Communications in
Number Theory and Physics
Volume 10, Number 3, 373–431, 2016

Vertical sheaves and Fourier-Mukai transform on elliptic Calabi-Yau threefolds

Duiliu E. Diaconescu

This paper studies the action of the Fourier-Mukai transform on moduli spaces of vertical torsion sheaves on elliptic Calabi-Yau threefolds in Weierstrass form. Moduli stacks of semistable one dimensional sheaves on such threefolds are identified with open and closed substacks of moduli stacks of vertical semistable two dimensional sheaves on their Fourier-Mukai duals. In particular, this yields explicit conjectural results for Donaldson-Thomas invariants of vertical two dimensional sheaves on K3-fibered elliptic Calabi-Yau threefolds.

1. Introduction

Starting with Mukai's work on the subject [43, 44], Fourier-Mukai functors have played a central role in the study of moduli spaces of stable sheaves on algebraic varieties. An incomplete list of applications of Fourier-Mukai functors to moduli spaces of torsion free sheaves on surfaces includes [4, 5, 8, 10, 30, 49, 54, 56–59]. Further applications to moduli spaces of torsion free sheaves on elliptic threefolds and higer dimensional elliptic fibrations include [3, 7, 12–14, 18, 19]. A comprehensive review of the subject and a more complete list of results can be be found in [6]. More recently, t-structures and moduli problems of Bridgeland stable objects in the derived category have been studied in [9, 15, 35–38, 41, 42, 60] using a similar approach.

Of particular importance for the present paper is the relative Fourier-Mukai transform for elliptic fibrations. This was constructed by Bartocci et al [8] and Bridgeland [10, 11] for elliptic surfaces and Friedman, Morgan and Witten [18, 19] for stable bundles on elliptic threefolds. The foundational results for elliptic threefolds used in this paper were proven by Bridgeland and Macciocia in [12]. The higher dimensional construction was carried out by Bartocci et al in [7].

An important problem in this framework is whether the Fourier-Mukai transform preserves Gieseker, or, for torsion sheaves, Simpson stability. In particular if Fourier-Mukai transform yields isomorphisms of moduli spaces of semistable sheaves. Several results obtained in the literature prove that this is the indeed the case for suitable open subspaces of moduli spaces parameterizing relatively semistable objects. However isomorphisms of proper moduli spaces are much harder to prove. One such result was obtained by Yoshioka in [56], showing that Fourier-Mukai transform identifies moduli spaces of semistable pure dimension one sheaves on an elliptic surface with moduli spaces of semistable torsion-free sheaves on the dual surface. The main goal of the present paper is to study the analogous problem for pure dimension one sheaves on elliptic threefolds. As explained in more detail below this problem is mainly motivated by applications to Donaldson-Thomas invariants [31, 51] of pure dimension two sheaves and modularity questions.

1.1. The main result

Let $p : X \to B$ be a smooth projective Weierstrass model with trivial canonical class over a smooth Fano surface B. The Mukai dual $\widehat{X}$ of X was constructed in [12] as a fine relative moduli space for rank one degree zero torsion free sheaves on the fibers of $p : X \to B$. For sufficiently generic X the dual $\widehat{X}$ is again a smooth Weietrstrass model $\hat{p} : \widehat{X} \to B$ and there is a canonical isomorphism $\widehat{X} \simeq X$ over B. Since $\widehat{X}$ is a fine moduli space, there is a (non-unique) universal Poincaré sheaf $\mathcal{P}$ on $\widehat{X} \times X$. The Fourier-Mukai functor $\Phi : D^b(\widehat{X}) \to D^b(X)$ with kernel $\mathcal{P}$ was proven in [12] to be an equivalence of derived categories. Moreover it was also shown there in that the inverse functor $\widehat{\Phi} : D^b(X) \to D^b(\widehat{X})$ is also a Fourier-Mukai transform whose kernel $\mathcal{Q}$ is the derived dual $\mathcal{P}$ up to a shift. A more detailed summary is provided in Section 3.1.

The main goal of this paper is to study the action of the above Fourier-Mukai functors on moduli stacks of Simpson semistable torsion sheaves on $X, \widehat{X}$. In order to formulate a concrete statement one first needs a concrete presentation of the Kähler cones and the homology groups of $X, \widehat{X}$. As shown in Lemma 2.1, one has an isomorphism

$$\mathrm{Pic}(X)/_{\mathrm{torsion}} \simeq \mathbb{Z}\langle\Theta\rangle \oplus p^*\mathrm{Pic}(B),$$

where Θ is the image of the canonical section $\sigma : B \to X$. Then any Kähler class $\omega \in \mathrm{Pic}_{\mathbb{R}}(X)$ can be written as $\omega = t\Theta + p^*\eta$, with $t \in \mathbb{R}$, $t > 0$ and $\eta \in \mathrm{Pic}_{\mathbb{R}}(B)$ a sufficiently ample Kähler class on B. In particular $\omega = t\Theta - sp^*K_B$ is a Kähler class on X for $s > t > 0$, where K_B is the canonical class of B. Lemma 2.1 also shows that there is a natural isomorphism $\mathrm{Pic}(X) \simeq H_4(X, \mathbb{Z})$ which will be used implicitely throughout this paper. In particular the pairing between Kähler classes and homology classes will be identified with the intersection product. Using Poincaré duality, Chern classes of sheaves on X will be also regarded as even homology classes. Finally, note the direct sum decomposition

$$H_2(X, \mathbb{Z})/_{\mathrm{torsion}} \simeq \mathbb{Z}\langle f\rangle \oplus \sigma_*H_2(B, \mathbb{Z})$$

where $\sigma : B \to X$ is the canonical section of the Weierstrass model and f is the elliptic fiber class. Of course, completely analogous statements hold for $\hat{p} : \widehat{X} \to B$, the notation being obvious.

