Notes on motivic periods

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The second part of a set of notes based on lectures given at the
IHES in 2015 on Feynman amplitudes and motivic periods.

These notes started out as an appendix to [13]. The aim is merely to pro-
vide some basic definitions and tools to describe a certain class of numbers
and functions defined by integrals of algebraic forms over algebraic domains.

1. Introduction

A period, according to an elementary definition of Kontsevich and Zagier, is a complex number whose real and imaginary parts are given by an integral of a rational function over a domain defined by polynomial inequalities [41].

An example is the number

$$\pi = \int_{x^2+y^2\leq 1} dxdy.$$

The number π is ubiquitous and clearly deserves a name of its own. In these notes, we seek to address the problem of how to describe general periods, and families of periods depending on parameters.

Following Grothendieck, periods can be viewed as the coefficients of a comparison isomorphism between two cohomology theories. For simplicity, consider a period $I = \int_\sigma \omega$, where σ is a closed cycle representing an element in the Betti homology $H_n(X(\mathbb{C});\mathbb{Q})$ of a smooth affine algebraic variety X over $\mathbb{Q}$, and ω is a regular differential form over $\mathbb{Q}$ representing an element in the algebraic de Rham cohomology $H^n_{dR}(X;\mathbb{Q})$. The de Rham-Grothendieck comparison isomorphism:

$$\mathrm{comp} : H^n_{dR}(X;\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H^n(X(\mathbb{C});\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$$

is induced by integration of algebraic differential forms over closed cycles. This data can be represented in a category $\mathcal{T}$ of triples (V_B, V_{dR}, c) where V_B, V_{dR} are finite-dimensional vector spaces over $\mathbb{Q}$, and $c : V_{dR} \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} V_B \otimes_{\mathbb{Q}} \mathbb{C}$ is an isomorphism. The period integral can be encoded by algebraic data: an object $(H^n(X(\mathbb{C});\mathbb{Q}), H^n_{dR}(X;\mathbb{Q}), \mathrm{comp})$ in the category $\mathcal{T}$, together with a class $[\sigma]$, an element of the dual of the first vector space, and $[\omega]$, an element of the second. Define a space of periods $\mathcal{P}^{\mathfrak{m}}_{\mathcal{T}}$ of $\mathcal{T}$ to be the $\mathbb{Q}$-vector space spanned by symbols

$$(1.1) \qquad ((V_B, V_{dR}, c), \sigma, \omega) \quad \text{where } \sigma \in V_B^\vee, \ \omega \in V_{dR}$$

modulo a certain equivalence relation (linearity in σ, ω, and functoriality with respect to morphisms in $\mathcal{T}$), which reflects the fact that periods can have different integral representations. The space $\mathcal{P}^{\mathfrak{m}}_{\mathcal{T}}$ forms a ring, and is equipped with a homomorphism called the period map

$$\mathrm{per} : \mathcal{P}^{\mathfrak{m}}_{\mathcal{T}} \longrightarrow \mathbb{C},$$

which sends the class of (1.1) to $\sigma(c(\omega))$. In this way, we obtain an element $I^{\mathfrak{m}}$ in the ring $\mathcal{P}_{\mathcal{T}}^{\mathfrak{m}}$ whose period $\mathrm{per}(I^{\mathfrak{m}}) = I$ is the integral we started off with.

The crucial point is that $\mathcal{T}$ is a Tannakian category, which automatically endows $\mathcal{P}_{\mathcal{T}}^{\mathfrak{m}}$ with the action of a group[1], whose action on elements such as $I^{\mathfrak{m}}$ is a prototype for a 'Galois group of periods'. In more classical language, $\mathcal{P}_{\mathcal{T}}^{\mathfrak{m}}$ is simply the affine ring of the scheme of tensor isomorphisms from the de Rham to Betti fiber functors on $\mathcal{T}$. The idea of a 'Galois theory of periods' has its origins in Grothendieck's Tannakian philosophy of mixed motives, and has been developed by Nori, Kontsevich, André, and most recently Ayoub, Huber and Müller-Stach. The more common, and most sophisticated, approach to this subject involves replacing $\mathcal{T}$ with a suitable category of mixed motives. Several different approaches are possible. In this context, Grothendieck's period conjecture states that the period map is injective.

The naive category $\mathcal{T}$ defined above is the simplest possible framework in which one can set up a working Galois theory of periods. However, much is gained by adding just a little more; namely the requirement that V_B, V_{dR} are equipped with filtrations forming a mixed Hodge structure. This leads to a category $\mathcal{H}$ of triples (V_B, V_{dR}, c) carrying some extra data (Hodge and weight filtrations, and a real Frobenius involution, which encodes the action of complex conjugation). This category was first introduced by Deligne who proved that it is Tannakian. The upshot is that one obtains a 'ring of $\mathcal{H}$-periods' $\mathcal{P}_{\mathcal{H}}^{\mathfrak{m}}$ defined entirely in terms of linear algebra which encapsulates many fundamental features of periods. A 'motivic' period, for us, is then an element of $\mathcal{P}_{\mathcal{H}}^{\mathfrak{m}}$ that comes from the cohomology of an algebraic variety in a specific way. We shall use the adjective 'motivic' for such a period, although much of this paper is in fact Hodge-theoretic. The ring $\mathcal{P}_{\mathcal{H}}^{\mathfrak{m}}$ has the universal property that the periods of any reasonable category of mixed motives admitting Betti and de Rham realisations will factor through it.

These notes explore some simple consequences of this general notion of $\mathcal{H}$-period, and explain how to compute using these objects. For this, one is obliged to use the language of Hopf algebras and matrix coefficients, since the fundamental objects are not the (motivic) Galois groups themselves but their affine rings. This elementary formalism already enables one to attach a panoply of invariants to an $\mathcal{H}$-period, such as its weight, rank, dimension, Hodge polynomial, and more elaborate notions such as its single-valued versions, unipotency filtration and Galois groups. A glossary of non-standard terms is given in §11. Much of this work is motivated by applications to

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[1]in fact, two groups, one for each fiber functor Betti or de Rham.

physics, where some of these concepts (such as the notion of 'transcendental weight') have already taken root and have several applications, and we felt there was a need to place these notions in a rigorous context.

One key point, that must be mentioned from the outset, is that all the concepts in these notes translate immediately into a suitable Tannakian subcategory of mixed motives over the rationals $\mathcal{MM}_\mathbb{Q}$, whenever it is defined. Any reasonable candidate for such a category has Betti and de Rham realisations, so we obtain a map of rings of periods (rings $\mathcal{P}_\bullet^\mathfrak{m} = \mathcal{O}(\mathrm{Isom}_\bullet^\otimes(\omega_{dR}, \omega_B))$, with $\bullet = \mathcal{MM}(\mathbb{Q}), \mathcal{H}$)

$$(1.2) \qquad\qquad \mathcal{P}_{\mathcal{MM}_\mathbb{Q}}^\mathfrak{m} \longrightarrow \mathcal{P}_\mathcal{H}^\mathfrak{m}.$$

All the constructions in this paper can be pulled back to the ring on the left without any difficulty. Possible choices of categories include Nori's category of motives, or the abelian category of mixed Tate motives over number fields [42]. If one wants to prove *independence* of periods, then it is enough to work in the elementary category $\mathcal{H}$, and defining invariants of $\mathcal{P}_\mathcal{H}^\mathfrak{m}$ provides tools to do precisely that.[2]

Another important point is that if the Hodge realisation $\mathcal{MM}_\mathbb{Q} \to \mathcal{H}$ is fully faithful, as one hopes, then (1.2) is injective and the action of the motivic Galois group of $\mathcal{MM}_\mathbb{Q}$ is already correctly calculated by the action of the 'elementary' Galois group of $\mathcal{H}$. This is the case for mixed Tate motives over number fields, so we can identify motivic periods in $\mathcal{P}_{\mathcal{MT}(\mathbb{Q})}^\mathfrak{m}$, for example, with their image in $\mathcal{P}_\mathcal{H}^\mathfrak{m}$ with impunity. For these reasons, we choose to work unconditionally in $\mathcal{H}$, whilst waiting for the dust to settle on the final definition of $\mathcal{MM}_\mathbb{Q}$.

1.1. Contents

In §2 we gather some properties of Tannakian categories, matrix coefficients and unipotent algebraic groups needed for the rest of the notes. This section, which is one of the most technical, can be referred to when needed. If the reader is only interested in periods over $\mathbb{Q}$, then §2.3 can be simplified by taking the rings B_1, B_2, k to be equal to $\mathbb{Q}$. In §3 we attach some basic invariants to elements of $\mathcal{P}_\mathcal{H}^\mathfrak{m}$. Section 4 defines further concepts including

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[2]the relations should come through the back door as a consequence of bounds on algebraic K-theory and the Tannakian formalism. The theory of multiple zeta values provides many examples of relations between periods which can be proved by analytic methods but for which a proof using algebraic correspondences are not presently known.

single-valued versions of de Rham $\mathcal{H}$-periods generalising the single-valued multiple zeta values of [12], and a certain projection map, which can be used, for example, to infer results about p-adic periods of mixed Tate motives from their complex periods. Section 5 offers some basic examples and can be read in parallel with the previous sections for illustration. In §6, we study a decomposition map which enables us to break up an arbitrary $\mathcal{H}$-period into elementary pieces. It takes the form of a canonical isomorphism

$$\Phi : \mathrm{gr}_\bullet^C \mathcal{P}_\mathcal{H}^{\mathrm{m}} \xrightarrow{\sim} \mathcal{P}_{\mathcal{H}^{ss}}^{\mathrm{m}} \otimes_\mathbb{Q} T^c(H)$$

where $\mathcal{P}_{\mathcal{H}^{ss}}^{\mathrm{m}}$ is the ring of pure or semi-simple periods; T^c is the tensor coalgebra (or shuffle algebra) graded by length of tensors; H is a certain explicitly-defined vector space which is a direct sum of pure Hodge structures; and C (for coradical) is a certain filtration on $\mathcal{H}$-periods by unipotency degree.

Example 1.1. The map Φ is a generalisation to all periods of the 'highest length' part of the map ϕ of [15], which assigns to any motivic multiple zeta value an element of a shuffle algebra on certain symbols. For example,

$$\Phi(\zeta^{\mathrm{m}}(2n+1)) = 1 \otimes f_{2n+1} \quad \text{for all } n \geq 1$$

where $f_{2n+1} \in H$ are certain elements which span a copy of $\mathbb{Q}(-2n-1)$. Let $\mathbb{L}^{\mathrm{m}}$ denote the motivic period corresponding to $2\pi i$. Since it is the period of a pure object, it satisfies $\Phi(\mathbb{L}^{\mathrm{m}}) = \mathbb{L}^{\mathrm{m}} \otimes 1$. From these ingredients, one can then decompose the $\mathcal{H}$-periods corresponding to multiple zeta values. For example, one finds

$$\Phi(\zeta^{\mathrm{m}}(2,3)) = 3(\mathbb{L}^{\mathrm{m}})^2 \otimes f_3 \quad \text{and} \quad \Phi(\zeta^{\mathrm{m}}(3,2)) = -2(\mathbb{L}^{\mathrm{m}})^2 \otimes f_3,$$

Similarly, $\zeta^{\mathrm{m}}(3,5) \in C_2 \mathcal{P}_\mathcal{H}^{\mathrm{m}}$ satisfies $\Phi(\zeta^{\mathrm{m}}(3,5)) = -5 \otimes (f_5 \otimes f_3)$.

The map Φ may provide a useful model with which to think about the structure of periods. In Section 7, we very briefly describe how one might set up similar notions for period integrals depending on parameters. The notion of 'families of $\mathcal{H}$-periods' is very rich, and we barely do it justice. In Section 9 we associate various notions of symbol associated to a family of $\mathcal{H}$-periods. For the symbol to exist, the family must underly a (globally) unipotent vector bundle with integrable connection, which always holds in the mixed Tate case. The symbol is a tensor product of differential 1-forms modulo some relations. In the words of one of the referees, this should clarify many clumsy approaches in the recent physics literature. One can think

of the decomposition map as a kind of analogue of a symbol for constant periods. We also define single-valued versions of families of $\mathcal{H}$-periods, again with applications to physics in mind. Finally, in §10 we provide some geometric examples and prove some technical results required for [13]. Lastly, we discuss some examples in the case of the projective line minus three points for illustrative purposes.

Further background about periods can be found in the book project [38], the surveys [41] and [4] and references therein. An obvious omission from this paper is a discussion of Tannakian ideas relating to limiting mixed Hodge structures and the regularisation of divergent (motivic) periods.

Since these notes are based on lectures, there is not much in the way of technical argumentation. In order to keep the length of the paper down, we provide essential technical arguments only where constructions are new or absent from the literature, and omit proofs which are straightforward or well-known.

2. Generalities

2.1. Recap on Tannakian categories

Let k be a field. Following [26] §1.2, a *tensor category* $\mathcal{T}$ over k is a k-linear rigid abelian tensor category, which is ACU and satisfies $k \xrightarrow{\sim} \text{End}(1)$ ([26], §§2.1, 2.7, 2.8). A *fiber functor* from $\mathcal{T}$ to a scheme S over k is an exact k-linear functor from $\mathcal{T}$ to the category of quasi-coherent sheaves on S, which is compatible with the tensor product and the ACU structures. A *Tannakian category* is a tensor category equipped with a fiber functor ω to a non-empty scheme S. If $S = \text{Spec}(B)$ is affine, then due to rigidity, ω necessarily lands in the category of projective B-modules of finite type.

Theorem 2.1. *([45], corrected in [26]). Let $\mathcal{T}$ be a Tannakian category with a fiber functor to S, a non-empty scheme over k. Then the groupoid of tensor automorphisms $\text{Aut}_{\mathcal{T}}^{\otimes}(\omega)$ is faithfully flat on $S \times S$, and ω defines an equivalence of categories from $\mathcal{T}$ to the category of representations of $\text{Aut}_{\mathcal{T}}^{\otimes}(\omega)$.*

In the applications, $k = \mathbb{Q}$ and all Tannakian categories we consider will be neutralised by the Betti realisation. They will possess a fiber functor ω_B to the category of vector spaces over $\mathbb{Q}$, and a second fiber functor ω_{dR} to a smooth scheme S over $\mathbb{Q}$. The space $S(\mathbb{C})$ will be the domain for a family of periods.

2.2. Matrix coefficients.

The following construction is paraphrased from [26] §4.7. Let $\mathcal{T}$ be a (small) category over k, and let B_1, B_2 be two k-algebras, not necessarily commutative. The following discussion is excessively general for our purposes, but this actually simplifies the presentation. Let ω_i, for $i = 1, 2$, be a functor from $\mathcal{T}$ to the category of right projective B_i modules of finite type.

Definition 2.2. A *matrix coefficient* in $\mathcal{T}$ is a triple (M, f, v), where M is an object of $\mathcal{T}$, $f \in \omega_1(M)^\vee$ and $v \in \omega_2(M)$. Consider the following k-vector space

$$\mathcal{P}_{\mathcal{T}}^{\omega_1, \omega_2} = \langle (M, f, v) \rangle_k / \sim$$

spanned by symbols (M, f, v) modulo the following relations:

(i) (Bimodule structure). For all $\lambda_1, \lambda_2 \in B_2$, and $\mu_1, \mu_2 \in B_1$,

$$(M, f, v_1 \lambda_1 + v_2 \lambda_2) \sim (M, f, v_1)\lambda_1 + (M, f, v_2)\lambda_2$$
$$(M, \mu_1 f_1 + \mu_2 f_2, v) \sim \mu_1(M, f_1, v) + \mu_2(M, f_2, v)$$

Furthermore, if $\lambda \in k$, then $(M, f, v)\lambda \sim \lambda(M, f, v)$. Thus $\mathcal{P}_{\mathcal{T}}^{\omega_1, \omega_2}$ is a left B_1-module and right B_2-module, whose induced k-vector space structures coincide. We shall call such an object a (B_1, B_2)-bimodule over k.

(ii) (Morphisms). If $\phi : M_1 \to M_2$ is a morphism in $\mathcal{T}$ then

$$(M_1, f_1, v_1) \sim (M_2, f_2, v_2)$$

whenever $v_2 = \omega_2(\phi)(v_1)$ and $f_1 = (\omega_1(\phi))^t f_2$, where t is the transpose.

Denote the equivalence class of (M, f, v) by $[M, f, v]_{\mathcal{T}}^{\omega_1, \omega_2}$, or simply $[M, f, v]^{\omega_1, \omega_2}$ when there is no ambiguity about the ambient category.

The space $\mathcal{P}_{\mathcal{T}}^{\omega_1, \omega_2}$ is denoted by $L_k(\omega_1, \omega_2)$ in [26]. We use the letter $\mathcal{P}$ because we shall eventually think of elements of $\mathcal{P}_{\mathcal{T}}^{\omega_1, \omega_2}$ as periods. Condition (i) implies that for all objects M in $\mathcal{T}$, there is a morphism $f \otimes v \mapsto [M, f, v]^{\omega_1, \omega_2}$

(2.1) $$\omega_1(M)^\vee \otimes_k \omega_2(M) \longrightarrow \mathcal{P}_{\mathcal{T}}^{\omega_1, \omega_2}$$

of (B_1, B_2)-bimodules over k, which is functorial in M by (ii). This is a universal property satisfied by $\mathcal{P}_{\mathcal{T}}^{\omega_1, \omega_2}$: any such collection of functorial maps

from $\omega_1(M)^\vee \otimes_k \omega_2(M)$ for all M into a (B_1, B_2)-bimodule over k factors through $\mathcal{P}_{\mathcal{T}}^{\omega_1, \omega_2}$.

There is a natural k-linear map $k \to \omega_2(M) \otimes_{B_2} \omega_2(M)^\vee$, which sends 1 to the element corresponding to the identity via the isomorphism

$$\omega_2(M) \otimes_{B_2} \omega_2(M)^\vee \xrightarrow{\sim} \operatorname{Hom}_{B_2}(\omega_2(M), \omega_2(M)).$$

Writing $\omega_1(M)^\vee \otimes_k \omega_2(M) = \omega_1(M)^\vee \otimes_k k \otimes_k \omega_2(M)$ we deduce a map

$$\omega_1(M)^\vee \otimes_k \omega_2(M) \longrightarrow \omega_1(M)^\vee \otimes_k \omega_2(M) \otimes_{B_2} \omega_2(M)^\vee \otimes_k \omega_2(M)$$

which in turn induces a morphism of (B_1, B_2)-bimodules over k:

$$(2.2) \qquad \Delta : \mathcal{P}_{\mathcal{T}}^{\omega_1, \omega_2} \longrightarrow \mathcal{P}_{\mathcal{T}}^{\omega_1, \omega_2} \otimes_{B_2} \mathcal{P}_{\mathcal{T}}^{\omega_2, \omega_2}.$$

One verifies that it defines a right coaction of $\mathcal{P}_{\mathcal{T}}^{\omega_2, \omega_2}$ on $\mathcal{P}_{\mathcal{T}}^{\omega_1, \omega_2}$. Since $\omega_2(M)$ is projective of finite type, we can write $\omega_2(M)$ as a direct summand of B_2^n for some n and let e_i (respectively $e_i^\vee$) for $1 \le i \le n$ be coordinates of $B_2^n \to \omega_2(M)$ (respectively $\omega_2(M) \to B_2^n$). The element $\sum_{i=1}^n e_i \otimes e_i^\vee$ represents the identity on $\omega_2(M)$. This gives the following formula for (2.2) on the level of matrix coefficients:

$$(2.3) \qquad \Delta[M, f, v]^{\omega_1, \omega_2} = \sum_i [M, f, e_i]^{\omega_1, \omega_2} \otimes [M, e_i^\vee, v]^{\omega_2, \omega_2}$$

In a similar way, the space $\mathcal{P}_{\mathcal{T}}^{\omega_1, \omega_1}$ naturally coacts on $\mathcal{P}_{\mathcal{T}}^{\omega_1, \omega_2}$ on the left.

2.3. Tannakian case

Now suppose that B_1, B_2 are commutative, $\mathcal{T}$ is a tensor category and ω_i is a fiber functor to $\operatorname{Spec}(B_i)$, for $i = 1, 2$. The tensor structure on $\mathcal{T}$ implies that $\mathcal{P}_{\mathcal{T}}^{\omega_1, \omega_2}$ is a commutative k-algebra. In formulae:

$$[M_1, f_1, v_1]^{\omega_1, \omega_2} \times [M_2, f_2, v_2]^{\omega_1, \omega_2} = [M_1 \otimes M_2, f_1 \otimes f_2, v_1 \otimes v_2]^{\omega_1, \omega_2},$$

which is well-defined as one easily checks.

Consider the affine scheme over $B = B_1 \otimes_k B_2$ defined by

$$\operatorname{Hom}_{\mathcal{T}}^{\otimes}(\omega_2, \omega_1) = \operatorname{Spec} \mathcal{P}_{\mathcal{T}}^{\omega_1, \omega_2}.$$

If R is any commutative B-algebra then its R-points are given ([26], Proposition 6.6) by collections of homomorphisms of R-modules

$$(2.4) \qquad \phi_M : R \otimes_B \left(B_1 \otimes_k \omega_2(M) \right) \longrightarrow \left(\omega_1(M) \otimes_k B_2 \right) \otimes_B R$$

which are functorial in M and respect the tensor product. The corresponding homomorphism $\phi : \mathcal{P}_{\mathcal{T}}^{\omega_1,\omega_2} \to R$ is given on matrix coefficients by the formula

$$\phi[M,f,v]^{\omega_1,\omega_2} = f(\phi_M(v)).$$

It follows from the existence and properties of duals in $\mathcal{T}$ that the ϕ_M's are automatically isomorphisms and therefore $\mathrm{Isom}_{\mathcal{T}}^{\otimes}(\omega_2,\omega_1) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{T}}^{\otimes}(\omega_2,\omega_1)$ is an isomorphism and we indeed have $\mathcal{P}_{\mathcal{T}}^{\omega_1,\omega_2} = \mathcal{O}(\mathrm{Isom}_{\mathcal{T}}^{\otimes}(\omega_2,\omega_1))$.

Applying the above in the case when both fiber functors are equal implies that for $i = 1, 2$, $\mathcal{P}_{\mathcal{T}}^{\omega_i,\omega_i}$ is a commutative bialgebra over B_i. It has an antipode, which on matrix coefficients is the involution $S : [M,f,v]^{\omega_i,\omega_i} \to [M^\vee, v, f]^{\omega_i,\omega_i}$, unit $[1,1,1]^{\omega_i,\omega_i}$ and counit $\varepsilon : [M,f,v]^{\omega_i,\omega_i} \mapsto f(v)$, and is a commutative Hopf algebra with respect to these structures. It therefore defines an affine group scheme $\mathrm{Aut}_{\mathcal{T}}^{\otimes}(\omega_i)$ over B_i which we shall denote by

$$G_{\mathcal{T}}^{\omega_i} = \mathrm{Spec}\, \mathcal{P}_{\mathcal{T}}^{\omega_i,\omega_i}.$$

If R is a commutative B_i-algebra, then an R-valued point $g \in G_{\mathcal{T}}^{\omega_i}(R)$ can be viewed as a functorial collection of isomorphisms

$$g_M : R \otimes_{B_i} \omega_i(M) \xrightarrow{\sim} \omega_i(M) \otimes_{B_i} R$$

for every object M of $\mathcal{T}$, which are compatible with the tensor product. The homomorphism from $P_{\mathcal{T}}^{\omega_i,\omega_i}$ to R is defined by the formula $g[M,f,v]^{\omega_i,\omega_i} = f(g_M v)$, and every such homomorphism arises in this way.

We shall use the main theorem 2.1 in the following case:

Theorem 2.3. *The functor* $\omega_i : \mathcal{T} \to \mathrm{Rep}\,(G_{\mathcal{T}}^{\omega_i})$ *is an equivalence of categories.*

The scheme $\mathrm{Isom}_{\mathcal{T}}^{\otimes}(\omega_2,\omega_1) = \mathrm{Spec}\, \mathcal{P}_{\mathcal{T}}^{\omega_1,\omega_2}$ is a $G_{\mathcal{T}}^{\omega_1} \times G_{\mathcal{T}}^{\omega_2}$ - bitorsor, with $G_{\mathcal{T}}^{\omega_1}$ acting on the left, $G_{\mathcal{T}}^{\omega_2}$ on the right. The action of $G_{\mathcal{T}}^{\omega_2} \times G_{\mathcal{T}}^{\omega_1}$ on $\mathcal{P}_{\mathcal{T}}^{\omega_1,\omega_2}$ is given as follows. The right coaction (2.2) is equivalent to a left action

of $G_{\mathcal{T}}^{\omega_2}$

(2.5) $$G_{\mathcal{T}}^{\omega_2} \times \mathcal{P}_{\mathcal{T}}^{\omega_1,\omega_2} \longrightarrow \mathcal{P}_{\mathcal{T}}^{\omega_1,\omega_2}$$

via the formula $g(\xi) = (\mathrm{id} \otimes g)\Delta(\xi)$. We shall call (2.5) the 'Galois action' on $\mathcal{P}_{\mathcal{T}}^{\omega_1,\omega_2}$. Concretely, if R is a B_2-algebra, then a point $g \in G_{\mathcal{T}}^{\omega_2}(R)$ acts by

$$g[M, f, v]^{\omega_1,\omega_2} = [M, f, g_M(v)]^{\omega_1,\omega_2}.$$

There is correspondingly a right action on $\mathcal{P}_{\mathcal{T}}^{\omega_1,\omega_2}$ by $G_{\mathcal{T}}^{\omega_1}$ which acts on the element $f \in \omega_1(M)^{\vee}$ on the right. Depending on the situation, it can happen that one or other action of $G_{\mathcal{T}}^{\omega_i}$ on $\mathcal{P}_{\mathcal{T}}^{\omega_1,\omega_2}$ plays a more important role than the other.

2.4. Minimal objects

Return to the situation of §2.2. The following useful lemma will be used to attach quantities to motivic periods.

Suppose that $\mathcal{T}$ is an abelian category such that every object of $\mathcal{T}$ has finite length, and suppose that the functors ω_1, ω_2 are exact.

Lemma 2.4. *Let $\xi \in \mathcal{P}_{\mathcal{T}}^{\omega_1,\omega_2}$. There exists a smallest object $M(\xi)$ of $\mathcal{T}$, unique up to isomorphism, such that ξ is a matrix coefficient of $M(\xi)$, i.e., every object of $\mathcal{T}$ of which ξ is a matrix coefficient admits a subquotient isomorphic to $M(\xi)$.*

Proof. First observe that if $[M_1, f_1, v_1]$ and $[M_2, f_2, v_2]$ are two matrix coefficients in $\mathcal{T}$, then the relations imply that

$$[M_1, f_1, v_1] + [M_2, f_2, v_2] = [M_1 \oplus M_2, (f_1, f_2), (v_1, v_2)].$$

Similarly, every linear combination of elements $\xi \in \mathcal{P}_{\mathcal{T}}^{\omega_1,\omega_2}$ can be represented by a single matrix coefficient $[M, f, v]$. In fact, $\mathcal{P}_{\mathcal{T}}^{\omega_1,\omega_2}$ can alternatively be defined as the set of equivalence classes (M, f, v) with respect to relation (ii) of §2.2, equipped with a well-defined bimodule structure (i).

Now let M be an object of $\mathcal{T}$, and $f \in \omega_1(M)^{\vee}, v \in \omega_2(M)$. Define M_v to be the smallest subobject of M such that v is in the image of $\omega_2(M_v)$. It exists, because if every subobject $N \subsetneq M$ satisfies $v \notin \mathrm{Im}\,\omega_2(N)$, then $M_v = M$, otherwise we can replace M by any subobject N such that $v \in \mathrm{Im}\,\omega_2(N)$, and proceed by induction on the length. It is unique up to isomorphism; if $v \in \omega_2(N_1)$ and $v \in \omega_2(N_2)$, where N_1, N_2 are both minimal subobjects of

M, then by writing $N_1 \cap N_2 = \ker(N \to N/N_1 \oplus N/N_2)$ and using the exactness of ω_2, it follows that $v \in \omega_2(N_1 \cap N_2) \cong \omega_2(N_1) \cap \omega_2(N_2)$ and hence by minimality $N_1 \cong N_1 \cap N_2 \cong N_2$. Similarly, let $_fM$ be the smallest quotient object of M such that $f \in \omega_1(_fM)^\vee$.

Now consider a morphism $\phi : M \to M'$. We first show that

$$\phi(M_v) \cong \phi(M_v)_{\phi(v)} \cong M'_{\phi(v)}.$$

The second isomorphism holds since $\phi(M_v)$ is a subobject of M'. Note that $\omega_2(\phi(M_v))$ contains $\phi(v)$. We show that $\phi(M_v)$ is minimal for this property. For if $N \subset \phi(M_v)$ is a subobject such that $\phi(v) \in \omega_2(N)$, then $\phi^{-1}(N) := \ker(M_v \to \phi(M_v)/N)$ satisfies $v \in \omega_2(\phi^{-1}(N))$, and so by definition of M_v we have $\phi^{-1}(N) \cong M_v$, and $N \cong \phi(M_v)$. This proves the first isomorphism.

It follows that if $\phi : M \to M'$ is surjective, so too is $\phi : M_v \to M'_{\phi(v)}$. It follows from the definition that if ϕ is injective, $M_v \cong M'_{\phi(v)}$ is an isomorphism. The same statement holds for $_fM$ with the words injective and surjective interchanged.

Now apply the first remark to the surjective map $M_v \twoheadrightarrow {}_f(M_v)$. Denote the image of v in $\omega_2(_f(M_v))$ by v also. We obtain a commutative diagram

$$
\begin{array}{ccc}
(M_v)_v & \twoheadrightarrow & (_f(M_v))_v \\
\downarrow & & \downarrow \\
M_v & \twoheadrightarrow & {}_f(M_v)
\end{array}
$$

where the two vertical maps are injective. Since $(M_v)_v = M_v$, it follows that the vertical map on the right is an isomorphism. On the other hand, applying the second remark to $M_v \hookrightarrow M$, we obtain an injection $_f(M_v) \hookrightarrow {}_fM$, and hence an isomorphism $(_f(M_v))_v \cong (_fM)_v$. We conclude that

$$_f(M_v) \cong (_fM)_v$$

and we shall subsequently denote this object simply by $_fM_v$.

We now show that the map which assigns to a matrix coefficient (M, f, v) the isomorphism class of $_fM_v$ respects the equivalence relation (ii). Consider an equivalence $[M, f, v] = [N, f', v']$ arising from a morphism $\phi : M \to N$. In other words, $f' = \phi^t(f)$ and $v' = \phi(v)$. If ϕ is injective, $M_v \cong N_{v'}$ by definition and hence $_fM_v \cong {}_f(M_v) \cong {}_{f'}(N_{v'}) \cong {}_{f'}N_{v'}$. When ϕ is surjective, we again have $_fM_v \cong {}_{f'}N_{v'}$ by a similar argument. Since any morphism can be expressed as a composition of an injection and surjection, we deduce that the isomorphism class of $M(\xi) := {}_fM_v$ only depends on the equivalence class

$\xi = [M, f, v]$. It follows from the definitions that the morphisms $M \to {}_fM \leftarrow {}_fM_v$ define an equivalence $\xi = [{}_fM_v, f, v]$. $\square$

Now suppose that $\mathcal{T}$ satisfies the more stringent conditions of §2.3, and is in particular Tannakian. This implies that every object has finite length.

Let $\xi \in \mathcal{P}_{\mathcal{T}}^{\omega_1, \omega_2}$. The $G_{\mathcal{T}}^{\omega_2}$ representation it generates is the ω_2-image of an object of $\mathcal{T}$, by the Tannaka theorem. We show that it is isomorphic to $M(\xi)$.

Corollary 2.5. *Consider $\xi \in \mathcal{P}_{\mathcal{T}}^{\omega_1, \omega_2}$. Let $\langle G_{\mathcal{T}}^{\omega_2}\xi\rangle_{B_2}$ (respectively $\langle \xi\, G_{\mathcal{T}}^{\omega_1}\rangle_{B_1}$) denote the representation of $G_{\mathcal{T}}^{\omega_2}$ (resp. $G_{\mathcal{T}}^{\omega_1}$) it generates. Then*

$$(2.6) \qquad \omega_2(M(\xi)) \cong \langle G_{\mathcal{T}}^{\omega_2}\xi\rangle_{B_2} \quad and \quad \omega_1(M(\xi)) \cong \langle \xi G_{\mathcal{T}}^{\omega_1}\rangle_{B_1}.$$

Proof. The morphisms (2.1) induce functorial morphisms

$$\omega_1(M)^\vee \otimes \omega_2(M) \longrightarrow \mathcal{P}_{\mathcal{T}}^{\omega_1, \omega_2}$$

in the (ind)-category of left $G_{\mathcal{T}}^{\omega_2}$-representations. By the Tannaka theorem, there exists an ind-object $\mathcal{P}_{\mathcal{T}}^{\omega_1, \bullet}$ of $\mathcal{T}$ whose ω_2-image is $\mathcal{P}_{\mathcal{T}}^{\omega_1, \omega_2}$, and the above morphisms are the ω_2-image of a family of functorial morphisms

$$\omega_1(M)^\vee \otimes M \longrightarrow \mathcal{P}_{\mathcal{T}}^{\omega_1, \bullet}$$

of ind-objects of $\mathcal{T}$. Apply ω_1 to obtain a family of functorial morphisms

$$\omega_1(M)^\vee \otimes \omega_1(M) \longrightarrow \omega_1(\mathcal{P}_{\mathcal{T}}^{\omega_1, \bullet}),$$

and by the universal property we deduce that $\omega_1(\mathcal{P}_{\mathcal{T}}^{\omega_1, \bullet}) \cong \mathcal{P}_{\mathcal{T}}^{\omega_1, \omega_1}$.

Now suppose that $\xi = [M, f, v]^{\omega_1, \omega_2}$. By the previous lemma, we can assume that $M = {}_fM_v$. Consider the morphism

$$(2.7) \qquad \eta \mapsto [M, f, \eta]^{\omega_1, \omega_2} : \omega_2(M) \longrightarrow \mathcal{P}_{\mathcal{T}}^{\omega_1, \omega_2}.$$

It respects the action of $G_{\mathcal{T}}^{\omega_2}$ on both sides. Therefore by the Tannaka theorem it is the ω_2-image of a morphism

$$e_f : M \longrightarrow \mathcal{P}_{\mathcal{T}}^{\omega_1, \bullet}$$

of ind-objects of $\mathcal{T}$. Its ω_1-image is

$$p \mapsto [M, f, p]^{\omega_1, \omega_1} : \omega_1(M) \xrightarrow{\omega_1(e_f)} \mathcal{P}_{\mathcal{T}}^{\omega_1, \omega_1}.$$

Under the dual map, the counit $\varepsilon : \mathcal{P}_\mathcal{T}^{\omega_1,\omega_1} \to k$ maps to $f \in \omega_1(M)^\vee$. Incidentally, this shows that e_f defines an equivalence $\xi = [\mathcal{P}_\mathcal{T}^{\omega_1,\bullet}, \varepsilon, \xi]^{\omega_1,\omega_2}$.

Let $N \subset \mathcal{P}_\mathcal{T}^{\omega_1,\bullet}$ be the sub object such that $\omega_2(N) \subset \mathcal{P}_\mathcal{T}^{\omega_1,\omega_2}$ is the $G_\mathcal{T}^{\omega_2}$-representation generated by ξ. We obtain a commutative diagram

$$
\begin{array}{ccc}
\omega_2(M) & \xrightarrow{(2.7)} & \mathcal{P}_\mathcal{T}^{\omega_1,\omega_2} \\
\cup & & \cup \\
\langle G_\mathcal{T}^{\omega_2} v \rangle & \longrightarrow & \omega_2(N)
\end{array}
$$

By assumption, $M = M_v$ and hence the vertical inclusion on the left is an equality. The map along the bottom is surjective since $\omega_2(N)$ contains ξ, the image of v under (2.7). Therefore $\omega_2(N)$ is isomorphic to the image of $\omega_2(M)$ in $\mathcal{P}_\mathcal{T}^{\omega_1,\omega_2}$, and the map e_f factors as

$$
e_f : M \longrightarrow\!\!\!\!\rightarrow N \subset \mathcal{P}_\mathcal{T}^{\omega_1,\bullet}.
$$

On the other hand, the image of the counit $\varepsilon \in \mathcal{P}_\mathcal{T}^{\omega_1,\omega_1}$ in $\omega_1(N)$ maps to $f \in \omega_1(M)^\vee$. Since $M = {}_f M$, it has no non-trivial such quotients, and $M \cong N$. This proves that $\omega_2(M(\xi)) = \langle G_\mathcal{T}^{\omega_2} \xi \rangle_{B_2}$. The other statement is similar. □

2.5. Coradical filtration and decomposition

Let U be a pro-unipotent algebraic group over a field k of characteristic 0, and let M be a left U-module over k, i.e., M is a right $\mathcal{O}(U)$-comodule. Denote the coaction by $\Delta : M \to M \otimes_k \mathcal{O}(U)$, and let $\mathcal{O}(U)_+$ be the kernel of the augmentation $\varepsilon : \mathcal{O}(U) \to k$.

Define a filtration $C_i M$ on M by $C_{-1} M = 0$, and

$$
C_i M = \{x \in M \; : \; \Delta(x) = x \otimes 1 \pmod{C_{i-1} M}\}
$$

Equivalently, $C_i M$ is the fastest increasing filtration on M by U-submodules such that $C_{-1} M = 0$ and such that U acts trivially on $\mathrm{gr}^C M$. In particular, $C_0 M = M^U$. This filtration is functorial with respect to morphisms of U-modules. It exhausts M, i.e., $M = \bigcup_{i \geq 0} C_i M$. This follows from Engel's theorem in the case when M is of finite type over k, for then the action of U on M factors through a unipotent algebraic matrix group, and M has a non-trivial fixed vector $v \in C_0 M$. Replacing M with $M/C_0 M$ and by induction on the dimension of M, we deduce that $M = C_n M$ for some n. The general case follows from the fact that M is the inductive limit of its sub $\mathcal{O}(U)$-comodules of finite type.

