Communications in
Number Theory and Physics
Volume 10, Number 3, 571–585, 2016

A rigid Calabi-Yau manifold with Picard number two

Eberhard Freitag

We study a projective Calabi–Yau threefold $\mathcal{Y}^+$ which has been constructed in [FS]. It is rigid ($h^{12} = 0$) and has Picard number $h^{11} = 2$. We construct a pair of divisors $\mathcal{D}^\pm$ which give a basis of $\mathrm{Pic}(\mathcal{Y}^+) \otimes_{\mathbb{Z}} \mathbb{Q}$ and determine all intersection numbers $\mathcal{D}^\pm \cdot \mathcal{D}^\pm \cdot \mathcal{D}^\pm$.

Introduction

Basic for our example is a certain complete intersection $\mathcal{X}$ of four quadrics introduced in the paper [GN] of van Geemen and Nygaard:

$$(\mathcal{X}:)\qquad \begin{aligned} Y_0^2 &= X_0^2 + X_1^2 + X_2^2 + X_3^2,\\ Y_1^2 &= X_0^2 - X_1^2 + X_2^2 - X_3^2,\\ Y_2^2 &= X_0^2 + X_1^2 - X_2^2 - X_3^2,\\ Y_3^2 &= X_0^2 - X_1^2 - X_2^2 + X_3^2. \end{aligned}$$

The variety $\mathcal{X}$ has 96 isolated singularities which are ordinary double points (nodes).

In the paper [CM] it has been pointed out that the results of [GN] imply that $\mathcal{X}$ admits a resolution that is a (projective) Calabi–Yau threefold. The basic result — essentially due to van Geemen and Nygaard [GN] — is the following theorem.

Theorem. *The Hodge numbers of a Calabi–Yau desingularization of $\mathcal{X}$ are*

$$h^{11} = 32, \qquad h^{12} = 0.$$

Hence this Calabi–Yau manifold is rigid.

In the paper [FS], we constructed a certain group G of order 16 of biholomorphic automorphisms of $\mathcal{X}$ that acts freely on $\mathcal{X}$. The basic thing is that there exists a *projective* small resolution $\mathcal{X}^+$ such that G extends as group of biholomorphic mappings to $\mathcal{X}^+$. The quotient $\mathcal{Y}^+ = \mathcal{X}^+/G$ then is a projective resolution of $\mathcal{Y} := \mathcal{X}/G$ in form of a rigid Calabi-Yau manifold whose Picard number is two. The varieties $\mathcal{X}$ and $\mathcal{Y}$ are Siegel modular threefolds. But for this paper the modular background is not necessary. Everything can be done by using the explicit equations.

We recall the definition of the group G. Let $P_1, \dots, P_8$ be homogenous polynomials in $\mathbb{C}[Y_0, \dots, Y_3, X_0, \dots, X_3]$ of the same degree such that not all of them vanish. Then we can consider the rational map from P^7 into itself,

$$(y_0, \dots, y_3, x_0, \dots, x_3) \longmapsto (P_1(y_0, \dots, x_3), \dots, P_8(y_0, \dots, x_3)).$$

We denote this map symbolically by

$$(P_1, \dots, P_8).$$

Lemma. *The two transformations*

$$\begin{gathered}(-Y_2, \mathrm{i}Y_3, \mathrm{i}Y_0, Y_1, -\mathrm{i}X_3, -\mathrm{i}X_2, X_1, -X_0),\\ (\mathrm{i}Y_1, Y_0, -\mathrm{i}Y_3, Y_2, X_1, \mathrm{i}X_0, X_3, -\mathrm{i}X_2)\end{gathered}$$

define biholomorphic transformations of $\mathcal{X}$ onto itself. The group G which is generated by the two biholomorphic transformations is isomorphic to $\mathbb{Z}/4 \times \mathbb{Z}/4$ and acts freely on $\mathcal{X}$.

We recall some facts about projective resolutions of threefolds X with nodal singularities. Let D be an effective Weil divisor on X. Then we associate the ideal sheaf $\mathcal{I}(D)$ of all holomorphic functions on open subsets that satisfy $(f) \geq D$ on this subset. The blow up of D, by definition, is the blow up of the ideal sheaf $\mathcal{I}(D)$. Let D, D' be two divisors such that $D - D'$ is a Cartier divisor, then the two blow ups agree. Since for every Weil divisor D there exists a Cartier divisor D' such that $D + D'$ is effective, we can define the blow up of D for arbitrary Weil divisors. The blow ups of D and nD, $n > 0$, agree. With the results of the paper [FS], where the divisor class group has been described by means of explicit generators, one can check the following theorem.

