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A descendent tropical Landau-Ginzburg potential for $\mathbb{P}^2$

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Following work of Gross [13], a family of Landau-Ginzburg potentials for $\mathbb{P}^2$ is defined using counts of tropical objects analogous to holomorphic disks with descendants. Oscillatory integrals of this family compute an enhancement of Givental's J-function, encoding many descendent Gromov-Witten invariants. This construction can be seen as yielding a canonical family of Landau-Ginzburg potentials on a refinement of a sector of the big phase space, and the resulting descendent J-function is the natural lift given by the constitutive equations of Dijkgraaf and Witten [8] to this setting.

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1. Introduction

As introduced by Givental [12], the mirror dual of a toric Fano manifold X is a Landau-Ginzburg model $(\check{X}, W)$, a pair consisting of a variety $\check{X}$ and a potential $W : \check{X} \to \mathbb{C}$. This relationship was extended by Barannikov [1] to give an isomorphism between certain A and B-model Frobenius manifolds. The former can be regarded as parametrizing quantum cohomology, a deformation of the cup product given by enumerative data, while the latter parametrizes an *unfolding* of perturbations of W. Oscillatory integrals on the B-model side can be used to identify this isomorphism and recover the data encoded in the A-model. The usual basis of perturbations of W, however, makes the translation to quantum cohomology difficult.

A more direct connection between the geometry of this pair was suggested by work of Cho and Oh [5], who give a correspondence between terms of W and counts of certain holomorphic disks in X. This observation leads to a natural strategy for defining a perturbation of W: select a set A of general points in X, consider families of disks passing through subsets of A, and deform W in terms given by counts of such families. This idea was explored independently by Fukaya, Oh, Ohta, and Ono [11] (see [10] for an overview) and Gross [13], in symplectic and tropical languages, respectively. In both cases, the structure of quantum cohomology is apparent on the unfolding of W.

Away from a distinguished root vertex, the tropical disks considered by Gross are piecewise linear embeddings of trivalent graphs into the real plane. By extending the relevant definitions to allow for higher valence vertices, one can instead define perturbations of W in terms of counts of *descendent* tropical disks, fragments of the tropical descendent curves considered by Markwig and Rau [28]. This construction gives a fundamentally new structure in mirror symmetry: a natural family of Landau-Ginzburg potentials on (a refinement of) a sector of the big phase space. In its natural coordinates, oscillatory integrals on this family compute a descendent enhancement of Givental's J-function. More precisely:

Theorem 1.1. *For any general choice of a countably infinite set of points*

$$A := \{Q, P_1, \ldots, \} \subset \mathbb{R}^2$$

there is a family $\check{\mathcal{X}}_{\mathrm{desc}} \xrightarrow{\kappa} \mathcal{M}_{\mathrm{desc}}$ *and tropically defined regular function*

$$W_{\mathrm{desc}}(A) : \check{\mathcal{X}}_{\mathrm{desc}} \to \mathbb{C}$$

giving a formal perturbation of W. These data gives rise to a local system $\mathcal{R}$ on $\mathcal{M}_{\mathrm{desc}} \otimes \operatorname{Spec}\mathbb{C}[\hbar, \hbar^{-1}]$, whose fiber over $(\kappa, \hbar)$ is $H_2((\check{\mathcal{X}}_{\mathrm{desc}})_\kappa$, $\mathrm{Re}(\mathrm{W}_{\mathrm{desc}}/\hbar) \ll 0)$. There exists a multi valued basis Ξ_0, Ξ_1, Ξ_2 of $\mathcal{R}$ such that

$$\sum_{i=0}^{2} \alpha^i \int_{\Xi_i} e^{W_{\mathrm{desc}}(A)/\hbar} f\Omega = \hbar^{-3\alpha} \sum_{j=0}^{2} (\alpha\hbar)^j \, \phi_j$$

where Ω is a canonical relative two-form of $\check{\mathcal{X}}_{\mathrm{desc}}$ over $\mathcal{M}_{\mathrm{desc}}$, we have identified a fiber of $\mathcal{R}^\vee$ with $\mathbb{C}[\alpha]/(\alpha^3)$ (with α^i dual to Ξ_i), f is a regular function on $\check{\mathcal{X}}_{\mathrm{desc}} \times \operatorname{Spec}\mathbb{C}[\hbar]$ defined in terms of tropical data, and ϕ_j is defined below.

Let T_i generate $H^{2i}(\mathbb{P}^2, \mathbb{Z})$, $[l] \in H_2(\mathbb{P}^2, \mathbb{Z})$ be the class of a line, and

$$\gamma := y_{0,0}T_0 + y_{1,0}T_1 + y_{2,0}T_2 + y_{2,1}\psi^1 T_2 + y_{2,2}\psi^2 T_2 + \cdots$$

be a formal expression for insertion into Gromov-Witten invariants. Then

$$\phi_j := \delta_{j,0} + \sum_{d,w\geq 0} \frac{\hbar^{-1}}{w!} \left\langle \frac{T_{2-j}}{\hbar - \psi}, \gamma^w, T_0 \right\rangle_{0,d[l]} e^{dy_{1,0}},$$

where

$$y_{0,0}, y_{1,0}, y_{2,0}, y_{2,1}, \ldots .$$

are natural coordinates on $\mathcal{M}_{\mathrm{desc}}$.

As a side effect, a new correspondence is proven between invariants of the type appearing above and certain counts of tropical curves.

The context of this result becomes clearer through a study of the relevant parameter spaces. Let $\mathcal{M}_B$ be the formal universal unfolding of W and $\omega : \mathcal{M}_{\mathrm{desc}} \to \mathcal{M}_B$ the unique map induced by W_{desc}. Under ω, the flat coordinates $\tilde{y}_i$ of $\mathcal{M}_B$ are taken to formal series in $y_{i,j}$ whose coefficients are given by descendent Gromov-Witten invariants. The equality of the pull-back of Givental's J-function under ω (as given by mirror symmetry) and the generating function of Theorem 1.1 encodes an application of the topological recursion relation.

A parallel phenomenon was long ago observed on the opposite side of the mirror. $\mathcal{M}_{\mathrm{desc}}$ can be naturally identified with a refinement of a formal neighborhood in a sector of the big phase space $\mathcal{M}_\infty$, encoding descendent insertions on point-class conditions. The constitutive equations of Dijkgraaf and Witten introduced a natural lift of vector fields from $\mathcal{M}$ to $\mathcal{M}_\infty$ in [8], which is reproduced in ω. Lifts of many structures of the small phase space to $\mathcal{M}_\infty$ have been explored in [7] and elsewhere; it would be interesting to pursue a B-model interpretation of this data. It is tempting to imagine the construction of a mirror relationship that bypasses descent to the small phase space.

The construction studied in this paper is not limited in relevance to the tropics or $\mathbb{P}^2$. Indeed, we expect it to transfer to the setting investigated by Fukaya, Oh, Ohta, and Ono, yielding a novel symplectic formulation of mirror symmetry involving descendent holomorphic disks.

Likewise, the tropical methods used here are immediately applicable to other mirror pairs. As shown in [29], Gross's tropical mirror symmetry construction can be carried out on more complicated affine manifolds. In principle, such modifications can be patched to explain enumerative dualities in the wide range of mirror partners generated by the Gross-Siebert construction (see [21]).

As in all approaches to tropical curve counting, the concept of multiplicity plays a central role here. In contrast to Mikhalkin's foundational multiplicity, those encountered in this paper and in [13] are still mysterious, but may help to build stronger connections between the tropical and classical world. In particular, they may potentially be understood through a log-geometric construction for $\mathbb{P}^2$ analogous to that for $\mathbb{P}^1$ in [4], linking the appropriate classical and tropical moduli spaces of curves. For recent progress in this direction, see [33].

Wall crossing structures and scattering diagrams figure significantly in this work, generalizations of those found in [13]. It seems clear that there are many other similar enhancements, and a framework for classifying these may help to uncover some sort of limiting enumerative scattering structure. Related constructions have now been explored in depth in [3] [16] [17] [18] and elsewhere; it would be interesting to formalize the relationship of these works to this paper. An appealing modification is suggested by the results of Filippini and Stoppa [9]; Block-Göttsche multiplicity should allow the definition of a q-refined analogue of our Landau-Ginzburg potential, and thus, under the appropriate oscillatory integral formalism, a q-refined J-function. See [26] for some recent related work. This construction is extensible to the descendent context of this paper through the multiplicity of [34].

2. Background and structure

Mathematical interest in the enumerative aspects of mirror symmetry arose in 1991 [2], when counts of rational curves (now known as genus-0 Gromov-Witten invariants) in the quintic threefold $X \subset \mathbb{P}^4$ were predicted to coincide with period integrals on a "mirror partner" manifold, $\check{X}$. Much progress has since been made toward an understanding of an underlying geometric phenomenon, notably in pursuit of Kontsevich's homological mirror symmetry conjecture (see [24]) and the eponymous conjecture of Strominger, Yau, and Zaslow [36].

The latter suggests the relevance of real, piecewise linear (or *tropical*) geometry in this duality. In particular, a real (affine) n-manifold B is thought to serve as an intermediary between mirror-dual complex n-manifolds X and $\check{X}$; indeed, the Gross-Siebert program (as announced in [20]) continues to have great success in pushing this idea forward. Enumerative information in the flavour of [2] is then expected to be reflected in B; for example, counts of curves in X and of piecewise linear graphs (tropical curves) in B should be related. See [15] for an introduction to this circle of ideas.

A major step in this direction came with Mikhalkin's result [30] equating counts of degree d, genus g curves in $\mathbb{P}^2$ through $3d + g - 1$ general points with those of tropical curves satisfying analogous degree, genus, and incidence conditions in the real plane. A tropical B-model mirroring this A-model data arrived with Gross's explicit tropical construction [13]. As $X = \mathbb{P}^2$ is Fano, rather than Calabi-Yau, its mirror partner is not merely another manifold but a Landau-Ginzburg model. Such a Landau-Ginzburg model is a pair consisting of a manifold $\check{X}$ and a regular function $W : \check{X} \to \mathbb{C}$.

Following work of Givental [12], Barannikov generalizes work of Saito [35] to prove mirror symmetry for $\mathbb{P}^n$ as an isomorphism between two Frobenius manifolds, $\mathcal{M}_A$ and $\mathcal{M}_B$ [1]. The former parametrizes the deformations to the cohomology ring of $\mathbb{P}^2$ given by big quantum cohomology, while the latter parametrizes perturbations of W.

Quantum cohomology is governed by Gromov-Witten theory, the relevant details of which we review below. For any $g, n, d \geq 0$, there exists a compactified moduli space $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^2, d[l])$ of genus g, n-pointed, stable maps to $\mathbb{P}^2$ representing $d[l] \in H_2(\mathbb{P}^2, \mathbb{Z})$ (where $[l]$ is the class of a line). This space carries natural line bundles $\mathcal{L}_i$ for $1 \leq i \leq n$. A point of the moduli space can be thought of as a map $f : (C, x_1, \ldots, x_n) \to X$ such that $f_*[C] = \beta$; the fiber of $\mathcal{L}_i$ at this point is the cotangent line $\mathfrak{m}_{x_i}/\mathfrak{m}_{x_i}^2$, where $\mathfrak{m}_{x_i} \subset \mathcal{O}_{C,x_i}$ is the maximal ideal. One can then define $\psi_i := c_1(\mathcal{L}_i)$.

For $\alpha_i \in H^*(\mathbb{P}^2, \mathbb{Q})$, we can define descendent Gromov-Witten invariants

$$\begin{aligned}
&\langle \psi^{i_1}\alpha_1, \ldots, \psi^{i_n}\alpha_n\rangle_{0,d[l]} \\
&:= \int_{[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^2, d[l])} \psi_1^{i_1} \cup ev_1^*(\alpha_1) \cup \cdots \cup \psi_n^{i_n} \cup ev_n^*(\alpha_n) \in \mathbb{Q}.
\end{aligned}$$

Such quantities with $i_j = 0$ for all j are simply known as Gromov-Witten invariants. Though it may be non-integral and even negative, the quantity above is usually thought of as a count of curves of genus g, representing class β and intersecting the Poincaré duals of the classes α_i with tangency conditions prescribed by the ψ classes.

For $0 \leq i \leq 2$, let T_i be a positive generator of $H^{2i}(\mathbb{P}^2, \mathbb{Z})$ and $\gamma := T_0 y_0 + T_1 y_1 + T_2 y_2 \in H*(\mathbb{P}^2, \mathbb{C}[[y_0, y_1, y_2]])$. The *prepotential* of $\mathbb{P}^2$ is then written as

$$\Phi := \sum_{d,k=0}^{\infty} \frac{1}{k!} \langle \gamma^k \rangle_{0,d[l]} \in H^*(\mathbb{P}^2, \mathbb{C}[[y_0, y_1, y_2]]),$$

where the notation γ^k refers to k insertions of γ into the Gromov-Witten invariant, which is distributive in each entry. The structure constants of the so-called quantum cohomology product on $H^*(X, \mathbb{C})$ are encoded in the coefficients of Φ, and the manifold $\mathcal{M}_A := \operatorname{Spec} \mathbb{C}[[y_0, y_1, y_2]]$ can be seen as parametrizing the deformations they specify, making it a Frobenius manifold (see [27] for much information on these interesting objects).

The ideas of quantum cohomology can be extended with descendants to the *big phase space* $\mathcal{M}_\infty := \prod_{n=0}^{\infty} \mathcal{M}_A$, an infinite dimensional complex manifold with natural coordinates $y_{i,j}$ for $0 \leq i \leq 2$ and $0 \leq j$. For each $g \geq 0$, there is a descendent prepotential function on $\mathcal{M}_\infty$ encoding genus-g enumerative information.

$$F_g := \sum_{k \geq 0} \sum_{\beta \in H_2(\mathbb{P}^2, \mathbb{Z})} \sum_{a_1, b_1, \ldots, a_k, b_k} \frac{1}{k} \langle \psi^{a_1} T_{b_1}, \ldots, \psi^{a_k} T_{b_k} \rangle_{g,\beta} y_{b_1,a_1} \cdots y_{b_k,a_k}.$$

There are many interesting structures on $\mathcal{M}_\infty$, some of which are lifts of those found on $\mathcal{M}_A$; see [7].

The mirror partner of $\mathbb{P}^2$ is given by $\check{X} := V(x_0 x_1 x_2 - 1) \subset \mathbb{C}^3$ and $W = x_0 + x_1 + x_2$. Structures on $\mathcal{M}_A$ can be identified on the manifold $\mathcal{M}_B$ parametrizing a universal unfolding of W through the evaluation of certain oscillatory integrals. Following the distillation given in §1 of [13], one can consider the following:

- A local system $\mathcal{R}$ on $\mathcal{M}_B \times \mathbb{C}^*$ whose fiber at $(b, \hbar)$ is given by potentially unbounded, rapid decay cycles in $H_2(X, \Re(W_b/\hbar) \ll 0, \mathbb{C})$, where W_b is the deformation of W given by $b \in \mathcal{M}_B$.
- A section s of the dual local system $\mathcal{R}^\vee$, whose value on Ξ is given by

$$\int_\Xi \exp(W_b/\hbar) f\Omega,$$

where Ω is a nonvanishing holomorphic 2-form on $\check{X}$ and $f : \check{X} \times \mathcal{M}_B \times \mathbb{C}$ is a regular function with $f_{\check{X} \times \{1\} \times \mathbb{C}^*} = 1$.

If these data are chosen to satisfy certain conditions (M4 of loc. cit.), one can identify the flat coordinates y_i of $\mathcal{M}_A$ on $\mathcal{M}_B$ and extract Givental's J-function, a particular generating function of genus-0 Gromov-Witten invariants. For much more detail on this mirror relationship, see Chapter 2 of [14]. If the basis of deformations of W is chosen arbitrarily, the change of coordinates will be too complex to yield structural or enumerative insight.

Guided by insights of Cho and Oh [5] and Nishinou [31], Gross defines a tropical deformation of W that readily matches the data on the other side of the mirror. For any arrangement of $k \geq 0$ general points in the plane, he defines a tropical Landau-Ginzburg potential W_k whose terms (and thus a basis of deformations) are given by counts of tropical disks passing through the selected points. These tropical disks are merely pieces of Mikhalkin's tropical curves, and are trivalent.

Using a powerful construction known as a scattering diagram, a variant of that introduced by Gross and Siebert [21] along with Kontsevich and Soibelman [25], the integrals of W_k were shown to be independent of the general choice of k points and expressible as a sum of counts of tropical curves. Furthermore, the mirror map becomes trivial, and thus the counts of tropical curves computed in the integral are known to correspond to certain classical Gromov-Witten invariants of $\mathbb{P}^2$. While encoding all of the genus-0 invariants studied by Mikhalkin, Givental's J-function also includes a limited range of descendent Gromov-Witten invariants. Those with descendent insertions on point class conditions were given tropical analogues Markwig and Rau [28], but many have no *a priori* tropical interpretation. Therefore, Gross's work expands the correspondence between tropical and classical Gromov-Witten invariants, but depends on prior proofs of mirror symmetry.

Tropical geometry enjoys an attractive immediacy that suggests many natural extensions. An excellent example of this is found in the multiplicity of Block and Göttsche [23], yielding an invariant whose classical meaning is still being explored. The author's thesis [32], which has some overlap with this paper, explored another tropically inspired extension: the mirror symmetry of $\mathbb{P}^2$ to a descendent setting. By expanding the definition of the tropical disk to allow vertices of valence greater than three, one can define an alternative tropical *descendent* family of Landau-Ginzburg potentials W_{desc} on $\mathcal{M}_{\text{desc}}$. The oscillatory integrals of W_{desc} can calculated in a straightforward extension of [13] and again interpreted as counts of tropical curves, now reminiscent of those appearing in the work of Markwig and Rau [28] on tropical descendent Gromov-Witten theory. As in Gross's construction, these counts have no *a priori* classical enumerative interpretation, but under mirror symmetry their generating function is found to be the natural lift the J function to a certain sector of the big phase space. See Remark 9.9.

2.1. Outline

We begin with a set of preliminary definitions in Section 3. The reader is advised to pay close attention to $\mathcal{B}_k$, a fairly intricate bookkeeping structure with some unusual operations that is necessary to index the valences of tropical disks and curves.

In §4, we define the tropical objects necessary for our construction. Our tropical curves are generalizations of those found in [14] and [28], for they are designed to calculate genus zero invariants (see Theorem 1.1 for notation) of type

$$\langle \psi^{\nu} T_i, T_0^k, \psi^{r_1} T_2, \ldots, \psi^{r_n} T_2 \rangle_{0,d[l]}.$$

In [14], Gross gives tropical methods to calculate invariants of type

$$\langle \psi^{\nu} T_i, T_2, \ldots, T_2 \rangle_{0,d[l]},$$

while Markwig and Rau [28] use an intersection theory to give tropical versions of

$$\langle T_0^k, T_1^l, \psi^{r_1} T_2, \ldots, \psi^{r_n} T_2 \rangle_{0,d[l]}.$$

From a combinatorial perspective, the insertion of a ψ class in a tropical Gromov-Witten invariant is reflected by an increment in the required valency of a vertex in the corresponding tropical curves.

We will also make use of a modification of the concept of the *tropical disk* found in [13]; these should be understood as fragments of descendent tropical curves broken apart at a vertex. Instead of restricting to trivalent disks, we will allow higher valence vertices to occur at marked points. The valences are recorded using $\mathcal{B}_k$.

The relevant tropical objects are compiled into moduli spaces of predictable dimension, which are used to define certain counts of tropical curves as putative tropical descendent Gromov-Witten invariants. Their invariance and relation to classical Gromov-Witten theory is justified in later sections.

In §6, we introduce the B-model moduli relevant to our problem. The tropical Landau-Ginzburg potential of [13] is defined as a sum of monomials defined by trivalent disks passing through a selection of k points in the plane, while the sum for our descendent potential runs over disks with higher valence vertices. The oscillatory integrals of [13] are adapted to this setting in §7, which has some overlap with [32]. The process involves a generalization of the *scattering diagrams* and *broken lines*.

We next show that our descendent Landau-Ginzburg potential exhibits certain wall-crossing behaviour with respect to the scattering diagram. The resulting automorphisms are used to prove that the integrals do not depend on the choice of our k points, and that they extract a generating function whose coefficients are the descendent tropical Gromov-Witten invariants defined in §4. See Theorem 7.2. Some of the necessary arguments are unpleasantly technical, and have therefore been shunted to §10 for the interested reader.

Section 8 treats a number of formal manipulations on generating series of tropical and classical descendent Gromov-Witten invariants, useful for satisfying the conditions necessary to apply mirror symmetry. The generating function $\mathbb{T}_{\mathrm{trop}}$ defined by the integrals can then be related to a pullback $\mathbb{J}$ of the J-function by identifying flat coordinates on the B-model moduli. See Theorem 9.5. The induced change of coordinates can be written in terms of classical Gromov-Witten invariants, yielding an expression for $\mathbb{T}_{\mathrm{trop}}$ entirely in these terms. Finally, the axioms of Gromov-Witten theory are applied to show that $\mathbb{J}$ is equal to the classical counterpart $\mathbb{T}$ of $\mathbb{T}_{\mathrm{trop}}$, thus proving Theorem 1.1 and the classical relevance of our tropical descendent invariants.

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3. Preliminary definitions and notation

3.1. Toric notation

Set $M := \mathbb{Z}^2$, $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$, $N := \mathrm{Hom}(M, \mathbb{Z})$, and $\langle \cdot, \cdot \rangle : N \times M \to \mathbb{Z}$ as the usual pairing. Let Σ be a toric fan for $X_\Sigma := \mathbb{P}^2$ in $M_{\mathbb{R}}$, with each ray $\rho_i \in \Sigma^{[1]}$ generated by a primitive element $\mathbf{m}_i \in M$ for $0 \leq i \leq 2$, and define $\sigma_{i,j} \subset M_{\mathbb{R}}$ to be the cone generated by $\mathbf{m}_i$ and $\mathbf{m}_j$. Let S_i be the union of the dimension i closed cones of Σ.

