The Galois coaction on the electron anomalous magnetic moment

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Recently S. Laporta published a partial result on the fourth order QED contribution to the electron anomalous magnetic moment $g - 2$ [17]. This result contains explicit polylogarithmic parts with fourth and sixth roots of unity. In this note we convert Laporta's result into the motivic 'f alphabet'. This provides a much shorter expression which makes the Galois structure visible. We conjecture the $\mathbb{Q}$ vector spaces of Galois conjugates of the QED $g - 2$ up to weight four. The conversion into the f alphabet relies on a conjecture by D. Broadhurst that iterated integrals in certain Lyndon words provide an algebra basis for the extension of multiple zeta values (MZVs) by sixth roots of unity. We prove this conjecture in the motivic setup.

1. Introduction

In [17] S. Laporta published a partial result (and a high precision numerical evaluation) of the fourth order QED contribution to the electron anomalous magnetic moment. The result consists of a polylogarithmic part, an elliptic part, and a yet unknown part. The polylogarithmic part is explicitly given in terms of iterated integrals [11]. The letters in the iterated integrals are from two alphabets with fourth and sixth roots of unity, respectively. We analyze the motivic Galois structure of Laporta's result.

Classical Galois theory associates a finite group to a polynomial $p \in \mathbb{Z}[x]$. The elements of the Galois group act as permutation on the roots of p while maintaining the structure of the field obtained by adding the roots of p to $\mathbb{Q}$. Because roots of polynomials in one variable are points, the classical Galois group acts on zero-dimensional objects. In the second half of the 20th century A. Grothendieck envisioned a generalization of Galois theory to higher dimensional varieties. While the idea was (and still is) developed by many outstanding mathematicians it turned out that dualizing the concept of a group action is both easier to handle and more powerful.

The natural concept of dualizing varieties is the integral. In the algebraic context—which is necessary for a Galois theory to exist—the domain of integration is a variety (in general with boundary) over $\mathbb{Q}$, i.e. it is given by polynomial inequalities with integer coefficients. Likewise the integrand needs to be a rational function with integer coefficients. (One may replace the integers by algebraic numbers in $\mathbb{C}$ without changing the concept.) The variety associated to the differential form is the zero locus of the denominator. In the case that the result is a mere number it is called a 'period' in [16]. Periods form a $\mathbb{Q}$ algebra which is denoted by $\mathbb{P}$. In general, one also wants to allow the polynomials to have variables which are not integrated. In this case the result will be a function on these variables. Here, however, we can restrict ourselves to periods.

Clearly, the integral dualizes these varieties: It is a map from the domain and the differential form into the complex numbers. By dualizing the Galois action becomes a coaction,

$$(1) \qquad\qquad \Delta \colon \mathbb{P} \longrightarrow \mathbb{P}^{\partial \mathfrak{r}} \otimes_{\mathbb{Q}} \mathbb{P}.$$

The dual of the Galois group is the Hopf algebra $\mathbb{P}^{\partial \mathfrak{r}}$ of deRham periods. One may equivalently swap the sides of the tensor product and co-act with $\mathbb{P}^{\partial \mathfrak{r}}$ to the right.

It should be mentioned that strictly speaking the Galois coaction is an object in algebraic geometry. It needs a sophisticated mathematical theory (the theory of 'motives'). One main conjecture in the theory is that it works identically with the mere numbers in $\mathbb{P}$. The conjecture can only fail if there exist some unknown identities in $\mathbb{P}$ which have no algebraic origin. Still, any formula we obtain from using the Galois coaction would be valid in $\mathbb{P}$ because all algebraic relations in the motive are true in $\mathbb{P}$. Here, we do not make the distinction between numbers and their motivic counterparts. One other subtlety is that there exist different versions of the Galois coaction. Here we use the simplest one, the unipotent coaction. In the context of polylogarithms (the 'mixed Tate' case) the unipotent coaction is equivalent to the full coaction.

The concept of classical Galois conjugates is best translated into $\mathbb{Q}$ vector spaces. In our notation with the left coaction we have periods on the right hand side of the tensor product. Without giving a precise definition we consider the $\mathbb{Q}$ vector space of theses periods as Galois conjugates. The simplest cases are

$$\Delta 1 = 1^{\partial \mathfrak{r}} \otimes 1, \quad \text{and} \quad \Delta \pi = 1^{\partial \mathfrak{r}} \otimes \pi.$$

So, the vector space of Galois conjugates of 1 is $\mathbb{Q}$ while $\mathbb{Q}\pi$ is conjugate to π.[1] From

$$\Delta \log(a) = \log(a)^{\partial\mathfrak{r}} \otimes 1 + 1^{\partial\mathfrak{r}} \otimes \log(a), \quad 1 \neq a \in \mathbb{Q}$$

we obtain that a logarithm has a two-dimensional vector space of Galois conjugates: $\mathbb{Q} + \mathbb{Q}\log(a)$. One finds more explicit examples e.g. in references [5, 18] where the term coaction is often used without referring to Galois.

A difficulty handling polylogarithmic numbers is that they obey many relations over $\mathbb{Q}$ (i.e. with rational coefficients). In the case of multiple zeta values (MZVs) it is conjectured that these relations are exhausted by the 'generalized double shuffle relations'. Using these relations is still difficult at high weights because it is impossible to—a priori—give a sequence of equations that leads to the reduction of a specific MZV to a $\mathbb{Q}$ basis (see e.g. [1, 2] for this approach). For other types of polylogarithmic numbers there do not even conjecturally exist complete sets of equations.