Theorem. *There exists an effective Weil divisor D on $\mathcal{X}$ whose class is G-invariant and such that its blow up gives a projective resolution $\mathcal{X}^+$ of $\mathcal{X}$ with an action of G. Moreover, the divisor $-D$ will give a second projective resolution $\mathcal{X}^-$ with G-action. These two resolutions are unique in* *the following sense. If D' is any G-invariant Weil divisor whose blow up is a resolution, then there exists an integer $n \neq 0$ such that $D' - nD$ is a Cartier divisor. The quotients $\mathcal{Y}^+ = \mathcal{X}^+/G$ and $\mathcal{Y}^- = \mathcal{X}^-/G$ are projective resolutions of $\mathcal{Y}$.*

We will describe an explicit example for such a divisor in Theorem 4.1. Before we continue, we must choose one of the two resolutions $\mathcal{X}^+$ and $\mathcal{X}^-$. They are opposite resolutions in the sense that all nodes are flopped. To distinguish both, it is sufficient to fix the ruling of one node. We take the node $(\sqrt{2}, 0, \sqrt{2}, 0, -1, 0, 0, 1)$. It is contained in the following divisors $D^\pm$. They are cut out inside $\mathcal{X}$ by two equations.

$$D^\pm : \quad X_0 + X_1 + X_2 + X_3 = Y_1(X_1 + X_3) \pm (\sqrt{2}/2)Y_2Y_3 = 0.$$

To be precise, the ideals $\mathcal{I}(D^\pm)$ of these divisors are given by the radical of the ideals generated by the generating ideal $\mathcal{I}$ of $\mathcal{X}$ and these two elements.

Definition. *The desingularization $\mathcal{X}^+$ is chosen such that the ruling of the node $(\sqrt{2}, 0, \sqrt{2}, 0, -1, 0, 0, 1)$ is obtained by blowing up the divisor D^+.*

This means that the surface D^+ is blown up in $\mathcal{X}^+$ over the node, but D^- remains unchanged.

The determination of the intersection numbers uses computer calculations. For this one needs a base field. We take $K = \mathbb{Q}(\zeta)$ where ζ is an 8th root of unity. The varieties $D^\pm$ are irreducible over K. We think that they are irreducible over $\mathbb{C}$ but it is not necessary to know this.

In particular, we have to use a program that decides whether for a given $g \in G$ and a given node a on $g(D^+)$ the surface $g(D^+)$ is blown up in $\mathcal{X}^+$ over a or not. This program uses the description of the Picard group in [FS]. This paper contains an appendix which contains a rather detailed description of the way how the computer computations have been organized. In particular, there have been constructed 32 divisors which give generators of the Picard group (tensored with $\mathbb{Q}$). This description includes the action of the automorphism group on this Picard group.

1. Intersection numbers away from nodes

In the paper [FS], 188 two-dimensional subvarieties of $\mathcal{X}$ have been defined. We use here two of them. They have been introduced already in the introduction. They are components of the hyperplane section $X_0 + \cdots + X_3 = 0$.

One of them is cut out by the equations

$$D^+: \quad X_0 + X_1 + X_2 + X_3 = Y_1(X_1 + X_3) + (\sqrt{2}/2)Y_2Y_3 = 0.$$

The vanishing ideal of this variety is the radical of the ideal generated by the defining ideals of $\mathcal{X}$ and D^+. It can be computed as

$$\begin{aligned}
&Y_0^2 - 2X_1^2 - 2X_1X_2 - 2X_1X_3 - 2X_2^2 - 2X_2X_3 - 2X_3^2,\\
&Y_1^2 - 2X_1X_2 - 2X_1X_3 - 2X_2^2 - 2X_2X_3,\\
&Y_1Y_2 + \sqrt{2}Y_3X_1 + \sqrt{2}Y_3X_2,\\
&Y_1Y_3 + \sqrt{2}Y_2X_2 + \sqrt{2}Y_2X_3,\\
&Y_1X_1 + Y_1X_3 + 1/2\sqrt{2}Y_2Y_3,\\
&Y_2^2 - 2X_1^2 - 2X_1X_2 - 2X_1X_3 - 2X_2X_3,\\
&Y_3^2 - 2X_1X_2 - 2X_1X_3 - 2X_2X_3 - 2X_3^2,\\
&X_0 + X_1 + X_2 + X_3.
\end{aligned}$$

Replacing $\sqrt{2}$ by $-\sqrt{2}$ we get a complementary ideal

$$D^-: \quad X_0 + X_1 + X_2 + X_3 = Y_1(X_1 + X_3) - (\sqrt{2}/2)Y_2Y_3 = 0.$$

Lemma 1.1. *The transformation σ that changes the signs of the x_i and fixes the y_i is contained in the normalizer of the group G. It maps D^+ to D^- and it induces biholomorphic maps*

$$\sigma : \mathcal{X}^+ \xrightarrow{\sim} \mathcal{X}^-, \quad \mathcal{Y}^+ \xrightarrow{\sim} \mathcal{Y}^-.$$

We recall some basics about divisors and their intersection numbers.

