

Calabi-Yau modular forms in limit: Elliptic fibrations

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We study the limit of Calabi-Yau modular forms, and in particular, those resulting in classical modular forms. We then study two parameter families of elliptically fibred Calabi-Yau fourfolds and describe the modular forms arising from the degeneracy loci. In the case of elliptically fibred Calabi-Yau threefolds our approach gives a mathematical proof of many observations about modularity properties of topological string amplitudes starting with the work of Candelas, Font, Katz and Morrison. In the case of Calabi-Yau fourfolds we derive new identities not computed before.

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2010 Mathematics Subject Classification: 14N35, 14J15, 32G20.

Key words and phrases: modular forms, Hodge filtration, Picard-Fuchs system.

1. Introduction

Theoretical Physics and in particular string theory has provided mathematicians with many q -expansions which at first glance look like modular forms. This is actually the case for some examples of such q -expansions, however, in general they transcend the world of modular and automorphic forms. The case of the mirror quintic is the most well-known one, and it is argued in [Mov15b, Mov15a, AMSY14] that there is a parallel modular form theory in this case. These are called Calabi-Yau modular forms. In this paper we gather further evidence that Calabi-Yau modular forms are natural generalizations of classical automorphic forms. It is a well-known fact that some automorphic forms are the limit of others. We would like to study these phenomena in the context of Calabi-Yau modular forms for the case of elliptically fibred Calabi-Yau manifolds. Here, as first observed in [CFKM94], the corresponding limit for many examples are modular forms for $SL(2, \mathbb{Z})$. This observation has ultimately led to a reformulation of the topological string partition function for this class of Calabi-Yau manifolds in terms of meromorphic Jacobi forms which has culminated in the first all-genus results for the Gromov-Witten theory of compact versions of these manifolds [HKK15]. In the case of compact elliptically fibred Calabi-Yau fourfolds, which are the focus of the present paper, Gromov-Witten invariants have been computed up to genus one [KP08] which is the highest non-vanishing genus for fourfolds. However, a reformulation of the generating functions for these invariants in terms of classical modular forms is still lacking. One goal of the present paper is to remedy this gap by expressing generating functions for the genus zero Gromov-Witten invariants in terms of $SL(2, \mathbb{Z})$ modular forms. In the case of non-rigid Calabi-Yau manifolds of dimension ≥ 3 we do not have an underlying Hermitian symmetric domain and so we have to rephrase our problem in terms of Picard-Fuchs systems. Below we describe the general setting together with the main statement of our results and elaborate on the motivation from Physics and in particular string theory.

1.1. Main statement

We start with two parameter families of elliptically fibred Calabi-Yau n -folds X_z , $z \in (\mathbb{C}^2, 0)$. These are constructed in the framework of toric geometry,

see §2.1. For the construction of the field of Calabi-Yau modular forms, one can skip such geometric considerations, and one can start with the corresponding Picard-Fuchs system:

$$\begin{aligned}
 (1) \quad L_1 &:= -n \cdot \theta_1 \theta_2 + \theta_1^2 - a_0 z_1 (\theta_1 + a_1) (\theta_1 + a_2) = 0, \\
 (2) \quad L_2 &:= \theta_2^n - (-1)^n z_2 (n \cdot \theta_2 - \theta_1) (n \cdot \theta_2 - \theta_1 + 1) \cdots (n \cdot \theta_2 - \theta_1 + n - 1) \\
 &= 0,
 \end{aligned}$$

where n, a_0, a_1, a_2 are parameters of the system. The relevant cases to String Theory are the cases $n = 3, 4$ and (a_0, a_1, a_2) as in the Table 1. If we define $\mathcal{L}_n \subset [z_1, z_2, \theta_1, \theta_2]$, with $\theta_i = z_i \partial_{z_i}$, to be the differential left ideal generated by the operators L_1 and L_2 , then \mathcal{L}_n annihilates the periods of a $(n, 0)$ -forms $\omega^{(n,0)}$ in X_z . The system \mathcal{L}_n has one holomorphic solution $\Pi^0 = O(1)$ and logarithmic solutions $\Pi^a = \Pi^0 \log(z_a) + O(1)$, $a = 1, 2$. The field M_n of differential Calabi-Yau modular forms in these situations is the field extension M_n of \mathbb{C} generated by

$$\begin{aligned}
 (3) \quad & z_1, z_2, \quad \theta_1^i \theta_2^j \Pi^0, \quad \theta_1^i \theta_2^j \left(\Pi^0 \theta_a \Pi^b - \Pi^b \theta_a \Pi^0 \right), \\
 & a, b = 1, 2, \dots, \mathfrak{h} := h^{12}(X_z) = 2, \quad i, j \in \mathbb{N}_0.
 \end{aligned}$$

Elements of M_n are by definition rational functions in (3) with coefficients in \mathbb{C} . One talks about the field of Calabi-Yau modular forms because constructing a graded algebra in this case, similar to the algebra of modular forms, demands a more elaborate analysis which is beyond the scope of this work. We refer for a discussion on these issues for the case of the mirror quintic to [Mov15b, Mov15a]. The field M_n is finitely generated, for instance, it is shown in [AMSY14] that for $n = 3$ one actually needs only $\frac{3h^2+7h+4}{2} = 15$ elements in the list (3) in order to generate M_n . The modular expressions of the elements of M_n are obtained after inserting the mirror map $(\tau_1, \tau_2) = \left(\frac{\Pi^1}{\Pi^0}, \frac{\Pi^2}{\Pi^0} \right)$ or using the $(q_1, q_2) = (e^{\tau_1}, e^{\tau_2})$ coordinates. From now on we will use the same name for an element $f(x)$ of M_n when working with different coordinate systems $x = (z_1, z_2), (\tau_1, \tau_2)$ or (q_1, q_2) . The main result of the present paper is the following

Theorem 1. *Let $f(q_1, q_2) \in M_n$ and assume that it is of the form*

$$f = f_0(q_1) + f_1(q_1)(q_2 q_1^{\frac{n}{2}}) + \cdots + f_i(q_1)(q_2 q_1^{\frac{n}{2}})^i + \cdots$$

where $q_1^{\frac{n}{2}} = e^{\tau_1 \cdot \frac{n}{2}}$. Then for arbitrary n and (a_0, a_1, a_2) as in Table 1 all $f_i(e^{\tau_1})$'s are algebraic over the field of meromorphic quasi-modular forms on

(a_0, a_1, a_2)	Group	Modular forms
$(432, 5/6, 1/6)$	$SL(2, \mathbb{Z})$	$E_4(\tau), E_6(\tau)$
$(64, 3/4, 1/4)$	$\Gamma_0(2)$	$E_2(\tau) - 2E_2(2\tau), E_4(\tau)$
$(27, 2/3, 1/3)$	$\Gamma_0(3)$	$E_2(\tau) - 3E_2(3\tau), E_4(\tau), E_6(\tau)$
$(16, 1/2, 1/2)$	$\Gamma(2)$	θ_2^4, θ_3^4

Table 1: Modular groups

the upper half plane $\tau_1 \in \mathbb{H}$ for the subgroup of $SL(2, \mathbb{Z})$ listed in the same table.

Note that, by definition, an element $f \in M_n$ might have poles at many different points of the z_1 -plane, and hence, as a function of τ_1 it might have poles at many points on the upper half plane including $i\infty$. Writing f in terms of q_1 , M_n becomes a sub-field of the field of convergent Laurent series in q_1 with a finite pole order and with a convergence radius which is not necessarily equal to one (due to the appearance of other poles). In the present paper we perform explicit computations for the $SL(2, \mathbb{Z})$ case and all $f_i(q_1)$ we encounter in this case for an example of f are of the form

$$(4) \quad f_i(q_1) = \frac{P(E_2, E_4, E_6)}{\eta^{48i}},$$

where $P(E_2, E_4, E_6)$ is a polynomial in the Eisenstein series E_2 , E_4 and E_6 , see §6. This functional form is a subset of the allowed f_i appearing in Theorem 1. It would be possible to analyze the proof of Theorem 1 and obtain a finer statement but we do not attempt to do this in the current paper.

There is a tremendous amount of computation in the Physics literature confirming our main theorem for $n = 3$ and $(a_1, a_2) = (1/6, 5/6)$, see §1.2. It does not seem to us that there is any Physics for $n \geq 5$. The case $n = 4$, $(a_1, a_2) = (1/6, 5/6)$ is the main motivation for us. In this case we have a collection of four-point functions $C_{abcd}^{(1,1,1,1)} \in M_n$, $a, b, c, d = 1, 2$ which are invariant under index permutations, for definitions see (57) and (64). For instance, we derive the following identity for the four-point function

$$\begin{aligned}
 (5) \quad C_{2222}^{(1,1,1,1)} &= -q_2 \left(\frac{q_1^2}{\eta^{48}} \right) \left[\frac{5}{9} E_4 E_6 (35 E_4^3 + 37 E_6^2) \right] \\
 &\quad - q_2^2 \left(\frac{q_1^4}{\eta^{96}} \right) \left[\frac{5}{602208} (12377569 E_4^9 + 1960000 E_2 E_4^7 E_6 \right. \\
 &\quad \left. + 85433141 E_4^6 E_6^2 + 4144000 E_2 E_4^4 E_6^3 + 86392307 E_4^3 E_6^4 \right. \\
 &\quad \left. + 2190400 E_2 E_4 E_6^5 + 11544823 E_6^6) \right] + \mathcal{O}(q_2^3).
 \end{aligned}$$

There is no enumerative geometry attached to this function. However, if we write it in terms of three-point functions $C_{ab\gamma_i}^{(1,1,2)}$, $a, b = 1, i = 1, 2$

$$(6) \quad C_{abcd}^{(1,1,1,1)} = -4C_{ab\gamma_1}^{(1,1,2)} C_{cd\gamma_1}^{(1,1,2)} + C_{ab\gamma_2}^{(1,1,2)} C_{cd\gamma_1}^{(1,1,2)} + C_{ab\gamma_1}^{(1,1,2)} C_{cd\gamma_2}^{(1,1,2)},$$

then from $C_{ab\gamma_i}^{(1,1,2)}$ we can derive the Gromov-Witten potentials $F^0(\gamma_i)$, $i = 1, 2$:

$$(7) \quad C_{ab\gamma_i}^{(1,1,2)} = \partial_{\tau_a} \partial_{\tau_b} F^0(\gamma_i), \quad a, b, i = 1, 2.$$