Denote by T_Σ the free abelian group generated by the set of rays $\Sigma^{[1]}$ of Σ and $T_\Sigma^+ \subseteq T_\Sigma$ its associated semigroup. For $\rho_i \in \Sigma^{[1]}$, denote by t_i the corresponding generator in T_Σ. We will make use of the surjective map $r : T_\Sigma \to M$ defined by $r(t_i) = \mathbf{m}_i$,

Define $\overline{z} = \sum_{\rho_i \in \Sigma^{[1]}} t_i$, and for $z = \sum_{\rho_i \in \Sigma^{[1]}} a_i t_i \in T_\Sigma$, let $|z| := \sum_{\rho_i \in \Sigma^{[1]}} a_i \in \mathbb{Z}$.

3.2. Technical tools

We set $k \in \mathbb{Z}_{>0}$, which will serve as an "order of approximation" and allow us to avoid issues of infinity in our tropical structures.

An ordered set of points $A := \{Q, P_1, P_2, \ldots, P_k\} \subset M_{\mathbb{R}}$ will be called an *arrangement*. For an arrangement A and $Q' \in M_{\mathbb{R}}$, denote by $A(Q')$ the arrangement formed by replacing $Q \in A$ by Q'. For an arrangement A, define $S_i(A)$ to be the translation of S_i centered at $Q \in A$.

We will often need a notion of generality, which depends on context. In this paper, generality will always (in a fairly obvious way) refer to conditions defined by the complements of finite sets of tropical curve-like objects. We leave it to readers to satisfy themselves with the details.

Let $\mathcal{B}_k := \prod_{i=1}^{k}\{0, 1, \ldots, k\}$. For a *vector* $b = (b_1, b_2, \ldots, b_k) \in \mathcal{B}_k$, denote by b_i the i-th entry and $\#(b)$ the number of non-zero entries of b. Furthermore, let $b\{i\}$ indicate the position and $b(i) = b_{b\{i\}}$ the value of i-th non-zero entry in b for $1 \le i \le \#(b)$. Define $|b| := \sum_{j=1}^{k} b_i = \sum_{i=1}^{\#(b)} b(i)$. We occasionally need component-wise operations for $b, c \in \mathcal{B}_k$: $bc := (b_1c_1, b_2c_2, \ldots, b_kc_k)$ and $b + c := (b_1 + c_1, b_2 + c_2, \ldots, b_k + c_k)$. Let $0 \in \mathcal{B}_k$ be the additive identity. We say $b, c \in \mathcal{B}_k$ are disjoint if $bc = 0$ and $b \le c$ if, for all $1 \le i \le k$, $b_i \le c_i$. Furthermore, we say c dominates b (written $b \prec c$) if $b \le c$ and $b_i > 0$ if $c_i > 0$ for all $1 \le i \le k$. If $b \le c$, we define $c - b \in \mathcal{B}_k$ by $(c - b)_i = c_i - b_i$ if $b_i \ne 0$ and 0 otherwise. Set $\binom{|b|}{b} := \binom{|b|}{b_1,\ldots,b_k}$. For $1 \le i \le k$, let e^i denote the elementary vector with an i-th entry of 1 and 0 elsewhere.

We will also need an index set containing three distinct types of elements:

$$\mathcal{I} := \{x, p_1, p_2, \ldots, q_1, q_2, \ldots\}.$$

$\mathcal{I}$ will be used to label the three types of marked points encountered in our construction.

4. Tropical objects

Our construction requires modest modifications of the parametrized tropical curves and disks found in [14]. The disks will glue together to form the curves.

4.1. Definitions

A *metric graph* is a topological realization of a graph with possible non-compact edges attached to a single vertex, and a coordinate function (homeomorphism onto its image) $l_E : E \to \mathbb{R}_{\ge 0}$ for each edge E, with l_E surjective when E is non-compact. We will call a finite (here referring to the number of edges and vertices), connected genus-0 metric graph a *frame*. For a frame Γ, let $\Gamma^{[1]}$ be the set of edges, $\Gamma^{[1]}_\infty$ the set of non-compact edges, $\Gamma^{[0]}$ the set of vertices, and $\Gamma^{[0]}_i$ the set of i-valent vertices.

4.1.1. Curves. Let Γ be a frame for which $\Gamma^{[0]}_1 = \Gamma^{[0]}_2 = \varnothing$. Assign a weight function $w : \Gamma^{[1]} \to \mathbb{Z}_{\ge 0}$ such that $w(\Gamma^{[1]}_\infty) \subseteq \{0, 1\}$ and $w^{-1}(0) \subseteq \Gamma^{[1]}_\infty$, defining a weighted frame (Γ, w). A *marking* will be a bijection *marks* from a subset $H \subseteq \mathcal{I}$ of the form $\{x, p_1, \ldots, p_n, q_1, \ldots, q_m\}$ or $\{p_1, \ldots, p_n, q_1, \ldots, q_m\}$ to $w^{-1}(0)$. We will write $marks(t) \in \Gamma^{[1]}_\infty$ as E_t for $t \in H$. The data $(\Gamma, w, marks, H)$ constitutes a *marked, weighted frame*. We will suppress the dependence on the map $marks$, simply writing $(\Gamma, w, \{x, n, m\})$ when $H = \{x, p_1, \ldots, p_n, q_1, \ldots, q_m\}$ and $(\Gamma, w, \{n, m\})$ when $H = \{p_1, \ldots, p_n, q_1, \ldots, q_m\}$.

A *parametrized tropical curve* $(\Gamma, w, h, \{x, n, m\})$ is a marked, weighted frame $(\Gamma, w, \{x, n, m\})$ and a continuous map $h : \Gamma \to M_{\mathbb{R}}$, smooth on the interior of each edge of weight greater than 0, satisfying:

- At any point on the interior of a given edge E, $h_*(\partial_x) = w(E)\mathbf{v}_E$, where x is the coordinate given by l_E and $\mathbf{v}_E$ is a primitive vector in M.
- (Balancing condition) Let $V \in \Gamma^{[0]}$, and $E_1, \ldots, E_j$ be the edges adjacent to V. Let $\mathbf{m}_{E_i} = \pm\mathbf{v}_{E_i} \in M$ be a primitive vector pointing away from $h(V)$ along the direction of $h(E_i)$. Then
$$\sum_{i=1}^{m} w(E_i)\mathbf{m}_{E_i} = 0.$$

A *tropical curve* is an equivalence class of parametrized tropical curves where $\mathcal{C} = (\Gamma, w, h, \{x, n, m\})$ is equivalent to $\mathcal{C}' = (\Gamma', w', h', \{x', n', m'\})$ if there exists an isometry $\phi : \Gamma \to \Gamma'$ respecting the marking and weight data, smooth on the interior of each edge, and with $\phi \circ h' = h$. A tropical curve $\mathcal{C} = [(\Gamma, w, h, \{x, n, m\})]$ is *in* X_Σ if, for each unmarked $E \in \Gamma^{[1]}_\infty$, $h(E)$ is a translation of some $\rho_i \in \Sigma^{[1]}$. In this case we can define its degree as
$$\Delta(\mathcal{C}) := \sum_{\rho_i \in \Sigma^{[1]}} d_i t_i \in T^+_\Sigma$$
where d_i is the number of unbounded edges of Γ that are mapped to translations of ρ_i by h.

The *combinatorial type* of a tropical curve $\mathcal{C} = [(\Gamma, w, h, \{x, n, m\})]$ is defined as the homeomorphism class of Γ, the markings, weights, and the data $\mathbf{m}_E$ for each edge E. Note that the combinatorial type and metric structure of the underlying frame determine the image of a tropical curve up to translation in $M_{\mathbb{R}}$.

4.1.2. Disks. Our strategy for counting these curves involves a similar object, the *tropical disk* (modified from the definition in [14]). A tropical disk (or simply *disk*) $\mathcal{D} = [(\Gamma, w, h, \{n, m\})]$ is defined by the same collection of data as a tropical curve, except we require that the underlying frame Γ has precisely one univalent vertex, V_{out}. The (unique) edge of Γ attached to

V_{out} will be called E_{out}. We impose the balancing condition at every vertex but V_{out}. Note that $x \in \mathcal{I}$ will not be used as a marking for any edge. Disks should be thought of as pieces of tropical curves that have been broken apart at a vertex; the point of attachment becomes V_{out}.

Define $\mathbf{m}(\mathcal{D}) := w(E_{out})\mathbf{m}^{prim}(\mathcal{D}) = -r(\Delta(\mathcal{D})) \in M_{\mathbb{R}}$, where $\mathbf{m}^{prim}(\mathcal{D})$ is a primitive vector tangent to $h(E_{out})$ pointing toward $h(V_{out})$. The formalism we used to treat tropical curves can be extended to disks in an obvious fashion.

4.2. Collections

As is customary for counting problems, we assemble our objects into moduli spaces.

4.2.1. Curves.

Definition 4.1. Let A be an arrangement, $\Delta \in T_{\Sigma}^{+}$, $S \subseteq M_{\mathbb{R}}$, $m, \nu \in \mathbb{Z}_{\geq 0}$ and $b \in \mathcal{B}_k$. Then we define $\mathcal{M}_{\Delta}^{\text{curve}}(A, b, T_{0,tr}^{m}, \psi^{\nu} S)$ to be the moduli space of tropical curves

$$\mathcal{C} = [(\Gamma, w, h, \{x, p_1, \ldots, p_{\#(b)}, q_1, \ldots, q_m\})]$$

in X_{Σ}, such that

1) $h(E_{p_j}) = P_{b\{j\}}$.
2) If E_x shares a vertex V_l with E_{p_l} for $1 \leq l \leq \#(b)$, then

$$Val(V_l) = 2 + b(l) + \nu$$

   and the valence of the vertex V_j attached to E_{p_j} for $j \in \{1, \ldots, \#(b)\} \setminus \{l\}$ is given by

$$Val(V_j) = 2 + b(j)$$

3) Otherwise, the valence of the vertex V_x attached to E_x is $\nu + 3$ and $Val(V_j) = 2 + b(j)$ for $1 \leq j \leq \#(b)$.
4) $h(E_x) \in S$.
5) $\Delta(\mathcal{C}) = \Delta$

Lemma 4.2. *Let $b \in \mathcal{B}_k$, $\Delta \in T_{\Sigma}^{+}$ and A be a general arrangement. For $0 \leq l \leq 2$ and $r(\Delta) = 0$, $\mathcal{M}_{\Delta}^{\text{curve}}(A, b, T_{0,tr}^{m}, \psi^{\nu} S_l)$ is a polyhedral complex of* *dimension* $|\Delta| + m - \nu - |b| + l - 2$*. By the generality of the points* $P_i \in A$*, the same result holds if we replace* S_l *with* $S_l(A)$*.*

Proof. This follows from the argument of Lemma 5.11 in [14], changing the number of bounded edges to be $|\Delta| + m + \#(b) + 1 - (3 + \nu + \sum_{j=1}^{\#(b)} [b(j) - 1])$. □

4.2.2. Disks.

Definition 4.3. Let A be an arrangement, $m \in \mathbb{Z}_{\geq 0}$ and $b \in \mathcal{B}_k$. Then we define $Disk(A, b, T^m_{0,tr})$ to be the set of tropical disks

$$\mathcal{D} = [(\Gamma, w, h, \{p_1, \ldots, p_{\#(b)}, q_1, \ldots, q_m\})]$$

(note that we do not mark disks with $x \in \mathcal{I}$) in X_Σ, such that

1) $h(E_{p_j}) = P_{b\{j\}}$.
2) The valence of each vertex V is:
   - 1 if $V = V_{out}$
   - 3 if $V \neq V_{out}$ is not attached to E_{p_i} for any p_i
   - $2 + b(j)$ if $V \neq V_{out}$ is attached to E_{p_j}

Definition 4.4. Define $\mathrm{RootDisk}(A, b, T^m_{0,tr}) \subseteq Disk(A, b, T^m_{0,tr})$ to be the subset of disks with $h(V_{out}) = Q$. We define $Disk(A, T^m_{0,tr})$ to be the union over all $b \in \mathcal{B}_k$ of the sets $Disk(A, b, T^m_{0,tr})$, with related subset $\mathrm{RootDisk}(A, T^m_{0,tr})$.

Definition 4.5. Let $\mathcal{D} = [(\Gamma, w, h, \{p_1, \ldots, p_{\#(b)}, q_1, \ldots, q_m\})]$ be a tropical disk in $\mathrm{Disk}(A, b, T^m_{0,tr})$. Define the *flexibility* of $\mathcal{D}$ as

$$F(\mathcal{D}) := |\Delta(\mathcal{D})| + m - |b|.$$

Lemma 4.6. *Let* A *be general. The set of disks* $\mathcal{D}$ *in* $\mathrm{RootDisk}(A, T^m_{0,tr})$ *with* $F(\mathcal{D}) = n$ *is an* $n - 1$ *dimensional polyhedral complex. The set of such disks in* $\mathrm{Disk}(A, T^m_{0,tr})$ *is an* $n + 1$ *dimensional polyhedral complex.*

Proof. This follows from the argument of Lemma 5.6 in [14], replacing the idea of Maslov index with flexibility and adjusting the number of bounded edges as above. □

Definition 4.7. Let $\mathcal{D}$ be a tropical disk in $Disk(A, b, T^m_{0,tr})$. We say $\mathcal{D}$ is *semirigid* if $F(\mathcal{D}) = 1$ and *rigid* if $F(\mathcal{D}) = 0$. By Lemma 4.6, the set of semirigid disks in $\mathrm{RootDisk}(A, T^m_{0,tr})$ is 0-dimensional. Note, as one degenerate example, the single semirigid disk $\mathcal{D} \in \mathrm{RootDisk}(A, 0, T^1_{0,tr})$.

5. Multiplicity

An essential element of tropical invariant theory is *multiplicity*, the amount by which a tropical curve contributes to a particular invariant. In contrast to Mikhalkin's work, the multiplicities we require vary depending on the invariant to which a curve contributes. As our descendent Landau-Ginzburg potential is defined by counts of tropical disks, we will also require a notion of multiplicity for these objects.

5.1. Disks

We will have slightly different definitions of multiplicity for semirigid and rigid disks, closely related to Mikhalkin's famous multiplicity. This approach was inspired by the methods of [28].

Definition 5.1. Let A be a general arrangement and $\mathcal{D}$ a semirigid tropical disk in $\mathrm{RootDisk}(A, b, T^m_{0,tr})$. Then $\mathcal{D}$ can be considered as a point on the interior of a moduli space $\mathcal{M}_\mathcal{D}$ of tropical disks in X_Σ of the same combinatorial type with $h(V_{out}) = Q$ and no constraint on the image of the collapsed edges E_{p_i} under h. There are natural coordinates on this moduli space given by the lengths of the bounded edges $E \in \Gamma$. Define $ev(\mathcal{D}) : \mathcal{M}_\mathcal{D} \to M_\mathbb{R}^{\#(b)}$ by

$$ev(h) = \big(h(p_1), \ldots, h(p_{\#(b)})\big).$$

For each vertex $V \in \Gamma^{[0]}$, define $n_i(V)$ to be the number of unbounded rays radiating from V in the direction $\mathbf{m}_i$. Define

$$\mathrm{Aut}(\mathcal{D}) := \prod_{V \in \Gamma^{[0]}} \frac{1}{n_0(V)!n_1(V)!n_2(V)!}$$

and

$$\mathrm{Mult}(h) := |\det(ev)|\mathrm{Aut}(\mathcal{D}),$$

where $\det(ev)$ is the determinant of the linear part of ev and we set $|\det(ev)| := 1$ if $|\#(b)| = 0$.

Lemma 4.6 gives an equality of dimension between the source and target of ev. In particular, as $\mathcal{D}$ is semirigid, its number of compact edges is equal to twice $\#(b)$. One can consider the (0-dimensional) cell corresponding to $\mathcal{D}$ in $\mathrm{RootDisk}(A, T^m_{0,tr})$ as being cut out of $\mathcal{M}_\mathcal{D}$ by $\#(b)$ general 2-dimensional constraints.

Definition 5.2. Let $\mathcal{D}$ be a rigid tropical disk in $Disk(A, b, T^m_{0,tr})$, with $\#(b)$ necessarily greater than 0. We modify the definition above by placing $\mathcal{D}$ into a moduli space $\mathcal{M}_{\mathcal{D}_{rigid}}$ of tropical disks sharing the same combinatorial type, length of E_{out}, and image $h(E_{p_1}) \in M_{\mathbb{R}}$. The lengths of the rest of the bounded edges give a set of coordinates. We define $ev'(\mathcal{D}) : \mathcal{M}_{\mathcal{D}_{rigid}} \to M_{\mathbb{R}}^{\#(b)-1}$ by

$$ev'(h) = \big(h(p_2), \ldots, h(p_{\#(b)})\big)\,.$$

As before

$$\mathrm{Mult}(\mathcal{D}) := |\det(ev')|\mathrm{Aut}(\mathcal{D}),$$

where $\det(ev')$ is the determinant of the linear part of ev' and $|\det(ev')| := 1$ if $|\#(b)| = 1$.

As in Definition 5.1, this expression is meaningful due to the dimensional restrictions of Lemma 4.6.

5.2. Curves

Definition 5.3. Let $S \subseteq M_{\mathbb{R}}$ and

$$\mathcal{C} = [(\Gamma, w, h, \{x, p_1, \ldots, p_{\#(b)}, q_1, \ldots, q_m\})] \in \mathcal{M}^{\mathrm{curve}}_{\Delta}(A, b, T^m_{0,tr}, \psi^\nu S).$$

Denote by $\Gamma_1, \ldots, \Gamma_w$ the closures of each of the connected components of $\Gamma \setminus E_x$, with h_i being the restriction of h to Γ_i.

Each disk $\mathcal{D}_i$ defined by Γ_i and h_i is viewed as being marked by those points $p \in \{p_1, \ldots, p_{\#(b)}\}$ and $q \in \{q_1, \ldots, q_m\}$ whose corresponding edges belong to Γ_i. That is, $\mathcal{D}_i \in Disk(A, s^i, T^{m_i}_{0,tr})$ where m_i counts the number of marked points q_j in Γ_i and $b^i \in \mathcal{B}_k$ is the vector of values of b corresponding to the marked points p_j in Γ_i. Note that $\sum_i m_i = m$, the vectors $b^j \in \mathcal{B}_k$ are pairwise disjoint, and $\sum_j b^j = b$. Denote by

$$\overline{\mathrm{Dec}}(\mathcal{C}) := \{\mathcal{D}_1, \ldots, \mathcal{D}_w\}$$

the *decomposition* of $\mathcal{C}$, define $\mathrm{Dec}(\mathcal{C}) \subset \overline{\mathrm{Dec}}(\mathcal{C})$ to be the subset of disks that do not consist of a single marked edge, and $\mathrm{simpDec}(\mathcal{C}) \subset \mathrm{Dec}(\mathcal{C})$ to be the subset of disks consisting of a single unmarked, unbounded edge.

We can now state the precise relationship between curves and disks.

Lemma 5.4. *Let $S \subseteq M_{\mathbb{R}}$ be a subset. Let*

$$\mathcal{C} = [(\Gamma, w, h, \{x, p_1, \dots, p_{\#(b)}, q_1, \dots, q_m\})] \in \mathcal{M}_{\Delta}^{\text{curve}}(A, b, T_{0,tr}^m, \psi^\nu S).$$

1) *If $S = M_{\mathbb{R}}$ and $|b| = |\Delta| - \nu + m$, then either:*
   a) *E_x does not share a vertex with any E_{p_i}. In this case, all but two of the disks $\mathcal{D} \in \mathrm{Dec}(\mathcal{C})$ are semirigid, and the remaining two are rigid.*
   b) *E_x shares a vertex with E_{p_j}. In this case, $\mathcal{D}_i \in \mathrm{Dec}(\mathcal{C})$ is semirigid for all choices of i.*
2) *If $S = C$, a general translation of S_1, and $|b| = |\Delta| - \nu + m - 1$, then all but one of the disks $\mathcal{D} \in \mathrm{Dec}(\mathcal{C})$ are semirigid, and the remaining one is rigid.*
3) *If $S = Q'$, a general point in $M_{\mathbb{R}}$, and $|b| = |\Delta| - \nu + m - 2$, all disks $\mathcal{D} \in \mathrm{Dec}(\mathcal{C})$ are semirigid.*

Proof. This follows from the argument of Lemma 5.12 in [14], adjusting the dimensional requirements as dictated by Lemma 4.2. □

The following, rather mysterious, multiplicities (taken from [13]) are necessary for defining our tropical invariants. Let $\mathcal{C}$ be a tropical curve, with vertex V_x attached to E_x. Define:

$$\begin{aligned}
\mathrm{Mult}_x^0(\mathcal{C}) &= \frac{1}{n_0(V_x)!n_1(V_x)!n_2(V_x)!} \\
\mathrm{Mult}_x^1(\mathcal{C}) &= -\frac{\sum_{j=1}^{n_0(V_x)} \frac{1}{j} + \sum_{j=1}^{n_1(V_x)} \frac{1}{j} \sum_{j=1}^{n_2(V_x)} \frac{1}{j}}{n_0(V_x)!n_1(V_x)!n_2(V_x)!} \\
\mathrm{Mult}_x^2(\mathcal{C}) &= \frac{\left(\sum_{l=0}^{2} \sum_{j=1}^{n_l(V_x)} \frac{1}{j}\right)^2 + \sum_{l=0}^{2} \sum_{j=1}^{n_l(V_x)} \frac{1}{j^2}}{2n_0(V_x)!n_1(V_x)!n_2(V_x)!},
\end{aligned}$$

where the terms $n_i(V_x)$ are as in Definition 5.1.

Definition 5.5. Fix a general arrangement $A = \{Q, P_1, \dots, P_k\}$. Let $b \in \mathcal{B}_k$, $n = \#(b)$, and $a_i := b(i) - 1$. Recall the definition $\overline{z} = t_0 + t_1 + t_2 \in T_\Sigma$. We now define tropical curve counts that we will call *descendent tropical invariants*, though they are not *a priori* independent of the chosen arrangement.