This paper will concrentrate on the relation between pure dimension one sheaves on $\widehat{X}$ and vertical pure dimension two sheaves on X. According to Definition 2.3.i, a sheaf E on X of pure dimension two is vertical if $\mathrm{ch}_1(E) \cdot f = 0$ and $\mathrm{ch}_2(E)$ is a multiple of the fiber class f. The discrete invariants of a sheaf $\widehat{F}$ of pure dimension one on $\widehat{X}$ are given by an element

$$\hat{\gamma} = (\hat{\gamma}_i)_{1\leq i\leq 3} \in H_2(B, \mathbb{Z}) \oplus \mathbb{Z} \oplus \mathbb{Z}$$

where

$$\mathrm{ch}_2(\widehat{F}) = \hat{\sigma}_*(\hat{\gamma}_1) + \hat{\gamma}_2\hat{f}, \qquad \chi(\widehat{F}) = \hat{\gamma}_3.$$

The discrete invariants of a vertical sheaf E on X of pure dimension two are given by

$$\gamma = (\gamma_i)_{1\leq i\leq 3} \in H_2(B, \mathbb{Z}) \oplus (1/2)\mathbb{Z} \oplus \mathbb{Z},$$

where

$$\mathrm{ch}_1(E) = p^*\gamma_1, \qquad \mathrm{ch}_2(E) = \gamma_2 f, \qquad \mathrm{ch}_3(E) = -\gamma_3\mathrm{ch}_3(\mathcal{O}_x)$$

with $x \in X$ an arbitrary closed point. According to equations (3.8), (3.9), the induced action of Fourier-Mukai transform on topological invariants is encoded in the group isomorphism

$$\phi : H_2(B,\mathbb{Z}) \oplus \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\sim} H_2(B,\mathbb{Z}) \oplus (1/2)\mathbb{Z} \oplus \mathbb{Z},$$
$$\phi(\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3) = (\hat{\gamma}_1, \hat{\gamma}_3 + K_B \cdot \hat{\gamma}_1/2, \hat{\gamma}_2).$$

Here $\cdot$ denotes the intersection product on B.

Note also that Definition 2.3.ii introduces a notion of adiabatic stability for vertical sheaves on X which plays an important part in this paper. Given a Kähler class $\omega = t\Theta + p^*\eta$, a vertical pure dimension two sheaf E is called ω-adiabatically semistable if and only if it is Simpson semistable with respect to all Kähler classes $\omega' = t'\Theta + p^*\eta$, where $0 < t' \leq t$.

Given Kähler classes $\omega = t\Theta - sp^*K_B$, $\hat{\omega} = \widehat{\Theta} - s\hat{p}^*K_B$ with $s > t > 0$, $s > 1$, let $\mathcal{M}_{\hat{\omega}}(\widehat{X}, \hat{\gamma})$, $\mathcal{M}_\omega(X,\gamma)$ denote the moduli stacks of Simpson semistable sheaves with $\hat{\gamma}$, γ on $\widehat{X}$, X, respectively. Let $\mathcal{M}^{\mathsf{ad}}_\omega(X,\gamma) \subset \mathcal{M}_\omega(X,\gamma)$ be the substack of adiabatically semistable sheaves as defined in 2.3.ii. Then the main result of the present paper is

Theorem 1.1. *Let $\hat{\gamma} \in H_2(B,\mathbb{Z}) \oplus \mathbb{Z} \oplus \mathbb{Z}$ be fixed topological invariants such that $\hat{\gamma}_3 > 0$. Then there exists a constant $s_1(\hat{\gamma}) \in \mathbb{R}$, $s_1(\hat{\gamma}) > 1$, depending on $\hat{\gamma}$, such that for any $s \in \mathbb{R}$, $s > s_1(\hat{\gamma})$, there exists a second constant $t_1(\hat{\gamma}, s) \in \mathbb{R}$, $0 < t_1(\hat{\gamma}, s) < 1$, depending on $(\hat{\gamma}, s)$, such that the following statements hold for any $t \in \mathbb{R}$, $0 < t < t_1(s, \hat{\gamma})$.*

(i) The Fourier-Mukai transform Φ yields an isomorphism of moduli stacks

$$\varphi : \mathcal{M}_{\hat{\omega}}(\widehat{X}, \hat{\gamma}) \xrightarrow{\sim} \mathcal{M}^{\mathsf{ad}}_\omega(X,\gamma),$$

where $\hat{\omega} = \Theta - s\hat{p}^*K_B$, $\omega = t\Theta - sp^*K_B$ and $\gamma = \phi(\hat{\gamma})$.

(ii) The substack $\mathcal{M}^{\mathsf{ad}}_\omega(X,\gamma) \subset \mathcal{M}_\omega(X,\gamma)$ is open and closed in $\mathcal{M}_\omega(X,\gamma)$.

The proof of Theorem 1.1 is given in Section 3 and requires some preliminary results proven in Section 2. In comparison with the analogous result for elliptic surfaces [56, Thm. 3.15], one needs to introduce a suitable notion of generic stability for vertical pure dimension two sheaves in Definition 2.5. Then one has to further check that generic stability is equivalent to adiabatic stability in Lemmas 2.11 and 2.12. The proof is then given step-by-step in Section 3. In contrast with [56, Thm 3.15], one cannot rule out non-adiabatic components of the moduli stack of semistable pure dimension two sheaves on a threefold by taking an appropriate limit of the Kähler class. However, as shown below, such components can be ruled out for elliptic threefolds which also admit a $K3$-fibration structure, and for two dimensional sheaves supported on the $K3$ fibers.

1.2. Sheaf counting on elliptic $K3$ pencils

As stated in the second paragraph of the introduction, Theorem 1.1 is mainly motivated by applications to Donaldson-Thomas invariants of pure dimension two sheaves on elliptic Calabi-Yau threefolds. These are counting invariants defined in [51] for stable sheaves and generalized in [31, 34] for semistable ones. Generating series of Donaldson-Thomas invariants for pure dimension two sheaves have been conjectured to have modular properties in [16, 20]. In the mathematics literature, this conjecture has been proven for certain cases in [21, 22, 53]. In particular explicit results for Donaldson-Thomas invariants of such sheaves on $K3$ fibered Calabi-Yau threefolds were obtained by Gholampour and Sheshmani in [21]. For nodal $K3$ pencils these results are restricted to rank one torsion free sheaves on reduced $K3$ fibers.

On the other hand, string theoretic arguments [33, 40] lead to a conjectural identification of Donaldson-Thomas invariants for vertical pure dimension two sheaves on an elliptic threefold X with genus zero Gopakumar-Vafa invariants on its dual $\widehat{X}$. This correspondence was first conjectured in [40] for sheaves supported on a rational elliptic surface inside X. As observed in [26], in that case this follows from the results of [56]. As it stands, Theorem 1.1 does not prove such an identification for general vertical sheaves because the moduli stack $\mathcal{M}_\omega(X,\gamma)$ can in principle have other components in addition to $\mathcal{M}^{\mathsf{ad}}_\omega(X,\gamma)$. From a string theory point of view it is natural to conjecture that such components are absent for sufficiently small $t_1(\hat{\gamma}, s)$, but mathematically this is an open problem.