Thankfully another deep mathematical input resolves this problem. For many polylogarithmic numbers there exist an (at least) conjectural isomorphism into an algebra where the only relations are shuffles. This shuffle algebra is composed of words in letters of certain weights. For MZVs one has one letter for each odd weight ≥ 3. It is easy to check that words in these letters span weight spaces of the desired (conjectural) dimensions. Another benefit of this so called 'f alphabet' is that the coaction becomes simple deconcatenation,

$$\Delta w = \sum_{w=uv} u \otimes v.$$

This, in particular, makes it trivial to re-construct the original word from a known term in the coaction. Moreover, it is clear that the Galois conjugates of a word w are spanned by its right factors (v if $w = uv$). The general strategy to handle polylogarithmic numbers it to convert them into an f alphabet. The method for this conversion is the decomposition algorithm by F. Brown [5].

The only grain of salt is that the conversion is non-canonical in the sense that it depends on a choice of an algebra basis for the numbers considered. In many cases such a basis is not even conjecturally known. For all (known) polylogarithmic numbers in quantum field theory (QFT), however, we have

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[1] The idea that multiples of π are Galois conjugates of π can be motivated by an analogy to classical Galois theory: Multiples of π are zeros of a power series with rational coefficients: $\prod_{n=1}^{\infty}(1 - (\frac{x}{n\pi})^2) = \frac{\sin(x)}{x} \in \mathbb{Q}[[x]]$. Strictly speaking, zero is not a Galois conjugate of π.

such bases and the conversion is possible. In general, also non polylogarithmic numbers have f alphabets (see e.g. [9]). Note that the vector spaces of Galois conjugates are canonical although the conversion into the f alphabet is not.

In this note we convert the polylogarithmic part of the QED contribution to $g - 2$ into the f alphabet. This conversion simplifies the result. The $\mathbb{Q}$ vector spaces of Galois conjugates happen to be of intriguingly small dimensions.

It is proved for generic kinematics in [7, 8] that the $\mathbb{Q}$ vector spaces of Galois conjugates of fixed weights are finite dimensional in QFT. It is conjectured and mostly proved that at weight n all Galois conjugates come from sub-quotient graphs with at most $2n + 1$ edges. (Note that the Hodge weight used in [7, 8] is twice the transcendental weight for MZVs.) Our data suggest that the space of weight n Galois conjugates is exhausted by the n-loop contribution to the electron $g - 2$. The small dimensions of the $\mathbb{Q}$ vector spaces are further evidence for a coaction structure in QED similar to the one conjectured in ϕ^4 theory [18].

In this note we mostly process a result by S. Laporta. We try to keep this note short to emphasize that the results should be mostly attributed to S. Laporta.

For the extension MZV(4) of MZVs by fourth roots of unity (see Section 2 for details) P. Deligne gave a basis in 2010 [12]. Its restriction to Lyndon words provides an algebra basis of MZV(4). In this note we conveniently use a 'parity' version of Deligne's basis [18]. In its f alphabet one has one letter at each positive integer weight (see Table 2),

$$(2) \qquad f_n^4 \cong \begin{cases} 2\operatorname{Re}\operatorname{Li}_n(\mathrm{i}), & \text{if } n \text{ is odd,} \\ 2\mathrm{i}\operatorname{Im}\operatorname{Li}_n(\mathrm{i}), & \text{if } n \text{ is even.} \end{cases}$$

We use the sign $\cong$ to emphasize that, strictly speaking, the right hand side is not equal to the left hand side. It is obtained by applying the composition of a non-canonical (motivic f alphabet) isomorphism with the conjectural period isomorphism to the left hand side.

In the case of sixth roots of unity P. Deligne only gave a basis in [12] for the subspace of MZV(6) that is constructed by using the three letters $0, 1, \xi_6$, where

$$\xi_N = \exp(2\pi\mathrm{i}/N)$$

is a primitive Nth root of unity. For the conversion of the fourth order contribution to $g - 2$ this is not sufficient. In general, we need the full alphabet

of $N + 1$ letters (for $N = 6$)

$$(3) \qquad\qquad X_N = \{0,\ \xi_N^k, k = 0, \ldots, N - 1\}.$$

In 2015, D. Broadhurst defined a set of iterated integrals with words in X_6 (see Definition 4) and conjectured that this set provides an algebra basis for MZV(6) [3]. For mere numbers it seems impossible to prove Broadhurst's conjecture. However, in the (conjecturally equivalent) motivic setup one can use the Galois coaction to give a (rather simple) proof.

Theorem 1. *The motivic Broadhurst set of iterated integrals in sixth roots of unity is an algebra basis for motivic MZV(6).*

The f alphabet for MZV(6) has two generators for weight one and one generator for every weight greater than one. Here, we again use a parity version of Broadhurst's basis. The letters are

$$(4) \qquad\qquad f_n^6 \cong \begin{cases} -2\mathrm{Re}\,\mathrm{Li}_n(\xi_3), & \text{if } n \text{ is odd} \\ -2\mathrm{iIm}\,\mathrm{Li}_n(\xi_3), & \text{if } n \text{ is even} \end{cases} \quad n \geq 1, \text{ and}$$

$$(5) \qquad\qquad g_1^6 \cong 2\log 2.$$

Note that in Laporta's result the two alphabets MZV(4) and MZV(6) do not mix. This makes it possible to avoid the use of twelfth roots of unity. Also note that (see Prop. 8)

$$(6) \quad g_1^6 \cong -2f_1^4 \quad \text{and} \quad \frac{3^{n-1}}{3^{n-1} - 1} f_n^6 \cong -\frac{4^{n-1}}{2^{n-1} - 1} f_n^4 \quad \text{for odd } n \geq 3.$$

Although these letters are related they must be distinguished because they belong to different alphabets. However, it is the benefit of the parity basis that words in $\{f_1^4, f_3^4, f_5^4, \ldots\}$ and $\{g_1^6, f_3^6, f_5^6, \ldots\}$ can (non-trivially) be converted into each other. We have, e.g., the identity

$$(7) \qquad\qquad g_1^6 f_3^6 \cong \frac{256}{27} f_1^4 f_3^4 - \frac{139}{28350}\pi^4.$$