Let X be a normal irreducible algebraic variety of dimension n over $\mathbb{C}$. A divisor on X is a formal sum of irreducible closed subvarieties of codimension one. Let $Y \subset X$ be a subvariety of everywhere codimension 1. The associated divisor is the sum of all irreducible components (with multiplicities 1). To every rational function its principal divisor can be associated. Two divisors are called equivalent if their difference is principal. Since the singular locus of X has codimension ≥ 2, the divisors (divisor classes) on X are in one-to-one correspondence with the divisors (divisor classes) on the regular locus. Let $\pi : \tilde{X} \to X$ be a small resolution. Small means that there exists a closed subvariety $T \subset \tilde{X}$ of codimension ≥ 2 such that π defines a biholomorphic map of $\tilde{X} - T$ onto the regular locus of X. The divisors (divisor classes) of X are in one-to-one correspondence with those of $\tilde{X}$.

Now we assume that X is projective and non-singular. Then the intersection number $D_1 \cdots D_n$ of n divisors $D_1, \dots, D_n$ can be defined. It is invariant under equivalence and it is $\mathbb{Z}$-multilinear. In the case that the divisors are effective and intersect only in finitely many points $P_1, \dots, P_m$, the intersection number is the sum of the multiplicities of P_i in the scheme theoretic intersection $D_1 \cap \cdots \cap D_n$. For details we refer to [Sh].

We can consider divisors D_1, D_2, D_3 on $\mathcal{X}$ as divisors on $\mathcal{X}^+$ (also on $\mathcal{X}^-$) and then study intersection numbers of three divisors. These numbers may be different for $\mathcal{X}^+$ or $\mathcal{X}^-$. If one of the three divisors is equivalent to a divisor which avoids all nodes, then the intersection is independent on the choice of $\mathcal{X}^+$ or $\mathcal{X}^-$. For example, this is the case for hyperplane sections. They are all equivalent and there is one which avoids a finite number of given points.

In the following, the notation $D_1 \cdot D_2 \cdot D_3$ means the intersection number in $\mathcal{X}^+$ if not something else is stated. We have

$$D_1 \cdot D_2 \cdot D_3 = g(D_1) \cdot g(D_2) \cdot g(D_3) \quad \text{for all } g \in G.$$

We consider the divisors D^+ and D^- on $\mathcal{X}$. Their sum H is the divisor of a hyperplane section. In this section we want to compute the intersection numbers of three of the divisors H, D^+, D^- where one of them must be H. As we explained, they to not depend on the choice of the small resolution.

Proposition 1.2. *Let H be a divisor on $\mathcal{X}$ that represents a hyperplane section, i.e. $H \sim (X_0)$. Then*

$$H \cdot H \cdot H = 16.$$

(This is the degree of the complete intersection $\mathcal{X}$.)

Proof. The hyperplane sections (X_0), (X_1), (X_2) have (on $\mathcal{X}$) 16 intersection points,

$$(\pm 1, \pm \mathrm{i}, \pm \mathrm{i}, \pm 1, 0, 0, 0, 1).$$

None of them is a node. Their multiplicity is one. □

Proposition 1.3. *Let H be a divisor on $\mathcal{X}$ that represents a hyperplane section, i.e. $H \sim (X_0)$. Then*

$$H \cdot H \cdot D^+ = H \cdot H \cdot D^- = 8.$$

Proof. The biholomorphic transformation σ of $\mathcal{X}$ fixes the divisor class of H and D^+ and D^- are interchanged. Hence $H \cdot H \cdot D^+ = H \cdot H \cdot D^-$. Their sum is 16 by Proposition 1.2. (We should mention that the biholomorphic transformation σ extends to a biholomorphic map $\mathcal{X}^+ \to \mathcal{X}^-$. This shows that $H \cdot H \cdot D^+$, computed on $\mathcal{X}^+$, equals $H \cdot H \cdot D^-$, computed on $\mathcal{X}^-$. But as we noted already these intersection numbers are independent of the choice of a small resolution, since one of them is a hyperplane section.) □