We find that (6) together with (7) allows us to solve for the functions $C_{22\gamma_i}^{(1,1,2)}$ at least to low orders in an expansion in q_2 which determines the potentials $F^0(\gamma_i)$ in such an expansion as follows¹

$$(8) \quad F^0(\gamma_1) = q_2 \left(\frac{q_1^2}{\eta^{48}} \right) \left[\frac{5}{18} E_4 E_6 (35 E_4^3 + 37 E_6^2) \right] + \mathcal{O}(q_2^2),$$

$$\begin{aligned}
 (9) \quad F^0(\gamma_2) &= 1 + q_2 \left(\frac{q_1^2}{\eta^{48}} \right) \left[\frac{5}{10368} (10321 E_4^6 + 1680(-24 + E_2) E_4^4 E_6 \right. \\
 &\quad \left. + 59182 E_4^3 E_6^2 + 1776(-24 + E_2) E_4 E_6^3 + 9985 E_6^4) \right] + \mathcal{O}(q_2^2).
 \end{aligned}$$

We now explain the enumerative geometry of the coefficients of

$$\begin{aligned}
 f_1 &= \frac{5}{18} \frac{1}{\eta^{48}} (E_4 E_6 (35 E_4^3 + 37 E_6^2)) \\
 &= -20 q_1^{-2} + 7680 q_1^{-1} - 1800000 + 278394880 q_1 + \dots \\
 &\quad + N_{0,d_1,1}(\gamma_1) q_1^{d_1-2} + \dots
 \end{aligned}$$

¹In all following appearances of $F^0(\gamma_i)$ we will suppress terms logarithmic in the q_i as these only contain information about classical intersection numbers.

For further details see [KP08]. The B -model Calabi-Yau fourfold X_z underlying the Picard-Fuchs system \mathcal{L}_n , $n = 4$, is mirror dual to a Calabi-Yau fourfold \tilde{X} which is the resolution of the degree 24 hypersurface in $\mathbb{P}^5(1, 1, 1, 1, 8, 12)$. The resolution is done by blowing-up once at the unique singular point $x_1 = x_2 = x_3 = x_4 = 0$. Let $\tilde{D}_1 \cong \mathbb{P}^3$ be the corresponding exceptional divisor. The variety \tilde{X} has the Hodge numbers $h^{0,0} = h^{4,0} = 1$, $h^{1,1} = 2$, $h^{3,1} = 3878$, $h^{2,2} = 15564$ and its elliptic fibration is given by $\tilde{X} \rightarrow \mathbb{P}^3$ which is a projection to the first four coordinates. Let D_2 be the divisor in \tilde{X} which is a pull-back of a linear $\mathbb{P}^2 \subset \mathbb{P}^3$ and $D_1 = 4D_2 + \tilde{D}_1$. For $\beta \in H_2(\tilde{X}, \mathbb{Z})$ and $\gamma \in H^4(\tilde{X}, \mathbb{Z})$ we have the Gromov-Witten invariants

$$(10) \quad N_{g,\beta}(\gamma) = \int_{[\bar{M}_{g,1}(\tilde{X},\beta)]^{\text{virt}}} \text{ev}^*(\gamma),$$

where $\bar{M}_{g,1}(\tilde{X}, \beta)$ is the moduli space of genus g , 1-pointed stable maps to \tilde{X} representing the class β and $\text{ev} : M_{g,1}(\tilde{X}, \beta) \rightarrow \tilde{X}$ is the evaluation map. We take a basis $[E], [\mathbb{P}^1] \in H_2(\tilde{X}, \mathbb{Z})$, where $[E]$ is the homology class of fibers of $\tilde{X} \rightarrow \mathbb{P}^3$ and $[\mathbb{P}^1]$ is the homology class of a line \mathbb{P}^1 inside \tilde{D}_1 . We write $N_{g,d_1,d_2}(\gamma) := N_{g,d_1[E]+d_2[\mathbb{P}^1]}(\gamma)$. In our formula (8), γ_1 is the Poincaré dual to D_2^2 and γ_2 is dual to a linear combination of D_2^2 and $D_1 D_2$. Our modular expressions for the Gromov-Witten generating functions are proved by using the B -model side of mirror symmetry, and showing such statements for the A -model side by using the definition (10) are highly non-trivial open problems.

1.2. Motivation

Recently, there has been a lot of progress and activity in solving the topological string on elliptic Calabi-Yau threefolds [KMW12, AS12, HIK⁺15, HKLV14, HLV14, KKL⁺14, CHS15, HKLV15, HKK15, GHK⁺15, KKL15]. In the case of non-compact Calabi-Yau three-folds these results lead to the computation of refined stable pair invariants [CKK14] and translate on the physics side to partition functions of 6d SCFTs. In the compact case the topological string partition function is the generating function of Gromov-Witten invariants and on the physics side leads to the computation of the entropy of black holes [HMOV15]. In all these cases one can observe that topological string free energies are fully expressible in terms of classical modular forms. We review these results here where we confine ourselves to the case of compact Calabi-Yau three-folds \tilde{X} with a complex two-dimensional

base B and elliptic fibre E^2 . Here one can define a generating function for the Gromov-Witten invariants in terms of a genus expansion in a parameter λ

$$(11) \quad F(\lambda, \underline{q}) = \sum_{g=0}^{\infty} \lambda^{2g-2} F^{(g)}(\underline{q}),$$

where the upper index g indicates the genus. According to the split of the cohomology $H_2(\tilde{X}, \mathbb{Z})$ into the base and the fibre cohomology, we define $q_B^\beta = \prod_{k=1}^{b_2(B)} \exp(2\pi i \int_\beta i\omega + b)$, where $\beta \in H_2(B, \mathbb{Z})$, and $q = \exp(2\pi i \int_f i\omega + b)$ with f being the curve representing the fibre³. We now define

$$(12) \quad F_\beta^{(g)}(q) = \text{Coeff}(F^{(g)}(\underline{q}), q_B^\beta).$$

Then one observes [KMW12] that the $F_\beta^{(g)}(q)$ have distinguished modular properties and can be written as

$$(13) \quad F_\beta^{(g)} = \left(\frac{q^{\frac{1}{24}}}{\eta} \right)^{12 \sum_i c_i \beta^i} P_{2g+6 \sum_i c_i \beta^i - 2}(E_2, E_4, E_6),$$

where $P_{2g+6 \sum_i c_i \beta^i - 2}(E_2, E_4, E_6)$ are (quasi)-modular forms of weight $2g + 6 \sum_i c_i \beta^i - 2$ and the c_i are integer coefficients depending on the base B .

As was first observed in [HIK⁺15], the above modularity properties can be repackaged in the topological string partition function leading to a sum over meromorphic Jacobi forms:

$$(14) \quad Z(\underline{q}, \lambda) = \exp(F(\lambda, \underline{q})) = \sum_{\beta} q_B^\beta Z_\beta(q, \lambda),$$

where Z_β are Jacobi forms of weight zero with index a quadratic form on $H_2(B, \mathbb{Z})$. This repackaging has led to the first all-genus solutions of the topological string on compact Calabi-Yau manifolds [HKK15].

Motivated by these results, our objectives for the present paper are to give mathematical proofs for modularity properties of topological string amplitudes for elliptic Calabi-Yau n -folds with $n \geq 3$.

²In general the fibre can degenerate over co-dimension one loci in the base and lead to more cohomology classes whose intersection matrices are given by ADE dynkin diagrams as described by Kodaira. Here we limit ourselves to the case where there is only one such cohomology class.

³ $i\omega + b$ denotes the complexified Kähler form.

Acknowledgements

We are thankful to Stefan Reiter who helped us regarding many properties of linear differential equations. Thanks go to Viktor Levandovskyy who thought us how to use the libraries `nctools.lib` and `dmodapp.lib` for non-commutative rings in `Singular`.

2. Preliminaries

2.1. Toric geometry of elliptically fibred Calabi-Yau varieties

In this paper we confine ourselves to the class of elliptically fibred Calabi-Yau n -folds over \mathbb{P}^{n-1} . The elliptic fibre can be one of four types depending on the weighted projective space in which it is realized. Denote by $\mathbb{P}^2(w_1, \dots, w_r)$ a projective bundle over the base $B = \mathbb{P}^{n-1}$. The four classes are given by four choices of weights $(w_1, \dots, w_r) = \{(1, 2, 3), (1, 1, 2), (1, 1, 1), (1, 1, 1, 1)\}$ leading to elliptic curves which are hypersurfaces in the first three cases and a complete intersection in the last case. The Calabi-Yau manifolds corresponding to the first three cases can be realized as hypersurfaces in toric ambient spaces. The corresponding polyhedron with the Mori cone vectors is given by [KMW12]:

$$(15) \quad \begin{array}{c} D_0 \\ D_1 \\ \vdots \\ D_n \\ D_z \\ D_x \\ D_y \end{array} \left| \begin{array}{cccccc} 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & & & & e_1 & e_2 \\ & & \Delta_B & & \vdots & \vdots \\ 1 & & & & e_1 & e_2 \\ 1 & 0 & \cdots & 0 & e_1 & e_2 \\ 1 & 0 & \cdots & 0 & 1 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 1 \end{array} \right. \begin{array}{cc} l^{(1)} & l^{(2)} \\ \sum_i e_i - 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \\ 1 & -n \\ -e_1 & 0 \\ -e_2 & 0. \end{array}$$

In the above Δ_B represents the toric polyhedron of the base which in our case is \mathbb{P}^{n-1} :

$$(16) \quad \begin{array}{c} D_1 \\ D_2 \\ \vdots \\ D_{n-1} \\ D_n \end{array} \left| \begin{array}{cccc} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & 1 \\ -1 & -1 & \cdots & -1 \end{array} \right.$$