As shown in below, there is however one situation where such extra components can be ruled out. Excepting $\mathbb{P}^2$, all smooth Fano surfaces B have a natural projection $\rho : B \to \mathbb{P}^1$, which induces a projection $\pi = \rho \circ p : X \to \mathbb{P}^1$. The generic fiber of ρ is a smooth reduced elliptic $K3$ surface on X. Moreover if B is a Hirzebruch surface $\mathbb{F}_a$, $0 \leq a \leq 1$, for sufficiently generic X, all fibers are reduced irreducible $K3$ surfaces with at most nodal singularities. Under this assumptions, Proposition 1.2 shows that no extra components are present in the moduli space of semistable vertical sheaves supported on $K3$ fibers for suitable Kähler classes. Therefore, in such cases Theorem 1.1 yields explicit conjectural results for generalized Donaldson-Thomas invariants of two dimensional sheaves supported on the $K3$ fibers, verifying the modularity conjecture.

In more detail, suppose B is the total space of the projective bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a))$ with $0 \leq a \leq 1$. Let $\rho : B \to \mathbb{P}^1$ and $\pi = \rho \circ p : X \to \mathbb{P}^1$. For sufficiently generic X the fibers of π are reduced irreducible $K3$-surfaces with at most nodal singularities. Recall that $\hat{\sigma} : B \to \widehat{X}$ denotes the canonical section and $\hat{f}$ denotes the fiber class of $\hat{p} : \widehat{X} \to B$. Let Ξ denote the fiber class of $\rho : B \to \mathbb{P}^1$. The $K3$ fiber class on X is $D = p^*\Xi$. Then note the following.

Proposition 1.2. *Let $\omega = t\Theta - sp^*K_B$ with $s, t \in \mathbb{R}$, $s > t > 0$. Let $\gamma = (rD, l, m)$ be arbitrary discrete invariants with $r, l, m \in \mathbb{Z}$, $r \geq 1$. Then there exists a constant $t_2(\gamma, s) \in \mathbb{R}$, $t_2(\gamma, s) > 0$ such that $\mathcal{M}^{\mathsf{ad}}_\omega(X, \gamma) = \mathcal{M}_\omega(X, \gamma)$ for any $0 < t < t_2(\gamma, s)$ such that $t/s \in \mathbb{R} \setminus \mathbb{Q}$.*

The proof of Proposition 1.2 is given in Section 4. It should be noted that similar results for torsion free sheaves on elliptic surfaces have been obtained before in [17, Thm. I.3.3], [46, Prop. I.1.6] and [55, Lemma 1.2]. Here one has to generalize these results to semistable pure dimension two sheaves supported on scheme theoretic thickenings of divisors in the $K3$ pencil, including nodal fibers. This requires a careful reduction to the reduced smooth surface case via Jordan-Hölder filtrations and blow-ups.

Next consider topological invariants $\hat{\gamma} = (r\Xi, n, k)$ in Theorem 1.1, where $r, n, k \in \mathbb{Z}$, $r, k \geq 1$, $n \geq 0$. Then equations (3.9) yield

$$\gamma_1 = rD, \qquad \gamma_2 = (k - r)f, \qquad \gamma_3 = -n. \tag{1.1}$$

Let $DT_{\hat{\omega}}(\widehat{X}; r, n, k) \in \mathbb{Q}$ denote the generalized Donaldson-Thomas invariants counting $\hat{\omega}$-semistable pure dimension one sheaves on $\widehat{X}$ constructed in [31]. According to [31, Thm 6.16.a], these invariants are independent of $\hat{\omega}$, hence the subscript will be dropped in the following. Moreover it is conjectured in [31, Conj. 6.12] that there exist integral invariants $\Omega(\widehat{X}; r, n, k) \in \mathbb{Z}$ related to the rational ones by the multicover formula

$$DT(\widehat{X}; r, n, k) = \sum_{\substack{m \in \mathbb{Z},\ m \geq 1, \\ m | (r,n,k)}} \frac{1}{m^2} \Omega(\widehat{X}; r/m, n/m, k/m). \tag{1.2}$$

Alternatively the integral invariants can be conjecturally defined directly by specialization of the motivic invariants of Kontsevich and Soibelman [34] as explained in Section 7.1 of loc.cit.

For a primitive vector $\hat{\gamma} = (r, n, k)$ there are no strictly semistable objects, and the invariants $DT(\widehat{X}; r, n, k) = \Omega(\widehat{X}; r, n, k)$ specialize to the integral virtual cycle invariants defined in [51]. In particular this holds for $k = 1$.

Then the resulting invariants were conjecturally identified with genus zero Gopakumar-Vafa invariants in [32],

$$\Omega(\widehat{X}, r, n, 1) = n_0(\widehat{X}, r, n) \tag{1.3}$$

for any $(r,n) \in \mathbb{Z}^2$, $r, n \geq 0$, $(r,n) \neq (0,0)$. Here $n_0(r,n)$ denotes the genus zero Gopakumar-Vafa invariant for curve class $r\hat{\sigma}(\Xi) + n\hat{f}$. As shown in [52], equation (1.3) follows from the GW/stable pair correspondence conjectured in [47] provided that the integral invariants $\Omega(\widehat{X}; r, n, k) \in \mathbb{Z}$ are independent of k for fixed (r,n). Independence of k is a special case of [52, Conj. 6.13].

Let $DT_\omega(X, r, l, m)$ denote the generalized Donaldson-Thomas invariants counting ω-semistable vertical two dimensional sheaves with invariants $\gamma = (r\Xi, l, m)$. The wallcrossing formulas of [31, 34] imply easily that the invariants $DT_\omega(X, r, l, m)$ are independent of ω, hence the subscript may be dropped. Again, Conjecture 6.12 in [31] states the existence of integral invariants $\Omega(X, r, l, m)$ related to the rational ones by a multicover formula of the form (1.2).

Then Theorem 1.1 and Proposition 1.2 imply that

$$DT(\widehat{X}; r, n, k) = DT(X, r, k-r, n)$$

for any $r, n, k \in \mathbb{Z}$, $r, k \geq 1$, $n \geq 0$. Granting the existence of integral invariants, they will be also related by

$$\Omega(\widehat{X}; r, n, k) = \Omega(X, r, k-r, n).$$

However note that there is an isomorphism of moduli stacks $\mathcal{M}_\omega(X, r, l, m) \simeq \mathcal{M}_\omega(X, r, l-r, m)$ for any (r, l, m). This is obtained by taking tensor product by the line bundle $p^*\mathcal{O}_B(-C_0)$, where C_0 is a section of the ruling $\rho : B \to \mathbb{P}^1$. For concreteness let C_0 be the unique section with $C_0^2 = -1$ for $B = \mathbb{F}_1$ and an arbitrary section with $C_0^2 = 0$ for $B = \mathbb{F}_0$. Therefore

$$\Omega(\widehat{X}; r, n, k) = \Omega(X, r, k, n). \tag{1.4}$$

for any (r, n, k), $r, k \geq 1$, $n \geq 0$.

