nu^2_{2,2}(u) & \nu^2_{2,3}(u) & \nu^2_{2,4}(u)\\ \sqrt{u}\,\mathbf{I}\acute{\mathbf{K}}\mathbf{M}(0,6;3|u) & \sqrt{u}\acute{\nu}^2_{2,2}(u) & \sqrt{u}\acute{\nu}^2_{2,3}(u) & \sqrt{u}\acute{\nu}^2_{2,4}(u) \end{pmatrix}
\end{aligned}
$$

in the $u\to 0^+$ limit.

Proposition 5.10 (Factorization of $\breve{\omega}_4(0^+)$). *We have*

(5.80)
$$
\begin{aligned}
\breve{c}_4 = 3^4 5^4 \lim_{u\to 0^+} u^4\breve{\omega}_4(u) &= -\frac{45\sqrt{15}\pi^3}{32}\det\check{\mathbf{M}}_2\\
&= -\frac{3\pi^6}{16} m(1+x_1+x_2+x_3+x_4).
\end{aligned}
$$

Consequently, we have

(5.81)
$$
\begin{aligned}
&2^{12}3^2\lim_{u\to 1^-}(1-u)^2\breve{\omega}_4(u)\\
&= -\frac{3\pi^6}{16} m(1+x_1+x_2+x_3+x_4) - \frac{3\pi^6}{16}\int_0^\infty \frac{1-J_0(t)}{t}[J_0(t)]^5\,\mathrm{d}\,t\\
&= -\frac{3\pi^6}{16} m(1+x_1+x_2+x_3+x_4+x_5).
\end{aligned}
$$

Proof. Using methods in Proposition 3.3, we can show that

(5.82)
$$
\begin{aligned}
2^6u^4\breve{\omega}_4(u) &= \det\begin{pmatrix} O(\log u) & \mu^1_{2,1}{+}O(u) & \mu^1_{2,2}{+}O(u) & O(\log u)\\ \sqrt{u}\,\mathbf{I}\acute{\mathbf{K}}\mathbf{M}(0,6;1|u) & O(u) & O(u) & \sqrt{u}\acute{\nu}^1_{2,4}(u)\\ O(\log u) & \mu^2_{2,1}{+}O(u) & \mu^2_{2,2}{+}O(u) & O(\log u)\\ \sqrt{u}\,\mathbf{I}\acute{\mathbf{K}}\mathbf{M}(0,6;3|u) & O(u) & O(u) & \sqrt{u}\acute{\nu}^2_{2,4}(u) \end{pmatrix}\\
&= -\det\begin{pmatrix}\mu^1_{2,1} & \mu^1_{2,2}\\ \mu^2_{2,1} & \mu^2_{2,2}\end{pmatrix}\det\begin{pmatrix}\sqrt{u}\,\mathbf{I}\acute{\mathbf{K}}\mathbf{M}(0,6;1|u) & \sqrt{u}\acute{\nu}^1_{2,4}(u)\\ \sqrt{u}\,\mathbf{I}\acute{\mathbf{K}}\mathbf{M}(0,6;3|u) & \sqrt{u}\acute{\nu}^2_{2,4}(u)\end{pmatrix}\\
&\quad + O(u^2\log^2 u), \quad \text{as } u\to 0^+,
\end{aligned}
$$

and

$$\det\begin{pmatrix}\sqrt{u}\,\mathbf{I\acute{K}M}(0,6;1|u) & \sqrt{u}\acute{\nu}^1_{2,4}(u)\\ \sqrt{u}\,\mathbf{I\acute{K}M}(0,6;3|u) & \sqrt{u}\acute{\nu}^2_{2,4}(u)\end{pmatrix} \tag{5.83}$$
$$=\det\begin{pmatrix}-\mathbf{IKM}(0,5;1)+o(1) & -\mathbf{IKM}(2,3;1)+o(1)\\ -\mathbf{IKM}(0,5;3)+o(1) & -\mathbf{IKM}(2,3;3)+o(1)\end{pmatrix}$$
$$=\det\check{\mathbf{M}}_2+o(1),\quad \text{as } u\to0^+.$$

The rest of our claims then follow from familiar arguments in §5.2. □

To wrap up this section, we reduce $\check\omega_4(u), u\to1^-$ to $\det\check{\mathbf{N}}_2$.