Proposition 1.4. *Let H be a hyperplane section, i.e. $H \sim (Y_0)$. Then*

$$H \cdot D^+ \cdot D^- = 12, \quad H \cdot D^+ \cdot D^+ = H \cdot D^- \cdot D^- = -4.$$

Proof. The divisors (Y_0), D^+, D^- have 12 intersection points. None of them is a node. The multiplicities are 1. The rest follows with the help of Proposition 1.3. □

2. Intersections in exceptional fibres

We have to recall some details about the resolutions of a node. First, one can consider the blow up of a node. The exceptional fibre is biholomorphic equivalent to a $P^1 \times P^1$. After a biholomorphic map has been chosen, we can talk about horizontal lines $P^1 \times \{a_2\}$ and vertical lines $\{a_1\} \times P^1$. A different choice of the biholomorphic map can preserve "horizontal" and "vertical" or exchanges them. One can contract one of the two types of lines to obtain a small resolution with exceptional fibre P^1. Hence there exist two essentially different small resolutions. The choice of the small resolution is called a ruling of the node. We recall the following fact. Let $Y \subset \mathcal{X}$ be a pure two dimensional subvariety which is smooth at a node. Then the blow-up of Y gives one of the two small resolutions. The strict transform in the blow up of the node is a horizontal or vertical line.

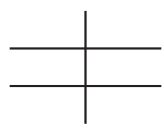
The following Lemma has been communicated to us by S. Cynk. It follows from the detailed description of the blow up of nodes in [Cl]. Let C be a smooth surface through a three dimensional node. Then the strict transform of C in the blow up of a is the blow up of C at a. From this one can deduce the following lemma.

Lemma 2.1. *Assume that a is a node in $\mathcal{X}$ and that Y_1, Y_2 are two smooth surfaces which contain a. In the case that the node a is not a (maybe embedded) component of the scheme theoretic intersection $Y_1 \cap Y_2$, the corresponding lines in $P^1 \times P^1$ must be parallel (including equal), i.e. both horizontal,* *or both vertical. In the other cases the two lines intersect properly (one horizontal, one vertical).*

Additional Remark. In the case of parallel lines, the two lines are equal if and only if the tangent planes of Y_1, Y_2 at a are the same.

Assume that we have three surfaces Y_1, Y_2, Y_3 which have the node a as isolated intersection point and which are smooth at a. We consider their strict transforms in the exceptional fibre $P^1 \times P^1$ of the blow up of the node. These can be three parallel lines (equal lines are considered to be parallel) or two parallel lines and a further line intersecting them.

Definition 2.2. *Let Y_1, Y_2, Y_3 be three surfaces in $\mathcal{X}$ which have the node a as isolated intersection point and which are smooth at a. They are called in good position (at a) if their strict transform in the blow up of the node consists of two different parallel lines and one which intersects them.*



We contract this figure in horizontal and vertical direction.

In the horizontal direction we see one line and two points on it. The intersection of these three is empty. In the vertical direction we see one line and one point on it. The intersection is one point.

Next we want to compute $D^+ \cdot D^+ \cdot D^-$. We replace D^+, D^+, D^- by linearly equivalent divisors D_1, D_2, D_3 which intersect in only finitely many points (moving lemma). Let μ be the number of intersection points in regular points. We notice that each node that is contained in D^+ is also contained in D^-, The reason is that $D^+ + D^-$ is locally principal close to a node, but D^+ is not. The same argument gives that D_1, D_2, D_3 meet in all the 24 nodes which are contained in D^+. The computation of the intersection number can be done analytically, (see [FS] for the description of the local Picard group at a node.) One can assume that D_1, D_2, D_3 are smooth surfaces which are in good position in the sense of Definition 2.2. Now we can use Lemma 2.1

to prove

$$D^+ \cdot D^+ \cdot D^- = \mu + 18.$$

The same argument gives

$$D^+ \cdot D^- \cdot D^- = \nu + 6.$$

Since the number of interior points agree, $\mu = \nu$ (use σ), we get for the difference

$$D^+ \cdot D^+ \cdot D^- - D^+ \cdot D^- \cdot D^- = 12.$$

From Proposition 1.4 we get

$$D^+ \cdot D^+ \cdot D^- + D^+ \cdot D^- \cdot D^- = 12.$$

Summing up the two equations and taking the difference we get

$$D^+ \cdot D^+ \cdot D^- = 12, \quad D^+ \cdot D^- \cdot D^- = 0.$$

We also have

$$\begin{aligned} D^+ \cdot D^+ \cdot D^+ &= D^+ \cdot \ D^+ \cdot H - D^+ \cdot D^+ \cdot \ D^- = -16, \\ D^- \cdot D^- \cdot D^- &= D^- \cdot \ D^- \cdot H - D^+ \cdot D^- \cdot \ D^- = -4. \end{aligned}$$

So we have proved the following result.