Now let

$$Z_{X,r,k}(q) = \sum_{n\in\mathbb{Z}} \Omega(X, r, k, n) q^{n-r/2}.$$

and suppose for concreteness that $B = \mathbb{F}_1$. Then, granting the invariance of $DT(\widehat{X}, r, n, k)$ under translations $k \mapsto k+1$ and the identification (1.3) one obtains

$$Z_{X,r,k}(q) = \sum_{n\geq 0} n_0(\widehat{X}, r, n) q^{n-r/2}$$

for any $r, k \in \mathbb{Z}$, $r, k \geq 1$.

Granting the identification (1.3), an explicit formula for the series $Z_{X,r,k}(q)$ follows from the work of Maulik and Pandharipande [39] on Gopakumar-Vafa invariants of K3 pencils. The explicit computation for the Weierstrass model over $\mathbb{F}_1$ was done by Rose and Yui [48]. The formula obtained in [48, Thm. 7.5] is written in terms of a certain transformation of modular forms defined in [48, Def. 7.1]. Let

$$f(z) = \sum_n a_n z^n$$

be modular form for $SL(2,\mathbb{Z})$ and $r, k \in \mathbb{Z}$, $r \geq 1$. Then define $f_{r,k}(z)$ by

$$f_{r,k}(z) = \sum_{n=0}^{\infty} a_{rn+k'} z^{rn+k'}$$

where $0 \leq k' < r$ is the unique integer in this range such that $k' \equiv k$ (mod r). Note that this is modular form for the subgroup $\Gamma_1(r^2) \subset SL(2,\mathbb{Z})$ of the same weight as $f(z)$. Then identity (1.4) and [48, Thm 7.5] yield the following conjectural formula:

$$Z_{X,r,k}(q) = -2 \sum_{\ell=0}^{r-1} \left(\frac{1}{\Delta(u)}\right)_{r,\ell-1} E_{10}(u)_{r,1-\ell} \tag{1.5}$$

where $q = u^r$. Here $\Delta(u) = u \prod_{r=1}^{\infty}(1-u^r)^{24}$ is the discriminant cusp form of weight 12 and $E_{10}(u)$ is the weight 10 Eisenstein series.

To conclude, note two natural open problems emerging from the present work. One open question in the context of Theorem 1.1 is whether there exists a sufficiently small constant $t_1(\hat{\gamma}, s)$ such that the moduli stack $\mathcal{M}_\omega(X,\gamma)$ coincides with the substack of adiabatically semistable objects. String theoretic arguments [33, 40] lead to the conjecture that this is indeed the case. As shown in Proposition 1.2, this holds in the special case of vertical sheaves on elliptic K3 pencils. The proof given in Section 4 relies on Bogomolov inequality and the algebraic Hodge theorem for surfaces. This leads to the interesting question whether analogous tools can be developed in general for vertical sheaves on elliptic threefolds.

The second open problem is whether formula (1.5) can be given a direct proof using degeneration techniques as in [21].

Acknowledgements. This work is motivated by a related project initiated in collaboration with Vincent Bouchard, Thomas Creutzig, Chuck Doran, Terry Gannon and Callum Quigley. I am very grateful to them for very stimulating conversations and sharing their insights with me. I would also like to thank Lev Borisov, Ugo Bruzzo, Ron Donagi, Tony Pantev and Noriko Yui for very helpful discussions. The author also acknowledges the partial support of NSF grant DMS-1501612 during the completion if this work.

2. Vertical sheaves and adiabatic stability

This section introduces adiatically semistable vertical sheaves on elliptic Weierstrass models, and shows that adiabatic stability is equivalent to a natural notion of generic stability.

2.1. Basics of Weirestrass models

Let B be a smooth projective del Pezzo surface. Let $p : X \to B$ be a smooth generic Weierstrass model with canonical section $\sigma : B \to X$. Let $\Theta \subset X$ denote the image of the canonical section. Then Θ determines a homology class in $H_4(X, \mathbb{Z})$ as well as a divisor class in $\mathrm{Pic}(X)$. Let $f \in H_2(X, \mathbb{Z})$ denote the class of the elliptic fiber. The same notation $\cdot$ will be used for the intersection product on X, as well as B. The distinction will be clear from the context.

Lemma 2.1. *There are direct sum decompositions*

$$H_4(X, \mathbb{Z})/_{\mathrm{torsion}} \simeq \mathbb{Z}\langle\Theta\rangle \oplus p^*H_2(B, \mathbb{Z}) \qquad (2.1)$$
$$H_2(X, \mathbb{Z})/_{\mathrm{torsion}} \simeq \mathbb{Z}\langle f\rangle \oplus \sigma_*H_2(B, \mathbb{Z}),$$

Moreover, there is an isomorphism $\mathrm{Pic}(X) \simeq H^4(X, \mathbb{Z})$.

Proof. One proceeds by analogy with [48, Lemma 6.1]. Note that $h^{0,i}(X) = 0$, $i \in \{1, 2\}$, and $h^{1,1}(X) = h^{1,1}(B) + 1$ according to [27, Sect. 11]. This implies that there is an isomorphism $\mathrm{Pic}(X) \simeq H^2(X, \mathbb{Z})$. By Alexander-Lefschetz duality, there is also an isomorphism $H^2(X, \mathbb{Z}) \simeq H_4(X, \mathbb{Z})$. Next recall that $\mathrm{Pic}(B) \simeq H_2(B, \mathbb{Z})$ is freely generated by rational curve classes $C_1, \ldots, C_{h^{1,1}(B)}$ such that the intersection matrix $I_B = (C_i \cdot C_j)_{1\leq i,j\leq h^{1,1}(B)}$ has determinant $|\det(I_B)| = 1$. Let $D_i = p^*C_i \in \mathrm{Pic}(X) \simeq H_4(X, \mathbb{Z})$, $1 \leq i \leq h^{1,1}(B)$. Let I_X denote the intersection matrix between the divisor classes

$\Theta, D_1, \ldots, D_{h^{1,1}(B)}$ and the curve classes $f, \sigma_*(C_1), \ldots, \sigma_*(C_{h^{1,1}(B)})$ on X. Straightfowrard intersection computations show that $|\det(I_X)| = 1$ as well. This implies the isomorphisms claimed above. □

As explained in Section 1.1, the isomorphism $\mathrm{Pic}(X) \simeq H_4(X, \mathbb{Z})$ following from Lemma 2.1 will be implicitely used throughout this paper. Moreover, Chern classes of sheaves on X will be identified with homology classes by Poincaré duality. Then note the following.