The result for the QED contribution a_e to the electron $g - 2$ is

$$
\begin{aligned}
(8) \quad a_e \cong{} & \frac{1}{2}\left(\frac{\alpha}{\pi}\right) + \left(\frac{197}{144} + \frac{1}{12}\pi^2 + \frac{27}{32}f_3^6 - \frac{1}{4}g_1^6\pi^2\right)\left(\frac{\alpha}{\pi}\right)^2 \\
& + \left(\frac{28259}{5184} + \frac{17101}{810}\pi^2 + \frac{139}{16}f_3^6 - \frac{149}{9}g_1^6\pi^2 - \frac{525}{32}g_1^6 f_3^6\right. \\
& \quad \left.+ \frac{1969}{8640}\pi^4 - \frac{1161}{128}f_5^6 + \frac{83}{64}f_3^6\pi^2\right)\left(\frac{\alpha}{\pi}\right)^3 \\
& + \left(\frac{1243127611}{130636800} + \frac{30180451}{155520}\pi^2 - \frac{255842141}{2419200}f_3^6 - \frac{8873}{36}g_1^6\pi^2\right. \\
& \quad + \frac{126909}{2560}\frac{f_4^6}{i\sqrt{3}} - \frac{84679}{1280}g_1^6 f_3^6 + \frac{169703}{3840}\frac{f_2^6\pi^2}{i\sqrt{3}} + \frac{779}{108}g_1^6 g_1^6\pi^2 \\
& \quad + \frac{112537679}{3110400}\pi^4 - \frac{2284263}{25600}f_5^6 + \frac{8449}{96}g_1^6 g_1^6 f_3^6 - \frac{12720907}{345600}f_3^6\pi^2 \\
& \quad - \frac{231919}{97200}g_1^6\pi^4 + \frac{150371}{256}\frac{f_6^6}{i\sqrt{3}} + \frac{313131}{1280}g_1^6 f_5^6 - \frac{121383}{1280}f_2^6 f_4^6 \\
& \quad - \frac{14662107}{51200}f_3^6 f_3^6 + \frac{8645}{128}\frac{f_2^6 g_1^6 f_3^6}{i\sqrt{3}} - \frac{231}{4}g_1^6 g_1^6 g_1^6 f_3^6 - \frac{16025}{48}\frac{f_4^6\pi^2}{i\sqrt{3}} \\
& \quad + \frac{4403}{384}g_1^6 f_3^6\pi^2 - \frac{136781}{1920}f_2^6 f_2^6\pi^2 + \frac{7069}{75}f_2^4 f_2^4\pi^2 \\
& \quad - \frac{1061123}{14400}f_3^6 g_1^6\pi^2 + \frac{1115}{72}\frac{f_2^6 g_1^6 g_1^6\pi^2}{i\sqrt{3}} + \frac{781181}{20736}\frac{f_2^6\pi^4}{i\sqrt{3}} \\
& \quad - \frac{4049}{1080}g_1^6 g_1^6\pi^4 + \frac{90514741}{54432000}\pi^6 - \frac{95624828289}{2050048}f_7^6 \\
& \quad - \frac{29295}{512}g_1^6 f_2^6 f_4^6 + \frac{107919}{512}g_1^6 f_3^6 f_3^6 + \frac{337365}{256}f_3^6 g_1^6 f_3^6 \\
& \quad - \frac{55618247}{409600}f_5^6\pi^2 - \frac{1055}{256}g_1^6 f_2^6 f_2^6\pi^2 + \frac{26}{3}f_1^4 f_2^4 f_2^4\pi^2 \\
& \quad + \frac{553}{4}g_1^6 f_3^6 g_1^6\pi^2 - \frac{35189}{1024}f_3^6 g_1^6 g_1^6\pi^2 + \frac{79147091}{2211840}f_3^6\pi^4 \\
& \quad \left.- \frac{3678803}{4354560}g_1^6\pi^6 + \sqrt{3}(E_{4a} + E_{5a} + E_{6a} + E_{7a})\right. \\
& \quad \left.+ E_{6b} + E_{7b} + U\right)\left(\frac{\alpha}{\pi}\right)^4 .
\end{aligned}
$$

All calculations were done using the Maple package `HyperlogProcedures` [20]. Galois conjugates are listed in Table 1.

wt.	dim.	words					
0	1	1					
1	0	—					
2	1	π^2					
3	2	f_3^6	$g_1^6\pi^2$				
4	6	f_4^6	$g_1^6 f_3^6$	$f_2^6\pi^2$	$f_2^4\pi^2$	$g_1^6 g_1^6\pi^2$	π^4
5	4	f_5^6	$g_1^6 g_1^6 f_3^6$	$f_3^6\pi^2$	$g_1^6\pi^4$		
6	15	f_6^6	$g_1^6 f_5^6$	$f_2^6 f_4^6$	$f_3^6 f_3^6$	$f_2^6 g_1^6 f_3^6$	$g_1^6 g_1^6 g_1^6 f_3^6$	$f_4^6\pi^2$
		$g_1^6 f_3^6\pi^2$	$f_2^6 f_2^6\pi^2$	$f_2^4 f_2^4\pi^2$	$f_3^6 g_1^6\pi^2$	$f_2^6 g_1^6 g_1^6\pi^2$	$f_2^6\pi^4$	$g_1^6 g_1^6\pi^4$ $\quad\pi^6$
7	11	f_7^6	$g_1^6 f_2^6 f_4^6$	$g_1^6 f_6^6 f_3^6$	$f_3^6 g_1^6 f_3^6$	$f_5^6\pi^2$	$g_1^6 f_3^6 g_1^6\pi^2$	$g_1^6 f_2^6 f_2^6\pi^2$
		$f_1^4 f_2^4 f_2^4\pi^2$	$f_3^6 g_1^6 g_1^6\pi^2$	$f_3^6\pi^4$	$g_1^6\pi^6$			

Table 1: A basis of the Galois conjugates in (8). In general, higher loop orders in a_e will provide new Galois conjugates which are not listed in the table. However, we expect that all new Galois conjugates have at least weight five. So, up to weight four the above table should be comprehensive.