Now consider $\mathcal{O}(U)$, viewed as a right $\mathcal{O}(U)$-comodule in the natural way, and denote by $\Delta : \mathcal{O}(U) \to \mathcal{O}(U) \otimes_k \mathcal{O}(U)$ the coproduct dual to the multiplication in U. The above construction defines an increasing filtration $C_i\mathcal{O}(U)$. It satisfies $C_0\mathcal{O}(U) = k$, because $\mathrm{id} = (\varepsilon \otimes \mathrm{id})\Delta$ in any Hopf algebra and hence $x \in C_0\mathcal{O}(U)$ satisfies $x = \varepsilon(x)$ and is constant, via $\mathcal{O}(U) = \mathcal{O}(U)_+ \oplus k$.

Now let us denote by $\Delta^r = \Delta - \mathrm{id} \otimes 1 : M \to M \otimes_k \mathcal{O}(U)$, and

$$\Delta' = \Delta - \mathrm{id} \otimes 1 - 1 \otimes \mathrm{id} : \mathcal{O}(U) \to \mathcal{O}(U) \otimes_k \mathcal{O}(U).$$

Note that $x \in C_n M$ if and only if $(\Delta^r)^{n+1}x = 0$. If $M = \mathcal{O}(U)$ and $n \geq 1$, $x \in C_n\mathcal{O}(U)_+$ if and only if $(\Delta')^n x = 0$. This follows since

$$\Delta^r x = \Delta' x + 1 \otimes x \equiv \Delta' x \quad (\mathrm{mod}\ C_{i-1}\mathcal{O}(U))$$

for all $x \in C_i\mathcal{O}(U)$, and $i \geq 1$. In particular, $\mathrm{gr}_1^C \mathcal{O}(U) = (C_1\mathcal{O}(U))_+$ is the space of primitive elements in $\mathcal{O}(U)$, which we shall occasionally denote by $\mathrm{Prim}(\mathcal{O}(U))$.

Lemma 2.6. *The coaction satisfies*

$$\Delta C_n(M) \subseteq \sum_{i+j=n} C_i(M) \otimes_k C_j(\mathcal{O}(U)).$$

Proof. The coassociativity $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$ implies, by substituting in the definitions of Δ' and Δ^r, the following identity

$$(2.8) \qquad\qquad (\Delta^r \otimes \mathrm{id})\Delta^r = (\mathrm{id} \otimes \Delta')\Delta^r.$$

Now let $x \in C_n(M)$, and write using a variant of Sweedler's notation

$$\Delta^r x = \sum_{0 \leq i \leq n} x_i \otimes \alpha_i$$

where the x_i are in $C_i(M)$ and are linearly independent modulo $C_{i-1}(M)$. Since $x \in C_n(M)$, we have $(\Delta^r)^{k+1}x \in C_{n-k-1}M$. By coassociativity (2.8)

$$(\Delta^r)^{k+1}x = (\mathrm{id} \otimes \Delta')^k \Delta^r x = \sum_{0 \leq i \leq n} x_i \otimes (\Delta')^k \alpha_i.$$

By the independence assumption on the x_i, we must have $(\Delta')^k \alpha_i = 0$ whenever $i \geq n - k$, and hence $\alpha_{n-k} \in C_k$. $\qquad\qquad\square$

It follows from the lemma that

$$\Delta^r C_n M \subseteq C_{n-1}M \otimes_k C_1\mathcal{O}(U) + C_{n-2}M \otimes_k \mathcal{O}(U)$$

and taking the quotient modulo $C_{n-2}M \otimes_k \mathcal{O}(U)$ defines a map

$$\delta : \mathrm{gr}^C_n M \longrightarrow \mathrm{gr}^C_{n-1}M \otimes_k C_1\mathcal{O}(U)$$

which is injective by definition of the filtration C. The exact sequence

$$0 \longrightarrow C_0\mathcal{O}(U) \longrightarrow C_1\mathcal{O}(U) \longrightarrow \mathrm{gr}^C_1\mathcal{O}(U) \longrightarrow 0$$

splits via the augmentation map ε which sends $C_1\mathcal{O}(U) \to C_0\mathcal{O}(U) = k$. Thus $C_1\mathcal{O}(U) = \mathrm{gr}^C_1\mathcal{O}(U) \oplus k$. Since in any Hopf algebra $(\mathrm{id} \otimes \varepsilon)\Delta^r = 0$, the map δ lands in the kernel of $\mathrm{id} \otimes \varepsilon$, namely $\mathrm{gr}^C_{n-1}M \otimes_k \mathrm{gr}^C_1\mathcal{O}(U)$.

Definition 2.7. Iterating δ we deduce an injective map

$$(2.9) \qquad \Phi : \mathrm{gr}^C M \longrightarrow M^U \otimes_k T^c(\mathrm{gr}^C_1\mathcal{O}(U))$$

where $T^c(V) = \bigoplus_{n \geq 0} V^{\otimes n}$ denotes the tensor coalgebra on V, graded by the length of tensors. Recall that $M^U = \mathrm{gr}^C_0 M$. The map Φ respects the grading on both sides. We shall call this map the *decomposition into primitives*.

In the case $M = \mathcal{O}(U)$, $M^U = k$ and (2.9) yields an injective graded map

$$\Phi : \mathrm{gr}^C_n \mathcal{O}(U) \longrightarrow T^c(\mathrm{gr}^C_1\mathcal{O}(U)).$$

Equip the tensor coalgebra $T^c(V)$ with the deconcatenation coproduct

$$\Delta^{\mathrm{dec}} : T^c(V) \quad \longrightarrow \quad T^c(V) \otimes_k T^c(V)$$
$$v_1 \otimes \cdots \otimes v_n \quad \mapsto \quad \sum_{k=0}^{n}(v_1 \otimes \cdots \otimes v_k) \otimes (v_{k+1} \otimes \cdots \otimes v_n).$$

Lemma 2.8. *The following diagram commutes*

$$
\begin{array}{ccc}
\mathrm{gr}^C M & \xrightarrow{\mathrm{gr}^C\Delta} & \mathrm{gr}^C M \otimes_k \mathrm{gr}^C\mathcal{O}(U) \\
\downarrow & & \downarrow \\
M^U \otimes_k T^c(\mathrm{gr}^C_1\mathcal{O}(U)) & \xrightarrow{\mathrm{id}\otimes\Delta^{\mathrm{dec}}} & M^U \otimes_k T^c(\mathrm{gr}^C_1\mathcal{O}(U)) \otimes_k T^c(\mathrm{gr}^C_1\mathcal{O}(U))
\end{array}
$$

where the vertical map on the left is Φ, and that on the right is $\Phi \otimes \Phi$.

Proof. The lemma can be proved directly from the recursive definition of Φ in terms of δ. It is more instructive to give a conceptual proof by interpreting Φ in the following way. Consider the sequence

$$0 \longrightarrow \mathrm{gr}^C_{n-1} M \longrightarrow C_n M / C_{n-2} M \longrightarrow \mathrm{gr}^C_n M \longrightarrow 0.$$

Since U acts trivially on $\mathrm{gr}^C M$, it follows that the action of U on $C_n M / C_{n-2} M$ factors through U^{ab}. If we write $\mathfrak{u} = \mathrm{Lie}\, U$, then we deduce a map

$$\mathfrak{u}^{ab} \times \mathrm{gr}^C_n M \longrightarrow \mathrm{gr}^C_{n-1} M$$

and hence, denoting $\mathbb{L}(\mathfrak{u}^{ab})$ the free Lie algebra on $\mathfrak{u}^{ab}$, we have

$$(2.10) \qquad\qquad \mathbb{L}(\mathfrak{u}^{ab}) \times \mathrm{gr}^C M \longrightarrow \mathrm{gr}^C M.$$

The affine ring of $\mathbb{L}(\mathfrak{u}^{ab})$ is the tensor coalgebra $T^c(\mathrm{gr}^C_1 \mathcal{O}(U))$, so the dual of the previous map is $\mathrm{gr}^C M \to \mathrm{gr}^C M \otimes_k T^c(\mathrm{gr}^C_1 \mathcal{O}(U))$. The map Φ is obtained by projecting onto the component of $\mathrm{gr}^C M$ of degree zero. The fact that $\mathbb{L}(\mathfrak{u}^{ab}) \times \mathrm{gr}^C M \to \mathrm{gr}^C M$ is a left action can be expressed as a commutative diagram. Dualising it and projecting onto $C_0 M$ gives precisely the statement of the lemma. $\qquad\square$

2.5.1. Multiplicative structure. Now suppose in addition that M is a commutative k-algebra, and that the action of U respects the multiplication μ on M. An induction on the indices i, j shows that $\mu(C_i M \times C_j M) \subseteq C_{i+j} M$. In particular $C_0 M = M^U$ is a subalgebra of M. Recall that the tensor coalgebra $T^c(V)$ is equipped with a commutative product m called the shuffle product.

Lemma 2.9. *If M is a commutative k-algebra, then Φ is a homomorphism of graded commutative k-algebras, where $T^c(V)$ is equipped with the shuffle product.*

Proof. Since Δ is a homomorphism,

$$\Delta^r(xy) = \Delta^r(x)(y \otimes 1) + (x \otimes 1)\Delta^r(y) + \Delta^r(x)\Delta^r(y)$$

for all $x, y \in M$. It follows that δ is a derivation

$$(2.11) \qquad\qquad \delta(xy) = (x \otimes 1)\delta(y) + \delta(x)(y \otimes 1)$$

for $x, y \in \mathrm{gr}^C M$, where multiplication by 1 denotes the identity on $\mathcal{O}(U)_+$. Now denote by ∂ the right deconcatenation map $\partial : T^c(V) \to T^c(V) \otimes V$

which is given by the formula $\partial(v_1 \otimes \cdots \otimes v_n) = (v_1 \otimes \cdots \otimes v_{n-1}) \otimes v_n$. It follows from the definition of the map Φ as the iteration of δ that

$$\partial \Phi = (\Phi \otimes \mathrm{id})\delta.$$

Suppose that $\Phi(ab) = \Phi(a) \, \underline{\mathrm{m}} \, \Phi(b)$ for all $a, b \in \mathrm{gr}^C M$ of total degree $< n$. If a or b is in $\mathrm{gr}_0^C M$ then the statement is trivial. Then for $x, y \in \mathrm{gr}^C M$ of total degree n and degree ≥ 1, we have

$$\partial \Phi(xy) = (\Phi \otimes \mathrm{id})\delta(xy) = (\Phi \otimes \mathrm{id})\big((x \otimes 1)\delta(y) + \delta(x)(y \otimes 1)\big).$$

By induction hypothesis applied to $\Phi \otimes \mathrm{id}$, the right hand side is

$$(\Phi \otimes \mathrm{id})(x \otimes 1) \, \underline{\mathrm{m}} \, (\Phi \otimes \mathrm{id})\delta(y) + (\Phi \otimes \mathrm{id})(y \otimes 1) \, \underline{\mathrm{m}} \, (\Phi \otimes \mathrm{id})\delta(x)$$

which by the previous equation gives $\partial \Phi(xy) = \Phi(x) \, \underline{\mathrm{m}} \, \partial \Phi(y) + \Phi(y) \, \underline{\mathrm{m}} \, \partial \Phi(x)$. This is in fact one of the many equivalent definitions of the shuffle product, and proves, by the injectivity of ∂, that $\Phi(xy) = \Phi(x) \, \underline{\mathrm{m}} \, \Phi(y)$. $\qquad \square$

Another way to see this lemma is simply to note that the action (2.10) respects the multiplication on gr_M^C and to encode this by a commutative diagram. The dual diagram, after projecting to $C_0 M$ in the appropriate place implies the lemma.

Remark 2.10. One can think of $C_i \mathcal{O}(U)$ as functions of 'unipotent monodromy' of degree i in the following way. For any $f \in \mathcal{O}(U)$, viewed as a function on groups $U(R)$, for all commutative k-algebras R, define a new function $M_u f$ by $(M_u f)(x) = f(ux) - f(x)$. An $f \in C_i(\mathcal{O}(U))$ satisfies $M_{u_1} \cdots M_{u_n} f = 0$ for all $u_1, \ldots, u_n \in U(R)$ whenever $n \geq i + 1$. In particular, elements of $C_0 \mathcal{O}(U)$ are constant, and elements of $C_1 \mathcal{O}(U)$ are functions f satisfying $f(ab) = f(a) + f(b)$.

2.5.2. Cohomological interpretation. Let M, U be as above. Consider the normalised cobar complex, dual to ([49], page 283),

$$0 \to M \to M \otimes_k \mathcal{O}(U) \to M \otimes_k \mathcal{O}(U)_+ \otimes_k \mathcal{O}(U)$$
$$\to M \otimes_k \mathcal{O}(U)_+^{\otimes 2} \otimes_k \mathcal{O}(U) \to \cdots$$

where the $(n+1)^{\mathrm{th}}$ arrow is given by

$$d_n = \sum_{i=0}^{n-1} (-i)^i \, \mathrm{id}^{\otimes i} \otimes \Delta \otimes \mathrm{id}^{\otimes n-1-i},$$

(e.g., $d_1 = \Delta$, and $d_2 = \Delta \otimes \mathrm{id} - \mathrm{id} \otimes \Delta$), followed by projection onto $\mathcal{O}(U)_+$ in the appropriate places. It is a resolution of M in the category of $\mathcal{O}(U)$-comodules. Derive the functor $\otimes_{\mathcal{O}(U)} k$, where k is viewed as an $\mathcal{O}(U)$-module via the augmentation map, by applying it to the previous complex with the first two terms removed. It defines a complex

$$\mathcal{R}_M : 0 \longrightarrow M \xrightarrow{\Delta} M \otimes_k \mathcal{O}(U)_+ \longrightarrow M \otimes_k \mathcal{O}(U)_+^{\otimes 2} \longrightarrow \cdots ,$$

with essentially the same differentials d_n as before. Since the category of $\mathcal{O}(U)$-comodules is equivalent to the category of representations of U of finite type, we deduce that

$$H^n(\mathcal{R}_M) = \mathrm{Ext}^n_{\mathcal{O}(U)-\mathrm{comod}}(k, M) = \mathrm{Ext}^n_{\mathrm{Rep}(U)}(k, M) = H^n(U; M).$$

Therefore $H^0(U; M) = H^0(\mathcal{R}_M) = C_0 M$, since the image of Δm vanishes in $M \otimes \mathcal{O}(U)_+$ if and only if $\Delta m = m \otimes 1$. In the special case $M = k$, we have

$$\mathcal{R}_k : 0 \longrightarrow k \xrightarrow{0} \mathcal{O}(U)_+ \xrightarrow{\Delta'} \mathcal{O}(U)_+^{\otimes 2} \longrightarrow \cdots$$

and it follows that $H^0(U; k) = k$, and

$$(2.12) \qquad\qquad H^1(U; k) = \mathrm{gr}_1^C(\mathcal{O}(U)).$$

This is a cohomological interpretation for the terms in the right-hand side of the decomposition map Φ. The map δ can in fact be viewed as a differential in a certain spectral sequence, as we shall show below.

Lemma 2.11. *Viewing $\mathcal{O}(U)$ as a left U-module (right $\mathcal{O}(U)$-comodule)*

$$H^0(U; \mathcal{O}(U)) = k \quad and \quad H^n(U; \mathcal{O}(U)) = 0 \text{ for all } n \geq 1.$$

Proof. Take the cobar resolution with $M = k$ and reverse all tensors to give

$$0 \to k \to \mathcal{O}(U) \to \mathcal{O}(U) \otimes_k \mathcal{O}(U)_+ \to \mathcal{O}(U) \otimes_k \mathcal{O}(U)_+^{\otimes 2} \to \cdots$$

It is a resolution for the same reasons as the cobar resolution. It agrees with $\mathcal{R}_{\mathcal{O}(U)}$ from the second term onwards, and has the same differentials up to a possible overall sign, so we can read off the cohomology of $\mathcal{O}(U)$. $\square$

Proposition 2.12. *Suppose that U has cohomological dimension 1. Then*

$$0 \longrightarrow \mathrm{gr}_n^C M \xrightarrow{\;\delta\;} \mathrm{gr}_{n-1}^C M \otimes_k H^1(U; k) \longrightarrow \mathrm{gr}_{n-1}^C H^1(U; M) \longrightarrow 0$$

is exact for all $n \geq 1$.

Proof. Filter the complex $\mathcal{R}_M$ by $F^p \mathcal{R}_M = \mathcal{R}_{C_{-p}M}$. It defines a spectral sequence with $E^0_{p,q} = \mathrm{gr}_F^p \mathcal{R}_M^{p+q}$ and $E^1_{p,q} = H^{p+q}(U; \mathrm{gr}_{-p}^C M)$, and converges to $\mathrm{gr}_F^p H^{p+q}(U; M)$. Since $\mathrm{gr}^C M$ is a trivial U-module we have

$$E^1_{p,q} = \mathrm{gr}_{-p}^C M \otimes_k H^{p+q}(U; k).$$

The differential $d^1 : E^1_{-p,p} \to E^1_{1-p,p}$ is the operator δ defined earlier. Since $H^j(U; k)$ vanishes for all $j \geq 2$, $E^1_{p,q}$ vanishes unless $p + q \in \{0, 1\}$ and the spectral sequence degenerates. Therefore the following sequence is exact:

$$0 \to \mathrm{gr}_F^{-n} H^0(U; M) \to \mathrm{gr}_n^C M \xrightarrow{\;\delta\;} \mathrm{gr}_{n-1}^C M \otimes_k H^1(U; k)$$
$$\to \mathrm{gr}_F^{1-n} H^1(U; M) \to 0.$$

The result follows since $H^0(U; M) = C_0 M$ and $\mathrm{gr}_F^{-n} H^0(U; M) = 0$ if $n \geq 1$. $\qquad\square$

Corollary 2.13. *Suppose that $M = T \otimes_k \mathcal{O}(U)$, where T is a trivial U-module, and U has cohomological dimension 1. Then the map Φ is an isomorphism*

$$\Phi : \mathrm{gr}^C M \xrightarrow{\;\sim\;} T \otimes_k T^c(H^1(U; k)).$$

Proof. By the previous lemma and proposition, $H^1(U; M) = H^1(U; \mathcal{O}(U)) \otimes_k T = 0$, and therefore $\delta : \mathrm{gr}_n^C M \to \mathrm{gr}_{n-1}^C M \otimes_k H^1(U; k)$ is surjective as the last term in the exact sequence of the previous proposition vanishes. The iterations of δ are therefore also surjective, hence so is Φ. $\qquad\square$

3. Motivic periods over $\mathbb{Q}$

For the rest of this section, we only require the results of §2.2 and §2.3 in the case $k = B_1 = B_2 = \mathbb{Q}$.

3.1. Category $\mathcal{H}$ of Betti and de Rham realisations

Based on [21] §1.10, consider the category $\mathcal{H}$ whose objects are triples (V_B, V_{dR}, c) consisting of the following data:

1) A finite-dimensional $\mathbb{Q}$-vector space V_B with a finite increasing (weight) filtration $W_\bullet V_B$.

2) A finite-dimensional $\mathbb{Q}$-vector space V_{dR} with a finite increasing (weight) filtration $W_\bullet V_{dR}$ and a finite decreasing (Hodge) filtration $F^\bullet V_{dR}$.

3) An isomorphism

$$c : V_{dR} \otimes_\mathbb{Q} \mathbb{C} \xrightarrow{\sim} V_B \otimes_\mathbb{Q} \mathbb{C}$$

which respects the filtrations $W_\bullet$ on both sides.

4) A linear involution $F_\infty : V_B \xrightarrow{\sim} V_B$ called the real Frobenius.

This data is subject to the conditions:

- if c_{dR} (resp. c_B) is the $\mathbb{C}$-antilinear involution on $V_{dR} \otimes \mathbb{C}$ (resp. $V_B \otimes \mathbb{C}$) given by $x \otimes \lambda \mapsto x \otimes \overline{\lambda}$, then the following diagram commutes:

$$
\begin{array}{ccc}
V_{dR} \otimes \mathbb{C} & \xrightarrow{c} & V_B \otimes \mathbb{C} \\
\downarrow{c_{dR}} & & \downarrow{F_\infty \otimes c_B} \\
V_{dR} \otimes \mathbb{C} & \xrightarrow{c} & V_B \otimes \mathbb{C}
\end{array}
$$

In particular, $c\, c_{dR}\, c^{-1}$ preserves the lattice $V_B \subset V_B \otimes \mathbb{C}$.

- that V_B, equipped with the weight filtration $W_\bullet$ and Hodge filtration $cF^\bullet$ on $V_\mathbb{C} = V_B \otimes_\mathbb{Q} \mathbb{C}$, is a $\mathbb{Q}$-mixed Hodge structure. Writing F instead of cF, this is equivalent to $\mathrm{gr}_n^W V_\mathbb{C} = \bigoplus_{p+q=n} F^p \cap \overline{F}^q$. We assume furthermore that this mixed Hodge structure is graded-polarizable.

The morphisms in the category $\mathcal{H}$ are given by morphisms of triples respecting the above data. It is shown in [21], 1.10, that $\mathcal{H}$ is a Tannakian category, the essential point being that mixed Hodge structures form an abelian category [23]. This category could be further enriched by adding more realisations.

By (3), the weight filtration defines a filtration on (V_B, V_{dR}, c) by subobjects:

$$(3.1) \qquad W_n(V_B, V_{dR}, c) = (W_n V_B, W_n V_{dR}, c|_{W_n}).$$

Denote the *Hodge numbers* of an object $V = (V_B, V_{dR}, c)$ in $\mathcal{H}$ by

$$(3.2) \qquad h_{p,q}(V) = \dim_\mathbb{Q}(\mathrm{gr}_{p+q}^W \cap F^p)V^{dR} = \dim_\mathbb{C}(F^p \cap \overline{F}^q)V_\mathbb{C}.$$

The Tate objects $\mathbb{Q}(n)$, where $n \in \mathbb{Z}$, are the unique triples $(\mathbb{Q}, \mathbb{Q}, c)$ such that $c(1) = (2\pi i)^{-n}$, with weight $-2n$ and Hodge filtration on the second vector space $\mathbb{Q}$ defined by $F^{-n}\mathbb{Q} = \mathbb{Q}$, $F^{1-n}\mathbb{Q} = 0$.

3.2. The ring of $\mathcal{H}$-periods

The category $\mathcal{H}$ has two fiber functors

$$\omega_\bullet : \mathcal{H} \longrightarrow \mathrm{Vec}_\mathbb{Q} \qquad \bullet = B \text{ or } dR$$
$$(V_B, V_{dR}, c) \mapsto V_\bullet,$$

so we can apply §2.2 with $\mathcal{T} = \mathcal{H}$, $k = \mathbb{Q}$. Let us write[3]

$$\mathfrak{m} = (\omega_B, \omega_{dR}) \quad \text{and} \quad \mathfrak{dr} = (\omega_{dR}, \omega_{dR}).$$

This defines rings $\mathcal{P}_\mathcal{H}^{\mathfrak{m}}$ and $\mathcal{P}_\mathcal{H}^{\mathfrak{dr}}$ as in §2.2 and §2.3, and a canonical element

$$c \in \mathrm{Isom}_\mathcal{H}^\otimes(\omega_{dR}, \omega_B)(\mathbb{C})$$

which is given by the data (3).

Definition 3.1. The ring of $\mathcal{H}$-periods is $\mathcal{P}_\mathcal{H}^{\mathfrak{m}}$. It is equipped with

- a period homomorphism

$$\mathrm{per} : \mathcal{P}_\mathcal{H}^{\mathfrak{m}} \longrightarrow \mathbb{C}$$

  which sends $[(V_B, V_{dR}, c), \sigma, \omega]^{\mathfrak{m}}$ to $\sigma c(\omega)$.

- an increasing weight filtration

$$W_n \mathcal{P}_\mathcal{H}^{\mathfrak{m}} = \langle [(V_B, V_{dR}, c), \sigma, \omega]^{\mathfrak{m}} : \text{ where } \omega \in W_n V_{dR} \rangle_\mathbb{Q}$$

  Equivalently, this is the subspace spanned by $[M, \sigma, \omega]^{\mathfrak{m}}$ for objects M in $\mathcal{H}$ satisfying $W_n M = M$ where W_n was defined in (3.1).

- a right coaction $\Delta^{\mathfrak{m}} : \mathcal{P}_\mathcal{H}^{\mathfrak{m}} \longrightarrow \mathcal{P}_\mathcal{H}^{\mathfrak{m}} \otimes_\mathbb{Q} \mathcal{P}_\mathcal{H}^{\mathfrak{dr}}$ or equivalently, a left action

$$G_\mathcal{H}^{dR} \times \mathcal{P}_\mathcal{H}^{\mathfrak{m}} \longrightarrow \mathcal{P}_\mathcal{H}^{\mathfrak{m}}.$$

  It respects the weight filtration on $\mathcal{P}_\mathcal{H}^{\mathfrak{m}}$ by (3.1). Likewise, $\mathcal{P}_\mathcal{H}^{\mathfrak{m}}$ admits a right action of the Betti Galois group $G_\mathcal{H}^B$ which also preserves W.

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[3]In [12, 16] I wrote $\mathfrak{m} = (\omega_{dR}, \omega_B)$, and put the coaction on the left, for purely psychological reasons. The corresponding rings of periods are identical.

• the real Frobenius involution

$$F_\infty : \mathcal{P}_\mathcal{H}^\mathfrak{m} \xrightarrow{\sim} \mathcal{P}_\mathcal{H}^\mathfrak{m}$$

defined by $F_\infty[M, \sigma, \omega]^\mathfrak{m} = [M, \sigma \circ F_\infty, \omega]^\mathfrak{m}$. It has the property that

$$\mathrm{per}(F_\infty \xi) = \overline{\mathrm{per}(\xi)},$$

where the bar denotes complex conjugation. In particular, F_∞-invariant motivic periods have periods in $\mathbb{R}$. The $G_\mathcal{H}^{dR}$-action commutes with F_∞.

Note that, since $\mathcal{P}_\mathcal{H}^\mathfrak{m}$ admits a $G_\mathcal{H}^{dR}$-action, it is the ω_{dR} image of an (ind-)object of the category $\mathcal{H}$ via Theorem 2.3. Therefore $\mathcal{P}_\mathcal{H}^\mathfrak{m}$ also carries, in addition to the weight filtration, a decreasing Hodge filtration F. The subspace $F^n \mathcal{P}_\mathcal{H}^\mathfrak{m}$ is spanned by the $[M, \sigma, v]^\mathfrak{m}$ with $v \in F^n M_{dR}$. The Hodge filtration is not preserved by the group $G_\mathcal{H}^{dR}$, but will nonetheless play a role later on, and is of course preserved by the (right) action of the Betti Galois group $G_\mathcal{H}^B = Aut_\mathcal{H}^\otimes(\omega_B)$.

3.3. Some variants

Denote the subspace $\mathcal{P}_\mathcal{H}^{\mathfrak{m},+} \subset \mathcal{P}_\mathcal{H}^\mathfrak{m}$ of *effective $\mathcal{H}$-periods* to be the subspace of $\mathcal{H}$-periods of objects with non-negative Hodge numbers

$$\mathcal{P}_\mathcal{H}^{\mathfrak{m},+} = \langle [M, \sigma, \omega]^\mathfrak{m} : M \in Ob(\mathcal{H}) \text{ such that } h_{p,q}(M) = 0 \text{ unless } p, q \geq 0 \rangle_\mathbb{Q}$$

It forms a ring and is stable under the action of $G_\mathcal{H}^{dR} \times G_\mathcal{H}^B$.

Similarly, the ring of *mixed Artin-Tate $\mathcal{H}$-periods*[4] is the subspace

$$\mathcal{P}_{HT}^\mathfrak{m} = \langle [M, \sigma, \omega]^\mathfrak{m} : M \in Ob(\mathcal{H}) \text{ such that } F^p W_{p+q} M_{dR} = 0 \text{ if } p > q \rangle_\mathbb{Q}.$$

It is again a ring and is stable under the action of $G_\mathcal{H}^{dR} \times G_\mathcal{H}^B$. The notation is justified as follows: let $HT \subset \mathcal{H}$ be the full Tannakian subcategory of $\mathcal{H}$ consisting of triples (V_B, V_{dR}, c), whose underlying mixed Hodge structure has Hodge numbers $h_{p,q} = 0$ if $p \neq q$. Its ring of motivic periods is $\mathcal{P}_{HT}^\mathfrak{m}$. Suppose that M is any object of $\mathcal{H}$ of this type. Then the weight filtration

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[4]the ring of mixed Tate periods corresponds to the $\mathcal{H}$-periods of mixed Tate objects: those whose associated weight-graded is a direct sum of Tate objects $\mathbb{Q}(n)$

on its de Rham vector space splits via

$$\mathrm{gr}^W_{2n} M_{dR} = W_{2n} \cap F^n M_{dR}.$$

It follows that $\mathcal{P}^{\mathfrak{m}}_{HT}$ is also graded by the weight and we can write

$$\mathcal{P}^{\mathfrak{m}}_{HT} = \bigoplus_{n \in \mathbb{Z}} \mathrm{gr}^W_{2n} \mathcal{P}^{\mathfrak{m}}_{HT}.$$

Note that the weight grading is not preserved by $G^{dR}_{\mathcal{H}}$, only the weight filtration. The weight grading is, however, preserved by $G^B_{\mathcal{H}}$ since the latter preserves the Hodge filtration.

Combining the two, we have a ring of *effective mixed Artin Tate $\mathcal{H}$-periods*

$$\mathcal{P}^{\mathfrak{m},+}_{HT} = \mathcal{P}^{\mathfrak{m},+}_{\mathcal{H}} \cap \mathcal{P}^{\mathfrak{m}}_{HT}.$$

One can show that it is the largest subalgebra of $\mathcal{P}^{\mathfrak{m}}_{HT}$ which is stable under the action of $G^{dR}_{\mathcal{H}}$ and has non-negative weights.

The ring of $\mathcal{H}$-*de Rham* periods $\mathcal{P}^{\mathfrak{dr}}_{\mathcal{H}}$ has similar properties to $\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$ (weight filtration, left $G^{dR}_{\mathcal{H}}$ action), except that it does not have a real Frobenius involution, and the period map is replaced by the evaluation map (counit)

$$\begin{aligned} \mathrm{ev} : \mathcal{P}^{\mathfrak{dr}}_{\mathcal{H}} &\longrightarrow k \\ [M, e, v]^{\mathfrak{dr}} &\mapsto e(v), \end{aligned}$$

which is nothing other than evaluation on the element $1 \in G^{dR}_{\mathcal{H}}(k)$. There are analogous effective and Artin-Tate versions.

Remark 3.2. The fact that the weight filtration is strict on the category of mixed Hodge structures [23] implies that the functor

$$(V_B, V_{dR}, c) \mapsto \mathrm{gr}^W_\bullet(V_B, V_{dR}, c) = (\mathrm{gr}^W V_B, \mathrm{gr}^W V_{dR}, \mathrm{gr}^W c)$$

is exact. Thus, by composing with ω_{dR} or ω_B one obtains new fiber functors we denote by $\underline{\omega_{dR}} = \omega_{dR}\,\mathrm{gr}^W$ and $\underline{\omega_B} = \omega_B\,\mathrm{gr}^W$.

3.4. Weight filtrations on $\mathcal{P}^{\mathfrak{dr}}_{\mathcal{H}}$

The ring $\mathcal{P}^{\mathfrak{dr}}_{\mathcal{H}}$ is none other than $\mathcal{O}(G^{dR}_{\mathcal{H}})$ viewed as a left $G^{dR}_{\mathcal{H}}$-module (or right $\mathcal{O}(G^{dR}_{\mathcal{H}})$-comodule). It could be written $\mathcal{P}^{l,\mathfrak{dr}}_{\mathcal{H}}$ to distinguish from $\mathcal{P}^{r,\mathfrak{dr}}_{\mathcal{H}}$ and $\mathcal{P}^{c,\mathfrak{dr}}_{\mathcal{H}}$ which are the same vector space, but considered with the right

(respectively conjugation) action of $G_{\mathcal{H}}^{dR}$. We shall never consider $\mathcal{P}^{r,\partial\mathfrak{r}}$, except to remark that the antipode (inversion in $G_{\mathcal{H}}^{dR}$) interchanges $\mathcal{P}_{\mathcal{H}}^{l,\partial\mathfrak{r}}$ and $\mathcal{P}_{\mathcal{H}}^{r,\partial\mathfrak{r}}$. By the Tannaka theorem, these rings all define ind-objects in $\mathcal{H}$ and in particular are equipped with weight filtrations. It is important to note that, since the $G_{\mathcal{H}}^{dR}$-action is different in each case, these structures are distinct.

To avoid ambiguity, one can distinguish the following two coactions:

$$\Delta^{\mathfrak{m},l} : \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}} \longrightarrow \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}} \otimes_{\mathbb{Q}} \mathcal{P}_{\mathcal{H}}^{l,\partial\mathfrak{r}}$$
$$\Delta^{\mathfrak{m},c} : \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}} \longrightarrow \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}} \otimes_{\mathbb{Q}} \mathcal{P}_{\mathcal{H}}^{c,\partial\mathfrak{r}}$$

They are given by an identical formula, namely (2.2), but differ in that the interpretation of the right-hand side is slightly different. If G is a group acting on a set X, the former corresponds to the action of G on $G \times X$ via $g(h,x) = (gh,x)$, where $g,h \in G$ and $x \in X$. This is the usual formula for a left action. It satisfies $\Delta^{\mathfrak{m},l}(g\xi) = (\mathrm{id} \otimes g)\Delta^{\mathfrak{m},l}\xi$ for $g \in G_{\mathcal{H}}^{dR}$ and hence $\Delta^{\mathfrak{m},l}W_n\mathcal{P}_{\mathcal{H}}^{\mathfrak{m}} \subset \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}} \otimes_{\mathbb{Q}} W_n\mathcal{P}_{\mathcal{H}}^{l,\partial\mathfrak{r}}$.

The second coaction $\Delta^{\mathfrak{m},c}$ corresponds to the action of G on $G \times X$ given by the formula $g(h,x) = (ghg^{-1}, gx)$. Therefore

$$\Delta^{\mathfrak{m},c}(g\xi) = (g \otimes c_g)\Delta^{\mathfrak{m},c}\xi \quad \text{for } g \in G_{\mathcal{H}}^{dR}$$

where c_g is conjugation by g. In this case we have

$$\Delta^{\mathfrak{m},c}W_n\mathcal{P}_{\mathcal{H}}^{\mathfrak{m}} \subset \sum_{i+j=n} W_i\mathcal{P}_{\mathcal{H}}^{\mathfrak{m}} \otimes_{\mathbb{Q}} W_j\mathcal{P}_{\mathcal{H}}^{c,\partial\mathfrak{r}}.$$

Note that the ring $\mathcal{P}_{\mathcal{H}}^{l,\partial\mathfrak{r}}$ has elements in negative weights, but that

$$(3.3) \qquad\qquad W_{-1}\mathcal{P}_{\mathcal{H}}^{c,\partial\mathfrak{r}} = 0.$$

To see this, observe that the canonical map (2.1)

$$(3.4) \qquad\qquad \omega_{dR}(M)^{\vee} \otimes_{\mathbb{Q}} \omega_{dR}(M) \to \mathcal{P}_{\mathcal{H}}^{c,\partial\mathfrak{r}}$$

is compatible with the action of $G_{\mathcal{H}}^{dR}$, and hence respects the weight filtration on both sides. The left-hand side can be identified with $\mathrm{End}(\omega_{dR}(M))^{\vee}$. Since elements of $G_{\mathcal{H}}^{dR}$ preserve the weight filtration on $\omega_{dR}(M)$ the natural transformation $G_{\mathcal{H}}^{dR} \to \mathrm{End}(\omega_{dR}(M))$ of functors from commutative $\mathbb{Q}$-algebras to sets, lands in $W_0\mathrm{End}(\omega_{dR}(M))$, so dually, the image of $W_{-1}(\omega_{dR}(M)^{\vee} \otimes_{\mathbb{Q}} \omega_{dR}(M))$ under (3.4) is zero. Since the direct sum of the maps (3.4) generates $\mathcal{P}_{\mathcal{H}}^{c,\partial\mathfrak{r}}$, it follows that $W_{-1}\mathcal{P}_{\mathcal{H}}^{c,\partial\mathfrak{r}} = 0$.

3.5. Some periods which could be called motivic

The period homomorphism per $: \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}} \to \mathbb{C}$ is surjective since any complex number can be obtained as a period of a mixed Hodge structure. A *motivic period* will be an element of $\mathcal{P}_{\mathcal{H}}^{\mathfrak{m}}$ which comes from the cohomology of an algebraic variety in a precise way. The following family of examples is sufficient for our purposes and covers many cases considered in [41].

Example 3.3. Let X be a smooth scheme over $\mathbb{Q}$ and $D \subset X$ a normal crossing divisor over $\mathbb{Q}$. Consider the triple consisting of

- relative Betti cohomology $H_B^n(X, D) = H^n(X(\mathbb{C}), D(\mathbb{C}); \mathbb{Q})$. Since complex conjugation on the topological spaces $X(\mathbb{C}), D(\mathbb{C})$ is continuous it defines an involution $F_\infty : H_B^n(X, D) \xrightarrow{\sim} H_B^n(X, D)$.

- relative algebraic de Rham cohomology $H_{dR}^n(X, D)$. It is the hypercohomology of the sheaf of Kähler differential forms on a cosimplicial variety constructed out of the irreducible components of D.

