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On cubic Hodge integrals and random matrices

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A conjectural relationship between the GUE partition function with even couplings and certain special cubic Hodge integrals over the moduli spaces of stable algebraic curves is under consideration.

1. Introduction

1.1. Cubic Hodge partition function

Let $\overline{\mathcal{M}}_{g,k}$ denote the Deligne–Mumford moduli space of stable curves of genus g with k distinct marked points. Denote by $\mathcal{L}_i$ the i^{th} tautological line bundle over $\overline{\mathcal{M}}_{g,k}$, and $\mathbb{E}_{g,k}$ the rank g Hodge bundle. Let $\psi_i := c_1(\mathcal{L}_i)$, $i = 1, \dots, k$, and let $\lambda_i := c_i(\mathbb{E}_{g,k})$, $i = 0, \dots, g$. Recall that the Hodge integrals over $\overline{\mathcal{M}}_{g,k}$, *aka* the intersection numbers of ψ- and λ-classes, are integrals of the form

$$\int_{\overline{\mathcal{M}}_{g,k}} \psi_1^{i_1} \cdots \psi_k^{i_k} \cdot \lambda_1^{j_1} \cdots \lambda_g^{j_g}, \qquad i_1, \dots, i_k, j_1, \dots, j_g \geq 0.$$

Note that the dimension-degree matching implies that the above integrals vanish unless

$$3g - 3 + k = (i_1 + i_2 + \cdots + i_k) + (j_1 + 2\, j_2 + 3\, j_3 + \cdots + g\, j_g).$$

The particular case of *cubic Hodge integrals* of the form

$$(1) \qquad \int_{\overline{\mathcal{M}}_{g,k}} \Lambda_g(p)\, \Lambda_g(q)\, \Lambda_g(r)\, \psi_1^{i_1} \cdots \psi_k^{i_k}, \qquad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 0$$

was intensively studied after the formulation of the celebrated Gopakumar–Mariño–Vafa conjecture [21, 28] regarding the Chern–Simons/string duality.

Here we denote

$$\Lambda_g(z) = \sum_{i=0}^{g} \lambda_i \, z^i$$

the Chern polynomial of $\mathbb{E}_{g,k}$. A remarkable expression for the cubic Hodge integrals of the form

$$\int_{\overline{\mathcal{M}}_{g,k}} \frac{\Lambda_g(p)\Lambda_g(q)\Lambda_g(r)}{(1 - x_1\,\psi_1)\cdots(1 - x_k\,\psi_k)}, \qquad k \geq 0$$

conjectured in [28] was proven in [25, 30]; for more about cubic Hodge integrals see in the subsequent papers [9, 26, 27, 34].

In the present paper we will deal with the specific case of Hodge integrals (1) with a pair of equal parameters among p, q, r; without loss of generality[1] one can assume that $p = q = -1$, $r = 1/2$. So, the *special cubic Hodge integrals* of the form

$$\int_{\overline{\mathcal{M}}_{g,k}} \Lambda_g(-1)\,\Lambda_g(-1)\,\Lambda_g\left(\frac{1}{2}\right)\,\psi_1^{i_1}\cdots\psi_k^{i_k} \tag{2}$$

will be considered. Denote

$$\begin{aligned}\mathcal{H}(\mathbf{t};\epsilon) = \sum_{g\geq 0}\epsilon^{2g-2}\sum_{k\geq 0}\frac{1}{k!}\sum_{i_1,\dots,i_k\geq 0} t_{i_1}\cdots t_{i_k} \\ \times \int_{\overline{\mathcal{M}}_{g,k}} \Lambda_g(-1)\,\Lambda_g(-1)\,\Lambda_g\left(\frac{1}{2}\right)\,\psi_1^{i_1}\cdots\psi_k^{i_k}\end{aligned} \tag{3}$$

the generating function of these integrals. Here and below $\mathbf{t} = (t_0, t_1, \dots)$ are independent variables, ϵ is a parameter. The exponential $e^{\mathcal{H}} =: Z_{\mathbb{E}}$ is called the cubic Hodge partition function while $\mathcal{H}(\mathbf{t};\epsilon)$ is the cubic Hodge

[1]Indeed, the general situation under consideration is $p = q = -2s$, $r = s$, $s \neq 0$. Similarly as (3)–(4), one can define $\mathcal{H}_g(\mathbf{t};s)$, $g \geq 0$; see also [9]. Then $\mathcal{H}_0(\mathbf{t};s)$ does not depend on s, and for $g \geq 1$, the dependence on s for $\mathcal{H}_g(\mathbf{t};s)$ can be obtained through a rescaling of $v := \partial_{t_0}^2\mathcal{H}_0(\mathbf{t})$. Hence, the "one-parameter family" is essentially a single "point". Our choice $s = 1/2$, however, is the simplest/best choice, which avoids a rescaling of v in the comparison between the Hodge integrals and matrix integrals.

free energy. It can be written in the form of genus expansion

$$\mathcal{H}(\mathbf{t};\epsilon) = \sum_{g\geq 0} \epsilon^{2g-2}\mathcal{H}_g(\mathbf{t}) \tag{4}$$

where $\mathcal{H}_g(\mathbf{t})$ is called the genus g part of the cubic Hodge free energy, $g \geq 0$. Clearly $\mathcal{H}_0(\mathbf{t})$ coincides with the Witten–Kontsevich generating function of genus zero intersection numbers of ψ-classes

$$\begin{aligned} \mathcal{H}_0(\mathbf{t}) &= \sum_{k\geq 0} \frac{1}{k!} \sum_{i_1,\dots,i_k\geq 0} t_{i_1}\cdots t_{i_k} \int_{\overline{\mathcal{M}}_{0,k}} \psi_1^{i_1}\cdots\psi_k^{i_k} \\ &= \sum_{k\geq 3} \frac{1}{k(k-1)(k-2)} \sum_{i_1+\cdots+i_k=k-3} \frac{t_{i_1}}{i_1!}\cdots\frac{t_{i_k}}{i_k!}. \end{aligned} \tag{5}$$

We note that an efficient algorithm for computing $\mathcal{H}_g(\mathbf{t})$, $g \geq 1$ was recently proposed in [9].

1.2. GUE partition function with even couplings

Let $\mathcal{H}(N)$ denote the space of $N \times N$ Hermitean matrices. Denote

$$dM = \prod_{i=1}^N dM_{ii} \prod_{i<j} d\mathrm{Re}M_{ij}\, d\mathrm{Im}M_{ij}$$

the standard unitary invariant volume element on $\mathcal{H}(N)$. The most studied Hermitean random matrix model is governed by the following GUE partition function with even couplings

$$Z_N(\mathbf{s};\epsilon) = \frac{(2\pi)^{-N}}{\mathrm{Vol}(N)} \int_{\mathcal{H}(N)} e^{-\frac{1}{\epsilon}\operatorname{tr} V(M;\mathbf{s})} dM. \tag{6}$$

Here, $V(M;\mathbf{s})$ is an **even** polynomial of M

$$V(M;\mathbf{s}) = \frac{1}{2}M^2 - \sum_{j\geq 1} s_j\, M^{2j}, \tag{7}$$

or, more generally, a power series, by $\mathbf{s} = (s_1, s_2, s_3, \dots)$ we denote the collection of coefficients[2] of $V(M)$, and by $\mathrm{Vol}(N)$ the volume of the quotient

[2]The notation here is slightly different from that of [8, 11] where the coefficient of M^{2j} was denoted by s_{2j}.

of the unitary group over the maximal torus $[U(1)]^N$

$$\mathrm{Vol}(N) = \mathrm{Vol}\left(U(N)/\,[U(1)]^N\right) = \frac{\pi^{\frac{N(N-1)}{2}}}{G(N+1)}, \quad G(N+1) = \prod_{n=1}^{N-1} n!. \tag{8}$$