Furthermore, e_1 and e_2 are determined by the three types of elliptic curves which are realized as hypersurfaces:

$$(17) \quad \{(e_1, e_2)\} = \{(-2, -3), (-1, -2), (-1, -1)\}.$$

Using the Mori cone vectors $l^{(1)}$ and $l^{(2)}$ one derives (see [HKTY95]) the Picard-Fuchs system \mathcal{L}_n in (1) and (2). It depends on the Euler number of the base $\chi = n$. The vector (e_1, e_2) determines \mathcal{L}_n with:

$$(18) \quad \begin{aligned} (e_1, e_2) = (-2, -3) &\Rightarrow (a_0, a_1, a_2) = (432, 5/6, 1/6) \\ (e_1, e_2) = (-1, -2) &\Rightarrow (a_0, a_1, a_2) = (64, 3/4, 1/4) \\ (e_1, e_2) = (-1, -1) &\Rightarrow (a_0, a_1, a_2) = (27, 2/3, 1/3). \end{aligned}$$

We also include the last case where the fibre elliptic curve is realized as a complete intersection in \mathbb{P}^3 [KMW12]:

$$(19) \quad (a_0, a_1, a_2) = (16, 1/2, 1/2).$$

3. Non-commutative rings

Let $\mathbb{C}[z, \theta] = \mathbb{C}[z_1, z_2, \dots, z_h, \theta_1, \theta_2, \dots, \theta_h]$ be a non-commutative ring with non-commutative relations

$$\theta_i z_i = z_i(\theta_i + 1).$$

Here, the variable $\theta_i := z_i \frac{\partial}{\partial z_i}$ can be interpreted as the logarithmic derivation. Let also \mathcal{L} be a finitely generated left ideal of $\mathbb{C}[z, \theta]$. For a fixed coordinate z_2 , the restriction of I to $z_2 = 0$ is defined to be

$$\mathcal{L} |_{z_2=0} := \left\{ A \in \mathbb{C}[\hat{z}, \hat{\theta}] \mid \exists B_1, B_2 \in \mathbb{C}[z, \theta], A + z_2 B_1 + \theta_2 B_2 \in I \right\}.$$

Here, \hat{z} (resp. $\hat{\theta}$) is z (resp. θ) with z_2 (resp. θ_2) removed. The computer algebra SINGULAR, see [GPS01], has two libraries nctools.lib, dmodapp.lib for dealing with non-commutative ideals and their restrictions. If Π^0 is a holomorphic solution of \mathcal{L} then $\Pi^0 |_{z_2=0}$ is a holomorphic solution of $\mathcal{L} |_{z_2=0}$. We are mainly interested in the case where $h = 2$. In this paper we only need the following

Proposition 1. *Let $\mathcal{L}_n \subset \mathbb{C}[z_1, z_2, \theta_1, \theta_2]$ be the left ideal generated by L_1 and L_2 in (1) and (2). The restriction $\mathcal{L}_n |_{z_2=0}$ is generated by*

$$(20) \quad L := \theta_1^2 - a_0 z (\theta_1 + a_1) (\theta_1 + a_2).$$

$(1/2, 1/2), (2/3, 1/3), (3/4, 1/4), (5/6, 1/6),$ $(1/6, 1/6), (1/3, 1/6), (1/2, 1/6), (1/3, 1/3), (2/3, 2/3),$ $(1/4, 1/4), (1/2, 1/4), (3/4, 1/2), (3/4, 3/4), (1/2, 1/3),$ $(2/3, 1/6), (2/3, 1/2), (5/6, 1/3), (5/6, 1/2), (5/6, 2/3),$ $(5/6, 5/6), (3/8, 1/8), (5/8, 1/8), (7/8, 3/8), (7/8, 5/8),$ $(5/12, 1/12), (7/12, 1/12), (11/12, 5/12), (11/12, 7/12)$

Table 2: N -integral hypergeometric mirror maps.

Proof. This follows immediately from the explicit form of L_1 and L_2 . \square

From now on we write z, θ etc. instead of z_1, θ_1 in situations where we have taken the limit $z_2 \rightarrow 0$. In Appendix C we have computed more restrictions of non-commutative ideals.

3.1. Modular forms and Gauss hypergeometric equation

In the literature, we can find many examples of modular forms derived from the solutions of the Gauss hypergeometric equation (20) and for particular values of a_0, a_1, a_2 , however, a uniform approach for arbitrary parameters a_i has been recently developed in [DGMS13] and [MS14]. In [Mov15a] page 155 we have shown that the mirror map/Schwarz map of (20) has integral q -coefficients if and only if the pair a_1, a_2 belongs to the class of 28 elements in Table 2:

For the proof of Theorem 1 we will need the condition $a_1 + a_2 = 1$. This reduces our table above to the four cases of (a_1, a_2) shown in Table 1. The parameter a_0 is just a rescaling of z_1 and n can be any positive integer. For all 28 examples in the Table 2 one can determine an arithmetic group Γ , which is basically the monodromy group of L , and the corresponding algebra of modular forms. For our purposes we only need the four cases relevant for this article and gathered in Table 1. In this table E_i 's and θ_i 's are classical Eisenstein and theta series, respectively. The quasi-modular forms in each case is given by the \mathbb{C} -algebra generated by E_2 and the modular forms in the third column. In the last row note that $\theta_4^4 = \theta_3^4 - \theta_2^4$. In the third row we have a polynomial relation between the three modular forms there, see for instance the last section of [Mov15c].

3.2. Hypergeometric functions

In this section we first recall some well-known properties of the hypergeometric function

$$F(a, b|z) = {}_pF_q(a_1, a_2, \dots, a_p, b_1, b_2, \dots, b_q|z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k k!} z^k,$$

$$|z| < 1, \quad b_i \neq 0, -1, -2, \dots$$

which satisfies the linear differential equation $L(a, b)F(a, b|z) = 0$, where

$$(21) \quad \begin{aligned} L(a, b) &= \theta(\theta + b_1 - 1)(\theta + b_2 - 1) \cdots (\theta + b_q - 1) \\ &\quad - z(\theta + a_1)(\theta + a_2) \cdots (\theta + a_p) \\ &= 0 \end{aligned}$$

$(a_i)_k = a_i(a_i + 1)(a_i + 2) \cdots (a_i + k - 1)$, $(a_i)_0 = 1$ is the Pochhammer symbol and $\theta = z \frac{d}{dz}$. For simplicity, for a function f and a linear differential operator L when we say that f satisfies $L = 0$ we mean $L(f) = 0$. For $q = p - 1$ and $b_1 = b_2 = \cdots = b_q = 1$, we have also the following logarithmic solution $G(a, 1|z) + F(a, 1|z) \log z$, where

$$(22) \quad G(a, 1|z) = \sum_{k=1}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(k!)^p} \left[\sum_{j=1}^p \sum_{i=0}^{k-1} \left(\frac{1}{a_j + i} - \frac{1}{1 + i} \right) \right] z^k.$$

We would like to find solutions of $L(a, b)$ when some of the b_i 's are negative integers or zero. Let F be any solution of $L(a, b)$. We note that $z^a F$ satisfies

$$\begin{aligned} &(\theta - a)(\theta + b_1 - 1 - a)(\theta + b_2 - 1 - a) \cdots (\theta + b_q - 1 - a) \\ &\quad - z(\theta + a_1 - a)(\theta + a_2 - a) \cdots (\theta + a_p - a) = 0 \end{aligned}$$

and so $z^{b_1-1} F$ satisfies

$$L(a_1 - b_1 + 1, \dots, a_p - b_1 + 1; 2 - b_1, b_2 - b_1 + 1, \dots, b_q - b_1 + 1) = 0.$$

We will also need the following

$$(23) \quad \begin{aligned} &\frac{\partial}{\partial z} F(a_1, \dots, a_p, b_1, \dots, b_q|z) \\ &= \frac{a_1 a_2 \cdots a_p}{b_1 b_2 \cdots b_q} F(a_1 + 1, \dots, a_p + 1, b_1 + 1, \dots, b_q + 1|z). \end{aligned}$$

Let us proceed to the discussion for the case of the classical Gauss hypergeometric equation with $p = q + 1 = 2$. We conclude that two solutions of

$$(\theta - n - 1)\theta + z(\theta + a_1)(\theta + a_2) = 0, \quad n \in \mathbb{N}_0$$

are given by $z^{n+1}F(a_1 + n + 1, a_2 + n + 1, n + 2|z)$, where F here refers to two solutions of $L(a_1 + n + 1, a_2 + n + 1, n + 2)$. For the holomorphic solution F this can be also seen using the limit

$$\begin{aligned} & \lim_{b_1 \rightarrow -n} \frac{F(a_1, a_2, b_1 | z)}{\Gamma(b_1)} \\ &= \frac{(a_1)_{n+1}(a_2)_{n+1}}{(n+1)!} z^{n+1}F(a_1 + n + 1, a_2 + n + 1; n + 2|z). \end{aligned}$$

4. Proof of Theorem 1

The proof of Theorem 1 is given at the level of periods or solutions of linear differential equations. More precisely, we prove that for $f(z_1, z_2) \in \mathbb{M}_n$ of the form

$$f = f_0(z_1) + f_1(z_1)(q_2q_1^{\frac{n}{2}}) + \cdots + f_i(z_1)(q_2q_1^{\frac{n}{2}})^i + \cdots,$$

all $f_i(z)$ are in the field $\mathbb{C}(z, F, \theta F)$, where F is the Gauss hypergeometric function. After inserting the mirror map in f_i 's one gets the main result as it is announced, see §3.1. Note that we further need the modular expression of each z , F and θF . For instance, for $\text{SL}(2, \mathbb{Z})$ we have

$$\begin{aligned} (24) \quad F(z) \rightarrow F(z(\tau)) &= (E_4)^{\frac{1}{4}}, \quad z \rightarrow z(\tau) = \frac{1}{864}(1 - \sqrt{1 - 1728/J}), \\ \theta F(z) \rightarrow \theta F(z(\tau)) &= \frac{E_4^{1/4}(E_2E_4 - E_6)}{6(E_4^{\frac{3}{2}} + E_6)}. \end{aligned}$$

Similar expressions for the other three hypergeometric functions in Table 1 can be obtained by using the definition of Eisenstein series for triangle groups (see pages 706-707 of [DGMS13] together with their modular expressions in Section 4.2 for $\Gamma(2)$ and Section 4.3 for $\Gamma(m)$, $m = 2, 3$ of the same article).