- the comparison isomorphism ([34])

$$\mathrm{comp}_{B,dR} : H_{dR}^n(X, D) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H_B^n(X, D) \otimes_{\mathbb{Q}} \mathbb{C}.$$

It follows from the existence of a natural mixed Hodge structure [23, 24] that

$$(3.5) \qquad H^n(X, D) := (H_B^n(X, D), H_{dR}^n(X, D), \mathrm{comp}_{B,dR})$$

is an object in the category $\mathcal{H}$. Given a cohomology class $\omega \in H_{dR}^n(X, D)$ and a relative homology cycle $\sigma \in (H_B^n(X, D))^\vee$ we can define the *motivic period* associated to this data to be the matrix coefficient

$$(3.6) \qquad [H^n(X, D), \sigma, \omega]^{\mathfrak{m}} \in P_{\mathcal{H}}^{\mathfrak{m}}.$$

Its period $\sigma(\mathrm{comp}_{B,dR} \, \omega) \in \mathbb{C}$ could be written $\int_\sigma \omega$ and is given by a linear combination of integrals. The Hodge numbers $h_{p,q}$ of $H^n(X, D)$ are all zero unless $0 \leq p, q \leq 2n$, so (3.6) is effective and lies in $W_{2n}\mathcal{P}_{\mathcal{H}}^{\mathfrak{m},+}$.

Definition 3.4. The space of *effective motivic periods* $\mathcal{P}^{\mathfrak{m},+}$ is the subspace of $\mathcal{P}_{\mathcal{H}}^{\mathfrak{m},+}$ spanned by the elements (3.6).

We shall not need to define a ring of motivic periods which are not effective (we shall never write $\mathcal{P}^{\mathfrak{m}}$ except in this sentence).

This working definition of effective motivic periods amply suffices for many purposes (e.g., the constant cosmic Galois group [13]). By the Künneth formula, $\mathcal{P}^{\mathfrak{m},+}$ is closed under multiplication and it is immediate from (2.2) that it is closed under the action of $G_{\mathcal{H}}^{dR}$ and $G_{\mathcal{H}}^{B}$. Likewise, the ring of effective de Rham periods $\mathcal{P}^{\mathfrak{dr},+} \subset \mathcal{P}_{\mathcal{H}}^{\mathfrak{dr},+}$ is the subspace spanned by $[H^n(X, D), v, \omega]^{\mathfrak{dr}}$, where $v \in H_{dR}^n(X, D)^{\vee}$.

Now define G^{dR} to be the quotient of $G_{\mathcal{H}}^{dR}$ by the subgroup which acts trivially on the ring $\mathcal{P}^{\mathfrak{m},+}$ of effective motivic periods. The affine group scheme G^{dR} acts faithfully on $\mathcal{P}^{\mathfrak{m},+}$, and is an approximation to a motivic Galois group. Its category of representations $\operatorname{Rep} G^{dR}$ is a crude version of a Tannakian category of mixed motives. A key point is that the groups G^{dR} and $G_{\mathcal{H}}^{dR}$ act in an identical manner on $\mathcal{P}^{\mathfrak{m},+}$.

A folklore version of Grothendieck's period conjecture states that

Conjecture 1. The period homomorphism per $: \mathcal{P}^{\mathfrak{m},+} \to \mathbb{C}$ is injective.

Note that this conjecture is weaker than the analoguous conjecture for the motivic periods of the category of Nori motives, for example.

3.6. Some terminology

We list a sample of possible quantities to describe $\mathcal{H}$ periods. For any object M in $\mathcal{H}$ one can make the following definitions.

- Let M_B^+, M_B^- denote the $\pm$ eigenspaces for F_∞, and set $\operatorname{rank}^{\pm} M = \dim_{\mathbb{Q}} M_B^{\pm}$. The comparison $M_{dR} \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} M_B \otimes_{\mathbb{Q}} \mathbb{C}$ implies that

$$\operatorname{rank}(M) = \operatorname{rank}^+(M) + \operatorname{rank}^-(M) = \dim_{\mathbb{Q}} M_{dR}.$$

- Define the *de Rham Galois group* $G^{dR}(M)$ of M to be the largest quotient of $G_{\mathcal{H}}^{dR}$ which acts faithfully on M_{dR}. Equivalently, it is the de Rham Tannaka group of the full Tannakian subcategory of $\mathcal{H}$ generated by M. Define the *Betti Galois group* in the same way on replacing dR by B. The comparison map gives a canonical isomorphism

$$G^B(M) \times \mathbb{C} \xrightarrow{\sim} G^{dR}(M) \times \mathbb{C}.$$

Define the *transcendence dimension* of M to be

$$\dim_{\operatorname{tr}}(M) = \dim G^{dR}(M) = \dim G^B(M).$$

Define the *component group* $\pi_0^\bullet(M)$ to be $\pi_0(G^\bullet(M))$, where $\bullet = B, dR$ and π_0 is the étale group scheme whose affine ring $\mathcal{O}(\pi_0^\bullet(M))$ is the largest separable subalgebra of $\mathcal{O}(G^\bullet(M))$ (see [48], §6.5-7). Since π_0 commutes with change of base field, the comparison isomorphism gives

$$\pi_0^B(M) \times \mathbb{C} \xrightarrow{\sim} \pi_0^{dR}(M) \times \mathbb{C}.$$

- Define the *Hodge polynomial* of M to be (see (3.2))

$$h(M)(r,s) = \sum_{r,s} h_{p,q}(M)\, r^p s^q \ \in \ \mathbb{Z}[r^\pm, s^\pm]$$

- Define the class of a *period matrix* of M as follows. Let $M = (M_B, M_{dR}, c_M)$, and $r = \operatorname{rank}(M)$. Choose isomorphisms $M_B \cong \mathbb{Q}^r$ and $M_{dR} \cong \mathbb{Q}^r$, which are adapted to the weight (resp. Hodge and weight) filtrations, and write c_M in this basis. This gives a well-defined element

$$[c_M] \ \in \ W_0\mathrm{GL}(M_B)\backslash W_0\mathrm{GL}(\mathbb{C}^r)/F^0 W_0\mathrm{GL}(M_{dR})$$

where $W_0\mathrm{GL}(M_B)$ denotes the subgroup of $\mathrm{GL}(M_B)(\mathbb{Q})$ which preserves W, and so on. It is an equivalence class of square $d \times d$ matrices of complex numbers, where d is the rank of M. The matrix c_M in fact lies in the subspace satisfying $F_\infty c_M = \bar{c}_M$. Thus if we furthermore choose our Betti basis $M_B \cong \mathbb{Q}^r$ to be compatible with the decomposition $M_B = M_B^+ \oplus M_B^-$, then the rows corresponding to M_B^+ have entries in $\mathbb{R}$, and those corresponding to M_B^- have entries in $i\mathbb{R}$.

   The *determinant* $\det(M)$ is defined to be $\det(c_M)$ in $\mathbb{C}/\mathbb{Q}^\times$.

Definition 3.5. Now let $\xi \in \mathcal{P}_\mathcal{H}^\mathfrak{m}$ and denote its minimal object (§2.4) by $M(\xi)$. This enables us to attach the following invariants to ξ:

1) Define the *space of de Rham Galois conjugates* of ξ to be the right $\mathcal{O}(G_\mathcal{H}^{dR})$-subcomodule of $\mathcal{P}_\mathcal{H}^\mathfrak{m}$ generated by ξ. It is isomorphic to $M(\xi)_{dR}$ by Corollary 2.5.

   Likewise, define the *space of Betti Galois conjugates* to be the left $\mathcal{O}(G_\mathcal{H}^B)$-subcomodule of $\mathcal{P}_\mathcal{H}^\mathfrak{m}$ generated by ξ. It is isomorphic to $M(\xi)_B$ by Corollary 2.5.

   Define the *space of biconjugates* of ξ to be the left $\mathcal{O}(G_\mathcal{H}^B)$ and right $\mathcal{O}(G_\mathcal{H}^{dR})$-subcomodule of $\mathcal{P}_\mathcal{H}^\mathfrak{m}$ generated by ξ. It is the vector space spanned by the set of all matrix coefficients $[M(\xi), M(\xi)_B^\vee, M(\xi)_{dR}]^\mathfrak{m}$.

Define the *algebra of de Rham/Betti/bi-conjugates of* ξ to be the subalgebras of $\mathcal{P}_{\mathcal{H}}^{\mathfrak{m}}$ generated by the above vector spaces.[5]

2) Define the $\mathrm{rank}(\xi) := \mathrm{rank}\, M(\xi)$ (similarly $\mathrm{rank}^{\pm}(\xi) = \mathrm{rank}^{\pm} M(\xi)$).

3) Define the de Rham (resp. Betti) *Galois group* $G_{\xi}^{\bullet} = G^{\bullet}(M(\xi))$, where $\bullet \in \{B, dR\}$. Define the *transcendence dimension* of ξ to be

$$\dim_{\mathrm{tr}}(\xi) = \dim G_{\xi}^{dR} = \dim G_{\xi}^{B}.$$

Define the *component group* to be $\pi_0^{\bullet}(\xi) := \pi_0(G_{\xi}^{\bullet})$, and the *degree* (for want of a better word) of ξ to be $\deg(\xi) = \left| \pi_0^{\bullet}(\xi)(\mathbb{C}) \right| \in \mathbb{N}$.

4) Define the *Hodge polynomial* of ξ to be $h(\xi)(r,s) = h(M(\xi))(r,s)$.

5) Define the class of a *period matrix* of ξ to be $[c_{\xi}] = [c_{M(\xi)}]$, and define the *determinant* to be $\det(\xi) = \det(c_{\xi})$.

Many of the above definitions go through for an $\mathcal{H}$-de Rham period $\xi \in \mathcal{P}_{\mathcal{H}}^{\mathfrak{dr}}$ with the obvious changes, which we leave to the reader.

The Hodge polynomial $h(\xi)$ is symmetric in r, s and satisfies $\mathrm{rank}\, \xi = h(\xi)(1,1)$. The element ξ is *effective* if and only if $h(\xi) \in \mathbb{Z}[r,s]$, and is *mixed Artin-Tate* (lies in $\mathcal{P}_{HT}^{\mathfrak{m}}$) if and only if $\xi \in \mathbb{Z}[(rs)^{\pm 1}]$.

It follows from the formulae for sums and products of matrix coefficients that if $\xi_1, \xi_2 \in \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}}$ then $M_{\xi_1 + \xi_2}$ is a subquotient of $M_{\xi_1} \oplus M_{\xi_2}$ and $M_{\xi_1 \xi_2}$ is a subquotient of $M_{\xi_1} \otimes M_{\xi_2}$. Therefore the rank satisfies

$$\mathrm{rank}(\xi_1 + \xi_2) \leq \mathrm{rank}(\xi_1) + \mathrm{rank}(\xi_2)$$
$$\text{and} \quad \mathrm{rank}(\xi_1 \xi_2) \leq \mathrm{rank}(\xi_1)\, \mathrm{rank}(\xi_2).$$

More generally, the Hodge polynomial satisfies

$$h(\xi_1 + \xi_2) \preccurlyeq h(\xi_1) + h(\xi_2) \quad \text{and} \quad h(\xi_1 \xi_2) \preccurlyeq h(\xi_1) h(\xi_2)$$

where $\preccurlyeq$ means that the inequality $\leq$ holds coefficient by coefficient.

Remark 3.6. The transcendence dimension of ξ is $\dim G_{\xi}^{dR} = \deg_{\mathrm{tr}} \mathcal{O}(G_{\xi}^{dR})$. The latter is the ring generated by the 'de Rham biconjugates' $[M(\xi), v, w]^{\mathfrak{dr}}$ for all $v \in M(\xi)_{dR}^{\vee}$ and $w \in M(\xi)_{dR}$. Applying the comparison map, this is

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[5]One might be tempted, by analogy with algebraic numbers, to define notions of Betti/de Rham/biconjugates of ξ by considering the orbits of ξ under the group of R-points of the corresponding groups $G_{\mathcal{H}}^{B/dR}(R)$, for R a commutative $\mathbb{Q}$-algebra. We shall not.

isomorphic to the ring generated by the biconjugates $[M(\xi), \sigma, w]^{\mathfrak{m}}$ where $\sigma \in M(\xi)_B^\vee$ and $w \in M(\xi)_{dR}$, tensored with $\mathbb{C}$. It follows from this argument that

$$(3.7) \qquad \dim_{\mathrm{tr}}(\xi) = \deg_{\mathrm{tr}}(\text{Ring of biconjugates of } \xi)$$

Therefore the period conjecture 1 implies the following (compare [1]):

Conjecture 2. Let $\xi \in \mathcal{P}^{\mathfrak{m},+}$ be a motivic period. Let $P_\xi \subset \mathbb{C}$ be the $\mathbb{Q}$-algebra generated by the images of the Galois biconjugates of ξ under the period homomorphism. Then the transcendence degree of P_ξ satisfies $\deg_{\mathrm{tr}} P_\xi = \dim_{\mathrm{tr}}(\xi)$.

3.7. Semi-simple and unipotent periods

Let $\mathcal{H}^{ss}$ denote the full Tannakian subcategory of $\mathcal{H}$ generated by semi-simple objects. Define the *ring of semi-simple (or pure) periods* to be $\mathcal{P}_{\mathcal{H}^{ss}}^{\mathfrak{m}}$, and respectively $\mathcal{P}_{\mathcal{H}^{ss}}^{\mathfrak{dr}}$ its de Rham version.

Every object of $\mathcal{H}^{ss}$ is graded by the weight filtration. It follows that $\mathcal{P}_{\mathcal{H}^{ss}}^{\mathfrak{m}}$ is also graded by the weight filtration. The action of the group $G_{\mathcal{H}}^{dR}$ on $\mathcal{P}_{\mathcal{H}^{ss}}^{\mathfrak{m}}$ factors through a quotient we denote by $S_{\mathcal{H}}^{dR}$. It is a projective limit of reductive affine algebraic groups over $\mathbb{Q}$, and there is an exact sequence

$$(3.8) \qquad 1 \longrightarrow U_{\mathcal{H}}^{dR} \longrightarrow G_{\mathcal{H}}^{dR} \longrightarrow S_{\mathcal{H}}^{dR} \longrightarrow 1$$

where $U_{\mathcal{H}}^{dR}$ is pro-unipotent. Define the ring of *unipotent de Rham periods* to be

$$\mathcal{P}_{\mathcal{H}}^{\mathfrak{u}} = \mathcal{O}(U_{\mathcal{H}}^{dR}),$$

equipped with the conjugation action of $G_{\mathcal{H}}^{dR}$. The left action of $U_{\mathcal{H}}^{dR}$ on $\mathcal{P}_{\mathcal{H}}^{\mathfrak{m}}$ is equivalent to a right coaction, which we call the *unipotent de Rham coaction*

$$(3.9) \qquad \Delta^u : \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}} \longrightarrow \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}} \otimes_{\mathbb{Q}} \mathcal{P}_{\mathcal{H}}^{\mathfrak{u}}.$$

It is given by the same formula as (2.2), where elements on the right hand side of the tensor product are viewed as functions on $U_{\mathcal{H}}^{dR}$. This coaction is equivariant with respect to the action of $G_{\mathcal{H}}^{dR}$, i.e., $\Delta^u(g\xi) = (g \otimes c_g)\Delta^u\xi$, where c_g denotes conjugation by $g \in G_{\mathcal{H}}^{dR}$. In particular, $\Delta^u W_n \subset \sum_{i+j=n} W_i \otimes W_j$.

The ring of unipotent periods $\mathcal{P}_{\mathcal{H}}^{\mathfrak{u}}$ is equipped with an *antipode*

$$S : \mathcal{P}_{\mathcal{H}}^{\mathfrak{u}} \to \mathcal{P}_{\mathcal{H}}^{\mathfrak{u}},$$

which is dual to inversion in the group $U_{\mathcal{H}}^{dR}$. Using notation introduced earlier, the restriction gives a natural surjective map $\mathcal{P}_{\mathcal{H}}^{c,\partial\mathfrak{r}} \to \mathcal{P}_{\mathcal{H}}^{\mathfrak{u}}$ which is $G_{\mathcal{H}}^{dR}$ equivariant, and it follows from a previous calculation (3.3) that $\mathcal{P}_{\mathcal{H}}^{\mathfrak{u}}$ has non-negative weights:

$$W_{-1} \mathcal{P}_{\mathcal{H}}^{\mathfrak{u}} = 0.$$

Remark 3.7. If one replaces the de Rham functor with the graded de Rham functor ω_{dR}, then the analogous sequence to (3.8) is canonically split, since gr^W is a fiber functor from $\mathcal{H}$ to $\mathcal{H}^{ss}$ which is the identity on $\mathcal{H}^{ss}$. Thus $G_{\mathcal{H}}^{dR} = U_{\mathcal{H}}^{dR} \rtimes S_{\mathcal{H}}^{dR}$.

Proposition 3.8. *There is a non-canonical isomorphism of algebras*

$$\mathcal{P}_{\mathcal{H}}^{\mathfrak{m}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \cong \mathcal{P}_{\mathcal{H}^{ss}}^{\mathfrak{m}} \otimes_{\mathbb{Q}} \mathcal{P}_{\mathcal{H}}^{\mathfrak{u}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}.$$

It does not respect the coalgebra structure.

Proof. Choose points in $\mathrm{Isom}_{\mathcal{H}}(\omega_B, \omega_{dR})(\overline{\mathbb{Q}})$ and $\mathrm{Isom}_{\mathcal{H}}(\omega_{dR}, \omega_{\underline{dR}})(\overline{\mathbb{Q}})$. They induce isomorphisms over $\overline{\mathbb{Q}}$

$$\mathrm{Isom}_{\mathcal{H}}(\omega_{dR}, \omega_B) \times \overline{\mathbb{Q}} \xrightarrow{\sim} \mathrm{Isom}_{\mathcal{H}}(\omega_{dR}, \omega_{dR}) \times \overline{\mathbb{Q}} \cong \mathrm{Isom}_{\mathcal{H}}(\omega_{\underline{dR}}, \omega_{\underline{dR}}) \times \overline{\mathbb{Q}}.$$

The group in the middle is $G_{\mathcal{H}}^{dR} \times \overline{\mathbb{Q}}$, and the one on the right is $G_{\mathcal{H}}^{dR} \times \overline{\mathbb{Q}}$ which splits canonically since $G_{\mathcal{H}}^{dR} \cong U_{\mathcal{H}}^{dR} \rtimes S_{\mathcal{H}}^{dR}$ by Remark 3.7. On the level of affine rings we deduce isomorphisms of algebras $\mathcal{O}(G_{\mathcal{H}}^{dR}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \cong$ $\mathcal{O}(G_{\mathcal{H}}^{dR}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} = \mathcal{O}(S_{\mathcal{H}}^{dR}) \otimes_{\mathbb{Q}} \mathcal{O}(U_{\mathcal{H}}^{dR}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$. This gives a non-canonical isomorphism of algebras

$$\mathcal{P}_{\mathcal{H}}^{\mathfrak{m}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \cong \mathcal{O}(S_{\mathcal{H}}^{dR}) \otimes_{\mathbb{Q}} \mathcal{O}(U_{\mathcal{H}}^{dR}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}},$$

which, on taking $U_{\mathcal{H}}^{dR}$-invariants, induces $\mathcal{P}_{\mathcal{H}^{ss}}^{\mathfrak{m}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \cong \mathcal{O}(S_{\mathcal{H}}^{dR}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$. The statement follows from the identification $\mathcal{P}_{\mathcal{H}}^{\mathfrak{u}} = \mathcal{O}(U_{\mathcal{H}}^{dR})$. $\square$

3.8. Filtration by unipotency and decomposition

The existence of the weight filtration implies that we can apply the constructions of §2.5 to

$$U = U_{\mathcal{H}}^{dR} \quad \text{and} \quad M = \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}},$$

where $\mathcal{P}_{\mathcal{H}}^{\mathfrak{m}}$ is equipped with the comodule structure $\Delta^u : M \to M \otimes_{\mathbb{Q}} \mathcal{O}(U)$.

Definition 3.9. We shall say that an element ξ in $\mathcal{P}_{\mathcal{H}}^{\mathfrak{m}}$ is of *unipotency degree* or *coradical degree* $\leq i$ if it lies in $C_i \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}}$.

An element $\xi \in \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}}$ is of unipotency degree zero if and only if $\Delta(\xi) = \xi \otimes 1$, so it is $U_{\mathcal{H}}^{dR}$-invariant and hence semi-simple:

$$C_0 \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}} = \mathcal{P}_{\mathcal{H}^{ss}}^{\mathfrak{m}}.$$

An element ξ of unipotency degree at most one corresponds to a period of a simple extension. This is discussed in further detail in §6.

Recall that $\Delta^{u,r} = \Delta^u - \mathrm{id} \otimes 1$. Then $\xi \in C_i \mathcal{P}_H^{\mathfrak{m}}$ if and only if

$$(\Delta^{u,r})^{i+1} \xi = 0.$$

As in §2.5, we deduce the existence of a derivation

$$\delta : C_n \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}} \longrightarrow C_{n-1} \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}} \otimes_{\mathbb{Q}} H^1(U).$$

Definition 3.10. The *decomposition into primitives map*

$$\Phi : \mathrm{gr}_{\bullet}^{C} \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}} \longrightarrow \mathcal{P}_{\mathcal{H}^{ss}}^{\mathfrak{m}} \otimes_{\mathbb{Q}} T^c(H^1(U))$$

is defined by iterating δ.

The map Φ is a homomorphism of $S_{\mathcal{H}}^{dR}$-modules. To see this, recall that the coaction Δ^u is equivariant with respect to the action of $G_{\mathcal{H}}^{dR}$ on the left on $\mathcal{P}_{\mathcal{H}}^{\mathfrak{m}}$, and by conjugation on $\mathcal{O}(U)$. Therefore so is $\Delta^{u,r}$, and likewise δ. On the other hand, U acts trivially on both $\mathrm{gr}^C \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}}$ and $H^1(U)$, so the action of $G_{\mathcal{H}}^{dR}$ factors through its quotient $S_{\mathcal{H}}^{dR}$. Hence δ is $S_{\mathcal{H}}^{dR}$-equivariant, and by iteration, so is Φ.

The map Φ, together with the invariants defined above, give the first steps towards a classification of motivic periods by group theory and provides a tool for proving linear or algebraic independence of motivic periods (e.g.,

[16]). This is discussed in §6, where we shall also show that Φ is in fact an isomorphism. Note that Φ is not to be confused with the notion of symbol (§9).

4. Further remarks on motivic periods

The paragraphs below are independent from each other and can be skipped.

4.1. Universal period matrix and 'single-valued' periods

Let M be an object of $\mathcal{H}$. Then there is a canonical morphism

$$(4.1) \qquad M_{dR} \longrightarrow M_B \otimes_{\mathbb{Q}} \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}}$$

obtained by composing the natural map

$$\delta^{\vee} \otimes \mathrm{id} : \mathbb{Q} \otimes_{\mathbb{Q}} M_{dR} \to M_B \otimes_{\mathbb{Q}} M_B^{\vee} \otimes_{\mathbb{Q}} M_{dR}$$

with the map (2.1) $M_B^{\vee} \otimes_{\mathbb{Q}} M_{dR} \to \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}}$. It is given by the formula

$$v \mapsto \sum_i e_i \otimes [M, e_i^{\vee}, v]^{\mathfrak{m}}$$

where e_i (resp. $e_i^{\vee}$) is a basis (resp. dual basis) of M_B. Extending scalars from $\mathbb{Q}$ to $\mathcal{P}_{\mathcal{H}}^{\mathfrak{m}}$, it defines an isomorphism

$$c_M^{\mathfrak{m}} : M_{dR} \otimes_{\mathbb{Q}} \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}} \xrightarrow{\sim} M_B \otimes_{\mathbb{Q}} \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}}$$

which is functorial in M and which we think of as a *universal comparison map*. It is equal to the isomorphism of fiber functors ι_M (the notation is defined in (2.4); set $B_1 = B_2 = k = \mathbb{Q}$ and $R = \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}}$), where

$$\iota \in \mathrm{Isom}_{\mathcal{H}}^{\otimes}(\omega_{dR}, \omega_B)(\mathcal{P}_{\mathcal{H}}^{\mathfrak{m}})$$

is the element corresponding to the identity on $\mathcal{P}_{\mathcal{H}}^{\mathfrak{m}}$. Since

$$c \in \mathrm{Isom}_{\mathcal{H}}^{\otimes}(\omega_{dR}, \omega_B)(\mathbb{C})$$

is, by definition of the period homomorphism, equal to $\mathrm{per}(\iota)$, it follows that the comparison map $c_M : M_{dR} \to M_B \otimes_{\mathbb{Q}} \mathbb{C}$ is obtained from (4.1) by applying the period homomorphism; i.e., $c_M = (\mathrm{id} \otimes \mathrm{per}) c_M^{\mathfrak{m}}$.

As a first application, the universal coaction (4.1) defines a lift of the period matrix of M to the ring of $\mathcal{H}$-periods:

$$[c_M^{\mathfrak{m}}] \in W_0\mathrm{GL}(M_B)\backslash W_0\mathrm{GL}(\mathcal{P}_{\mathcal{H}}^{\mathfrak{m}}, r)/F^0 W_0\mathrm{GL}(M_{dR})$$

where $r = \mathrm{rank}\, M$. We have $[c_M] = \mathrm{per}[c_M^{\mathfrak{m}}]$ since the period homomorphism is $\mathbb{Q}$-linear. Applying this to the minimal object $M = M(\xi)$ defines an invariant $[c_\xi^{\mathfrak{m}}]$ of any element $\xi \in \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}}$. Its determinant is an element $\det(c_\xi^{\mathfrak{m}}) \in \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}}/\mathbb{Q}^\times$.

Another application is to construct single-valued versions of $\mathcal{H}$-periods, inspired by [8]. We only need the fact that the real Frobenius F_∞ defines a $\mathbb{Q}$-linear involution on $\mathcal{P}_{\mathcal{H}}^{\mathfrak{m}}$ which commutes with the action of $G_{\mathcal{H}}^{dR}$. Let

$$f \in \mathrm{Isom}_{\mathcal{H}}^{\otimes}(\omega_{dR}, \omega_B)(\mathcal{P}_{\mathcal{H}}^{\mathfrak{m}})$$

correspond to $F_\infty : \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}} \xrightarrow{\sim} \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}}$. It satisfies $f_M = (F_\infty \otimes \mathrm{id})\iota_M = (\mathrm{id} \otimes F_\infty)\iota_M$. Since $\mathrm{Isom}_{\mathcal{H}}^{\otimes}(\omega_{dR}, \omega_B)$ is a right $G_{\mathcal{H}}^{dR}$-torsor, there is a unique element

$$(4.2) \qquad \mathbf{s} \in G_{\mathcal{H}}^{dR}(\mathcal{P}_{\mathcal{H}}^{\mathfrak{m}}) \quad \text{such that} \quad f\,\mathbf{s} = \iota$$

which is computed explicitly below. This gives rise to a homomorphism (*single-valued map*)

$$(4.3) \qquad \mathbf{s}^{\mathfrak{m}} : \mathcal{P}_{\mathcal{H}}^{\mathfrak{dr}} \longrightarrow \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}}$$

which is $G_{\mathcal{H}}^{dR}$-equivariant if one equips the left-hand side $\mathcal{P}_{\mathcal{H}}^{\mathfrak{dr}} = \mathcal{O}(G_{\mathcal{H}}^{dR})$ with the action of $G_{\mathcal{H}}^{dR}$ by conjugation.

Remark 4.1. In [12] a slightly different single-valued map $\mathrm{sv}^{\mathfrak{m}}$ was defined on the ring of mixed Tate periods, which is graded by weight. It is defined by a similar formula on replacing F_∞ by F_∞ twisted by $(-1)^n$ in weight $2n$.

The situation is summarised by the following commutative diagram

$$
\begin{array}{ccc}
M_{dR} \otimes_{\mathbb{Q}} \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}} & \xrightarrow{\ \iota_M\ } & M_B \otimes_{\mathbb{Q}} \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}} \\
\downarrow{\scriptstyle \mathbf{s}_M^{\mathfrak{m}}} & & \uparrow{\scriptstyle \mathrm{id}\otimes F_\infty} \\
M_{dR} \otimes_{\mathbb{Q}} \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}} & \xrightarrow{\ \iota_M\ } & M_B \otimes_{\mathbb{Q}} \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}}
\end{array}
$$

where all maps are isomorphisms, which is functorial in M.

The *single-valued $\mathcal{H}$-period matrix* $s_M^{\mathfrak{m}} : M_{dR} \to M_{dR} \otimes_{\mathbb{Q}} \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}}$ can be computed as follows. Since $F_\infty^{-1} = F_\infty$, it follows from the above that $s_M^{\mathfrak{m}}$ is given

by the composite $\iota_M^{-1} \circ (\mathrm{id} \otimes F_\infty)^{-1} \circ \iota_M = \iota_M^{-1} \circ (\mathrm{id} \otimes F_\infty) \circ \iota_M$ which is explicitly

$$M_{dR} \xrightarrow{c_M^{\mathfrak{m}}} M_B \otimes_{\mathbb{Q}} \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}} \xrightarrow{\mathrm{id} \otimes F_\infty} M_B \otimes_{\mathbb{Q}} \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}} \xrightarrow{(c_M^{\mathfrak{m}})^{-1}} M_{dR} \otimes_{\mathbb{Q}} \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}}.$$

Thus if C_M is a matrix representing the map $c_M^{\mathfrak{m}}$ with respect to some choice of bases for M_{dR}, M_B, then $s_M^{\mathfrak{m}}$ is represented by $(F_\infty C_M)^{-1} C_M$. This is indeed invariant under change of basis for M_B, which amounts to replacing C_M with $P C_M$ for some $P \in \mathrm{GL}(M_B; \mathbb{Q})$; the quantity $(F_\infty C_M)^{-1} C_M$ is unchanged since F_∞ acts trivially on the coefficients of P because they are rational.

Finally, the *single-valued period matrix* s_M is obtained by applying the period map to $s_M^{\mathfrak{m}}$, and is given directly from the usual comparison map by $s_M = \overline{C}_M^{-1} C_M$, where $c_M = \mathrm{per}\, C_M$ denotes the period matrix.

4.2. Motivic philosophy

It is hoped that there exists a neutral Tannakian category $\mathcal{MM}_{\mathbb{Q}}$ of mixed motives over $\mathbb{Q}$ equipped, in particular, with Betti and de Rham realisations, and hence a functor $\mathcal{MM}_{\mathbb{Q}} \to \mathcal{H}$ and thus a homomorphism

$$(4.4) \qquad\qquad \mathcal{P}_{\mathcal{MM}_{\mathbb{Q}}}^{\mathfrak{m}} \longrightarrow \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}}.$$

The elements (3.5) should certainly be in its image, and the following diagram

$$
\begin{array}{ccc}
\mathcal{P}_{\mathcal{MM}_{\mathbb{Q}}}^{\mathfrak{m}} & \longrightarrow & \mathcal{P}_{\mathcal{MM}_{\mathbb{Q}}}^{\mathfrak{m}} \otimes_{\mathbb{Q}} \mathcal{P}_{\mathcal{MM}_{\mathbb{Q}}}^{\mathfrak{dr}} \\
\downarrow & & \downarrow \\
\mathcal{P}_{\mathcal{H}}^{\mathfrak{m}} & \longrightarrow & \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}} \otimes_{\mathbb{Q}} \mathcal{P}_{\mathcal{H}}^{\mathfrak{dr}},
\end{array}
$$

where the horizontal maps are given by the coactions (2.2), would commute. Therefore the action of $G_{\mathcal{H}}^{dR}$ on the de Rham realisation M_{dR} of an object $M \in \mathcal{MM}_{\mathbb{Q}}$ would be motivic, i.e., would factor through $G_{\mathcal{H}}^{dR} \to G_{\mathcal{MM}_{\mathbb{Q}}}^{dR}$. Grothendieck's period conjecture states that the period $\mathrm{per} : \mathcal{P}_{\mathcal{MM}_{\mathbb{Q}}}^{\mathfrak{m}} \to \mathbb{C}$ is injective, which motivates many classical conjectures in transcendence theory [1, 2, 7]. Since per factors through (4.4) this would imply Conjecture 1 and a much weaker conjecture: namely that the homomorphism (4.4) is injective. In this case, relations between elements of $\mathcal{P}_{\mathcal{MM}_{\mathbb{Q}}}^{\mathfrak{m}}$ could be detected in $\mathcal{P}_{\mathcal{H}}^{\mathfrak{m}}$, and $G_{\mathcal{H}}^{dR} \to G_{\mathcal{MM}_{\mathbb{Q}}}^{dR}$ would have the same image in $\hat{A}ut(M_{dR})$ for every

object $M \in \mathcal{MM}_{\mathbb{Q}}$ (and likewise for Betti). It is for this reason that we allow ourselves to call the periods (3.6) 'motivic'.

One important situation in which much of the above certainly works is the case $\mathcal{MT}(\mathbb{Q})$ of mixed Tate (or Artin-Tate) motives over $\mathbb{Q}$[6] [27, 42]. One then has a morphism $\mathcal{P}_{\mathcal{MT}(\mathbb{Q})}^{\mathfrak{m}} \to \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}}$ which is known to be injective (by the full faithfulness of the Hodge realisation). Borel's deep results on the rational algebraic K-theory of $\mathbb{Q}$ (see Example 6.6 below) give a precise upper bound for the size of the ring $\mathcal{P}_{\mathcal{MT}(\mathbb{Q})}^{\mathfrak{m}}$. Several applications of motives to number theory rely in an essential way on this upper bound. Note that even if one has the 'right' definition of $\mathcal{MM}_{\mathbb{Q}}$, this upper bound is not available in general.

Remark 4.2. Defining mixed motives as a full subcategory of realisations (à la Jannsen, Deligne) as opposed to by explicit generators and relations (à la Nori) would not give the same answer if the realisation functors are not fully faithful. Furthermore, theorems about the *independence* of motivic periods proved in the ring $\mathcal{P}_{\mathcal{H}}^{\mathfrak{m}}$ will carry over unconditionally to any reasonable definition of a category of mixed motives (irrespective of whether (4.4) is injective or not). On the other hand, when proving *relations* between motivic periods, it is preferable to prove them using morphisms of mixed Hodge structures which come from geometry, in which case they would also carry over to any suitably defined $\mathcal{P}_{\mathcal{MM}_{\mathbb{Q}}}^{\mathfrak{m}}$.

4.3. Projection map

An inconvenience of working with de Rham periods is the lack of a (complex) period homomorphism. One way around this is to construct single-valued periods as we did in §4.1. Another approach is to write de Rham periods as images of motivic periods. The latter works particularly well in the mixed Artin-Tate case as we now explain.

Proposition 4.3. *Every effective motivic period of weight zero is a motivic algebraic number. The period map gives an isomorphism*

$$\mathrm{per} : W_0 \mathcal{P}^{\mathfrak{m},+} \xrightarrow{\sim} \overline{\mathbb{Q}}.$$

Proof. See §10.2 below. □

---

[6] these exist over any number field, but for the time being we are working only over $\mathbb{Q}$

Suppose that M is an object of $\mathcal{H}$ which is effective (all Hodge numbers $h_{p,q}(M)$ vanish unless $p, q \geq 0$). Say that M is *separated* if

$$W_0 M_{dR} \longrightarrow M_{dR} \longrightarrow M_{dR}/F^1 M_{dR}$$

is an isomorphism. This implies that there is a splitting

$$(4.5) \qquad\qquad M_{dR} = W_0 M_{dR} \oplus F^1 M_{dR}.$$

Equivalently, $h_{p,q}(M) = 0$ unless $(p,q) = (0,0)$ or $p, q > 0$.

Define a comparison map $c_0^t : M_B^\vee \otimes_{\mathbb{Q}} \mathcal{P}^{\mathfrak{m}}(W_0 M) \to M_{dR}^\vee \otimes_{\mathbb{Q}} \mathcal{P}^{\mathfrak{m}}(W_0 M)$ to be the dual of the composition c_0 of the maps

$$M_{dR} \longrightarrow M_{dR}/F^1 M_{dR} \cong W_0 M_{dR} \xrightarrow{c_M^{\mathfrak{m}}} W_0 M_B \otimes_{\mathbb{Q}} \mathcal{P}^{\mathfrak{m}}(W_0 M)$$
$$\subset M_B \otimes_{\mathbb{Q}} \mathcal{P}^{\mathfrak{m}}(W_0 M)$$

where $\mathcal{P}^{\mathfrak{m}}(W_0 M)$ is the vector space of motivic periods of $W_0 M$. By the previous proposition, these are algebraic motivic periods (see §5.1) in the case where M is an object of the form (3.5). In this case, we obtain a linear map from the motivic periods of M to its de Rham periods:

$$[M, \sigma, \omega]^{\mathfrak{m}} \mapsto [M, c_0^t(\sigma), \omega]^{\mathfrak{dr}} : \mathcal{P}^{\mathfrak{m}}(M) \longrightarrow \mathcal{P}^{\mathfrak{dr}}(M) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}.$$

If M is of Artin-Tate type (Hodge numbers equal to (p,p) only) and effective, then it is necessarily separated. So, writing $\mathcal{P}_{HT}^{\bullet,+} = \mathcal{P}^{\bullet,+} \cap \mathcal{P}_{HT}^{\bullet}$ for $\bullet \in \{\mathfrak{m}, \mathfrak{dr}\}$, we obtain a linear map

$$(4.6) \qquad\qquad \pi_{\mathfrak{dr},\mathfrak{m}+} : \mathcal{P}_{HT}^{\mathfrak{m},+} \longrightarrow \mathcal{P}_{HT}^{\mathfrak{dr},+} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}.$$

Another way to define (4.6) is by the coaction

$$\mathcal{P}_{HT}^{\mathfrak{m},+} \xrightarrow{\Delta} \mathcal{P}_{HT}^{\mathfrak{m},+} \otimes_{\mathbb{Q}} \mathcal{P}_{HT}^{\mathfrak{dr},+} \longrightarrow \overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathcal{P}_{HT}^{\mathfrak{dr},+}$$

where the second map is the projection of $\mathcal{P}_{HT}^{\mathfrak{m},+}$ onto its weight 0 component (recall that it is graded by the weight and has non-negative degrees). Note that the projection map, restricted to the subring of motivic periods of mixed Tate motives, lands in $\mathcal{P}^{\mathfrak{dr},+}$, i.e., without tensoring with $\overline{\mathbb{Q}}$.