1) When $3d-2-\nu+m-|b|=0$, we define

$$\langle\psi^{a_1}P_{b\{1\}}\ldots,\psi^{a_n}P_{b\{n\}},T^m_{0,tr},\psi^\nu S_0(A)\rangle^{\text{trop}}_{0,d}$$

to be

$$\sum_{\mathcal{C}}\mathrm{Mult}(\mathcal{C})$$

where the sum is over all $\mathcal{C}\in\mathcal{M}^{\text{curve}}_{d\bar{z}}(A,b,T^m_{0,tr},\psi^\nu S_0(A))$ with

$$\mathrm{Mult}(\mathcal{C}):=\mathrm{Mult}^0_x(\mathcal{C})\prod_{\mathcal{D}_i\in\mathrm{Dec}(\mathcal{C})}\mathrm{Mult}(\mathcal{D}_i).$$

2) When $3d-1-\nu+m-|b|=0$, we define

$$\langle\psi^{a_1}P_{b\{1\}},\ldots,\psi^{a_n}P_{b\{n\}},T^m_{0,tr},\psi^\nu S_1(A)\rangle^{\text{trop}}_{0,d}$$

to be

$$\sum_{\mathcal{C}}\mathrm{Mult}(\mathcal{C})$$

where the sum is over all marked tropical rational curves satisfying one of the following conditions:

a) $\nu\geq 0$,

$$\mathcal{C}\in\mathcal{M}^{\text{curve}}_{d\bar{z}}(A,b,T^m_{0,tr},\psi^\nu S_1(A)),$$

and no $\mathcal{D}\in\mathrm{simpDec}(\mathcal{C})$ maps into the connected component of $S_1(A)\setminus\{Q\}$ containing $h(E_x)$. By Lemma 5.4, there is precisely one rigid $\hat{\mathcal{D}}\in\mathrm{Dec}(\mathcal{C})$. Suppose that the connected component of $S_1(A)\setminus\{Q\}$ is $Q+\mathbb{R}_{\geq 0}\mathbf{m}_i$. Then we define:

$$\mathrm{Mult}(\mathcal{C}):=|\mathbf{m}(\hat{\mathcal{D}})\wedge\mathbf{m}_i|\mathrm{Mult}^0_x(\mathcal{C})\prod_{\mathcal{D}\in\mathrm{Dec}(\mathcal{C})}\mathrm{Mult}(\mathcal{D}).$$

b) $\nu\geq 1$ and

$$\mathcal{C}\in\mathcal{M}^{\text{curve}}_{d\bar{z}}(A,b,T^m_{0,tr},\psi^{\nu-1}S_0(A))$$

In this case,

$$\mathrm{Mult}(\mathcal{C}):=\mathrm{Mult}^1_x(\mathcal{C})\prod_{\mathcal{D}\in\mathrm{Dec}(\mathcal{C})}\mathrm{Mult}(\mathcal{D})$$

3) When $3d - \nu + m - |b| = 0$, we define

$$\langle \psi^{a_1} P_{b\{1\}}, \ldots, \psi^{a_n} P_{b\{n\}}, T^m_{0,tr}, \psi^\nu S_2(A) \rangle^{\mathrm{trop}}_{0,d}$$

to be

$$\sum_{\mathcal{C}} \mathrm{Mult}(\mathcal{C})$$

where the sum is over all marked tropical curves $\mathcal{C}$ satisfying one of the following conditions:

a) $\nu \geq 0$,

$$\mathcal{C} \in \mathcal{M}^{\mathrm{curve}}_{d\overline{z}}(A, b, T^m_{0,tr}, \psi^\nu S_2(A))$$

and E_x does not share a vertex with any of the E_{p_i}'s. Furthermore, no $\mathcal{D} \in \mathrm{simpDec}(\mathcal{C})$ maps into the connected component of $S_2(A) \setminus S_1(A)$ containing $h(E_x)$. By Lemma 5.4, there are precisely two rigid disks in $\mathrm{Dec}(\mathcal{C})$, which we call $\mathcal{D}_1$ and $\mathcal{D}_2$. Then

$$\mathrm{Mult}(\mathcal{C}) := |\mathbf{m}(\mathcal{D}_1) \wedge \mathbf{m}(\mathcal{D}_2)| \mathrm{Mult}^0_x(\mathcal{C}) \prod_{\mathcal{D} \in \mathrm{Dec}(\mathcal{C})} \mathrm{Mult}(\mathcal{D}).$$

b) $\nu \geq 0$,

$$\mathcal{C} \in \mathcal{M}^{\mathrm{curve}}_{d\overline{z}}(A, b, T^m_{0,tr}, \psi^\nu S_2(A))$$

and E_x shares a vertex with E_{p_i}. Suppose l elements of $\mathrm{simpDec}(\mathcal{C})$ map into the connected component of $S_2(A) \setminus S_1(A)$ containing $h(E_x)$. Then we define:

$$\mathrm{Mult}(\mathcal{C}) := \binom{a_i + \nu - l}{\nu} \mathrm{Mult}^0_x(\mathcal{C}) \prod_{\mathcal{D} \in \mathrm{Dec}(\mathcal{C})} \mathrm{Mult}(\mathcal{D}).$$

c) $\nu \geq 1$ and

$$\mathcal{C} \in \mathcal{M}^{\mathrm{curve}}_{d\overline{z}}(A, b, T^m_{0,tr}, \psi^{\nu-1} S_1(A))$$

Furthermore, no $\mathcal{D} \in \mathrm{simpDec}(\mathcal{C})$ maps into the connected component of $S_1(A) \setminus S_0(A)$ containing $h(E_x)$, which we suppose to be $Q + \mathbb{R}_{\geq 0}\mathbf{m}_i$. By Lemma 5.4, there is precisely one rigid $\hat{\mathcal{D}} \in \mathrm{Dec}(\mathcal{C})$. Then we define:

$$\mathrm{Mult}(\mathcal{C}) := |\mathbf{m}(\hat{\mathcal{D}}) \wedge \mathbf{m}_i| \mathrm{Mult}^1_x(\mathcal{C}) \prod_{\mathcal{D} \in \mathrm{Dec}(\mathcal{C})} \mathrm{Mult}(\mathcal{D}).$$

d) $\nu \geq 2$ and

$$\mathcal{C} \in \mathcal{M}^{\mathrm{curve}}_{d\bar{z}}(A, b, T^m_{0,tr}, \psi^{\nu-2} S_0(A))$$

In this case,

$$\mathrm{Mult}(\mathcal{C}) := \mathrm{Mult}^2_x(\mathcal{C}) \prod_{\mathcal{D} \in \mathrm{Dec}(\mathcal{C})} \mathrm{Mult}(\mathcal{D}).$$

If $3d - \nu + m - |b| + i \neq 2$ (we will call this *incompatible dimension*), we define

$$\langle \psi^{a_1} P_{b\{1\}}, \dots, \psi^{a_n} P_{b\{n\}}, T^m_{0,tr}, \psi^\nu S_i(A) \rangle^{\mathrm{trop}}_{0,d} := 0$$

Refinements of these counts based on the image of E_x under h will be useful in the remainder.

Definition 5.6. For $\sigma \in \Sigma$, define

$$\langle \psi^{a_1} P_{b\{1\}}, \dots, \psi^{a_n} P_{b\{n\}}, T^m_{0,tr}, \psi^\nu S_i(A) \rangle^{\mathrm{trop}}_{d,\sigma}$$

to be the contribution to $\langle \psi^{a_1} P_{b\{1\}}, \dots, \psi^{a_n} P_{b\{n\}}, T^m_{0,tr}, \psi^\nu S_i(A) \rangle^{\mathrm{trop}}_{0,d}$ from tropical curves with $h(E_x)$ mapping to the interior of $Q + \sigma$.

The following property of these curve counts will be important in what follows.

Lemma 5.7. *The descendent tropical Gromov-Witten invariants described above satisfy a tropical fundamental class axiom:*

$$\begin{aligned}
&\langle \psi^{a_1} P_{b\{1\}}, \dots, \psi^{a_n} P_{b\{n\}}, T^m_{0,tr}, \psi^\nu S_i(A) \rangle^{\mathrm{trop}}_{0,d} \\
= &\sum_{j=1}^{n} \langle \psi^{a_1} P_{b\{1\}}, \dots, \psi^{a_j - 1} P_{b\{j\}}, \dots, \psi^{a_n} P_{b\{n\}}, T^{m-1}_{0,tr}, \psi^\nu S_i(A) \rangle^{\mathrm{trop}}_{0,d} \\
&+ \langle \psi^{a_1} P_{b\{1\}}, \dots, \psi^{a_n} P_{b\{n\}}, T^{m-1}_{0,tr}, \psi^{\nu-1} S_i(A) \rangle^{\mathrm{trop}}_{0,d},
\end{aligned}$$

where the above counts are taken to be zero if any of the exponents on ψ are negative.

Proof. This is immediate if any (and thus all) of the counts appearing are of incompatible dimension. Otherwise, it can be seen by removing the edge E_{q_m} from each of the curves contributing to the invariant on the left hand side, thereby generating curves contributing to invariants appearing on the right hand side. When $i \leq 1$, E_{q_m} shares a vertex V with exactly one of the marked edges $E_V \in \{E_x, E_{p_1}, \ldots, E_{p_n}\}$. When E_{q_m} is removed, the valency of V is decreased by one, and the resulting curve then contributes to exactly one of the tropical invariants appearing on the right hand side, with an unchanged multiplicity. On the other hand, each tropical curve appearing in the counts on the right hand side can be uniquely modified, through the addition of an edge E_{q_m}, to give a curve appearing on the left. Again, the multiplicity is unchanged, so the lemma in this case follows from a bijection of curves.

The same argument holds for most of the curves appearing when $i = 2$. The case that requires more care occurs when E_{q_m} shares a vertex V with E_x and some E_{p_j}. The removal of E_{q_m} then yields a curve contributing to two invariants appearing on the right hand side, those for which the power of ψ is decremented for $P_{b\{j\}}$ or $S_2(A)$. The multiplicities of these curves is dictated by part 3b of Definition 5.5, and the equality of their contribution to the left and right hand sides of the lemma follows from the familiar identity

$$\binom{a+1}{b} = \binom{a}{b} + \binom{a}{b-1}.$$

See Figure 5.1 for an example. $\square$

6. Tropical B-model

With the necessary tropical machinery in place, we can begin to define the elements of the mirror relationship.

6.1. Descendent Landau-Ginzburg potential

Let A be a general arrangement. We generalize the methods of [13] to produce a finer perturbation of the Landau-Ginzburg potential. The resulting oscillatory integrals recover tropical versions of a broader class of Gromov-Witten invariants.

Definition 6.1. To $P_i \in A$ associate the variables $u_{i,j}$ in the ring:

$$R'_k := \frac{\mathbb{C}[\{u_{i,j}\}_{i,j}]}{I}$$

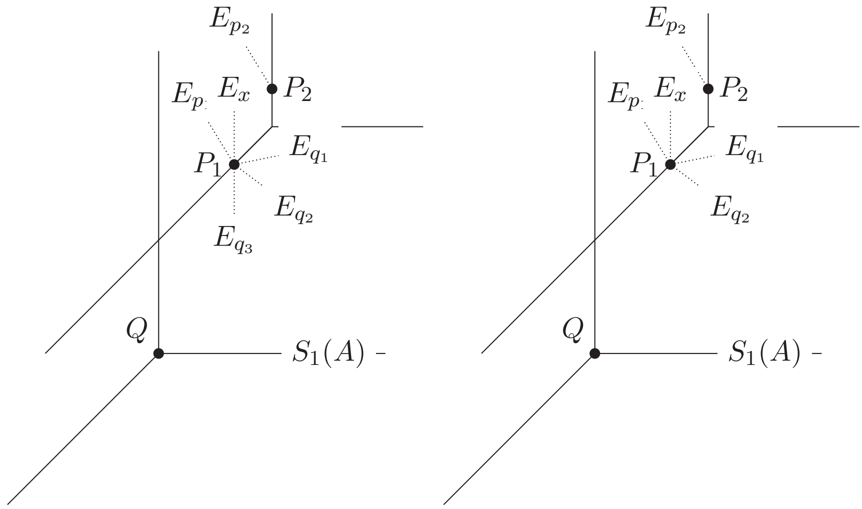

Figure 5.1. Two tropical curves. The dotted lines indicate unbounded edges of weight zero. By Definition 5.5 3b, the curve on the left contributes $\binom{4}{2}$ to $\langle \psi^2 P_1, P_2, T^3_{0,tr}, \psi^2 S_2(A)\rangle^{\text{trop}}_{0,1}$, while the curve on the right contributes $\binom{3}{2}$ to $\langle \psi^1 P_1, P_2, T^2_{0,tr}, \psi^2 S_2(A)\rangle^{\text{trop}}_{0,1}$ and $\binom{3}{1}$ to $\langle \psi^2 P_1, P_2, T^2_{0,trop}, \psi^1 S_2(A)\rangle^{\text{trop}}_{0,1}$.

with $j \in \mathbb{Z}_{\geq 0}$ and $i \in \mathbb{Z}_{>0}$, where I is the ideal generated by the set

$$\{u_{i,j}u_{i,j'} | 1 \leq i \leq k,\ 0 \leq j \leq j' \leq k\} \cup \{u_{i,j} | i > k \text{ or } j \geq k\}$$

Let $m \in \mathbb{Z}_{\geq 0}$ and define

$$R_{k,m} := R_k[y_{0,0}]/(y_{0,0}^{m+1}).$$

For $b \in \mathcal{B}_k$, define

$$u_b := \prod_{i=1}^{\#(b)} u_{b\{i\},b(i)-1},$$

and

$$y_{2,j} := \sum_{i=1}^{k} u_{i,j}.$$

Note that $a \in R_{k,m}$ can be uniquely represented as

$$a = \sum_{\substack{0 \le m' \le m \\ b \in \mathcal{B}_k}} a_{b,m'} u_b y_{0,0}^{m'}$$

with $a_{b,m'} \in \mathbb{C}$, where we abuse notation by writing $u_b y_{0,0}^{m'}$ to denote its equivalence class in $R_{k,m}$.

Definition 6.2. Let $\mathcal{D}$ be a tropical disk in $Disk(A, b, T_{0,tr}^{w})$. Define $u_{\mathcal{D}} := u_b$ and $y_{0,0}^{\mathcal{D}} := \frac{y_{0,0}^{w}}{w!}$.

If $\mathcal{D}$ is rigid or semirigid disk, we define

$$\mathrm{Mono}(\mathcal{D}) := \mathrm{Mult}(\mathcal{D}) u_{\mathcal{D}} z^{\Delta(\mathcal{D})} y_{0,0}^{\mathcal{D}} \in \mathbb{C}[T_\Sigma] \otimes_{\mathbb{C}} R_{k,m}$$

where $z^{\Delta(\mathcal{D})} \in \mathbb{C}[T_\Sigma]$ is the monomial associated to $\Delta(\mathcal{D}) \in T_\Sigma$. We will write $x_i = z^{t_i}$, so $z^{n_0 t_0 + n_1 t_1 + n_2 t_2} = x_0^{n_0} x_1^{n_1} x_2^{n_2}$.

Definition 6.3. The (k, m)-descendent Landau-Ginzburg potential associated to A is defined as

$$W_{k,m}(A) := \sum_{\mathcal{D}} \mathrm{Mono}(\mathcal{D})$$

where the sum is over all semirigid disks $\mathcal{D} \in \mathrm{RootDisk}(A, b, T_{0,tr}^{m'})$ for any $b \in \mathcal{B}_k$ and $m' \le m$.

Definition 6.4.

$$W_{\mathrm{basic}}(A) := x_0 + x_1 + x_2$$

6.2. B-model moduli

Here we give the necessary modification of the construction found in [13]. The surjective map $r : T_\Sigma \to M$ defined by $r(t_i) = \mathbf{m}_i$ yields

$$0 \to K_\Sigma \to T_\Sigma \xrightarrow{r} M \to 0,$$

with K_Σ defined as the kernel of r. Dualizing over $\mathbb{Z}$ gives

$$0 \to \mathrm{Hom}(M, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(T_\Sigma, \mathbb{Z}) \to \mathrm{Pic} X_\Sigma \to 0$$

Tensoring with $\mathbb{C}^\times$ gives the sequence

$$0 \to \mathrm{Hom}(M, \mathbb{Z}) \otimes \mathbb{C}^\times \to \mathrm{Hom}(T_\Sigma, \mathbb{C}^\times) \xrightarrow{\kappa} \mathrm{Pic} X_\Sigma \otimes \mathbb{C}^\times \to 0$$

defining κ, which provides the family of mirrors to X_Σ. Set

$$\check{\mathcal{X}} := \mathrm{Hom}(T_\Sigma, \mathbb{C}^\times) = \mathrm{Spec}\, \mathbb{C}[T_\Sigma].$$

The *Kähler moduli space* of X_Σ is defined to be

$$\mathcal{M}_\Sigma := \mathrm{Pic} X_\Sigma \otimes \mathbb{C}^\times = \mathrm{Spec}\, \mathbb{C}[K_\Sigma].$$

In our case, $K_\Sigma \cong \mathbb{Z}$. Note that κ, by definition, is now a map

$$\kappa : \mathrm{Spec}\, \mathbb{C}[T_\Sigma] \to \mathcal{M}_\Sigma.$$

A fiber of κ over a closed point of $\mathcal{M}_\Sigma$ is isomorphic to $\mathrm{Spec}\, \mathbb{C}[M] = (\mathbb{C}^*)^2$. Define the (k, m)-order thickening of the Kähler moduli space by

$$\mathcal{M}_{k,m} := \mathcal{M}_\Sigma \times \mathrm{Spec}\, R_{k,m}$$

and likewise

$$\check{\mathcal{X}}_{\Sigma,k,m} := \check{\mathcal{X}}_\Sigma \times \mathrm{Spec}\, R_{k,m}.$$

This yields a family

$$\kappa : \check{\mathcal{X}}_{\Sigma,k,m} \to \mathcal{M}_{\Sigma,k,m}$$

By construction, $W_{k,m}(A)$ is a regular function on $\check{X}_{\Sigma,k,m}$, and should be considered as a family of Landau-Ginzburg potentials.

7. Integrals

Oscillatory integrals of $W_{k,m}(A)$ can be understood in terms of counts of tropical curves glued from the disks whose monomials define its summands. In this section, we will give the main result of [32] in this direction and a summary of the methods used in its proof. Elements of the argument easily generalized from those found in [13] are given with a reference to the relevant result, while subtler points are presented in more detail.

Definition 7.1. Economical presentation of the integral will rely on a few formal expressions. Let

$$\frac{S_i(A)}{\hbar-\psi} := S_i(A)\hbar^{-1} + S_i(A)\hbar^{-2}\psi + S_i(A)\hbar^{-3}\psi^2 + \cdots$$

and

$$\gamma_{a,tr} := \sum_{1\le v+1,w\le k} \psi^v P_w u_{w,v}.$$

These will be inserted into tropical invariants, which are then understood as being expanded linearly. For example,

$$\langle \gamma_{a,tr}, S_2(A)\rangle^{\mathrm{trop}}_{0,d} := \sum_{1\le j+1,i\le k} \langle \psi^j P_i, S_2(A)\rangle^{\mathrm{trop}}_{0,d} u_{i,j}.$$

Theorem 7.2. *A choice of a general arrangement A gives rise to a function $W_{k,0}(A)\in\mathbb{C}[T_\Sigma]\otimes_{\mathbb{C}} R_{k,0}$, and hence a family of Landau-Ginzburg potentials on the family $\check{\mathcal{X}}_{\Sigma,k,0}\xrightarrow{\kappa}\mathcal{M}_{\Sigma,k,0}$ with a relative nowhere-vanishing two-form Ω. This data gives rise to a local system $\mathcal{R}$ on $\mathcal{M}_{\Sigma,k,0}\otimes \operatorname{Spec}\mathbb{C}[\hbar,\hbar^{-1}]$, whose fiber over $(\kappa,\hbar)$ is $H_2((\check{\mathcal{X}}_{\Sigma,k})_\kappa, \mathrm{Re}(\mathrm{W}_{\mathrm{basic}}(\mathrm{A})/\hbar)\ll 0)$. Letting $y_{1,0}:=\log(\kappa)$, there exists a multi valued basis Ξ_0,Ξ_1,Ξ_2 of $\mathcal{R}$ satisfying the requirements of §1 of [13] such that*

$$\sum_{i=0}^{2}\alpha^i\int_{\Xi_i} e^{W_{k,0}(A)/\hbar}\Omega = \hbar^{-3\alpha}\sum_{j=0}^{2}(\alpha\hbar)^j\, e^{y_{1,0}\alpha}\Theta_j$$

where we have identified a fiber of $\mathcal{R}^\vee$ with $\mathbb{C}[\alpha]/(\alpha^3)$, with α^i dual to Ξ_i. Then

$$\begin{aligned}
\Theta_0 &:= 1 + \sum_{d>0,w\ge 0}\frac{\hbar^{-1}}{w!}\left\langle \frac{S_0(A)}{\hbar-\psi}, \gamma^w_{a,tr}\right\rangle^{\mathrm{trop}}_{0,d} e^{dy_{1,0}}\\
\Theta_1 &:= \sum_{d>0,w\ge 0}\frac{\hbar^{-1}}{w!}\left\langle \frac{S_1(A)}{\hbar-\psi}, \gamma^w_{a,tr}\right\rangle^{\mathrm{trop}}_{0,d} e^{dy_{1,0}}\\
\Theta_2 &:= \hbar^{-1}\sum_{j=0}^{k} y_{2,j}(-\hbar)^j + \sum_{d>0,w\ge 0}\frac{\hbar^{-1}}{w!}\left\langle \frac{S_2(A)}{\hbar-\psi}, \gamma^w_{a,tr}\right\rangle^{\mathrm{trop}}_{0,d} e^{dy_{1,0}}.
\end{aligned}$$

Furthermore, the result does not depend on the choice of A.