We will analyze the equality

$$f_i(z_1) = \frac{1}{i!} \left(q_1^{\frac{n}{2}} \frac{\partial}{\partial q_1} \right)^{(i)} f.$$

4.1. Solutions of \mathcal{L}_n

Let us consider the Picard Fuchs system. Let L_1 and L_2 be as in (1) and (2) and let L be the Gauss hypergeometric equation (20). It is also useful to define

$$(25) \quad L^m := L - m\theta_1.$$

We have $L_1 = L - n\theta_1\theta_2$. We have three solutions of \mathcal{L}_n of the form:

$$\begin{aligned} \Pi^0 &= 1 + \sum_{i=1}^{\infty} \Pi_i^0(z_1)z_2^i \\ \Pi^a &= \Pi^0 \ln(z_a) + \sum_{i=1}^{\infty} \Pi_i^a(z_1)z_2^i \quad a = 1, 2. \end{aligned}$$

We need to analyze the following Wronskians in the limit $z_2 = 0$:

$$(26) \quad W^{a,b} := \det \begin{pmatrix} \Pi^0 & \theta_a \Pi^0 \\ \Pi^b & \theta_a \Pi^{1b} \end{pmatrix}, \quad a, b = 1, 2.$$

All $W_{a,b}$'s satisfy Picard-Fuchs differential equations of higher orders. We will write

$$(27) \quad W^{a,b} = \sum_{i=0}^{\infty} W_i^{a,b}(z_1)z_2^i.$$

In what follows we will use the derivation of differential operators with respect to the differentiation variable, for instance

$$\frac{\partial L^m}{\partial \theta} = 2(1 - a_0z)\theta - a_0z(a_1 + a_2) - m.$$

Proposition 2. *We have*

$$(28) \quad L^{n \cdot i} \Pi_i^0 = 0,$$

$$(29) \quad L^{n \cdot i} (\Pi_i^0 \log(z_1) + \Pi_i^1) = 0,$$

$$(30) \quad L^{n \cdot i} \Pi_i^1 = -\frac{L^{n \cdot i}}{\partial \theta} \Pi_i^0,$$

$$(31) \quad L^{n \cdot i} \Pi_i^2 = n \cdot \theta_1 \Pi_i^0.$$

Proof. We just apply the operator $L - n\theta_1\theta_2$ to Π^0, Π^2 and Π^1 , respectively, and we arrive at the above equalities. Note that the third one is the reformulation of the second one using the equality

$$(32) \quad L(f \log z) = L(f) \log z + \frac{\partial L}{\partial \theta}(f).$$

In general, for two holomorphic functions f and g in z and a differential operator L of order k in z, θ , we have used

$$(33) \quad L(fg) = L(f) \cdot g + \frac{\partial L}{\partial \theta}(f) \cdot \theta g + \dots + \frac{\partial^k L}{\partial \theta^k}(f) \cdot \theta^k g.$$

We can verify this easily for $L = \theta^n$ by induction on n . □

Proposition 3. *The Wronskian of the differential operator L^m in (25) (up to multiplication with a constant) is*

$$(1 - a_0z)^{-a_1-a_2} \left(\frac{z}{1 - a_0z} \right)^m.$$

Proof. We use the differential equation of the Wronskian W

$$\theta W = \frac{a_0(a_1 + a_2)z + m}{1 - a_0z} W.$$

□

Using the properties of hypergeometric functions introduced in §3.2 we get the following:

Proposition 4. *We have*

$$(34) \quad \Pi_i^0 = c_i^0 z^{n \cdot i} \frac{\partial^{n \cdot i}}{\partial z^{n \cdot i}} F(a_1, a_2, 1|z),$$

$$(35) \quad \begin{aligned} \Pi_i^0 \log(z_1) + \Pi_i^1 &= c_i^1 z^{n \cdot i} \frac{\partial^{n \cdot i}}{\partial z^{n \cdot i}} (F(a_1, a_2, 1|z) \ln(z_1) + G(a_1, a_2, 1|z)) \\ &+ \tilde{c}_i^1 \Pi_i^0, \end{aligned}$$

where $c_i^0, c_i^1, \tilde{c}_i^1$ are constants.

The constants $c_i^0, c_i^1, \tilde{c}_i^1$ can be computed after applying the second operator L_2 to Π^0, Π^1 . For the mathematical proof of Theorem 1 we do not need to compute them, however, for explicit verifications of Theorem 1 one

must compute them. From (34) it follows that Π_i^0 is in the field $\mathbb{C}(z, F, \theta F)$. Note that

$$\theta^2 F = \frac{a_0(a_1 + a_2)z}{1 - a_0z} \theta F + \frac{a_0 a_1 a_2 z_1}{1 - a_0z} F.$$

Proposition 5. *The quantities $W_i^{a,1}$, $a = 1, 2$ are in the field $\mathbb{C}(z, F, \theta F)$.*

Proof. In (35) we use

$$F(a_1, a_2, 1|z) \ln(z) + G(a_1, a_2, 1|z) = F(a_1, a_2, 1|z) \log(q)$$

and

$$(36) \quad \frac{\partial \log(q)}{\partial z} = \frac{(1 - a_0z)^{-a_1 - a_2} z^{-1}}{F^2}$$

and we write

$$\Pi_i^0 \log(z) + \Pi_i^1 = \Pi_i^0 \log(q) + A_i$$

where $A_0 = 0$. We claim that A_i is in $\mathbb{C}(z, F, \theta F)$. We can see this in two different ways. First, by using (35) and (36), second, by applying the second differential operator L_2 on $\Pi_i^0 \log q + A_i$ which gives a recursion for the A_i 's fixing them without ambiguity. \square

4.2. Nonhomogeneous differential equations

We would like to solve the non-homogeneous equation (31). In general, if we are given a second order linear differential operator $L = \theta^2 + p(z)\theta + q(z)$ with two linearly independent solutions y_1, y_2 , then a solution of the non-homogeneous differential equation $L = g(z)$ is given by $u_1 y_1 + u_2 y_2$, where

$$u_1 = - \int \frac{y_2 g}{W(y_1, y_2)} dz, \quad u_2 := \int \frac{y_1 g}{W(y_1, y_2)} dz$$

and $W(y_1, y_2) = y_1 \theta y_2 - y_2 \theta y_1 = e^{-\int p(x)}$ is the Wronskian. We apply this to the non-homogeneous differential equation (31) and obtain

$$y_1(u_1 + \frac{y_2}{y_1} u_2) = n \cdot \Pi_i^0 \left(- \int \tilde{\Pi}_i^1 \theta \Pi_i^0 (1 - a_0z)^{a_1 + a_2 - 1} \left(\frac{z}{1 - a_0z} \right)^{-n \cdot i} \frac{dz}{z} \right. \\ \left. + \frac{\tilde{\Pi}_i^1}{\Pi_i^0} \int \Pi_i^0 \theta \Pi_i^0 (1 - a_0z)^{a_1 + a_2 - 1} \left(\frac{z}{1 - a_0z} \right)^{-n \cdot i} \frac{dz}{z} \right)$$

where $\tilde{\Pi}_i^1$ is a second solution of $L^{ni} = 0$. Note that by (28) a first solution is given by Π_i^0 . For

$$(37) \quad a_1 + a_2 = 1$$

and $i = 0$ we can solve these integrals and we get

$$\Pi_0^2 = -\frac{n}{2}\Pi_0^0 \log\left(\frac{1 - a_0z}{z}\right).$$

This is defined up to addition of a linear combination of Π_0^0 and $\tilde{\Pi}_0^1$. We know that the original Π_0^2 arising from the solution Π^2 of \mathcal{L}_n is holomorphic at $z_1 = 0$. Therefore, we add a multiple of $\tilde{\Pi}_0^1$ to the expression above and arrive at

$$(38) \quad \Pi_0^2 = -\frac{n}{2}\Pi_0^0 \log\left(q\frac{1 - a_0z}{z}\right),$$

where $q = q_1|_{z_2=0}$. Note that for $i = 0$, $\tilde{\Pi}_i^0$ is the logarithmic solution of the Gauss hypergeometric equation. We can add a multiple of Π^0 to Π^2 and assume that Π_0^2 is divisible by z . In this way the formula of Π_0^2 in (38) becomes unique.