One possible application of the projection map is to prove identities between de Rham periods of mixed Tate motives using the complex period map. One can even deduce identities between p-adic periods using complex analysis. The idea is the following. Take a relation $P(\xi_1, \ldots, \xi_n) = 0$

between motivic periods in, say $\mathcal{P}^{\mathfrak{m},+}_{\mathcal{MT}(\mathbb{Z})}$ for simplicity. Such a relation can be proved by combining the coaction and complex analysis [15]. By applying the projection map we deduce a polynomial identity between de Rham periods. Finally, take the p-adic period to deduce $P(\xi^{(p)}_1, \ldots, \xi^{(p)}_n) = 0$ where $\xi^{(p)}_i = \mathrm{per}_p \pi_{\mathfrak{dr},\mathfrak{m}+} \xi_i$. This answers a question of Yamashita ([50], Remark 3.9): the motivic Drinfeld associator $\mathcal{Z}^{\mathfrak{m}}$ defined in [12] provides a common source for relations between both the complex and p-adic multiple zeta values via the period map per for the former, and via $\mathrm{per}_p \pi_{\mathfrak{dr},\mathfrak{m}+}$ for the latter. The fact that $\mathbb{L}^{\mathfrak{dr}} \neq 0$ but $\pi_{\mathfrak{dr},\mathfrak{m}+} \mathbb{L}^{\mathfrak{m}} = 0$ explains the confusing fact that it is sometimes stated in the literature that '$2\pi i = 0$' and sometimes that '$2\pi i = 1$' in this context.

Stated differently, let $\overline{G}^{\mathfrak{m}}_{MT}$ and $\overline{G}^{dR}_{MT}$ be the affine (monoid) schemes defined by the spectra of $\mathcal{P}^{\mathfrak{m},+}_{MT}$ and $\mathcal{P}^{\mathfrak{dr},+}_{MT} \subset \mathcal{P}^{\mathfrak{dr}}_{\mathcal{MT}}$. Then the projection is a morphism

$$G^{dR}_{MT} \longrightarrow \overline{G}^{dR}_{MT} \longrightarrow \overline{G}^{\mathfrak{m}}_{MT}$$

and a Frobenius element $F_p \in G^{dR}_{MT}(\mathbb{Q}_p)$ maps to a $\mathbb{Q}_p$-valued point on $\overline{G}^{\mathfrak{m}}_{MT}$.

5. Some basic examples of motivic periods

Before proceeding further with the discussion, we list some very simple examples of motivic periods to illustrate the concepts introduced earlier.

5.1. Algebraic numbers

This is the study of Artin motives ([21], 1.16) which in principle reduces to Grothendieck's version of Galois theory. However, the point of view of motivic periods leads to some interesting twists on this well-known tale. Let $P \in \mathbb{Q}[x]$ be an irreducible polynomial, set $F = \mathbb{Q}[x]/(P)$, and apply Example 3.3 with $X = \mathrm{Spec}\, F$, $D = \emptyset$, and $n = 0$. The object $H^0(X)$ is (the realization of) an Artin motive. Its de Rham and Betti realizations are $H^0_{dR}(X) = F$, and $H^0_B(X) = H_0(X(\mathbb{C}); \mathbb{Q})^{\vee} = \mathrm{Hom}(F, \mathbb{C})^{\vee}$. Let $\overline{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Given $\alpha \in \overline{\mathbb{Q}}$ such that $P(\alpha) = 0$, denote by $\sigma_\alpha : F \hookrightarrow \mathbb{C}$ the unique embedding of F such that $\sigma_\alpha(x) = \alpha$. Define a *motivic algebraic number*

$$\alpha^{\mathfrak{m}} = [H^0(X), \sigma_\alpha, x]^{\mathfrak{m}} \in W_0 \mathcal{P}^{\mathfrak{m},+}.$$

Its period is by definition $\mathrm{per}(\alpha^{\mathfrak{m}}) = \alpha$. The diagonal $X \to X \times X$ gives rise to a morphism $H^0(X) \otimes H^0(X) \to H^0(X)$ in the category $\mathcal{H}$. Using this and

the defining relations between matrix coefficients, we deduce that

$$f(\alpha^{\mathrm{m}}) = [H^0(X), \sigma_\alpha, f(x)]^{\mathrm{m}}$$

for $f = x^n$ and by additivity, for any polynomial $f \in \mathbb{Q}[x]$. By embedding $\mathbb{Q}(\alpha), \mathbb{Q}(\beta)$ into $\mathbb{Q}(\alpha, \beta)$, we deduce that $\alpha \mapsto \alpha^{\mathrm{m}} : \overline{\mathbb{Q}} \to \mathcal{P}^{\mathrm{m},+}$ is a homomorphism. It follows that the period map is an isomorphism

$$(5.1) \qquad\qquad \langle \alpha^{\mathrm{m}} : \alpha \in \overline{\mathbb{Q}} \rangle_{\mathbb{Q}} \xrightarrow{\mathrm{per}} \overline{\mathbb{Q}} \subset \mathbb{C}$$

so we can identify algebraic numbers with their motivic versions. Note that the minimal object $M_{\alpha^{\mathrm{m}}}$ associated to α^{m} is a strict subquotient of $H^0(X)$ whenever $\alpha \notin \mathbb{Q}$, since it is a factor of $\mathrm{coker}(H^0(\mathrm{Spec}\,\mathbb{Q}) \to H^0(\mathrm{Spec}\,F))$. The category of Artin motives $AM_{\mathbb{Q}}$ over $\mathbb{Q}$ is equivalent to the full tensor subcategory of $\mathcal{H}$ generated by the objects $H^0(X)$, and we could take this as its definition.

It is customary to consider only the Betti Galois group. The absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the left on $\mathrm{Hom}(F, \mathbb{C}) = \mathrm{Hom}(F, \overline{\mathbb{Q}})$ via its action on $\overline{\mathbb{Q}}$ and gives an automorphism of the Betti fiber functor. Indeed, as is well-known, the Betti Galois group $G^B_{AM(\mathbb{Q})}$ is the constant group scheme over $\mathbb{Q}$ corresponding to $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Therefore the (right) action of $G^B_{AM_{\mathbb{Q}}}(\mathbb{Q})$ on motivic algebraic numbers is equivalent to the (left) action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\overline{\mathbb{Q}}$ via the isomorphism (5.1). The action of real Frobenius F_∞ on the latter corresponds to complex conjugation on the former. The story usually ends here.

Now consider, somewhat unconventionally, the case of the de Rham Galois group. Its action on $H^0_{dR}(\mathrm{Spec}\,F)$ respects the diagonal map and hence the multiplication on F. Furthermore, it preserves $H^0_{dR}(\mathrm{Spec}\,K)$ for all subfields $K \subset F$.

Definition 5.1. Consider the functor

$$\mathcal{A}_F(R) = \{\alpha \in \mathrm{Aut}_R(F \otimes_{\mathbb{Q}} R)$$
$$\text{such that } \alpha(K \otimes_{\mathbb{Q}} R) \subset K \otimes_{\mathbb{Q}} R \text{ for all } K \subset F\}$$

from commutative $\mathbb{Q}$-algebras R to groups. Define the *group of field automorphisms* to be the projective limit over all field extensions $F/\mathbb{Q}$ of finite type

$$\mathcal{A}_{\overline{\mathbb{Q}}} = \varprojlim_F \mathcal{A}_F.$$

It follows from the fact that all algebraic relations between α^{m} are induced by linearity, inclusions of fields $K \subset F$ and diagonals $\operatorname{Spec} F \to \operatorname{Spec} F \times \operatorname{Spec} F$ that

$$G^{dR}_{AM(\mathbb{Q})} \cong \mathcal{A}_{\overline{\mathbb{Q}}}$$

(which shows in particular that the right-hand side is representable: its affine ring is generated by matrix coefficients $[H^0(\operatorname{Spec} F), f, v]^{\mathfrak{dr}}$, where $f \in F^\vee$ and $v \in F$). The comparison isomorphism implies that

$$G^B_{AM(\mathbb{Q})} \times \mathbb{C} \xrightarrow{\sim} G^{dR}_{AM(\mathbb{Q})} \times \mathbb{C}$$

so the usual absolute Galois group can be retrieved as the complex points (or $\overline{\mathbb{Q}}$-points) of the de Rham Galois group.

Now consider a motivic algebraic number α^{m}, for $\alpha \in \overline{\mathbb{Q}}$. The *degree* of α can be retrieved as the number of connected components of $G^\bullet_{\alpha^{\mathrm{m}}}$ for $\bullet = B, dR$. Its *minimal object* $M(\alpha^{\mathrm{m}})$ is an object of $AM_{\mathbb{Q}}$. Its periods generate the Galois closure F' of $\mathbb{Q}(\alpha)$. Its de Rham group scheme is $G^{dR}_{\alpha^{\mathrm{m}}} = \mathcal{A}_{\mathbb{Q}(\alpha)}$, and its Betti group $G^B_{\alpha^{\mathrm{m}}}$ is the constant group scheme of $\operatorname{Gal}(F'/\mathbb{Q})$. The quantity $\operatorname{rank}(\alpha^{\mathrm{m}}) = \dim_{\mathbb{Q}} M_B(\alpha^{\mathrm{m}})$ is the dimension of the $\mathbb{Q}$-vector space spanned by the Galois conjugates of α over $\mathbb{Q}$. This is called the *conjugate dimension* of α and, surprisingly, was introduced only very recently (see [9] and references therein).

Note that although the Betti orbit $G^B(\mathbb{Q})\alpha^{\mathrm{m}}$ of α^{m} corresponds to the usual notion of Galois conjugates of α, the de Rham orbit $G^{dR}(R)\alpha^{\mathrm{m}}$ is sensitive to R.

5.2. Motivic $2\pi i$ (Lefschetz motive)

Let $X = \mathbb{P}^1 \backslash \{0, \infty\}$, $D = \emptyset$. Consider $H^1(X) = \mathbb{Q}(-1)$ in Example 3.3. Its de Rham version is $H^1_{dR}(X; \mathbb{Q}) = \mathbb{Q}[\frac{dx}{x}]$ and its Betti version is $H_1(X(\mathbb{C})) = \mathbb{Q}[\gamma_0]$ where γ_0 is a small loop winding around 0 in the positive direction. Define the Lefschetz motivic period[7]

$$\mathbb{L}^{\mathrm{m}} = [H^1(X), [\gamma_0], [dx/x]]^{\mathrm{m}} \in W_2 P^{\mathrm{m},+}.$$

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[7]In [12] the Lefschetz motivic period was viewed as an object in $\mathcal{P}^{\mathrm{m}}_{\mathcal{MT}(\mathbb{Z})}$, where $\mathcal{MT}(\mathbb{Z})$ is the category of mixed Tate motives unramified over $\mathbb{Z}$. There is an injection $\mathcal{P}^{\mathrm{m}}_{\mathcal{MT}(\mathbb{Z})} \to \mathcal{P}^{\mathrm{m},+} \subset \mathcal{P}^{\mathrm{m}}_{\mathcal{H}}$, and the object $\mathbb{L}^{\mathrm{m}}$ defined here is its image. A similar remark applies for the later examples.

It satisfies $F_\infty \mathbb{L}^{\mathfrak{m}} = -\mathbb{L}^{\mathfrak{m}}$. By Cauchy's theorem, its period is

$$\text{per}(\mathbb{L}^{\mathfrak{m}}) = \int_{\gamma_0} \frac{dx}{x} = 2\pi i.$$

It is the 'motivic version' of $2\pi i$. Since $H^1_{dR}(X)$ is a one-dimensional representation of G^{dR}, we obtain a homomorphism of affine group schemes

$$\lambda : G^{dR} \longrightarrow \mathbb{G}_m.$$

Thus the group $G^{dR}(\mathbb{Q})$ acts upon $\mathbb{L}^{\mathfrak{m}}$ by multiplication

$$g(\mathbb{L}^{\mathfrak{m}}) = \lambda(g)\mathbb{L}^{\mathfrak{m}},$$

where $\lambda(g) \in \mathbb{Q}^\times$. The character λ_g is non-trivial: if $H^1_{dR}(X)$ were the trivial representation, then by Theorem 2.3, $H^1(X)$ would be equivalent to the trivial object $\mathbb{Q}(0) = H^0(pt)$ which has rational periods. Since

$$\text{per}(\mathbb{L}^{\mathfrak{m}}) = 2\pi i \notin \mathbb{Q}$$

is irrational, we conclude that λ is non-trivial (this also follows from the fact that the Hodge structure on $H^1(X)$ is $\mathbb{Q}(-1)$ which is pure of weight 2). It follows that $\mathbb{L}^{\mathfrak{m}}$ is transcendental: if there were a polynomial $P \in \mathbb{Q}[x]$ such that $P(\mathbb{L}^{\mathfrak{m}}) = 0$, then every conjugate $\lambda(g)\mathbb{L}^{\mathfrak{m}}$ would also be a root of P. Since a non-zero polynomial has only finitely many roots, it would follow that $P = 0$.

On the other hand, it is convenient to define the de Rham version of $\mathbb{L}^{\mathfrak{m}}$, denoted $\mathbb{L}^{\mathfrak{dr}} \in \mathcal{P}^{\mathfrak{dr}}_{\mathcal{H}} = \mathcal{O}(G^{dR})$, to be the matrix coefficient

$$\mathbb{L}^{\mathfrak{dr}} = [H^1(X), [dx/x]^\vee, [dx/x]]^{\mathfrak{dr}}.$$

The coaction $\Delta(\mathbb{L}^{\mathfrak{m}}) = \mathbb{L}^{\mathfrak{m}} \otimes \mathbb{L}^{\mathfrak{dr}}$ is given by application of (2.2), and it follows that $g(\mathbb{L}^{\mathfrak{dr}}) = \lambda(g)\mathbb{L}^{\mathfrak{dr}}$, and $\text{ev}(g(\mathbb{L}^{\mathfrak{dr}})) = \lambda(g)$, since $\text{ev}(\mathbb{L}^{\mathfrak{dr}}) = 1$.

5.3. Motivic logarithms (Kummer motive)

Let $X = \mathbb{P}^1 \backslash \{0, \infty\}$ and $D = \{1, \alpha\}$ for some $1 < \alpha \in \mathbb{Q}$. Consider the object in $\mathcal{H}$ known as a Kummer motive

$$K_\alpha = H^1(\mathbb{P}^1 \backslash \{0, \infty\}, \{1, \alpha\}).$$

It sits in an exact sequence $0 \to \mathbb{Q}(0) \to K_\alpha \to H^1(X) \to 0$. A basis for the de Rham cohomology $(K_\alpha)_{dR}$ is given by the relative cohomology classes of

the forms $\frac{dx}{x}$ and $\frac{dx}{\alpha-1}$, which vanish along D. Let γ_0 be as in §5.2, and γ_1 denote the interval $[1, \alpha] \subset X(\mathbb{R})$. Their boundaries are contained in $D(\mathbb{C})$, and they form a basis for $(K_\alpha)_B^\vee$. The comparison isomorphism is represented by the matrix

$$(5.2) \qquad \begin{pmatrix} \int_{\gamma_1} \frac{dx}{\alpha-1} & \int_{\gamma_1} \frac{dx}{x} \\ \int_{\gamma_0} \frac{dx}{\alpha-1} & \int_{\gamma_0} \frac{dx}{x} \end{pmatrix} = \begin{pmatrix} 1 & \log(\alpha) \\ 0 & 2\pi i \end{pmatrix}$$

with respect to this choice of basis. Define the motivic logarithm to be

$$\log^{\mathrm{m}}(\alpha) = [K_\alpha, [\gamma_1], [\tfrac{dx}{x}]]^{\mathrm{m}} \in W_2 \mathcal{P}^{\mathrm{m},+}.$$

Its period is $\mathrm{per}(\log^{\mathrm{m}}(\alpha)) = \log(\alpha)$, and $F_\infty \log^{\mathrm{m}} \alpha = \log^{\mathrm{m}} \alpha$. The group G^{dR} acts on $(K_\alpha)_{dR}$, fixing the subspace $\mathbb{Q}(0)_{dR}$ and acting on the quotient $H^1_{dR}(X) = \mathbb{Q}(-1)_{dR}$ via λ_g as in the previous example. Thus we have a homomorphism

$$(\nu_\alpha, \lambda) : G^{dR} \longrightarrow \mathbb{G}_a \rtimes \mathbb{G}_m.$$

Equivalently, the de Rham action is given for $g \in G^{dR}(\mathbb{Q})$ by

$$(5.3) \qquad g \log^{\mathrm{m}}(\alpha) = \lambda(g) \log^{\mathrm{m}}(\alpha) + \nu_\alpha(g).$$

For illustration, we can prove the functional equation of the motivic logarithm as follows. Let $1 < \beta \in \mathbb{Q}$, and consider the morphisms of pairs of spaces

$$(\mathbb{G}_m, \{1, \alpha\}) \xrightarrow{\times\beta} (\mathbb{G}_m, \{\beta, \alpha\beta\}) \subseteq (\mathbb{G}_m, \{1, \beta, \alpha\beta\})$$
$$(\mathbb{G}_m, \{1, z\}) \subseteq (\mathbb{G}_m, \{1, \beta, \alpha\beta\}) \quad \text{for } z \in \{\beta, \alpha\beta\}.$$

Since $\frac{dx}{x}$ is invariant under multiplication, these give relations

$$\log^{\mathrm{m}}(\alpha) = [H^1(\mathbb{G}_m, \{1, \beta, \alpha\beta\}), [\beta, \alpha\beta], [\tfrac{dx}{x}]]^{\mathrm{m}}$$
$$\log^{\mathrm{m}}(z) = [H^1(\mathbb{G}_m, \{1, \beta, \alpha\beta\}), [1, z], [\tfrac{dx}{x}]]^{\mathrm{m}} \quad \text{for } z \in \{\beta, \alpha\beta\}.$$

Finally use additivity with respect to Betti classes $[1, \alpha\beta] = [1, \beta] + [\beta, \alpha\beta]$ to obtain the expected relation between the three motivic periods

$$\log^{\mathrm{m}}(\alpha\beta) = \log^{\mathrm{m}}(\alpha) + \log^{\mathrm{m}}(\beta).$$

It follows that the motivic logarithms over $\mathbb{Q}$ are linear combinations of the motivic periods $\log^{\mathrm{m}}(p)$ for $p \geq 2$ prime. Since $\log : \mathbb{R}_{>0} \to \mathbb{R}$ has a unique

zero at $x = 1$, the functional equation of the logarithm implies that the numbers $\log(p)$ are linearly independent over $\mathbb{Q}$, and *a fortiori* the $\log^{\mathfrak{m}}(p)$. By (5.3) we have

$$\Delta^u \log^{\mathfrak{m}}(p) = \log^{\mathfrak{m}}(p) \otimes 1 + 1 \otimes \nu_p$$

where ν_p is viewed in $\mathcal{O}(U_{\mathcal{H}}^{dR})$. We deduce that the decomposition map satisfies

$$\Phi(\log^{\mathfrak{m}}(p)) = 1 \otimes \nu_p \ \in \ \mathcal{P}_{\mathcal{H}^{ss}}^{\mathfrak{m}} \otimes_{\mathbb{Q}} H^1(U_{\mathcal{H}}^{dR}),$$

and the ν_p are independent, since Φ is injective. Since it is a homomorphism,

$$\Phi\left((\mathbb{L}^{\mathfrak{m}})^k \prod_i \log^{\mathfrak{m}}(p_i)^{n_i} \right) = (\mathbb{L}^{\mathfrak{m}})^k \otimes \prod_i (\nu_{p_i})^{\text{Ш } n_i}$$

where the products on the right-hand side are with respect to the shuffle product, p_i are a finite set of primes, and $n_i \geq 0$.

Corollary 5.2. *Since Φ is injective, the set of elements $\{\mathbb{L}^{\mathfrak{m}}, \log^{\mathfrak{m}}(p)$ for p prime$\}$ are algebraically independent over $\mathbb{Q}$.*

This completes the description of all algebraic relations between the motivic periods $\log^{\mathfrak{m}}(\alpha)$, for $\alpha \in \mathbb{Q}$, and $\mathbb{L}^{\mathfrak{m}}$.

5.3.1. Single-valued versions. Define the de Rham version

$$\log^{\mathfrak{dr}}(\alpha) = [K_\alpha, [\tfrac{dx}{\alpha-1}]^\vee, [\tfrac{dx}{x}]]^{\mathfrak{dr}} \in W_2 \mathcal{P}^{\mathfrak{dr}},$$

where $[\tfrac{dx}{\alpha-1}]^\vee \in (K_\alpha)_{dR}^\vee$ takes the value 0 on $[\tfrac{dx}{x}]$ and 1 on $[\tfrac{dx}{\alpha-1}]$. For $\alpha = p$ prime, its image in $\mathcal{O}(U^{dR})$ is precisely ν_p. The coaction formula 2.2 gives

$$\Delta \log^{\mathfrak{m}}(\alpha) = \log^{\mathfrak{m}}(\alpha) \otimes \mathbb{L}^{\mathfrak{dr}} + 1^{\mathfrak{m}} \otimes \log^{\mathfrak{dr}}(\alpha),$$

which is equivalent to (5.3). It follows from the computations above that $\log^{\mathfrak{dr}}(p)$ for p prime are also algebraically independent over $\mathbb{Q}$ (use the de

Rham version of Φ). The motivic period matrix associated to $\log^{\mathrm{m}} \alpha$ is

$$(5.4) \qquad\qquad C^{\mathrm{m}} = \begin{pmatrix} 1 & \log^{\mathrm{m}}(\alpha) \\ 0 & \mathbb{L}^{\mathrm{m}} \end{pmatrix}.$$

The real Frobenius F_∞ acts by -1 on the second row. Therefore

$$(F_\infty C^{\mathrm{m}})^{-1} C^{\mathrm{m}} = \begin{pmatrix} 1 & 2\log^{\mathrm{m}}(\alpha) \\ 0 & -1 \end{pmatrix}$$

and we deduce that $\mathbf{s}^{\mathrm{m}}(\mathbb{L}^{\partial\mathfrak{r}}) = -1$ and $\mathbf{s}^{\mathrm{m}}(\log^{\partial\mathfrak{r}}(\alpha)) = (1+F_\infty)\log^{\mathrm{m}}(\alpha) = 2\log^{\mathrm{m}}(\alpha)$.

5.4. Motivic multiple zeta values

Iterated integrals on the punctured projective line provide a class of motivic periods for which one knows how to compute the motivic coaction. This is most developed in the case of multiple zeta values. For any $n_1, \ldots, n_{r-1} \geq 1$ and $n_r \geq 2$, there are motivic multiple zeta values

$$(5.5) \qquad\qquad \zeta^{\mathrm{m}}(n_1, \ldots, n_r) \in \mathcal{P}_{HT}^{\mathrm{m}} \cap \mathcal{P}^{\mathrm{m},+} \subset \mathcal{P}^{\mathrm{m},+}$$

of weight $2n_1 + \cdots + 2n_r$ (recall $\mathcal{P}_{HT}^{\mathrm{m},+}$ is graded by W) whose periods are

$$\mathrm{per}(\zeta^{\mathrm{m}}(n_1, \ldots, n_r)) = \zeta(n_1, \ldots, n_r) = \sum_{1 \leq k_1 < \cdots < k_r} \frac{1}{k_1^{n_1} \cdots k_r^{n_r}}.$$

They are defined as follows. Let $X = \mathbb{P}^1 \backslash \{0, 1, \infty\}$ and set

$$\zeta^{\mathrm{m}}(n_1, \ldots, n_r) = [\mathcal{O}(\pi_1^{\mathrm{m}}(X, \vec{1}_0, -\vec{1}_1)), \mathrm{dch}, w]^{\mathrm{m}}$$

where $\pi_1^{\mathrm{m}}(X, \vec{1}_0, -\vec{1}_1)$ is the motivic torsor of paths on X [27] from the unit tangent vector at 0 to minus the unit tangent vector at 1, w is the word $w = e_0^{n_1-1} e_1 \cdots e_0^{n_r-1} e_1$ in $e_0 = \frac{dx}{x}$, and $e_1 = \frac{dx}{1-x}$, and dch is the Betti image of the straight line path from 0 to 1. For further details, see [14]. This actually defines the motivic multiple zeta values as motivic periods of the category $\mathcal{MT}(\mathbb{Z})$ of mixed Tate motives over $\mathbb{Z}$. The latter admits a fully faithful functor to the category $\mathcal{H}$ [27], and so the ring of periods $\mathcal{P}_{\mathcal{MT}(\mathbb{Z})}^{\mathrm{m}}$ injects into $\mathcal{P}_{\mathcal{H}}^{\mathrm{m}}$ and we can identify it with its image. Furthermore, Beilinson's construction of the motivic torsor of path given in [27] can be

realised in the form of Example 3.3, so we can also view the images of the $\zeta^{\mathrm{m}}(n_1, \ldots, n_r) \in \mathcal{P}^{\mathrm{m}}_{\mathcal{H}}$ as elements of $\mathcal{P}^{\mathrm{m},+}$ as claimed above.

The depth of (5.5) is defined to be r. The fact that the depth filtration is motivic implies the following bound for the unipotency degree

$$\mathrm{u.\ d.}(\zeta^{\mathrm{m}}(n_1, \ldots, n_r)) \leq r.$$

The unipotency degree has sometimes been referred to as the 'motivic depth'. A fascinating feature of multiple zeta values is the existence of a discrepancy between the unipotency degree and the depth, related to modular forms for $\mathrm{SL}_2(\mathbb{Z})$.

The simplest examples are the motivic zeta values, $\zeta^{\mathrm{m}}(2n+1) \in C_1 \mathcal{P}^{\mathrm{m},+}$, for $n \geq 1$, which admit a motivic period matrix of the expected form

$$\begin{pmatrix} 1 & \zeta^{\mathrm{m}}(2n+1) \\ 0 & (\mathbb{L}^{\mathrm{m}})^{2n+1} \end{pmatrix}$$

Let us define some symbols $f_{2n+1} \in H^1(U^{dR}_{\mathcal{H}})$, for $n \geq 1$, as the images of the motivic zeta values under the decomposition map

$$\Phi(\zeta^{\mathrm{m}}(2n+1)) = 1 \otimes f_{2n+1}$$

Each f_{2n+1}, for $n \geq 1$ spans a copy of $\mathbb{Q}(-2n-1)$ and has weight $4n+2$. The interpretation of these elements will be explained in §6 below. Either from the explicit formula for the coaction on motivic multiple zeta values, or from the results of §6, the decomposition map gives an injective homomorphism

$$\Phi : \mathrm{gr}^C \mathcal{P}^{\mathrm{m},+}_{\mathcal{MT}(\mathbb{Z})} \longrightarrow \mathbb{Q}[\mathbb{L}^{\mathrm{m}}] \otimes_{\mathbb{Q}} \mathbb{Q}\langle f_3, f_5, \ldots \rangle$$

where the right-hand side denotes the shuffle algebra (tensor coalgebra) on symbols f_{2n+1} over $\mathbb{Q}$. In this case, it is in fact known that Φ is an isomorphism (Theorem 6.3). Now, the main result of [16] is a computation of the image under Φ of the elements $\zeta^{\mathrm{m}}(n_1, \ldots, n_r)$ where $n_i \in \{2, 3\}$, and a proof that their images are linearly independent.[8] Thus we can use these elements to split the coradical filtration C on $\mathcal{P}^{\mathrm{m}}_{\mathcal{MT}(\mathbb{Z})}$, and deduce the existence of a

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[8]The use of the decomposition map Φ considerably simplifies many of the arguments of [16]

canonical isomorphism [15]

$$\phi : \mathcal{P}^{\mathfrak{m},+}_{\mathcal{MT}(\mathbb{Z})} \cong \mathrm{gr}^C \mathcal{P}^{\mathfrak{m},+}_{\mathcal{MT}(\mathbb{Z})} \xrightarrow{\Phi} \mathbb{Q}[\mathbb{L}^{\mathfrak{m}}] \otimes_{\mathbb{Q}} \mathbb{Q}\langle f_3, f_5, \ldots \rangle.$$

One could use a different splitting of the coradical filtration C, which would lead to a different choice of isomorphism ϕ.

It is hard to understand Galois aspects of multiple zeta values without some sort of model of this kind. Indeed, using this model we can easily write down the invariants defined earlier. If $\xi \in \mathcal{P}^{\mathfrak{m},+}_{\mathcal{MT}(\mathbb{Z})}$ corresponds to $(\mathbb{L}^{\mathfrak{m}})^k f_{a_1} f_{a_2} \cdots f_{a_r}$ under ϕ, then the representation generated by ξ is the vector space

$$M(\xi)_{dR} = \langle \ell^k f_{a_1} \cdots f_{a_i} : \text{ for } 0 \le i \le r \rangle_{\mathbb{Q}}$$

obtained by slicing off letters from the right. A representative for the period matrix for $\xi = (\mathbb{L}^{\mathfrak{m}})^k f_{a_1} f_{a_2}$ is

$$\mathrm{per} \begin{pmatrix} (\mathbb{L}^{\mathfrak{m}})^k & (\mathbb{L}^{\mathfrak{m}})^k f_{a_1} & (\mathbb{L}^{\mathfrak{m}})^k f_{a_1} f_{a_2} \\ 0 & (\mathbb{L}^{\mathfrak{m}})^{k+a_1} & (\mathbb{L}^{\mathfrak{m}})^{k+a_1} f_{a_2} \\ 0 & 0 & (\mathbb{L}^{\mathfrak{m}})^{k+a_1+a_2} \end{pmatrix}$$

which means the top left-hand entry is $\mathrm{per}((\mathbb{L}^{\mathfrak{m}})^k) = (2\pi i)^k$, and so on. The general pattern is clear from this example.

Applying the projection map π to motivic multiple zeta values leads to *de Rham* multiple zeta values $\zeta^{\mathfrak{dr}}(n_1, \ldots, n_r) = \pi_{\mathfrak{dr},\mathfrak{m}+}\zeta^{\mathfrak{m}}(n_1, \ldots, n_r)$. It is proved in [16] that the kernel of $\pi_{\mathfrak{dr},\mathfrak{m}+}$ on the ring generated by motivic multiple zeta values is the ideal generated by $\zeta^{\mathfrak{m}}(2)$. Thus de Rham multiple zeta values are motivic MZV's modulo $\zeta^{\mathfrak{m}}(2)$. The former have single-valued periods, and a calculation using the period matrix for $\zeta^{\mathfrak{m}}(2n+1)$ similar to the one for the logarithm gives the single-valued versions $\mathbf{s}^{\mathfrak{m}}(\zeta^{\mathfrak{dr}}(2n+1)) = 2\zeta^{\mathfrak{m}}(2n+1)$. The de Rham versions of multiple zeta values also have p-adic periods, which can be thought of as follows. There are canonical Frobenius elements [50]

$$F_p \in G^{dR}_{\mathcal{MT}(\mathbb{Z})}(\mathbb{Q}_p),$$

and hence homomorphisms $\mathrm{per}_p : \mathcal{P}^{\mathfrak{dr}}_{\mathcal{MT}(\mathbb{Z})} = \mathcal{O}(G^{dR}_{\mathcal{MT}(\mathbb{Z})}) \to \mathbb{Q}_p$. The projection map enables us to associate p-adic periods to motivic multiple zeta values, which are a certain kind of p-adic multiple zeta values.[9]

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[9]This point of view quickly leads to new constructions. For example, one can consider curious hybrid quantities defined by the convolution of per with per_p:

$$\zeta_{\mathbb{R}*p}(n_1, \ldots, n_r) := m(\mathrm{per} \otimes \mathrm{per}_p \pi_{\mathfrak{dr},\mathfrak{m}+})\Delta\zeta^{\mathfrak{m}}(n_1, \ldots, n_r) \in \mathbb{R} \otimes_{\mathbb{Q}} \mathbb{Q}_p,$$

5.5. Motivic Euler sums

Euler sums are defined by the nested sums

$$\zeta(n_1, \ldots, n_r) = \sum_{1 \leq k_1 < \cdots < k_r} \frac{\text{sign}(n_1)^{k_1} \cdots \text{sign}(n_r)^{k_r}}{k_1^{|n_1|} \cdots k_r^{|n_r|}}$$

where $n_i \in \mathbb{Z} \backslash \{0\}$ and $n_r \neq 1$. Their depth is defined to be the quantity r. They can be written as iterated integrals on $X = \mathbb{P}^1 \backslash \{0, \pm 1, \infty\}$ from 0 to 1, which leads to a definition of motivic Euler sums

$$\zeta^{\mathfrak{m}}(w) = [\mathcal{O}(\pi_1^{\mathfrak{m}}(X, \vec{1}_0, -\vec{1}_1)), \text{dch}, w]^{\mathfrak{m}}$$

where w is a certain word in $e_0 = \frac{dx}{x}$ and $e_{\pm 1} = \frac{dx}{x \pm 1}$, and dch is as above. These are motivic periods of the category $\mathcal{MT}(\mathbb{Z}[\frac{1}{2}])$ of mixed Tate motives ramified at 2. The decomposition map now provides an injective homomorphism

$$\Phi : \text{gr}^C \mathcal{P}_{\mathcal{MT}(\mathbb{Z}[\frac{1}{2}])}^{\mathfrak{m},+} \longrightarrow \mathbb{Q}[\mathbb{L}^{\mathfrak{m}}] \otimes_{\mathbb{Q}} \mathbb{Q}\langle \nu_2, f_3, f_5, \ldots \rangle,$$

where ν_2, corresponding to the logarithm of 2, was defined earlier. It is an isomorphism by Theorem 6.3. The results of Deligne [22] can be translated into this setting. He proves that $\zeta^{\mathfrak{m}}(n_1, \ldots, n_{r-1}, -n_r)$ where the n_i are odd ≥ 1 and form a Lyndon word, are algebraically independent. An important difference with the case of multiple zeta values, which considerably simplifies matters, is that the depth filtration in this case coincides with the unipotency filtration. We can construct a splitting of the coradical filtration using this basis and hence an isomorphism

$$\phi^{(2)} : \mathcal{P}_{\mathcal{MT}(\mathbb{Z}[\frac{1}{2}])}^{\mathfrak{m},+} \cong \text{gr}^C \mathcal{P}_{\mathcal{MT}(\mathbb{Z}[\frac{1}{2}])}^{\mathfrak{m},+} \xrightarrow{\Phi} \mathbb{Q}[\mathbb{L}^{\mathfrak{m}}] \otimes_{\mathbb{Q}} \mathbb{Q}\langle \nu_2, f_3, f_5, \ldots \rangle.$$

The periods of $\mathcal{MT}(\mathbb{Z})$ correspond to elements with no ν_2 in their $\phi^{(2)}$-image. Note, however, that the maps ϕ and $\phi^{(2)}$ are not compatible. To remedy this, one could replace ϕ with the restriction of $\phi^{(2)}$ on $\mathcal{P}_{\mathcal{MT}(\mathbb{Z})}^{\mathfrak{m},+} \subset \mathcal{P}_{\mathcal{MT}(\mathbb{Z}[\frac{1}{2}])}^{\mathfrak{m},+}$, but this does not quite lead to an explicit basis for the periods of $\mathcal{MT}(\mathbb{Z})$. These ideas are studied in Glanois' thesis [29], who also constructed a new basis for the motivic periods of $\mathcal{MT}(\mathbb{Z})$ using certain modified Euler sums where the summation involves non-strict inequalities, weighted with certain powers of 2.

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where m is multiplication, and Δ the coaction (2.2).

6. Towards a classification of motivic periods

We can use the decomposition into primitives to classify $\mathcal{H}$-periods up to elements of lower unipotency degree. In this section, we shall drop the superscript dR and subscript $\mathcal{H}$ and write S, U, G instead of $S_{\mathcal{H}}^{dR}, U_{\mathcal{H}}^{dR}, G_{\mathcal{H}}^{dR}$.

The decomposition map involves a space $\mathrm{gr}_1^C(\mathcal{O}(U))$, which is exactly

$$\mathrm{Prim}(\mathcal{O}(U)) := \{f \in \mathcal{O}(U) : \Delta f = f \otimes 1 + 1 \otimes f\}$$

In this paragraph we analyse this space in some detail, which leads to further invariants of motivic periods, and a first step towards their classification.

6.1. Cohomology of U

The exact sequence (3.8) will now be written

$$1 \longrightarrow U \longrightarrow G \longrightarrow S \longrightarrow 1.$$

Proposition 6.1. *Let $n \geq 0$. There is an isomorphism of (right) S-modules*

$$H^n(U) \cong \bigoplus_{M \in \mathrm{Irr}(\mathcal{H}^{ss})} \mathrm{Ext}_{\mathcal{H}}^n(\mathbb{Q}, M) \otimes_{\mathrm{End}(M)} M_{dR}^{\vee},$$

where $\mathrm{Irr}(\mathcal{H}^{ss})$ denotes a set of representatives of isomorphism classes of simple objects in $\mathcal{H}^{ss}$ (or equivalently, of irreducible $\mathcal{O}(S)$-comodules).