Proposition 2.3. *We have*

$$\begin{aligned} D^+ \cdot D^+ \cdot D^+ &= -16, \quad D^- \cdot D^- \cdot D^- = -4, \\ D^+ \cdot D^+ \cdot D^- &= 12, \quad D^+ \cdot D^- \cdot D^- = 0. \end{aligned}$$

3. Intersection numbers between translates

We want to determine all intersection numbers $g_1(D^\pm) \cdot g_2(D^\pm) \cdot g_3(D^\pm)$ for $g_i \in G$.

Proposition 3.1. *We have*

$$H \cdot H \cdot g(D^+) = H \cdot H \cdot g(D^-) = 8.$$

This follows from Proposition 1.3. □

Proposition 3.2. *We have*

$$H \cdot g(D^+) \cdot g(D^-) = 12, \quad H \cdot g(D^+) \cdot g(D^+) = H \cdot g(D^-) \cdot g(D^-) = -4.$$

This follows from Proposition 1.4. □

Proposition 3.3. *We have*

$$H \cdot g(D^+) \cdot h(D^+) = H \cdot g(D^+) \cdot h(D^-) = H \cdot g(D^-) \cdot h(D^-) = 4$$

for all $g \neq h$ in G.

Proof. We can assume that h but not g is the identity element. The intersection of $Y_0 = 0$, D^+ and $g(D^+)$ consists of 4 points as a computation shows. For each g they belong to the following list of 12 points.

$$\begin{array}{lll}
(0,0,0,-2,-1,-\mathrm{i},\mathrm{i},1), & (0,0,-2\mathrm{i},0,\mathrm{i},-\mathrm{i},-1,1), & (0,0,-2\mathrm{i},0,-\mathrm{i},\mathrm{i},-1,1),\\
(0,0,2\mathrm{i},0,\mathrm{i},-\mathrm{i},-1,1), & (0,2\mathrm{i},0,0,\mathrm{i},-1,-\mathrm{i},1), & (0,0,0,2,-1,-\mathrm{i},\mathrm{i},1),\\
(0,0,2\mathrm{i},0,-\mathrm{i},\mathrm{i},-1,1), & (0,-2\mathrm{i},0,0,-\mathrm{i},-1,\mathrm{i},1), & (0,0,0,2,-1,\mathrm{i},-\mathrm{i},1),\\
(0,2\mathrm{i},0,0,-\mathrm{i},-1,\mathrm{i},1), & (0,0,0,-2,-1,\mathrm{i},-\mathrm{i},1), & (0,-2\mathrm{i},0,0,\mathrm{i},-1,-\mathrm{i},1).
\end{array}$$

None of them is a node. They have multiplicity one. This shows $H \cdot g(D^+) \cdot h(D^+) = 4$. The rest follows from Proposition 3.1. □

Lemma 3.4. *Let $g \in G$ be different from the unit element. We denote by $\alpha(g)$ the contribution of the nodes to the intersection number of D^+, D^-, $g(D^+)$ and similarly by $\beta(g)$ the contribution of D^+, D^-, $g(D^-)$. We have*

$$(\alpha(g), \beta(g)) = \begin{cases} (2,2) & \text{for } 6 \text{ choices,} \\ (4,0) & \text{for } 6 \text{ choices,} \\ (6,2) & \text{for } 3 \text{ choices (of } g\text{).} \end{cases}$$

The proof rests on the computation of the rulings of all nodes which uses the program mentioned at the end of the introduction. □

Sinc $g(H)$ is equivalent to H, from Proposition 1.4 we get

$$D^+ \cdot D^- \cdot g(D^+) + D^+ \cdot D^- \cdot g(D^-) = 12.$$

The contribution of the interior points (away from the nodes) to $D^+ \cdot D^- \cdot g(D^+)$ and $D^+ \cdot D^- \cdot g(D^-)$ is the same number μ (use σ). Hence we get

$$D^+ \cdot D^- \cdot g(D^+) - D^+ \cdot D^- \cdot g(D^-) = \alpha(g) - \beta(g).$$

By means of the values of Lemma 3.4 and Proposition 3.3 we get the following result.