The integral will be considered as a formal saddle point expansion with respect to the small parameter ϵ. Introduce the *'t Hooft coupling* parameter x by

$$x := N\,\epsilon.$$

Reexpanding the free energy $\mathcal{F}_N(\mathbf{s};\epsilon) := \log Z_N(\mathbf{s};\epsilon)$ in powers of ϵ and replacing the Barnes G-function by its asymptotic expansion [1, 20, 32]

$$\log G(N+1) \sim \left(\frac{N^2}{2} - \frac{1}{12}\right)\log N - \frac{3}{4}N^2 + \zeta'(-1) + \frac{N}{2}\log(2\pi) + \sum_{g\geq 2} \frac{B_{2g}}{4g(g-1)N^{2g-2}}, \qquad N\to\infty.$$

yields[3]

$$\mathcal{F}(x,\mathbf{s};\epsilon) := \mathcal{F}_N(\mathbf{s};\epsilon)|_{N=\frac{x}{\epsilon}} - \frac{1}{12}\log\epsilon = \sum_{g\geq 0}\epsilon^{2g-2}\mathcal{F}_g(x,\mathbf{s}). \tag{9}$$

Here, B_k, $k\geq 0$ denote the Bernoulli numbers defined through

$$\frac{z}{e^z-1} = \sum_{k\geq 0}\frac{B_k}{k!}z^k.$$

The GUE free energy $\mathcal{F}(x,\mathbf{s};\epsilon)$ can be represented [2, 23, 24] in the form

$$\begin{aligned}\mathcal{F}(x,\mathbf{s};\epsilon) = {} & \frac{x^2}{2\epsilon^2}\left(\log x - \frac{3}{2}\right) - \frac{1}{12}\log x + \zeta'(-1) \\ & + \sum_{g\geq 2}\epsilon^{2g-2}\frac{B_{2g}}{4g(g-1)x^{2g-2}} \\ & + \sum_{g\geq 0}\epsilon^{2g-2}\sum_{k\geq 0}\sum_{i_1,\dots,i_k\geq 1} a_g(i_1,\dots,i_k)\, s_{i_1}\dots s_{i_k}\, x^{2-2g-(k-|i|)},\end{aligned} \tag{10}$$

$$a_g(i_1,\dots,i_k) = \sum_{\Gamma}\frac{1}{\#\,\mathrm{Sym}\,\Gamma} \tag{11}$$

[3]It is often called $1/N$-expansion as $\epsilon = \mathcal{O}(1/N)$.

where the last summation is taken over all connected oriented ribbon graphs Γ of genus g with k unlabelled vertices of valencies $2i_1, \ldots, 2i_k$ and with labelled half-edges at every vertex, $\#\,\mathrm{Sym}\,\Gamma$ is the order of the symmetry group of Γ, and $|i| := i_1 + \cdots + i_k$ (see details in [2, 17, 18, 22–24, 29])[4].

Our goal is to compare the expansions (3) and (10).

1.3. From cubic Hodge integrals to random matrices. Main Conjecture.

It was already observed by E. Witten [33] that the GUE partition function with an even polynomial $V(M)$ is tau-function of a particular solution to the Volterra (also called the discrete KdV) hierarchy. Recall that the first equation of the hierarchy (the Volterra lattice equation) reads

$$\dot{w}_n = w_n\,(w_{n+1} - w_{n-1})$$

where

$$w_n = \frac{Z_{n+1} Z_{n-1}}{Z_n^2},$$

the time derivative is with respect to the variable $t = N\,s_1$. Other couplings s_k are identified with the time variables of higher flows of the hierarchy. On another side, the study [9] of integrable systems associated with the Hodge integrals[5] suggested the following conjectural statement: the Hodge partition function $Z_{\mathbb{E}} = e^{\mathcal{H}}$ of the form (3) as function of independent parameters t_i is also a tau-function of the Volterra hierarchy. This observation provides a motivation for the main conjecture of the present paper.

It will be convenient to change normalisation of the GUE couplings. Put

$$\bar{s}_k := \binom{2k}{k} s_k.$$

[4]The rational numbers $a_g(i_1, \ldots, i_k)$ have also the following alternative expression

$$a_g(i_1, \ldots, i_k) = \prod_{j=1}^{k} 2i_j \cdot \sum_G \frac{1}{\#\,\mathrm{Sym}\,G} \tag{12}$$

where the summation is taken over connected oriented ribbon graphs G of genus g with unlabelled half-edges and unlabelled vertices of valencies $2i_1, \ldots, 2i_k$.

[5]The first example of an integrable system associated with *linear* Hodge integrals was investigated by A. Buryak. In this case the integrable system was proved to be Miura equivalent to the Intermediate Long Wave equation [4].

Conjecture 1 (Main Conjecture). *The following formula holds true*

$$\sum_{g=0}^{\infty} \epsilon^{2g-2} \mathcal{F}_g(x, \mathbf{s}) + \epsilon^{-2} \Bigg(-\frac{1}{2} \sum_{k_1, k_2 \geq 1} \frac{k_1 \, k_2}{k_1 + k_2} \, \bar{s}_{k_1} \, \bar{s}_{k_2} + \sum_{k \geq 1} \frac{k}{1+k} \, \bar{s}_k - x \sum_{k \geq 1} \bar{s}_k - \frac{1}{4} + x \Bigg) = \cosh\left(\frac{\epsilon \, \partial_x}{2} \right) \left[\sum_{g=0}^{\infty} \epsilon^{2g-2} \, 2^g \, \mathcal{H}_g \left(\mathbf{t}(x, \mathbf{s}) \right) \right] + \zeta'(-1). \tag{13}$$

where

$$t_i(x, \mathbf{s}) := \sum_{k \geq 1} k^{i+1} \bar{s}_k - 1 + \delta_{i,1} + x \cdot \delta_{i,0}, \qquad i \geq 0. \tag{14}$$

Remark 2. Both sides of the conjectural identity (13) can be considered as living in the formal power series ring

$$\epsilon^{-2} \mathbb{C} \left[\epsilon^2 \right] [[x-1, s_1, s_2, \dots]].$$

Expanding both sides of (13) near $\mathbf{s} = \mathbf{0}$, $x = 1$ one obtains a series of interesting identities relating counting numbers of ribbon graphs and Hodge integrals, as simple consequences of the Main Conjecture. The simplest of them valid for any $g \geq 2$ reads

$$2^g \sum_{\mu \in \mathbb{Y}} \frac{(-1)^{\ell(\mu)}}{m(\mu)!} \int_{\overline{\mathcal{M}}_{g,\ell(\mu)}} \Lambda_g(-1) \, \Lambda_g(-1) \, \Lambda_g\left(\frac{1}{2}\right) \prod_{i=1}^{\ell(\mu)} \psi_i^{\mu_i+1} = \frac{1}{2g(2g-1)(2g-2)} \sum_{g'=0}^{g} (2g'-1) \binom{2g}{2g'} \frac{E_{2g-2g'} \, B_{2g'}}{2^{2g-2g'}}. \tag{15}$$

Here, $\mathbb{Y}$ denotes the set of partitions; for $\mu \in \mathbb{Y}$, $\ell(\mu)$ denotes the length of μ, $m_i(\mu)$ denotes the multiplicity of i in μ, $m(\mu)! := \prod_{i=1}^{\infty} m_i(\mu)!$. And E_k are the Euler numbers, defined via

$$\frac{1}{\cosh z} = \sum_{k=0}^{\infty} \frac{E_k}{k!} z^k.$$

To the best of our knowledge such identities even the simplest one (15) never appeared in the literature. We would like to mention that another interesting consequence of the Main Conjecture is recently obtained in [13].