Proof. First of all, we can write $U = \varprojlim U_n$ as a projective limit of unipotent affine group schemes U_n of finite type (unipotent algebraic matrix groups). Likewise, a representation V of U is an inductive limit $V = \varinjlim V_n$ of U_n-representations. Since $\varinjlim H^i(U_n, V_n) \overset{\sim}{\to} H^i(U, V)$, the arguments which follow can be deduced from well-known results for matrix groups and by taking limits.

Let M be an irreducible object of $\mathcal{H}^{ss}$. Then M_{dR} is an irreducible $\mathcal{O}(S)$-comodule. A Hochschild-Serre spectral sequence gives

$$H^p(S, H^q(U, M_{dR})) \Rightarrow H^{p+q}(G, M_{dR})$$

and one knows that S is of cohomological dimension 0, since it is proreductive. Therefore since U acts trivially on M_{dR},

$$H^0(S, H^n(U, M_{dR})) = H^n(U, M_{dR})^S \cong (H^n(U) \otimes_{\mathbb{Q}} M_{dR})^S,$$

where $H^n(U)$ denotes $H^n(U;\mathbb{Q})$, and we deduce that

$$\left(H^n(U) \otimes_{\mathbb{Q}} M_{dR}\right)^S \cong H^n(G, M_{dR}).$$

Let M, N be irreducible S-modules. Then $(N_{dR}^\vee \otimes_{\mathbb{Q}} M_{dR})^S = \mathrm{End}_S(M_{dR})$ if N and M are isomorphic, and zero otherwise, by Schur's lemma. It follows that

$$H^n(U) \cong \bigoplus_{M \in \mathrm{Irr}(\mathcal{H}^{ss})} H^n(G, M_{dR}) \otimes_{\mathrm{End}_S(M_{dR})} M_{dR}^\vee.$$

Note that since U acts trivially on M_{dR}, we have $\mathrm{End}_G(M_{dR}) = \mathrm{End}_S(M_{dR})$. Since $\omega_{dR} : \mathcal{H} \to \mathrm{Rep}(G)$ is an equivalence, we deduce that

$$H^n(G, M_{dR}) = \mathrm{Ext}^n_{\mathrm{Rep}(G)}(\mathbb{Q}, M_{dR}) = \mathrm{Ext}^n_{\mathcal{H}}(\mathbb{Q}(0), M),$$

and $\mathrm{End}_S(M_{dR}) = \mathrm{End}(M)$. $\qquad\square$

It is a well-known fact due to Beilinson that

$$\mathrm{Ext}^n_{\mathcal{H}}(\mathbb{Q}, M) = 0 \quad \text{for } n \geq 2.$$

Corollary 6.2. *The cohomology $H^n(U; M)$ vanishes for all $n \geq 2$.*

Recall that

$$H^1(U) \cong \mathrm{gr}_1^C \mathcal{O}(U) \cong C_1 \mathcal{O}(U)_+ \cong \mathrm{Prim}(\mathcal{O}(U)).$$

Theorem 6.3. *The decomposition map*

$$\Phi : \mathrm{gr}_\bullet^C \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}} \longrightarrow \mathcal{P}_{\mathcal{H}^{ss}}^{\mathfrak{m}} \otimes_{\mathbb{Q}} T^c(\mathrm{gr}_1^C \mathcal{O}(U))$$

is an isomorphism of S-modules.

Proof. By Proposition 3.8, there is a non-canonical isomorphism $\mathcal{P}_{\mathcal{H}}^{\mathfrak{m}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \cong \mathcal{P}_{\mathcal{H}^{ss}}^{\mathfrak{m}} \otimes_{\mathbb{Q}} \mathcal{O}(U) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$. The group U is of cohomological dimension 1, by the previous corollary. Now apply Corollary 2.13 with $T = \mathcal{P}_{\mathcal{H}^{ss}}^{\mathfrak{m}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ to conclude that the decomposition map, after extending scalars to $\overline{\mathbb{Q}}$, is an isomorphism. Since it was already injective over $\mathbb{Q}$, it follows that it is surjective over $\mathbb{Q}$. $\qquad\square$

Remark 6.4. One can view $T^c(H^1(U))$ as the associated graded, for the length filtration, of $H^0(\mathbb{B}(N))$ where N is a DGA which computes the cohomology of U, and $\mathbb{B}$ is the (reduced) bar construction. The decomposition

map Φ therefore resembles the bar construction of a fibration, and suggests thinking about elements of $\mathcal{P}^{\mathfrak{m}}_{\mathcal{H}^{ss}}$ as functions on a 'base' corresponding to S, and $T^c(H^1(U))$ as iterated integrals on a 'fiber' corresponding to U. From this point of view, δ can be thought of as a kind of Gauss-Manin connection. If one wants to copy this setup for mixed Tate motives over number fields rather than mixed Hodge structures, this suggests replacing N with Bloch's cycle complex $\mathcal{N}$ and echoes the construction of [11].

6.2. Primitives in $\mathcal{O}(U)$

We analyse the statement of Proposition 6.1 in more detail in the case $n = 1$. It gives an isomorphism of S-modules

$$(6.1) \qquad \mathrm{Prim}(\mathcal{O}(U)) \xrightarrow{\sim} \bigoplus_{M \in \mathrm{Irr}(\mathcal{H}^{ss})} \mathrm{Ext}^1_{\mathcal{H}}(\mathbb{Q}(0), M^{\vee}) \otimes_{\mathrm{End}(M)} M_{dR}.$$

First of all, observe from Remark 2.10 that

$$\mathrm{Prim}(\mathcal{O}(U^{ab})) \xrightarrow{\sim} \mathrm{Prim}(\mathcal{O}(U)).$$

The action of S by conjugation on U^{ab} induces an action of S on $\mathcal{O}(U^{ab})$, and preserves the space of primitive elements. Since S is the (de Rham) Tannaka group of $\mathcal{H}^{ss}$, the S-module generated by any element $f \in \mathrm{Prim}(\mathcal{O}(U))$ defines a representation of S, and hence an object of $\mathcal{H}^{ss}$ by Theorem 2.3.

Definition 6.5. For any $f \in \mathrm{Prim}(\mathcal{O}(U))$, let M_f denote the associated object of $\mathcal{H}^{ss}$. Its de Rham vector space is the $\mathcal{O}(S)$-comodule generated by f. It comes equipped with a distinguished element $f \in (M_f)_{dR}$.

Let $f \in \mathrm{Prim}(\mathcal{O}(U))$. One associates an extension to f as follows. Consider the short exact sequence of right U-modules

$$0 \longrightarrow C_0\mathcal{O}(U) \longrightarrow C_1\mathcal{O}(U) \longrightarrow \mathrm{Prim}(\mathcal{O}(U)) \longrightarrow 0.$$

It is not split, although the underlying sequence of $\mathbb{Q}$-vector spaces is split by the augmentation map. By Remark 2.10, it can be rewritten

$$0 \longrightarrow \mathbb{Q} \longrightarrow C_1\mathcal{O}(U^{ab}) \longrightarrow \mathrm{Prim}(\mathcal{O}(U^{ab})) \longrightarrow 0.$$

It is an exact sequence in the category of right $U^{ab} \rtimes S$-modules. We can pull back this extension along the inclusion $(M_f)_{dR} \subseteq \mathrm{Prim}(\mathcal{O}(U^{ab}))$ to obtain

$$0 \longrightarrow \mathbb{Q} \longrightarrow \mathcal{E}_{dR} \longrightarrow (M_f)_{dR} \longrightarrow 0.$$

Now choose an isomorphism $G/[U,U] \to U^{ab} \rtimes S$, i.e., a splitting of $1 \to U^{ab} \to G/[U,U] \to S \to 1$. It exists by Levi's (Mostow's) theorem. Via this isomorphism, the previous exact sequence can be viewed in the category of G-modules, and hence, via the Tannaka theorem, as an exact sequence in $\mathcal{H}$:

$$(6.2) \qquad\qquad 0 \longrightarrow \mathbb{Q} \longrightarrow \mathcal{E} \longrightarrow M_f \longrightarrow 0.$$

Another choice of isomorphism $G/[U,U] \cong U^{ab} \rtimes S$ yields an isomorphic extension. The dual extension $0 \to M_f^\vee \to \mathcal{E}^\vee \to \mathbb{Q} \to 0$, together with the vector $f \in (M_f)_{dR}$, defines a class

$$[\mathcal{E}^\vee] \otimes f \ \in \ \mathrm{Ext}^1_\mathcal{H}(\mathbb{Q}, M_f^\vee) \otimes_\mathbb{Q} (M_f)_{dR}$$

as required. By decomposing M_f into S-isotypical components, we can project this element into the right-hand side of (6.1).

Definition 6.6. Let us denote the extension class of (6.2) by $\mathcal{E}_f$.

In the other direction, consider an extension in $\mathcal{H}$

$$0 \longrightarrow M^\vee \longrightarrow \mathcal{E} \longrightarrow \mathbb{Q} \longrightarrow 0,$$

and a vector $v \in M_{dR}$, where M is a simple object. Choose a lift of the element $1 \in \mathbb{Q}_{dR}$ to $f \in \mathcal{E}_{dR}$, and a lift of v to $\widetilde{v} \in \mathcal{E}_{dR}^\vee$ along the map $\mathcal{E}_{dR}^\vee \to M_{dR}$. The image of the following unipotent matrix coefficient (§3.7) in $\mathcal{O}(U)_+$

$$\xi = [\mathcal{E}, \widetilde{v}, f]^{\mathfrak{u}} \in \mathcal{O}(U)_+$$

does not depend on the choices of $\widetilde{v}, f$. For instance, if f' is another lift of 1, then $f - f' \in M_{dR}^\vee$, and $[\mathcal{E}, \widetilde{v}, f - f']^{\mathfrak{u}}$ is equivalent to $[M^\vee, v, f - f']^{\mathfrak{u}}$, which is constant because U acts trivially on M_{dR}. Similarly, if $\widetilde{v}'$ is a lift of v then $\widetilde{v}' - \widetilde{v} \in \mathbb{Q}_{dR}$ and $[\mathcal{E}, \widetilde{v}' - \widetilde{v}, f]^{\mathfrak{u}}$ is equivalent to a unipotent period of $\mathbb{Q}(0)$, hence constant. By Schur's lemma, a non-zero endomorphism $\alpha : M \to M$ is an automorphism. If $\mathcal{E}^\alpha$ denotes the extension $\mathcal{E}$ twisted by $\alpha^\vee$, then the identity map $\mathcal{E} \xrightarrow{\sim} \mathcal{E}^\alpha$ gives an equivalence of matrix coefficients $\xi = [\mathcal{E}^\alpha, \widetilde{v}, f]$. It is straightforward to check using (2.2) and the formulae which follow that ξ is a primitive element. This construction provides an inverse to (6.1).

6.3. Extensions in $\mathcal{H}$

The contents of this section are standard and well-known. Let $M = (M_B, M_{dR}, c)$ be an object in $\mathcal{H}$. The following complex

$$W_0 M_B^+ \oplus F^0 W_0 M_{dR} \xrightarrow{\mathrm{id} - c} (W_0 M_B \otimes_{\mathbb{Q}} \mathbb{C})^{c_{dR}}$$

represents $R\mathrm{Hom}_{\mathcal{H}}(\mathbb{Q}(0), M)$. Recall that c_{dR} is complex conjugation on the right-hand factor of $M_{dR} \otimes_{\mathbb{Q}} \mathbb{C}$. Its action on $M_B \otimes_{\mathbb{Q}} \mathbb{C}$ is $c\, c_{dR}\, c^{-1} = F_\infty \otimes c_B$ where c_B is complex conjugation on the right-hand factor of $M_B \otimes_{\mathbb{Q}} \mathbb{C}$.

The kernel of the above complex is

$$\mathrm{Hom}_{\mathcal{H}}(\mathbb{Q}(0), M) \xrightarrow{\sim} W_0 M_B^+ \cap c(F^0 W_0 M_{dR})$$

and the map is given by the image of $1 \in \mathbb{Q}_{dR} \overset{c}{\cong} \mathbb{Q}_B$. The cokernel is [18]

$$(6.3) \qquad \mathrm{Ext}^1_{\mathcal{H}}(\mathbb{Q}, M) \xrightarrow{\sim} W_0 M_B^+ \backslash (W_0 M_B \otimes_{\mathbb{Q}} \mathbb{C})^{c_{dR}} / c(F^0 W_0 M_{dR}).$$

The map is given as follows. If $\mathcal{E}$ is an extension of $\mathbb{Q}(0)$ by M in $\mathcal{H}$, it gives rise, after applying a fiber functor $\bullet = B/dR$, to two exact sequences

$$0 \longrightarrow M_\bullet \longrightarrow \mathcal{E}_\bullet \longrightarrow \mathbb{Q}_\bullet \longrightarrow 0.$$

Choose B and dR splittings by choosing a lift of $1_B \in \mathbb{Q}_B$ to $1_B \in W_0 \mathcal{E}_B^+$ and of $1_{dR} \in \mathbb{Q}_{dR}$ to $1_{dR} \in F^0 W_0 \mathcal{E}_{dR}$. Then $1_B - c(1_{dR})$ gives a well-defined element in the right-hand side of (6.3). Note that (6.3) is uncountably generated. Let $\mathcal{H}(\mathbb{R})$ denote the category of triples (M_B, M_{dR}, c) where now M_B, M_{dR} are vector spaces over $\mathbb{R}$ (replace the ground field $\mathbb{Q}$ by $\mathbb{R}$). There is a functor $\otimes \mathbb{R} : \mathcal{H} \to \mathcal{H}(\mathbb{R})$, sending (M_B, M_{dR}, c) to $(M_B \otimes \mathbb{R}, M_{dR} \otimes \mathbb{R}, c \otimes \mathrm{id})$.

Corollary 6.7. *Suppose that $W_{-1} M = M$. Then*

$$(6.4) \qquad \dim_{\mathbb{R}} \mathrm{Ext}^1_{\mathcal{H}(\mathbb{R})}(\mathbb{R}(0), M \otimes \mathbb{R}) = \dim_{\mathbb{Q}} M_B^- - \dim_{\mathbb{Q}} F^0 M_{dR}$$

Proof. Since c_{dR}, F_∞ act trivially on $c(F^0 M_{dR}) \cap M_B^+$, so too must c_B, since by §3.1 we have $F_\infty \otimes c_B = cc_{dR}c^{-1}$. It follows that $c(F^0 M_{dR}) \cap M_B^+ \subset F^0 \cap \overline{F}^0 = 0$, since M has weights ≤ -1. Now $(M_B \otimes_{\mathbb{Q}} \mathbb{C})^{c_{dR}} = (M_B^+ \otimes \mathbb{R}) \oplus (M_B^- \otimes i\mathbb{R})$, and conclude using (6.3) together with the fact that $W_0 M = M$. $\qquad \square$

The formula (6.3), together with (6.1) and Theorem 6.3 provides a complete description of $\mathcal{H}$-periods, graded for the coradical filtration, in terms of semi-simple objects in $\mathcal{H}^{ss}$. In practice, we often wish to fix a full Tannakian subcategory of pure objects in $\mathcal{H}^{ss}$ (such as the one generated by Tate objects $\mathbb{Q}(n)$), and consider all $\mathcal{H}$-periods of objects whose semi-simplifications are of this type (periods of mixed Tate objects, in this case). The above results give a precise description for the structure of $\mathcal{H}$-periods of this type.

In order for this to be an accurate reflection of the structure of motivic periods, we need to know something about the image of the decomposition map Φ on the subspace $\mathcal{P}^{\mathfrak{m},+}$, which we address presently.

6.4. Speculation and context

Recall that $\mathcal{P}^{\mathfrak{m},+} \subset \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}}$ was the ring of motivic periods, i.e., those which come from the cohomology of an algebraic variety, and G^{dR} is the quotient of $G^{dR}_{\mathcal{H}}$ acting faithfully on $\mathcal{P}^{\mathfrak{m},+}$. Let U^{dR} denote its unipotent radical. Let $\mathcal{P}^{\mathfrak{m},+}_{ss} = (\mathcal{P}^{\mathfrak{m},+})^{U^{dR}}$ denote the invariants under U^{dR}, and set

$$\mathcal{M} = C^1 \mathcal{O}(U^{dR})_+ = \mathrm{Prim}(\mathcal{O}(U^{dR})).$$

Via (6.1) we think of $\mathcal{M}$ as 'motivic' extension classes.

Conjecture 3. The decomposition map induces an isomorphism

$$\Phi : \mathrm{gr}^{\mathcal{C}} \mathcal{P}^{\mathfrak{m},+} \longrightarrow \mathcal{P}^{\mathfrak{m},+}_{ss} \otimes_{\mathbb{Q}} T^c(\mathcal{M}).$$

This conjecture is a generalisation of Goncharov's freeness conjecture for mixed Tate motives, and states that there should be no relations between the decompositions of motivic periods. We can now try to describe the constituent pieces. This is all conjectural, so we shall be brief.

The putative Tannakian category of mixed motives over $\mathbb{Q}$ should have a functor

$$\mathcal{MM}_{\mathbb{Q}} \xrightarrow{h} \mathcal{H}$$

where $h = (\omega_B, \omega_{dR}, \mathrm{comp}_{B,dR})$ is fully faithful and hence morphisms

$$\mathrm{Ext}^1_{\mathcal{MM}_{\mathbb{Q}}}(\mathbb{Q}, M) \hookrightarrow \mathrm{Ext}^1_{\mathcal{H}}(\mathbb{Q}, h(M)) \xrightarrow{\otimes \mathbb{R}} \mathrm{Ext}^1_{\mathcal{H}(\mathbb{R})}(\mathbb{R}, h(M) \otimes \mathbb{R}).$$

One expects $\mathcal{P}^{\mathfrak{m},+}_{ss}$ to be generated by the cohomology of smooth projective algebraic varieties X over $\mathbb{Q}$. There is a definition for the group $\mathrm{Ext}^1_{\mathcal{MM}_{\mathbb{Q}}}(\mathbb{Q}(0), M)$, when $M = H^p(X)(q)$, in terms of motivic cohomology

[43]. Thus we expect $\mathcal{M}$ to be generated by the image of $H^{p+1}_{\mathcal{M}}(X, \mathbb{Q}(q)) \otimes H^p_{dR}(X)^\vee(-q)$ in the right-hand side of (6.1). Here, motivic cohomology $H^{p+1}_{\mathcal{M}}(X, \mathbb{Q}(q))$ can be defined either as a piece of the Adams grading of the algebraic K-theory of X, or via Bloch's higher Chow groups. Finally, Beilinson's conjectures predict the rank of these groups. In the simplest possible case when M is pure and of weight ≤ -3, then the image of $\mathrm{Ext}^1_{\mathcal{MM}_\mathbb{Q}}(\mathbb{Q}, M)$ in $\mathrm{Ext}^1_{\mathcal{H}(\mathbb{R})}(\mathbb{R}, h(M) \otimes \mathbb{R})$ should be a lattice and its rank given by (6.4). See [44] for further details.

Putting these conjectural pieces together gives a fairly complete but highly speculative picture for the structure of the ring of motivic periods. In particular, we obtain a precise prediction for the 'size' of the ring of motivic periods of given types. A strategy that one can pursue is to fix a given tensor category of pure motives (for example Tate motives), and try to construct geometrically the iterated extensions (or equivalently, their motivic periods). The decomposition map is a tool to show, by computing periods, that the extensions one has are independent (§5.4, §5.5)

Remark 6.8. A more detailed account of this subject would include a discussion of regulators and special values of L-functions. In the present framework, one can translate Deligne's conjecture on critical L-values as giving a formula for certain motivic periods of unipotency degree 0. Beilinson's conjectures give a formula for certain determinants of motivic periods of unipotency degree 1. It is tantalising to speculate that this might be the beginning of a tower of conjectural formulae describing certain periods of all higher unipotency degrees.

6.5. Representatives for primitive elements

In this section we return to general $\mathcal{H}$-periods. One would like to represent elements of $\mathrm{gr}^C_1 \mathcal{O}(U^{dR}_\mathcal{H})$ as concretely as possible. It follows from Corollary 2.13 that the decomposition in degree one

$$\Phi : \mathrm{gr}^C_1 \mathcal{P}^{\mathfrak{m}}_\mathcal{H} \longrightarrow \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}^{ss}} \otimes_\mathbb{Q} \mathrm{gr}^C_1 \mathcal{O}(U^{dR}_\mathcal{H})$$

is surjective. However, there is no canonical map from $\mathrm{gr}^C_1 \mathcal{O}(U^{dR}_\mathcal{H})$ to $\mathcal{P}^{\mathfrak{m}}_\mathcal{H}$, nor can we assign a period to an element of $\mathrm{gr}^C_1 \mathcal{O}(U^{dR}_\mathcal{H})$ in any obvious way. In some special cases, one can in fact assign numbers to primitive elements via the following two constructions:

1) Call $f \in \mathrm{Prim}(\mathcal{O}(U^{dR}_\mathcal{H}))$ *stable* if $F^1 M_f = M_f$ and M_f is effective. This implies that all its Hodge numbers $h_{p,q}$ vanish for $p \leq 0$ or $q \leq 0$. In

this case, a representative for the extension class $\mathcal{E}_f$ is separated (§4.3), and its de Rham realisation splits by (4.5) :

$$(\mathcal{E}_f)_{dR} \cong \mathbb{Q} \oplus (M_f)_{dR}.$$

Thus we can view $f \in (M_f)_{dR} = F^1(\mathcal{E}_f)_{dR}$ and $1 \in \mathbb{Q}_{dR}^\vee = F^0 \mathcal{E}_{dR}^\vee$ and define a canonical de Rham period

$$\xi_f = [\mathcal{E}_f, 1, f]^{\mathfrak{dr}} \in \mathcal{P}_{\mathcal{H}}^{\mathfrak{dr}}.$$

Its single-valued version $\mathbf{s}^{\mathfrak{m}}(\xi_f)$ lies in $\mathcal{P}_{\mathcal{H}}^{\mathfrak{m}}$ and we can take its period to obtain a number. The action of $G_{\mathcal{H}}^{dR}$ on $\mathbf{s}^{\mathfrak{m}}(\xi_f)$ is compatible with the conjugation action of $G_{\mathcal{H}}^{dR}$ on ξ_f.

2) As in (1), but also assume that $(M_f)_B^+ = 0$. Then $(\mathcal{E}_f^\vee)_B^+ \cong \mathbb{Q}_B^\vee$, and $1 \in \mathbb{Q}_B^\vee$ lifts to $1 \in (\mathcal{E}_f^\vee)_B^+$. We can directly define a motivic period $[\mathcal{E}_f, 1, f]^{\mathfrak{m}} \in \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}}$. Taking its period assigns a number to such a primitive element.

6.6. Example: Mixed Tate motives over $\mathbb{Q}$

One of the few situations in which Beilinson's conjectures are completely known is the category $\mathcal{MT}(\mathbb{Q})$ of mixed Tate motives over $\mathbb{Q}$. Its simple objects are Tate motives $\mathbb{Q}(n)$. The real Frobenius F_∞ acts on $\mathbb{Q}(n)_B$ by $(-1)^n$. Thus

$$\mathrm{Ext}^1_{\mathcal{MT}(\mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(n)) \longrightarrow \mathrm{Ext}^1_{\mathcal{H}}(\mathbb{Q}(0), \mathbb{Q}(n))$$

whose image has rank one if n is odd, and zero if n is even. In this case, Beilinson's conjecture is known as a consequence of deep theorems due to Borel, and we have

$$(6.5) \qquad \mathrm{Ext}^1_{\mathcal{MT}(\mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(n)) \cong K_{2n-1}(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

which has rank 1 for $n \geq 3$ odd and rank 0 for n even. For $n = 1$,

$$(6.6) \qquad \mathrm{Ext}^1_{\mathcal{MT}(\mathbb{Q})}(\mathbb{Q}(0), \mathbb{Q}(1)) \cong K_1(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}^* \otimes_{\mathbb{Z}} \mathbb{Q}$$

is isomorphic to the infinite dimensional $\mathbb{Q}$-vector space with one generator for every prime p. Furthermore, all higher Ext groups vanish. It follows that

$$H^1(U_{\mathcal{MT}(\mathbb{Q})}^{dR}) = \bigoplus_{n \geq 1} \left(K_{2n-1}(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q}(-n)_{dR} \right).$$

Let $\mathcal{P}^{\mathrm{m},+}_{\mathcal{MT}(\mathbb{Q})}$ denote the ring of effective periods of $\mathcal{MT}(\mathbb{Q})$. The subspace of semi-simple periods is generated by $\mathbb{L}^{\mathrm{m}}$. The decomposition is an isomorphism

$$(6.7) \qquad \mathrm{gr}^{C}_{\bullet} \mathcal{P}^{\mathrm{m},+}_{\mathcal{MT}(\mathbb{Q})} \xrightarrow{\sim} \mathbb{Q}[\mathbb{L}^{\mathrm{m}}] \otimes T^{c}\left(\bigoplus_{n \geq 1} K_{2n-1}(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q}(-n)\right).$$

The Tate objects $\mathbb{Q}(n)$ satisfy both conditions (1) and (2) of §6.5. A generator f_{2n-1} of the image of (6.5) in $\mathrm{Ext}^{1}_{\mathcal{H}}$ gives a rational multiple of $\zeta^{\mathrm{m}}(2n-1)$ under the second prescription. Choose the rational multiple to be one.[10] Similarly, choose generators ν_{p} of (6.6) which correspond under (2) to $\log^{\mathrm{m}}(p)$ for p prime.

With this choice of generators, there is an isomorphism

$$T^{c}(H^{1}(U^{dR}_{\mathcal{MT}(\mathbb{Q})})) \cong \mathbb{Q}\langle \nu_{p}, f_{3}, f_{5}, f_{7}, \ldots \rangle$$

where the right-hand side denotes the shuffle algebra (tensor coalgebra) on generators ν_{p}, for p prime, which span a copy of $\mathbb{Q}(-1)$ of weight 2, and f_{2n+1}, for $n \geq 1$ which span a copy of $\mathbb{Q}(-1-2n)$ of weight $4n+2$.

Theorem 6.9. *The decomposition into primitives* (6.7) *gives an isomorphism*

$$(6.8) \qquad \mathrm{gr}^{C}\mathcal{P}^{\mathrm{m},+}_{\mathcal{MT}(\mathbb{Q})} \xrightarrow{\sim} \mathbb{Q}[\mathbb{L}^{\mathrm{m}}] \otimes \mathbb{Q}\langle \nu_{p}, f_{3}, f_{5}, f_{7}, \ldots \rangle,$$

where $\mathbb{L}^{\mathrm{m}}$ is the Lefschetz period §5.2.

Since in this case we can associate to each f_{2n-1} its canonical period (2) of §6.5 which equals $\zeta(2n-1)$, this gives an elementary way to think about periods of mixed Tate motives over $\mathbb{Q}$ as formal words in tensor products of odd zeta values $\zeta(2n-1)$ and logarithms of prime numbers. See [15] for examples.

Remark 6.10. Beilinson's conjectures hold more generally for mixed Tate motives over number fields. Since for the time being we are considering only periods over $\mathbb{Q}$, this discussion is postponed to §10.1.

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[10]Note that prescription (1) applied to f_{2n-1} gives the single-valued motivic zeta value which is exactly double that, namely $2\zeta^{\mathrm{m}}(2n-1)$.

7. Families of periods

We now sketch a possible formalism for studying periods varying in a family, and explain how this generalises several concepts which have been used in the physics literature in the case of iterated integrals and polylogarithms.

7.1. Vector bundles and local systems

See [21], §10.24-10.52. Let S be a smooth geometrically connected scheme over a field $k \subset \mathbb{C}$. An algebraic vector bundle V on S is a sheaf of locally free $\mathcal{O}_S$-modules of finite type. Denote the corresponding analytic vector bundle on S^{an} by $V^{\mathrm{an}} = \mathcal{O}_{S^{\mathrm{an}}} \otimes_{\mathcal{O}_S} V$. Consider the category

$$\mathcal{A}(S) = \text{Algebraic vector bundles on } S \text{ equipped with an}$$
$$\text{integrable connection with regular singularities at infinity.}$$

Let ω denote the functor which to any object of $\mathcal{A}(S)$ associates the underlying vector bundle and forgets the connection. The category $\mathcal{A}(S)$ is a Tannakian category over k, and ω is a fiber functor over S.

Definition 7.1. The de Rham algebraic fundamental groupoid is the groupoid (in the category of schemes over k, acting on S) defined by

$$\pi_1^{\mathrm{alg},dR}(S) = \mathrm{Aut}_{\mathcal{A}(S)}^{\otimes}(\omega).$$

Consider also the category

$$\mathcal{L}(S) = \text{Local systems of finite-dimensional } k\text{-vector spaces on } S(\mathbb{C}).$$

For any complex point $t \in S(\mathbb{C})$, the 'fiber at t' defines a functor $\omega_t : \mathcal{L}(S) \to \mathrm{Vec}_k$, and $\mathcal{L}(S)$, equipped with ω_t, is a neutral Tannakian category over k.

Definition 7.2. The Betti algebraic fundamental group is the affine group scheme over k defined by the Tannaka group of $\mathcal{L}(S)$

$$\pi_1^{\mathrm{alg},B}(S, t) = \mathrm{Aut}_{\mathcal{L}(S)}^{\otimes}(\omega_t).$$

Given two complex points $t, t' \in S(\mathbb{C})$, the fibers of the Betti algebraic groupoid over (t, t') are $\pi_1^{\mathrm{alg},B}(S, t, t') = \mathrm{Isom}_{\mathcal{L}(S)}^{\otimes}(\omega_t, \omega_{t'})$.

Denote the ordinary topological fundamental group of $S(\mathbb{C})$ at a point $t \in S(\mathbb{C})$ by $\pi_1^{\text{top}}(S(\mathbb{C}), t)$, and recall that there is an equivalence of categories

$$\mathcal{L}(S) \xrightarrow{\sim} \text{Finite-dimensional representations over } k \text{ of } \pi_1^{\text{top}}(S(\mathbb{C}), t),$$

which to a local system associates its fiber at t together with its action of the topological fundamental group. Thus every element of $\pi_1^{\text{top}}(S(\mathbb{C}), t)$ defines an automorphism of the fiber functor ω_t, giving a natural homomorphism

$$(7.1) \qquad\qquad \pi_1^{\text{top}}(S(\mathbb{C}), t) \longrightarrow \pi_1^{\text{alg}, B}(S, t)(k)$$

which is Zariski-dense. Similarly there is a natural morphism of groupoids

$$\pi_1^{\text{top}}(S(\mathbb{C}), t, t') \longrightarrow \pi_1^{\text{alg}, B}(S, t, t')(k)$$

where $\pi_1^{\text{top}}(S(\mathbb{C}), t, t')$ are homotopy classes of paths from t to t' in $S(\mathbb{C})$.

Recall that the Riemann-Hilbert correspondence [25], [21] 10.32(a), is an equivalence of categories $\mathcal{A}(S \times \mathbb{C}) \to \mathcal{L}(S \times \mathbb{C})$ over $\mathbb{C}$ ([21] 10.35). To a complex vector bundle with integrable connection (V, ∇) it assigns the locally constant sheaf of flat sections $(V^{\text{an}})^{\nabla}$ of the corresponding analytic bundle. Thus there is an isomorphism of affine groupoid schemes over $\mathbb{C}$

$$\pi_1^{\text{alg}, B}(S \times \mathbb{C}, a, b) \xrightarrow{\sim} \pi_1^{\text{alg}, dR}(S \times \mathbb{C}, a, b),$$

where $\pi_1^{\text{alg}, dR}(S, a, b)$ denotes the fiber of $\pi_1^{\text{alg}, dR}(S \times \mathbb{C})$ over $(a, b) \in (S \times S)(\mathbb{C})$, or equivalently the affine group scheme over $\mathbb{C}$ given by $\text{Isom}^{\otimes}_{\mathcal{A}(S \times \mathbb{C})}(\omega_a, \omega_b)$ where ω_a, ω_b denote the functors 'fiber at a, b' respectively.

Remark 7.3. The basepoint s can be replaced by any simply-connected subset $X \subset S(\mathbb{C})$. For any point $x \in X$, the fiber at x defines a fiber functor $\omega_x : \mathcal{L}(S) \to \text{Vec}_k$. A path γ from x to x' defines an isomorphism of fiber functors $\omega_x \xrightarrow{\sim} \omega_{x'}$. Since X is simply connected, such a path is unique up to homotopy, and the isomorphism $\omega_x = \omega_{x'}$ is canonical. Thus ω_x depends only on x up to unique isomorphism. We shall abusively denote the resulting fiber functor, for any choice of $x \in X$, by ω_X. Likewise, one can define the fundamental group of $S(\mathbb{C})$ based at X, which we denote by $\pi_1^{\text{top}}(S(\mathbb{C}), X)$.

7.2. A category of realizations

Let $k = \mathbb{Q}$ and S be as in the previous paragraph. Based on [21] §1.21, consider the category $\mathcal{H}(S)$ consisting of triples

$$(\mathcal{V}_B, \mathcal{V}_{dR}, c)$$

given by the following data:

1) A local system $\mathcal{V}_B$ of finite-dimensional $\mathbb{Q}$-vector spaces over $S(\mathbb{C})$, equipped with a finite increasing filtration $W_\bullet \mathcal{V}_B$ of local sub-systems.

2) An algebraic vector bundle $\mathcal{V}_{dR}$ on S in $\mathcal{A}(S)$ equipped with an integrable connection $\nabla : \mathcal{V}_{dR} \to \mathcal{V}_{dR} \otimes_{\mathcal{O}_S} \Omega^1_S$ with regular singularities at infinity, a finite increasing filtration $W_\bullet \mathcal{V}_{dR}$ of $\mathcal{V}_{dR}$ by sub-objects in $\mathcal{A}(S)$, and a finite decreasing filtration $F^\bullet$ of algebraic sub-bundles satisfying Griffiths transversality $\nabla : F^p \mathcal{V}_{dR} \subset F^{p-1} \mathcal{V}_{dR} \otimes_{\mathcal{O}_S} \Omega^1_S$.

3) An isomorphism of analytic vector bundles with connexion

$$c : \mathcal{V}_{dR}^{\mathrm{an}} \overset{\sim}{\longrightarrow} \mathcal{V}_B \otimes_{\mathbb{Q}} \mathcal{O}_{S^{\mathrm{an}}},$$

which respects the filtrations W, and where the connexion on $\mathcal{V}_B \otimes_{\mathbb{Q}} \mathcal{O}_{S^{\mathrm{an}}}$ is the one for which sections of $\mathcal{V}_B$ are flat. This is equivalent to an isomorphism $(\mathcal{V}_{dR}^{\mathrm{an}})^\nabla \cong \mathcal{V}_B \otimes_{\mathbb{Q}} \mathbb{C}$ of local systems of complex vector spaces on $S(\mathbb{C})$ which respects the weight filtrations on both sides.

4) The data $\mathcal{V}_B, c$ is functorial in the choice of algebraic closure $\mathbb{C}$ of $\mathbb{R}$. In particular, there is an isomorphism of local systems

$$F_\infty : \mathcal{V}_B \overset{\sim}{\longrightarrow} \sigma^* \mathcal{V}_B$$

where $\sigma : S(\mathbb{C}) \overset{\sim}{\to} S(\mathbb{C})$ is induced by complex conjugation.

This data is subject to the following conditions:

- At each point $t \in S(\mathbb{C})$, the vector space $(\mathcal{V}_B)_t$ equipped with the filtration W and cF on $(\mathcal{V}_B)_t \otimes_{\mathbb{Q}} \mathbb{C}$ is a graded-polarisable mixed Hodge structure.

- We shall not consider taking limits in these notes, but if one wishes to, one should add further constraints [46] to demand that $\mathcal{V}_B$ defines an admissible variation of mixed Hodge structures, has locally quasi-unipotent monodromy, admits relative weight filtrations, and so on.

- Let $\mathcal{O}_{\overline{S}^{\mathrm{an}}}$ denote the sheaf of antiholomorphic functions on S^{an}. Pulling back the comparison (3) to $\overline{S}(\mathbb{C})$ via σ^* induces an $\mathcal{O}_{\overline{S}^{\mathrm{an}}}$-linear isomorphism $\overline{c} : \mathcal{V}_{dR} \otimes_{\mathcal{O}_S} \mathcal{O}_{\overline{S}^{\mathrm{an}}} \xrightarrow{\sim} \sigma^*(\mathcal{V}_B) \otimes_{\mathbb{Q}} \mathcal{O}_{\overline{S}^{\mathrm{an}}}$. The following diagram commutes:

$$
\begin{array}{ccc}
c : \mathcal{V}_{dR} \otimes_{\mathcal{O}_S} \mathcal{O}_{S^{\mathrm{an}}} & \xrightarrow{\sim} & \mathcal{V}_B \otimes_{\mathbb{Q}} \mathcal{O}_{S^{\mathrm{an}}} \\
\downarrow & & \downarrow \\
\overline{c} : \mathcal{V}_{dR} \otimes_{\mathcal{O}_S} \mathcal{O}_{\overline{S}^{\mathrm{an}}} & \xrightarrow{\sim} & \sigma^*(\mathcal{V}_B) \otimes_{\mathbb{Q}} \mathcal{O}_{\overline{S}^{\mathrm{an}}}
\end{array}
$$

where the vertical map on the left (resp. right) is the identity on $\mathcal{V}_{dR}$ (resp. $F_\infty : \mathcal{V}_B \to \sigma^*(\mathcal{V}_B)$) tensored with the map $f \mapsto \overline{f} : \mathcal{O}_{S^{\mathrm{an}}} \to \mathcal{O}_{\overline{S}^{\mathrm{an}}}$.

The morphisms in $\mathcal{H}(S)$ respect the above data.