Proof. See [32] and below. □

7.1. Scattering diagrams

The first step in the proof of Theorem 7.2 is to construct a set of structures that govern the combinatorics of $W_{k,0}(A)$. These methods are part of a larger theory developed by Kontsevich, Soibelman, Gross, Siebert, and a number of others, with deep and unexpected links to other areas of mathematics (see [17]).

One can form an object $\mathcal{T}$ called a *tropical tree* from a rigid tropical disk $\mathcal{D} = [(\Gamma, w, h, \{p_1, \ldots, p_{\#(b)}, q_1, \ldots, q_m\})]$ in $Disk(A, b, T^m_{0,tr})$ by deleting the vertex V_{out} from the underlying frame (thereby creating a non-compact edge E_{out} with $w(E_{out}) \in \mathbb{Z}_{>0}$) and modifying h by extending the image of E_{out} to be an unbounded ray in $M_{\mathbb{R}}$. We denote by $Tree(A, b, T^m_{0,tr})$ the set of such trees derived from rigid disks in $Disk(A, b, T^m_{0,tr})$. Note that the one dimensional subset of rigid disks in $Disk(A, b, T^m_{0,tr})$ has a natural fibration over the (finite) zero dimensional set $Tree(A, b, T^m_{0,tr})$, and we associate the same multiplicity $\mathrm{Mult}(\mathcal{T})$, monomial $\mathrm{Mono}(\mathcal{T})$, and flexibility $F(\mathcal{T}) = 0$ to a tree as we do to any of its overlying disks. There is a finite set of trees associated to a general arrangement A:

$$\mathfrak{T}(A)_{k,m} = \bigcup_{w \leq m} Tree(A, b, T^m_{0,tr}).$$

If one represents the set $\mathfrak{T}(A)_{k,m}$ in $M_{\mathbb{R}}$ by drawing the outgoing edge corresponding to each rigid tree, a striking pattern emerges. The points at which pairs of these outgoing edges intersect have rays sprouting from them, as their corresponding rigid trees can be glued at such a point to form a "child" tree. The weight and direction of the outgoing edge of the child is, by the balancing condition, determined by the weights and directions of its parents' outgoing edges. Similarly, the multiplicity and monomial of the child tree are simply determined by those of its parents. This "scattering" at points of intersection gives our tool its name.

We hereafter will specialize our scattering diagrams to the case of $m = 0$, which was addressed in [32], although it is straightforward to generalize to $m > 0$.

Definition 7.3. The following definition is from [13].

1) A *ray* or *line* is a pair $(\mathfrak{d}, f_{\mathfrak{d}})$ such that
   - $\mathfrak{d} \subseteq M_{\mathbb{R}}$ is given by

$$\mathfrak{d} = \mathbf{m}_{\text{init}} - \mathbb{R}_{\geq 0} r(m_0)$$

if $\mathfrak{d}$ is a ray and

$$\mathfrak{d} = \mathbf{m}_{\text{init}} - \mathbb{R}r(m_0)$$

if $\mathfrak{d}$ is a line, where $\mathbf{m}_{\text{init}} \in M_{\mathbb{R}}$ with $m_0 \in T_\Sigma$ satisfying

$$-\mathbf{m}_{\mathfrak{d}} := r(m_0) \neq 0.$$

The set $\mathfrak{d}$ is the *support* of the ray or line. If $\mathfrak{d}$ is a ray, then $\mathbf{m}_{\text{init}}$ is called the *initial point* and is denoted $\text{Init}(\mathfrak{d})$.

- $f_{\mathfrak{d}} \in \mathbb{C}[z^{m_0}] \otimes_{\mathbb{C}} R_{k,m}$
- $f_{\mathfrak{d}} \equiv 1 \mod (\{u_{i,j}\}_{i,j})z^{m_0}$

2) A *scattering digram* $\mathfrak{D}$ is a finite collection of lines and rays.

We will sometimes write $w(\mathfrak{d}) := w(E_{out})$ for walls $\mathfrak{d}$ in $\mathfrak{D}$.

If $\mathfrak{D}$ is a scattering diagram, we write

$$\text{Supp}(\mathfrak{D}) := \bigcup_{\mathfrak{d} \in \mathfrak{D}} \mathfrak{d} \subseteq M_{\mathbb{R}}$$

and

$$\text{Sing}(\mathfrak{D}) := \bigcup_{\mathfrak{d} \in \mathfrak{D}} \partial\mathfrak{d} \cup \bigcup_{\substack{\mathfrak{d}_1, \mathfrak{d}_2 \\ \dim \mathfrak{d}_1 \cap \mathfrak{d}_2 = 0}} \mathfrak{d}_1 \cap \mathfrak{d}_2$$

where $\partial\mathfrak{d} = \{\text{Init}(\mathfrak{d})\}$ if $\mathfrak{d}$ is a ray, and is empty if it is a line.

Definition 7.4. $\mathfrak{D}(A)_{k,0}$ is assembled from the outgoing edges of the trees in $\mathfrak{T}(A)_{k,0}$. The ray in $\mathfrak{D}(A)_{k,0}$ corresponding to a tree $\mathcal{T} \in \mathfrak{T}(A)_{k,0}$ is of the form $(\mathfrak{d}, f_{\mathfrak{d}})$, where

- $\mathfrak{d} = h(E_{out})$
- $f_{\mathfrak{d}} = 1 + w(E_{out})\text{Mult}(\mathcal{T})z^{\Delta(\mathcal{T})}u_{\mathcal{T}}$

Definition 7.5. Given a scattering diagram $\mathfrak{D}$ and smooth immersion $\xi : [0,1] \to M_{\mathbb{R}} \setminus \text{Sing}(\mathfrak{D})$ with endpoints not in $\text{Supp}(\mathfrak{D})$ and intersecting $\text{Supp}(\mathfrak{D})$ transversally, one can define an associated ring automorphism $\theta_{\xi,\mathfrak{D}}$ of $\mathcal{R}_{k,0}$. Find numbers

$$0 < s_1 \leq s_2 \leq \cdots \leq s_n < 1$$

and elements $\mathfrak{d}_i$ such that $\xi(s_i) \in \mathfrak{d}_i$, $\mathfrak{d}_i \neq \mathfrak{d}_j$ if $i \neq j$ and n is taken to be as large as possible to account for all walls of $\mathfrak{D}$ that are crossed by ξ. For each

$i \in \{1, \ldots, n\}$, define $\theta_{\xi,\mathfrak{d}_i}$ to be the automorphism with action

$$\begin{aligned}\theta_{\xi,\mathfrak{d}_i}(z^w) &= z^w f_{\mathfrak{d}_i}^{\langle \mathbf{n}_0, r(w)\rangle}\\ \theta_{\xi,\mathfrak{d}_i}(a) &= a\end{aligned}$$

for $w \in T_\Sigma$, $a \in R_{k,0}$, where $\mathbf{n}_0 \in N$ is chosen to be primitive, annihilating the tangent space to $\mathfrak{d}_i$, and satisfying

$$\langle \mathbf{n}_0, \xi'(s_i)\rangle < 0$$

Then $\theta_{\xi,\mathfrak{D}} := \theta_{\xi,\mathfrak{d}_n} \circ \cdots \circ \theta_{\xi,\mathfrak{d}_1}$, where composition is taken from right to left.

The reproductive process discussed for $\mathcal{T}_{k,m}$ gives rise to a useful property of $\mathcal{D}_{k,0}$ that distinguishes it from scattering diagrams encountered in other contexts, for example [19].

Lemma 7.6. *If $P \in \mathrm{Sing}(\mathfrak{D}(A)_{k,0})$ is a singular point with $P \notin A$ and ξ_p is a small loop around P, then*

$$\theta_{\xi_p,\mathfrak{D}(A)_{k,0}} = \mathrm{Id}.$$

Proof. See [14], Proposition 5.28. □

These automorphisms of $\mathbb{C}[T_\Sigma] \otimes_{\mathbb{C}} \mathcal{R}_{k,0}$ belong to $\mathbb{V}_{\Sigma,k}$, a group originally defined in [25] as a set of Hamiltonian symplectomorphisms (see [14], 5.4.2). Their significance in this context is the preservation the choice of $\Omega := \frac{dx_1 \wedge dx_2}{x_1 x_2}$ referenced in Theorem 7.2 and the oscillatory integrals of $W_{k,0}(A)$ under their action.

Lemma 7.7. *Let $\theta \in \mathbb{V}_{\Sigma,k}$, $(w,h) \in \mathcal{M}_{\Sigma,k} \times \mathbb{C}^\times$ and suppose that f is in the ideal generated by $\{u_{i,j}\}$ in $\mathbb{C}[T_\Sigma] \otimes_{\mathbb{C}} \mathcal{R}_{k,0}$. Then, for any cycle*

$$\Xi \in H_2(\kappa^{-1}(w), \mathrm{Re}(W_{\mathrm{basic}}/\hbar) \ll 0, \mathbb{C}),$$

we have

$$\int_\Xi e^{\frac{W_{\mathrm{basic}}+f}{\hbar}}\Omega = \int_\Xi e^{\frac{\theta(W_{\mathrm{basic}}+f)}{\hbar}}\Omega.$$

Proof. See [14] Lemma 5.40. □

7.1.1. Broken lines. The technique of *broken lines* connects $\mathfrak{D}_{k,0}(A)$ to the potential $W_{k,0}(A)$. Every semirigid disk in $\mathrm{RootDisk}(A, T^0_{0,tr})$ can be uniquely described as a central (infinitely long) stem onto which a number of rigid disks are grafted. At each attachment point, the stem bends in a way dictated by the balancing condition. This is easily understood in terms of the scattering diagram, because the possible points at which any particular tree can be attached (as a rigid disk) to a stem are given by the wall it contributes to $\mathfrak{D}(A)_{k,0}$. Therefore, in order to understand semirigid disks contributing to $W_{k,0}(A)$, it is sufficient to analyze the behavior of these stems (broken lines) with respect to the scattering diagram. We use the following definition, adapted from [13].

Definition 7.8. A *broken line* with basepoint $Q' \in M_\mathbb{R}$ is a continuous proper piecewise linear map

$$\beta : (-\infty, 0] \to M_\mathbb{R}$$

with endpoint $Q' = \beta(0)$, along with some additional data. Let

$$-\infty = s_0 < s_1 < \cdots < s_n = 0$$

be the smallest set of real numbers such that $\beta|_{(s_{i-1},s_i)}$ is linear. Then, for each $1 \le i \le n$, we are given the additional data of a monomial $c_i z^{w_i^\beta} \in \mathbb{C}[T_\Sigma] \otimes_\mathbb{C} R_{k,0}$ with $w_i^\beta \in T_\Sigma \setminus K_\Sigma$, satisfying:

1) For each i, $r(w_i^\beta) = -\beta'(s)$ for $s \in (s_{i-1}, s_i)$.
2) $w_1^\beta = t_i$ for some $0 \le i \le 2$ and $c_1 = 1$.
3) $\beta(s_i) \in \mathrm{Supp}(\mathfrak{D}(A)_{k,0}) \setminus \mathrm{Sing}(\mathfrak{D}(A)_{k,0})$ for $1 \le i \le n$.
4) If $\beta(s_i) \in \mathfrak{d}_1 \cap \cdots \cap \mathfrak{d}_n$, then $c_{i+1} z^{w_{i+1}^\beta}$ is a term in

$$(\theta_{\beta,\mathfrak{d}_1} \circ \cdots \circ \theta_{\beta,\mathfrak{d}_n})(c_i z^{w_i^\beta})$$

More explicitly, suppose that for $1 \le j \le n$, $f_{\mathfrak{d}_j} = 1 + c_{\mathfrak{d}_j} z^{w_{\mathfrak{d}_j}}$, with $c^2_{\mathfrak{d}_j} = 0$, and $\mathbf{n} \in N$ is primitive, orthogonal to all of the $\mathfrak{d}_j$'s, and chosen so that

$$(\theta_{\beta,\mathfrak{d}_1} \circ \cdots \circ \theta_{\beta,\mathfrak{d}_n})(c_i z^{w_i^\beta}) = c_i z^{w_i^\beta} \prod_{j=1}^{n} (1 + c_{\mathfrak{d}_j} z^{w_{\mathfrak{d}_j}})^{\langle \mathbf{n}, r(w_i^\beta)\rangle}$$
$$= c_i z^{w_i^\beta} \prod_{j=1}^{n} (1 + \langle \mathbf{n}, p(w_i^\beta)\rangle c_{\mathfrak{d}_j} z^{m_{\mathfrak{d}_j}}).$$

Then we must have

$$c_{i+1} z^{w_{i+1}^\beta} = \prod_{j \in J} \langle \mathbf{n}, r(w_i^\beta)\rangle c_{\mathfrak{d}_j} z^{m_{\mathfrak{d}_j}}$$

for some $J \subseteq \{1, \ldots, n\}$. We interpret this as β being bent at time s_i by $\mathfrak{d}_j$ for $j \in J$.

Proposition 7.9. *If A is general, there is a one-to-one correspondence between broken lines with endpoint Q and semirigid disks in* $\mathrm{RootDisk}(A, T^0_{0,tr})$. *In addition, if β is a broken line corresponding to a disk $\mathcal{D}$, and cz^w is the monomial associated to the last segment of β, then*

$$cz^w = \mathrm{Mono}(\mathcal{D})$$

Proof. See Proposition 5.32 of [14]. □

7.2. Wall crossing and evaluation of integrals

To evaluate the integral appearing in Theorem 7.2, we must first show that changing the arrangement A transforms $W_{k,0}(A)$ by the action of an element of $\mathbb{V}_{\Sigma,k}$, and will thus leave the integral unchanged. Examining the change in the integral effected by replacing A by $A(Q')$ while moving Q' out to infinity in a particular direction, it becomes clear that the contribution to the integral from terms with certain classes of monomials vanishes. One can thus understand the contribution of these monomials to the integral associated to A by considering the wall crossing automorphisms we encounter as we move Q' back to Q. These automorphisms, and thus the oscillatory integrals, can be interpreted in terms of tropical curves. Using this technique, Theorem 7.2 was proven by Gross in the non-descendent case (in our notation, concerning only tropical objects with $b = (b_1, \ldots, b_n)$ with $b_i \leq 1$) in [14]. The same techniques are modified to treat the descendent case (arbitrary b_i) in [32]. This modification is straightforward in most cases, as the relevant scattering diagrams have identical structure away from the points of A.

7.3. Wall crossing

In order to characterize the way in which $W_{k.0}(A)$ depends on Q, it is essential to understand the interplay between disks, broken lines, and trees at marked points.

Lemma 7.10. *Let $Q' \in M_{\mathbb{R}}$ be very near P_l and, for $1 \leq j \leq n$, let $\mathcal{D}_j \in \mathrm{RootDisk}(A(Q'), b^j, T^0_{0,tr})$ be semirigid disks such that the vectors b^j are pairwise disjoint and $e^l b^j = 0$ for all j . If $\sum_{j=1}^n \mathbf{m}(\mathcal{D}_j) \neq 0$, then the disks $\mathcal{D}_j$ can be joined at P_l to give a rigid tree $\mathcal{T} \in Tree(A, b, T^0_{0,tr})$ with outgoing edge $P_l + \mathbb{R}\sum_{j=1}^n \mathbf{m}(\mathcal{D}_j)$, where $b := ne^l + \sum_{j=1}^n b^j$. Let $M_i \subseteq \{\mathcal{D}_1, \ldots, \mathcal{D}_n\}$ be the set of the original disks that are simply outgoing edges in the direction $\mathbf{m}_i$. Then*

$$\mathrm{Mult}(\mathcal{T}) = \frac{u_{k,n-1}}{|M_0|!|M_1|!|M_2|!} \prod_{i \in \{1,\ldots,n\}} \mathrm{Mult}(\mathcal{D}_i)$$

Proof. Because the disks $\mathcal{D}_j$ are semirigid and both A and Q' are in general position, each belongs in the interior a 2 dimensional cell of the moduli space of disks $\mathrm{Disk}(A(Q'), b^j, T^0_{0,tr})$, allowing one to deform their endpoint from Q' to nearby P_l. After gluing these deformed disks and extending the necessary unbounded edge for balancing, it is easy to see that the resulting tree is rigid. The rest follows from linear algebra. □

Lemma 7.11. *Let $\mathcal{T} \in Tree(A, b, T^0_{0,tr})$ with $b_l = n + 1$. By splitting $\mathcal{T}$ at the vertex V mapping to P_l, we can form n semirigid tropical disks rooted at some $Q' \in M_{\mathbb{R}}$, chosen near P_l.*

Proof. Call the n tropical disks formed by the above procedure $\mathcal{D}_1, \ldots, \mathcal{D}_n$, with $\mathcal{D}_j \in \mathrm{RootDisk}(A(Q'), b^j, T^0_{0,tr})$. As $\mathcal{T}$ is rigid, $F(\mathcal{D}_j) \leq 1$ for each j. Note $F(\mathcal{T}) = |\Delta(\mathcal{T})| - |b| = 0$, $|b| = n + \sum_{j=1}^n |b^j|$ and $|\Delta(\mathcal{T})| = \sum_{j=1}^n |\Delta(\mathcal{D}_j)|$, so

$$\sum_{j=1}^n F(\mathcal{D}_j) = n.$$

Thus $F(\mathcal{D}_j) = 1$ for all $j \in \{1, \ldots, n\}$. □

Theorem 7.12. *If $A(Q)$ and $A(Q')$ are two general arrangements and ξ is a path connecting Q and Q' for which $\theta_{\xi,\mathfrak{D}(A)_{k,0}}$ is defined,*

$$\theta_{\xi,\mathfrak{D}(A)_{k,0}}(W_{k,0}(A(Q))) = W_{k,0}(A(Q')).$$

Proof. As Q is moved, its attached broken lines deform, with their bends moving along the walls of the scattering diagram. Occasionally, one of these bends will coincide with a point $P \in \mathrm{Sing}(\mathfrak{D}(A)_{k,0})$. Lemmas 7.10 and 7.11 control the wall crossing behavior when $P \in A$. Though the proof of this result is rather technical, the central idea is encapsulated in Figure 10.1. We refer the interested reader to §10.1. □

Theorem 7.13. *Let A and A' be two general arrangements.*

$$W_{k,0}(A') = \theta(W_{k,0}(A))$$

for some $\theta \in \mathbb{V}_{\Sigma,k}$.

Proof. This follows from Theorem 7.12 and a relatively straightforward generalization of the techniques of [13], Theorem 4.15. For details, see [32]. □

7.4. Evaluation of Integrals

We now have all the tools necessary for the calculation of the integrals.

Lemma 7.14. *For $\Xi \in H_2(\kappa^{-1}(u), \mathrm{Re}(W_{\mathrm{basic}}/\hbar) \ll 0, \mathbb{C})$, the integral*

$$\int_{\Xi} e^{\frac{W_{k,0}(A)}{\hbar}} \Omega$$

is independent of the choice of general arrangement A.

Proof. Follows from Lemma 7.7 along with Theorems 7.12 and 7.13. □

This allows the key tool we will use for its computation, observing the change in the contribution from specific terms as we move Q.

The following identities reduce the integral to a combinatorial question.

Lemma 7.15. *Restricting to $x_0x_1x_2 = \kappa$, we have*

$$\sum_{i=0}^{2} \alpha^i \int_{\Xi_i} e^{(x_0+x_1+x_2)/\hbar} x_0^{n_0} x_1^{n_1} x_2^{n_2} \Omega = \hbar^{-3\alpha} \kappa^{\alpha} \sum_{i=0}^{2} \Psi_i(n_0, n_1, n_2) \alpha^i$$

where α and Ξ_i are as defined in Theorem 7.2 and

$$\Psi_i(n_0, n_1, n_2) = \sum_{d=0}^{\infty} D_i(d, n_0, n_1, n_2) \hbar^{-(3d-n_0-n_1-n_2)} \kappa^d,$$

where the terms D_i are numerical quantities defined in [13], Lemma 5.2. For $w = n_0t_0 + n_1t_1 + n_2t_2 \in T_\Sigma$, we write $D_i(d, w) := D_i(d, n_0, n_1, n_2)$.

Proof. See [13], Lemma 5.2. □

Definition 7.16. Fix a general arrangement A. For $Q' \in M_\mathbb{R}$, let $S_{k,0}(Q')$ be the finite set of triples (c, ν, w) with $c \in R_{k,0}$, $\nu \geq 0$ an integer, and $w \in T_\Sigma$ such that:

$$e^{(W_{k,0}(A(Q')) - W_{\mathrm{basic}}(A(Q')))/\hbar} = \sum_{(c,\nu,w)\in S_{k,0}(Q')} c\hbar^{-\nu} z^w,$$

with each term $c\hbar^{-\nu} z^w$ of the form $\hbar^{-\nu} \prod_{i=1}^{\nu} \mathrm{Mono}(\mathcal{D}_i)$ for $\mathcal{D}_1, \ldots, \mathcal{D}_\nu$ semi-rigid disks with endpoint Q'.

Then

$$L_i^d(Q') := \sum_{(c,\nu,w)\in S_{k,0}(Q')} c\hbar^{-(3d+\nu-|w|)} D_i(d, w).$$

Lemma 7.17.

$$\sum_{i=0}^{2} \alpha^i \int_{\Xi_i} e^{W_{k,0}(A)/\hbar} = \hbar^{-3\alpha} \kappa^\alpha \sum_{i=0}^{2} \sum_{d\geq 0} L_i^d(Q) \kappa^d \alpha^i$$

Proof. Follows from definitions. □

Definition 7.18. For each cone $\sigma \in \Sigma$, σ is the image under p of a proper face $\tilde{\sigma}$ of the cone $C := T_\Sigma^+ \otimes \mathbb{R}$. For $d \geq 0$, define $C_d \subseteq C$ to be the cube

$$C_d = \left\{ \sum_{i=0}^{2} n_i t_i | 0 \leq n_i \leq d \right\}$$

and for $\sigma \in \Sigma$

$$\tilde{\sigma}_d := (\tilde{\sigma} + C_d) \setminus \bigcup_{\tau \subsetneq \sigma, \tau \in \Sigma} (\tilde{\tau} + C_d).$$

where + denotes the Minkowski sum.