Proposition 6. *The quantities $W_i^{a,2}$, $a = 1, 2$ are in the field $\mathbb{C}(z, F, \theta F)$.*

Proof. Imitating the case of Π_i^1 's, we write

$$(39) \quad \Pi_i^0 \log z_2 + \Pi_i^2 = \Pi_i^0 \log\left(z_2 q^{-\frac{n}{2}} \left(\frac{1 - a_0z_1}{z_1}\right)^{-\frac{n}{2}}\right) + B_i$$

After applying the second differential operator L_2 on the above expression we get a recursion for the B_i 's which shows that they are in the field $\mathbb{C}(z, F, \theta F)$. If we denote by \tilde{z}_2 the expression inside the logarithm in (39), then using (36) we have $\theta_1 \log(\tilde{z}_2) \in \mathbb{C}(z, F, \theta F)$ and $\theta_2 \log(\tilde{z}_2) = 1$. Note that $B_0 = 0$ and hence it is in $\mathbb{C}(z, F, \theta F)$. This is the main reason for defining the logarithmic expression (39). □

4.3. Differential field

The field M_n of Calabi-Yau modular forms defined in the introduction is by definition closed under derivations θ_1, θ_2 . We have

$$\begin{pmatrix} \frac{\partial z_1}{\partial \tau_1} & \frac{\partial z_1}{\partial \tau_2} \\ \frac{\partial z_2}{\partial \tau_1} & \frac{\partial z_2}{\partial \tau_2} \end{pmatrix} = \frac{1}{\frac{\partial \tau_1}{\partial z_1} \frac{\partial \tau_2}{\partial z_2} - \frac{\partial \tau_1}{\partial z_2} \frac{\partial \tau_2}{\partial z_1}} \begin{pmatrix} \frac{\partial \tau_2}{\partial z_2} & -\frac{\partial \tau_1}{\partial z_2} \\ -\frac{\partial \tau_2}{\partial z_1} & \frac{\partial \tau_1}{\partial z_1} \end{pmatrix}$$

and therefore is invariant under

$$(40) \quad \frac{\partial}{\partial \tau_2} = q_2 \frac{\partial}{\partial q_2} = \frac{(\Pi^0)^2}{W^{11}W^{22} - W^{21}W^{12}} (-W^{21}\theta_1 + W^{11}\theta_2)$$

$$(41) \quad \frac{\partial}{\partial \tau_1} = q_1 \frac{\partial}{\partial q_1} = \frac{(\Pi^0)^2}{W^{11}W^{22} - W^{21}W^{12}} (-W^{12}\theta_2 + W^{22}\theta_1)$$

This is still not enough to prove Theorem 1. Propositions 5 and 6 imply that the coefficients of the z_2 -expansion of elements of M_n are in the field $\mathbb{C}(z, F, \theta F)$.

Proposition 7. *We have*

$$(42) \quad \left(\frac{1 - a_0 z_1}{z_1} \right)^{\frac{n}{2}} \frac{q_2 q_1^{\frac{n}{2}}}{z_2} = 1 + \sum_{i=1}^{\infty} C_i z_2^i$$

and $C_i \in \mathbb{C}(z, F, \theta F)$.

Note that the quantity in (42) does not belong to M_n , however, its z_2 -expansion is similar to the z_2 -expansion of the elements of M_n .

Proof. It follows from (38) that the quantity X in (42) starts with 1. We have

$$\partial_2 X = X \partial_2 \cdot \log(X) = X \cdot \left(\frac{W^{22} + \frac{n}{2} W^{21}}{(\Pi^0)^2} - \frac{1}{z_2} \right)$$

Substituting the left hand side of (42) in the X of the above equality we get a recursion of C_i 's which proves the Proposition. □

Because of Proposition 7, it is natural to add the quantities

$$(43) \quad \left(\frac{1 - a_0 z_1}{z_1} \right)^{\frac{n}{2}}, \quad \text{and} \quad \frac{q_2 q_1^{\frac{n}{2}}}{z_2}$$

in (42) to M_n and define \tilde{M}_n to be the field generated by the elements of M_n and (43). Note that for n even, the first element is already in M_n . The

field \check{M}_n is defined here only for computational reasons, namely as a tool to extract coefficients of $(q_2 q_1^{\frac{n}{2}})^i$ in the field M_n .

Proposition 8. *The field \check{M}_n is invariant under the derivation $q_1^{-\frac{n}{2}} \frac{\partial}{\partial q_2}$.*

The field M_n is of course not invariant under $\frac{\partial}{\partial q_i}$. It is invariant under the operator $q_2 \frac{\partial}{\partial q_2}$, however, this operator cannot be used in order to compute the q_2 -coefficients of an element in M_n .

Proof. The proof follows from

$$q_1^{-\frac{n}{2}} \frac{\partial}{\partial q_2} = \left(\frac{q_2 q_1^{\frac{n}{2}}}{z_2} \right)^{-1} \frac{(\Pi^0)^2}{z_2(W^{11}W^{22} - W^{21}W^{12})} (-W^{21}\theta_1 + W^{11}\theta_2)$$

Note that

$$\frac{(\Pi^0)^2 W^{21}}{z_2(W^{11}W^{22} - W^{21}W^{12})} \Big|_{z_2=0} \quad \frac{(\Pi^0)^2 W^{11}}{(W^{11}W^{22} - W^{21}W^{12})} \Big|_{z_2=0}$$

are in the field $\mathbb{C}(z, F, \theta F)$. □

5. Yukawa couplings for elliptically fibred Calabi-Yau fourfolds

In this section we will focus on the class of elliptically fibred Calabi-Yau fourfolds. We will review mirror symmetry and proceed to compute 4-point functions which are also called Yukawa-couplings. Using the results from the previous sections we can express all Yukawa-couplings in terms of modular forms. This will provide the first example of a Calabi-Yau fourfold whose A-model correlation functions are expressed in terms of modular forms. Here we will review how to compute periods of a Calabi-Yau fourfold X and relate these to genus 0 Gromov-Witten potentials of the mirror Calabi-Yau fourfold \check{X} where we will follow the references [GMP95, KLR98, May97, GHKK10].

5.1. A-side of the Mirror Symmetry

In the case of fourfolds, in order to obtain zero virtual dimension for the moduli space of holomorphic maps, one needs to intersect the holomorphic curves with an extra four-cycle γ in the Calabi-Yau \check{X} . γ can be any homology class Poincare dual to a cohomology class in the primary vertical

subspace $H_V^{2,2}(\tilde{X})$. Here for a Calabi-Yau d -fold \tilde{X} , $H_V^{k,k}(\tilde{X})$ consists of elements of the form

$$(44) \quad \mathcal{O}_a^{(k)} = \sum_{i_1, \dots, i_k} \alpha_a^{i_1, \dots, i_k} J_{i_1} \wedge \dots \wedge J_{i_k} \in H^{k,k}(\tilde{X}).$$

where $a = 1, 2 \dots$ enumerates a class of elements of $H_V^{k,k}(\tilde{X})$. In the language of topological string theory the cohomology elements $\mathcal{O}_a^{(k)}$ are also called degree k A model operators. Among their non-zero correlation functions are the two-point functions

$$(45) \quad \eta_{ab}^{(k)} = \langle \mathcal{O}_a^{(k)} \mathcal{O}_b^{(d-k)} \rangle = \int_X \mathcal{O}_a^{(k)} \wedge \mathcal{O}_b^{(d-k)},$$

which do not receive any instanton corrections. In mathematical terms, all these quantities are still integer valued and no q -expansion is attached. However, the following three- and four-point functions do receive worldsheet instanton corrections

$$(46) \quad C_{ab\gamma}^{(1,1,2)} = \langle \mathcal{O}_a^{(1)} \mathcal{O}_b^{(1)} \mathcal{O}_\gamma^{(2)} \rangle, \quad C_{abcd}^{(1,1,1,1)} = \langle \mathcal{O}_a^{(1)} \mathcal{O}_b^{(1)} \mathcal{O}_c^{(1)} \mathcal{O}_d^{(1)} \rangle.$$

and hence depend on the q -parameter. The genus 0 Gromov-Witten potential is defined by

$$(47) \quad F^0(\gamma) = \sum_{\beta \in H_2(X, \mathbb{Z})} N_\beta^0(\gamma) q^\beta, \quad \partial_{\tau_a} \partial_{\tau_b} F^0(\gamma) = C_{ab\gamma}^{(1,1,2)}.$$

where $N_\beta^0(\gamma)$ are the Gromov-Witten invariants which are in general rational and one has $q^\beta = \prod_{i=1}^{h^{1,1}} e^{2\pi i \tau_i \beta_i}$, see [KP08]. The potential (47) also admits an expansion in terms of integer invariants $n_\beta^0(\gamma) \in \mathbb{Z}$ as follows.

$$(48) \quad F^0(\gamma) = \frac{1}{2} C_{ab\gamma}^{0(1,1,2)} \tau_a \tau_b + b_{a\gamma}^0 \tau_a + a_\gamma^0 + \sum_{\beta > 0} n_\beta^0(\gamma) \text{Li}_2(q^\beta),$$

where we have

$$(49) \quad C_{ab\gamma}^{0(1,1,2)} = \int_{\tilde{X}} \mathcal{O}_a^{(1)} \wedge \mathcal{O}_b^{(1)} \wedge \mathcal{O}_\gamma^{(2)},$$

and

$$(50) \quad \text{Li}_k(q) = \sum_{d=1}^{\infty} \frac{q^d}{d^k}.$$

5.2. B-side of the Mirror Symmetry

Let us now come to the B model. Here the operators are elements of the horizontal subspace of the cohomology of the mirror Calabi-Yau variety X . Contrary to the three-fold case variations of the $(4, 0)$ form Ω in the fourfold case do not span the full cohomology $H^4(X)$, but rather a subspace known as the horizontal subspace $H^4_H(X)$. By definition it is perpendicular to $H^4_V(X)$. It has the Hodge decomposition

$$(51) \quad H^4_H(X) = H^{4,0} \oplus H^{3,1} \oplus H^{2,2}_H \oplus H^{1,3} \oplus H^{0,4},$$

where $H^{2,2}_H$ is the subspace of $H^{2,2}$ generated solely from the second variation of Ω with respect to the complex structure of X . Periods are then defined in terms of a basis $\gamma_a^{(i)}$ of $H^4_H(X)$ as follows

$$(52) \quad \Pi^{(i)a} = \int_{\gamma_a^{(i)}} \Omega, \quad i = 0, \dots, 4,$$

where the cycles $\gamma_a^{(i)}$ are chosen such that they are dual to a basis $\hat{\gamma}_a^{(i)}$ of $H^{4-i,i}(X)$ with pairing

$$(53) \quad \int_{\gamma_a^{(i)}} \hat{\gamma}_a^{(i)} = \delta^{ij} \delta_{ab}.$$

Their z -expansion is of the form

$$(54) \quad \begin{aligned} \Pi^{(0)} &= 1 + c_a z_a + \mathcal{O}(z^2), \\ \Pi^{(1)a} &= d_a(z) + \log(z_a) \Pi^{(0)}(z), \\ \Pi^{(2)\gamma} &= \frac{1}{2} \sum_{a,b=1}^{h^{1,1}(X)} C_{ab\gamma}^{0(1,1,2)} \left(d_a(z) \log(z_b) + d_b(z) \log(z_a) \right. \\ &\quad \left. + \Pi^{(0)}(z) \log(z_a) \log(z_b) \right) + d_{h^{1,1}(X)+1}^\gamma, \end{aligned}$$

where the d_a are polynomials of the form

$$(55) \quad \begin{aligned} d_1 &= d_{1,a}^1 z_a + d_{2,a,b}^1 z_a z_b + \mathcal{O}(z^3), \\ &\vdots \\ d_{h^{1,1}(X)} &= d_{1,a}^{h^{1,1}(X)} z_a + d_{2,a,b}^{h^{1,1}(X)} z_a z_b + \mathcal{O}(z^3), \\ d_{h^{1,1}(X)+1}^\gamma &= 1 + \mathcal{O}(z). \end{aligned}$$

Furthermore, $C_{ab\gamma}^{0(1,1,2)}$ are constants defined in (49). Note that in Section 4.1 we have adopted the notation

$$\Pi^{(0)} = \Pi^0, \quad \Pi^{1(a)} = \Pi^a.$$

In order to introduce the prepotential (48) on the B model side we then have to use the following identities

$$(56) \quad F^0(\gamma) = \frac{\Pi^{(2)\gamma}}{\Pi^{(0)}}, \quad \tau_a = \frac{\Pi^{(1)a}}{\Pi^{(0)}}, \quad a = 1, \dots, h^{1,1}(\tilde{X}).$$

This justifies the Ansätze (54) for the B model periods.