Corollary 2.2. *(i) A real divisor class*

$$\omega = t\Theta + p^*\eta, \qquad t \in \mathbb{R},\ t > 0,$$

is ample if and only if $\eta + tK_B$ is an ample divisor class on B.

(ii) Let $C \in H_2(B, \mathbb{Z})$ be an arbitrary curve class and let $\Sigma = \sigma_*(C) + nf \in H_2(X, \mathbb{Z})/_{\mathsf{torsion}}$ with $n \in \mathbb{Z}$. Then Σ is an efective curve class if and only if C is effective and $n \geq 0$.

Proof. For (i) suppose Σ is an effective curve class on X which contains an irreducible curve. Let $\eta \in \mathrm{Pic}(B)$ be an ample class and note that

$$\Sigma \cdot p^*\eta = p_*\Sigma \cdot \eta \geq 0.$$

Since Σ contains an irreducible curve, one of the following cases must hold.

- (a) The set theoretic support of the irreducible curve in Σ is not contained in Θ. In this case $\Sigma \cdot \Theta \geq 0$.
- (b) The set theoretic support of the irreducible curve in Σ is contained in Θ. In this case $\Sigma \cdot \Theta < 0$ and $\Sigma = \sigma_*(C)$ with C an effective curve class on B. Moreover,

$$\Sigma \cdot \Theta = C \cdot K_B.$$

Then the claim (i) follows easily.

(ii) Let $\eta \in \mathrm{Pic}(B)$ be an arbitrary ample class. Note that for sufficiently large $k > 0$ there exists a divisor H in the linear system $|k\eta|$ such that $Z = p^{-1}(C)$ does not contain the set theoretic support of any of the irreducible components of Σ. Since Σ is effective, this implies $\Sigma \cdot \eta \geq 0$, hence

$$C \cdot \eta = p_*\Sigma \cdot \eta = \Sigma \cdot p^*\eta \geq 0.$$

Since this holds for any ample class η, it follows that C must be effective or zero. If $C = 0$, the claim is obvious. Suppose $C \neq 0$ and $n < 0$. Note that

$K_B \cdot C \neq 0$. Let $s \in \mathbb{R}$ be a real number such that

$$0 < s - 1 < \frac{|n|}{|K_B \cdot C|}. \tag{2.2}$$

Then $\eta = \Theta - sp^* K_B$ is an ample class on X, hence

$$0 < \eta \cdot \Sigma = (s-1)|K_B \cdot C| - |n|.$$

This contradicts the second inequality in (2.2). □

2.2. Adiabatic and generic stability

In order to fix terminology, recall Simpson and slope stability for two dimensional sheaves on X. The former is a natural generalization of Gieseker stability to torsion sheaves introduced in [50]. Let ω be an ample class on X. For any nonzero coherent sheaf E on X of dimension two let

$$\mu_\omega(E) = \frac{\omega \cdot \mathrm{ch}_2(E)}{\omega^2 \cdot \mathrm{ch}_1(E)/2}, \qquad \nu_\omega(E) = \frac{\chi(E)}{\omega^2 \cdot \mathrm{ch}_1(E)/2}.$$

Then Simpson (semi)stability with respect to ω is defined by the conditions

$$\mu_\omega(E') \ (\leq)\ \mu_\omega(E) \tag{2.3}$$

for any proper nonzero subsheaf $0 \subset E' \subset E$, and

$$\nu_\omega(E') \ (\leq)\ \nu_\omega(E) \tag{2.4}$$

if the slope inequality (2.3) is saturated. Recall that any Simpson semistable sheaf must be of pure dimension. Furthermore, a pure dimension two sheaf E is Simpson semistable if and only if the above inequalities are satisfied for saturated proper nonzero subsheaves i.e. E/E' pure of dimension two. In contrast, ω-slope (semi)stability is defined by imposing only condition (2.3) with respect to nonzero proper saturated subsheaves.

For completeness, recall that the ω-slope of a nonzero pure dimension one sheaf E is defined by

$$\mu_\omega(E) = \frac{\chi(E)}{\omega \cdot \mathrm{ch}_2(E)}.$$

Such a sheaf is called Simpson ω-semistable if and only if

$$\mu_\omega(E') \ (\leq)\ \mu_\omega(E)$$

for any proper nontrivial subsheaf $E' \subset E$. In this case Simpson ω-(semi) stability and ω-slope (semi)stability coincide.

Throughout this paper Simpson stability relative to an ample class $\omega \in \mathrm{Pic}_{\mathbb{R}}(X)$ will be simply called ω-stability.

Definition 2.3. (i) A pure dimension two sheaf E on X will be called vertical if and only if

$$\mathrm{ch}_1(E) \in p^*\mathrm{Pic}(B), \qquad \mathrm{ch}_2(E) \in (1/2)\mathbb{Z}\langle f\rangle.$$

(ii) A vertical pure dimension two sheaf E on X will be called adiabatically ω-(semi)stable if and only if it is $(t'\Theta + p^*\eta)$-(semi)stable for all $0 < t' \leq t$.

(iii) A vertical pure dimension two sheaf E on X will be called adiabatically ω-slope (semi)stable if and only if it is $(t'\Theta + p^*\eta)$-slope (semi)stable for all $0 < t' \leq t$.

Note that the discrete invariants of a vertical sheaf E are given by

$$\mathrm{ch}_1(E) = p^*C, \qquad \mathrm{ch}_2(E) = kf, \qquad \mathrm{ch}_3(E) = -n\mathrm{ch}_3(\mathcal{O}_x) \tag{2.5}$$

where $C \in \mathrm{Pic}(B)$ is an effective divisor class on B, $k \in (1/2)\mathbb{Z}$ and $n \in \mathbb{Z}$. Using the isomorphism $\mathrm{Pic}(B) \simeq H_2(B, \mathbb{Z})$, this yields an element

$$\gamma = (C, k, n) \in H_2(B, \mathbb{Z}) \oplus (1/2)\mathbb{Z} \oplus \mathbb{Z},$$

as stated in Section 1.

Let H be a very ample divisor on B and $Z = p^{-1}(H)$. For sufficiently generic H in its linear system, Z is a smooth elliptic surface with reduced fibers. Furthermore if E is a vertical pure dimension one sheaf the restriction of $E|_Z$ is a one dimensional sheaf set theoretically suported on a finite union of elliptic fibers. Basically E will be said to be generically semistable if the restriction $E|_Z = E \otimes_X \mathcal{O}_Z$ is an $\omega|_Z$-semistable pure dimension one sheaf on Z for any sufficiently generic very ample divisor H on B. Technically, this notion requires a more careful definition.