1.4. Computational aspects of the Main Conjecture: how do we verify it?

We will check validity of the Main Conjecture for small genera. Begin with $g = 0$. Let us start with $\mathcal{H}_0(\mathbf{t})$. Instead of the explicit expansion (5) we use the following well known representation

$$\mathcal{H}_0 = \frac{v^3}{6} - \sum_{i\geq 0} t_i \frac{v^{i+2}}{i!(i+2)} + \frac{1}{2}\sum_{i,\,j\geq 0} t_i t_j \frac{v^{i+j+1}}{(i+j+1)\,i!\,j!} \tag{16}$$

where $v = v(\mathbf{t}) = t_0 + \cdots$ is the unique series solution to the equation

$$v = \sum_{i\geq 0} t_i \frac{v^i}{i!}. \tag{17}$$

Here we recall that

$$v = \frac{\partial^2 \mathcal{H}_0(\mathbf{t})}{\partial t_0^2} = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{i_1+\cdots+i_k=k-1} \frac{t_{i_1}}{i_1!} \cdots \frac{t_{i_k}}{i_k!} \tag{18}$$

is a particular solution to the Riemann–Hopf hierarchy

$$\frac{\partial v}{\partial t_k} = \frac{v^k}{k!}\frac{\partial v}{\partial t_0}, \qquad k = 0,\, 1,\, 2, \ldots.$$

For the genus zero GUE free energy $\mathcal{F}_0 = \mathcal{F}_0(x, \mathbf{s})$ one has a similar representation. Like above, introduce

$$u(x,\mathbf{s}) = \frac{\partial^2 \mathcal{F}_0(x,\mathbf{s})}{\partial x^2} \tag{19}$$

and put

$$w(x,\mathbf{s}) = e^{u(x,\mathbf{s})}. \tag{20}$$

Proposition 3. *The function $w = w(x, \mathbf{s})$ is the unique series solution to the equation*

$$w = x + \sum_{k\geq 1} k\, \bar{s}_k\, w^k, \qquad \bar{s}_k := \binom{2k}{k} s_k, \quad w(x,\mathbf{s}) = x + \cdots. \tag{21}$$

The genus zero GUE free energy $\mathcal{F}_0$ *with even couplings has the following expression*

$$\mathcal{F}_0 = \frac{w^2}{4} - x\,w + \sum_{k\geq 1} \bar{s}_k \left(x\,w^k - \frac{k}{k+1} w^{k+1} \right) + \frac{1}{2} \sum_{k_1,k_2\geq 1} \frac{k_1 k_2}{k_1+k_2} \bar{s}_{k_1} \bar{s}_{k_2} w^{k_1+k_2} + \frac{x^2}{2} \log w. \tag{22}$$

The proof of this proposition will be given in Sect. 2.

Clearly w also satisfies the Riemann–Hopf hierarchy in a different normalization

$$\frac{\partial w}{\partial \bar{s}_k} = k\,w^k \frac{\partial w}{\partial x}, \qquad k \geq 1.$$

The solution can be written explicitly in the form essentially equivalent to (18)

$$w = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{i_1+\cdots+i_n=n-1} \mathrm{wt}(i_1)\cdots \mathrm{wt}(i_n)\, \bar{s}_{i_1}\cdots \bar{s}_{i_n}$$

where we put $\bar{s}_0 = x$ and denote

$$\mathrm{wt}(i) = \begin{cases} 1, & i=0 \\ i, & \text{otherwise.} \end{cases}$$

It is now straightforward to verify that the substitution (14) yields

$$e^{v(\mathbf{t}(x,\mathbf{s}))} = w(x,\mathbf{s}), \qquad \text{i.e. } v\left(\mathbf{t}(x,\mathbf{s})\right) = u(x,\mathbf{s}) \tag{23}$$

and

$$\mathcal{H}_0\left(\mathbf{t}(x,\mathbf{s})\right) = \mathcal{F}_0\left(x,\mathbf{s}\right) - \frac{1}{2}\sum_{k_1,k_2\geq 1} \frac{k_1\,k_2}{k_1+k_2}\,\bar{s}_{k_1}\,\bar{s}_{k_2} + \sum_{k\geq 1} \frac{k}{1+k}\,\bar{s}_k - x\sum_{k\geq 1}\bar{s}_k - \frac{1}{4} + x. \tag{24}$$

See in Sect. 3 for the details of this computation.

In order to proceed to higher genera we will use the method that goes back to the paper [6] by R. Dijkgraaf and E. Witten. The idea of this method is to express the positive genus free energy terms via the genus zero. Let us first explain this method for the Hodge free energy.

Theorem 4 ([9]). *There exist functions $H_g(v, v_1, v_2, \ldots, v_{3g-2})$, $g \geq 1$ of independent variables v, v_1, v_2, $\ldots$ such that*

$$\mathcal{H}_g(\mathbf{t}) = H_g\left(v(\mathbf{t}), \frac{\partial v(\mathbf{t})}{\partial t_0}, \ldots, \frac{\partial^{3g-2} v(\mathbf{t})}{\partial t_0^{3g-2}}\right), \quad g \geq 1. \tag{25}$$

Here $v(\mathbf{t})$ is given by eq. (18). Moreover, for any $g \geq 2$ the function H_g is a polynomial in the variables v_2, $\ldots$, v_{3g-2} with coefficients in $\mathbb{Q}\left[v_1, v_1^{-1}\right]$ (independent of v).

Explicitly,

$$H_1(v, v_1) = -\frac{1}{16}v + \frac{1}{24}\log v_1 \tag{26}$$

$$\begin{aligned} H_2(v_1, v_2, v_3, v_4) = &\frac{7v_2}{2560} - \frac{v_1^2}{11520} + \frac{v_4}{1152v_1^2} - \frac{v_3}{320v_1} \\ &+ \frac{v_2^3}{360v_1^4} + \frac{11v_2^2}{3840v_1^2} - \frac{7v_3v_2}{1920v_1^3}, \end{aligned} \tag{27}$$

etc. The algorithm for computing the functions H_g can be found in [9]. They were used in the construction of the associated integrable hierarchy via the quasi-triviality transformation approach [14].

Let us now proceed to the higher genus terms for the random matrix free energy (recall that only even couplings are allowed).

Theorem 5. *There exist functions $F_g(v, v_1, \ldots, v_{3g-2})$, $g \geq 1$ of independent variables v, v_1, v_2, $\ldots$ such that*

$$\mathcal{F}_g(x, \mathbf{s}) = F_g\left(u(x, \mathbf{s}), \frac{\partial u(x, \mathbf{s})}{\partial x}, \ldots, \frac{\partial^{3g-2} u(x, \mathbf{s})}{\partial x^{3g-2}}\right), \quad g \geq 1. \tag{28}$$

Here

$$u(x, \mathbf{s}) = \frac{\partial^2 \mathcal{F}_0(x, \mathbf{s})}{\partial x^2} = \log w(x, \mathbf{s}).$$

Recall that the function $w(x, \mathbf{s})$ is determined from eq. (21)*.*

Explicitly

$$F_1(v, v_1) = \frac{1}{12}\log v_1 + \text{const} \tag{29}$$

with const=$\zeta'(-1)$,

$$F_2(v_1, v_2, v_3, v_4) = -\frac{v_2}{480} - \frac{v_1^2}{2880} + \frac{v_4}{288\, v_1^2} - \frac{v_3}{480\, v_1} + \frac{v_2^3}{90\, v_1^4} + \frac{v_2^2}{960\, v_1^2} - \frac{7 v_3 v_2}{480\, v_1^3} \tag{30}$$

etc. For any $g \geq 2$ the function F_g is a polynomial in the variables $v_2, \ldots, v_{3g-2}$ with coefficients in $\mathbb{Q}\left[v_1, v_1^{-1}\right]$.

Using the fact that $\partial_{t_0} = \partial_x$ (see Section 3.2 below) along with the standard expansion

$$\cosh\left(\frac{\epsilon\, \partial_x}{2}\right) = 1 + \sum_{n\geq 1} \frac{1}{(2n)!} \left(\frac{\epsilon}{2}\right)^{2n} \partial_x^{2n}$$

we recast the Main Conjecture for $g \geq 1$ into a sequence of the following relationships between the functions F_g and H_g

$$F_1 = 2H_1 + \frac{v}{8} + \text{const} \tag{31}$$

and, for $g \geq 2$

$$F_g(v_1, \ldots, v_{3g-2}) = \frac{v_{2g-2}}{2^{2g}\,(2g)!} + \frac{D_0^{2g-2} H_1(v; v_1)}{2^{2g-3}\,(2g-2)!} + \sum_{m=2}^{g} \frac{2^{3m-2g}}{(2g-2m)!} D_0^{2(g-m)} H_m(v_1, \ldots, v_{3m-2}) \tag{32}$$

where the operator D_0 is defined by

$$D_0 = v_1 \frac{\partial}{\partial v} + \sum_{k\geq 1} v_{k+1} \frac{\partial}{\partial v_k}.$$

For example,

$$F_2(v_1, v_2, v_3, v_4) = 4H_2(v_1, v_2, v_3, v_4) + \frac{1}{4} D_0^2 H_1 + \frac{1}{384} v_2. \tag{33}$$

Equations (31), (33) can be easily verified (see below). In order to verify validity of eqs. (32) for any $g \geq 2$ we write a conjectural explicit expression for the functions $F_g(v_1, \ldots, v_{3g-2})$ responsible for the genus g random matrix free energies. This will be done in the next subsection.