5.3. Yukawa couplings from Picard-Fuchs equations

Yukawa-couplings are defined through the holomorphic $(4, 0)$ -form Ω as follows:

$$(57) \quad C_{abcd}^{(1,1,1,1)} = \int_{X_z} \Omega \wedge \partial_a \partial_b \partial_c \partial_d \Omega,$$

where $a, b, c, d \in \{1, \dots, h^{3,1}(X)\}$ are complex structure moduli. We will utilize Griffiths transversality and the Picard-Fuchs equation to compute these four-point functions. Griffith transversality amounts to the following constraints:

$$(58) \quad \int_{X_z} \Omega \wedge \partial_1^{i_1} \partial_2^{i_2} \dots \partial_{h^{3,1}}^{i_{h^{3,1}}} \Omega = 0, \quad i_1 + \dots + i_{h^{3,1}} < 4$$

We will now present a formalism to compute four-point functions. In order to proceed we restrict our attention to Calabi-Yau fourfolds with a maximal number of 3 complex structure moduli and define the functions

$$(59) \quad W^{(i,j,k)} = \int_{X_z} \Omega \partial_1^i \partial_2^j \partial_3^k \Omega.$$

Note that (58) is equivalent to

$$(60) \quad \begin{aligned} W^{(i,j,k)} &= 0 \quad \text{for } i + j + k < 4, \\ W^{(i,j,k)} &= C_{\underbrace{1 \dots 1}_{i \text{ times}} \underbrace{2 \dots 2}_{j \text{ times}} \underbrace{3 \dots 3}_{k \text{ times}}}^{(1,1,1,1)} \quad \text{for } i + j + k = 4. \end{aligned}$$

Moreover, we arrive at further constraints by rewriting

$$(61) \quad \prod_{m=1}^3 \partial_m^{i_m} \int \Omega \wedge \prod_{m=1}^3 \partial_m^{j_m} \Omega = 0 \quad \text{for } \sum_m i_m + j_m = 5 \text{ and } \sum_m j_m < 4,$$

as first order differential equations in the four-point functions:

$$(62) \quad \begin{aligned} W^{(4,1,0)} &= \frac{1}{2}(\partial_2 W^{(4,0,0)} + 4\partial_1 W^{(3,1,0)}), \\ W^{(5,0,0)} &= \frac{5}{2}\partial_1 W^{(4,0,0)}, \\ W^{(3,2,0)} &= \frac{1}{2}(2\partial_2 W^{(3,1,0)} + 3\partial_1 W^{(2,2,0)}), \\ W^{(2,2,1)} &= \frac{1}{2}(\partial_3 W^{(2,2,0)} + 2\partial_2 W^{(2,1,1)} + 2\partial_1 W^{(1,2,1)}), \\ W^{(3,1,1)} &= \frac{1}{2}(\partial_3 W^{(3,1,0)} + \partial_2 W^{(3,0,1)} + 3\partial_1 W^{(2,1,1)}), \end{aligned}$$

and all permutations of these. For the differential operator $L_k = \sum_j f_k^{(j)} \partial^j$ which annihilates Ω we have also

$$(63) \quad \sum_{\mathbf{j}} f_k^{\mathbf{j}} W^{(\mathbf{j})} = 0.$$

This is obtained after taking the wedge product of the original equation with Ω and then integrating it over X . These equations can be supplemented further by applying more derivatives on the Picard-Fuchs operators so that one obtains algebraic equations relating the four-point functions. Acting with yet another derivative and using (62) it is possible to obtain first order differential equations for the four-point functions which together with the algebraic constraints are enough to fix those up to a constant. The constant can then be fixed in terms of the classical intersection numbers of the mirror geometry as follows. Consider transforming the Yukawa-coupling to the mirror coordinates $\underline{\tau}$:

$$(64) \quad \begin{aligned} C_{abcd}^{(1,1,1,1)}(\underline{\tau}) &= \sum_{e,f,g,h} \frac{1}{(\Pi^{(0)})^2} C_{efgh}^{(1,1,1,1)}(\underline{z}) \frac{\partial z_e(\tau_i)}{\partial \tau_a} \frac{\partial z_f(\tau_i)}{\partial \tau_b} \frac{\partial z_g(\tau_i)}{\partial \tau_c} \frac{\partial z_h(\tau_i)}{\partial \tau_d} \\ &= C_{abcd}^{0(1,1,1,1)} + \mathcal{O}(\tau_i), \end{aligned}$$

where the $C_{abcd}^{0(1,1,1,1)}$ are the classical intersection numbers of the mirror Calabi-Yau manifold. Using (40) and (41), we can see that the Yukawa-couplings $C_{abcd}^{(1,1,1,1)}(\mathcal{T})$ are elements of the ring M_n . These couplings are related to the three-point functions through the identities:

$$(65) \quad C_{abcd}^{(1,1,1,1)}(\mathcal{T}) = C_{ab\gamma}^{(1,1,2)}(\mathcal{T}) \left(\eta^{(2)^{-1}}\right)^{\gamma\delta} C_{\delta cd}^{(2,1,1)}(\mathcal{T}).$$

In the next section we will provide explicit examples for a particular family of Calabi-Yau fourfolds.

6. Main example

In this section we focus on the particular example of an elliptic fibration over \mathbb{P}^3 as also studied in [KP08].

6.1. Toric data

The Mori cone vectors are given by

$$(66) \quad \begin{aligned} l^{(1)} &= (-6, 0, 0, 0, 0, 2, 3, 1) \\ l^{(2)} &= (0, 1, 1, 1, 1, 0, 0, -4). \end{aligned}$$

From these we deduce the Picard-Fuchs operators (1) and (2) with $n = 4, a_0 = 432, a_1 = \frac{1}{6}, a_2 = \frac{5}{6}$. In this example $H^{1,1}(\tilde{X})$ is generated by two elements J_1 and J_2 which are Poincaré dual to D_1 and D_2 introduced in the Introduction. We take the following linearly independent elements of $H_V^{2,2}$:

$$\gamma_1 := J_2^2, \quad \gamma_2 := \frac{1}{17}(4J_1^2 + J_1J_2)$$

(In [KP08] we have also the notation $D_1 = E$ and $D_2 = B, E$ standing for the elliptic fibre and B standing for base). The A -model notation for these objects that we used in §5.1 is $\gamma_i := \mathcal{O}_i^{(2)}, i = 1, 2$. The inverse of the intersection matrix in this basis is

$$[\gamma_i \cdot \gamma_j] = (\eta^{(2)})^{-1} = \begin{pmatrix} -4 & 1 \\ 1 & 0 \end{pmatrix}$$

Furthermore, we have

$$\int_{\tilde{X}} J_1^4 = 64, \quad \int_{\tilde{X}} J_1^3 J_2 = 16, \quad \int_{\tilde{X}} J_1^2 J_2^2 = 4, \quad \int_{\tilde{X}} J_1 J_2^3 = 1$$

All other integrations of combinations of J_i 's over \tilde{X} is zero, see [KP08] for the details of this computation. The BPS numbers for this particular Calabi-Yau manifold can be found in [KP08] and we also include them in the appendix of this paper.