First note that given any very ample line bundle L on B the projection formula yields an isomorphism $H^0(X, p^*L) \simeq H^0(B, L)$ since $p_*\mathcal{O}_X \simeq \mathcal{O}_B$. Therefore the linear system $|L|$ parametrizes simultaneously divisors $H \subset B$ as well as vertical divisors $Z = p^{-1}(H)$ in X. Let $S_{\mathsf{sm}} \subset |L|$ denote the open subset parametrizing smooth divisors H_s such that $Z_s = p^{-1}(H_s)$ is a smooth elliptic surface with reduced fibers.

Since E is vertical of pure dimension two, its scheme theoretic support will be a divisor D_E on X of the form

$$D_E = \sum_{i=1}^{k} \ell_i p^{-1}(C_i)$$

where $\ell_i \in \mathbb{Z}$, $\ell_i \geq 1$, and C_i is a reduced irreducible divisor on B for $1 \leq i \leq k$. Given any very ample line bundle L on X there is a nonempty open subset $S_{E,\mathsf{tr}} \subset S_{\mathsf{sm}}$ such that the following hold for any closed point $s \in S_{E,\mathsf{tr}}$:

$(T.1)$ the corresponding divisor H_s intersects each C_i transversely at finitely many smooth points of C_i, for $1 \leq i \leq k$ and

$(T.2)$ H_s also intersects the discriminant $\Delta \subset B$ of the map $p : X \to B$ transversely at finitely many smooth points of Δ. This implies that the elliptic fibration $p|_{Z_s} : Z_s \to H_s$ will be a Weierstrass model with at most nodal fibers.

Furthermore, according to [28, Lemma 1.1.13], there exists a second nonempty open subset $S_{E,\mathsf{pure}} \subset |L|$ such that $E|_{Z_s}$ is a pure dimension one sheaf for any closed point $s \in S_{E,\mathsf{pure}}$.

Before defining generic stability note the following lemma. The proof is straightforward and will be omitted.

Lemma 2.4. *Let H be a smooth projective curve and $p_Z : Z \to B$ a smooth Weierstrass model over H. Let G be a pure dimension one sheaf on Z with set theoretic support on a finite union of elliptic fibers. Let ω_Z, ω'_Z be arbitrary Kähler classes on Z. Then G is ω_Z-semistable if and only if it is ω'_Z semistable.*

In the situation of Lemma 2.4, the sheaf G will be said to be semistable if it is ω_Z-semistable for some arbitrary polarization of Z. Given a vertical pure dimension two sheaf E and a divisor Z_s corresponding to $s \in S_{E,\mathsf{tr}} \cap S_{E,\mathsf{pure}}$ the sheaf $E|_{Z_s}$ is set theoretically supported on a finite union of elliptic fibers. Therefore one can formulate:

Definition 2.5. A vertical pure dimension two sheaf E will be called generically ω-semistable if and only if for any very ample linear system $\Pi = |L|$ on B there exists a nonempty open subset $S_E \subset S_{E,\mathsf{tr}} \cap S_{E,\mathsf{pure}} \subset \Pi$ such that the restriction $E|_{Z_s}$ a semistable sheaf on Z_s for any closed point $s \in S_E$.

In the remaining part of this section it will be shown that adiabatic semistability is equivalent to generic semistability for vertical semistable pure dimension two sheaves. Since the proof is fairly long, it will be divided into several shorter steps.

Lemma 2.6. *Let F be an arbitrary pure dimension two sheaf on X and $D \subset X$ a divisor such that $F|_D$ is a one dimensional sheaf on X. Then $\mathcal{T}or_1^X(\mathcal{O}_D, F) = 0$ and there is an exact sequence*

$$0 \to F(-D) \to F \to F|_D \to 0.$$

where $F(-D) = F \otimes_X \mathcal{O}_X(-D)$.

Proof. This follows immediately from the standard exact sequence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

taking a tensor product by F. Under the current assumptions, the sheaf $\mathcal{T}or_1^X(\mathcal{O}_D, F)$ is one dimensional, hence it must vanish since $F(-D)$ is pure of dimension two. □

Let F be a nonzero pure dimension two sheaf on X with $\mathrm{ch}_1(F) \in p^*\mathrm{Pic}(B)$. The second Chern class of F is of the form

$$\mathrm{ch}_2(F) = \sigma_*(\alpha_F) + k_F f$$

where α_F is a curve class on B and $k_F \in (1/2)\mathbb{Z}$. Let $H \subset B$ a sufficiently generic very ample divisor on B such that $Z = p^{-1}(H)$ is smooth, and $F|_Z$ is a pure dimension one sheaf on Z.

Lemma 2.7. *Suppose $\chi(F|_Z) > 0$. Then*

$$H \cdot \alpha_F > 0.$$

Proof. Using Lemma 2.6 and the Rieman-Roch theorem, one has

$$\chi(F|_Z) = \chi(F) - \chi(F(-Z)) = Z \cdot \mathrm{ch}_2(F).$$

Then the conclusion follows. □

Lemma 2.8. *Let $\omega = t\Theta - sp^*K_B,\ s, t \in \mathbb{R},\ s > t > 0$. Suppose E is a nonzero adiabatically ω-slope semistable vertical pure dimension two sheaf* *on X with topological invariants*

$$\mathrm{ch}_1(E) = p^*C, \qquad \mathrm{ch}_2(E) = kf, \qquad \mathrm{ch}_3(E) = -n\mathrm{ch}_3(\mathcal{O}_x)$$

where C is a nonzero effective divisor class on B, $k \in (1/2)\mathbb{Z}$ and $n \in \mathbb{Z}$. Let $F \subset E$ be a nonzero proper subsheaf with topological invariants

$$\mathrm{ch}_1(F) = p^*C_F, \qquad \mathrm{ch}_2(F) = \sigma_*(\alpha_F) + k_F f$$

where C_F is a nonzero effective divisor class on B, α_F is an arbitrary divisor class on B and $k_F \in (1/2)\mathbb{Z}$. Then $K_B \cdot \alpha_F \geq 0$.

Proof. Note that

$$\mu_\omega(E) = \frac{k}{(s-t/2)|K_B \cdot C|}, \qquad \mu_\omega(F) = \frac{(1-s/t)K_B \cdot \alpha_F + k_F}{(s-t/2)|K_B \cdot C_F|}.$$

Therefore F destabilizes E for sufficiently small $t > 0$ unless $K_B \cdot \alpha_F \geq 0$. □

The proof that adiabatic stability implies generic stability uses the same geometric construction as the proof of the Grauert-Mülich Theorem in [28, Sect 3.1].