1.5. An explicit expression for F_g

We first recall some notations. $\mathbb{Y}$ will denote the set of all partitions. For any partition $\lambda \in \mathbb{Y}$ denote by $\ell(\lambda)$ the *length* of λ, by $\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)}$ the non-zero components, $|\lambda| = \lambda_1 + \cdots + \lambda_{\ell(\lambda)}$ the *weight*, and by $m_i(\lambda)$ the *multiplicity* of i in λ. Put $m(\lambda)! := \prod_{i\geq 1} m_i(\lambda)!$. The set of all partitions of weight k will be denoted by $\mathbb{Y}_k$. For an arbitrary sequence of variables $v_1, v_2, \dots,$ denote $v_\lambda = v_{\lambda_1} \cdots v_{\lambda_{\ell(\lambda)}}$.

Conjecture 6. *For any $g \geq 2$, the genus g GUE free energy F_g has the following expression*

$$
\begin{aligned}
(34) \qquad & F_g(v_1, \dots, v_{3g-2}) \\
&= \frac{v_{2g-2}}{2^{2g}\,(2g)!} + \frac{1}{2^{2g-3}\,(2g-2)!} D_0^{2g-2}\left(-\frac{1}{16} v + \frac{1}{24}\log v_1\right) \\
&+ \sum_{m=2}^{g} \frac{2^{3m-2g}}{(2g-2m)!} \sum_{k=0}^{3m-3} \sum_{\substack{k_1+k_2+k_3=k \\ 0\leq k_1,k_2,k_3\leq m}} \frac{(-1)^{k_2+k_3}}{2^{k_1}} \\
&\times \sum_{\rho,\mu\in\mathbb{Y}_{3m-3-k}} \frac{\langle \lambda_{k_1}\lambda_{k_2}\lambda_{k_3}\tau_{\rho+1}\rangle_g}{m(\rho)!} Q^{\rho\mu}\, D_0^{2g-2m}\left(\frac{v_{\mu+1}}{v_1^{\ell(\mu)+m-1-k}}\right)
\end{aligned}
$$

where for a partition $\mu = (\mu_1, \dots, \mu_\ell)$, $\mu + 1$ denotes the partition $(\mu_1 + 1, \dots, \mu_\ell + 1)$, $Q^{\rho\mu}$ is the so-called Q-matrix defined by

$$
Q^{\rho\mu} = (-1)^{\ell(\rho)} \sum_{\substack{\mu^1\in\mathbb{Y}_{\lambda_1},\dots,\mu^{\ell(\rho)}\in\mathbb{Y}_{\lambda_{\ell(\rho)}} \\ \cup_{q=1}^{\ell(\rho)}\mu^q=\mu}} \prod_{q=1}^{\ell(\rho)} \frac{(\rho_q + \ell(\mu^q))!\,(-1)^{\ell(\mu^q)}}{m(\mu^q)!\prod_{j=1}^{\infty}(j+1)!^{m_j(\mu^q)}}.
$$

In this formula we have used the notation

$$
\langle \lambda_{k_1}\lambda_{k_2}\lambda_{k_3}\tau_\nu\rangle_g := \int_{\overline{\mathcal{M}}_{g,\ell}} \lambda_{k_1}\lambda_{k_2}\lambda_{k_3}\psi_1^{\nu_1}\cdots\psi_\ell^{\nu_\ell}, \qquad \forall\, \nu = (\nu_1, \dots, \nu_\ell) \in \mathbb{Y}.
$$

Details about Q-matrix can be found in [12]. Conj. 6 indicates that the the special cubic Hodge integrals (2) naturally appear in the expressions for the higher genus terms of GUE free energy.

Organization of the paper. In Sect. 2 we review the approach of [8, 14] to the GUE free energy, and prove Prop. 3 and Thm. 5. In Sect. 3 we verify Conj. 1 and Conj. 6 up to the genus 2 approximation, and give explicit formulae of $\mathcal{F}_g$ for $g = 3, 4, 5$.

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2. GUE free energy with even valencies

2.1. Calculating the GUE free energy from Frobenius manifold of $\mathbb{P}^1$ topological σ-model

It is known that the GUE partition function Z_N (with even and odd couplings) is the tau-function of a particular solution to the Toda lattice hierarchy (see e.g. Proposition A.2.3 in [11], where one can also find a detailed proof). Using this fact, one of the authors in [8] developed an efficient algorithm of calculating of GUE free energy, which is an application of the general approach of [7, 14] for the particular example of the two-dimensional Frobenius manifold with potential

$$F = \frac{1}{2} u\, v^2 + e^u.$$

(Warning: only in this section, the notation v is different from that of the Introduction.)

More precisely, let $\mathcal{F}^{\mathbb{P}^1}$ denote the following generating series of Gromov–Witten invariants of $\mathbb{P}^1$

$$\begin{aligned}
\mathcal{F}^{\mathbb{P}^1} &:= \sum_{g\geq 0} \epsilon^{2g-2} \mathcal{F}_g^{\mathbb{P}^1},\\
\mathcal{F}_g^{\mathbb{P}^1} &:= \sum_{k\geq 0} \frac{1}{k!} \sum_{\alpha_1,\dots,\alpha_k=1}^{2} \sum_{p_1,\dots,p_k\geq 0} t^{\alpha_1}_{p_1}\cdots t^{\alpha_k}_{p_k}\\
&\quad \times \sum_{\beta\in H_2(\mathbb{P}^1;\mathbb{Z})} \int_{\left[\overline{\mathcal{M}}_{g,k}(\mathbb{P}^1,\beta)\right]^{\rm virt}} \mathrm{ev}_1^*(\phi_{\alpha_1})\cdots \mathrm{ev}_k^*(\phi_{\alpha_k})\cdot \psi_1^{p_1}\cdots\psi_k^{p_k}.
\end{aligned}$$

Here, $\phi_1 := 1 \in H^0(\mathbb{P}^1;\mathbb{C})$, $\phi_2 \in H^2(\mathbb{P}^1;\mathbb{C})$ is the Poincaré dual of a point normalized by

$$\int_{\mathbb{P}^1} \phi_2 = 1,$$

$\overline{\mathcal{M}}_{g,k}(\mathbb{P}^1,\beta)$ denotes the moduli space of stable maps of curves of genus g, degree β with k marked points to the target $\mathbb{P}^1$, ev_i denote the i-th evaluation map and ψ_i the first Chern class of the i-th tautological line bundle on $\overline{\mathcal{M}}_{g,k}(\mathbb{P}^1, \beta)$. It has been observed in [8] that

$$\mathcal{F} = \mathcal{F}^{\mathbb{P}^1}\Big|_{t_0^1=x,\, t_1^1=1,\, t^1_{p\geq 2}=0,\, t^2_{2p-1}=(2p)!s_p-\delta_{p,1},\, t^2_{2p}=0}$$

where $\mathcal{F}$ is the GUE free energy with even valencies (see (10)). Hence one can apply the general approach in [14] for computing $\mathcal{F}$, for which we will now give a brief reminder referring the reader to [8, 14, 15] for more details.

Introduce two analytic functions $\theta_1(u, v; z), \theta_2(u, v; z)$ as follows

$$\theta_1(u,v;z) = -2\, e^{zv} \sum_{m=0}^{\infty} \left(-\frac{1}{2}u + c_m\right) e^{mu} \frac{z^{2m}}{m!^2} =: \sum_{p\geq 0} \theta_{1,p}(u,v)\, z^p \tag{35}$$

$$\theta_2(u,v;z) = z^{-1} \left(\sum_{m\geq 0} e^{mu+zv} \frac{z^{2m}}{(m!)^2} - 1 \right) =: \sum_{p\geq 0} \theta_{2,p}(u,v)\, z^p. \tag{36}$$

Here $c_m = \sum_{k=1}^{m} \frac{1}{k}$ denotes the m-th harmonic number.