6.2. Period expansions

Next, we want to use the results of Proposition 1–8 to express the periods and Yukawa-couplings in terms of $SL(2, \mathbb{Z})$ modular forms. In this case we have (24). As a first step we solve for the constants of Proposition 4, we find:

$$\begin{aligned}
 c_0^0 &= c_0^1 = 1 \\
 c_1^0 &= c_1^1 = 1 \\
 c_2^0 &= c_2^1 = \frac{1}{16} \\
 c_3^0 &= c_3^1 = \frac{1}{1296} \\
 c_4^0 &= c_4^1 = \frac{1}{331776} \\
 c_5^0 &= c_5^1 = \frac{1}{207360000}
 \end{aligned}
 \tag{67}$$

Regarding the constants \tilde{c}_i^1 we find that all of these are zero. Next, we compute the logarithmic periods and find that the quantities A_i in Proposition 5 are given by:

$$\begin{aligned}
 A_0 &= 0, \\
 A_1 &= -\frac{6(1 - 1688z + 1067904z^2 - 307556352z^3)}{(1 - 432z)^4 F(z)}, \\
 &\vdots
 \end{aligned}
 \tag{68}$$

In particular, all $A_i \in \mathbb{Q}(z, F, \theta F)$ and have the form

$$A_i = \frac{P_i(z)}{(1 - 432z)^{4i} F(z)},
 \tag{69}$$

where $P_i(z)$ are polynomials in z . For the B_i which appear in Proposition 6 we find

$$B_i = \frac{Q_i(z, F, \theta F)}{(1 - 432z)^{4i} F(z)},
 \tag{70}$$

with polynomials Q_i . For example, we have

$$(71) \quad B_1(z) = \frac{1}{F(z)(1 - 432z)^4} \times 4(3 - 5064z + 3203712z^2 - 922669056z^3 + 3F(z)^2(1 - 1708z + 1075344z^2 - 291589632z^3 + 62983360512z^4) - 5F(z)\theta F(z)(1 - 2184z + 1907712z^2 - 828610560z^3 + 143183904768z^4)).$$

Regarding Proposition 7, we have

$$(72) \quad C_i(z) = \frac{R_i(z, F, \theta F)}{(1 - 432z)^{4i} F}.$$

The R_i are polynomials, the first of which is given by:

$$(73) \quad R_1(z) = 4(3F(z)(1 - 1708z + 1075344z^2 - 2915896332z^3 + 62983360512z^4) - 5\theta F(1 - 2184z + 1907712z^2 - 828610560z^3 + 143183904768z^4)).$$

In order to be able to apply the derivation defined in Proposition 8 we further need to compute

$$(74) \quad \left. \frac{(\Pi^0)^2 W^{21}}{z_2(W^{11}W^{22} - W^{21}W^{12})} \right|_{z_2=0} = -\frac{6(-1 + 1688z - 1067904z^2 + 3075563z^3)}{(-1 + 432z)^3},$$

$$(75) \quad \left. \frac{(\Pi^0)^2 W^{11}}{(W^{11}W^{22} - W^{21}W^{12})} \right|_{z_2=0} = 1.$$

6.3. Yukawa couplings and modularity

Using the above results together with Propositions 1–8 we can now express all 4-point functions defined in (64) in terms of modular forms. In order to proceed we first write down the Yukawa-couplings on the B-model side as rational functions in the complex structure moduli:

$$\begin{aligned}
 W^{(4,0)} &= -\frac{64}{z(1)^4\Delta_1} \\
 W^{(3,1)} &= \frac{16(-1 + 432z_1)}{z_1^3z_2\Delta_1} \\
 W^{(2,2)} &= -\frac{4(1 - 432z_1)^2}{z_1^2z_2^2\Delta_1} \\
 W^{(1,3)} &= \frac{(-1 + 432z_1)^3}{z_1z_2^2\Delta_1} \\
 (76) \quad W^{(0,4)} &= \frac{64(-1 + 1728z_1 - 1119744z_1^2 + 322486272z_1^3)}{z_2^3\Delta_1\Delta_2},
 \end{aligned}$$

where Δ_1, Δ_2 are given by

$$\begin{aligned}
 \Delta_1 &= -1 + 1728z_1 - 1119744z_1^2 + 322486272z_1^3 \\
 &\quad + 34828517376z_1^4(-1 + 256z_2), \\
 (77) \quad \Delta_2 &= -1 + 256z_2.
 \end{aligned}$$

We now want to compute $C_{abcd}^{(1,1,1,1)}(\underline{\tau})$ as an expansion in $q_2 = e^{-\tau_2}$. Applying the derivation of Proposition 8 to (64) we find after using (24):

$$\begin{aligned}
 (78) \quad C_{2222}^{(1,1,1,1)} &= -q_2 \left(\frac{q_1^2}{\eta^{48}} \right) \left[\frac{5}{9} E_4 E_6 (35 E_4^3 + 37 E_6^2) \right] \\
 &\quad - q_2^2 \left(\frac{q_1^4}{\eta^{96}} \right) \left[\frac{5}{602208} (12377569 E_4^9 + 1960000 E_2 E_4^7 E_6 \right. \\
 &\quad \quad + 85433141 E_4^6 E_6^2 + 4144000 E_2 E_4^4 E_6^3 + 86392307 E_4^3 E_6^4 \\
 &\quad \quad \left. + 2190400 E_2 E_4 E_6^5 + 11544823 E_6^6) \right] + \mathcal{O}(q_2^3).
 \end{aligned}$$

Notice that $C_{2222}^{(1,1,1,1)}$ is of modular weight -2^4 and if we define

$$(79) \quad Y^{(1)} = - \left(\frac{q_1^2}{\eta^{48}} \right) \frac{5}{9} E_4 E_6 (35 E_4^3 + 37 E_6^2),$$

⁴We assign weight 0 to the combination $q_2 q_1^2$.

then equation (78) can be written as

$$(80) \quad C_{2222}^{(1,1,1,1)} = q_2 Y^{(1)} + q_2^2 \left[\left(\frac{q_1^4}{\eta^{96}} \right) \left(-\frac{5}{602208} (12377569 E_4^9 + 85433141 E_4^6 E_6^2 + 86392307 E_4^3 E_6^4) \right) + \frac{5}{12} E_2 \left(Y^{(1)} \right)^2 \right] + \mathcal{O}(q_2^3)$$

This structure is reminiscent to the “holomorphic anomaly” observed in [HST99] in the case of elliptic Calabi-Yau threefolds and it would be very interesting to explore the significance of such an anomaly equation for the case of elliptic Calabi-Yau fourfolds further. In this paper we find evidence for such an anomaly structure also for the Gromov-Witten potential $F^0(\gamma_1)$ which derive in the following. Using the identity

$$(81) \quad C_{2222}^{(1,1,1,1)} = -4C_{22\gamma_1}^{(1,1,2)} C_{22\gamma_1}^{(1,1,2)} + C_{22\gamma_2}^{(1,1,2)} C_{22\gamma_1}^{(1,1,2)} + C_{22\gamma_1}^{(1,1,2)} C_{22\gamma_2}^{(1,1,2)}$$

and the expansions

$$(82) \quad C_{22\gamma_1}^{(1,1,2)} = 0 + \mathcal{O}(q_2), \quad C_{22\gamma_2}^{(1,1,2)} = 1 + \mathcal{O}(q_2),$$

we derive

$$(83) \quad F^0(\gamma_1) = q_2 \left(\frac{q_1^2}{\eta^{48}} \right) \left[\frac{5}{18} E_4 E_6 (35 E_4^3 + 37 E_6^2) \right] + \mathcal{O}(q_2^2).$$

We observe that $F^0(\gamma_1)$ has modular weight -2 . In order to derive the second order term q_2 we now impose an anomaly structure of the form

$$(84) \quad F^0(\gamma_1) = -q_2 Y^{(1)} + q_2^2 \left[\left(\frac{q_1^4}{\eta^{96}} \right) P_{46}(E_4, E_6) + k E_2 \left(Y^{(1)} \right)^2 \right] + \mathcal{O}(q_2^3),$$

where $P_{46}(E_4, E_6)$ is a polynomial of weight 46 in E_4 and E_6 and k is a constant. We find

$$(85) \quad k = -\frac{1}{6},$$

$$P_{46}(E_4, E_6) = -\frac{5}{373248} E_4 E_6 (29908 E_4^9 + 207234483 E_4^6 E_6^2 + 208392741 E_4^3 E_6^4 + 27245569 E_6^6).$$

Using these results together with the identity (81) we can solve for $F(\gamma_2)$ to first order in q_2 :

$$(86) \quad F^0(\gamma_2) = 1 + q_2 \left(\frac{q_1^2}{\eta^{48}} \right) \left[\frac{5}{10368} (10321E_4^6 + 1680(-24 + E_2)E_4^4E_6) \right. \\ \left. + 59182E_4^3E_6^2 + 1776(-24 + E_2)E_4E_6^3 + 9985E_6^4 \right] + \mathcal{O}(q_2^2)$$

Note that $F^0(\gamma_2)$ is not a modular form of a definite weight but rather consists of pieces with weights -2 and 0 .

Appendix A. Table of BPS numbers for the main example

$d_1 \setminus d_2$	0	1	2	3	4
0	0	-20	-820	-68060	-7486440
1	0	7680	491520	56256000	7943424000
2	0	-1800000	-159801600	-24602371200	-4394584496640
3	0	278394880	35703398400	7380433205760	1662353371955200
4	0	623056099920	-6039828417600	-1683081588149760	-478655396625235200
5	0	97531011394560	2356890607411200	388243145737128960	119544387620870983680

Table A1: $n_{d_1, d_2}^0(\gamma_1)$.

$d_1 \setminus d_2$	0	1	2	3	4
0	0	0	0	0	0
1	960	5760	181440	13791360	1458000000
2	1920	-1817280	-98640000	-10715760000	-1476352644480
3	2880	421685760	29972448000	4447212981120	783432258136320
4	3840	2555202430080	-6353500619520	-1273702762398720	-285239128072550400
5	4800	506461104057600	4042353816604800	373520266906348800	86478430090747622400

Table A2: $n_{d_1, d_2}^0(\gamma_2)$

Appendix B. More on Mirror Symmetry

In this section we want to elaborate on details of period constructions on the B-side of the Mirror Symmetry and construct a more complete map between A-model and B-model quantities. We refer to the original references for a more thorough review.