Let L be a very ample line bundle on B, let $V = H^0(B, L)$ and $\Pi = \mathbb{P}(V)$ denote the associated linear system. By convention, $\mathbb{P}(V) = \mathrm{Proj}(S^\bullet(V^\vee))$ such that $H^0(\Pi, \mathcal{O}_\Pi(1)) \simeq V^\vee$. Let $\mathcal{K}$ be the kernel of the evaluation map $\mathsf{ev} : V \otimes \mathcal{O}_B \twoheadrightarrow L$, which is a locally free sheaf on B. According to [28, Sect 3.1], the total space $\mathcal{H}$ of the projective bundle $\mathbb{P}(\mathcal{K})$ parametrizes pairs (H, b) with $H \in \Pi$ and $b \in H$ a closed point. In more detail, note that the evaluation map determines tautologically a section θ of the line bundle $\pi^*L \otimes \pi^*\mathcal{O}_\Pi(1)$, where $\pi : \Pi \times B \to B$ is the canonical projection. Then $\mathcal{H}$ is the divisor $\theta = 0$ in $\Pi \times B$. In particular there are natural projections

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\;q\;} & B \\ {\scriptstyle\rho}\big\downarrow & & \\ \Pi & & \end{array}$$

Moreover, for any closed point $s \in \Pi$ the scheme theoretic inverse image $\rho^{-1}(s)$ is the divisor $\theta|_{B_s} = 0$ in $B_s = B \times \{s\} \subset B \times \Pi$. Given the construction of θ, it follows that the restriction $q|_{\rho^{-1}(s)}$ maps $\rho^{-1}(s)$ isomorphically onto H_s. Let $q_s : \rho^{-1}(s) \xrightarrow{\sim} H_s$ denote the resulting isomorphism.

For future reference it will be useful to provide an explicit construction for the inverse morphism $q_s^{-1} : H_s \to \rho^{-1}(s)$. By restriction to H_s one obtains an exact sequence

$$0 \to \mathcal{K}|_{H_s} \to V \otimes \mathcal{O}_{H_s} \xrightarrow{\mathsf{ev}_s} L|_{H_s} \to 0$$

where $\mathsf{ev}_s = \mathsf{ev}|_{H_s}$. Let $0 \neq z_s \in V$ be a defining section of H_s. Then $\mathsf{ev}_s(z_s \otimes 1) = 0$, hence there is a section $y_s \in H^0(H_s, \mathcal{K}|_{H_s})$ such that the following diagram is commutative

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{H_s} & \xrightarrow{1} & \mathcal{O}_{H_s} & \longrightarrow & & & 0 \\ & & \downarrow{\scriptstyle y_s} & & \downarrow{\scriptstyle f_s} & & & & \downarrow \\ 0 & \longrightarrow & \mathcal{K}|_{H_s} & \longrightarrow & V \otimes \mathcal{O}_{H_s} & \xrightarrow{\mathsf{ev}_s} & L|_{H_s} & \longrightarrow & 0, \end{array}$$

where $f_s(1) = z_s \otimes 1$. Then the snake lemma yields an exact sequence

$$0 \to \mathrm{Coker}(y_s) \to \mathrm{Coker}(f_s) \to L|_{H_s} \to 0$$

where $\mathrm{Coker}(f_s)$ is locally free since f_s is injective on fibers. This implies that $\mathrm{Coker}(y_s)$ is also locally free, hence y_s is injective on fibers. Therefore y_s determines a section $\xi_s : H_s \to \mathbb{P}(\mathcal{K}|_{H_s})$. The scheme theoretic image of ξ_s coincides tautologically with $\rho^{-1}(s)$ and

$$q_s \circ \xi_s = 1_{H_s}.$$

Note that $H^0(B, \mathcal{K}) = 0$, hence the section y_s does not extend to B. However, the following lemma shows that ξ_s can be extended to a certain open subset $U \subset B$.

Lemma 2.9. *There exists an open subscheme $U \subset B$ and a section $\xi : U \to q^{-1}(U) \subset \mathcal{H}$ such that*

$$\xi|_{H_s \cap U} = \xi_s|_{H_s \cap U}. \tag{2.6}$$

Furthermore suppose $C \subset B$ is a fixed effective divisor such that the set theoretic intersection $C \cap H_s$ is a finite set of closed points. Then the open subscheme U can be chosen such that $C \subset U$.

Proof. Let M be a very ample line bundle on B such that $H^1(B, \mathcal{K} \otimes_B L^{-1} \otimes_B M) = 0$. The exact sequence

$$0 \to \mathcal{K} \otimes_B L^{-1} \otimes_B M \to \mathcal{K} \otimes_B M \to (\mathcal{K} \otimes_B M)|_{H_s} \to 0$$

yields a surjective map

$$H^0(B, \mathcal{K} \otimes_B M) \twoheadrightarrow H^0(H_s, (\mathcal{K} \otimes_B M)|_{H_s}). \tag{2.7}$$

Since M is very ample, $H^0(H_s, M|_{H_s})$ is nontrivial. Let $\psi : \mathcal{O}_{H_s} \to M|_{H_s}$ be a nonzero section of $M|_{H_s}$ and let $U_\psi \subset H_s$ be the complement of the zero divisor of ψ. Then

$$y_s \otimes \psi|_{U_\psi} : \mathcal{O}_{U_\psi} \to \mathcal{K}|_{U_\psi} \otimes_{U_\psi} M|_{U_\psi}$$

determines a section of $\mathbb{P}(\mathcal{K}|_{U_\psi})$ which coincides with $\xi_s|_{U_\psi}$. Since the map (2.7) is surjective, there exists a nonzero section $y : \mathcal{O}_B \to \mathcal{K} \otimes_B M$ such that $y|_{H_s} = y_s \otimes \psi$. Let $I \subset M$ be the image of the morphism $\mathcal{K}^\vee \to M$ determined by y. Then $I \simeq \mathcal{I}_Y \otimes M$, where $\mathcal{I}_Y$ is the ideal sheaf of a zero dimensional subscheme $Y \subset B$. Let $U \subset B$ be the complement of Y. Then $I|_U$ is locally free, hence it determines a section $\xi : U \to \mathbb{P}(\mathcal{K})$ which agrees with ξ_s over U_ψ.