Note that, as in the Introduction, we will only consider the GUE partition function with *even* couplings. The corresponding (genus zero) Euler–Lagrange equation [7, 14, 15] (see the Proposition 6.1 in [7] or see the eq. (3.6.78) in [14]) reads

$$x - w + \sum_{k\geq 1} (2k)!\, s_k \sum_{m=1}^{k} m\, w^m \frac{v^{2k-2m}}{(2k-2m)!\, m!^2} = 0 \tag{37}$$

$$-v + \sum_{k\geq 1} (2k)!\, s_k \sum_{m=0}^{k-1} w^m \frac{v^{2k-1-2m}}{(2k-1-2m)!\, m!^2} = 0 \tag{38}$$

where $w = e^u$ (as in the Introduction). Note that we are only interested in the unique series solution $(v(x, \mathbf{s}), w(x, \mathbf{s}))$ of (37), (38) such that $v(x, \mathbf{0}) = 0$, $w(x, \mathbf{0}) = x$. It is then easy to see from eq. (38) that

$$v = v(x, \mathbf{s}) \equiv 0.$$

And eq. (37) becomes

$$x - w + \sum_{m\geq 1} s_m\, m\, w^m \frac{(2m)!}{m!^2} = 0. \tag{39}$$

Define a family of analytic functions $\Omega_{\alpha,p;\beta,q}(u,v)$ by the following generating formula

$$\sum_{p,q\geq 0}\Omega_{\alpha,p;\beta,q}z^p y^q = \frac{1}{z+y}\left[\frac{\partial\theta_\alpha(z)}{\partial v}\frac{\partial\theta_\beta(y)}{\partial u}+\frac{\partial\theta_\alpha(z)}{\partial u}\frac{\partial\theta_\beta(y)}{\partial v}-\delta_{\alpha+\beta,3}\right], \quad \alpha,\beta=1,2. \tag{40}$$

The genus zero GUE free energy $\mathcal{F}_0(x,\mathbf{s})$ then has the following expression

$$\begin{aligned}\mathcal{F}_0 = {} & \frac{1}{2}\sum_{p,q\geq 2}(2p)!(2q)!\, s_p s_q\, \Omega_{2,2p-1;2,2q-1} \\ & + x\sum_{q\geq 1}(2q)!\, s_q\, \Omega_{1,0;2,2q-1} - x\,\Omega_{1,0;2,1} + \frac{1}{2}(1-2s_1)^2\,\Omega_{2,1;2,1} \\ & + \sum_{q\geq 2}(2s_1-1)\,(2q)!\, s_q\, \Omega_{2,1;2,2q-1} + \frac{1}{2}x^2\,\Omega_{1,0;1,0}.\end{aligned} \tag{41}$$

The higher genus terms in the $1/N$ expansion of the GUE free energy can be determined recursively from the *loop equation* [8, 14] for a sequence of functions

$$F_g = F_g(u,v,u_1,v_1,\dots,v_{3g-2},u_{3g-2}),\quad g\geq 1.$$

This equation has the following form

$$\begin{aligned} & \sum_{r\geq 0}\left[\frac{\partial\Delta\mathcal{F}}{\partial v_r}\left(\frac{v-\lambda}{D}\right)_r - 2\frac{\partial\Delta\mathcal{F}}{\partial u_r}\left(\frac{1}{D}\right)_r\right] \\ & + \sum_{r\geq 1}\sum_{k=1}^{r}\binom{r}{k}\left(\frac{1}{\sqrt{D}}\right)_{k-1}\left[\frac{\partial\Delta\mathcal{F}}{\partial v_r}\left(\frac{v-\lambda}{\sqrt{D}}\right)_{r-k+1} - 2\frac{\partial\Delta\mathcal{F}}{\partial u_r}\left(\frac{1}{\sqrt{D}}\right)_{r-k+1}\right] \\ = {} & D^{-3}e^u\left(4\,e^u+(v-\lambda)^2\right) \\ & -\epsilon^2\sum_{k,l}\left[\frac{1}{4}S(\Delta\mathcal{F},v_k,v_l)\left(\frac{v-\lambda}{\sqrt{D}}\right)_{k+1}\left(\frac{v-\lambda}{\sqrt{D}}\right)_{l+1}\right. \\ & \qquad - S(\Delta\mathcal{F},v_k,u_l)\left(\frac{v-\lambda}{\sqrt{D}}\right)_{k+1}\left(\frac{1}{\sqrt{D}}\right)_{l+1} \\ & \qquad \left. + S(\Delta\mathcal{F},u_k,u_l)\left(\frac{1}{\sqrt{D}}\right)_{k+1}\left(\frac{1}{\sqrt{D}}\right)_{l+1}\right] \\ & -\frac{\epsilon^2}{2}\sum_k\left[\frac{\partial\Delta\mathcal{F}}{\partial v_k}\frac{4\,e^u(v-\lambda)\,u_1 - T\,v_1}{D^3}+\frac{\partial\Delta\mathcal{F}}{\partial u_k}\frac{4\,(v-\lambda)\,v_1 - T\,u_1}{D^3}\right](e^u)_{k+1}\end{aligned} \tag{42}$$

where $\triangle\mathcal{F}=\sum_{g\geq 1}\epsilon^{2g}F_g$, $D=(v-\lambda)^2-4e^u$, $T=(v-\lambda)^2+4e^u$, $S(f,a,b):=\frac{\partial^2 f}{\partial a\partial b}+\frac{\partial f}{\partial a}\frac{\partial f}{\partial b}$, and f_r stands for $\partial_x^r(f)$. Solution $\Delta\mathcal{F}$ of (42) exists and is unique up to an additive constant. F_g is a polynomial in $u_2, v_2, \ldots, u_{3g-2}, v_{3g-2}$. For $g\geq 2$, F_g is a rational function of u_1, v_1. Then [14] the genus g term in the expansion (9), in the particular case of even couplings only, reads

$$\mathcal{F}_g(x,\mathbf{s})=F_g\left(u(x,\mathbf{s}), v=0, \frac{\partial u(x,\mathbf{s})}{\partial x}, v_1=0,\ldots,\frac{\partial^{3g-2}u(x,\mathbf{s})}{\partial x^{3g-2}}, v_{3g-2}=0\right),\quad g\geq 1.$$

It should be noted that the reason that one can take $v_1=0$ is due to a careful analysis of the rational dependence of v_1, u_1 in F_g, where the Corollary 3.10.22 from [14] would be helpful.

This procedure will be used in the next subsection.

2.2. Proof of Prop. 3, Thm. 5

Proof of Prop. 3. Noting that

$$\theta_2(u,0,z)=z^{-1}\Big(\sum_{m\geq 0} w^m\frac{z^{2m}}{(m!)^2}-1\Big),$$

$$\partial_v\theta_2(u,0,z)=\sum_{m\geq 0} w^m\frac{z^{2m}}{(m!)^2},\qquad \partial_u\theta_2(u,0,z)=\sum_{m\geq 0} m w^m\frac{z^{2m-1}}{(m!)^2}$$

and using (40) we have

$$\sum_{p,q\geq 0}\Omega_{2,p;2,q}z^p y^q=\frac{\sum_{m\geq 0}w^m\frac{z^{2m}}{(m!)^2}\sum_{m\geq 0}mw^m\frac{y^{2m-1}}{(m!)^2}}{z+y}+\frac{\sum_{m\geq 0}mw^m\frac{z^{2m-1}}{(m!)^2}\sum_{m\geq 0}w^m\frac{y^{2m}}{(m!)^2}}{z+y}. \tag{43}$$

It follows that if $p+q$ is odd then $\Omega_{2,p;2,q}$ vanishes; otherwise, we have

$$\Omega_{2,p;2,q}=\frac{w^{\frac{p+q}{2}+1}}{\left(1+\frac{p+q}{2}\right)\left[\left(\frac{p}{2}\right)!\right]^2\left[\left(\frac{q}{2}\right)!\right]^2},\qquad p,q\text{ are both even;} \tag{44}$$

$$\Omega_{2,p;2,q}=\frac{\frac{p+1}{2}\frac{q+1}{2}w^{\frac{p+q}{2}+1}}{\left(1+\frac{p+q}{2}\right)\left[\left(\frac{p+1}{2}\right)!\right]^2\left[\left(\frac{q+1}{2}\right)!\right]^2},\qquad p,q\text{ are both odd.} \tag{45}$$