Definition 7.19. For $\sigma \in \Sigma$ and $Q' \in M_\mathbb{R}$, define

$$L_{i,\sigma}^d(Q') := \sum_{(c,\nu,w)\in S_{k,0}(Q'),\, w\in\tilde{\sigma}_d} c\hbar^{-(3d+\nu-|w|)} D_i(d, w).$$

Lemma 7.20. $L_i^d(Q') = \sum_{\sigma\in\Sigma} L_{i,\sigma}^d(Q')$.

Proof. Follows immediately from definitions. □

Lemma 7.21. *Let $\{0\} \neq \sigma \in \Sigma$, and $\mathbf{v} \in \sigma$ be non-zero. Then*

$$\lim_{s\to\infty} L_{i,\sigma}^d(Q + s\mathbf{v}) = 0.$$

Proof. See [14], Lemma 5.51. □

Definition 7.22. Let $\mathfrak{D} = \mathfrak{D}(A)_{k,0}$. Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be connected components of $M_{\mathbb{R}} \setminus \mathfrak{D}$ with $dim(\bar{\mathbf{C}}_1 \cap \bar{\mathbf{C}}_2) = 1$. Pick general points Q_i in $\mathbf{C}_i$, and let ξ be a general path from Q_1 to Q_2 intersecting $\mathrm{Supp}(\mathfrak{D})$ exactly once at $\xi(s_0)$, a nonsingular point of $\mathrm{Supp}(\mathfrak{D})$. Let $\mathfrak{d} \in \mathfrak{D}$ contain $\xi(s_0)$, and let $\mathbf{n}_{\mathfrak{d}}$ be a primitive vector perpendicular to $\mathfrak{d}$ pointing toward Q_1.

Suppose that $f_{\mathfrak{d}} = 1 + c_{\mathfrak{d}} z^{w_{\mathfrak{d}}}$. Select $\alpha, \tau \in \Sigma$ with $dim(\tau) = dim(\alpha) + 1$ and $\alpha \subseteq \tau$. Note that there is a unique index $j(\alpha, \tau) \in \{0, 1, 2\}$ such that $\mathbf{m}_j \in \tau$ but $\mathbf{m}_j \notin \alpha$.

Define

$$L_{i,\mathfrak{d},\xi,\alpha\to\tau}^d := \sum_{(c,\nu,w)} cc_{\mathfrak{d}} \langle \mathbf{n}_{\mathfrak{d}}, \mathbf{m}_{j(\alpha,\tau)}\rangle D_i(d, w + w_{\mathfrak{d}} + t_{j(\alpha,\tau)}) h^{-(\nu+3d-|w+w_{\mathfrak{d}}|)},$$

where we sum over all (c, ν, w) in $S_{k,0}(Q_1)$ satisfying $w + w_{\mathfrak{d}} \in \tilde{\alpha}_d$ but $w + w_{\mathfrak{d}} + t_{j(\alpha,\tau)} \in \tilde{\tau}_d$. If (c, ν, w) satisfies these conditions, then we say that $c\hbar^{-\nu} z^w$ *contributes* to $L_{i,\mathfrak{d},\xi,\alpha\to\tau}^d$. Define

$$L_{i,\xi,\alpha\to\tau}^d := \sum_{\mathfrak{d}} L_{i,\mathfrak{d},\xi,\alpha\to\tau}^d$$

where $\mathfrak{d}$ ranges over all rays of $\mathfrak{D}$ containing $\xi(s_0)$. In order to define this operation for a general path ξ, one can break it up into segments of the type outlined above.

Lemma 7.23. *Let ξ_j be the straight path joining Q with $Q + s\mathbf{m}_j$ for $s \gg 0$. Let $\xi_{j,j+1}$ be the loop based at Q which passes linearly from Q to $Q + s\mathbf{m}_j$, takes a large circular arc to $Q + s\mathbf{m}_{j+1}$, and then proceeds linearly from $Q + s\mathbf{m}_{j+1}$ to Q. Here we take j modulo 3, and $\xi_{j,j+1}$ is always a counterclockwise loop. Then*

$$L_i^d(Q) = L_{i,\{0\}}^d(Q) - \sum_{j=0}^{2} L_{i,\xi_j,\{0\}\to\rho_j}^d - \sum_{j=0}^{2} L_{i,\xi_{j,j+1},\rho_{j+1}\to\sigma_{j,j+1}}^d.$$

Proof. See Lemma 5.15 of [13]. □

Each of the contributions in Lemma 7.23 can then be interpreted in terms of counts of descendent tropical curves. Unfortunately, this process is rather technical, and we refer the interested reader to §10.2 for details. From these identifications, we obtain the following lemma, from which Theorem 7.2 follows directly.

Lemma 7.24.

$$L_0^d(Q) = \delta_{0,d} + \sum_{w\geq 0} \frac{\hbar^{-1}}{w!} \left\langle \frac{S_0(A)}{\hbar - \psi}, \gamma_{a,tr}^w \right\rangle_{0,d}^{\mathrm{trop}} \kappa^d$$

$$L_1^d(Q) = \sum_{w\geq 0} \frac{1}{w!} \left\langle \frac{S_1(A)}{\hbar - \psi}, \gamma_{a,tr}^w \right\rangle_{0,d}^{\mathrm{trop}} \kappa^d$$

$$L_2^d(Q) = \delta_{0,d}\hbar \sum_{j=0}^{k} y_{2,j}(-\hbar)^j + \sum_{w\geq 0} \frac{\hbar}{w!} \left\langle \frac{S_2(A)}{\hbar - \psi}, \gamma_{a,tr}^w \right\rangle_{0,d}^{\mathrm{trop}} \kappa^d.$$

8. Formal operations

It is possible to enhance the previous arguments to directly evaluate the integrals of $W_{k,m}(A)$ for $m > 0$, but it is more convenient to use the axioms of Gromov-Witten theory to assemble it from the results of Theorem 7.2.

Remark 8.1. As the integral is independent of the general arrangement A chosen, we will write $W_{k,m}(A)$ as $W_{k,m}$ in the following.

The pair of operators on $\mathbb{C}[T_\Sigma] \otimes_{\mathbb{C}} R_{k,m}$ defined below are closely related to the *fundamental class axiom* of Gromov-Witten theory and will allow us to calculate oscillatory integrals of $W_{k,m}$ for $m > 0$.

Definition 8.2.

$$\mathrm{op} := \sum_{1\leq j,l\leq k} u_{j,l} \frac{\partial}{\partial u_{j,l-1}}$$

$$\tilde{\mathrm{op}} := \exp(y_{0,0}\,\mathrm{op}) = \sum_{j=0}^{\infty} \frac{y_{0,0}^j}{j!}\,\mathrm{op}^j\,.$$

Lemma 8.3.

$$W_{k,m} = y_{0,0} + \tilde{\mathrm{op}}(W_{k,0})$$

Proof. Let $\mathcal{D}$ be a disk in $\mathrm{RootDisk}(A, b, T^m_{0,tr})$. Two disks are *similar* if they differ by a permutation of the markings on the collapsed edges E_{q_i}. For $1 \le j \le k$, let g_j denote the number of edges marked by elements of $\{q_1, \ldots, q_m\}$ that map to P_j under h. Then there are $\binom{m}{g_1,\ldots,g_k}$ similar disks associated to $\mathcal{D}$ which contribute a total of

$$\frac{y^m_{0,0}}{g_1! \cdots g_k!} \mathrm{Mult}(\mathcal{D}) u_{\mathcal{D}} z^{\Delta(\mathcal{D})} \tag{8.1}$$

to $W_{k,m}$. Define $\mathcal{D}' \in \mathrm{RootDisk}(A, b', T^0_{0,tr})$ to be the result of removing the edges marked by $q_1, \ldots, q_m$ from $\mathcal{D}$ and adjusting the entries of the vector b in the necessary way (removing the edges E_{q_i} reduces the valences of the vertices to which they are attached). This disk contributes

$$\mathrm{Mult}(\mathcal{D}') u_{\mathcal{D}'} z^{\Delta(\mathcal{D}')} = \mathrm{Mult}(\mathcal{D}) u_{\mathcal{D}'} z^{\Delta(\mathcal{D})} \tag{8.2}$$

to $W_{k,0}$. The term $\frac{y^m_{0,0}}{m!} \mathrm{op}^m$ in $\tilde{\mathrm{op}}$ will create summands of the same multi-degree as (8.1) when acting on (8.2), and the contribution of these terms to $\tilde{\mathrm{op}}(W_{k,0})$ is easily seen to equal (8.1). On the other hand, one can associate a set of similar disks to any term appearing in the expansion $\tilde{\mathrm{op}}(W_{k,0})$ by adding marked edges E_{q_i} to the associated disk in $W_{k,0}$. Finally, the term $y_{0,0}$ in the RHS of the lemma corresponds to the semirigid disk consisting of a single q_1-marked edge mapping to Q. □

Lemma 8.4. $e^{\tilde{\mathrm{op}}(W_{k,0})/\hbar} = \tilde{\mathrm{op}}\left(e^{W_{k,0}/\hbar}\right)$

Proof. Set $b \in \mathcal{B}_k$. Let $\mathcal{D}_j \in \mathrm{RootDisk}(A, b^j, T^0_{0,tr})$ for $1 \le j \le \nu$, with b^j pairwise disjoint and b dominating $\sum_{j=1}^{\nu} b^j := b'$. These disks contribute to $W_{k,0}$ and its exponential, and thus to the quantities appearing on either side of the lemma. We will compare their contribution on either side of the desired equality to terms of multi-degree $u_b y_0^{|b-b'|} \hbar^{-\nu}$. On the LHS, this is given by

$$\prod_{j=1}^{\nu} \frac{y^{|b-b^j|}_{0,0}}{|b-b^j|!} \binom{|b-b^j|}{b-b^j} \mathrm{Mono}(\mathcal{D}_j)$$

(recall the definitions given in §3), while on the RHS it is given by

$$\frac{y^{|b-b'|}_{0,0}}{|b-b'|!} \binom{|b-b'|}{b-b'} \prod_{j=1}^{\nu} \mathrm{Mono}(\mathcal{D}_j).$$

Because $\sum_{j=1}^{\nu} b - b^j = b - b'$, the two expressions are equal. All terms appearing on either side of the desired equality result from such choices of sets of disks, and the lemma is proven. $\square$

Together, Lemmas 8.3 and 8.4 yield:

Corollary 8.5. $e^{W_{k,m}/\hbar} = \tilde{\mathrm{op}}\left(e^{(y_{0,0}+W_{k,0})/\hbar}\right)$.

Remark 8.6. If we, by abuse of notation, extend op and $\tilde{\mathrm{op}}$ to their obvious operators on $\mathbb{C}[[y_{1,0}]] \otimes_{\mathbb{C}} R_{k,m}$, their actions commute with the integration of Theorem 7.2.

Definition 8.7. Let

$$\gamma_{b,tr} := T_{0,tr} y_{0,0} + \gamma_{a,tr}$$

be a formal expression as in Definition 7.1.

Corollary 8.8.

$$\sum_{i=0}^{2} \alpha^i \int_{\Xi_i} e^{W_{k,m}/\hbar}\Omega = \hbar^{-3\alpha} \sum_{j=0}^{2} (\alpha\hbar)^j \, e^{y_{1,0}\alpha} \tilde{\Theta}_j$$

where

$$\tilde{\Theta}_0 := e^{y_{0,0}/\hbar} + \sum_{d>0,w\geq 0} \frac{\hbar^{-1}}{w!} \left\langle \frac{S_0(A)}{\hbar-\psi}, \gamma_{b,tr}^w \right\rangle_{0,d}^{\mathrm{trop}} e^{dy_{1,0}}$$

$$\tilde{\Theta}_1 := \sum_{d>0,w\geq 0} \frac{\hbar^{-1}}{w!} \left\langle \frac{S_1(A)}{\hbar-\psi}, \gamma_{b,tr}^w \right\rangle_{0,d}^{\mathrm{trop}} e^{dy_{1,0}}$$

$$\tilde{\Theta}_2 := \hbar^{-1} e^{y_{0,0}/\hbar} \sum_{j=0}^{k} (-\hbar)^j \sum_{l=0}^{m} \frac{y_{0,0}^l}{l!} y_{2,l+j} + \sum_{d>0,w\geq 0} \frac{\hbar^{-1}}{w!} \left\langle \frac{S_2(A)}{\hbar-\psi}, \gamma_{b,tr}^w \right\rangle_{0,d}^{\mathrm{trop}} e^{dy_{1,0}}$$

in $R_{k,m}[[y_1]]$.

Proof. By Corollary 8.5 and Remark 8.6,

$$\sum_{i=0}^{2} \alpha^i \int_{\Xi_i} e^{W_{k,m}/\hbar}\Omega = e^{y_{0,0}/\hbar} \, \tilde{\mathrm{op}} \left(\sum_{i=0}^{2} \alpha^i \int_{\Xi_i} e^{W_{k,0}/\hbar}\Omega \right).$$

Then, by Theorem 7.2,

$$\begin{aligned}
\tilde{\Theta}_0 &:= e^{y_{0,0}/\hbar} + \sum_{d>0, w\geq 0} e^{y_{0,0}/\hbar}\,\tilde{\text{op}}\left(\frac{\hbar^{-1}}{w!}\left\langle \frac{S_0(A)}{\hbar-\psi}, \gamma^w_{a,tr}\right\rangle^{\text{trop}}_{0,d} e^{dy_{1,0}}\right)\\
\tilde{\Theta}_1 &:= \sum_{d>0, w\geq 0} e^{y_{0,0}/\hbar}\,\tilde{\text{op}}\left(\frac{\hbar^{-1}}{w!}\left\langle \frac{S_1(A)}{\hbar-\psi}, \gamma^w_{a,tr}\right\rangle^{\text{trop}}_{0,d} e^{dy_{1,0}}\right)\\
\tilde{\Theta}_2 &:= e^{y_{0,0}/\hbar}\hbar^{-1}\,\tilde{\text{op}}\left(\sum_{j=0}^{k} y_{2,j}(-\hbar)^j\right)\\
&\quad + \sum_{d>0, w\geq 0} e^{y_{0,0}/\hbar}\,\tilde{\text{op}}\left(\frac{\hbar^{-1}}{w!}\left\langle \frac{S_2(A)}{\hbar-\psi}, \gamma^w_{a,tr}\right\rangle^{\text{trop}}_{0,d} e^{dy_{1,0}}\right).
\end{aligned}$$

Select $d, \nu \in \mathbb{Z}_{>0}$, $b \in \mathcal{B}_k$ with $n := \#(b)$, and $l \in \mathbb{Z}_{\geq 0}$. We wish to find the coefficient of $\frac{y_{0,0}^l}{l!} u_b e^{dy_{1,0}} \hbar^{-(\nu+2)}$ in

$$e^{y_{0,0}/\hbar}\,\tilde{\text{op}}\left(\sum_{d>0, w\geq 0}\frac{\hbar^{-1}}{w!}\left\langle \frac{S_0(A)}{\hbar-\psi}, \gamma^w_{a,tr}\right\rangle^{\text{trop}}_{0,d} e^{dy_{1,0}}\right).$$

This can be seen to be equal to

$$\begin{aligned}
&\sum_{i=0}^{\min(l,\nu)} \frac{l!}{i!} \sum_{\substack{b' \prec b\\ |b-b'|=l-i}} \frac{1}{|b-b'|!}\binom{|b-b'|}{b-b'} \langle \psi^{b'(1)-1}P_{b\{1\}}, \ldots, \psi^{b'(n)-1}P_{b\{n\}}, \psi^{\nu-i}S_0(A)\rangle^{\text{trop}}_{0,d}\\
&= \sum_{i+w_1+\cdots+w_n=l} \binom{l}{i, w_1, \ldots, w_n} \langle \psi^{b(1)-w_1-1}P_{b\{1\}}, \ldots, \psi^{b(n)-w_n-1}P_{b\{n\}}, \psi^{\nu-i}S_0(A)\rangle^{\text{trop}}_{0,d},
\end{aligned}$$

where the above invariants are interpreted as zero if they contain any negative powers of ψ. The first line is a count of the ways in which the various terms of $\sum_{d>0,w\geq 0}\frac{\hbar^{-1}}{w!}\langle\frac{S_0(A)}{\hbar-\psi}, \gamma^w_{a,tr}\rangle^{\text{trop}}_{0,d} e^{dy_{1,0}}$ can be upgraded under the action of $e^{y_{0,0}/\hbar}\,\tilde{\text{op}}$ to have the desired coefficient; the multinomial factor results from the coefficient of the relevant summand of $\tilde{\text{op}}$. An iteration of the tropical fundamental class axiom (Lemma 5.7) shows the equality of the above expression and

$$\langle \psi^{b(1)-1}P_{b\{1\}}, \ldots, \psi^{b(n)-1}P_{b\{n\}}, T^l_{0,tr}, \psi^{\nu}S_0(A)\rangle^{\text{trop}}_{0,d}.$$

Of course, the same result holds when replacing $S_0(A)$ with $S_1(A)$ or $S_2(A)$. $\square$

Next, we normalize the integral from the above corollary to satisfy the conditions of §1 of [13], allowing us to apply mirror symmetry.

Lemma 8.9. *Let* $\Xi \in H_2(\kappa^{-1}(u), \mathrm{Re}(W_{\text{basic}}/\hbar) \ll 0, \mathbb{C})$. *Then*

$$\int_\Xi e^{W_{k,m}/\hbar}(\mathrm{op}(W_{k,m}))\Omega = \hbar\, \mathrm{op}\left(\int_\Xi e^{W_{k,m}/\hbar}\Omega\right)$$

in $\mathbb{C}[T_\Sigma] \otimes_{\mathbb{C}} R_{k,m}$.

Proof.

$$\begin{aligned}\int_\Xi e^{W_{k,m}/\hbar}(\mathrm{op}(W_{k,m}))\Omega &= \hbar \int_\Xi \mathrm{op}(e^{W_{k,m}/\hbar})\Omega \\ &= \hbar\, \mathrm{op}\left(\int_\Xi e^{W_{k,m}/\hbar}\Omega\right).\end{aligned}$$

□

Combining Lemma 8.9, Corollary 8.8, and the tropical fundamental class axiom (Lemma 5.7), we achieve the following result.

Corollary 8.10. *Let* $f := 1 + \mathrm{op}(W_{k,m})$. *Then*

$$\sum_{i=0}^{2} \alpha^i \int_{\Xi_i} e^{W_{k,m}/\hbar} f\Omega = \hbar^{-3\alpha} \sum_{j=0}^{2} (\alpha\hbar)^j\, e^{y_{1,0}\alpha} L_j,$$

where

$$\begin{aligned}
L_0 &:= e^{y_{0,0}/\hbar} + \sum_{d>0, w\geq 0} \frac{1}{w!} \left\langle \frac{S_0(A)}{\hbar - \psi}, T_{0,tr}, \gamma_{b,tr}^{w} \right\rangle_{0,d}^{\text{trop}} e^{dy_{1,0}} \\
L_1 &:= \sum_{d>0, w\geq 0} \frac{1}{w!} \left\langle \frac{S_1(A)}{\hbar - \psi}, T_{0,tr}, \gamma_{b,tr}^{w} \right\rangle_{0,d}^{\text{trop}} e^{dy_{1,0}} \\
L_2 &:= \hbar^{-1} e^{y_{0,0}/\hbar} \sum_{l=0}^{m} \frac{y_{0,0}^l}{l!} y_{2,l} + \sum_{d>0, w\geq 0} \frac{1}{w!} \left\langle \frac{S_2(A)}{\hbar - \psi}, T_{0,tr}, \gamma_{b,tr}^{w} \right\rangle_{0,d}^{\text{trop}} e^{dy_{1,0}}.
\end{aligned}$$

If we define ϕ_i *by rewriting*

$$\hbar^{-3\alpha} \sum_{j=0}^{2} (\alpha\hbar)^j\, e^{y_{1,0}\alpha} L_j = \hbar^{-3\alpha} \sum_{j=0}^{2} (\alpha\hbar)^j\, \phi_j,$$

we see that

$$\begin{aligned}\phi_0 &:= L_0\\ \phi_1 &:= y_{1,0}\hbar^{-1}L_0 + L_1\\ \phi_2 &:= \frac{y_{1,0}^2\hbar^{-2}}{2}L_0 + y_{1,0}\hbar^{-1}L_1 + L_2.\end{aligned}$$

If we write $\phi_i := \sum_{j=0}^{\infty}\hbar^{-j}\phi_{i,j}$ *with* $\phi_{i,j} \in \mathbb{C}[[y_{1,0}]] \otimes_{\mathbb{C}} R_{k,m}$,

$$\begin{aligned}\phi_{i,0} &= \delta_{i,0}\\ \phi_{0,1} &= y_{0,0} + \tilde{K}_2\\ \phi_{1,1} &= y_{1,0} + \tilde{K}_1\\ \phi_{2,1} &= \sum_{l=0}^{m}\frac{y_{0,0}^l}{l!}y_{2,l} + \tilde{K}_0,\end{aligned}$$

where

$$\tilde{K}_i := \sum_{d>0,w\geq 0}\frac{1}{m!}\langle S_{2-i}(A), T_{0,tr}, \gamma_{b,tr}^w\rangle_{0,d}^{\text{trop}} e^{dy_{1,0}}.$$

Proof. This follows from some bookkeeping. The tropical fundamental class axiom yields the insertion of $T_{0,tr}$ that distinguishes the sum over $d > 0, w \geq 0$ in L_i from that appearing in $\tilde{\Theta}_i$, while the simplification in the first term of L_2 results from a cancellation of terms of opposite sign. The description of ϕ_i in terms of L_j follows easily from its definition, while the final set of identities results from a check of the maximal powers of $\hbar$ appearing in each term. □

Definition 8.11. By Theorem 7.2, the expressions above are independent of the choice of arrangement A. Thus, we can simply write

$$\langle\psi^{a_1}T_{2,tr}\ldots,\psi^{a_n}T_{2,tr}, T_{0,tr}^m, \psi^\nu T_{2-i}\rangle_{0,d}^{\text{trop}}$$

in place of

$$\langle\psi^{a_1}P_{r\{1\}}\ldots,\psi^{a_n}P_{r\{n\}}, T_{0,tr}^m, \psi^\nu S_i(A)\rangle_{0,d}^{\text{trop}}.$$

With this observation, we write

$$\gamma_{b,tr} := T_{0,tr}y_{0,0} + T_{2,tr}y_{2,0} + \psi T_{2,tr}y_{2,1} + \cdots + \psi^{k-1}T_{2,tr}y_{2,k-1}.$$

Markwig and Rau have shown that the tropical invariants appearing in the definition of $\tilde{K}_i$ are equal to certain classical Gromov-Witten invariants of $\mathbb{P}^2$. In particular

$$\langle \psi^{a_1} T_{2,tr} \ldots, \psi^{a_n} T_{2,tr}, T_{0,tr}^m, S_i(A) \rangle_{0,d}^{\text{trop}} = \langle \psi^{a_1} T_2 \ldots, \psi^{a_n} T_2, T_0^m, T_{2-i} \rangle_{0,d[l]},$$

where T_i is a positive generator of $H^{2i}(\mathbb{P}^2, \mathbb{Z})$ and $[l] \in H_2(\mathbb{P}^2, \mathbb{Z})$ the class of a line.