Proposition 5.11 (Factorization of $\check\omega_4(1^-)$). *We have the following factorization*

$$\lim_{u\to1^-}(1-u)^2\check\omega_4(u)=-\frac{\pi^2}{2^{11}}\det\check{\mathbf{N}}_2 \tag{5.84}$$

so that

$$\det\check{\mathbf{N}}_2:=\det\begin{pmatrix}\mathbf{IKM}(0,6;1) & \mathbf{IKM}(0,6;3)\\ \mathbf{IKM}(2,4;1) & \mathbf{IKM}(2,4;3)\end{pmatrix} \tag{5.85}$$
$$=\frac{\pi^4}{96}m(1+x_1+x_2+x_3+x_4+x_5).$$

Proof. Akin to Proposition 3.2, we have

(5.86)

$$2^6u^2(1-u)^2\check\omega_4(u)$$
$$=\det\begin{pmatrix}\mathbf{IKM}(0,6;1)+\circ & \nu^1_{2,2}(1)+\circ & \circ & \nu^1_{2,4}(1)+\circ\\ \sharp & \acute\nu^1_{2,2}(1)+\circ & \circ & \acute\nu^1_{2,4}(1)+\circ\\ \mathbf{IKM}(0,6;3)+\circ & \nu^2_{2,2}(1)+\circ & \circ & \nu^1_{2,4}(1)+\circ\\ \sharp & \sharp & \frac12+\circ & \sharp\end{pmatrix}$$
$$=-\frac12\det\begin{pmatrix}\mathbf{IKM}(0,6;1)+\circ & \mathbf{IKM}(2,4;1)+\circ & \mathbf{IKM}(2,4;1)+\circ\\ \sharp & \mathbf{\acute{I}KM}(2,4;1|1)+\circ & \mathbf{I\acute{K}M}(2,4;1|1)+\circ\\ \mathbf{IKM}(0,6;3)+\circ & \mathbf{IKM}(2,4;3)+\circ & \mathbf{IKM}(2,4;3)+\circ\end{pmatrix}$$
$$+o(1)$$

where a hash (resp. circle) stands for a bounded (resp. infinitesimal) quantity, as u approaches 1 from below. Using the fact that $\mathbf{I\acute{K}M}(2,4;1|1)-$

$\mathbf{\acute{I}KM}(2,4;1|1) = -\,\mathbf{IKM}(1,3;1) = -\frac{\pi^2}{2^4}$, we can compute

$$
\begin{aligned}
(5.87)\qquad & 2^6 \lim_{u\to 1^-} u^2(1-u)^2\check{\omega}_4(u) \\
&= -\frac{1}{2}\det\begin{pmatrix} \mathbf{IKM}(0,6;1) & \mathbf{IKM}(2,4;1) & 0 \\ \mathbf{I\acute{K}M}(0,6;1|1) & \mathbf{\acute{I}KM}(2,4;1|1) & -\frac{\pi^2}{2^4} \\ \mathbf{IKM}(0,6;3) & \mathbf{IKM}(2,4;3) & 0 \end{pmatrix} \\
&= -\frac{\pi^2}{2^5}\det\check{\mathbf{N}}_2,
\end{aligned}
$$

so our conclusion follows immediately. □

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Program in Applied and Computational Mathematics (PACM)
Princeton University, Princeton, NJ 08544, USA
and Academy of Advanced Interdisciplinary Studies (AAIS)
Peking University, Beijing 100871, P. R. China
E-mail address: yajunz@math.princeton.edu, yajun.zhou.1982@pku.edu.cn

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