The category $\mathcal{H}(S)$ is Tannakian and has exact, faithful, tensor functors:

$$
\begin{array}{ccccccc}
\omega_{dR} : \mathcal{H}(S) & \longrightarrow & \mathcal{A}(S) & , & \omega_B : \mathcal{H}(S) & \longrightarrow & \mathcal{L}(S) \\
(\mathcal{V}_B, \mathcal{V}_{dR}, c) & \mapsto & \mathcal{V}_{dR} & & (\mathcal{V}_B, \mathcal{V}_{dR}, c) & \mapsto & \mathcal{V}_B.
\end{array}
$$

One can think of c as an isomorphism of functors (not strictly speaking fiber functors) from $\omega_{RH} \circ \omega_{dR}$ to $\omega_B \otimes \mathbb{C}$, where $\omega_{RH} : \mathcal{A}(S) \to \mathcal{L}(S) \otimes \mathbb{C}$ is $\mathcal{V} \mapsto (\mathcal{V}^{\mathrm{an}})^\nabla$.

7.2.1. Fiber functors.
In order to motivate the following definitions, note that:

- families of periods (such as the dilogarithm $\mathrm{Li}_2(x)$) are multivalued functions, i.e., functions on a universal covering space of $S(\mathbb{C})$. In applications, we are often given a region $X \subset S(\mathbb{C})$ where the function has a prescribed branch, or, for example, a local Taylor expansion (the dilogarithm has an expansion $\sum_{n \geq 1} \frac{x^n}{n^2}$ which converges on $X = \{x \in \mathbb{C} : |x| < 1\}$). This is the region where the chain of integration is unambiguous.

- in some applications, including Feynman integrals, we cannot control the poles of the integrand. However we may be given a region $Y \subset S(\mathbb{C})$ which is guaranteed to be free of poles. In this region, the integrals are finite and can be evaluated (bearing in mind that the integrals may also be multi-valued). We shall allow the region Y to be empty.

Having made these preliminary comments, we now define Betti and de Rham fiber functors relative to some extra data X, Y, as follows. First of all,

for any simply-connected $X \subset S(\mathbb{C})$ we have, by Remark 7.3, a fiber functor

$$\omega_B^X = \omega_X \omega_B : \mathcal{H}(S) \longrightarrow \mathrm{Vec}_{\mathbb{Q}}$$

which neutralizes $\mathcal{H}(S)$ over $\mathbb{Q}$. On the other hand, the functor

$$\omega_{dR} : \mathcal{H}(S) \longrightarrow S$$

which to a triple $(\mathcal{V}_B, \mathcal{V}_{dR}, c)$ associates the vector bundle underlying $\mathcal{V}_{dR}$ and forgets the connection, is a fiber functor according to the definition §2. For any morphism $u : T \to S$ of schemes over $\mathbb{Q}$, the composite of ω followed by u^* defines a fiber functor over T. Now consider a region $Y \subset S(\mathbb{C})$, and furthermore assume that Y is contained in some $U(\mathbb{C})$ where $U \subset S$ is affine. Then we can consider the ring $\mathcal{O}_{S,Y} = \varinjlim_U \mathcal{O}_U$ where the limit is over all open affine $U \subset S$ such that $Y \subset U(\mathbb{C})$. It is non-zero by assumption. Define a fiber functor by pulling back along the morphism $u_Y : \mathrm{Spec}\,(\mathcal{O}_{S,Y}) \to S$, which we denote by

$$\omega_{dR}^Y = u_Y^* \omega_{dR} : \mathcal{H}(S) \longrightarrow \mathrm{Proj}(\mathcal{O}_{S,Y})$$

and takes values in the category of projective modules of finite type over $\mathcal{O}_{S,Y}$. The fiber functor ω_{dR}^Y is simply $\Gamma(\mathcal{O}_{S,Y}, \bullet)$. We shall mainly consider:

1) $Y = \emptyset$. Then u_Y is the generic point of S, and $\mathcal{O}_{S,Y} = K_S$, where K_S is the field of fractions of S. Our fiber functor is

$$\omega_{dR}^{\mathrm{gen}} : \mathcal{H}(S) \longrightarrow \mathrm{Vec}_{K_S}.$$

2) Let $Y = \{s\}$ where $s \in S(\mathbb{Q}) \subset S(\mathbb{C})$ is a rational point of S. Then $\mathcal{O}_{Y,S} = \mathcal{O}_s$ is the local ring of S at s. The fiber functor

$$\omega_{dR}^s : \mathcal{H}(S) \longrightarrow \mathrm{Proj}(\mathcal{O}_s),$$

takes values in projective (hence free) modules over $\mathcal{O}_s$ of finite type.

3) $S = \mathrm{Spec}\,B$ is affine, and $Y = S(\mathbb{C})$. Then $\mathcal{O}_{S,Y} = \mathcal{O}_S = B$, and the fiber functor $\omega_{dR}^S : \mathcal{H}(S) \to \mathrm{Proj}(B)$ is the global sections functor $\Gamma(S, \omega(\mathcal{V}_{dR}))$.

Denote the corresponding Tannaka groups by $G^B_{\mathcal{H}(S),X} = Aut^{\otimes}_{\mathcal{H}(S)}(\omega^X_B)$, and $G^{dR}_{\mathcal{H}(S),Y} = Aut^{\otimes}_{\mathcal{H}(S)}(\omega^Y_{dR})$. The functor ω_{dR} gives a morphism

$$(7.2) \qquad \pi_1^{\mathrm{alg},dR}(S, \omega^Y_{dR}) \longrightarrow G^{dR}_{\mathcal{H}(S),Y}$$

of affine group schemes over $\mathcal{O}_{S,Y}$, where $\pi_1^{\mathrm{alg},dR}(S, \omega^Y_{dR}) = Aut^{\otimes}_{\mathcal{A}(S)}(\omega^Y_{dR})$. Similarly, the functor ω^X_B defines a morphism $\pi_1^{\mathrm{alg},B}(S, X) \to G^B_{\mathcal{H}(S),X}$ of affine group schemes over $\mathbb{Q}$, and in particular a *monodromy homomorphism*:

$$\pi_1^{\mathrm{top}}(S(\mathbb{C}), X) \longrightarrow G^B_{\mathcal{H}(S),X}(\mathbb{Q}).$$

7.3. Ring of $\mathcal{H}(S)$-periods.

Let $X, Y \subset S(\mathbb{C})$ be as above, and let ω^X_B, ω^Y_{dR} be the corresponding fiber functors on $\mathcal{H}(S)$. Define a ring

$$\mathcal{P}^{\mathfrak{m},X,Y}_{\mathcal{H}(S)} = \mathcal{O}(\mathrm{Isom}^{\otimes}_{\mathcal{H}(S)}(\omega^Y_{dR}, \omega^X_B))$$

of matrix coefficients (denoted $\mathcal{P}^{\omega^X_B, \omega^Y_{dR}}_{\mathcal{H}(S)}$ as in §2.2), where $B_1 = \mathbb{Q}, B_2 = \mathcal{O}_{S,Y}$, and $k = \mathbb{Q}$. It is a $\mathbb{Q} \otimes_{\mathbb{Q}} \mathcal{O}_{S,Y}$-bimodule, and generated by matrix coefficients

$$(7.3) \qquad [(\mathcal{V}_B, \mathcal{V}_{dR}, c), \sigma, \omega]^{\mathfrak{m}}$$

where $\sigma \in \omega^X_B \mathcal{V}^{\vee}_B$, $\omega \in \omega^Y_{dR} \mathcal{V}_{dR}$. Similarly, define a ring of 'de Rham periods' to be $\mathcal{P}^{\mathfrak{dr},Y}_{\mathcal{H}(S)} = Aut^{\otimes}_{\mathcal{H}(S)}(\omega^Y_{dR})$. It is an $\mathcal{O}_{S,Y} \otimes_{\mathbb{Q}} \mathcal{O}_{S,Y}$-bimodule. It is generated by matrix coefficients $[(\mathcal{V}_B, \mathcal{V}_{dR}, c), v, \omega]^{\mathfrak{m}}$ where ω is as above and $v \in \omega^Y_{dR} \mathcal{V}^{\vee}_{dR}$.

These rings are functorial in the following way. Let S' be a smooth geometrically connected scheme over $\mathbb{Q}$, and $f : S \to S'$ a smooth morphism. Let $X \subset S'(\mathbb{C})$ be simply-connected such that $f(X) \subseteq X'$, and let $Y' \subset S'(\mathbb{C})$ such that $f(Y) \subseteq Y'$. The pull-back defines a functor $f^* : \mathcal{H}(S') \to \mathcal{H}(S)$ and hence a morphism

$$(7.4) \qquad f^* : \mathcal{P}^{\mathfrak{m},X',Y'}_{\mathcal{H}(S')} \to \mathcal{P}^{\mathfrak{m},X,Y}_{\mathcal{H}(S)}$$

and a similar map on replacing $\mathfrak{m}$ by $\mathfrak{dr}$ and making the obvious changes.

7.3.1. Constants. Now apply this to $S' = \mathrm{Spec}\, \mathbb{Q}$, $X' = Y' = S'(\mathbb{C})$ (example (3) of the previous section) and $f : S \to \mathrm{Spec}\,(\mathbb{Q})$ the structural map. One checks that the category $\mathcal{H}(S')$ is equivalent to $\mathcal{H}$, and that the Betti and de Rham fiber functors on $\mathcal{H}$ and $\mathcal{H}(S')$ coincide. Therefore

$$\mathcal{P}^{\mathfrak{m},pt,pt}_{\mathcal{H}(\mathrm{Spec}\,(\mathbb{Q}))} = \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}},$$

and we obtain canonical homomorphisms (*'constant' maps*)

$$(7.5) \qquad\qquad \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}} \longrightarrow \mathcal{P}^{\mathfrak{m},X,Y}_{\mathcal{H}(S)} \quad \text{and} \quad \mathcal{P}^{\mathfrak{dr}}_{\mathcal{H}} \longrightarrow \mathcal{P}^{\mathfrak{dr},Y}_{\mathcal{H}(S)}$$

In this way, $\mathcal{H}$-periods can be viewed as 'constant' $\mathcal{H}(S)$-periods, since the functor $f^* : \mathcal{H} \to \mathcal{H}(S)$ associates to (V_B, V_{dR}, c) a triple $(\mathcal{V}_B, \mathcal{V}_{dR}, c)$ where $\mathcal{V}_{dR} = V_{dR} \otimes_{\mathbb{Q}} \mathcal{O}_S$ with $\nabla = \mathrm{id} \otimes d$, and $\mathcal{V}_B$ is the constant local system with fibres V_B.

7.3.2. Evaluation. Now suppose that there is a rational point

$$(7.6) \qquad\qquad t \in S(\mathbb{Q}) \quad \text{such that} \quad t \in X \cap Y.$$

There are *evaluation maps* at the point t

$$(7.7) \qquad\quad \mathrm{ev}_t : \mathcal{P}^{\mathfrak{m},X,Y}_{\mathcal{H}(S)} \longrightarrow \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}} \quad \text{and} \quad \mathrm{ev}_t : \mathcal{P}^{\mathfrak{dr},Y}_{\mathcal{H}(S)} \longrightarrow \mathcal{P}^{\mathfrak{dr}}_{\mathcal{H}},$$

which are induced by the functor 'fiber at t' from $\mathcal{H}(S) \to \mathcal{H}$ via $t : \mathcal{O}_{S,Y} \to \mathbb{Q}$. The constant maps (7.5) are sections of the evaluation maps (7.7). Note that one may wish to weaken the condition (7.6) if one bears in mind that for $t \notin X$ the evaluation map is not well-defined (it is ambiguous up to the action of monodromy), and for $t \notin Y$ the evaluation may be infinite due to the presence of poles.

7.4. Some properties of $\mathcal{H}(S)$-periods

The ring $\mathcal{P}^{\mathfrak{m},X,Y}_{\mathcal{H}(S)}$ has a left *Galois action* by the group $G^{dR,Y}_{\mathcal{H}(S)}$, or equivalently, a right coaction

$$(7.8) \qquad\qquad \Delta^{\mathfrak{m}} : \mathcal{P}^{\mathfrak{m},X,Y}_{\mathcal{H}(S)} \longrightarrow \mathcal{P}^{\mathfrak{m},X,Y}_{\mathcal{H}(S)} \otimes_{\mathcal{O}_{S,Y}} \mathcal{P}^{\mathfrak{dr},Y}_{\mathcal{H}(S)},$$

given by the same formula as (2.2). Since the fiber functor ω^Y_{dR} factors through ω_{dR}, the action of $G^{dR,Y}_{\mathcal{H}(S)}$ restricts to an action of the algebraic de

Rham fundamental group $\pi_1^{dR,\mathrm{alg}}(S, \omega_{dR}^Y)$ via the map (7.2). More generally, there is an action of the de Rham algebraic fundamental groupoid:

$$\pi_1^{dR,\mathrm{alg}}(S, \omega_{dR}^{Y_1}, \omega_{dR}^{Y_2}) \times \mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{m},X,Y_1} \longrightarrow \mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{m},X,Y_2}$$

for any $Y_1, Y_2 \subset S(\mathbb{C})$ as above, where the left-hand side is $\mathrm{Isom}_{\mathcal{A}(S)}^{\otimes}(\omega_{dR}^{Y_1}, \omega_{dR}^{Y_2})$.

The ring $\mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{m},X,Y}$ has an increasing *weight filtration* $W_{\bullet}$ which is inherited from the weight filtration on the category $\mathcal{H}(S)$ (specifically, W_n is generated by matrix coefficients (7.3) such that the de Rham class satisfies $\omega \in \omega_{dR}^Y W_n \mathcal{V}_{dR}$), and is preserved by $G_{\mathcal{H}(S)}^{dR,Y}$. The morphisms (7.5) and (7.7) preserve the weight filtrations. The same comments apply to the rings of de Rham periods $\mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{dr},Y}$. This notion, applied in the setting [13] gives a rigorous meaning to intuitive notions of 'transcendental weight' of functions in the physics literature.

The ring $\mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{m},X,Y}$ also has a right action by $\pi_1^{\mathrm{alg},B}(S, X)$, and in particular, a right action by the topological fundamental group or *monodromy action*:

$$(7.9) \qquad \mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{m},X,Y} \times \pi_1^{\mathrm{top}}(S(\mathbb{C}), X) \longrightarrow \mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{m},X,Y}.$$

It commutes with the action of $G_{\mathcal{H}(S)}^{dR,Y}$ and also respects W. The monodromy action can be read off matrix coefficients (7.3) by its action on the Betti class σ, since the $\mathbb{Q}$-vector space $(\mathcal{V}_B)_X$ naturally carries a right $\pi_1^{\mathrm{top}}(S(\mathbb{C}), X)$-action. More generally, for any $X_1, X_2 \subset S(\mathbb{C})$ simply connected we have an action of the topological fundamental groupoid or *continuation along paths*

$$(7.10) \qquad \mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{m},X_1,Y} \times \pi_1^{\mathrm{top}}(S(\mathbb{C}), X_1, X_2) \longrightarrow \mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{m},X_2,Y}.$$

These actions commute with the action of $G_{\mathcal{H}(S)}^{dR,Y}$ and respect the weight filtration, since the latter are defined entirely in terms of the de Rham class.

The following structures on $\mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{m},X,Y}$ are *not preserved* by the action of $G_{\mathcal{H}(S)}^{dR,Y}$. By the Tannaka theorem 2.3, $\mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{m},X,Y}$ is the ω_{dR}^Y-image of an ind-object in $\mathcal{H}(S)$. Denote its image under ω_{dR} by $\widetilde{\mathcal{P}}_{\mathcal{H}(S)}^{\mathfrak{m},X,Y}$. It is an (infinite-dimensional) algebraic vector bundle on S, or ind-object of $\mathcal{A}(S)$, whose image under u_Y^*, the restriction to $\mathrm{Spec}\,(\mathcal{O}_{S,Y})$, is $\mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{m},X,Y}$. Furthermore, it is equipped with an increasing weight filtration W, decreasing Hodge filtration F, and an integrable connection

$$\nabla : \widetilde{\mathcal{P}}_{\mathcal{H}(S)}^{\mathfrak{m},X,Y} \longrightarrow \widetilde{\mathcal{P}}_{\mathcal{H}(S)}^{\mathfrak{m},X,Y} \otimes_{\mathcal{O}_S} \Omega_S^1$$

which satisfies Griffiths transversality. Restricting to $\mathrm{Spec}\,(\mathcal{O}_{S,Y})$, we deduce the existence of a *Hodge filtration* $F^\bullet\mathcal{P}^{\mathrm{m},X,Y}_{\mathcal{H}(S)}$ and *connection*

$$(7.11) \qquad\qquad \nabla : \mathcal{P}^{\mathrm{m},X,Y}_{\mathcal{H}(S)} \longrightarrow \mathcal{P}^{\mathrm{m},X,Y}_{\mathcal{H}(S)} \otimes_{\mathcal{O}_{S,Y}} \Omega^1_{\mathcal{O}_{S,Y}},$$

where $\Omega^1_{\mathcal{O}_{S,Y}}$ is the ring of Kähler differentials on $\mathcal{O}_{S,Y}$. The connection (7.11) is integrable, respects W, and satisfies Griffiths transversality with respect to F. On matrix coefficients the connection (7.11) is given by

$$\nabla[(\mathcal{V}_B, \mathcal{V}_{dR}, c), \sigma, \omega]^{\mathrm{m}} = [(\mathcal{V}_B, \mathcal{V}_{dR}, c), \sigma, \nabla\omega]^{\mathrm{m}},$$

where $\nabla : u_Y^*(\mathcal{V}_{dR}) \to u_Y^*(\mathcal{V}_{dR}) \otimes_{\mathcal{O}_{S,Y}} \Omega^1_{S,Y}$ is the connection on $\mathcal{V}_{dR}$ restricted to $u_Y : \mathrm{Spec}\,(\mathcal{O}_{S,Y}) \to S$. The space $F^n\mathcal{P}^{\mathrm{m}}_{\mathcal{H}(S),X,Y}$ is generated by matrix coefficients $[(\mathcal{V}_B, \mathcal{V}_{dR}, c), \sigma, \omega]^{\mathrm{m}}$ where $\omega \in \omega^Y_{dR} F^n\mathcal{V}_{dR}$. There is an analogous connection on the ring of left de Rham periods $\mathcal{P}^{dR}_{\mathcal{H}(S)}$, where the connection acts on $[(\mathcal{V}_B, \mathcal{V}_{dR}, c), v, \omega]^{\mathfrak{dr}}$ through its action on ω. From formula (2.2), namely

$$\Delta[V, \sigma, \omega]^{\mathrm{m}} = \sum_i [V, \sigma, e_i]^{\mathrm{m}} \otimes [V, e_i^\vee, \omega]^{\mathfrak{dr}}$$

one can check that the following diagram commutes

$$
\begin{array}{ccc}
\mathcal{P}^{\mathrm{m},X,Y}_{\mathcal{H}(S)} & \overset{\Delta}{\longrightarrow} & \mathcal{P}^{\mathrm{m},X,Y}_{\mathcal{H}(S)} \otimes_{\mathcal{O}_{S,Y}} \mathcal{P}^{\mathfrak{dr},Y}_{\mathcal{H}(S)} \\
\downarrow & & \downarrow \\
\mathcal{P}^{\mathrm{m},X,Y}_{\mathcal{H}(S)} \otimes_{\mathcal{O}_{S,Y}} \Omega^1_{\mathcal{O}_{S,Y}} & \overset{\Delta\otimes\mathrm{id}}{\longrightarrow} & \mathcal{P}^{\mathrm{m},X,Y}_{\mathcal{H}(S)} \otimes_{\mathcal{O}_{S,Y}} \mathcal{P}^{\mathfrak{dr},Y}_{\mathcal{H}(S)} \otimes_{\mathcal{O}_{S,Y}} \Omega^1_{S,Y}
\end{array}
$$

where the vertical map on the left is ∇, and on the right is $\mathrm{id} \otimes \nabla$. Hence

$$(7.12) \qquad\qquad (\Delta \otimes \mathrm{id})\nabla = (\mathrm{id} \otimes \nabla)\Delta,$$

which relates the Galois coaction to the connection. Since the connection (7.11) only invokes the de Rham framing, it commutes with the monodromy action (7.9).

Remark 7.4. The previous remarks give a proof of two formulae (5.23) conjectured in [28] in the case of the multiple polylogarithms (iterated integrals on the moduli space of curves of genus 0 with n marked points), and a generalisation to all families of motivic periods (and in particular, motivic Feynman amplitudes).

Finally, let us suppose that X, Y are preserved by a subgroup A of the group of automorphisms of S. Then the functoriality (7.4) gives rise to an action

$$(7.13) \qquad A \times \mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{m}, X, Y} \longrightarrow \mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{m}, X, Y}$$

of A on $\mathcal{H}(S)$-periods, and similarly on $\mathcal{P}_{\mathcal{H}(S), Y}^{\mathfrak{dr}}$ under the weaker assumption that only Y is stable under A.

7.5. The period homomorphism

Let $X, Y \subset S(\mathbb{C})$ with X simply-connected. The period is defined by pairing the Betti and de Rham classes of a matrix coefficient (7.3) using the comparison c. In order to obtain a multi-valued function on the whole of $S(\mathbb{C})$, and not just on X, these classes must be suitably extended as follows. Let $\pi : \widetilde{S}(\mathbb{C})_X \to S(\mathbb{C})$ denote the universal covering space of $S(\mathbb{C})$ based at X, and let $M_{X, Y}(S(\mathbb{C}))$ denote the ring of meromorphic functions on $\widetilde{S}(\mathbb{C})_X$ which have no poles on $\pi^{-1}(Y)$. By this we mean that for every $f \in M_{X, Y}(S(\mathbb{C}))$, and any $x \in \widetilde{S}(\mathbb{C})_X$, there exists a g, an element in the fraction field of S, such that $f \times \pi^{-1}(g)$ is analytic in some open neighbourhood of x. If $x \in \pi^{-1}(Y)$ then we can take $g = 1$, and f is already analytic in some neighbourhood of x. Elements of $M_{X, Y}(S(\mathbb{C}))$ can be thought of as multivalued meromorphic functions on $S(\mathbb{C})$ with a prescribed branch on the set X, and poles outside Y.

Suppose that Y is contained in the complex points of some open affine subset of S, as earlier. The *period map* is then a homomorphism

$$\mathrm{per} : \mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{m}, X, Y} \longrightarrow M_{X, Y}(S(\mathbb{C})),$$

and is defined on matrix coefficients $[(\mathcal{V}_B, \mathcal{V}_{dR}, c), \sigma, v]^{\mathfrak{m}}$ as follows. The local system $\pi^*(\mathcal{V}_B^\vee)$ is trivial on the simply connected space $\widetilde{S}(\mathbb{C})_X$ and σ extends to a unique global section $\sigma \in \Gamma(\widetilde{S}(\mathbb{C})_X, \pi^*(\mathcal{V}_B^\vee))$. Let $x \in \widetilde{S}(\mathbb{C})_X$, and let N_x be a sufficiently small neighbourhood of x such that the restriction of π to N_x is an isomorphism. We obtain a local section $\sigma_x \in \Gamma(\pi(N_x), \mathcal{V}_B^\vee)$, defined by $\sigma_x = (\pi|_{N_x}^{-1})^* \sigma$.

On the other hand, by assumption on Y, there exists an open affine $U \subset S$ with $Y \subset U(\mathbb{C})$ such that $v \in \Gamma(U, \mathcal{V}_{dR})$. Let $W \subset S$ be an open affine such that $\pi(x) \in W(\mathbb{C})$ and the restriction of $\mathcal{V}_{dR}$ to W is trivial as a vector bundle. Since S is irreducible, $U \cap W \neq \emptyset$ and we have $v|_{U \cap W} \in \Gamma(U \cap W, \mathcal{V}_{dR}) = \Gamma(W, \mathcal{V}_{dR}) \otimes_{\mathcal{O}_W} \mathcal{O}_{U \cap W}$. It can have poles on $W \backslash U$. We can

'clear its denominator', since $\mathcal{V}_{dR}$ is of finite type: there exists an $\alpha \in \mathcal{O}_W$ such that $\alpha v \in \Gamma(W, \mathcal{V}_{dR})$. By making N_x smaller, we can assume that $W(\mathbb{C})$ contains $\pi(N_x)$, and we can view αv, by restriction and passing to the associated analytic vector bundle, as an element in $\Gamma(\pi(N_x), \mathcal{V}_{dR}^{an})$. We have defined

$$\sigma_x \in \Gamma(\pi(N_x), \mathcal{V}_B^\vee) \quad \text{and} \quad \alpha v \in \Gamma(\pi(N_x), \mathcal{V}_{dR}^{an}).$$

The comparison map $c : \mathcal{V}_{dR}^{an} \to \mathcal{V}_B \otimes_\mathbb{Q} \mathcal{O}_S^{an}$ yields an element

$$\sigma_x(c(\alpha v)) \in \Gamma(\pi(N_x), \mathcal{O}_S^{an})$$

which can be viewed as a locally analytic function on N_x. The period homomorphism is defined on the following matrix coefficient by

$$\text{per}([(\mathcal{V}_B, \mathcal{V}_{dR}, c), \sigma, \alpha v]^\mathfrak{m}) = \sigma_x(c(\alpha v)),$$

and the period of $[(\mathcal{V}_B, \mathcal{V}_{dR}, c), \sigma, v]^\mathfrak{m}$ is obtained by dividing by the rational function α. It is well-defined (does not depend on the representative for the matrix coefficient) because morphisms in $\mathcal{H}(S)$ respect the comparison c. Note that it locally has poles along the zeros of α. In the case when $\pi(x) \in Y \subset U(\mathbb{C})$, we may assume $W = U$ in the above and hence $\alpha = 1$, and the period has no poles. The period homomorphism therefore takes values in $M_{X,Y}(S(\mathbb{C}))$ as claimed.

The period map satisfies the following properties, which follow from the definitions. First of all, the period is functorial with respect to smooth morphisms, and in particular is compatible with the constant map (7.5). This means that the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{P}_\mathcal{H}^\mathfrak{m} & \xrightarrow{(7.5)} & \mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{m},X,Y} \\
\downarrow{\text{per}} & & \downarrow{\text{per}} \\
\mathbb{C} & \subset & M_{X,Y}(S(\mathbb{C}))
\end{array}
$$

where the inclusion on the bottom line is the inclusion of constant functions. The period is also compatible with monodromy; there is a commutative diagram

$$
\begin{array}{ccc}
\mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{m},X,Y} \times \pi_1^{\text{top}}(S(\mathbb{C}), X) & \longrightarrow & \mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{m},X,Y} \\
\downarrow{\text{per} \times \text{id}} & & \downarrow{\text{per}} \\
M_{X,Y}(S(\mathbb{C})) \times \pi_1^{\text{top}}(S(\mathbb{C}), X) & \longrightarrow & M_{X,Y}(S(\mathbb{C}))
\end{array}
$$

where the action of the topological fundamental group on $M_{X,Y}(S(\mathbb{C}))$ is induced by the action of the group of deck transformations on $\widetilde{S}(\mathbb{C})_X$. If

one thinks of elements of $M_{X,Y}(S(\mathbb{C}))$ as multivalued functions on an open subset of $S(\mathbb{C})$, this is just analytic continuation along loops. More generally, given two simply-connected subsets X_1, X_2 of $S(\mathbb{C})$, we have a compatibility of groupoid actions

$$
\begin{array}{ccc}
\mathcal{P}^{\mathfrak{m},X_1,Y}_{\mathcal{H}(S)} \times \pi_1^{\mathrm{top}}(S(\mathbb{C}), X_1, X_2) & \longrightarrow & \mathcal{P}^{\mathfrak{m},X_2,Y}_{\mathcal{H}(S)} \\
\downarrow \mathrm{per} \times \mathrm{id} & & \downarrow \mathrm{per} \\
M_{X_1,Y}(S(\mathbb{C})) \times \pi_1^{\mathrm{top}}(S(\mathbb{C}), X_1, X_2) & \longrightarrow & M_{X_2,Y}(S(\mathbb{C}))
\end{array}
$$

where the map along the bottom is defined by analytic continuation.

Now let $x \in \mathrm{Der}_{\mathbb{Q}}(\mathcal{O}_{S,Y})$ be a $\mathbb{Q}$-linear derivation of $\mathcal{O}_{S,Y}$. It defines a map $\Omega^1_{S,Y} \to \mathcal{O}_{S,Y}$. Let $\partial_x = (\mathrm{id} \otimes x)\nabla$, where ∇ is (7.11). We have a commutative diagram

$$
\begin{array}{ccc}
\mathcal{P}^{\mathfrak{m},X,Y}_{\mathcal{H}(S)} & \xrightarrow{\partial_x} & \mathcal{P}^{\mathfrak{m},X,Y}_{\mathcal{H}(S)} \\
\downarrow \mathrm{per} & & \downarrow \mathrm{per} \\
M_{X,Y}(S(\mathbb{C})) & \longrightarrow & M_{X,Y}(S(\mathbb{C}))
\end{array}
$$

where the map along the bottom is differentiation of locally analytic functions along the vector field defined by x on some open subset of $S(\mathbb{C})$ containing Y. Thus the connexion on the ring of periods corresponds to differentiation of functions.

The period map is functorial. Suppose we are in the situation described in the lines preceeding (7.4). Then there is a commutative diagram

$$
\begin{array}{ccc}
\mathcal{P}^{\mathfrak{m},X',Y'}_{\mathcal{H}(S')} & \xrightarrow{f_*} & \mathcal{P}^{\mathfrak{m},X,Y}_{\mathcal{H}(S)} \\
\downarrow \mathrm{per} & & \downarrow \mathrm{per} \\
M_{X',Y'}(S'(\mathbb{C})) & \xrightarrow{f_*} & M_{X,Y}(S(\mathbb{C}))
\end{array}
$$

where the map along the bottom is composition $\phi \mapsto \phi \circ f$.

Suppose that $t \in S(\mathbb{Q})$ is a rational point and the image of t in $S(\mathbb{C})$ lies in $X \cap Y$. Then the period map is well-defined at t, and we have a commutative diagram

$$
\begin{array}{ccc}
\mathcal{P}^{\mathfrak{m},X,Y}_{\mathcal{H}(S)} & \xrightarrow{\mathrm{ev}_t} & \mathcal{P}^{\mathfrak{m}}_{\mathcal{H}} \\
\downarrow \mathrm{per}_t & & \downarrow \mathrm{per} \\
\mathbb{C} & = & \mathbb{C}
\end{array}
$$

where per_t is evaluation of elements of $M_{X,Y}(S(\mathbb{C}))$ at t. Thus (7.7) corresponds to taking the value of a function at a point. In this manner many classical notions for multivalued functions have analogues on the ring of

motivic periods. Nonetheless, the ring of motivic periods has extra features such as the weight and Galois group which are invisible on functions.

7.6. Complex conjugation

Consider the category $\mathcal{H}(\overline{S})$ consisting of triples $(\mathcal{V}_B, \mathcal{V}_{dR}, \overline{c})$ defined in an identical manner as above, except that

$$\overline{c} : \mathcal{V}_{dR} \otimes_{\mathcal{O}_S} \mathcal{O}_{\overline{S}^{\mathrm{an}}} \xrightarrow{\sim} \mathcal{V}_B \otimes_{\mathbb{Q}} \mathcal{O}_{\overline{S}^{\mathrm{an}}}$$

is antiholomorphic (and respects W, etc). The real Frobenius defines an equivalence $F_\infty : \mathcal{H}(S) \to \mathcal{H}(\overline{S})$ which maps $(\mathcal{V}_B, \mathcal{V}_{dR}, c)$ to $(\sigma^* \mathcal{V}_B, \mathcal{V}_{dR}, \overline{c})$. We can form a ring of periods $\mathcal{P}_{\mathcal{H}(\overline{S})}^{\mathfrak{m}, X, Y}$ as before, and we have an isomorphism

$$F_\infty : \mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{m}, X, Y} \xrightarrow{\sim} \mathcal{P}_{\mathcal{H}(\overline{S})}^{\mathfrak{m}, \overline{X}, Y}.$$

Composing with the map $\sigma^* : \mathcal{P}_{\mathcal{H}(\overline{S})}^{\mathfrak{m}, \overline{X}, Y} \to \mathcal{P}_{\mathcal{H}(\overline{S})}^{\mathfrak{m}, X, Y}$, which sends $(\mathcal{V}_B, \mathcal{V}_{dR}, \overline{c})$ to $(\sigma^* \mathcal{V}_B, \mathcal{V}_{dR}, \overline{c}\sigma^*)$, gives an isomorphism $\sigma^* F_\infty : \mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{m}, X, Y} \xrightarrow{\sim} \mathcal{P}_{\mathcal{H}(\overline{S})}^{\mathfrak{m}, X, Y}$. The period map on $\mathcal{P}_{\mathcal{H}(\overline{S})}^{\mathfrak{m}, X, Y}$ takes values in the ring $\overline{M}_{X,Y}(S(\mathbb{C}))$ of antiholomorphic functions on $S(\mathbb{C})$ with prescribed branch on X. The following diagram commutes

$$
\begin{array}{ccc}
F_\infty \sigma^* : \mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{m}, X, Y} & \xrightarrow{\sim} & \mathcal{P}_{\mathcal{H}(\overline{S})}^{\mathfrak{m}, X, Y} \\
\downarrow{\mathrm{per}} & & \downarrow{\mathrm{per}} \\
M_{X,Y}(S(\mathbb{C})) & \xrightarrow{\sim} & \overline{M}_{X,Y}(S(\mathbb{C}))
\end{array}
$$

where the map along the bottom is complex conjugation $f \mapsto \overline{f}$.

8. Further remarks

There are many constructions that can be made involving families of periods. I shall only mention the minimum required for applications to [13], as well as single-valued functions and symbols which have independent applications to physics.

8.1. Some jargon

Most of the definitions of earlier paragraphs generalise in an evident way to the case of families of periods. I will not repeat all of them here except to mention that an element $\xi \in \mathcal{P}_{\mathcal{H}(S), X, Y}^{\mathfrak{m}}$ generates a representation of $G_{\mathcal{H}(S), Y}^{dR}$

which, by the Tannaka theorem, defines a (minimal) object $M(\xi)$ of $\mathcal{H}(S)$ (see §2.4). The *Hodge numbers* and *Hodge polynomial* of ξ are defined as the Hodge numbers and polynomial of the fiber of $M(\xi)$ at any point $t \in X$. Define the *local system associated to* ξ to be $M(\xi)_B$. It is equivalent to the *monodromy representation* of ξ which is the vector space $\omega_B^X(M(\xi)) \in \mathrm{Vec}_\mathbb{Q}$ together with its left $\pi_1^{\mathrm{top}}(S(\mathbb{C}), X)$-action. Likewise, define the *vector bundle with connexion* associated to $M(\xi)$ to be $M(\xi)_{dR}$, equipped with its integrable connexion ∇. The vector bundle and local system associated to ξ are equivalent under the Riemann-Hilbert correspondence after tensoring with $\mathbb{C}$.

8.2. Variant: families of periods with single-valued branch

For applications such as [13], we are often faced with the situation where we have a family of periods on some open subvariety S of a given variety Z, without knowing what S is. This is in fact the typical situation; S will be the complement of a discriminant locus which is complicated and not computable in practice. We often have some further information, namely that the periods are well-defined on some connected open subset $U \cap S(\mathbb{C}) \subset S(\mathbb{C})$ for the analytic topology. This motivates the following definition.

Let Z be a smooth, geometrically connected algebraic variety over $\mathbb{Q}$, and let $U \subset Z(\mathbb{C})$ be a connected open analytic subset. For any geometrically connected $S \subset Z$, consider the ring of periods $\mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{m},s,\emptyset}$ for any $s \in S(\mathbb{C}) \cap U$. For any two points $s_1, s_2 \in U \cap S(\mathbb{C})$, analytic continuation along paths gives

$$\mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{m},s_1,\emptyset} \times \pi_1^{\mathrm{top}}(U \cap S(\mathbb{C}), s_1, s_2) \overset{\sim}{\longrightarrow} \mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{m},s_2,\emptyset}$$

and in particular, since $U \cap S(\mathbb{C})$ is connected, a canonical isomorphism

$$\left(\mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{m},s_1,\emptyset}\right)^{\pi_1(U \cap S(\mathbb{C}),s_1)} = \left(\mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{m},s_2,\emptyset}\right)^{\pi_1(U \cap S(\mathbb{C}),s_2)}.$$

Thus, by moving the base-point s if necessary, we can define

$$\mathcal{P}_{\mathcal{H}(Z)}^{\mathfrak{m},U} = \varinjlim_{S} \left(\mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{m},s,\emptyset}\right)^{\pi_1(U \cap S(\mathbb{C}),s)}$$

where the limit is over all such $S \subset Z$, since any two such opens $S_1, S_2 \subset Z$ have a non-empty intersection $U \cap S_1 \cap S_2(\mathbb{C})$. The periods of elements of $\mathcal{P}_{\mathcal{H}(Z)}^{\mathfrak{m},U}$ restrict to single-valued meromorphic functions on U. The ring $\mathcal{P}_{\mathcal{H}(Z)}^{\mathfrak{m},U}$ has similar properties to the rings of periods discussed earlier, except for the monodromy action.