Remark 2.   1) In (8) the letters $f_1^6 \cong \log 3$ and f_4^4 are absent in all words. In particular the absence of the weight one letter f_1^6 drastically reduces the number of allowed words. Note that the absence of f_1^6 as a letter is much stronger than the absence of f_1^6 as a Galois conjugate (g_1^6 is present as a letter but not as a Galois conjugate).

The absence of the letter $\log 3$ in the parity Broadhurst basis of $\mathrm{MZV}(6)$ can be given the following description: An iterated integral $I(1, w, 0)$ with a word w in the three letters $0, -1, \xi_3^2$ (or ξ_3) is free of the letter $\log 3$ if and only if w is of the following type: (a) w has no letter ξ_3^2 or (b) w has a single letter ξ_3^2 and no letter -1 and weight ≥ 2 or (c) w does neither begin or end in ξ_3^2 and the deletion of all letters 0 in w leads to one of the three structures $\xi_3^2 - 1 \ldots - 1$, $-1 \ldots - 1\xi_3^2$, or $\xi_3^2 - 1 \ldots - 1\xi_3^2$. In particular, w can have at most two letters ξ_3^2. Regretfully, these words are not stable under shuffle nor does the restriction of the Broadhurst basis to $\log 3$ free elements give an algebra basis for $\log 3$ free $\mathrm{MZV}(6)$. It would be desirable to get a more natural description of $\log 3$ free $\mathrm{MZV}(6)$.

2) There are no odd powers of π in (8).

3) At odd weights all Galois conjugates have an odd number of letters (odd coradical depth, see Section 2).

4) The structure of numbers in (8) (i.e. $\mathrm{MZV}(N)$ for $N = 1, 2, 4, 6$) equals the structure of numbers found (or expected) in primitive graphs

of massless ϕ^4 theory. However, in massless ϕ^4 theory these numbers are found at much higher loop orders (loop orders $3, 9, 8, 7$ for $N = 1, 2, 4, 6$, respectively).

Note that in ϕ^4 theory the loop order for first Euler sums, $N = 2$, is artificially high by a subtle mechanism connected to the coaction conjectures [18]. Naively one would expect Euler sums at loop order 7 in ϕ^4).

The connection between massive and massless QFTs may hint to the existence of a universal structure which severely restricts the type of numbers in any QFT calculation (see [10] for a comparison between ϕ^4 and unphysical non-ϕ^4 numbers). By time of writing there exists nothing concrete about this potential structure (other than that it should be related to the Galois coaction).

5) The coaction principle works graph by graph, so it is not surprising to see it in a partial result. It may be possible, however, that the picture changes slightly when a full result is known.

6) Even weight letters in f^6 need to have a factor of $1/\sqrt{3}i$ because the letters are imaginary but a_e is real. Because all integrals of a_e are defined over $\mathbb{Q}$ imaginary numbers come as $\sqrt{-3}$ in MZV(6).

7) The elliptic and the unknown contributions E and U also have Galois conjugates.

8) We expect that the $\mathbb{Q}$ vector space of Galois conjugates of weight n is completely determined by loop orders $\leq n$. In particular, we expect that in Table 1 the list of polylogarithmic (mixed Tate) Galois conjugates up to weight four is complete to all loop orders in a_e.

9) The rich structure in (8) supports the validity of Laporta's result.

2. Proof of theorem 1

2.1. Iterated integrals

Iterated integrals were defined on manifolds by Chen [11]. Here, we only fix the notation in the elementary case of a punctured sphere. Let $a_1, \ldots, a_n, z \in \mathbb{C}$, then the iterated integral $I(z, a_n \ldots a_1, 0)$ is recursively defined by

$I(z, 0) = 1$ and

$$\frac{\partial}{\partial z} I(z, a_n a_{n-1} \ldots a_1, 0) = \frac{1}{z - a_n} I(z, a_{n-1} \ldots a_1, 0),$$

(9) $\qquad I(0, a_n a_{n-1} \ldots a_1, 0) = 0 \quad \text{if } n \geq 1.$

Iterated integrals with fixed endpoints $0, z$ form a shuffle algebra in the words $a_n \ldots a_1$,

(10) $\qquad\qquad\qquad I(z, u, 0) I(z, v, 0) = I(z, u \,\text{ш}\, v, 0),$

where iterated integrals are extended to sums of words by linearity.

Iterated integrals converge (are regular) if and only if $a_1 \neq 0$ and $a_n \neq z$. The definition of iterated integrals can be extended to all words by the following procedure: For $z \neq 0$ we set

(11) $\qquad\qquad I(z, 0^{\{n\}}, 0) = (-1)^n I(z, z^{\{n\}}, 0) = \log(|z|)^n/n!,$

where $a^{\{n\}}$ is a sequence of n letters a. The absolute value in the logarithm originates from a tangential base point regularization prescription applied to straight paths. We obtain the following 'un-shuffle' formula:

Lemma 3. *For any letters a, b and any word w we have*

(12) $\qquad I(z, wba^{\{n\}}, 0) = \sum_{i=0}^{n} (-1)^i I(z, a^{\{n-i\}}, 0) I(z, [w \,\text{ш}\, a^{\{i\}}]b, 0).$

Proof. Straight forward induction over n using the identity

(13) $\; I(z, a, 0) I(z, wba^{\{n-1\}}, 0) = n I(z, wba^{\{n\}}, 0) + I(z, [w \,\text{ш}\, a]ba^{\{n-1\}}, 0).$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

We use (12) and the analogous formula for reversed words to express non-convergent (singular) iterated integrals as polynomials in $\log(|z|)$ with coefficients which are sums of regular iterated integrals. If $|z| = 1$—which we have in the context of this article—only the constant term $i = n$ survives. In the proof of Theorem 1 we will need the fact that this constant term is an integer linear combination of regular iterated integrals.