We start by recalling that we can choose a dual basis $\hat{\gamma}_a^{(i)}$ of $H_H^{4-i,i}(X)$ (where $i = 0, \dots, 4$ is the grading) with pairing

$$(B.1) \quad \int_{\gamma_a^{(i)}} \hat{\gamma}_b^{(j)} = \delta^{ij} \delta_{ab}.$$

The holomorphic four-form is then expanded as $\Omega = \sum_i \Pi^{(i)a} \hat{\gamma}_a^{(i)}$. Denoting the complex structure moduli space of X by \mathcal{M} we find that for $z \in \mathcal{M}$ the horizontal parts of $F^k = \bigoplus_{p=0}^k H^{4-p,p}(X_z)$ form holomorphic vector bundles for which one can introduce frames $\beta_a^{(k)}$ with the basis expansion

$$(B.2) \quad \beta_a^{(k)} = \hat{\gamma}_a^{(k)} + \sum_{p>k} \Pi_a^{(p,k)} c(z) \hat{\gamma}_c^{(p)}.$$

These $\beta_a^{(k)}$ are the basic operators of the B model and under mirror symmetry we have the exchange

$$(B.3) \quad \mathcal{O}_a^{(k)} \mapsto \beta_a^{(k)} \Big|_{z=0}.$$

The dependence of the $\Pi_a^{(p,k)}$ on z is captured by the Picard-Fuchs operators $L_a(\underline{\theta}, \underline{z})$. These are written in terms of the logarithmic derivatives $\theta_a = z_a \frac{\partial}{\partial z_a}$ with respect to the canonical complex variables z_a defined at the large complex structure point. Define the formal limits

$$(B.4) \quad L_i^{\text{lim}}(\underline{\theta}) = \lim_{z_i \rightarrow 0} L_i(\underline{\theta}, \underline{z}), i = 1, \dots, r,$$

and consider the algebraic ring

$$(B.5) \quad \mathcal{R} = \mathbb{C}[\underline{\theta}] / \left(\mathcal{J} = \left\{ L_1^{\text{lim}}, \dots, L_r^{\text{lim}} \right\} \right).$$

One can define a grading for this ring by taking the ring at grade k , $\mathcal{R}^{(k)}$ to be generated by a basis of degree k polynomials whose number is given by $h_{4-k,k}^H(X) = h_{k,k}^V(\tilde{X})$ for $k = 0, \dots, 4$. There is a one-to-one map between the ring $\mathcal{R}^{(k)}$ and solutions of the Picard-Fuchs equations at large radius. A given ring element of the form $\mathcal{R}^{(k)a} = \sum_{|\underline{\alpha}|=k} \frac{1}{(2\pi i)^k} m_{\underline{\alpha}}^a \theta_1^{\alpha_1} \dots \theta_h^{\alpha_h}$ is mapped to a solution of the form

$$(B.6) \quad \tilde{\Pi}^{(k)a} = X_0(\underline{z}) \left[\mathbb{L}^{(k)a} + \mathcal{O}(\log(z)^{|\alpha|-1}) \right],$$

where

$$(B.7) \quad \mathbb{L}^{(k) a} = \sum_{|\alpha|=k} \frac{1}{(2\pi i)^k} \tilde{m}_{\alpha}^a \log^{\alpha_1}(z_1) \cdots \log^{\alpha_h}(z_h),$$

and $\tilde{m}_{\alpha}^a(\prod_i \alpha_i!) = m_{\alpha}^a$. Using the metric (45) to move indices down we furthermore demand

$$(B.8) \quad \mathcal{R}_a^{(k)} \mathbb{L}^{(k)b} = \delta_a^b.$$

With these definitions mirror symmetry, i.e. exchange of A and B model, is triggered by the identifications

$$(B.9) \quad \theta_i \leftrightarrow J_i, \quad \beta_a^{(k)} \Big|_{z=0} = \mathcal{R}_a^{(k)} \Omega \Big|_{z=0}.$$

Appendix C. Other families of elliptically fibred Calabi-Yau varieties

In our way to reformulate our main results for the Calabi-Yau n -folds with the Picard-Fuchs system (1) and (2), we studied also many other elliptically fibred Calabi-Yau varieties and computed the corresponding Picard-Fuchs systems. For future investigation we have collected our computations in the table bellow. In this table \mathbb{F}_i 's are Hirzebruch surfaces. The limit Picard-Fuchs equation in the variable z_1 means that the limit is taken with respect to all other variables except z_1 .

No.	CY	Base	Fibre	PF-system	$\lim_{z_i \rightarrow 0}$ PF-system
0	3-fold	\mathbb{P}^2	Elliptic	$L_1 = \theta_1(\theta_1 - 3\theta_2) - 12z_1(6\theta_1 + 1)(6\theta_1 + 5)$ $L_2 = \theta_3^2 + z_2(3\theta_2 - \theta_1 + 0)(3\theta_2 - \theta_1 + 1)(3\theta_2 - \theta_1 + 2)$	$\mathcal{L}_{z_1=0} = \theta_3^2 + 3z_2\theta_2(3\theta_2 + 1)(3\theta_2 + 2)$ $\mathcal{L}_{z_2=0} = \theta_3^2 - 43z_2z_1(\theta_1 + \frac{1}{6})(\theta_1 + \frac{5}{6})$
1		\mathbb{P}^1	K3 ($d = 4$)	$L_1 = \theta_1^2(\theta_1 - 2\theta_2) - 4z_1(4\theta_1 + 3)(4\theta_1 + 2)(4\theta_1 + 1)$ $L_2 = \theta_2^2 - z_2(2\theta_2 - \theta_1 + 1)(2\theta_2 - \theta_1)$	$\theta_1^3 - 4z_1(4\theta_1 + 1)(4\theta_1 + 2)(4\theta_1 + 3)$ $\theta_2^2 - 2z_2\theta_2(2\theta_2 + 1)$
2	3	\mathbb{P}^3	Elliptic	$L_1 = \theta_1(\theta_1 - 4\theta_2) - 12z_1(6\theta_1 + 5)(6\theta_1 + 1)$ $L_2 = \theta_2^4 - z_2(4\theta_2 - \theta_1)(4\theta_2 - \theta_1 + 1)(4\theta_2 - \theta_1 + 2)(4\theta_2 - \theta_1 + 3)$	$\theta_1^2 - 12z_1(6\theta_1 + 5)(6\theta_1 + 1)$ $\theta_2^4 - 4z_2\theta_2(4\theta_2 + 1)(4\theta_2 + 2)(4\theta_2 + 3)$
3		\mathbb{F}_0		$L_1 = \theta_1^2(\theta_1 - 2\theta_2 - 2\theta_3) - 8z_1(1 + 2\theta_1)(1 + 4\theta_1)(3 + 4\theta_1)$ $L_{2,0} = \theta_2^2 - z_2(-1 + \theta_1 - 2\theta_2 - 2\theta_3)(\theta_1 - 2\theta_2 - 2\theta_3)$ $L_{3,0} = \theta_3^2 - z_3(-1 + \theta_1 - 2\theta_2 - 2\theta_3)(\theta_1 - 2\theta_2 - 2\theta_3)$	$\theta_1^3 - 4z_1(4\theta_1 + 1)(4\theta_1 + 2)(4\theta_1 + 3)$ $\theta_2^2 - 4z_2\theta_2(\theta_2 + \frac{1}{3})$ $\theta_3^2 - 4z_3\theta_3(\theta_3 + \frac{1}{3})$
4	4	\mathbb{F}_1	K3 ($d = 4$)	$L_1 = \theta_1^2(\theta_1 - 2\theta_2 - 2\theta_3) - 8z_1(1 + 2\theta_1)(1 + 4\theta_1)(3 + 4\theta_1)$ $L_{2,1} = \theta_2^2 + z_2(\theta_1 - \theta_2 - 2\theta_3)(\theta_2 - \theta_3)$ $L_{3,1} = \theta_3^2 - z_3(-1 + \theta_1 - 2\theta_2 - 2\theta_3)(\theta_1 - 2\theta_2 - 2\theta_3)$	$\theta_1^3 - 4z_1(4\theta_1 + 1)(4\theta_1 + 2)(4\theta_1 + 3)$ $\theta_2^2 - 2z_2\theta_2^2$
5		4-fold	\mathbb{F}_2	$L_1 = \theta_1^2(\theta_1 - 2\theta_2 - 2\theta_3) - 8z_1(1 + 2\theta_1)(1 + 4\theta_1)(3 + 4\theta_1)$ $L_{2,2} = \theta_2^2 - z_2(2\theta_2 - \theta_3)(1 + 2\theta_2 - \theta_3)$ $L_{3,2} = \theta_3^2 - z_3(-1 + \theta_1 - 2\theta_2 - 2\theta_3)(\theta_1 - 2\theta_2 - 2\theta_3)$	$\theta_1^3 - 4z_1(4\theta_1 + 1)(4\theta_1 + 2)(4\theta_1 + 3)$ $\theta_2^2 - 4z_2\theta_2(\theta_2 + \frac{1}{3})$ $\theta_3^2 - 2z_3\theta_3(2\theta_3 + 1)$
6	6	\mathbb{P}^2	K3 ($d = 4$)	$L_1 = \theta_1^2(\theta_1 - 3\theta_2) - 8z_1(1 + 2\theta_1)(1 + 4\theta_1)(3 + 4\theta_1)$ $L_2 = \theta_3^2 - z_2(-2 + \theta_1 - 3\theta_2)(-1 + \theta_1 - 3\theta_2)(\theta_1 - 3\theta_2)$	$\theta_1^3 - 4z_1(4\theta_1 + 1)(4\theta_1 + 2)(4\theta_1 + 3)$ $\theta_3^2 + 3z_2\theta_2(3\theta_2 + 1)(3\theta_2 + 2)$
7			K3 ($d = 6$)	$L_1 = \theta_1^2(\theta_1 - 3\theta_2) + 6z_1(1 + 2\theta_1)(1 + 3\theta_1)(2 + 3\theta_1)$ $L_2 = \theta_3^2 - z_2(-2 + \theta_1 - 3\theta_2)(-1 + \theta_1 - 3\theta_2)(\theta_1 - 3\theta_2)$	$\theta_1^3 + 6z_1(3\theta_1 + 1)(3\theta_1 + 2)(2\theta_1 + 1)$ $\theta_3^2 + 3z_2\theta_2(3\theta_2 + 1)(3\theta_2 + 2)$
8	8	\mathbb{P}^2	K3 ($d = 8$)	$L_1 = \theta_1^2(\theta_1 - 3\theta_2) - 8z_1(1 + 2\theta_1)^3$ $L_2 = \theta_3^2 - z_2(-2 + \theta_1 - 3\theta_2)(-1 + \theta_1 - 3\theta_2)(\theta_1 - 3\theta_2)$	$\theta_1^3 - 8z_1(2\theta_1 + 1)^3$ $\theta_3^2 + 3z_2\theta_2(3\theta_2 + 1)(3\theta_2 + 2)$

Table C1: PF-system.

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RECEIVED FEBRUARY 12, 2016

ACCEPTED JULY 13, 2017