Definition 8.12. Let

$$\gamma_{b,cl} := T_0 y_{0,0} + T_2 y_{2,0} + \psi T_2 y_{2,1} + \cdots + \psi^{k-1} T_2 y_{2,k-1}$$

and

$$\gamma_{c,cl} := T_0 y_{0,0} + T_1 y_{1,0} + T_2 y_{2,0} + \psi T_2 y_{2,1} + \cdots + \psi^{k-1} T_2 y_{2,k-1}$$

be expressions for insertion into classical Gromo-Witten invariants.

This notation, along with the tropical correspondence result mentioned above, allows us to compactly express the generating functions $\tilde{K}_i$ as a .

Lemma 8.13.

$$\tilde{K}_i = \sum_{d>0, w\geq 0} \frac{1}{w!} \langle T_i, T_0, \gamma_{c,cl}^w \rangle_{0,d}$$

Proof.

$$\begin{aligned}
\tilde{K}_i &= \sum_{d>0, w\geq 0} \frac{1}{w!} \langle T_{i,tr}, T_{0,tr}, \gamma_{b,tr}^w \rangle_{0,d}^{\text{trop}} e^{dy_{1,0}} \\
&= \sum_{d>0, w\geq 0} \frac{1}{w!} \langle T_i, T_0, \gamma_{b,cl}^w \rangle_{0,d} e^{dy_{1,0}} \\
&= \sum_{d>0, w\geq 0} \frac{1}{w!} \langle T_i, T_0, \gamma_{c,cl}^w \rangle_{0,d},
\end{aligned}$$

The first equality follows from the observation above, while the second follows from the divisor axiom (see [6], Chapter 10). $\square$

Definition 8.14. Let

$$K_i := \sum_{d,w\geq 0} \frac{1}{w!}\langle T_i, T_0, \gamma_{c,cl}^w\rangle_{0,d}$$

(we now include degree 0 invariants).

Lemma 8.15. $\phi_{i,1} = K_{2-i}$

Proof. This follows from the fundamental class and point mapping axioms of Gromov-Witten theory when applied to the degree 0 ($d = 0$) pieces of K_i. For example,

$$\begin{aligned}\deg_0(K_2) &= \sum_{w\geq 0} \frac{1}{w!}\langle T_2, T_0, \gamma_{c,cl}^w\rangle_{0,0}\\ &= \langle T_2, T_0, \gamma_{c,cl}\rangle_{0,0}\\ &= \langle T_2, T_0, y_{0,0}T_0\rangle_{0,0}\\ &= y_{0,0}.\end{aligned}$$

The case of K_0 is slightly more interesting:

$$\begin{aligned}\deg_0(K_0) &= \sum_{w\geq 0} \frac{1}{w!}\langle T_0, T_0, \gamma_{c,cl}^w\rangle_{0,0}\\ &= \sum_{l\geq 1}\langle T_0, T_0^{l+1}, \psi^l T_2\rangle_{0,0}\frac{y_0^l y_{2,l-1}}{l!}\\ &= \sum_{l=0}^{m} \frac{y_{0,0}^l}{l!} y_{2,l}.\end{aligned}$$

The descendent point mapping axiom (see Chapter 10 of [6]) ensures that the only nonzero, degree-0 invariant appearing with at most 3 entries is $\langle T_0, T_0, T_2\rangle_{0,0} = 1$. The divisor axiom demonstrates that all invariants containing entries of T_1 are 0. A dimensional argument shows that all of the remaining invariants can be reduced, through the fundamental class axiom, to non-descendent invariants. By the classical point mapping axiom, all such invariants evaluate to zero except for the cases appearing in the above sum. □

9. Mirror symmetry

We will now use mirror symmetry to relate the integral to Givental's J-function, and thus classical Gromov-Witten theory. In contrast to the case explored by Gross in [14], mirror symmetry does not immediately yield an equality between the descendent tropical invariants appearing in the integrals and classical Gromov-Witten invariants for $\mathbb{P}^2$. Indeed, we will see that such a relationship would be impossible as our period integrals calculate a larger class of Gromov-Witten invariants than appear in the J-function.

9.1. J-function

We begin by rewriting J in a more convenient format.

Definition 9.1. Consider Givental's J function as an element

$$J_{\mathbb{P}^2} \in \mathbb{C}[[\tilde{y_0}, \tilde{y_1}, \tilde{y_2}, \hbar^{-1}]] \otimes H^*(\mathbb{P}^2, \mathbb{Z}),$$

defined as in [22], up to some minor rearrangement, as

$$\begin{aligned} J_{\mathbb{P}^2} = e^{(T_0\tilde{y_0}+T_1\tilde{y_1})/\hbar} \cup \Bigg(& T_0 + \tilde{y_2} T_2 \\ & + \sum_{i=0}^{2} \Bigg(\sum_{d\geq 1, \nu \geq 0} \langle T_2^{3d+i-2-\nu}, \psi^\nu T_{2-i}\rangle_{0,d} \hbar^{-(\nu+2)} e^{d\tilde{y_1}} \frac{\tilde{y_2}^{3d+i-2-\nu}}{(3d+i-2-\nu)!} \Bigg) T_i \Bigg). \end{aligned}$$

Define J_i to be the T_i component of $J_{\mathbb{P}^2}$.

Lemma 9.2. *Let $\gamma := T_0\tilde{y_0} + T_1\tilde{y_1} + T_2\tilde{y_2}$. Then*

$$J_{\mathbb{P}^2} = T_0 + \sum_{w,d=0}^{\infty} \sum_{i=0}^{2} \frac{1}{w!} \left\langle \frac{T_{2-i}}{\hbar - \psi}, T_0, \gamma^w \right\rangle_{0,d} T_i.$$

Proof. By the proof of Proposition 2.23 of [14], J_i is equal to the coefficient of T_2 in $T_i + \sum_{w,d=0}^{\infty} \sum_{j=0}^{2} \frac{1}{w!} \left\langle \frac{T_i}{\hbar-\psi}, T_j, \gamma^w \right\rangle_{0,d} T_{2-j}$. Therefore,

$$J_0 = 1 + \sum_{w,d=0}^{\infty} \sum_{j=0}^{2} \frac{1}{w!} \left\langle \frac{T_2}{\hbar - \psi}, T_0, \gamma^w \right\rangle_{0,d}$$
$$J_1 = \sum_{w,d=0}^{\infty} \sum_{j=0}^{2} \frac{1}{w!} \left\langle \frac{T_1}{\hbar - \psi}, T_0, \gamma^w \right\rangle_{0,d}$$
$$J_2 = \sum_{w,d=0}^{\infty} \sum_{j=0}^{2} \frac{1}{w!} \left\langle \frac{T_0}{\hbar - \psi}, T_0, \gamma^w \right\rangle_{0,d}$$
□

9.2. Mirror map

In this section, we use Barannikov mirror construction [1] (as applied to $\mathbb{P}^2$ in §1 of [13]) to identify flat coordinates on the descendent perturbation space of W. To simplify our expressions, we begin by passing to a limit of our coordinate ring and allow arbitrary powers of y_0.

Definition 9.3.

$$R_k := R'_k[[y_{0,0}]] = \varprojlim R_{k,m}$$

Definition 9.4. Let

$$\Phi : \mathbb{C}[[\tilde{y_0}, \tilde{y_1}, \tilde{y_2}, \hbar^{-1}]] \otimes H^*(\mathbb{P}^2, \mathbb{Z}) \to R_k[[y_{1,0}, \hbar^{-1}]] \otimes H^*(\mathbb{P}^2, \mathbb{Z})$$

be induced by $\tilde{y}_i \mapsto K_{2-i}$ for $0 \leq i \leq 2$.

Further, let

$$\mathbb{T}_{\text{trop}} := \sum_{i=0}^{2} \phi_i T_i$$

and

$$\mathbb{J} := \Phi(J_{\mathbb{P}^2}).$$

Theorem 9.5. *Let $\mathfrak{M}_{\Sigma,k}$ be the formal spectrum of the completion of $\mathbb{C}[K_\Sigma] \otimes_{\mathbb{C}} R_k$ at the maximal ideal $(y_{0,0}, \kappa - 1, \{u_{i,j}\}_{i,j})$. The completion is isomorphic to $\mathbb{C}[[y_{1,0}]] \otimes_{\mathbb{C}} R_k$ with $y_{1,0} := \log \kappa$, the latter expanded in* *a power series at $\kappa = 1$. Let*

$$\check{\mathfrak{X}}_{\Sigma,k} = \check{\mathcal{X}}_{\Sigma,k} \times_{\mathcal{M}_{\Sigma,k}} \mathfrak{M}_{\Sigma,k}.$$

The function $W_{k,m}$ is regular (for all m) on $\check{\mathfrak{X}}_{\Sigma,k}$ and restricts to $W_{\text{basic}} = x_0 + x_1 + x_2$ on the closed fiber of $\check{\mathfrak{X}}_{\Sigma,k} \to \mathfrak{M}_{\Sigma,k}$ and hence gives a deformation of this function over $\mathfrak{M}_{\Sigma,k}$. Thus we have a morphism ω from $\mathfrak{M}_{\Sigma,k}$ to the universal unfolding space $\mathcal{M} := \operatorname{Spec} \mathbb{C}[[\tilde{y_0}, \tilde{y_1}, \tilde{y_2}]]$. This map is given by:

$$\begin{aligned} \tilde{y_0} &\mapsto K_2 \\ \tilde{y_1} &\mapsto K_1 \\ \tilde{y_2} &\mapsto K_0. \end{aligned}$$

The morphism ω induces the map Φ defined above, and

$$\mathbb{T}_{\text{trop}} = \mathbb{J}.$$

Proof. Follows from the application of Corollary 8.10 and Lemma 8.15 to the obvious extension of Corollary 8.8 to the setting of $R_k[[y_{1,0}, \hbar^{-1}]]$, as our data satisfies the conditions set out in §1 of [13]. See Corollary 3.9 of ibid. for more details. □

9.3. Extension to formal series

For convenience, we again take an inverse limit of our results and definitions to remove our finite bounds on degree.

Definition 9.6. Let

$$R := \lim_{\leftarrow k} R_k$$

and

$$\tilde{R} := \mathbb{C}[[\hbar^{-1}, y_{0,0}, y_{1,0}, y_{2,0}, y_{2,1}, \ldots]].$$

Further, define W_{desc} to be the inverse limit of $W_{k,m}(A_k)$ with respect to k and m, where $A_k := (Q, P_1, \ldots, P_k) \subset (Q, P_1, \ldots,) := A_\infty$ for some general A_∞.

Remark 9.7. It's easy to see that the above results have immediate generalizations in which R_k has been replaced by R and $W_{k,m}$ by W_{desc}. Furthermore, note the natural inclusion of $\tilde{R}$ into $R[[y_{1,0}, \hbar^{-1}]]$ given by

$$y_{2,i} \mapsto \sum_j u_{j,i}.$$

Because the period integrals are symmetric with respect to point labelings, the limits of $\mathbb{T}_{\mathrm{trop}}$ and $\mathbb{J}$ in $R[[y_{1,0}, \hbar^{-1}]]$ are in the image of this inclusion. In the following, we restrict to this setting.

Corollary 9.8. *Defining $\gamma_J = T_0K_2 + T_1K_1 + T_2K_0$, we can write*

$$\mathbb{J} = T_0 + \sum_{i=0}^{2} \sum_{d,w \geq 0} \frac{1}{w!} \left\langle \frac{T_{2-i}}{\hbar - \psi}, T_0, \gamma_J^w \right\rangle_{0,d} T_i,$$

a generating function whose coefficients can be written entirely in terms of the classical Gromov-Witten invariants of $\mathbb{P}^2$.

Remark 9.9. If we define the stationary sector $\mathcal{M}_{\mathrm{sta}} \subset \mathcal{M}_\infty$ of the big phase space (see §2) to be the subspace defined by the vanishing of $y_{i,j}$ for $i \in \{0,1\}$ and $j > 0$, there is a natural surjective map from $\mathcal{M}_{\mathrm{desc}} \to \mathcal{M}_\infty$ for which the integrals of W_{desc} descend. Under this identification, the above is a restriction to $\mathcal{M}_{\mathrm{sta}}$ of the lift of Dijkgraaf and Witten [8] (as explored in [7]) of the J-function from the small to the big phase space.

9.4. Tropical-classical correspondence

Definition 9.10. Let $\mathbb{T}$ be the classical analogue of $\mathbb{T}_{\mathrm{trop}}$, where each tropical Gromov-Witten invariant is replaced by its classical counterpart, and define $\mathbb{T}_i$ by the equality $\mathbb{T} = \sum_{i=0}^{2} \mathbb{T}_i T_i$,

We will use induction to show that $\mathbb{T} = \mathbb{J}$, thus implying $\mathbb{T}_{\mathrm{trop}} = \mathbb{T}$. As $\mathbb{T}$ is classical, we can use the axioms of Gromov-Witten theory to rewrite it in a convenient form. We do so below.

Lemma 9.11.

$$\mathbb{T} = T_0 + \sum_{i=0}^{2} \sum_{d,w \geq 0} \frac{1}{w!} \left\langle \frac{T_{2-i}}{\hbar - \psi}, T_0, \gamma_{c,cl}^w \right\rangle_{0,d} T_i.$$

Proof. Observe the following simplification of $\mathbb{T}_0$:

$$
\begin{aligned}
\mathbb{T}_0 &= e^{y_{0,0}/\hbar} + \sum_{d>0,w\geq 0} \frac{1}{w!} \left\langle \frac{T_2}{\hbar-\psi}, T_0, \gamma_{b,cl}^{w} \right\rangle_{0,d} e^{dy_{1,0}} \\
&= T_0 + \sum_{v\geq 1} \langle \psi^{v-1} T_2, T_0, T_0^v \rangle_{0,0} \frac{y_{0,0}^v}{v!\hbar^v} \\
&\quad + \sum_{d>0,w\geq 0} \frac{1}{w!} \left\langle \frac{T_2}{\hbar-\psi}, T_0, \gamma_{b,cl}^{w} \right\rangle_{0,d} e^{dy_{1,0}} \\
&= T_0 + \sum_{d,w\geq 0} \frac{1}{w!} \left\langle \frac{T_2}{\hbar-\psi}, T_0, \gamma_{b,cl}^{w} \right\rangle_{0,d} e^{dy_{1,0}} \\
&= T_0 + \sum_{d,w\geq 0} \frac{1}{w!} \left\langle \frac{T_2}{\hbar-\psi}, T_0, \gamma_{c,cl}^{w} \right\rangle_{0,d}
\end{aligned}
$$

The second equality is by the fundamental class and point mapping axioms, while the last follows from the the divisor axiom. A parallel analysis (similar to that found in Lemma 9.2) can be applied to the other components. $\square$

The following operators are closely related to the dilaton axiom, and will allow us to decompose $\mathbb{J}$ and $\mathbb{T}$ to make way for induction.

Definition 9.12.

$$
\begin{aligned}
\mathrm{diff} &:= y_{0,0}\frac{\partial}{\partial y_{0,0}} + y_{1,0}\frac{\partial}{\partial y_{1,0}} + \sum_{i\geq 0} y_{2,i}\frac{\partial}{\partial y_{2,i}} \\
\mathrm{diff}_{>} &:= \sum_{i>0} y_{2,i}\frac{\partial}{\partial y_{2,i}}.
\end{aligned}
$$

Lemma 9.13.

$$
\sum_{j=0}^{2} \left(\frac{\partial}{\partial y_{j,0}} \mathbb{T} \right) K_{2-j} = \mathrm{diff}(\mathbb{T}).
$$

Proof.

$$
\begin{aligned}
&\sum_{j=0}^{2}\left(\frac{\partial}{\partial y_{j,0}}\mathbb{T}\right)K_{2-j}\\
&=\sum_{j,i=0}^{2}\left(\sum_{d,w\geq 0}\frac{1}{w!}\left\langle\frac{T_{2-i}}{\hbar-\psi},T_0,\gamma_{c,cl}^{w},T_j\right\rangle_{0,d}\sum_{d',w'\geq 0}\frac{1}{w'!}\left\langle T_{2-j},T_0,\gamma_{c,cl}^{w'}\right\rangle_{0,d'}\right)T_i\\
&=\sum_{i=0}^{2}\sum_{d,w\geq 0}\frac{1}{w!}\left\langle\frac{T_{2-i}}{\hbar-\psi},T_0,\gamma_{c,cl}^{w},\psi T_0\right\rangle_{0,d}T_i\\
&=\sum_{i=0}^{2}\mathrm{diff}\left(\sum_{d,w\geq 0}\frac{1}{w!}\left\langle\frac{T_{2-i}}{\hbar-\psi},T_0,\gamma_{c,cl}^{w}\right\rangle_{0,d}T_i\right)\\
&=\mathrm{diff}(\mathbb{T}).
\end{aligned}
$$

The second equality is due to the topological recursion relation (see [6]) , while the third is due to the dilaton axiom. □

Lemma 9.14.

$$\sum_{j=0}^{2}\left(\frac{\partial}{\partial y_{j,0}}\mathbb{J}\right)K_{2-j}=\mathrm{diff}(\mathbb{J}).$$

Proof.

$$
\begin{aligned}
&\sum_{j=0}^{2}\left(\frac{\partial}{\partial y_{j,0}}\mathbb{J}\right)K_{2-j}\\
&=\sum_{j,i=0}^{2}\frac{\partial}{\partial y_{j,0}}\left(\sum_{w,d=0}^{\infty}\frac{1}{w!}\left\langle\frac{T_{2-i}}{\hbar-\psi},T_0,\gamma_J^{w}\right\rangle_{0,d}\right)K_{2-j}T_i\\
&=\sum_{i,j,l=0}^{2}\sum_{w\geq 1,d\geq 0}\frac{1}{(w-1)!}\left\langle\frac{T_{2-i}}{\hbar-\psi},T_0,\gamma_J^{w-1},T_{2-l}\right\rangle_{0,d}\left(\frac{\partial}{\partial y_{j,0}}K_l\right)K_{2-j}T_i.
\end{aligned}
$$

For $0 \leq l \leq 2$,

$$\begin{aligned}
&\sum_{j=0}^{2}\left(\frac{\partial}{\partial y_{j,0}}K_l\right)K_{2-j}\\
&=\sum_{d,w\geq 0}\frac{1}{w!}\langle T_l, T_0, \gamma_{c,cl}^{w}, T_j\rangle_{0,d}\sum_{d',w'\geq 0}\frac{1}{w'!}\langle T_{2-j}, T_0, \gamma_{c,cl}^{w'}\rangle_{0,d'}\\
&=\sum_{d,w\geq 0}\frac{1}{w!}\langle T_l, T_0, \gamma_{c,cl}^{w}, \psi T_0\rangle_{0,d}\\
&=\mathrm{diff}(K_l),
\end{aligned}$$

where the above equalities follow from the reasoning used in the previous lemma. So we have

$$\begin{aligned}
&\sum_{j=0}^{2}\left(\frac{\partial}{\partial y_{j,0}}\mathbb{J}\right)K_{2-j}\\
&=\sum_{i,l=0}^{2}\sum_{w\geq 1, d\geq 0}^{\infty}\frac{1}{(w-1)!}\left\langle \frac{T_{2-i}}{\hbar-\psi}, T_0, \gamma_J^{w-1}, T_{2-l}\right\rangle_{0,d}\mathrm{diff}\left(K_l\right)T_i\\
&=\mathrm{diff}\left(\sum_{i=0}^{2}\sum_{w,d\geq 0}^{\infty}\frac{1}{w!}\left\langle\frac{T_{2-i}}{\hbar-\psi}, T_0, \gamma_J^{w}\right\rangle_{0,d}T_i\right).
\end{aligned}$$

□

We can now use induction to show $\mathbb{T} = \mathbb{J}$.

Definition 9.15. Define a $\mathbb{Z}\left[\frac{1}{3}\right]$ grading on the monomials of $\tilde{R}\otimes H^*(\mathbb{P}^2,\mathbb{Z})$ by

$$\mathrm{gr}(y_{0,0}^{j}y_{1,0}^{l}\hbar^{-\nu}\prod_m y_{2,m}^{a_m}T_i) := \frac{1}{3}\left(\nu - j - i + \sum_m a_m(m+1)\right) + \sum_{m>0} a_m.$$

Note that diff and $\mathrm{diff}_>$ preserve the gr-grading of monomials not sent to 0.