For completeness we give a formula for the simultaneous left-right un-shuffle. Let $a \neq b$, then

$$I(z, b^{\{m\}} w a^{\{n\}}, 0)$$
$$= \sum_{i=0}^{n} \sum_{j=0}^{m} (-1)^{i+j} I(z, b^{\{m-j\}}, 0) I(z, a^{\{n-i\}}, 0) I(z, [b^{\{j\}} \text{ ш } w \text{ ш } a^{\{i\}}]_{\text{reg}}, 0),$$

where the subscript 'reg' means that we keep only those words that neither begin in a nor end in b. The proof is by induction over m. We do not need this statement here and hence leave the details to the reader.

It is important to notice that the extension to singular iterated integrals is consistent with the shuffle product (10). A more detailed account on iterated integrals in this specific context can e.g. be found in [18, 19] and the references therein.

The $\mathbb{Q}$ algebra I of (motivic) iterated integrals has a coaction. The unipotent part of this coaction can be considered as a map

$$(14) \qquad\qquad \Delta I \longrightarrow (I \mod 2\pi\mathrm{i}) \otimes I.$$

An explicit formula for Δ co-acting on I was given by A. Goncharov and F. Brown [14], [4], [6].

2.2. Extensions of multiple zeta values

By geometric expansion a regular iterated integral can be expressed as a sum. For $a_i \in \mathbb{C} \setminus \{0\}$, $i = 1, \ldots, r$ we have

$$(15) \qquad (-1)^r I(z, 0^{\{n_r-1\}} a_r \ldots 0^{\{n_1-1\}} a_1, 0)$$

$$= \sum_{k_r > \ldots > k_1 \geq 1} \frac{(\frac{z}{a_r})^{k_r} \ldots (\frac{a_2}{a_1})^{k_1}}{n_r^{k_r} \ldots n_1^{k_1}} = \mathrm{Li}_{n_r, \ldots, n_1} \left(\frac{z}{a_r}, \ldots, \frac{a_2}{a_1} \right).$$

With (12) singular iterated integrals are polynomials in $\log(|z|)$ with coefficients given by sums of Lis.

In the special case $z = 1$ and $a_i \in X_N \setminus \{0\}$, $i = 1, \ldots, r$, we obtain extensions $\mathrm{MZV}(N)$ of MZVs by Nth roots of unity.

The motivic structure of $\mathrm{MZV}(N)$ is conjecturally understood if N is not a prime power p^n for $p \geq 5$ [13, 22]. In these cases there exists an (in most cases conjectural) isomorphism ψ into an f alphabet where multiplication is shuffle and the coaction is deconcatenation. The isomorphism ψ is non-canonical; it depends on a choice of an algebra basis of $\mathrm{MZV}(N)$.

N	weight 1	even weight	odd weight ≥ 3
1	0	0	1
2	1	0	1
$\neq 1, 2, p^n$ for $p \geq 5$	$\phi(N)/2 + \nu(N) - 1$	$\phi(N)/2$	$\phi(N)/2$

Table 2: Number of letters in the f alphabet of motivic MZV(N), where ϕ is Euler's totient function and $\nu(N)$ is the number of distinct primes in N. The last line in this table is mostly conjectural.

The number of letters in the f alphabet depends on N (see Table 2). The case $N = 1$ was solved by F. Brown [4] (with a contribution by D. Zagier [21]). In the case $N = 2, 3, 4, 8$ a vector space basis of MZV(N) was given by P. Deligne in [12]. This basis can trivially be transformed into an algebra basis by restriction to Lyndon words. In this section we address the case $N = 6$.

In [12] P. Deligne also considers the case $N = 6$. However, for $N = 6$ he restricts the alphabet in the iterated integrals to the three letters $0, 1, \xi_6$. In 2015, D. Broadhurst provided a conjectural algebra basis for the full MZV(6) (Definition 4) [3] (see also [15]). Theorem 1 proves that this basis is an algebra basis for motivic MZV(6). For other N some partial results were obtained by C. Glanois (private communication).

2.3. Sixth roots of unity and the Broadhurst conjecture

In [3] D. Broadhurst defines the following sets of numbers:

Definition 4. Consider the three numbers $0, \xi_3^2 = (-1 - \sqrt{3}\mathrm{i})/2, \xi_2 = -1$ as letters and order them by $0 \prec \xi_3^2 \prec \xi_2$. Then

$$B_1^6 = \{2\pi\mathrm{i}, I(1, \xi_3^2, 0), I(1, \xi_2, 0)\},$$
(16) $\quad B_n^6 = \{I(1, w, 0), w \text{ Lyndon word in } 0, \xi_3^2, \xi_2 \text{ with no } 0\xi_2 \text{ subsequence}\}.$

Example 5.

$$B_2^6 = \{I(1, 0\xi_3^2, 0), I(1, \xi_3^2\xi_2, 0)\}$$
(17) $\quad B_3^6 = \{I(1, 00\xi_3^2, 0), I(1, 0\xi_3^2\xi_3^2, 0),$
$$I(1, 0\xi_3^2\xi_2, 0), I(1, \xi_3^2\xi_3^2\xi_2, 0), I(1, \xi_3^2\xi_2\xi_2, 0)\}.$$

Remark 6. Because MZV(6) is stable under complex conjugation one could equally well use the letter ξ_3 instead of ξ_3^2. For the parity version of the Broadhurst basis (which we use for the QED $g-2$) complex conjugation merely flips the signs in all words with odd parity [18]. In the following we decided to choose the letter ξ_3^2 for the following three (minor) reasons: (1) Broadhurst uses the differential form $-\mathrm{d}(1-\xi_3 x) = -\mathrm{d}x/(x-\xi_3^2)$ in his original work [3]. This corresponds to the letter ξ_3^2. (2) With the letter ξ_3^2 words with no letter -1 are iterated integrals in $0,1$ with endpoint ξ_3 which is also used in Deligne's basis for MZV(3). (3) The author's Maple package [20] uses the letter ξ_3^2.