8.3. Single-valued versions

We can define single-valued versions of families of motivic periods in a similar way to §4.1, except that there are some slight differences. Let $M = (\mathcal{V}_B, \mathcal{V}_{dR}, c)$ be an object of $\mathcal{H}(S)$. As in §4.1, there is a universal comparison homomorphism

$$c_M^{\mathfrak{m}} \;:\; \omega_Y(\mathcal{V}_{dR}) \longrightarrow \omega_X(\mathcal{V}_B) \otimes_{\mathbb{Q}} \mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{m},X,Y}.$$

Applying $F_\infty \sigma^*$ to the right-hand factor gives a homomorphism

$$\overline{c_M}^{\mathfrak{m}} := F_\infty \sigma^* c_M^{\mathfrak{m}} \;:\; \omega_Y(\mathcal{V}_{dR}) \longrightarrow \omega_X(\mathcal{V}_B) \otimes_{\mathbb{Q}} \mathcal{P}_{\mathcal{H}(\overline{S})}^{\mathfrak{m},X,Y}.$$

Define a ring

$$\mathcal{P} = \mathcal{P}_{\mathcal{H}(\overline{S})}^{\mathfrak{m},X,Y} \otimes_{\mathbb{Q}} \mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{m},X,Y}.$$

Embed $\mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{m},X,Y}$ into $\mathcal{P}$ via $x \mapsto 1 \otimes x$, and $\mathcal{P}_{\mathcal{H}(\overline{S})}^{\mathfrak{m},X,Y}$ via $x \mapsto x \otimes 1$. Thus

$$c_M^{\mathfrak{m}} \quad , \quad \overline{c_M}^{\mathfrak{m}} \;\in\; \mathrm{Hom}(\omega_Y(\mathcal{V}_{dR}), \omega_X(\mathcal{V}_B)) \otimes_{\mathbb{Q}} \mathcal{P}$$

Finally, since $\det c_M^{\mathfrak{m}}$ is invertible in $\mathcal{P}$, define

$$\mathbf{s}_M^{\mathfrak{m}} = (\overline{c_M}^{\mathfrak{m}})^{-1} c_M^{\mathfrak{m}} \;\in\; \mathrm{End}(\omega_Y(\mathcal{V}_{dR})) \otimes_{\mathbb{Q}} \mathcal{P}.$$

We leave it to the reader to replace the above argument with universal arguments on torsors, exactly as in §4.1, to define a canonical element

$$\mathbf{s}^{\mathfrak{m}} \in G_{\mathcal{H}(S),Y}^{dR}(\mathcal{P})$$

or equivalently, a homomorphism

$$(8.1) \qquad\qquad \mathbf{s}^{\mathfrak{m}} : \mathcal{P}_{\mathcal{H}(S)}^{\partial\mathfrak{r},Y} \longrightarrow \mathcal{P}_{\mathcal{H}(\overline{S})}^{\mathfrak{m},X,Y} \otimes_{\mathbb{Q}} \mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{m},X,Y},$$

where the left (resp. right) action of $G_{\mathcal{H}(S),Y}^{dR}$ on $\mathcal{P}_{\mathcal{H}(S)}^{\partial\mathfrak{r},Y}$ corresponds to the left action of $G_{\mathcal{H}(S),Y}^{dR}$ on $\mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{m},X,Y}$ (resp. $\mathcal{P}_{\mathcal{H}(\overline{S})}^{\mathfrak{m},X,Y}$). It follows from the definition that the single-valued homomorphism (8.1) is compatible with the connexion:

$$(8.2) \qquad\qquad (\mathbf{s}^{\mathfrak{m}} \otimes \mathrm{id})\nabla(\xi) = (\mathrm{id} \otimes \nabla)\mathbf{s}^{\mathfrak{m}}(\xi).$$

This means that, after taking the period homomorphism, the single-valued map respects the *holomorphic* (and only the holomorphic) differential.

Remark 8.1. When $S = \mathrm{Spec}\,(\mathbb{Q})$ is a point, $\mathcal{P}^{\mathrm{m},X,Y}_{\mathcal{H}(\overline{S})} \cong \mathcal{P}^{\mathrm{m},X,Y}_{\mathcal{H}(S)} \cong \mathcal{P}^{\mathrm{m}}_{\mathcal{H}}$ and the earlier definition (4.3) is obtained as the composition

$$\mathcal{P}^{\partial\mathfrak{r}}_{\mathcal{H}} \xrightarrow{\mathbf{s}^{\mathrm{m}}} \mathcal{P}^{\mathrm{m}}_{\mathcal{H}} \otimes_{\mathbb{Q}} \mathcal{P}^{\mathrm{m}}_{\mathcal{H}} \longrightarrow \mathcal{P}^{\mathrm{m}}_{\mathcal{H}}$$

where the second map is multiplication.

Examples 8.2. Consider the dilogarithm motivic period on $S = \mathbb{P}^1\backslash\{0,1,\infty\}$, defined in (10.10) below. Here $X \subset S(\mathbb{C})$ is the open interval $(0,1)$. Its universal period matrix over a point $x \in X \subset S(\mathbb{C})$ is

$$c^{\mathrm{m}}_M = \begin{pmatrix} 1 & \mathrm{Li}^{\mathrm{m}}_1(x) & \mathrm{Li}^{\mathrm{m}}_2(x) \\ 0 & \mathbb{L}^{\mathrm{m}} & \mathbb{L}^{\mathrm{m}}\log^{\mathrm{m}}(x) \\ 0 & 0 & (\mathbb{L}^{\mathrm{m}})^2 \end{pmatrix}.$$

Let γ_0 (resp. γ_1) denote a small path around 0 (resp. 1) based at X. Under the monodromy homomorphism they act by left multiplication by

$$\rho(\gamma_0) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \rho(\gamma_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Denoting the image of $\mathrm{Li}^{\mathrm{m}}_k(x)$ under $F_\infty\sigma_*$ by $\overline{\mathrm{Li}}^{\mathrm{m}}_k(x)$ and similarly for $\log^{\mathrm{m}}(x)$, the matrix $\overline{c_M}^{\mathrm{m}}$ is given by

$$\overline{c_M}^{\mathrm{m}} = \begin{pmatrix} 1 & \overline{\mathrm{Li}^{\mathrm{m}}_1}(x) & \overline{\mathrm{Li}^{\mathrm{m}}_2}(x) \\ 0 & -\mathbb{L}^{\mathrm{m}} & -\mathbb{L}^{\mathrm{m}}\overline{\log^{\mathrm{m}}}(x) \\ 0 & 0 & (\mathbb{L}^{\mathrm{m}})^2 \end{pmatrix}.$$

By computing $(\overline{c_M}^{\mathrm{m}})^{-1}c^{\mathrm{m}}_M$ we find that $\mathbf{s}^{\mathrm{m}}(\mathbb{L}^{\partial\mathfrak{r}}) = (-1)$ (yet again),

$$\mathbf{s}^{\mathrm{m}}(\log^{\partial\mathfrak{r}}(x)) = \log^{\mathrm{m}}(x) + \overline{\log^{\mathrm{m}}}(x),$$

and similarly for $\mathrm{Li}^{\partial\mathfrak{r}}_1(x) = -\log^{\partial\mathfrak{r}}(1-x)$. The top-right corner gives

$$\mathbf{s}^{\mathrm{m}}(\mathrm{Li}^{\partial\mathfrak{r}}_2(x)) = \mathrm{Li}^{\mathrm{m}}_2(x) - \overline{\mathrm{Li}^{\mathrm{m}}_2}(x) + (\log^{\mathrm{m}}(x) + \overline{\log^{\mathrm{m}}}(x))\overline{\mathrm{Li}^{\mathrm{m}}_1}(x) \ \in \ \mathcal{P}$$

whose period is $2i$ times the Bloch-Wigner dilogarithm. Equivalent calculations for the associated period matrices were first carried out in [8] for the classical polylogarithms. For multiple polylogarithms, the computations are made much simpler using the language of non-commutative formal power series [12].

9. Symbols

We briefly indicate how a certain class of motivic periods give rise to invariants involving differential forms. There are several possible variations on this theme which all specialise (in the mixed Tate case) to the notion of 'symbol of differential forms' as currently used in the physics literature.

9.1. Some pitfalls

The symbol is commonly understood by physicists to be a tensor product of differential forms obtained by differentiating a family of period integrals with respect to a parameter. Consider the following examples:

1) The classical polylogarithm $\text{Li}_k(x) = \sum_{n \geq 1} \frac{x^n}{n^k}$, where $k \geq 1$, satisfies the following differential equation for all $k \geq 2$:

$$d\,\text{Li}_k(x) = \text{Li}_{k-1}(x)\,\frac{dx}{x}.$$

Furthermore $d\text{Li}_1(x) = \frac{dx}{1-x}$. The recipe in the physics literature for constructing the symbol is recursive by repeated differentiation. For example,

$$\text{symbol}(\text{Li}_1(x)) = \frac{dx}{1-x} \quad \text{and}$$

$$\text{symbol}(\text{Li}_k(x)) = \text{symbol}(\text{Li}_{k-1}(x)) \otimes \frac{dx}{x},$$

e.g.,

$$\text{symbol}(\text{Li}_2(x)) = \frac{dx}{1-x} \otimes \frac{dx}{x}.$$

Occasionally, the arguments $d\log f$ in the right-hand side are represented by their arguments f, and the symbol of $\text{Li}_2(x)$ is written $(1-x) \otimes x$. This notion is already very useful for encoding functional relations between polylogarithms and is ubiquitous in the literature. Note that the symbol captures the information that $\text{Li}_2(x)$ is an iterated integral of two one-forms on the punctured projective line, but some information is lost: one requires a path of integration to reconstruct the function $\text{Li}_2(x)$ from its symbol. The symbol is sometimes used to infer notions such as 'transcendental weight' and monodromy data about functions.

2) Simply differentiating functions is too naive, as one sees by considering examples such as $f(x)\mathrm{Li}_2(x) + g(x)\mathrm{Li}_1(x)\log(x)$, where f, g are rational functions. Repeated differentiation leads to an infinite sequence of more complicated functions. To make sense of the definition (1), one must find differential operators whose application successively decreases some quantity called the 'length' (in the examples in (1), this happens to coincide with the 'transcendental weight', although this will not necessarily be true in more general settings). This will depend on choices, and shows that the symbol should in fact be defined as a tensor product *modulo certain equivalence relations*. These are described in §9.3.1. This recursive structure will be encoded by the notion of a *unipotent connection*.

3) The symbol of a constant family of periods is necessarily zero since it is defined in terms of differentiation. However, the notion of symbol in the physics literature has somehow morphed into a version in which the arguments are formally allowed to be constants. This is the basis for the definition of 'motivic amplitudes' in [20]. One finds equations such as

$$\mathrm{symbol}(\mathrm{Li}_2(3)) = -2 \otimes 3,$$

which are obtained by specialising the earlier example to $x = 3$, and possibly based on Goncharov's notion of symbol (Remark 10.2). This notion does not go very far to capture constants, since most are zero under this map: for example, the corresponding notion of symbol for the quantity $\zeta(3)\mathrm{Li}_1(x)$ is zero.

4) A further problem with the formalism mentioned in (3), which is only defined in terms of de Rham classes and with no mention of Betti classes, is that the de Rham analogue of $\zeta(2)$, or $2\pi i$, is irretrievably zero, and so attempting to take the period leads to contradictions. This problem is fixed by replacing 'coproduct' with 'coaction', which is part of the structure of our definition of families of $\mathcal{H}$-periods.

5) There have been attempts to incorporate constants into a common 'symbolic' framework such as [28], which contains both the coaction for motivic multiple zeta values, and the symbols of polylogarithms. The main properties of this framework were mostly conjectural and shown to be equivalent to complicated combinatorial identities. We shall show how these problems can be easily overcome in our setting using the notion of 'symbol based at a point t', which has all the properties conjectured in [28].

Our notion of families of motivic periods fixes these problems and sub-
sumes all of the above notions. It also generalises these concepts to situa-
tions which are non-polylogarithmic. More precisely, the notions of symbol
presently found in the physics literature are derived from the coaction on
families of motivic periods and discarding more or less information. One
must bear in mind, however, that one side of the motivic coproduct involves
de Rham periods, not $\mathcal{H}$-periods, and the former do not have canonical
periods. They do, however, possess single-valued periods.

9.2. Abstract definition of the symbol

The symbol will be defined for a certain class of motivic periods. Recall that
an algebraic vector bundle with connection $(\mathcal{V}_{dR}, \nabla)$ on S is *unipotent* of
length n if there exists a filtration

$$(9.1) \qquad\qquad 0 = \mathcal{V}_{-1} \subset \mathcal{V}_0 \subset \cdots \subset \mathcal{V}_n = \mathcal{V}$$

by algebraic sub-bundles, with $\nabla : \mathcal{V}_k \to \mathcal{V}_k \otimes_{\mathcal{O}_S} \Omega^1_S$, such that each graded
quotient $\mathcal{V}_k/\mathcal{V}_{k-1}$ is isomorphic to a direct sum of trivial vector bundles
$(\mathcal{O}_S, d)$. Such a vector bundle automatically has regular singularities at in-
finity. The category of unipotent algebraic vector bundles with connection
forms a full Tannakian subcategory $\mathcal{A}^{un}(S)$ of $\mathcal{A}(S)$.

A local system V on S is *unipotent* if it admits a finite increasing filtra-
tion

$$(9.2) \qquad\qquad 0 = V_{-1} \subset V_0 \subset \cdots \subset V_n = V$$

of local sub-systems, such that each graded quotient V_k/V_{k-1} is trivial, i.e.,
constant. Equivalently, for any basepoint $s \in S(\mathbb{C})$, the representation $\omega_s(V)$
of $\pi_1(S(\mathbb{C}), s)$ associated to V is unipotent, i.e., admits a finite increasing fil-
tration such that the associated graded quotients are trivial representations.
The category of unipotent local systems forms a full Tannakian subcategory
$\mathcal{L}^{un}(S)$ of $\mathcal{L}(S)$. Since the category of unipotent vector bundles with connec-
tion (resp. local systems) behaves well with respect to extensions of scalars,
the Riemann-Hilbert correspondence induces an equivalence

$$\mathcal{A}^{un}(S) \otimes \mathbb{C} \sim \mathcal{L}^{un}(S) \otimes \mathbb{C}.$$

Definition 9.1. A family of motivic periods $\xi \in \mathcal{P}^{\mathfrak{m},X,Y}_{\mathcal{H}(S)}$ is *differentially
unipotent*, or has *unipotent monodromy*, if the associated vector bundle
with connection (§8.1) $M(\xi)_{dR}$ is unipotent. Equivalently, ξ is differentially

unipotent if the associated local system on $M(\xi)_B$ is unipotent. This means that ξ can be represented as a matrix coefficient (7.3) where one of $\mathcal{V}_B, \mathcal{V}_{dR}$ (and hence both) are unipotent.

We define a family of de Rham motivic periods $\xi \in \mathcal{P}_{\mathcal{H}(S)}^{\partial\mathfrak{r},Y}$ to be *differentially unipotent* in an identical manner. It follows immediately from the definition that the motivic coaction (7.8) preserves the quality of unipotency.

Note that the filtration involved in the definition of differential unipotency will not in general be the weight filtration.

The above categories give rise to Tannaka group schemes over $\mathbb{Q}$:

$$\pi_1^{un}(S(\mathbb{C}), s) = \operatorname{Aut}_{\mathcal{L}^{un}(S)}^{\otimes}(\omega_s) \quad \text{for } s \in S(\mathbb{C}),$$
$$\pi_1^{dR}(S, s) = \operatorname{Aut}_{\mathcal{A}^{un}(S)}^{\otimes}(\omega_s) \quad \text{for } s \in S(\mathbb{Q}).$$

The first group is the unipotent or Malčev completion of the topological fundamental group, the second is the de Rham fundamental group. They are naturally quotients of the Betti algebraic and de Rham algebraic fundamental groups.

A family of motivic or de Rham periods is differentially unipotent if and only if the natural actions by the Betti algebraic fundamental group, de Rham algebraic fundamental group, or topological fundamental group factor through their unipotent quotients. It follows that for a differentially unipotent de Rham period, its image under the dual of the natural map (7.2):

$$\mathcal{P}_{\mathcal{H}(S)}^{\partial\mathfrak{r},Y} = \mathcal{O}(G_{\mathcal{H}(S),Y}^{dR}) \quad \longrightarrow \quad \mathcal{O}(\pi_1^{\mathrm{alg},dR}(S, \omega_{dR}^Y)),$$
$$[(\mathcal{V}_B, \mathcal{V}_{dR}, c), v, \omega]^{\partial\mathfrak{r}} \quad \mapsto \quad (\alpha \mapsto v(\alpha(\omega))),$$

where $\alpha \in \pi_1^{\mathrm{alg},dR}(S, \omega_{dR}^Y)$, actually lands in the unipotent subspace

$$\mathcal{O}(\pi_1^{dR}(S, \omega_{dR}^Y)) \subset \mathcal{O}(\pi_1^{\mathrm{alg},dR}(S, \omega_{dR}^Y)).$$

Next restrict to the affine open $\operatorname{Spec}(\mathcal{O}_{S,Y})$ of S, which gives a map on affine rings

$$\mathcal{O}(\pi_1^{dR}(S, \omega_{dR}^Y)) \longrightarrow \mathcal{O}(\pi_1^{dR}(\operatorname{Spec}\mathcal{O}_{S,Y}, \omega_{dR}^Y)).$$

Finally it follows either from Chen's π_1-de Rham theorem (over the complex numbers), or from the universal properties of the reduced bar construction $\mathbb{B}$ (whose zero'th cohomology is the universal unipotent extension of $\mathcal{O}_{S,Y}$),

that the affine ring of the de Rham fundamental group on $\operatorname{Spec} \mathcal{O}_{S,Y}$ is described explicitly:

$$\mathcal{O}(\pi_1^{dR}(\operatorname{Spec} \mathcal{O}_{S,Y}, \omega_{dR}^Y)) \cong H^0(\mathbb{B}(\Omega_{S,Y})).$$

We shall recall the relevant definitions below.

Definition 9.2. We shall define the *symbol* of a differentially unipotent family of de Rham periods $\xi \in \mathcal{P}_{\mathcal{H}(S)}^{\partial\mathfrak{r},Y}$ to be its image:

$$(9.3) \qquad\qquad \operatorname{smb}(\xi) \in H^0(\mathbb{B}(\Omega_{S,Y})).$$

There is a natural *length filtration* on $H^0(\mathbb{B}(\Omega_{S,Y}))$. The *length* of ξ is bounded above by the length n in the filtration (9.2).

This is a generalisation of the notion of symbol as used by physicists. Our abstract definition of the symbol could be generalised further still. For example, one could also consider relative unipotent completion instead of unipotent completion.

9.3. Computing the symbol

We now explain how to compute the symbol in the spirit of the recursive differentiation procedure described in §9.1 (1). Consider a differentially unipotent de Rham period $\xi = [(\mathcal{V}_B, \mathcal{V}_{dR}, c), f, \omega]^{\partial\mathfrak{r}}$ in $\mathcal{P}_{\mathcal{H}(S)}^{\partial\mathfrak{r},Y}$, where $f \in \omega_Y(\mathcal{V}_{dR})^\vee$ and $\omega \in \omega_Y(\mathcal{V}_{dR})$, and let $\mathcal{V}$ denote the pull-back of $\mathcal{V}_{dR}$ to $\operatorname{Spec} \mathcal{O}_{S,Y}$. We shall assume that $H_{dR}^0(\mathcal{O}_{S,Y}) = \mathbb{Q}$. Since ξ is differentially unipotent, we can assume by equivalence of matrix coefficients that $\mathcal{V}_{dR}$ and hence $\mathcal{V}$ is unipotent. There is a filtration

$$(9.4) \qquad\qquad 0 = \mathcal{V}_{-1} \subset \mathcal{V}_0 \subset \cdots \subset \mathcal{V}_n = \mathcal{V}$$

by algebraic sub-bundles such that $\nabla : \mathcal{V}_k \to \mathcal{V}_k \otimes_{\mathcal{O}_{S,Y}} \Omega_{S,Y}^1$, and with respect to which ∇ is unipotent. It splits because $H^1(\operatorname{Spec} \mathcal{O}_{S,Y}, \mathcal{O}_{S,Y}) = 0$ since $\operatorname{Spec} \mathcal{O}_{S,Y}$ is affine and $\mathcal{O}_{S,Y}$ coherent. Choose a splitting of this filtration:

$$(9.5) \qquad\qquad \mathcal{V} \cong \operatorname{gr} \mathcal{V}.$$

The associated graded $(\operatorname{gr} \mathcal{V}, \operatorname{gr} \nabla)$ is a direct sum of trivial vector bundles $(\mathcal{O}_{S,Y}, d)$, and $\operatorname{gr} \mathcal{V} \cong \mathcal{O}_{S,Y} \otimes_{\mathbb{Q}} H^0(\operatorname{gr} \mathcal{V}, \operatorname{gr} \nabla)$ has a $\mathbb{Q}$-structure given by flat

sections. With respect to this choice of splitting, the connection map

$$N = \nabla - d \ \in \ \mathrm{Hom}_{\mathcal{O}_{S,Y}}(\mathcal{V}, \mathcal{V} \otimes_{\mathbb{Q}} \Omega^1_{S,Y})$$

satisfies the integrability condition

$$dN + N \wedge N = 0,$$

which is equivalent to $\nabla^2 = 0$, and also satisfies $N^m = 0$ for some m by the assumption of unipotency. It is an $\mathcal{O}_{S,Y}$-linear operator, where $\mathcal{V}$ and $\mathcal{V} \otimes_{\mathbb{Q}} \Omega^1_{S,Y}$ are left $\mathcal{O}_{S,Y}$-modules. For computations, let us write $V_k = \Gamma(\mathcal{O}_{S,Y}, \mathcal{V}_k)$. Then V_k/V_{k-1} is a free $\mathcal{O}_{S,Y}$-module and it follows by induction that the V_k are free $\mathcal{O}_{S,Y}$-modules too. We can choose a basis of V_n which is adapted to the filtration V_k, and is flat on the graded quotients $\mathrm{gr}\,V$, i.e., a basis element e of V_k satisfies $\nabla(e) \subset V_{k-1}$. Write the connection as $\nabla = d + N$ in this basis. Thus N is represented as an upper-triangular matrix of one-forms. Consider the element

$$\mathrm{smb}_N(\xi) = \sum_{k \geq 0} \langle f, N^k \omega \rangle \ \in \ \mathcal{O}_{S,Y} \otimes_{\mathbb{Q}} T^c(\Omega^1_{S,Y}),$$

where we recall that the tensor coalgebra $T^c(\Omega^1_{S,Y}) = \bigoplus_{k \geq 0}(\Omega^1_{S,Y})^{\otimes k}$, and tensors are over $\mathbb{Q}$. It depends on the choice of splitting (9.5) (resp. choice of basis of V).

One can think of the symbol in the following way. View N as an $n \times n$ matrix with coefficients in $T^c(\Omega^1_{S,\mathcal{U}})$, since its entries can be considered as tensors of length one. Consider the following matrix

(9.6) $1 + N + N^2 + N^3 + \cdots \ \in \ M_{n \times n}(T^c(\Omega^1_{S,\mathcal{T}})),$

where the multiplication of matrix entries is given by the (non-commutative) concatenation product in $T^c(\Omega^1_{S,\mathcal{U}})$. The series is finite by the nilpotence of N. The vector ω and covector f define an entry of this matrix, which is exactly $\mathrm{smb}_N(\xi)$.

9.3.1. Reduced bar construction.   Define an internal differential

$$d_I : T^c(\Omega^\bullet_{S,Y}) \to T^c(\Omega^\bullet_{S,Y})$$

by

$$d_I[\omega_1|\cdots|\omega_n] = \sum_{i=1}^{n}(-1)^i[j\omega_1|\cdots|j\omega_{i-1}|d\omega_i|\omega_{i+1}|\cdots|\omega_n]$$
$$+ \sum_{i=1}^{n-1}(-1)^{i+1}[j\omega_1|\cdots|j\omega_{i-1}|j\omega_i \wedge \omega_{i+1}|\omega_{i+2}|\cdots|\omega_n],$$

where j acts on $\Omega_{S,Y}^n$ by $(-1)^n$. Define a grading on $T^c(\Omega_{S,Y}^\bullet)$ by

$$\deg [\omega_1|\cdots|\omega_n] = \sum_{i=1}^{n}(\deg(\omega_i) - 1).$$

Consider the linear map $d_C : T^c(\Omega_{S,Y}^\bullet) \to \mathcal{O}_S \otimes_\mathbb{Q} T^c(\Omega_{S,Y}^\bullet)$ defined by

$$d_C[\omega_1|\cdots|\omega_n] = -\varepsilon(\omega_1)[\omega_2|\cdots|\omega_n] + (-1)^\nu \varepsilon(\omega_n)[\omega_1|\cdots|\omega_{n-1}]$$

where $\varepsilon : \Omega_{S,Y}^\bullet \to \mathcal{O}_{S,Y}$ is projection onto degree 0 and ν is given by

$$(\deg(\omega_n) - 1)\deg[\omega_1|\cdots|\omega_{n-1}].$$

One verifies that

$$d = \mathrm{id} \otimes d_I + \mathrm{id} \otimes d_C$$

satisfies $d^2 = 0$. See, for example, the presentation in [37] (3.4). Note that the signs simplify drastically when all ω_i are of degree one, which is the case we are mainly interested in. Consider the smallest subspace $\mathcal{R}$ in $\mathcal{O}_S \otimes_\mathbb{Q} T^c(\Omega_{S,Y})$ generated by

$$\mathcal{R} : [\omega_1|\cdots|\omega_i|f|\omega_{i+1}|\cdots|\omega_n]$$

where $f \in \mathcal{O}_{S,Y}$, and stable under the differential d. The quotient of $\mathcal{O}_{S,Y} \otimes_\mathbb{Q} T^c(\Omega_{S,Y})$ by $\mathcal{R}$ is a complex which we denote by $\mathbb{B}(\Omega_{S,Y}^\bullet)$. It is a close relative of Chen's reduced circular bar complex[11] on $\Omega_{S,Y}$.

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[11]In the usual formulation, due to Chen [19], one considers the tensor algebra $T^c(\Omega_{S,Y}^{\geq 1})$ and quotients out by a certain family of relations. The reader may like to check that Chen's relations are boundaries in the complex we have defined here and are therefore incorporated automatically.

Example 9.3. If $f \in \mathcal{O}_{S,Y}$ and $\omega \in \Omega^1_{S,Y}$ is closed we have:

$$d[f|\omega] = -[df|\omega] + [f\omega] - f[\omega]$$
$$d[\omega|f] = -[\omega|df] - [f\omega] + f[\omega].$$

The expressions on the right-hand side are therefore in $\mathcal{R}$.

9.3.2. The symbol. The integrability of N implies that the element $\mathrm{smb}_N(\xi)$ is *integrable*, i.e., is in the kernel of the differential d in $\mathbb{B}(\Omega^\bullet_{S,Y})$. Furthermore, it is of degree zero. Its cohomology class

$$\mathrm{smb}(\xi) \in H^0(\mathbb{B}(\Omega_{S,Y})),$$

is exactly the symbol as defined in (9.3). It is true, but not obvious from the computational procedure defined above, that the cohomology class of $\mathrm{smb}(\xi)$ is well-defined, i.e., independent of the choice of N. The integrability of $\mathrm{smb}(\xi)$ can be seen directly by writing the matrix (9.6) in the form

$$[1] + [N] + [N|N] + \cdots.$$

The equation $dN + N \wedge N = 0$ implies that it lies in the kernel of the differential d applied formally to the previous expression in the obvious way.

Remark 9.4. The bar complex is equipped with a shuffle product (with signs) which is compatible with d, and this induces a commutative algebra structure on its cohomology. It follows from the definition (9.3) that the symbol is a homomorphism:

$$\mathrm{smb}(\xi_1 \xi_2) = \mathrm{smb}(\xi_1) \, \text{ш} \, \mathrm{smb}(\xi_2).$$

The shuffle product restricted to $H^0(\mathbb{B}(\Omega_{S,Y}))$ has no signs.

Example 9.5. Suppose that $\mathrm{gr} \, \mathcal{V}$ has length two $(n = 2)$, so

$$0 = \mathcal{V}_{-1} \subset \mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 = \mathcal{V}$$

and that each graded quotient $\mathcal{V}_k / \mathcal{V}_{k-1}$ for $k = 0, 1, 2$ is a rank one $\mathcal{O}_{S,Y}$-module. Choose a basis e_0, e_1, e_2 of $V = \Gamma(\mathcal{O}_{S,Y}, \mathcal{V})$ such that $\nabla e_0 = 0$, $\nabla e_1 = e_0 \otimes \omega_1$ and $\nabla e_2 = e_1 \otimes \omega_2 + e_0 \otimes \omega_{12}$. In this basis, the matrix N

is

$$N = \begin{pmatrix} 0 & \omega_1 & \omega_{12} \\ 0 & 0 & \omega_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

For this to define a connection (or equivalently, $dN + N \wedge N = 0$), we must assume that ω_1, ω_2 are closed, and $d\omega_{12} + \omega_1 \wedge \omega_2 = 0$. If $\xi = [\mathcal{V}, e_0^\vee, e_2]^{\partial \mathfrak{r}}$, then

$$\begin{aligned} \mathrm{smb}_N(\xi) &= \langle e_0^\vee, e_2 \rangle + \langle e_0^\vee, N e_2 \rangle + \langle e_0^\vee, N^2 e_2 \rangle \\ &= 0 + \omega_{12} + \langle e_0^\vee, N e_1 \rangle \otimes \omega_2 + \langle e_0^\vee, N e_0 \rangle \otimes \omega_{12} \\ &= 0 + \omega_{12} + \omega_1 \otimes \omega_2 + 0. \end{aligned}$$

In bar notation, this is denoted by:

$$\mathrm{smb}_N(\xi) = [\omega_1 | \omega_2] + [\omega_{12}].$$

Now change basis to e_0', e_1', e_2' where $e_0' = e_0$, $e_1' = e_1$ and $e_2' = e_2 + f e_1$, where $f \in \mathcal{O}_{S,Y}$. In the new basis, the matrix N is replaced by

$$N' = \begin{pmatrix} 0 & \omega_1 & \omega_{12} + f\omega_1 \\ 0 & 0 & \omega_2 + df \\ 0 & 0 & 0 \end{pmatrix},$$

and since $e_2 = e_2' - f e_1$ one checks by a similar computation that

$$\mathrm{smb}_{N'}(\xi) = [\omega_1 | \omega_2 + df] + [\omega_{12} + f\omega_1] - f[\omega_1].$$

The difference between the two elements

$$\mathrm{smb}_{N'}(\xi) - \mathrm{smb}_N(\xi) = [\omega_1 | df] + [f\omega_1] - f[\omega_1]$$

which is exactly a boundary $-d([\omega_1 | f])$, by Example 9.3.

9.3.3. Variants. The above construction was defined for de Rham periods but can be embellished in any number of ways. Let $\xi \in \mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{m}, X, Y}$ be differentially unipotent. Then using the coaction (7.8) we can define

$$(\mathrm{id} \otimes \mathrm{smb}) \Delta(\xi) \ \in \ \mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{m}, X, Y} \otimes_{\mathcal{O}_{S,Y}} H^0(\mathbb{B}(\Omega_{S,Y})).$$

A further possibility is to introduce a base point as follows. Suppose that $t \in S(\mathbb{Q})$, whose image in $S(\mathbb{C})$ lies in $X \cap Y$ so that the evaluation (7.7) is

defined. Let $\xi \in \mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{m},X,Y}$ be differentially unipotent, and define the symbol 'based at t' by

$$\text{smb}_t(\xi) = (\text{ev}_t \otimes \text{smb})\Delta(\xi) \ \in \ \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}} \otimes_{\mathbb{Q}} H^0(\mathbb{B}(\Omega_{S,Y})).$$

This notion captures constants and satisfies similar properties to the symbol defined above. See for [28] for applications of such a construction.

Remark 9.6. Suppose that ξ is a period of a mixed Tate variation. By this we mean that we can write $\xi = [(\mathcal{V}_B, \mathcal{V}_{dR}, c), \sigma, \omega]^{\mathfrak{m}}$, where $\text{gr}_n^W(\mathcal{V}_B, \mathcal{V}_{dR}, c)$ is zero if n is odd, and isomorphic to a direct sum of constant Tate elements $\mathbb{Q}(-k)_{/S}$ if $n = 2k$ is even (images of the pull-back of the Tate objects $\mathbb{Q}(-k)$ in $\mathcal{H}$ along the structural map $\pi : S \to \text{Spec}(\mathbb{Q})$). The connection underlying $\mathbb{Q}(-k)_{/S}$ is isomorphic to $(\mathcal{O}_S, d)$ and is trivial. It follows that the element ξ is automatically differentially unipotent with respect to the weight filtration $\mathcal{V}_n = W_{2n}\mathcal{V}_{dR}$. If, furthermore, ξ is effective then we can apply a version of the projection map of §4.3 to associate to ξ a de Rham period $\xi^{\mathfrak{dr}}$, which is necessarily unipotent, and take its symbol $\text{smb}(\xi^{\mathfrak{dr}})$.

Thus we have shown that de Rham, and effective mixed Tate periods, always have symbols. All the examples used in physics so far seem to be of this special type.

9.4. Cohomological symbol

The bar complex is somewhat cumbersome. We can define a coarser version of a symbol of length n by passing to the associated length-graded of the bar complex.

Definition 9.7. Let $\xi \in \mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{dr},Y}$ be a differentially unipotent de Rham period of length $\leq n$. Recall that this means the filtration (9.2) satisfies $\mathcal{V}_n = \mathcal{V}$. Define its *cohomological symbol* in length n to be its class

$$\text{cmb}_n(\xi) = [\text{smb}(\xi)] \ \in \ \text{gr}_n^{\ell} H^0(\mathbb{B}(\Omega_{S,Y})),$$

where ℓ denotes the length filtration.

An Eilenberg-Moore spectral sequence implies that the associated graded for the length filtration $\text{gr}^{\ell} H^0(\mathbb{B}(\Omega_{S,Y})) \cong H^0(\mathbb{B}(H(\Omega_{S,Y})))$ is the bar complex on the cohomology of $\Omega_{S,Y}$, equipped with the trivial differential. Therefore

$$\text{cmb}_n(\xi) \in \mathcal{O}_{S,Y} \otimes_{\mathbb{Q}} H^1(\Omega_{S,Y})^{\otimes n},$$

and lies in the kernel of the map

$$H^1(\Omega_{S,Y})^{\otimes n} \longrightarrow \bigoplus_k H^1(\Omega_{S,Y})^{\otimes k-1} \otimes_{\mathbb{Q}} H^2(\Omega_{S,Y}) \otimes_{\mathbb{Q}} H^1(\Omega_{S,Y})^{\otimes n-k-1}$$

(9.7)
$$[\omega_1|\cdots|\omega_n] \mapsto \sum_k [\omega_1|\cdots|\omega_{k-1}|\omega_k \wedge \omega_{k+1}|\omega_{k+2}|\cdots|\omega_n].$$

Example 9.8. The cohomological symbol is the length n part of the symbol, after replacing forms with their cohomology classes. In our Example 9.5 it gives

$$\mathrm{cmb}_2(\xi) = \big[[\omega_1]\big|[\omega_2]\big].$$

This lies in the kernel of (9.7) since $[\omega_1] \wedge [\omega_2] = [\omega_1 \wedge \omega_2] = [-d\omega_{12}] = 0$.

This invariant can be computed directly and more simply in the following way. With the previous notations, consider the operator

$$\overline{N} = \mathrm{gr}_\bullet (\nabla - d) : \mathrm{gr}_\bullet \mathcal{V} \longrightarrow \mathrm{gr}_{\bullet-1} \mathcal{V} \otimes_{\mathbb{Q}} \Omega^1_{S,Y}.$$

Iterating it defines an operator

$$(\overline{N})^{\otimes n} : \mathrm{gr}_n \mathcal{V} \longrightarrow \mathrm{gr}_0 \mathcal{V} \otimes_{\mathbb{Q}} (\Omega^1_{S,Y})^{\otimes n}.$$

The vector ω defines a section in the associated graded via the map $\mathcal{V} \to \mathrm{gr}_n \mathcal{V}$. Similarly, since $\mathcal{V}_{-1} = 0$, we have $\mathrm{gr}_0 \mathcal{V} = \mathcal{V}_0 \subset \mathcal{V}$, and we can consider the image of the covector f along the dual map $\mathcal{V}^\vee \to \mathrm{gr}_0 \mathcal{V}^\vee$. Now consider

(9.8) $$\langle f, (\overline{N})^{\otimes n}\omega \rangle \ \in \ \mathcal{O}_{S,Y} \otimes_{\mathbb{Q}} (\Omega^1_{S,Y})^{\otimes n}.$$

Because $dN + N \wedge N = 0$, it follows that $d\overline{N} = 0$ and $\overline{N} \wedge \overline{N} = 0$.

Examples 9.9. Let $S = \mathbb{P}^n \backslash D$ where $D = \cup_{i=0}^m D_i$ is a union of $m + 1 \geq 1$ distinct hyperplanes over $\mathbb{Q}$ and $Y = S(\mathbb{C})$. Let $f_i = 0$ be an equation of D_i, where $f_i \in \mathbb{Q}[x_0, \ldots, x_n]$ is homogeneous of degree one. A basis for $H^1_{dR}(S)$ is given by the cohomology classes of forms

$$\omega_i = d\log\left(\frac{f_i}{f_0}\right) \quad \text{for } 1 \leq i \leq m.$$

A (cohomological) symbol is simply a linear combination of tensor products of ω_i which satisfies the integrability condition (9.7). Since this case is mixed Tate, the length filtration on the bar construction coincides with

(one half of) the weight filtration, and is canonically split by the Hodge filtration. It follows either from this, or from formality of the cohomology of a hypersurface complement [17] §3.2, that $H^0(\mathbb{B}(\Omega_S^\bullet)) \cong H^0(\mathbb{B}(H^\bullet(\Omega_S))$ and so there is no significant difference between symbols smb and their cohomological versions cmb_n. They are integrable words in the one-forms ω_i (resp. their cohomology classes $[\omega_i]$).[12] In the case $n = 1$, the integrability condition (9.7) is trivially satisfied. Example:

$$\mathrm{smb}(\mathrm{Li}_n^{\partial\mathfrak{r}}(x)) = [d\log(1-x)|d\log x| \cdots |d\log x].$$

This setting covers much of the recent work of physicists on symbols.