Proposition 3.5. *Let $g \in G$ be different from the unit element. We have*

$$\begin{aligned}
&6 \textit{ cases:} \quad D^+ \cdot D^- \cdot g(D^+) = 6, \quad D^+ \cdot D^- \cdot g(D^-) = 6,\\
&6 \textit{ cases:} \quad D^+ \cdot D^+ \cdot g(D^+) = -2, \quad D^+ \cdot D^+ \cdot g(D^-) = -2,\\
&6 \textit{ cases:} \quad D^- \cdot D^- \cdot g(D^+) = -2, \quad D^- \cdot D^- \cdot g(D^-) = -2,\\
&9 \textit{ cases:} \quad D^+ \cdot D^- \cdot g(D^+) = 8, \quad D^+ \cdot D^- \cdot g(D^-) = 4,\\
&9 \textit{ cases:} \quad D^+ \cdot D^+ \cdot g(D^+) = -4, \quad D^+ \cdot D^+ \cdot g(D^-) = 0,\\
&9 \textit{ cases:} \quad D^- \cdot D^- \cdot g(D^+) = -4, \quad D^- \cdot D^- \cdot g(D^-) = 0.
\end{aligned}$$

Next we intersect $D^\pm$ and $g(D^\pm)$ and $h(D^\pm)$ where $g \neq h$ and where both are different from the unit element.

Proposition 3.6. *Let (g, h) be two different elements of G. Both are assumed to be different from the unit element. There are $15 \cdot 14 = 210$ choices. In 174 cases the intersection points are not nodes. For them*

$$D^\pm \cdot g(D^\pm) \cdot h(D^\pm) = 2.$$

In the remaining 36 cases there are 4 intersection points in $\mathcal{X}$ which all are nodes. In these cases the intersection numbers in $\mathcal{X}^+$ are as follows.

$$D^\pm \cdot g(D^\pm) \cdot h(D^\pm) = 2 \textit{ in } 18 \textit{ cases.}$$

In the remaining 18 cases the situation is as follows.

$$D^\pm \cdot g(D^\pm) \cdot h(D^\pm) = \begin{cases} 4 & \textit{if the number of plus signs is odd,} \\ 0 & \textit{if the number of plus signs is even.} \end{cases}$$

Proof. There have to be treated the following cases. Let F be the subgroup of G that is generated by g and h.

1) The order of F is 16. There are 96 pairs with this property. There are two subcases:
   a) In 48 cases the intersection of $D^\pm$, $g(D^\pm)$, $h(D^\pm)$ consists of two points which are not nodes and which have multiplicity one and which both are defined over K.

   b) In 48 cases the intersection of $D^\pm$, $g(D^\pm)$, $h(D^\pm)$ consists of two points which are not nodes and which have multiplicity one and which both are *not* defined over K.

2) The order of F is 8. There are 72 pairs with this property. There are two subcases:
   a) In 36 cases there are two intersection points which are defined over $\mathbb{Q}(\mathrm{i})$. None of them is a node and the multiplicity is in each case 1.
   b) In 36 cases there are 4 intersection points in $\mathcal{X}$. They are all nodes. One has to take the intersection numbers in the small resolution $\mathcal{X}^+$. Here D^+, $g(D^+)$, $h(D^+)$ have intersection number 2 in 18 cases and 4 in 18 cases. The other intersection numbers $D^+ \cdot g(D^+) \cdot h(D^+)$ can be determined with the help of Proposition 3.3.

3) The order of F is 4. There are 42 cases. In all of them there are 2 intersection points which are defined over $\mathbb{Q}(\mathrm{i})$. They are not nodes and they have multiplicity 1.

In each of the cases only one sign combination, for example three plus signs, have to be treated. The other combinations then one can obtain with the help of Proposition 3.3.

The only really difficult case is 2b). In all other cases we have intersection points only outside of the nodes and we can work with the variety $\mathcal{X}$ and have not to desingularize it. As in Sect. 2, the intersection points can be computed in each case explicitly and after that one can compute their multiplicities.