Conjecture 7 (D. Broadhurst, 2015). *The Broadhurst set $B^6 = \{B_n^6, n \geq 1\}$ is an algebra basis for MZV(6).*

Because transcendentality statements are notoriously hard, it seems impossible to prove the above conjecture. However, in the conjecturally equivalent motivic setup where the coaction is available the conjecture can be proved (this is Theorem 1).

For the proof of Theorem 1 we need the following propositions:

Proposition 8. *Let ζ be the Riemann zeta function. We have*

$$\zeta(n) \in \frac{(1+(-1)^{n-1})3^{n-1}}{1-3^{n-1}}\mathrm{Li}_n(\xi_3) + \mathbb{Q}(2\pi\mathrm{i})^n \quad \textit{if } n \in \{2,3,4,\ldots\},$$

$$(18) \quad \mathrm{Li}_n(\xi_6) \in (2^{1-n} + (-1)^n)\mathrm{Li}_n(\xi_3) + \mathbb{Q}(2\pi\mathrm{i})^n \quad \textit{if } n \in \{1,2,3,4,\ldots\}.$$

Proof. This follows from Lemma 3.1 and Lemma 3.2 in [18]. □

The depth of a word is the number of its non-zero letters.

Proposition 9. *Let w be a word of weight n and depth one and let a,b letters, all in the alphabet $0, \xi_3^2, \xi_2$. Then*

$$(19) \qquad I(b,w,a) \in c\mathrm{Li}_n(\xi_3) + \mathbb{Q}(2\pi\mathrm{i})^n, \quad \textit{with } \mathbb{Q} \ni c \equiv 0 \mod 3.$$

Proof. We write $I(b,w,a) \equiv 0 \mod 2\pi\mathrm{i} \mod 3$ for (19). From (18) we have

$$\zeta(n) \equiv 0 \mod 2\pi\mathrm{i} \mod 3 \quad \text{for } n \geq 2,$$
$$\mathrm{Li}_n(\xi_6) \equiv 0 \mod 2\pi\mathrm{i} \mod 3 \quad \text{for } n \geq 1.$$

By Lemma 3.1 in [18] complex conjugation on $\mathrm{Li}_n(\xi_6)$ gives a factor of $(-1)^{n-1}$ modulo $2\pi\mathrm{i}$. Therefore we also have

$$\mathrm{Li}_n(\xi_6^5) \equiv 0 \quad \mathrm{mod}\ 2\pi\mathrm{i} \quad \mathrm{mod}\ 3 \text{ for } n \geq 1.$$

If w does not end in 0 we have four types of iterated integrals that begin in 0:

$$(20) \qquad I(\xi_2, 0^{\{n-1\}}\xi_2, 0) = I(\xi_3^2, 0^{\{n-1\}}\xi_3^2, 0)$$
$$= \begin{cases} -\zeta(n) & \text{if } n > 1 \\ 0 & \text{if } n = 1 \text{ (by Eq. (11))} \end{cases}$$

and

$$(21) \qquad I(\xi_2, 0^{\{n-1\}}\xi_3^2, 0) = -\mathrm{Li}_n(\xi_6^5), \quad I(\xi_3^2, 0^{\{n-1\}}\xi_2, 0) = -\mathrm{Li}_n(\xi_6).$$

These iterated integrals are 0 modulo $2\pi\mathrm{i}$ and modulo 3. Because $\log(|b|) = 0$ Lemma 3 extends this to all iterated integrals $I(b, w, 0)$. By path concatenation and path reversal for iterated integrals (see e.g. [19]) we have

$$(22) \qquad I(b, w, a) \equiv I(b, w, 0) + (-1)^n I(a, \widetilde{w}, 0) \quad \mathrm{mod\ products},$$

where $\widetilde{w}$ is w in reversed order. The words in the missing product terms have total depth one, implying that one factor has depth zero. By path concatenation, path reversal, and (11) this factor is zero. So, we can use (22) to complete the proof of the proposition. $\square$

2.4. Proof of Theorem 1

We use the ordered letters $g_1 \cong \log 2 \succ f_1 \cong -\mathrm{Li}_1(\xi_3) \succ f_2 \cong -\mathrm{Li}_2(\xi_3) \succ f_3 \cong -\mathrm{Li}_3(\xi_3) \succ \ldots$ for the f alphabet with respect to the Broadhurst set B^6. Let

$$(23) \qquad \mathcal{U}^6 := \langle\text{words in } g_1, f_n\rangle_{\mathbb{Q}}.$$

We endow $\mathcal{U}^6$ with the shuffle product and the deconcatenation coproduct. This makes $\mathcal{U}^6$ a Hopf algebra. The general theory on MZV(6) provides the

existence of an isomorphism [9]

$$\psi \,:\, \mathrm{MZV}(6) \longrightarrow \mathcal{U}^6 \otimes_{\mathbb{Q}} \mathbb{Q}[2\pi\mathrm{i}]. \tag{24}$$

The Hopf algebra $\mathcal{U}^6$ co-acts on $\mathcal{U}^6 \otimes_{\mathbb{Q}} \mathbb{Q}[2\pi\mathrm{i}]$ by deconcatenation on the left hand side of the tensor product. The isomorphism ψ can be constructed with the decomposition algorithm by F. Brown [5] (see also Example 2.1 in [18]).