When applied to a summand of $\mathbb{T}$,

$$\mathrm{gr}\left(\langle T_0, T_0^{r_{0,0}}, T_1^{r_{1,0}}, T_2^{r_{2,0}}, \ldots, (\psi^{k-1}T_2)^{r_{2,k-1}}, \psi^{\nu}T_{2-i}\rangle_{0,d}\hbar^{-(\nu+1)}T_i\prod_{a,b}\frac{y_{a,b}^{r_{a,b}}}{r_{a,b}!}\right)$$

$= d +$ the number of insertions with positive exponent on ψ, excluding the term whose power of ψ is recorded by the exponent of $\hbar$.

The implied integrality results from the dimensional requirements of nonzero invariants. Defining $\hat{K}_i := K_i T_0$, we examine the value of gr on a typical term.

$$\operatorname{gr}\left(\langle T_0, T_0^{r_{0,0}}, T_1^{r_{1,0}}, T_2^{r_{2,0}}, (\psi^1 T_2)^{r_{2,1}}, \ldots, (\psi^{k-1} T_2)^{r_{2,k-1}}, T_i\rangle_{0,d} T_0 \prod_{a,b} \frac{y_{a,b}^{r_{a,b}}}{r_{a,b}!}\right)$$
$$= d + \frac{1-i}{3} + \text{ the number of insertions with positive exponent on } \psi.$$

When applied to $\mathbb{J}$, gr admits a similar description. The coefficient of the monomial of a particular degree is a sum of products of Gromov-Witten invariants. When expressed in this form, the grading can be recovered from any summand of the coefficient as the sum of the degrees of the invariants in the product with the count of the total number of insertions with non-trivial ψ-classes, again excluding the term whose exponent of ψ is recorded by the power of $\hbar$.

Theorem 9.16. $\mathbb{T} = \mathbb{T}_{\text{trop}}$, *and thus the tropical descendent Gromov-Witten invariants of Definition 5.5 are equal to their intended classical counterparts.*

Proof. For $j \in \mathbb{Z}\left[\frac{1}{3}\right]$, define $\mathbb{J}_{[j]}$ and $\mathbb{T}_{[j]}$ to be the gr-degree j monomials of $\mathbb{J}$ and $\mathbb{K}$, respectively. Note that gr is integral and non-negative for all non-zero terms in $\mathbb{J}$ and $\mathbb{T}$. The case $\mathbb{J}_{[0]} = \mathbb{T}_{[0]}$ follows from the point mapping axiom. Let $n > 0 \in \mathbb{Z}$, and assume $\mathbb{J}_{[j]} = \mathbb{T}_{[j]}$ for all $j < n$.

We analyze the gr-degree n part of diff$(\mathbb{T})$ using Lemma 9.13 (recall $\hat{K}_i = K_i T_0$).

$$\begin{aligned}
\operatorname{diff}(\mathbb{T})_{[n]} &= \left(\sum_{j=0}^{2}\left(\frac{\partial}{\partial y_{j,0}}\mathbb{T}\right)\hat{K}_{2-j}\right)_{[n]} \\
&= \sum_{w=0}^{n}\sum_{j=0}^{2}\left(\frac{\partial}{\partial y_{j,0}}\mathbb{T}\right)_{[w+\frac{1-j}{3}]}\left(\hat{K}_{2-j}\right)_{[n-w-\frac{1-j}{3}]} \qquad (9.1)\\
&= \sum_{w=0}^{n}\sum_{j=0}^{2}\left(\frac{\partial}{\partial y_{j,0}}\mathbb{T}_{[w]}\right)\left(\hat{K}_{2-j}\right)_{[n-w-\frac{1-j}{3}]}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{2} \left(\sum_{w=0}^{n-1} \left(\frac{\partial}{\partial y_{j,0}} \mathbb{T}_{[w]} \right) \left(\hat{K}_{2-j} \right)_{[n-w-\frac{1-j}{3}]} \right. \\
&\qquad \left. + \left(\frac{\partial}{\partial y_{j,0}} \mathbb{T}_{[n]} \right) \left(\hat{K}_{2-j} \right)_{[-\frac{1-j}{3}]} \right) \\
&= \sum_{j=0}^{2} \left(\sum_{w=0}^{n-1} \left(\frac{\partial}{\partial y_{j,0}} \mathbb{J}_{[w]} \right) \left(\hat{K}_{2-j} \right)_{[n-w-\frac{1-j}{3}]} + \left(\frac{\partial}{\partial y_{j,0}} \mathbb{T}_{[n]} \right) T_0 y_{j,0} \right)
\end{aligned}
$$

The indices in equality 9.1 are due to the integrality of gr on monomials of $\mathbb{T}$ and the action of $\frac{\partial}{\partial y_{j,0}}$. Noting that the second summand of the last line is precisely the difference between $\mathrm{diff}_{>}(\mathbb{T})_{[n]}$ and $\mathrm{diff}(\mathbb{T})_{[n]}$,

$$
\begin{aligned}
\mathrm{diff}_{>}(\mathbb{T})_{[n]} &= \sum_{j=0}^{2} \left(\sum_{w=0}^{n-1} \left(\frac{\partial}{\partial y_{j,0}} \mathbb{J}_{[w]} \right) \left(\hat{K}_{2-j} \right)_{[n-w-\frac{1-j}{3}]} \right) \\
&= \sum_{j=0}^{2} \left(\sum_{w=0}^{n} \left(\frac{\partial}{\partial y_{j,0}} \mathbb{J}_{[w]} \right) \left(\hat{K}_{2-j} \right)_{[n-w-\frac{1-j}{3}]} \right. \\
&\qquad \left. - \left(\frac{\partial}{\partial y_{j,0}} \mathbb{J}_{[n]} \right) \left(\hat{K}_{2-j} \right)_{[-\frac{1-j}{3}]} \right) \\
&= \sum_{j=0}^{2} \left(\left(\frac{\partial}{\partial y_{j,0}} \mathbb{J} \right) \left(\hat{K}_{2-j} \right) \right)_{[n]} - \left(\frac{\partial}{\partial y_{j,0}} \mathbb{J}_{[n]} \right) T_0 y_{0,j} \qquad (9.2) \\
&= \mathrm{diff}_{>}(\mathbb{J})_{[n]},
\end{aligned}
$$

where (9.2) follows from Lemma 9.14. Therefore, $\mathrm{diff}_{>}(\mathbb{T}) = \mathrm{diff}_{>}(\mathbb{J})$. Of course, the above equality implies $\mathbb{T} = \mathbb{J}$ in all multi-degrees except those in the kernel of $\mathrm{diff}_{>}$. Terms in the kernel of $\mathrm{diff}_{>}$ are of degree 0 in $y_{2,j}$ for all $j > 0$. Then, in all multi-degrees encoding invariants with any ψ-class insertions present beyond those appearing in the usual J-function, $\mathbb{T} = \mathbb{J} = \mathbb{T}_{\mathrm{trop}}$. On the other hand, Gross has proven [13] that $\mathbb{T}_{\mathrm{trop}}$ agrees with $\mathbb{T}$ in all remaining multi-degrees. $\square$

We immediately obtain Theorem 1.1 and the following as corollaries.

Corollary 9.17. $\mathbb{T} = \mathbb{J}$.

10. Wall crossing and tropical technicalities

10.1. Proof of Theorem 7.12

Theorem 7.12, except for one case, follows from a straightforward modification of the argument found in [14], Theorem 5.35. The strategy is to analyze the behavior of so-called *degenerate* broken lines. These occur as the limits of deformations of ordinary broken lines; as one deforms the base point, two bends can can converge to a single point on the broken line, or one of the bends can approach a singular point of the scattering diagram. See Definition 5.34 of [14] for a rigorous definition. Let $\xi : [0,1] \to M_{\mathbb{R}}$ be a smooth path from Q to Q'. Subdivide the plane by a set of walls $\mathfrak{U}$ composed of those from $\mathfrak{D}_{0,k}(A)$ in addition to those formed by such degenerate broken lines; the change in $W_{k,0}\left(A\left(\xi(s)\right)\right)$ as $\xi(s)$ crosses one of these walls can be seen to be generated by an automorphism of $\mathbb{C}[T_\Sigma] \otimes_{\mathbb{C}} R_{k,0}$, a type of mutation process on the broken lines with endpoint $\xi(s)$.

For $\hat{Q} \in M_{\mathbb{R}}$, denote by $\mathfrak{B}(\hat{Q})$ the set of broken lines in $\mathfrak{D}(A)_{k,0}$ with endpoint $\hat{Q}$. Suppose $\xi(s_0)$ is in some wall L to which ξ is transverse, and for small $\epsilon > 0$, let $Q_1 := \xi(s_0 - \epsilon)$ and $Q_2 := \xi(s_0 + \epsilon)$. Let $\mathbf{n} \in N$ be a primitive vector annihilating the tangent space to L at $\xi(s_0)$ and taking a smaller value on Q_1 than Q_2. We decompose $\mathfrak{B}(Q_i)$ into $\mathfrak{B}^+(Q_i)$, $\mathfrak{B}^0(Q_i)$, and $\mathfrak{B}^-(Q_i)$, where the membership of $\beta \in \mathfrak{B}(Q_i)$ is determined the sign of $\langle \beta_*(-\partial/\partial s|_{s=0}), \mathbf{n}\rangle$.

These decompositions allows us to write

$$W_{k,0}(A(Q_i)) = W^-_{k,0}(A(Q_i)) + W^0_{k,0}(A(Q_i)) + W^+_{k,0}(A(Q_i)).$$

Following the techniques in [14], one can show

$$\theta_{\hat{\xi},\mathfrak{D}(A)_{k,0}}(W^\pm_{k,0}(A(Q_1))) = W^\pm_{k,0}(A(Q_2)),$$

where $\hat{\xi}$ is the segment of ξ joining Q_1 to Q_2.

For the remaining case, we will partition $\mathfrak{B}(Q_i)^0 = \bigsqcup_{j=1}^{l} \mathfrak{B}^i_j$ and show that for each $j \in \{1, \ldots, l\}$, $\mathfrak{B}^1_j$ and $\mathfrak{B}^2_j$ make equal contributions to $W_{k,0}(Q_1)$ and $W_{k,0}(Q_2)$, respectively. We will assume that a broken line with endpoint $\xi(s_0)$ passes through at most one singular point. The general case follows by an induction argument.

Suppose $\beta_1 \in \mathfrak{B}(Q_1)^0$ deforms continuously to $\beta_2 \in \mathfrak{B}(Q_2)^0$. In this case, each β_i will appear in a one element set $\mathfrak{B}^i_j$, and each $\mathfrak{B}^i_j$ will make the same contribution to $W_{k,0}(Q_i)$.

If $\beta \in \mathfrak{B}(Q_1)^0$ cannot be continuously deformed to an element of $\mathfrak{B}(Q_2)^0$, then it must deform to a degenerate broken line when the base point reaches $\xi(s_0)$. In other words, there is a map $B : (-\infty, 0] \times [0, s_0] \to M_{\mathbb{R}}$ such that $B|_{(-\infty,0]\times[0,s_0)}$ is a continuous deformation of β and $\beta' := B|_{(-\infty,0]\times\{s_0\}}$ is a degenerate broken line bending at $P \in \mathrm{Sing}(\mathfrak{D}(A)_{k,0})$ at time s'. There are two cases to examine: $P \in \{P_1, \ldots, P_k\}$ and $P \notin \{P_1, \ldots, P_k\}$. We explain the former, which requires a more sophisticated argument than that appearing in loc. cit.

Suppose $P = P_l$ and select $\hat{Q}$ very near P_l. We know that β bends along exactly one ray $\mathfrak{d}_0$ emanating from P_l whose attached function has a monomial containing $u_{l,w}$. By construction, $\mathfrak{d}_0$ is produced by a tropical tree, which, by Lemma 7.11, is constructed from perturbations of $w + 1$ semirigid descendent tropical disks with endpoint $\hat{Q}$. Call these disks $\mathcal{D}_1, \ldots, \mathcal{D}_{w+1}$, and define $\mathcal{D}_j$ as belonging to $\mathrm{RootDisk}(A(\hat{Q}), r^j, T^0_{0,tr})$. Also note that $B|_{(-\infty,s']\times\{s_0\}}$ is a broken line ending at P_l, corresponding to a semirigid disk $\mathcal{D}_0 \in \mathrm{RootDisk}(A(\hat{Q}), r^0, T^0_{0,tr})$. The vectors r^j are disjoint for all $0 \leq j \leq w + 1$. We can expect to form something like a tropical tree $\mathcal{T}_j$ for each $0 \leq j \leq w + 1$ by joining all of the $\mathcal{D}_i$ except for $\mathcal{D}_j$ at P_l and extending an unbounded outgoing edge $\mathfrak{d}_j$ as dictated by the balancing condition. See Figure 10.1. We may happen to have $\sum_{l\neq j} \mathbf{m}(\mathcal{D}_l) = 0$ and the result will not strictly qualify as a tropical tree, but these exceptional cases won't be problematic.

Let $M_i \subseteq \{\mathcal{D}_0, \ldots, \mathcal{D}_{w+1}\}$ be the subset of disks that are simply unbounded rays pointing in the direction $\mathbf{m}_i$ from P_l. Each choice of $0 \leq j \leq w + 1$ where $w(\mathfrak{d}_j) \neq 0$ gives rise to a broken line B_j bending at $\mathfrak{d}_j$ (the outgoing edge of $\mathcal{T}_j$, as discussed in the previous paragraph) constructed from the concatenation of a perturbation of a broken line defining $\mathcal{D}_j$ and that of $B|_{[s',0]\times\{s_0\}}$. We will show that the contributions to $W_{k,0}(Q_1)$ and $W_{k,0}(Q_2)$ from such broken lines are equal. We are analyzing behavior in the neighborhood of a wall of $\mathfrak{U}$ given by a union of degenerate broken lines bending at P_l. Notice that the side of the wall that each B_i inhabits (whether it can be deformed to a broken line in $W_{k,0}(Q_1)$ or $W_{k,0}(Q_2)$) is indicated by the sign of $\mathbf{m}_{\mathfrak{d}_i} \wedge \mathbf{m}(B|_{[s',0]\times\{s_0\}})$, where $\mathbf{m}(B|_{[s',0]\times\{s_0\}})$ gives the direction vector of β after bending at $\mathfrak{d}_0$ (considering β as coming in from infinity toward its end point). Furthermore, $\mathbf{m}(B|_{[s',0]} \times \{s_0\})$ is given by $\sum_{j=0}^{w+1} \mathbf{m}(\mathcal{D}_j)$.

The monomial obtained from the bend of B_i at $\mathfrak{d}_i$ is given by

$$\begin{aligned}
&w(\mathfrak{d}_i)\langle \mathbf{n}_i, \mathbf{m}(\mathcal{D}_i)\rangle \mathrm{Mono}(\mathcal{D}_i)\mathrm{Mono}(\mathcal{T}_i)\\
&= \frac{w(\mathfrak{d}_i)\langle \mathbf{n}_i, \mathbf{m}(\mathcal{D}_i)\rangle \mathrm{Mono}(\mathcal{T}_i)}{|M_0 \setminus \{\mathcal{D}_i\}|!|M_1 \setminus \{\mathcal{D}_i\}|!|M_2 \setminus \{\mathcal{D}_i\}|!} \prod_{n\neq i} \mathrm{Mono}(\mathcal{D}_n)\\
&= \frac{w(\mathfrak{d}_i)\langle \mathbf{n}_i, \mathbf{m}(\mathcal{D}_i)\rangle u_{l,w}}{|M_0 \setminus \{\mathcal{D}_i\}|!|M_1 \setminus \{\mathcal{D}_i\}|!|M_2 \setminus \{\mathcal{D}_i\}|!} \prod_{n} \mathrm{Mono}(\mathcal{D}_n),
\end{aligned}$$

where $\mathbf{n}_i \in N$ is orthogonal to $\mathfrak{d}_i$ and chosen so that

$$w(\mathfrak{d}_i)\langle \mathbf{n}_i, \mathbf{m}(\mathcal{D}_i)\rangle = \left| \left(\sum_{n\neq i} \mathbf{m}(\mathcal{D}_n) \right) \wedge \mathbf{m}(\mathcal{D}_i) \right|$$

(as $\mathbf{m}_{\mathfrak{d}_i}$ is given by $\sum_{n\neq i} \mathbf{m}(\mathcal{D}_n)$).
The result then follows from some basic observations. First,

$$\begin{aligned}
0 = \left(\sum_{j=0}^{w+1} \mathbf{m}(\mathcal{D}_j) \right)^{\wedge 2} &= \sum_{j=0}^{w+1} \mathbf{m}(\mathcal{D}_j) \wedge \left(\sum_{n=0}^{w+1} \mathbf{m}(\mathcal{D}_n) \right)\\
&= \sum_{j=0}^{w+1} \mathbf{m}(\mathcal{D}_j) \wedge \left(\sum_{n\neq j} \mathbf{m}(\mathcal{D}_n) \right).
\end{aligned}$$

Let $I^- := \{n \in \{0,\ldots,k+1\} \mid \left(\sum_{n\neq i} \mathbf{m}(\mathcal{D}_n)\right) \wedge \mathbf{m}(\mathcal{D}_i)) < 0\}$ under the identification of $\wedge^2 M_{\mathbb{R}}$ with $\mathbb{Z}$, with I^0 and I^+ defined analogously. Then

$$\begin{aligned}
0 &= \sum_{j\in I^-} \mathbf{m}(\mathcal{D}_j) \wedge \left(\sum_{n\neq j} \mathbf{m}(\mathcal{D}_n) \right) + \sum_{j\in I^+} \mathbf{m}(\mathcal{D}_j) \wedge \left(\sum_{n\neq j} \mathbf{m}(\mathcal{D}_n) \right)\\
&\quad + \sum_{j\in I^0} \mathbf{m}(\mathcal{D}_j) \wedge \left(\sum_{n\neq j} \mathbf{m}(\mathcal{D}_n) \right)\\
&= \sum_{j\in I^-} \mathbf{m}(\mathcal{D}_j) \wedge \left(\sum_{n\neq j} \mathbf{m}(\mathcal{D}_n) \right) + \sum_{j\in I^+} \mathbf{m}(\mathcal{D}_j) \wedge \left(\sum_{n\neq j} \mathbf{m}(\mathcal{D}_n) \right).
\end{aligned}$$

A series of implications follows:

$$-\sum_{j\in I^-}\mathbf{m}(\mathcal{D}_j)\wedge\left(\sum_{n\neq j}\mathbf{m}(\mathcal{D}_n)\right)=\sum_{j\in I^+}\mathbf{m}(\mathcal{D}_j)\wedge\left(\sum_{n\neq j}\mathbf{m}(\mathcal{D}_n)\right)$$
$$\sum_{j\in I^-}\left|\mathbf{m}(\mathcal{D}_j)\wedge\left(\sum_{n\neq j}\mathbf{m}(\mathcal{D}_n)\right)\right|=\sum_{j\in I^+}\left|\mathbf{m}(\mathcal{D}_j)\wedge\left(\sum_{n\neq j}\mathbf{m}(\mathcal{D}_n)\right)\right|$$
$$\sum_{j\in I^-}w(\mathfrak{d}_j)\langle\mathbf{n}_j,\mathbf{m}(\mathcal{D}_i)\rangle=\sum_{j\in I^+}w(\mathfrak{d}_j)\langle\mathbf{n}_j,\mathbf{m}(\mathcal{D}_i)\rangle.$$

Therefore

$$\begin{aligned}\sum_{j\in I^-}&\frac{w(\mathfrak{d}_j)\langle\mathbf{n}_j,\mathbf{m}(\mathcal{D}_i)\rangle u_{l,w}}{|M_0|!|M_1|!|M_2|!}\prod_n\mathrm{Mono}(\mathcal{D}_n)\\&=\sum_{j\in I^+}\frac{w(\mathfrak{d}_j)\langle\mathbf{n}_j,\mathbf{m}(\mathcal{D}_i)\rangle u_{l,w}}{|M_0|!|M_1|!|M_2|!}\prod_n\mathrm{Mono}(\mathcal{D}_n).\end{aligned}\tag{10.1}$$

Equation 10.1 closely resembles our desired result, as I^+ indexes disks related to broken lines contributing to one of $W_{k,0}(A(Q_1))$, $W_{k,0}(A(Q_2))$ and I^- indexes those which contribute to the other. To conclude, note that at most one broken line is produced for each set M_j, so we can say that the contribution from each B_i (where $\mathcal{D}_i\in M_j$) is just $\frac{1}{|M_j|}$ of the contribution from the unique broken line produced by M_j. That is, the contribution from $B_i\in M_j$ should be considered as

$$\begin{aligned}&\frac{1}{|M_j|}\frac{w(\mathfrak{d}_i)\langle\mathbf{n}_i,\mathbf{m}(\mathcal{D}_i)\rangle u_{l,w}}{|M_0\setminus\{\mathcal{D}_i\}|!|M_1\setminus\{\mathcal{D}_i\}|!|M_2\setminus\{\mathcal{D}_i\}|!}\prod_n\mathrm{Mono}(\mathcal{D}_n)\\&=\frac{w(\mathfrak{d}_i)\langle\mathbf{n}_i,p(\Delta(\mathcal{D}_i)\rangle u_{l,w}}{|M_0|!|M_1|!|M_2|!}\prod_n\mathrm{Mono}(\mathcal{D}_n)\end{aligned}$$

Of course, if $\mathcal{D}_i\notin\cup_j M_j$ then the contribution is

$$\begin{aligned}&\frac{w(\mathfrak{d}_i)\langle\mathbf{n}_i,\mathbf{m}(\mathcal{D}_i)\rangle u_{l,w}}{|M_0\setminus\{\mathcal{D}_i\}|!|M_1\setminus\{\mathcal{D}_i\}|!|M_2\setminus\{\mathcal{D}_i\}|!}\prod_n\mathrm{Mono}(\mathcal{D}_n)\\&=\frac{w(\mathfrak{d}_i)\langle\mathbf{n}_i,\mathbf{m}(\mathcal{D}_i)\rangle u_{l,w}}{|M_0|!|M_1|!|M_2|!}\prod_n\mathrm{Mono}(\mathcal{D}_n)\end{aligned}$$

Thus, 10.1 shows that the sum of the monomials generated by our set of broken lines on either side of the wall is equal. Deforming any of the B_i

to degenerate at P_l will result in the same scenario, showing that broken lines degenerating at P_l (for a particular deformation of Q) can be partitioned into sets which give equal contributions to $W_{k,0}(A(Q_1))$ and $W_{k,0}(A(Q_2))$. As $\theta_{\hat{\xi},\mathfrak{D}(A)_{k,0}}(W^0_{k,0}(A(Q_1))) = W^0_{k,0}(A(Q_1))$, we have proven that $\theta_{\hat{\xi},\mathfrak{D}(A)_{k,0}}(W_{k,0}(A(Q_1))) = W_{k,0}(A(Q_2))$.