We explain case 2b) for the rest of this section. We will consider two typical examples. The first example is

$$\begin{aligned} g &:= (-Y_0, Y_1, -Y_2, Y_3, X_0, -X_1, -X_2, X_3), \\ h &:= (\mathrm{i}Y_3, -\mathrm{i}Y_2, -Y_1, -Y_0, X_2, -\mathrm{i}X_3, \mathrm{i}X_0, X_1). \end{aligned}$$

Recall that D^+ is a component of the hyperplane section $X_0 + X_1 + X_2 + X_3 = 0$. The intersection of the three divisors

$$D^+, \quad g(D^+), \quad h(D^+),$$

considered in $\mathcal{X}$, consists of 4 nodes.

$$\begin{aligned} a_1 &:= (\sqrt{2}, 0, 0, -\mathrm{i}\sqrt{2}, 0, -1, 1, 0), \\ a_2 &:= (\sqrt{2}, 0, 0, \mathrm{i}\sqrt{2}, 0, -1, 1, 0), \\ a_3 &:= (-\sqrt{2}, 0, 0, -\mathrm{i}\sqrt{2}, 0, -1, 1, 0), \\ a_4 &:= (-\sqrt{2}, 0, 0, \mathrm{i}\sqrt{2}, 0, -1, 1, 0). \end{aligned}$$

Unfortunately the three divisors are not in good position with respect to all nodes $a_1, \dots, a_4$ in the sense of Definition 2.2. For example, at the second node a_2 the following happens. The curves $D^+ \cap h(D^+)$ and $g(D^+) \cap h(D^+)$ have a_2 as isolated point. But the intersection of D^+ and $h(D^+)$ in $\mathcal{X}$ is set theoretically an elliptic curve, cut out in $\mathcal{X}$ by the linear equations

$$Y_1 = Y_2 = X_0 + X_3 = X_1 + X_2.$$

But there are four embedded components in this curve, namely the 4 nodes. Hence the three lines in the exceptional $P^1 \times P^1$ of the node are parallel.

To remedy this situation, we replace the divisor D^+ by the other component D^-. From Proposition 3.3 we see that

$$D^+ \cdot g(D^+) \cdot h(D^+) + D^- \cdot g(D^+) \cdot h(D^+) = 4.$$

Hence we see

$$D^+ \cdot g(D^+) \cdot h(D^+) = 2 \Longleftrightarrow D^- \cdot g(D^+) \cdot h(D^+) = 2.$$

We use now the notation

$$D_1 = D^-, \quad D_2 = g(D^+), \quad D_3 = h(D^+).$$

a) The intersection of $D_1 \cap D_2 \cap D_3$ consists of the same 4 nodes $a_1, \dots, a_4$.

b) The three divisors D_1, D_2, D_3 are in good position at all 4 nodes. We have to compare the rulings of the nodes a_1, a_2, a_3, a_4. One checks

$$a_2 = g(a_1), \quad a_4 = g(a_3).$$

We want to compare the number of intersection points of D_1, D_2, D_3 over the node a_1 with those over a_2. Instead of this we can compare the number of intersection points of D_1, D_2, D_3 over the node a_1, the number of intersection points of $g^{-1}(D_1), g^{-1}(D_2), g^{-1}(D_2)$ over a_1. One can compute the positions of the corresponding line diagrams. It turns out that the two diagrams are different. So we obtain the following result.

The divisors D_1, D_2, D_3 have an intersection point in the small resolution over a_1 or a_2 but not over both. The same argument works for the pair a_3, a_4.

Hence we get

$$D_1 \cdot D_2 \cdot D_3 = 2.$$

We give a second example.

$$g := (\mathrm{i}Y_3, \mathrm{i}Y_2, -Y_1, Y_0, -X_2, -\mathrm{i}X_3, \mathrm{i}X_0, -X_1),$$
$$h := (\mathrm{i}Y_3, -\mathrm{i}Y_2, -Y_1, -Y_0, X_2, -\mathrm{i}X_3, \mathrm{i}X_0, X_1).$$

In this case one checks that D^+, $g(D^+)$, $h(D^+)$ are in good position at all 4 nodes in their intersection. We can check that in $\mathcal{X}^+$ over each node an intersection point survives. Hence we get

$$D^+ \cdot g(D^+) \cdot h(D^+) = 4$$

It follows (or can be computed in the same way)

$$D^- \cdot g(D^+) \cdot h(D^+) = 0$$

We settled two examples. Each other case can be treated in the same manner. This completes the proof of Proposition 3.6, 2b). □

4. Intersection numbers in the quotient

We recall from the paper [FS].

Theorem 4.1. *The orbit of the subvariety D^+ (the same is true for D^-) under the group G consists of 16 subvarieties of $\mathcal{X}$. They are non-singular. Each of the 96 nodes is contained in 4 of the 16 subvarieties. The blow up along the union of the 16 subvarieties (considered as reduced subvariety) is a smooth projective variety $\mathcal{X}^+$ with a free G-action. The quotient $\mathcal{Y}^+ = \mathcal{X}^+/G$ is a (projective) rigid Calabi-Yau manifold with $h^{11} = 2$ ($e = 4$).*

From [FS] we know the following result.