The decomposition algorithm inductively cuts off left letters in $\mathcal{U}^6$ by applying derivatives δ_{g_1} and δ_{f_m}, $m = 1, 2, 3, \ldots$. More precisely, the derivatives δ_x first map into $\mathcal{U}^6$, then cut off the left letter x in $\mathcal{U}^6$, and finally map back into $\mathrm{MZV}(6)$. In this construction words in $\mathcal{U}^6$ which do not begin in x are nullified. Schematically,

$$\delta_x : \mathrm{MZV}(6) \longrightarrow \mathrm{MZV}(6), \quad \delta_x = \psi^{-1} \circ \delta_x^{\mathcal{U}^6} \circ \psi.$$

If the letter x has weight m then one can reduce the calculation of δ_x on any iterated integral with weight $n > m$ to the case $n = m$ (see Eq. (2.13) in [18]):

$$\delta_x I(a_{n+1}, a_n \ldots a_1, a_0) = \tag{25}$$
$$\sum_{k=0}^{n-m} (\delta_x I(a_{k+m+1}, a_{k+m} \ldots a_{k+1}, a_k)) I(a_{n+1}, a_n \ldots a_{k+m+1} a_k \ldots a_1, a_0).$$

Note that $\delta_x I(a_{k+m+1}, a_{k+m} \ldots a_{k+1}, a_k) \in \mathbb{Q}$ is a number (because the weight of the iterated integral matches the weight of the letter x). By definition $\delta_x I = 0$ if I has weight less than m.

Lemma 10. *Let v be a word in the alphabet X_6. Then*

$$\delta_{f_m} I(1, 0^{\{n-1\}} \xi_3^2 v, 0)$$
$$\equiv \begin{cases} I(1, v, 0) & \textit{if } n = m \\ 0 & \textit{if } m < n \quad \bmod \textit{ lower depth} \quad \bmod 3, \\ * & \textit{if } m > n \end{cases}$$
$$\delta_{g_1} I(1, av, 0)$$
$$\equiv \begin{cases} I(1, v, 0) & \textit{if } a = \xi_2 \\ 0 & \textit{otherwise} \end{cases} \quad \bmod \textit{ lower depth} \quad \bmod 3. \tag{26}$$

The above equations mean that we first nullify all expressions which have lower depth than the words in the iterated integral of the left hand sides.

*Then we reduce the coefficients of all iterated integrals in (25) modulo 3
to obtain the right hand sides. The $*$ in the first equation merely means
that the reduction modulo 3 exists (no coefficient has a factor of 3 in the
denominator).*

Proof. By definition of the basis B^6 we have (in (4) we use the parity basis)

$$(27) \qquad \psi(I(1, 0^{\{n-1\}}\xi_3^2, 0)) = -\psi(\mathrm{Li}_n(\xi_3)) = f_n.$$

Applying this to (19) in Prop. 9 we get

$$(28) \qquad \psi(I(b, w, a)) \equiv 0 \quad \mathrm{mod}\ 2\pi\mathrm{i} \quad \mathrm{mod}\ 3$$

for any depth one word w and letters a, b in the alphabet $0, \xi_3^2, \xi_2$.

We use (25) on B^6. The words in the iterated integrals on the right
hand side give a partition of the word $a_n \ldots a_1$ on the left hand side. If the
word $a_{k+m} \ldots a_{k+1}$ has depth two or more then the right hand side vanishes
modulo lower depth.

By path concatenation, path reversal, and (11), iterated integrals of
depth zero words vanish for endpoints $0, \xi_3^2, \xi_2, 1$. So, we may assume that
$a_{k+m} \ldots a_{k+1}$ has depth one.

By (28) only the term $k = n - m$ contributes modulo $2\pi\mathrm{i}$ and modulo
3. By (22) and (19) we have $(a_{n+1} = 1)$

$$(29) \quad I(1, a_n \ldots a_{n-m+1}, a_{n-m}) = I(1, a_n \ldots a_{n-m+1}, 0)$$
$$\mathrm{mod\ products} \quad \mathrm{mod}\ 2\pi\mathrm{i} \quad \mathrm{mod}\ 3.$$

As in the proof of Prop. 9 the product terms vanish.

Equations (25) and (27) give the non-zero terms on the right hand sides
of (26).

If $m < n$ in the first equation or $a = 0$ in the second equation of (26) then
the word $a_n \ldots a_{n-m+1}$ has depth zero. The corresponding iterated integral
vanishes.

If $m > n$ in the first equation then the only possible depth one sequence
for $a_n \ldots a_{n-m+1}$ is $u = 0^{\{n-1\}}\xi_3^2 0^{\{m-n\}}$. The corresponding iterated integral
$I(1, u, 0)$ in (29) un-shuffles over $\mathbb{Z}$ (by (12)) to $I(1, 0^{\{m-1\}}\xi_3^2, 0)$ which is
mapped to f_m (by (27)). The integer coefficient has a reduction modulo 3.

If $a = \xi_3^2$ in the second equation of (26) then $m = 1$ and $I(a_{n+1}, a_n, 0) =
I(1, \xi_3^2, 0)$. By definition $\psi(I(1, \xi_3^2, 0)) = f_1$ with zero g_1 component. Hence
$\delta_{g_1}(I(1, \xi_3^2, 0)) = 0$ and (26) is proved. $\qquad\square$

We inductively define a 'naive' map ψ_0 from words in the alphabet X_6 into $\mathcal{U}^6$ which preserves *no* structure of MZV(6) but which may be considered as a conductor for the construction of ψ. For words u, v in the alphabet X_6 let

$$\psi_0(u\xi_2 v) = \psi_0(u)g_1\psi_0(v),$$

(30) $\psi_0(u0^{\{n-1\}}\xi_3^2 v) = \psi_0(u)f_n\psi_0(v)$ if the rightmost letter of u is not 0.