10.2. Results leading up to Lemma 7.24

Here we give some details leading up to Lemma 7.24, in which the integral is described in terms of counts of tropical curves. The first four lemmas below follow directly from the related results in [13], as the scattering diagrams in question have identical behavior away from the points P_i of A.

Definition 10.1. If $\mathcal{C}$ is a tropical curve contributing to

$$\langle \psi^{b(1)-1} P_{b\{1\}}, \ldots, \psi^{b(\#(b))-1} P_{b\{\#(b)\}}, \psi^{\nu} S_i(A) \rangle^{\text{trop}}_{d,0},$$

define $b^{\mathcal{C}} \in \mathcal{B}_k$ to be its corresponding vector and $u_{\mathcal{C}} := u_{b^{\mathcal{C}}}$.

Lemma 10.2.

$$\begin{aligned} &L^d_{i,\{0\}}(Q) = \delta_{0,d}\delta_{0,i} \\ &+ \sum_{\substack{\nu \geq i \\ b \in \mathcal{B}_k \\ |b| = 3d-2+i-\nu \\ d \geq 1}} \langle \psi^{b(1)-1} P_{b\{1\}}, \ldots, \psi^{b(\#(b))-1} P_{b\{\#(b)\}}, \psi^{\nu} S_i(A) \rangle^{\text{trop}}_{d,\{0\}} u_b h^{-(\nu+2-i)}. \end{aligned}$$

Proof. See Lemma 5.11 of ibid. □

Lemma 10.3.

$$\begin{aligned} &-L^d_{i,,\xi_j,\{0\}\to\rho_j} = \\ &\sum_{\substack{\nu \geq i-1 \\ b \in \mathcal{B}_k \\ |b| = 3d-2+i-\nu \\ d \geq 1}} \langle \psi^{b(1)-1} P_{b\{1\}}, \ldots, \psi^{b(\#(b))-1} P_{b\{\#(b)\}}, \psi^{\nu} S_i(A) \rangle^{\text{trop}}_{d,\rho_j} u_b h^{-(\nu+2-i)} \end{aligned}$$

Proof. See Lemma 5.16 of ibid. □

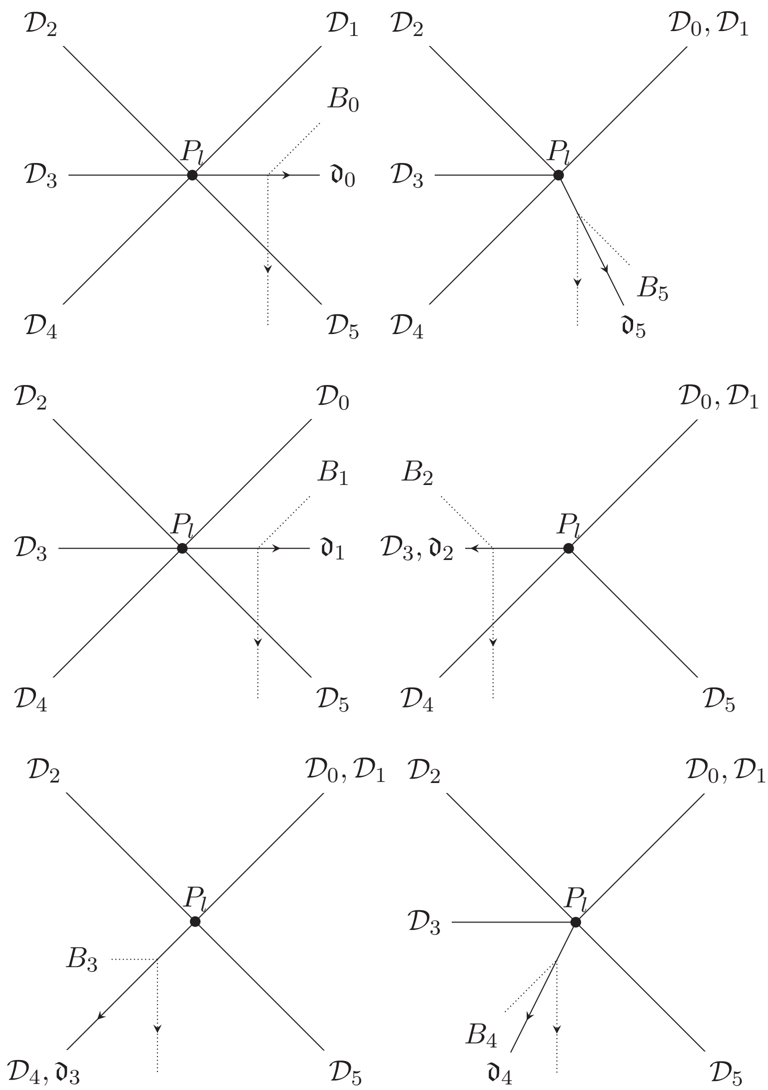

Figure 10.1. An example of the behavior encountered in the proof of Theorem 7.12. The first three broken lines are on the right hand side of the wall, while the last three are on the left.

Lemma 10.4. *For each point $P \in \mathrm{Sing}(\mathfrak{D})$, let ξ_P be a small counterclockwise loop around P, small enough so that it doesn't go around any other* *point of* $\mathrm{Sing}(\mathfrak{D})$*. Then*

$$L^d_{i,\xi_{j,j+1},\rho_{j+1}\to\sigma_{j,j+1}} = \sum_{P\in\mathrm{Sing}(\mathfrak{D})\cap(Q+\sigma_{j,j+1})} L^d_{i,\xi_P,\rho_{j+1}\to\sigma_{j,j+1}}$$

Proof. See Lemma 5.17 of ibid. □

Lemma 10.5. *Let* $P\in\mathrm{Sing}(\mathfrak{D})\cap(Q+\sigma_{j,j+1})$*, and suppose that*

$$P\notin A.$$

Then

$$-L^d_{i,\xi_P,\rho_{j+1}\to\sigma_{j,j+1}} = \sum_{\nu\geq 0}\sum_{\mathcal{C}} \mathrm{Mult}(\mathcal{C}) u_{\mathcal{C}} \hbar^{-(\nu+2-i)}$$

where the sum is over curves $\mathcal{C}$ *contributing to*

$$\langle \psi^{b(1)-1}P_{b\{1\}},\dots,\psi^{b(\#(b))-1}P_{b\{\#(b)\}},\psi^{\nu}S_i(A)\rangle^{\mathrm{trop}}_{d,\sigma_{j,j+1}}$$

for $d\geq 1$*,* $b\in\mathcal{B}_k$ *with* $|b| = 3d-2+i-\nu$ *and* $h(E_x)=P$*.*

Proof. See Lemma 5.17 of loc.cit. □

The following is the only place in the evaluation of the integral that requires a significant modification of Gross's techniques.

Lemma 10.6. *Let* $P\in\mathrm{Sing}(\mathfrak{D})\cap(Q+\sigma_{j,j+1})$*, and suppose that* $P=P_l\in A$*. Then*

$$-L^d_{i,\xi_P,\rho_{j+1}\to\sigma_{j,j+1}} = \sum_{w=1}^{k} u_{l,w-1}(-\hbar)^w\delta_{d,0}\delta_{2,i} + \sum_{\nu\geq 0}\sum_{\mathcal{C}} \mathrm{Mult}(\mathcal{C}) u_{\mathcal{C}} \hbar^{-(\nu+2-i)}$$

where the sum is over curves $\mathcal{C}$ *contributing to*

$$\langle \psi^{b(1)-1}P_{b\{1\}},\dots,\psi^{b(\#(b))-1}P_{b\{\#(b)\}},\psi^{\nu}S_i(A)\rangle^{\mathrm{trop}}_{d,\sigma_{j,j+1}}$$

for $d\geq 1$*,* $b\in\mathcal{B}_k$ *with* $|b| = 3d-2+i-\nu$ *and* $h(E_x)=P_l$*.*

Proof. Here we assume $i=2$, and write

$$L_{P,j} = L^d_{2,\xi_P,\rho_{j+1}\to\sigma_{j,j+1}}$$

Choose a basepoint Q' near P_l. As discussed in Lemmas 7.10 and 7.11, sets of $a+1$ semirigid disks with endpoint Q' not bending near P_l correspond to rays in $\mathfrak{D}$ based at P_l whose monomial contains $u_{l,a}$. More precisely, sets of semirigid disks $\{\mathcal{D}_1, \ldots, \mathcal{D}_{a+1}\}$ not bending near P_l with endpoint Q', $\prod_i \mathrm{Mono}(\mathcal{D}_i) \neq 0$, and $\sum_i p(\Delta(\mathcal{D}_i)) \neq 0$ are in one to one correspondence with rays in $\mathfrak{D}$ with attached monomial containing $u_{l,a}$. Such sets are naturally recovered from $\exp([W_{k,0}(A(Q')) - W_{\mathrm{basic}}(A(Q'))]/\hbar)$. Define $L_{P,j,a}$ to be the sum of monomials in $L_{P,j}$ that include the factor $u_{l,a}$.

To find terms from $\exp([W_{k,0}(A(Q')) - W_{\mathrm{basic}}(A(Q'))]/\hbar)$ that will contribute to $L_{P,j,a}$ upon crossing a wall radiating from P_l, we should examine those not containing the factor $u_{l,w}$ for any w. We consider a term $c\hbar^{-\nu} z^{\hat{n}}$ of the form:

$$c\hbar^{-\nu} z^{\hat{n}} = \hbar^{-\nu} \prod_{w=1}^{\nu} \mathrm{Mult}(\mathcal{D}_w) z^{\Delta(\mathcal{D}_w)} u_{\mathcal{D}},$$

where $\mathcal{D}_w \in \mathrm{RootDisk}(A(Q'), b^{\mathcal{D}_w}, T^0_{0,tr})$ for $1 \leq w \leq \nu$. As opposed to the case considered in [13] Lemma 5.17, we will have to consider the walls $\mathfrak{d}$ resulting from trees containing semirigid disks corresponding to unbounded rays (translated copies of ρ_i) emanating from P_l. Write $\hat{n} = \sum_{v=1}^{\nu} \Delta(\mathcal{D}_v) = \sum_{v=0}^{2} n_v t_v$ and choose the primitive normal vectors $\mathbf{n}_{\mathfrak{d}}$ to each ray $\mathfrak{d}$ issuing from P_l such that they point in the direction opposite to ξ'_P when ξ_P crosses $\mathfrak{d}$.

The term $c\hbar^{-\nu} z^{\hat{n}}$ can only contribute to $L_{P,j,a}$ when ξ_P crosses rays whose corresponding tree contains exactly $a+1$ semirigid disks joined at P_l. The relevant rays can be enumerated as follows. Select $\{\mathcal{D}_{i_1}, \ldots, \mathcal{D}_{i_s}\} \subseteq \{\mathcal{D}_1, \ldots, \mathcal{D}_\nu\}$ and M_v copies (here it's convenient to consider M_v as an integer rather than a set) of the simple disk composed of the ray parallel to ρ_v for $0 \leq v \leq 2$ such that $s + M_0 + M_1 + M_2 = a + 1$. Set $\mathbf{M} = \sum_{v=0}^{2} M_v t_v \in T_\Sigma$. Let $\tilde{n} := \sum_{v=1}^{s} \Delta(\mathcal{D}_{i_v}) + \mathbf{M}$ and $r(\tilde{n}) := w_{\tilde{n}} \mathbf{m}_{\tilde{n}}$, where $\mathbf{m}_{\tilde{n}}$ is primitive. These choices will produce a ray $\mathfrak{d} \in \mathfrak{D}$ with attached function

$$f_{\mathfrak{d}} = 1 + w_{\tilde{n}} \prod_{m=1}^{s} \mathrm{Mono}(\mathcal{D}_{i_m}) z^{\mathbf{M}} \frac{1}{M_0! M_1! M_2!}.$$

Let $c'\hbar^{-(\nu-s)} z^{n'} := \hbar^{-(\nu-s)} \prod_{\mathcal{D} \in \{\mathcal{D}_1, \ldots, \mathcal{D}_\nu\} \setminus \{\mathcal{D}_{i_1}, \ldots, \mathcal{D}_{i_s}\}} \mathrm{Mono}(\mathcal{D})$. This term will generate a contribution of $c\hbar^{-\nu} z^{\hat{n}}$ to $L_{P,j,a}$ upon crossing $\mathfrak{d}$, and this contribution will occur exactly when $n_{j+2} + M_{j+2} \leq d = n_j + M_j < n_{j+1} + M_{j+1}$. For simplicity of exposition, we set $j = 0$ in what follows. The quantity of the contribution is then, by definition,

$$w_{\tilde{n}} \langle \mathbf{n}_{\mathfrak{d}}, \mathbf{m}_0 \rangle D_2(d, n_0 + M_0 + 1, n_1 + M_1, n_2 + M_2) \frac{\hbar^{-(\nu - s + 3d - |\hat{n}| - |M|)}}{M_0! M_1! M_2!},$$

where D_2 is defined in [14], Lemma 5.43. Noting that $|\mathbf{M}| = a+1-s$ and recalling the isomorphism of $\bigwedge^2 M$ with $\mathbb{Z}$, we see that the above becomes

$$(r(\tilde{n}) \wedge \mathbf{m}_0)\, D_2(d, n_0 + M_0 + 1, n_1{+}M_1, n_2 + M_1) \frac{\hbar^{-(\nu-a-1+3d-|\hat{n}|)}}{M_0!M_1!M_2!}.$$

Our goal is to now sum this contribution over all choices of s, $\{\mathcal{D}_{i_1}, \ldots, \mathcal{D}_{i_s}\} \subseteq \{\mathcal{D}_1, \ldots, \mathcal{D}_\nu\}$, M_0, M_1, and M_2. These should exhaust the set of relevant rays emanating from P_l that ξ_P crosses, and should thus calculate the total contribution. After a little rearrangement (see [32]), the sum becomes the following, where $t := a + 1 - d + n_0$:

$$\begin{aligned}(10.2)\qquad \frac{\hbar^{-(\nu-a-1+3d-|\hat{n}|)}}{(d-n_0)!} \sum_{s=0}^{a+1} \sum_{M_1+M_2=t-s} &\frac{(-1)^{M_1+n_1+d+1}(n_1+M_1-d-1)!}{M_1!M_2!(d-n_2-M_2)!} \\ &\times \left(\binom{\nu-1}{s-1}(n_2-n_1) + \binom{\nu}{s}(M_2-M_1) \right),\end{aligned}$$

where we take any summands involving factorials with negative arguments to be 0.

Sublemma 10.7. *Let $d > 0$, $\nu, n_0, n_1, n_2, a \in \mathbb{Z}_{\geq 0}$ with $n_2, n_0 \leq d$. Set $t = a + 1 - d + n_0$, $|n| = n_0 + n_1 + n_2$. Then*

$$\begin{aligned}&\frac{1}{(d-n_0)!} \sum_{s=0}^{a+1} \sum_{M_1+M_2=t-s} \frac{(-1)^{M_1+n_1+d+1}(n_1+M_1-d-1)!}{M_1!M_2!(d-n_2-M_2)!} \\ &\qquad\qquad \times \left(\binom{\nu-1}{s-1}(n_2-n_1) + \binom{\nu}{s}(M_2-M_1) \right) \\ &= -\frac{1}{(d-n_0)!(d-n_1)!(d-n_2)!} \binom{\nu+3d-|n|-1-((d-n_0)+(d-n_1))}{a-((d-n_0)+(d-n_1))}\end{aligned}$$

where any summands involving factorials with negative arguments are taken to be 0 .

Proof. See Lemma 3.3.17 of [32]. □

Given a non-zero contribution to $-L_{P,j,a}$ of the term $c\hbar^{-\nu}z^{\hat{n}}$ in $\exp([W_{k,0}(A(Q')) - W_{\mathrm{basic}}(A(Q'))]/\hbar)$ (with $d > 0$ and $c\hbar^{-\nu}z^{\hat{n}}$ degree 0 in $u_{l,w}$ for all w), we can assemble a balanced tropical curve $\mathcal{C}$. Begin by gluing the disks $\mathcal{D}_1, \ldots, \mathcal{D}_\nu$ together by their outgoing vertices at P_l, add on $d - n_j$ unbounded edges in the direction $\mathbf{m}_j$ for $0 \leq j \leq 2$ and two additional edges

E_x and E_p that will be collapsed to mark x and P_l. Thus we have a tropical curve $\mathcal{C}$ with $h : \Gamma \to M_{\mathbb{R}}$ satisfying $h(E_x) = h(p) = h(V) = P_l$. Define $b := \sum_{m=1}^{\nu} b^{\mathcal{D}_m}$. This procedure yields a frame whose valence at the new vertex V is given by $Val(V) := \nu + 3d - |\hat{n}| + 2$. The previous sublemma describes the contribution to $-L_{P,j,a}$ of the term $c\hbar^{-\nu} z^{\hat{n}}$ upon crossing the corresponding rays radiating from P_l as

$$\begin{aligned}&\binom{Val(V) - 3 - ((d - n_0) + (d - n_1))}{a - ((d - n_0) + (d - n_1))}\\&\times \mathrm{Mult}_x^0(\mathcal{C}) \left(\prod_{l=1}^{\nu} \mathrm{Mult}(\mathcal{D}_l) u_{\mathcal{D}_l}\right) u_{l,a} \hbar^{-(\nu - a - 1 + 3d - |\hat{n}|)}.\end{aligned}$$

Suppose that $|b| = 3d - \nu'$ for some $\nu' \geq 0$. The quantity a should be thought of as specifying the number of ψ-classes associated to P_l, while ν' is the number of ψ-classes associated to E_x. By construction, $Val(V) = \nu + 3d - |\hat{n}| + 2$. On the other hand, because $\mathcal{C}$ is obtained by gluing $Val(V) - 2$ semirigid disks at V, we have

$$\begin{aligned}Val(V) - 2 &= \sum_{i=1}^{\nu} (|\Delta(\mathcal{D}_i)| - |b^{\mathcal{D}_i}|) + 3d - |\hat{n}|\\&= |\hat{n}| - [3d - \nu' - (a + 1)] + 3d - |\hat{n}|\\&= \nu' + a + 1.\end{aligned}$$

The first equality follows from $|\Delta(\mathcal{D}_i)| - |b^{\mathcal{D}_i}| = 1$ for each semirigid disk, while the second is due to Lemma 4.2. Therefore, the contribution to $-L_{P,j,a}$ from ξ_P crossing rays associated to this term is precisely the contribution of $\mathcal{C}$ to

$$\langle \psi^{b(1)-1} P_{r\{1\}}, \ldots, \psi^{b(\#(b))-1} P_{b\{\#(b)\}}, \psi^a P_l, \psi^{\nu'} S_2(A) \rangle^{\mathrm{trop}}_{d,\sigma_{j,j+1}} u_b u_{l,a} \hbar^{-\nu'}.$$

Conversely, it is easy to see that any such curve h contributing to the invariant will be accounted for by the integral by decomposing it into its constituent semirigid disks.

Suppose $d = 0$. An examination of Expression 10.2 shows that any nonzero contribution must occur when $n_0 = n_2 = 0$. In this case, $M_2 = 0$, which forces $M_1 = t - s = a + 1 - s$, so our quantity becomes

$$\hbar^{-(\nu-a-1-(n_1))} \sum_{s=0}^{a+1} \frac{(-1)^{a-s-n_1}(n_1+a-s)!}{(a+1-s)!} \times \left(\binom{\nu-1}{s-1}(-n_1) + \binom{\nu}{s}(-a-1+s) \right).$$

If $n_1 > d = 0$, then the argument applied in the first case of Lemma 10.7 shows that the above quantity is equal to 0. If $n_1 = 0$, then $\nu = 0$, and the above simplifies to

$$\hbar^{-(-a-1))} \sum_{s=0}^{a+1} \frac{(-1)^{a-s-n_1}(n_1+a-s)!}{(a+1-s)!} \left(\binom{0}{s}(-a-1+s) \right)$$
$$= \hbar^{a+1} \frac{(-1)^a(a)!}{(a+1)!} \left(\binom{0}{0}(-a-1) \right).$$

In this case the contribution to $-L_{P,j,a}$ from ξ_P is equal to $-(-\hbar)^{a+1}u_{l,a}$. $\square$

Lemma 10.8.

$$-L^d_{i,\xi_j,\rho_{j+1}\to\sigma_{j,j+1}} = \hbar \sum_{P_l \in Q+\sigma_{j,j+1}} \delta_{d,0}\delta_{2,i} \left(u_{l,0} - u_{l,1}\hbar + \cdots + u_{l,k}(-\hbar)^k \right)$$
$$+ \sum_{\substack{\nu \geq i-1 \\ b \in \mathcal{B}_k \\ |b| = 3d-2+i-\nu \\ d \geq 1}} \langle \psi^{b(1)-1}P_{b\{1\}}, \ldots, \psi^{b(\#(b))-1}P_{b\{\#(b)\}}, \psi^\nu S_i(A) \rangle^{\mathrm{trop}}_{d,\sigma_{j,j+1}j} u_r h^{-(\nu+2-i)}.$$

Proof. This follow from the previous lemmas. The first sum results from the previous remark as a is varied from 0 to $k-1$. $\square$

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