Proposition 4.2. *We denote by $\mathcal{D}^{\pm}$ the image of $D^{\pm}$ in $\mathcal{X}^+/G$. The two divisors give a $\mathbb{Q}$-basis of*

$$\mathrm{Pic}(\mathcal{X}^+/G) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

We want to compute the intersection numbers $\mathcal{D}^{\pm} \cdot \mathcal{D}^{\pm} \cdot \mathcal{D}^{\pm}$. These are essentially 4 cases, depending on the number of the plus (or minus) signs.

Let $\pi : \mathcal{X} \to \mathcal{X}/G$ be the canonical projection. We use the formula

$$16\mathcal{D}^\pm \cdot \mathcal{D}^\pm \cdot \mathcal{D}^\pm = \pi_*(\pi^*\mathcal{D}^\pm \cdot \pi^*\mathcal{D}^\pm \cdot \pi^*\mathcal{D}^\pm).$$

It implies

$$16\mathcal{D}^\pm \cdot \mathcal{D}^\pm \cdot \mathcal{D}^\pm = \pi^*\mathcal{D}^\pm \cdot \pi^*\mathcal{D}^\pm \cdot \pi^*\mathcal{D}^\pm.$$

We have

$$\pi^*\mathcal{D}^\pm = \sum_{g \in G} g(D^\pm).$$

We get

$$\begin{aligned} \mathcal{D}^\pm \cdot \mathcal{D}^\pm \cdot \mathcal{D}^\pm &= \frac{1}{16} \sum_{g_1, g_2, g_3} g_1(D^\pm) \cdot g_2(D^\pm) \cdot g_3(D^\pm) \\ &= \sum_{(g,h) \in G \times G} D^\pm \cdot g(D^\pm) \cdot h(D^\pm). \end{aligned}$$

With the results of the previous section we can derive now our main result.

Theorem 4.3. *The two divisors $\mathcal{D}^\pm$ of the rigid Calabi-Yau manifold $\mathcal{Y}^+ := \mathcal{X}^+/G$ give a basis of $\operatorname{Pic}\mathcal{Y}^+ \otimes_{\mathbb{Z}} \mathbb{Q}$. Their intersection numbers are*

$$\begin{aligned} \mathcal{D}^+ \cdot \mathcal{D}^+ \cdot \mathcal{D}^+ &= 296, \quad \mathcal{D}^- \cdot \mathcal{D}^- \cdot \mathcal{D}^- = 344, \\ \mathcal{D}^+ \cdot \mathcal{D}^+ \cdot \mathcal{D}^- &= 600, \quad \mathcal{D}^+ \cdot \mathcal{D}^- \cdot \mathcal{D}^- = 552. \end{aligned}$$

We recall that $\mathcal{Y}^-$ is biholomorph equivalent to $\mathcal{Y}^+$. The biholomorphic map σ between both interchanges $\mathcal{D}^+$ and $\mathcal{D}^-$.

Acknowledgements. We want to thank Sergey V. Ketov, who motivated us to determine the intersection numbers which play a role in string theory [Al]. We are grateful to S. Cynk who explained us several details about the resolution of nodes and we thank also R. Salvati Manni for widespread discussions and suggestions.

References

[Al] S. Alexandrov, *Twistor approach to string compactifications: a review*, Phys. Rept. **522** (2013), 1–57.

[Cl] C. H. Clemens, *Double solids*, Advances in Math. **47** (1983), 101–230.

[CM] S. Cynk and C. Meyer, *Modular Calabi-Yau threefolds of level eight*, Internat. J. Math. **18** (2007), no. 3, 331–347.

[FS] E. Freitag and R. Salvati-Manni, *On Siegel threefolds with a projective Calabi–Yau model*, Commun. Number Theory Phys. **5** (2011), no. 3, 713–750.

[GN] B. van Geemen and N. O. Nygaard, *On the geometry and arithmetic of some Siegel modular threefolds*, Journal of Number Theory **53** (1995), 45–87.

[Sh] I. R. Shafarevich, *Basic algebraic geometry*, Springer-Verlag, Berin / Heidelberg / New York, (1974)

Mathematisches Institut
Im Neuenheimer Feld 288
D69120 Heidelberg, Germany
E-mail address: freitag@mathi.uni-heidelberg.de

Received February 25, 2016