In the order of Example 5 we have $\psi_0(\text{words in } B_2^6) = \{f_2, f_1g_1\}$ and $\psi_0(\text{words in } B_3^6) = \{f_3, f_2f_1, f_2g_1, f_1f_1g_1, f_1g_1g_1\}$.

Lemma 11. *The map ψ_0 is one to one between Lyndon words in $B^6 \setminus \{2\pi i\}$ and Lyndon words in f, g.*

Proof. A word w is Lyndon if it is smaller than all its non-trivial rotations. Assume $w \neq 0^{\{n\}}$ is a word in X_6. Because 0 is the smallest letter in X_6, rotating a zero $w = 0v \mapsto v0$ always makes the word larger. To decide if w is Lyndon it therefore suffices to only check rotations of w which do not end in zero. If w is a word on which ψ_0 is defined (i.e. w has no subsequence $0\xi_2$ and w does not end in zero) then ψ_0 is defined on all its relevant rotations (not ending in zero).

It hence suffices to show that ψ_0 preserves the ordering on these words. We do this by induction over the number of letters in the image of ψ_0. Note that the ordering of the letters in $\mathcal{U}^6$ is defined such that the claim holds on them. Consider two distinct words $au, bv \in \mathcal{U}^6$ with initial letters a, b. Then $au \succ bv$ is equivalent to $a \succ b$ or $a = b$ and $u \succ v$. By induction, in both cases this is equivalent to $\psi_0^{-1}(au) \succ \psi_0^{-1}(bv)$. $\square$

By Radford's theorem Lyndon words generate the shuffle algebra $\mathcal{U}^6$. The above lemma implies that the number of weight n elements in $B^6 \setminus \{2\pi i\}$ matches the number of weight n generators in $\mathcal{U}^6$. In the isomorphism (24) the element $2\pi i \in B_1^6$ generates the polynomial algebra $\mathbb{Q}[2\pi i]$. So, from the existence of the isomorphism (24) it follows that the number of elements in B_n^6 matches the number of weight n algebra generators in MZV(6).

With these preparations we can now prove Theorem 1.

Proof of Theorem 1. The construction of ψ with the decomposition algorithm can only fail if bijectivity fails at some weight n. Because (24) simply maps $2\pi i$ onto $1 \otimes 2\pi i$ it suffices to show bijectivity modulo $2\pi i$. Because ψ (by construction) is compatible with the $\mathbb{Q}$ algebra structures it also suffices to show bijectivity modulo products. With both reductions the isomorphism

ψ induces a linear map ψ' from the $\mathbb{Q}$ span of $B^6 \setminus \{2\pi\mathrm{i}\}$ into the $\mathbb{Q}$ span of Lyndon words in $\mathcal{U}^6$.

By construction the weight is preserved by ψ and ψ'. So, we have to show that every weight n part ψ'_n of ψ' is a (finite dimensional) bijective linear map. (In fact with the statement after Lemma 11 it suffices to show injectivity or surjectivity of ψ'_n.) The existence of these bijective linear maps ψ'_n is a general feature of the decomposition algorithm.

The coradical depth of an $x \in \mathcal{U}^6$ is the maximum number of letters in any word of x. So, e.g., $2f_2 + 3f_1 g_1$ has coradical depth 2. The decomposition algorithm uses induction over the coradical depth. From the proof of Lemma 10 it is clear that ψ cannot increase the depth.

We use Lemma 10 to show that

$$(31) \quad \psi'_n(I(1,w,0)) \equiv \psi_0(w) + \sum_{\text{Lyndon } v \prec \psi_0(w)} c_v v \quad \text{mod lower depth} \quad \text{mod 3}$$

for any word w of weight n in letters X_6 (with coefficients $c_v \in \mathbb{Z}/3\mathbb{Z}$). Because the naive map ψ_0 preserves the depth, calculating modulo lower depth in Lemma 10 is equivalent to calculating modulo lower coradical depth in (31). So, the non-zero terms on the right hand sides of (26) iteratively generate the term $\psi_0(w)$ on the right hand side of (31).

From the existence of Eq. (26) we see by induction that the right hand side of (31) can be reduced modulo 3 after the reduction modulo lower depth.

Let y be the first letter of $\psi_0(w)$ and let $x \succ y$. Because g_1 is the largest letter we have $y = f_n$. Hence, the first letter of w is not ξ_2. In both cases—$x = f_m$ for $m < n$ and $x = g_1$—we have $\delta_x I(1,w,0) = 0$ modulo lower depth and modulo 3 from the zero terms on the right hand sides of (26). This implies by induction that $\psi'_n(I(1,w,0))$ cannot produce terms $\succ \psi_0(w)$ modulo lower depth and modulo 3. So, (31) is proved.

It follows from (31) that ψ'_n is block triangular in the depth. By induction over depth it suffices to show the bijectivity of ψ'_n modulo lower depth.

We sort the elements in the depth r part of B^6_n by $\prec$ and likewise sort the Lyndon words of coradical depth r in $\mathcal{U}^6$. In the proof of Lemma 11 we showed that ψ_0 preserves the ordering. From (31) it follows that the matrix of ψ'_n modulo lower depth becomes triangular modulo 3 with respect to the sorted bases. On the diagonal all entries are 1 modulo 3. Hence ψ'_n modulo lower depth is invertible modulo 3. This implies the bijectivity of ψ' and the theorem follows. $\qquad\square$

Altogether, in the proof of Theorem 1 we have reduced modulo $2\pi\mathrm{i}$, products, lower depths, and the number 3.

Note that we have also proved that the map ψ'_n (and hence also the isomorphism ψ) preserves the depth on B_n^6. In this sense B_n^6 is a basis of minimum depth.

Deligne's results for $N = 2, 3, 4, 8$ can also be proved with the method used in this section.

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