Indeed, in the case that p, q are both even, by comparing the coefficients of $z^p y^q$ of both sides of (43) we obtain that

$$\begin{aligned}
\Omega_{2,p;2,q} &= w^{\frac{p+q}{2}+1} \sum_{i=0}^{\frac{p}{2}} \frac{\frac{q}{2} - \frac{p}{2} + 2i + 1}{\left[(\frac{p}{2} - i)!\right]^2 \left[(\frac{q}{2} + i + 1)!\right]^2} \\
&= \frac{w^{\frac{p+q}{2}+1}}{\frac{q}{2} + \frac{p}{2} + 1} \sum_{i=0}^{\frac{p}{2}} \frac{(\frac{q}{2} + i + 1)^2 - (\frac{p}{2} - i)^2}{\left[(\frac{p}{2} - i)!\right]^2 \left[(\frac{q}{2} + i + 1)!\right]^2} \\
&= \frac{w^{\frac{p+q}{2}+1}}{\frac{q}{2} + \frac{p}{2} + 1} \sum_{i=0}^{\frac{p}{2}} \left[\frac{1}{\left[(\frac{p}{2} - i)!\right]^2 \left[(\frac{q}{2} + i)!\right]^2} \right. \\
&\qquad \left. - \frac{1}{\left[(\frac{p}{2} - i - 1)!\right]^2 \left[(\frac{q}{2} + i + 1)!\right]^2} \right] \\
&= \frac{w^{\frac{p+q}{2}+1}}{\left(1 + \frac{p+q}{2}\right) \left[(\frac{p}{2})!\right]^2 \left[(\frac{q}{2})!\right]^2}.
\end{aligned}$$

Here $1/(-1)! := 0$. In a similar way, for the case that p, q are both odd, one derives (45).

Substituting the expressions (44)–(45) in (41) we obtain

$$\begin{aligned}
\mathcal{F}_0 &= \frac{1}{2} x^2 u + \frac{1}{2} \sum_{k_1, k_2 \geq 0} (2k_1 + 2)!(2k_2 + 2)! s_{k_1+1} s_{k_2+1} \\
&\qquad \times \frac{(k_1 + 1)(k_2 + 1)\, w^{k_1+k_2+2}}{(k_1 + k_2 + 2) \left[(k_1 + 1)!\right]^2 \left[(k_2 + 1)!\right]^2} \\
&\quad + x \sum_{k \geq 0} (2k + 2)! s_{k+1} \frac{w^{k+1}}{(k+1)!^2} - x\, w + (1 - 4s_1) \frac{w^2}{4} \\
&\quad - \sum_{k \geq 1} (2k + 2)! s_{k+1} \frac{(k+1)\, w^{k+2}}{(k+2) \left[(k+1)!\right]^2}.
\end{aligned}$$

Equation (21) is already proved in (39). The proposition is proved. □

Proof of Theorem 5. For $g = 1, 2$, taking $v = v_1 = v_2 = \cdots = 0$ in the general expressions of $F_g(u, v, u_1, v_1, \ldots, u_{3g-2}, v_{3g-2})$ [14, 15] one obtains (29) and (30). For any $g \geq 1$, the existence of $F_g(u, u_1, \ldots, u_{3g-2})$ such that

$$\mathcal{F}_g(x, \mathbf{s}) = F_g\left(u(x, \mathbf{s}), \frac{\partial u(x, \mathbf{s})}{\partial x}, \ldots, \frac{\partial^{3g-2} u(x, \mathbf{s})}{\partial x^{3g-2}}\right)$$

is a direct result of [14, 15] when taking $v = v_1 = v_2 = \cdots = 0$ in $F_g(u, v, u_1, v_1, \ldots, u_{3g-2}, v_{3g-2})$. □

3. Verification of the Main Conjecture for low genera

3.1. Genus 0

Recall that the genus zero cubic Hodge free energy can be expressed as

$$\mathcal{H}_0(\mathbf{t}) = \frac{1}{2} \sum_{i,j\geq 0} \tilde{t}_i \, \tilde{t}_j \, \Omega_{i;j}(v(\mathbf{t})).$$

where $\tilde{t}_i = t_i - \delta_{i,1}$, $\mathbf{t} = (t_0, t_1, t_2, \ldots)$, $\Omega_{i;j}$ are polynomials in v given by

$$\Omega_{i;j}(v) = \frac{v^{i+j+1}}{(i+j+1)\, i!\, j!},$$

and $v(\mathbf{t})$ is the unique series solution to the following Euler–Lagrange equation of the one-dimensional Frobenius manifold

$$v = \sum_{i\geq 0} t_i \frac{v^i}{i!}.$$

(Warning: the above v is the flat coordinate of the one-dimensional Frobenius manifold; avoid confusing with v in Section 2 where (u, v) are flat coordinates of the two-dimensional Frobenius manifold of $\mathbb{P}^1$ topological σ-model.)

Let us consider the following substitution of time variables

$$t_i = \sum_{k\geq 1} k^{i+1} \bar{s}_k - 1 + \delta_{i,1} + x \cdot \delta_{i,0}, \qquad i \geq 0.$$

Note that with this substitution the cubic Hodge free energies will be considered to be expanded at $x = 1$. We have $\tilde{t}_i = \sum_{k\geq 1} k^{i+1} \bar{s}_k - 1 + x \cdot \delta_{i,0}$, and so

$$
\begin{aligned}
\mathcal{H}_0 &= \frac{1}{2} \sum_{i,j\geq 0} \tilde{t}_i\, \tilde{t}_j\, \Omega_{i;j}(v(\mathbf{t})) \\
&= \frac{1}{2} \sum_{i,j\geq 0} \left(\sum_{k_1\geq 1} k_1^{i+1} \bar{s}_{k_1} - 1 + x\cdot \delta_{i,0} \right) \left(\sum_{k_2\geq 1} k_2^{j+1} \bar{s}_{k_2} - 1 + x\cdot \delta_{j,0} \right) \\
&\quad \times \frac{v^{i+j+1}}{(i+j+1)\, i!\, j!} \\
&= \frac{1}{2} \sum_{i,j\geq 0} \sum_{k_1,k_2\geq 1} k_1^{i+1}\, k_2^{j+1}\, \bar{s}_{k_1}\, \bar{s}_{k_2}\, \frac{v^{i+j+1}}{(i+j+1)\, i!\, j!} \\
&\quad - \sum_{i,j\geq 0} \sum_{k_1\geq 1} k_1^{i+1}\, \bar{s}_{k_1}\, \frac{v^{i+j+1}}{(i+j+1)\, i!\, j!} \\
&\quad + x \sum_{i,j\geq 0} \sum_{k_1\geq 1} k_1^{i+1}\, \bar{s}_{k_1}\, \delta_{j,0}\, \frac{v^{i+j+1}}{(i+j+1)\, i!\, j!} + \frac{1}{2} \sum_{i,j\geq 0} \frac{v^{i+j+1}}{(i+j+1)\, i!\, j!} \\
&\quad - x \sum_{i,j\geq 0} \delta_{j,0}\, \frac{v^{i+j+1}}{(i+j+1)\, i!\, j!} + \frac{x^2}{2} \sum_{i,j\geq 0} \delta_{i,0}\, \delta_{j,0}\, \frac{v^{i+j+1}}{(i+j+1)\, i!\, j!}.
\end{aligned}
$$