Remark 9.10. The theory of iterated integrals enables us to construct a map in the opposite direction and associate a family of motivic periods to a symbol together with some extra data. This discussion would take us too far afield. A full treatment should also incorporate the mixed Hodge structure on the reduced bar construction [36] and a discussion of admissible variations and tangential base points.

It is important to remark that symbols *do not* have periods in their own right: to define periods one requires a path of integration, or at the very least a base-point (if one only wishes to define the associated single-valued periods).

10. Some geometric examples

Let S be as in §7 and let $\pi : S \to \mathrm{Spec}\,(\mathbb{Q})$ be the structural map. Denote the Tate variation on S by $\mathbb{Q}(-n)_{/S}$. It is the object of $\mathcal{H}(S)$ which is defined by $\pi^*\mathbb{Q}(-n)$. Concretely, it is the triple $(\mathcal{V}_B, \mathcal{V}_{dR}, c)$ where $\mathcal{V}_{dR} = (\mathcal{O}_S, d)$ is the trivial vector bundle with trivial connection; $\mathcal{V}_B$ is the constant local system $\mathbb{Q}$; and $c : \mathbb{C} = (\mathcal{V}_{dR}^{an})^\nabla \xrightarrow{\sim} \mathcal{V}_B \otimes \mathbb{C} = \mathbb{C}$ is multiplication by $(2\pi i)^n$. It has weight $2n$, and the Hodge filtration satisfies $F^n\mathcal{V}_{dR} = \mathcal{V}_{dR}$, $F^{n+1}\mathcal{V}_{dR} = 0$.

10.1. Mixed Tate motives over number fields

In the following example, let $S = \mathrm{Spec}\,(F)$ for F a number field. Since $S(\mathbb{C}) = \mathrm{Hom}(F, \mathbb{C})$, it is not geometrically connected and therefore does not

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[12]Furthermore, $H^1(\mathbb{P}^n; \mathcal{O}) = 0$, so the canonical extension $\overline{\mathcal{V}}$ of a unipotent vector bundle $\mathcal{V}$ is trivial, and we can take as de Rham fiber functor the global sections functor $\mathcal{V} \mapsto \Gamma(\mathbb{P}^n, \overline{\mathcal{V}}) \in \mathrm{Vec}_{\mathbb{Q}}$.

immediately fit into the framework of the previous paragraphs. However, one can define a category $H(S)$ of realisations without difficulty. An element of $H(S)$ consists of: $V_{dR} \in \mathrm{Vec}_F$ a vector space with the zero connexion; $\mathcal{V}_B$ a collection of vector spaces $V_\sigma \in \mathrm{Vec}_\mathbb{Q}$ for every $\sigma \in S(\mathbb{C})$; and a comparison $c = (c_\sigma)_\sigma$, where $c_\sigma : V_{dR} \otimes_{F,\sigma} \mathbb{C} \cong V_\sigma \otimes_\mathbb{Q} \mathbb{C}$ for every $\sigma \in S(\mathbb{C})$, together with filtrations W, F defined as before. There are Frobenius isomorphisms $F_\infty : V_\sigma \cong V_{\bar\sigma}$ which are compatible with the comparison isomorphisms via the action of complex conjugation. For example, the object $\mathbb{Q}(0)_{/S}$ is the triple $(\{V_\sigma\}, V_{dR}, c)$ where $V_{dR} = F$, and $V_\sigma = \mathbb{Q}$ for each σ, and the isomorphisms c_σ are the canonical ones. Taking $\alpha \in F$, and $\tau : F \hookrightarrow \mathbb{C}$, we can view algebraic numbers as $H(S)$-periods

$$\alpha^{\mathrm{m},\tau} = [\mathbb{Q}(0)_{/S}, \tau, \alpha]^{\mathrm{m}} \in \mathcal{P}_{H(S)}^{\mathrm{m},\tau,\mathrm{Spec}\ F}.$$

Its period $\mathrm{per}(\alpha^{\mathrm{m},\tau}) = \tau(\alpha) \in \mathbb{C}$. Note that the motivic Galois action is trivial on $\alpha^{\mathrm{m},\tau}$, since α^{m} is viewed as a family of periods over F (but observe that if F is Galois, the Galois action on algebraic numbers could be retrieved by considering the action of the automorphism group of $\mathrm{Spec}\,(F)$).

Now consider the category $\mathcal{MT}(\mathcal{O})$ of mixed Tate motives over a ring of integers $\mathcal{O}$ in F defined in [27]. The de Rham fiber functor is $\omega_{dR} : \mathcal{MT}(\mathcal{O}) \to \mathrm{Vec}_F$, and there is a Betti fiber functor $\omega_\sigma : \mathcal{MT}(\mathcal{O}) \to \mathrm{Vec}_\mathbb{Q}$ for every $\sigma \in S(\mathbb{C})$. Hence there is a functor $\mathcal{MT}(\mathcal{O}) \to H(S)$, which is known to be fully faithful [27]. In this manner, we can view the periods of $\mathcal{MT}(\mathcal{O})$ as families of periods over $\mathrm{Spec}\,(F)$.[13]

However, in this, mixed Tate, situation, the de Rham functor is in fact obtained from a canonical fiber functor $\omega : \mathcal{MT}(\mathcal{O}) \to \mathrm{Vec}_\mathbb{Q}$ by extension of scalars $\omega_{dR} = \omega \otimes_\mathbb{Q} F$. This leads to a slightly different point of view. Define, for every $\sigma \in S(\mathbb{C})$, a ring of motivic periods $\mathcal{P}_{\mathcal{MT}(\mathcal{O})}^{\mathrm{m}_\sigma}$ over $\mathbb{Q}$ where m_σ is the pair of fiber functors (ω_σ, ω) in the manner of §2.3. It is spanned by matrix coefficients $[M, x, v]^{\mathrm{m}}$ where $M \in \mathcal{MT}(\mathcal{O})$, $x \in M_\sigma^\vee$ and $v \in \omega(M)$. Every automorphism $\alpha \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ defines an isomorphism $\mathcal{P}_{\mathcal{MT}(\mathcal{O})}^{\mathrm{m}_\sigma} \cong \mathcal{P}_{\mathcal{MT}(\mathcal{O})}^{\mathrm{m}_{\sigma\alpha}}$ via its action on the Betti class.

Following an identical argument to Theorem 6.3, we deduce the following

Theorem 10.1. *The decomposition map gives a canonical isomorphism*

$$(10.1) \qquad \Phi : \mathrm{gr}^C \mathcal{P}_{\mathcal{MT}(\mathcal{O})}^{\mathrm{m}_\sigma,+} \xrightarrow{\sim} \mathbb{Q}[\mathbb{L}^{\mathrm{m}}] \otimes_\mathbb{Q} T^c\left(\bigoplus_n K_{2n-1}(\mathcal{O}) \otimes_\mathbb{Z} \mathbb{Q}(-n)_{dR}\right)$$

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[13]In order to capture better the idea of a family of periods ramified over certain primes, then the current set-up in which we only consider Betti and de Rham information is inadequate. One could proceed along the lines of [21], §1.18.

in an identical manner to §6.6, which describes the structure of its motivic periods. There is a de Rham version of the previous isomorphism:

$$(10.2) \quad \mathrm{gr}^C \mathcal{P}^{\mathfrak{dr}}_{\mathcal{MT}(\mathcal{O})} \xrightarrow{\sim} \mathbb{Q}[\mathbb{L}^{\mathfrak{dr}}, (\mathbb{L}^{\mathfrak{dr}})^{-1}] \otimes_{\mathbb{Q}} T^c(\bigoplus_n K_{2n-1}(\mathcal{O}) \otimes_{\mathbb{Z}} \mathbb{Q}(-n)_{dR})$$

where $\mathfrak{dr}$ means (ω, ω), for ω the canonical fiber functor. The projection map $\mathcal{P}^{\mathfrak{m}_\sigma, +}_{\mathcal{MT}(\mathcal{O})} \to \mathcal{P}^{\mathfrak{dr}}_{\mathcal{MT}(\mathcal{O})}$ corresponds to the homomorphism $\mathbb{L}^{\mathfrak{m}} \to 0$.

Remark 10.2. Goncharov considered the image (of what, in our language would be a $U^{dR}_{\mathcal{MT}(\mathcal{O})}$-period) in the de Rham version of (10.2) in the quotient

$$T^c(K_1(F) \otimes_{\mathbb{Z}} \mathbb{Q}(-1)) \cong \bigoplus_{n \geq 0}(K_1(F) \otimes_{\mathbb{Z}} \mathbb{Q})^{\otimes n} = \bigoplus_{n \geq 0}(F^{\times} \otimes_{\mathbb{Z}} \mathbb{Q})^{\otimes n},$$

(see discussion preceding Lemma 3.7 in [31]). One can ignore the coradical grading in this case, since in this particular quotient it is equivalent to the weight-grading (this is false in general). Thus Goncharov's notion of symbol is the homomorphism

$$\mathcal{O}(U^{dR}_{\mathcal{MT}(\mathcal{O})}) \longrightarrow \bigoplus_{n \geq 0}(F^{\times} \otimes_{\mathbb{Z}} \mathbb{Q})^{\otimes n}.$$

This map has a huge kernel and loses most of the information about periods. For example, if $\mathcal{O} = \mathbb{Z}$, then this homomorphism is identically zero and all (unipotent de Rham) multiple zeta values map to zero.

A version of this notion of symbol for variations, defined in [32], §1.3, is used in the physics literature, and the 'motivic amplitude' considered in [20] is defined by extrapolation as an element of $(F^{\times} \otimes_{\mathbb{Z}} \mathbb{Q})^{\otimes n}$ for F a certain field. For the reasons above, this notion loses information about periods and does not apply in the non-mixed Tate case. It is not to be confused with the notion of motivic periods defined here.

10.2. A family of examples

The following family of examples is sufficient for the purposes of [13]. Let $D \subset X$ be a family of simple normal crossing divisors relative to a smooth morphism $\pi : X \to S$, where S smooth over $\mathbb{Q}$ and geometrically connected. Furthermore, we assume that π is topologically trivial on the underlying analytic varieties (it is a locally trivial fibration of stratified varieties, according to [33]). Let $j : X \backslash D \hookrightarrow X$ be the inclusion. Define an object $H^n(X, D)_{/S}$

in the category $\mathcal{H}(S)$ as follows. Its Betti realisation is

$$H^n_B(X, D)_{/S} = R^n \pi_* j_! \mathbb{Q}$$

where $\mathbb{Q}$ is the constant sheaf on $(X \backslash D)(\mathbb{C})$. Since π is topologically trivial, this is a local system, and its fibres at a point $s \in S(\mathbb{C})$ are $H^n_B(X_s, D_s)$, where X_s, D_s denote the fibres of X, D. For de Rham, denote the irreducible components of D by D_i, for $i \in I$, where I is an ordered set, and write $D_J = \cap_{j \in J} D_j$ for any $\emptyset \neq J \subset I$. Consider the double complex of sheaves of relative differential forms on X

$$\Omega^\bullet_{D_\bullet/S} : \Omega^\bullet_{X/S} \longrightarrow \bigoplus_{|J|=1} \Omega^\bullet_{D_J/S} \longrightarrow \cdots \longrightarrow \bigoplus_{|J|=n} \Omega^\bullet_{D_J/S}$$

where the horizontal maps are determined by the usual rule: if i_j denotes the inclusion of $D_{J\backslash\{i_j\}} \hookrightarrow D_J$, where i_j is the kth element of J then i_j^* occurs with the sign $(-1)^k$. The $\Omega^\bullet_{D_J/S}$ denote the direct images of the corresponding sheaves on D_J, and vanish outside D_J.

Define the de Rham realisation by

$$H^n_{dR}(X, D)_{/S} = \mathbb{R}^n \pi_*(\Omega^\bullet_{D_\bullet/S}).$$

It is the sheaf associated to the presheaf whose sections over an affine open $U \subset S$ are the hypercohomology of $\Omega^\bullet_{D_\bullet/S}(\pi^{-1}(U))$. It is a locally free sheaf of $\mathcal{O}_S$-modules and its fibres at the point s are the relative de Rham cohomology groups:

$$(H^n_{dR}(X, D)_{/S})(s) = H^n_{dR}(X_s, D_s).$$

It admits an integrable connection ∇ by a relative version of [40]. To check the comparison isomorphism, denote by $\mathbb{Q}_J$ the constant sheaf $\mathbb{Q}$ on $D_J(\mathbb{C})$, extended by zero to the whole of $X(\mathbb{C})$. The complex of sheaves

$$\mathbb{Q}_{D_\bullet/S} : \mathbb{Q} \longrightarrow \bigoplus_{|J|=1} \mathbb{Q}_J \longrightarrow \bigoplus_{|J|=2} \mathbb{Q}_J \longrightarrow \cdots \longrightarrow \bigoplus_{|J|=n} \mathbb{Q}_J$$

where the sign conventions are exactly as defined for the complex $\Omega^\bullet_{D_\bullet/S}$, defines a resolution of $j_! \mathbb{Q}$. The analytification $\Omega^{\bullet, an}_{D_\bullet/S}$ of $\Omega^\bullet_{D_\bullet/S}$ is a resolution of $\mathbb{Q}_{D_\bullet/S} \otimes \mathbb{C}$ over $S^{an}(\mathbb{C})$. Using the triviality of π, and arguing as in [25],

Proposition 2.28, there is a natural isomorphism

$$c^{-1} : H_B^n(X, D)_{/S} \otimes_{\mathbb{Q}} \mathcal{O}_S^{an} \xrightarrow{\sim} (H_{dR}^n(X, D)_{/S})^{an}.$$

It is known that $H_B^n(X, D)_{/S}$, equipped with its weight filtration and Hodge filtration from $cH_{dR}^n(X, D)_{/S}$, is a variation of mixed Hodge structure. It is effective: the Hodge numbers on every fiber satisfy $h_{p,q} = 0$ if p or q are < 0.

10.3. Face maps

With the above notations, let $D_I = \cap_{i \in I} D_i$ denote an intersection of irreducible components of D of codimension k, and let $D^I = \bigcup_{j \notin I} D_j$ denote the union of all remaining irreducible components. The pair $D_I \supset D_I \cap D^I$ satisfies the conditions of the previous paragraph.

There are natural morphisms, that we shall call *face maps*, in the category $\mathcal{H}(S)$

$$(10.3) \qquad H^{n-k}(D_I, D_I \cap D^I)_{/S} \longrightarrow H^n(X, D)_{/S}.$$

For the de Rham (respectively Betti) realisation, this is given by the inclusion of complexes $\Omega_{D_I^\bullet/S}^\bullet \to \Omega_{D_\bullet/S}^\bullet$ (respectively $\mathbb{Q}_{D_I^\bullet/S} \to \mathbb{Q}_{D_\bullet/S}$).

On the other hand, let $0 \leq k \leq n - 1$ and let $D^{(k)} = \bigcup_{|I|=n-k} D_I$ denote the k-dimensional skeleton of D. Then $H^k(D^{(k)})_{/S}$ defines an object of $\mathcal{H}(S)$ given by truncating the complexes $\Omega_{D_\bullet/S}^\bullet$ and $\mathbb{Q}_{D_\bullet/S}$ on the left so that the non-zero components are $|J| \geq n - k$. The inclusion of these complexes similarly defines

$$(10.4) \qquad H^k(D^{(k)})_{/S} \longrightarrow H^n(X, D)_{/S}.$$

The case $k = n - 1$ is the boundary map in the relative cohomology sequence

$$\cdots \longrightarrow H^{n-1}(X)_{/S} \longrightarrow H^{n-1}(D^{(n-1)})_{/S}$$
$$\longrightarrow H^n(X, D)_{/S} \longrightarrow H^n(X)_{/S} \longrightarrow \cdots$$

The face maps (10.3) factor through (10.4).

10.4. Weight filtration

By strictness, apply the weight functor to the previous long exact cohomology sequence to obtain an exact sequence

$$\cdots \to W_k H^{n-1}(X)_{/S} \to W_k H^{n-1}(D^{(n-1)})_{/S}$$
$$\to W_k H^n(X,D)_{/S} \to W_k H^n(X)_{/S} \to \cdots$$

Since $H^n(X)$ has smooth fibers, it has weights concentrated in degrees between n and $2n$ (by [23] 8.2.4). It follows that

$$W_k H^{n-1}(D^{(n-1)})_{/S} \longrightarrow W_k H^n(X,D)_{/S}$$

is surjective if $k = n - 1$, and an isomorphism if $k < n - 1$. The following proposition generalises the previous fact, and is presumably well-known. We include a quick proof for completeness. It requires the following lemma.

Lemma 10.3. *Let $\phi : C \to C'$ be a morphism of cochain complexes such that $\phi_i : C_i \to C'_i$ is surjective, and $\phi_j : C_j \to C'_j$ are isomorphisms for all $j > i$. Then the induced maps on cohomology have the same properties: $H^i(C) \to H^i(C')$ is surjective, and $H^j(C) \overset{\sim}{\to} H^j(C')$ is an isomorphism for all $j > i$.*

Proof. Exercise. $\qquad\qquad\square$

Proposition 10.4. *Let $m < n$. The map*

$$W_k H^m(D^{(m)})_{/S} \longrightarrow W_k H^n(X,D)_{/S}$$

is surjective if $m = k$ and is an isomorphism if $k < m$.

Proof. Since the comparison isomorphism respects the weight filtration, it suffices to verify the statement in the Betti realisation. For this it is enough to check the statement on every fiber: for every $t \in S(\mathbb{C})$,

$$W_k H_B^m(D^{(m)})_t \longrightarrow W_k H_B^n(X,D)_t$$

is an isomorphism for $k < m$ and surjective for $k = m$. To alleviate the notation, write $B_I = (D_I)_t(\mathbb{C})$ and $Y = X_t(\mathbb{C})$. Consider

(10.5) $$H^m(B^{(m)}) \longrightarrow H^n(Y,B).$$

There are relative cohomology spectral sequences

$$(10.6) \qquad E_1^{p,q}(Y) = \bigoplus_{|J|=p} H^q(B_J) \quad \Longrightarrow \quad H^{p+q}(Y,B)$$

and

$$E_1^{p,q}(B^{(m)}) = \bigoplus_{|J|=p+n-m} H^q(B_J) \quad \Longrightarrow \quad H^{p+q}(B^{(m)})$$

The morphism (10.5) induces a map of spectral sequences

$$E_1^{p,q}(B^{(m)}) \longrightarrow E_1^{p+n-m,q}(Y)$$

which is the identity on each summand $H^q(B_J)$. Let $j \leq k$ and apply the functor gr_j^W. It is exact, giving a morphism of spectral sequences

$$(10.7) \qquad \mathrm{gr}_j^W E_r^{p,q}(B^{(m)}) \longrightarrow \mathrm{gr}_j^W E_r^{p+n-m,q}(Y).$$

Since B_J is smooth, $H^q(B_J)$ has weights in the interval $[q, 2q]$ by [23] 8.2.4, and therefore both sides of (10.7) vanish for all $q > j$. The entries of (10.7) for $r = 1$ are identical in the range $p \geq 0$. By running the spectral sequence, and applying the previous lemma, one verifies by induction on r that (10.7) is an isomorphism for $p \geq r - 1$ or $p + q \geq j + 1$ and surjective for other values of $p \geq 0$. □

The spectral sequence (10.6) implies the

Corollary 10.5. *Let $0 \leq k \leq n$. Then $\mathrm{gr}_k^W H^n(X,D)_{/S}$ is isomorphic to a subquotient of $\bigoplus_{|I| \geq n-k} \mathrm{gr}_k^W H^{n-|I|}(D_I)_{/S}$.*

Putting $k = 0$, $m = 0$ or $m = 1$ in the previous proposition gives the following corollary.

Corollary 10.6. *We have*

$$(10.8) \qquad W_0 H^n(X,D)_{/S} \cong \mathbb{Q}(0)_{/S}^{\oplus m}$$

where $m = \dim_{\mathbb{Q}} \widetilde{H}^{n-1}(D_t(\mathbb{C}))$ for any $t \in S(\mathbb{C})$. In particular, the motivic periods of $H^n(X,D)_{/S}$ of weight zero are constant and rational.

Proof. Note that if $|I| = n$ then $D_I \cong \mathrm{Spec}\, S$ and $H^0(D_I)_{/S} = \mathbb{Q}(0)_{/S}$, so (10.8) holds for some m. To determine m, pass to the Betti realisation at

the fiber s. With the notations of the previous proposition, we have

$$\operatorname{gr}_0^W H^n(Y, B) \cong E_2^{n,0}(Y) = \operatorname{coker}\left(\bigoplus_{|I|=n-1} H^0(B_I) \longrightarrow \bigoplus_{|I|=n} H^0(B_I) \right)$$

Since the B_I are connected, the dimension of this cokernel is the dimension of the reduced cohomology $\dim \widetilde{H}^{n-1}(B)$. $\square$

We considered earlier the case $S = \operatorname{Spec}\mathbb{Q}$, and D normal crossing, rather than simple normal crossing. Then a similar argument proves that the weight zero part of $H^n(X, D)_{/S}$ is the (realisation of a) constant Artin motive. The action of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ upon its Betti realisation is induced by the Galois action on the points $\bigcup_{|I|=n} D_I(\mathbb{C})$.

Under some further assumptions, the face maps provide information about the mixed Hodge structure on $H^n(X, D)_{/S}$ in low weights.

Proposition 10.7. *Suppose that $H^k((D_I)_s) = 0$ for all $k > n - |I|$ (for example, if the strata D_I have affine fibres of dimension $n - |I|$).*

Let $m < n$. Then for any $k \leq m$, the sum of face maps

$$\bigoplus_{|I|=n-m} W_k H^m(D_I, D_I \cap D^I)_{/S} \longrightarrow W_k H^n(X, D)_{/S}.$$

is surjective.

Proof. A similar spectral sequence argument as in the previous proposition. The assumption implies that $E_1^{p,q}$ vanishes above the diagonal, i.e., for all $p + q > n$. Now consider the map of spectral sequences induced by the sum of face maps. Take their fibers as in the previous proposition and apply gr_j^W for $j \leq k$. On $E_1^{p,q}$ terms we obtain the natural map

$$(10.9) \qquad \operatorname{gr}_j^W \bigoplus_{|I|=n-m} \bigoplus_{J \supset I, |J|=p} H^q(B_J) \longrightarrow \operatorname{gr}_j^W \bigoplus_{|J|=p} H^q(B_J).$$

This is an isomorphism above the diagonal $(p + q > n)$ since both sides are identically zero in that region by assumption. Furthermore, both sides of (10.9) vanish in the region $q > j$ (since the B_J are smooth and $H^q(B_J)$ has weights in the interval $[q, 2q]$) and therefore in particular in the region $q > m$. It follows that (10.9) is also surjective along the diagonal $p + q = n$. By induction on r and Lemma 10.3, the induced map is surjective on the diagonal for all r. $\square$

10.5. The periods

In the situation of §10.2, let $s \in S(\mathbb{C})$ and let $\sigma_s \subset X_s(\mathbb{C})$ be a topological n-chain whose boundary is contained in $D_s(\mathbb{C})$. It defines a relative homology class $[\sigma_s] \in H_B^n(X_s(\mathbb{C}), D_s(\mathbb{C}))^\vee = (H_B^n(X, D)_{/S})_s^\vee$. By local triviality, there exists a small neighbourhood $N \subset S(\mathbb{C})$ of s, and an isomorphism

$$(X(\mathbb{C}), D(\mathbb{C})) \cap \pi^{-1}(N) \cong N \times (X_s(\mathbb{C}), D_s(\mathbb{C})).$$

Via this isomorphism, the chain σ_s uniquely extends to a family of topological n-chains $\sigma_t \subset X_t(\mathbb{C})$ whose boundaries are contained in $D_t(\mathbb{C})$ for all $t \in N$.

For simplicity, let us consider the particular case when we are given a global form $\omega \in \Omega_{\pi^{-1}(U)/U}^n$ for some Zariski open $U \subset S$ and suppose that the fibres of X are of dimension n. Since the restrictions of ω to components of D vanish for reasons of dimension, it defines a relative class $[\omega] \in \Gamma(U, H_{dR}^n(X, D)_{/S})$, and

$$\xi = [H^n(X, D)_{|S}, [\sigma_s], [\omega]]^{\mathfrak{m}} \in \mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{m}, \{s\}, Y}$$

for any $Y \subset U(\mathbb{C})$. For any $t \in U(\mathbb{C})$, let ω_t denote the restriction of the form ω to a fiber t. The period is then

$$\mathrm{per}(\xi)(t) = \int_{\sigma_t} \omega_t$$

for all t in the open set $U(\mathbb{C}) \cap N$, and is extended by analytic continuation to a meromorphic function on the universal covering of $S(\mathbb{C})$ based at s.

10.6. Example: iterated integrals on the projective line minus 3 points

Let $S = \mathbb{P}^1 \backslash \{0, 1, \infty\}$ throughout this section.

10.6.1. Motivic logarithm. Let $Z = S \times \mathbb{G}_m$ and $\pi : Z \to S$ the projection onto the first factor. If x is the coordinate on S, and y the coordinate on $\mathbb{G}_m$, let $D = \{y = 1\} \cup \{y = x\}$. Let $X \subset S(\mathbb{C})$ denote the real interval $(0, 1)$, and for all $x \in X$, let $\sigma_x \subset \mathbb{G}_m(\mathbb{C})$ denote the straight path from 1 to x in the fiber over x. Its image is $\{x\} \times [1, x]$. Define the motivic logarithm

by

$$\log^{\mathrm{m}}(x) = [H^1(Z,D)_{|S}, [\sigma_x], [\tfrac{dy}{y}]]^{\mathrm{m}} \ \in \ \mathcal{P}^{\mathrm{m},X,Y}_{\mathcal{H}(S)}$$

where $Y = S(\mathbb{C})$. Note that $\mathcal{O}_{S,Y} = \mathcal{O}_S = \mathbb{Q}[x, x^{-1}, (1-x)^{-1}]$ (this was example (3) in §7.2.1). Its period is the logarithm as expected:

$$\operatorname{per}(\log^{\mathrm{m}}(x)) = \log(x) = \int_{\sigma_x} \frac{dy}{y} \quad \text{ for } x \in X.$$

A long exact relative cohomology sequence reduces to

$$0 \longrightarrow \mathbb{Q}(0)_{|S} \longrightarrow H^1(Z,D)_{|S} \longrightarrow \mathbb{Q}(-1)_{|S} \longrightarrow 0,$$

and with respect to the de Rham basis $[\tfrac{dy}{y}]$, $[\tfrac{dy}{x-1}]$ and Betti basis $[\sigma_x], [\gamma]$, where γ is a small loop around 0 in the fiber, the period matrix is given by the identical formula to (5.4), where $\mathbb{L}^{\mathrm{m}}$ is now viewed as a constant family of periods over S and $\alpha = x$. The coaction satisfies $\Delta \log^{\mathrm{m}}(x) = \log^{\mathrm{m}}(x) \otimes \mathbb{L}^{\eth\mathfrak{r}} + 1 \otimes \log^{\eth\mathfrak{r}}(x)$, and the Galois group is $\mathbb{G}_a \rtimes \mathbb{G}_m$, and is in fact defined over $\mathbb{Q}$ in this case. Recall that $\log^{\eth\mathfrak{r}}(x)$ does not have a period, but has a single-valued period $2\log|x|$.

10.6.2. Motivic fundamental groupoid. Now consider the ind-object of $\mathcal{H}(S)$

$$\mathcal{O}(\pi_1^{\mathcal{H}}(S, \overrightarrow{1_0}, \bullet)) := (\mathcal{O}(\pi_1^{un}(S, \overrightarrow{1_0}, \bullet)), \mathcal{O}(\pi_1^{dR}(S)) \otimes_{\mathbb{Q}} \mathcal{O}_S, \operatorname{comp})$$

where $\overrightarrow{1_0}$ is the tangential base-point at 0 with unit length. The first entry (Betti local system) is the affine ring of the unipotent completion of the torsor of paths beginning at $\overrightarrow{1_0}$ on $S(\mathbb{C})$ and defines a local system on $S(\mathbb{C})$: its fiber at a point $x \in S(\mathbb{C})$ is $\mathcal{O}(\pi_1^{un}(S, \overrightarrow{1_0}, x))$ with the action of $\pi_1^{\mathrm{top}}(S(\mathbb{C}), x)$. The second entry does not in fact depend on basepoints and is the affine ring of the (unipotent) de Rham fundamental group. It is a shuffle algebra on two generators

$$\mathcal{O}(\pi_1^{dR}(S)) \cong T^c(\mathbb{Q}e_0 \oplus \mathbb{Q}e_1)$$

where e_0, e_1 correspond to the one-forms $\tfrac{dx}{x}$ and $\tfrac{dx}{1-x}$. It defines a trivial vector bundle on S, and is equipped with the Kniznhik-Zamolodchikov connection

$$\nabla e_{i_1} \cdots e_{i_n} = \begin{cases} e_{i_1} \cdots e_{i_{n-1}} \otimes \frac{dx}{x} & \text{if } i_n = 0 \\ e_{i_1} \cdots e_{i_{n-1}} \otimes \frac{dx}{1-x} & \text{if } i_n = 1. \end{cases}$$

It is clearly unipotent with respect to the weight filtration. The weight filtration on $\mathcal{O}(\pi_1^{dR}(S))$ is in fact a grading in this case (it is split by the

Hodge filtration since it is mixed Tate), and the grading assigns degree two to e_0, e_1.

We take $X = (0, 1)$ and $Y = S(\mathbb{C})$ as before. Define the motivic multiple polylogarithm to be the family of motivic periods:

$$(10.10) \qquad \mathrm{Li}_w^{\mathfrak{m}}(x) = [\mathcal{O}(\pi_1^{\mathcal{H}}(S, \vec{1}_0, \bullet)), \sigma_x, w]^{\mathfrak{m}} \ \in \ \mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{m}, X, Y}$$

where w is a word in e_0, e_1 and σ_x is the straight line path from $\vec{1}_0$ to $x \in X$. The path σ_x is viewed as an element of $\mathcal{O}(\pi_1^{un}(S, \vec{1}_0, x))^\vee$ via the natural map

$$(10.11) \qquad \pi_1^{top}(S(\mathbb{C}), \vec{1}_0, x) \longrightarrow \pi_1^{un}(S, \vec{1}_0, x)(\mathbb{Q}).$$

The period of $\mathrm{Li}_w^{\mathfrak{m}}(x)$ is $\mathrm{Li}_w(x)$, which is the iterated integral $\int_{\sigma_x} w$, and we have $\mathrm{Li}_{e_0}^{\mathfrak{m}}(x) = \log^{\mathfrak{m}}(x)$. More generally we write $\mathrm{Li}_n^{\mathfrak{m}}(x)$ for $\mathrm{Li}_{e_1 e_0^{n-1}}^{\mathfrak{m}}(x)$, and call it the motivic classical polylogarithm. The connection satisfies

$$\nabla \mathrm{Li}_{we_s}^{\mathfrak{m}}(x) = (-1)^s \mathrm{Li}_w^{\mathfrak{m}}(x) \otimes \frac{dx}{x - s} \qquad \text{where } s \in \{0, 1\}, \ w \in \{e_0, e_1\}^\times$$

and $\mathrm{Li}_\emptyset^{\mathfrak{m}}(x)$ is the constant motivic period 1. The de Rham motivic multiple polylogarithms are defined by

$$\mathrm{Li}_w^{\mathfrak{dr}}(x) = [\mathcal{O}(\pi_1^{\mathcal{H}}(S, \vec{1}_0, \bullet)), \varepsilon, w]^{\mathfrak{m}} \ \in \ \mathcal{P}_{\mathcal{H}(S)}^{\mathfrak{dr}, Y}$$

where $\varepsilon : \mathcal{O}(\pi_1^{dR}(S)) \to \mathbb{Q}$ is the augmentation map (it sends every non-trivial word w to zero). The de Rham versions are the images of $\mathrm{Li}_w^{\mathfrak{m}}(x)$ under the projection map §4.3. Our definition of the symbol satisfies, as expected,

$$\mathrm{smb}(\mathrm{Li}_w^{\mathfrak{dr}}(x)) = w \ \in \ T^c(\Omega^1(S)) \cong H^0(\mathbb{B}(\Omega^1(S))).$$

The single-valued versions of $\mathrm{Li}_w^{\mathfrak{dr}}(x)$ are obtained in an identical way to [12], by simply writing a superscript $\mathfrak{m}$ everywhere (with a possible sign difference of $(-1)^{|w|}$).

Recall that for every word w we defined the (image in the ring of $\mathcal{H}$-periods) of the shuffle-regularised motivic multiple zeta values:

$$\zeta^{\mathfrak{m}}(w) = [\mathcal{O}(\pi_1^{\mathcal{H}}(S, \vec{1}_0, -\vec{1}_1)), \varepsilon, w]^{\mathfrak{m}} \ \in \ \mathcal{P}_{\mathcal{H}}^{\mathfrak{m}},$$

where $-\overrightarrow{1}_1$ is the tangent vector -1 at the point 1. Denote their pull-backs to $\mathcal{P}^{\mathsf{m},X,Y}_{\mathcal{H}(S)}$ via the structural map $\pi : S \to \mathrm{Spec}\,\mathbb{Q}$ by the same symbol. Let

$$\mathcal{P}^{\mathsf{m},+}_{\mathcal{M}_{0,4}} \subset \mathcal{P}^{\mathsf{m},X,Y}_{\mathcal{H}(S)}$$

denote the $\mathcal{O}_S$-module generated by the $\zeta^{\mathsf{m}}(w), \mathrm{Li}^{\mathsf{m}}_w(x)$ and $\mathbb{L}^{\mathsf{m}}$. Then $\mathcal{P}^{\mathsf{m},+}_{\mathcal{M}_{0,4}}$ is a ring of motivic periods which is stable under the monodromy action of the fundamental group of $S(\mathbb{C})$ at the base-point $\overrightarrow{1}_0$. More generally, one can consider in a similar manner the motivic periods given by iterated integrals on the moduli spaces of curves $\mathcal{M}_{0,n}$ whose periods are multiple polylogarithms in several variables. For example, the space of functions H used in the analytic bootstrap for the 6-point function in the planar limit of $N = 4$ SYM theory described in the introduction of [13] is contained in the space $\mathcal{P}^{\mathsf{m},+}_{\mathcal{M}_{0,6}}$.

Remark 10.8. These examples can be expressed geometrically in the spirit of §10.2 using Beilinson's construction for the unipotent fundamental group, and indeed defines a variation of mixed Tate motives in the sense of [27] 4.12.

If t is a rational point on S (or even a tangential base point), we can define the evaluation ev_t at the point t. In this situation, the de Rham coaction (7.8) commutes with evaluation at t:

$$\Delta\mathrm{ev}_t(\mathrm{Li}^{\mathsf{m}}) = (\mathrm{ev}_t \otimes \mathrm{ev}_t)\Delta\mathrm{Li}^{\mathsf{m}}(x).$$

The follows from the triviality of the de Rham vector bundle $\mathcal{O}(\pi_1^{dR}(S))$ and equation (2.2). The unipotent coaction can be computed by transposing a formula due to Goncharov [30] to this setting. By way of example, the de Rham coaction on the motivic dilogarithm satisfies

$$\Delta\mathrm{Li}^{\mathsf{m}}_2(x) = \mathrm{Li}^{\mathsf{m}}_2(x) \otimes (\mathbb{L}^{\mathfrak{dr}})^2 + \mathrm{Li}^{\mathsf{m}}_1(x) \otimes \mathbb{L}^{\mathfrak{dr}} \log^{\mathfrak{dr}}(x) + 1 \otimes \mathrm{Li}^{\mathfrak{dr}}_2(x)$$

and was discussed in further detail in Example 8.2. The unipotent coaction is obtained by replacing $\mathfrak{dr}$ in the right-hand terms by $\mathfrak{u}$, and noting that $\mathbb{L}^{\mathfrak{u}} = 1$.

In conclusion, the notion of families of motivic periods provides a framework for motivic multiple polylogarithms which includes constants and is sufficient for many applications to high-energy physics. It satisfies the properties conjectured in [28].

11. Glossary of non-standard terms

Types of periods:

$\mathcal{H}$-*periods* : page 577, §3.2 $\mathcal{H}$-*de Rham* : page 579, §3.3
motivic : page 581, §3.5   *effective* : page 578, §3.3
mixed Tate : page 578, §3.3   *single-valued*: page 589, §4.1
semi-simple: page 585, §3.7   *unipotent*: page 585, §3.7
primitive, stable: page 609, §1

Invariants of periods and auxiliary constructions:

conjugates: page 583, 1)   *rank*: page 584, §2
Hodge numbers: page 576 , (3.2)   (*polynomial*: page 583, §3.6)
period matrix: page 583, §3.6   (*single-valued*: page 589, §4.1)
determinant: page 583, §3.6   *unipotency degree*: page 587, §3.8
Transcendence dimension: page 582, §3.6 *Galois group*: page 582 , §3.6
Decomposition into primitives: page 571, 2.7; page 587, 3.10

Operations on periods

antipode: page 586, §3.7   *projection map*: page 591, §4.3

Families of periods

monodromy homomorphism/action: page 617, §7.2.1 , (7.9), page 619
constant map: page 618, §7.3.1   *evaluation map*: page 618, §7.3.2
Weight/Hodge filtration: page 619, §7.4   *connection*: page 620,
period homomorphism: page 621, §7.5

Symbols

symbol: page 632, §9.2   *symbol at a point*: page 637, §9.3.3
cohomological symbol: page 637, 9.7   *length*: page 632, §9.2
differentially unipotent/unipotent monondromy: page 630, 9.1

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