We simplify it term by term:

$$
\begin{aligned}
\frac{x^2}{2} \sum_{i,j\geq 0} \delta_{i,0}\, \delta_{j,0}\, \frac{v^{i+j+1}}{(i+j+1)\, i!\, j!} &= \frac{x^2}{2} v, \\
x \sum_{i,j\geq 0} \delta_{j,0}\, \frac{v^{i+j+1}}{(i+j+1)\, i!\, j!} &= x\, (e^v - 1), \\
\frac{1}{2} \sum_{i,j\geq 0} \frac{v^{i+j+1}}{(i+j+1)\, i!\, j!} &= \frac{1}{2} \sum_{\ell\geq 0} \sum_{i=0}^{\ell} \frac{v^{\ell+1} \ell!}{(\ell+1)!\, i!\, (\ell-i)!} \\
&= \frac{1}{2} \sum_{\ell\geq 0} \frac{v^{\ell+1}\, 2^\ell}{(\ell+1)!} = \frac{1}{4} (e^{2v} - 1), \\
x \sum_{i,j\geq 0} \sum_{k_1\geq 1} k_1^{i+1}\, \bar{s}_{k_1}\, \delta_{j,0}\, \frac{v^{i+j+1}}{(i+j+1)\, i!\, j!} &= x \sum_{i\geq 0} \sum_{k_1\geq 1} k_1^{i+1}\, \bar{s}_{k_1}\, \frac{v^{i+1}}{(i+1)!} \\
&= x \sum_{k\geq 1} \bar{s}_k\, (e^{kv} - 1),
\end{aligned}
$$

$$\sum_{i,j\geq 0}\sum_{k_1\geq 1} k_1^{i+1}\,\bar{s}_{k_1}\,\frac{v^{i+j+1}}{(i+j+1)\,i!\,j!}$$
$$=\sum_{k\geq 1} k\,\bar{s}_k\sum_{\ell\geq 0}(1+k)^\ell\,\frac{v^{\ell+1}}{(\ell+1)!}=\sum_{k\geq 1}\frac{k}{1+k}\,\bar{s}_k\,\left(e^{(1+k)v}-1\right),$$

$$\frac{1}{2}\sum_{i,j\geq 0}\sum_{k_1,k_2\geq 1} k_1^{i+1}\,k_2^{j+1}\,\bar{s}_{k_1}\,\bar{s}_{k_2}\,\frac{v^{i+j+1}}{(i+j+1)\,i!\,j!}$$
$$=\frac{1}{2}\sum_{k_1,k_2\geq 1} k_1\,k_2\,\bar{s}_{k_1}\,\bar{s}_{k_2}\sum_{\ell\geq 0}\frac{v^{\ell+1}}{(\ell+1)!}(k_1+k_2)^\ell$$
$$=\frac{1}{2}\sum_{k_1,k_2\geq 1}\frac{k_1\,k_2}{k_1+k_2}\,\bar{s}_{k_1}\,\bar{s}_{k_2}\,\left(e^{(k_1+k_2)v}-1\right).$$

Let $w=e^v$. We have

$$\mathcal{H}_0=\frac{1}{2}\sum_{k_1,k_2\geq 1}\frac{k_1\,k_2}{k_1+k_2}\,\bar{s}_{k_1}\,\bar{s}_{k_2}\,\left(w^{k_1+k_2}-1\right)-\sum_{k\geq 1}\frac{k}{1+k}\,\bar{s}_k\,\left(w^{1+k}-1\right)$$
$$+\,x\sum_{k\geq 1}\bar{s}_k\,(w^k-1)+\frac{1}{4}(w^2-1)-x\,(w-1)+\frac{x^2}{2}\log w.$$

On the other hand, recall from Prop. 3 that the genus zero GUE free energy with even couplings has the form

$$\mathcal{F}_0=\frac{w^2}{4}-x\,w+\sum_{k\geq 1}\bar{s}_k\left(x\,w^k-\frac{k}{k+1}w^{k+1}\right)$$
$$+\,\frac{1}{2}\sum_{k_1,k_2\geq 1}\frac{k_1k_2}{k_1+k_2}\bar{s}_{k_1}\bar{s}_{k_2}w^{k_1+k_2}+\frac{x^2}{2}\log w.$$

Here w is the power series solution to

$$w=x+\sum_{k\geq 1} k\,\bar{s}_k\,w^k.$$

Recall that $w=e^u$; so

$$e^u=x+\sum_{k\geq 1} k\,\bar{s}_k\,e^{ku}.$$

Namely,

$$1+\sum_{j\geq 1}\frac{u^j}{j!} = x+\sum_{k\geq 1} k\,\bar{s}_k\,\left(1+\sum_{j\geq 1}\frac{k^j\,u^j}{j!}\right).$$

It follows that

$$u(x,\mathbf{s}) = v(\mathbf{t}(x,\mathbf{s})). \tag{46}$$

We conclude that

$$\mathcal{H}_0(\mathbf{t}(x,\mathbf{s})) - \mathcal{F}_0(x,\mathbf{s}) = -\frac{1}{2}\sum_{k_1,k_2\geq 1}\frac{k_1\,k_2}{k_1+k_2}\,\bar{s}_{k_1}\,\bar{s}_{k_2} + \sum_{k\geq 1}\frac{k}{1+k}\,\bar{s}_k - x\sum_{k\geq 1}\bar{s}_k - \frac{1}{4}+x. \tag{47}$$

This finishes the proof of the genus zero part of the Main Conjecture.

3.2. Genus $1,2$

Note that the substitution (14)

$$(t_0,t_1,t_2,\dots)\mapsto(x,\bar{s}_1,\bar{s}_2,\dots)$$

satisfies that

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial t_0}, \tag{48}$$

$$\frac{\partial}{\partial \bar{s}_k} = \sum_{i\geq 0} k^{i+1}\frac{\partial}{\partial t_i}, \qquad k\geq 1. \tag{49}$$

In particular, we have

$$\frac{\partial v}{\partial t_0}(\mathbf{t}(x,\mathbf{s})) = \frac{\partial v(\mathbf{t}(x,\mathbf{s}))}{\partial x} = \frac{\partial u(x,\mathbf{s})}{\partial x}.$$

The last equality is due to (46).

Recall, from the algorithm of [9], that the genus 1 special cubic Hodge free energy is given by

$$H_1(v;v_1) = \frac{1}{24}\log v_1 - \frac{v}{16}.$$

So

$$2H_1(v;v_1) + \frac{v}{8} + \zeta'(-1) = \frac{1}{12}\log v_1 + \zeta'(-1).$$

This proves the genus 1 part of the Main Conjecture.

The genus 2 term of the special cubic Hodge free energy is given by

$$\begin{aligned} H_2(v_1,v_2,v_3,v_4) = {} & \frac{7\,v_2}{2560} - \frac{v_1^2}{11520} + \frac{v_4}{1152\,v_1^2} - \frac{v_3}{320\,v_1} \\ & + \frac{v_2^3}{360\,v_1^4} + \frac{11\,v_2^2}{3840\,v_1^2} - \frac{7\,v_3 v_2}{1920\,v_1^3}. \end{aligned}$$

So

$$\begin{aligned} & 4H_2 + \frac{1}{4}\,D_0^2\,H_1 + \frac{1}{384}v_2 \\ & = 4H_2(v_1,v_2,v_3,v_4) - \frac{5}{384}v_2 + \frac{1}{96}\left[\frac{v_3}{v_1} - \left(\frac{v_2}{v_1}\right)^2\right] \\ & = -\frac{v_2}{480} - \frac{v_1^2}{2880} + \frac{v_4}{288\,v_1^2} - \frac{v_3}{480\,v_1} + \frac{v_2^3}{90\,v_1^4} + \frac{v_2^2}{960\,v_1^2} - \frac{7v_3v_2}{480\,v_1^3} \\ & = F_2(v_1,v_2,v_3,v_4). \end{aligned}$$

This proves the genus 2 part of the Main Conjecture.

3.3. Genus 3, 4

Using the Main Conjecture along with the algorithm of [9], we obtain the following two statements.

Conjecture 7. *The genus* 3 *GUE free energy is given by*

$$
\begin{aligned}
(50)\qquad & F_3(u_1,\dots,u_7)\\
&= \frac{13u_4}{120960}+\frac{u_2^2}{24192}-\frac{u_1^4}{725760}+\frac{u_7}{10368u_1^3}-\frac{u_6}{5760u_1^2}-\frac{u_5}{13440u_1}\\
&\quad -\frac{103u_4^2}{60480u_1^4}+\frac{59u_3^3}{8064u_1^5}+\frac{u_3^2}{2688u_1^2}+\frac{u_3u_1}{12096}-\frac{5u_2^6}{81u_1^8}-\frac{13u_2^5}{1890u_1^6}\\
&\quad +\frac{5u_2^4}{5376u_1^4}-\frac{u_2^3}{9072u_1^2}-\frac{7u_6u_2}{5760u_1^4}-\frac{53u_3u_5}{20160u_1^4}+\frac{353u_5u_2^2}{40320u_1^5}+\frac{u_5u_2}{840u_1^3}\\
&\quad +\frac{89u_3u_4}{40320u_1^3}-\frac{83u_4u_2^3}{1890u_1^6}-\frac{211u_4u_2^2}{40320u_1^4}+\frac{u_4u_2}{2016u_1^2}+\frac{59u_3u_2^4}{378u_1^7}\\
&\quad +\frac{1993u_3u_2^3}{120960u_1^5}-\frac{u_3u_2^2}{576u_1^3}-\frac{83u_3^2u_2^2}{896u_1^6}+\frac{19u_3u_2}{120960u_1}-\frac{17u_3^2u_2}{2240u_1^4}\\
&\quad +\frac{1273u_3u_4u_2}{40320u_1^5}.
\end{aligned}
$$

Conjecture 8. *The genus 4 GUE free energy is given by*

$$
\begin{aligned}
(51)\qquad & F_4(u_1,\dots,u_{10})\\
&= \frac{1852u_2^9}{1215u_1^{12}}+\frac{151u_2^8}{675u_1^{10}}-\frac{101u_2^7}{12600u_1^8}-\frac{772u_3u_2^7}{135u_1^{11}}+\frac{9904u_4u_2^6}{6075u_1^{10}}\\
&\quad -\frac{1165u_2^6}{1161216u_1^6}-\frac{2851u_3u_2^6}{3600u_1^9}+\frac{14903u_3^2u_2^5}{2160u_1^{10}}+\frac{70261u_3u_2^5}{3225600u_1^7}\\
&\quad +\frac{2573u_4u_2^5}{10800u_1^8}+\frac{u_2^5}{7200u_1^4}-\frac{2243u_5u_2^5}{6480u_1^9}+\frac{195677u_3^2u_2^4}{230400u_1^8}+\frac{3197u_3u_2^4}{967680u_1^5}\\
&\quad +\frac{12907u_6u_2^4}{226800u_1^8}-\frac{10259u_4u_2^4}{1935360u_1^6}-\frac{22153u_5u_2^4}{414720u_1^7}-\frac{101503u_3u_4u_2^4}{32400u_1^9}\\
&\quad +\frac{1823u_4^2u_2^3}{5670u_1^8}+\frac{415273u_3u_5u_2^3}{829440u_1^8}+\frac{97u_5u_2^3}{120960u_1^5}+\frac{26879u_6u_2^3}{2903040u_1^6}\\
&\quad +\frac{u_2^3}{7257600}-\frac{49u_3u_2^3}{138240u_1^3}-\frac{5137u_4u_2^3}{4354560u_1^4}-\frac{877u_3^2u_2^3}{57600u_1^6}\\
&\quad -\frac{812729u_3u_4u_2^3}{2073600u_1^7}-\frac{212267u_7u_2^3}{29030400u_1^7}-\frac{305129u_3^3u_2^3}{103680u_1^9}+\frac{u_1^2u_2^2}{460800}\\
&\quad +\frac{1379u_4^2u_2^2}{34560u_1^6}+\frac{13138507u_3^2u_4u_2^2}{9676800u_1^8}+\frac{2417u_3u_4u_2^2}{537600u_1^5}+\frac{17u_4u_2^2}{138240u_1^2}
\end{aligned}
$$

$$+\frac{2143u_3u_5u_2^2}{34560u_1^6}+\frac{449u_5u_2^2}{1451520u_1^3}+\frac{2323u_8u_2^2}{3225600u_1^6}-\frac{2623u_3^2u_2^2}{967680u_1^4}-\frac{443u_6u_2^2}{9676800u_1^4}$$
$$-\frac{667u_7u_2^2}{537600u_1^5}-\frac{192983u_3^3u_2^2}{691200u_1^7}-\frac{60941u_3u_6u_2^2}{1075200u_1^7}-\frac{171343u_4u_5u_2^2}{1935360u_1^7}$$
$$+\frac{22809u_3^4u_2}{71680u_1^8}+\frac{1747u_3^3u_2}{806400u_1^5}+\frac{7u_3^2u_2}{38400u_1^2}+\frac{9221u_5^2u_2}{1935360u_1^6}+\frac{17u_1u_3u_2}{3225600}$$
$$+\frac{78533u_3^2u_4u_2}{691200u_1^6}+\frac{18713u_3u_4u_2}{14515200u_1^3}+\frac{15179u_4u_6u_2}{1935360u_1^6}+\frac{20639u_3u_7u_2}{4838400u_1^6}$$
$$+\frac{37u_8u_2}{302400u_1^4}-\frac{u_4u_2}{86400}-\frac{11u_5u_2}{362880u_1}-\frac{923u_6u_2}{14515200u_1^2}-\frac{113u_7u_2}{9676800u_1^3}$$
$$-\frac{55u_4^2u_2}{387072u_1^4}-\frac{419u_3u_5u_2}{1935360u_1^4}-\frac{1411u_4u_5u_2}{138240u_1^5}-\frac{7u_9u_2}{138240u_1^5}-\frac{1751u_3u_6u_2}{268800u_1^5}$$
$$-\frac{12035u_3^2u_5u_2}{96768u_1^7}-\frac{44201u_3u_4^2u_2}{276480u_1^7}+\frac{1549u_3^4}{115200u_1^6}+\frac{937u_3^3}{2903040u_1^3}+\frac{229u_4^3}{62208u_1^6}$$
$$+\frac{19u_5^2}{46080u_1^4}+\frac{u_1^3u_3}{691200}+\frac{949u_3u_4u_5}{55296u_1^6}+\frac{59u_3^2u_6}{10752u_1^6}+\frac{73u_4u_6}{107520u_1^4}+\frac{1777u_3u_7}{4838400u_1^4}$$
$$+\frac{143u_7}{14515200u_1}+\frac{31u_8}{9676800u_1^2}+\frac{u_{10}}{497664u_1^4}-\frac{u_3^2}{115200}-\frac{u_1u_5}{138240}$$
$$-\frac{73u_6}{29030400}-\frac{u_1^6}{43545600}-\frac{19u_1^2u_4}{87091200}-\frac{137u_3u_4}{2073600u_1}-\frac{239u_3u_5}{1451520u_1^2}$$
$$-\frac{661u_4^2}{5806080u_1^2}-\frac{u_9}{138240u_1^3}-\frac{17u_4u_5}{387072u_1^3}-\frac{89u_3u_6}{3225600u_1^3}-\frac{709u_3^2u_4}{3225600u_1^4}$$
$$-\frac{1291u_3u_4^2}{138240u_1^5}-\frac{1001u_3^2u_5}{138240u_1^5}-\frac{197u_5u_6}{387072u_1^5}-\frac{163u_3u_8}{967680u_1^5}-\frac{2069u_4u_7}{5806080u_1^5}$$
$$-\frac{2153u_3^3u_4}{28800u_1^7}.$$

We also computed the genus 5 free energy; it can be found in the Appendix to the preprint version arXiv: 1606.03720 of the present paper.

For the particular examples of enumerating squares, hexagons, octagons on a genus g Riemann surface ($g = 3, 4, 5$), one can use (50), (51), as well as the equation (A.0.1) of the arXiv preprint version to obtain the combinatorial numbers. We checked that these numbers agree with those in [11]. This gives some evidences of validity of the Main Conjecture for $g = 3, 4, 5$.

Remark 9. The genus 1, 2, 3 terms of the GUE free energy with even couplings were also derived in [16, 17, 31] for the particular case of only one nonzero coupling (i.e., in the framework of enumeration of $2m$-gons). To the best of our knowledge, explicit formulae for higher genus ($g \geq 4$) terms, even in the case of the particular examples, were not available in the literature.

Note added. A proof of the Main Conjecture was recently obtained in [10].

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