Suppose $C \subset B$ is a fixed effective divisor which intersects H_s at finitely many points. Then for sufficiently ample M the section ψ can be chosen such that U contains the set theoretic intersection $C \cap H_s$. Moreover the extension y can be chosen such that the support of Y is disjoint from the support of C. □

Analogous considerations apply to the linear system $|p^*L|$ on X. Note that $H^0(X, p^*L) \simeq H^0(B, L) = V$ since $p_*p^*L \simeq L$ and there is an exact sequence

$$0 \to p^*\mathcal{K} \to V \otimes \mathcal{O}_X \to p^*L \to 0.$$

Therefore the total space $\mathcal{Z}$ of $\mathbb{P}(p^*\mathcal{K})$ parametrizes pairs (Z, x) with $Z = p^{-1}(H)$ for some H in the linear system Π, and $x \in Z$ a closed point. Note that $\mathcal{Z} \simeq \mathcal{H} \times_B X$ and there are natural projections

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{q_Z} & X. \\ {\scriptstyle \rho_Z}\downarrow & & \\ \Pi & & \end{array}$$

For any closed point $s \in \Pi$ there is an isomorphism $\rho_Z^{-1}(s) \simeq \rho^{-1}(s) \times_B X$ and the restriction $q_Z|_{\rho_Z^{-1}(s)}$ maps $\rho_Z^{-1}(s)$ isomorphically to Z_s. The inverse morphism is given by the section $\zeta_s : Z_s \to q_Z^{-1}(Z_s)$,

$$\zeta_s = \xi_s \times_B 1_X.$$

Now let $\mathcal{F} = q_Z^* E$ and let $\mathcal{F}_S = \mathcal{F}|_{\pi^{-1}(S)}$ for any open subset $S \subset \Pi$. Then one has the following lemma.

Lemma 2.10. *The following statements hold for any vertical pure dimension two sheaf E on X.*

(i) There is a nonempty open subset $S_{\mathsf{fl}} \subset \Pi$ such that the restrictions $\rho\big|_{\rho^{-1}(S_{\mathsf{fl}})}$ and $\rho_Z\big|_{\rho_Z^{-1}(S_{\mathsf{fl}})}$ are flat and the fibers $\rho^{-1}(s)$, $\rho_Z^{-1}(s)$, $s \in S_{\mathsf{fl}}$ are normal irreducible divisors in X, B respectively.

(ii) There is a nonempty open subset $S_{E,\mathsf{fl}} \subset S_{\mathsf{fl}}$ such that $\mathcal{F}_S$ is flat over S and $E|_{Z_s}$ is pure one dimensional for any $s \in S_{E,\mathsf{fl}}$.

(iii) There exists a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_j = \mathcal{F}. \tag{2.8}$$

of $\mathcal{F}$ by coherent sheaves on $\mathcal{Z}$ which restricts to a relative Harder-Narasimhan fibration over $\rho_Z^{-1}(S_{\mathsf{hn}}) \subset \mathcal{Z}$ for a suitable nonempty open subset $S_{E,\mathsf{hn}} \subset S_{E,\mathsf{fl}}$. In particular the restrictions $\big(\mathcal{F}_i/\mathcal{F}_{i-1}\big)_{S_{E,\mathsf{hn}}}$, $1 \leq i \leq j$, are flat over $S_{E,\mathsf{hn}}$.

Proof. The first two statements are completely analogous to [28, Lemma 3.1.1]. For the third statement note that [28, Thm 2.3.2] implies the existence of a filtration of the form (2.8) over the open subset $\rho_Z^{-1}(S_{E,\mathsf{fl}}) \subset \mathcal{Z}$. However, this filtration can be extended to a filtration of sheaves on $\mathcal{Z}$ by successive applications of [25, Ex. 5.15.(d)]. □

Now one can finally prove:

Lemma 2.11. *Let $\omega = t\Theta - sp^*K_B$, $t, s \in \mathbb{R}$, $0 < t < s$, and suppose E is an adiabatically ω-slope semistable vertical pure dimension two sheaf on X. Then E is generically semistable.*

Proof. According to Definition 2.5 one has to prove the existence of a nonempty open subset $S_E \subset S_{E,\mathsf{tr}} \cap S_{E,\mathsf{pure}}$ such that the restriction $E|_{Z_s}$ is a semistable pure dimension one sheaf on Z_s for any closed point $s \in S_E$. Note that in Lemma 2.10 one has $S_{E,\mathsf{hn}} \subset S_{E,\mathsf{pure}}$ by construction.

If $\mathcal{F}_1 = \mathcal{F}$ in the filtration (2.8) it follows that $E|_{Z_s}$ is semistable for any $s \in S_{E,\mathsf{hn}}$, hence the claim follows. Suppose this is not the case i.e. the filtration (2.8) has length at least two. Let $s \in S_{E,\mathsf{hn}} \cap S_{E,\mathsf{tr}}$ be a closed point. Hence $H_s \subset B$ intersects the effective divisor $C_E = \sum_{i=1}^k \ell_i C_i$ at finitely many points. Let $\xi : U \to \rho^{-1}(U)$ be a section as in Lemma 2.9 such that $C_E \subset U$. Then relation (2.6) holds:

$$\xi|_{H_s \cap U} = \xi_s|_{H_s \cap U}.$$

Let $Z_s = p^{-1}(H_s)$. Recall that the projection $q_Z : \mathcal{Z} \to X$ maps $\rho_Z^{-1}(s)$ isomorphically to Z_s and the inverse morphism $\zeta_s : Z_s \to q_Z^{-1}(Z_s)$ is given by

$$\zeta_s = \xi_s \times_B 1_X.$$

Let $X_U = p^{-1}(U)$. Then $\zeta = \xi \times_U 1_{X_U}$ is a section of q_Z over X_U such that

$$\zeta|_{Z_s \cap X_U} = \zeta_s|_{Z_s \cap X_U}. \tag{2.9}$$

Moreover $D_E = p^{-1}(C_E)$ is a subscheme of X_U.

Let $\zeta_E : D_E \to \mathcal{Z}$ be the restriction of ζ to D_E. Let $\varphi : \mathcal{F}_1 \hookrightarrow \mathcal{F}$ be the first term in the filtration (2.8). By construction, $\varphi|_{\rho_Z^{-1}(s)}$ is injective. Since D_E is a subscheme of X_U, using equation (2.9), one obtains isomorphisms

$$\begin{aligned}\zeta_E^* \mathcal{F}_1 \otimes_X \mathcal{O}_{Z_s} &\simeq \zeta^* \mathcal{F}_1|_{X_U} \otimes_{X_U} \mathcal{O}_{Z_s \cap X_U} \otimes_{X_U} \mathcal{O}_{D_E} \\ &\simeq \zeta_s^* \big(\mathcal{F}_1|_{\rho_Z^{-1}(s)}\big)\big|_{Z_s \cap X_U} \otimes_{X_U} \mathcal{O}_{D_E}.\end{aligned} \tag{2.10}$$

However $\mathcal{F}_1|_{\rho_Z^{-1}(s)}$ is a subsheaf of $\mathcal{F}|_{\rho_Z^{-1}(s)} \simeq q_{Z,s}^*(E|_{Z_s})$, where $q_{Z,s} : \rho_Z^{-1}(s) \to Z_s$ denotes the natural projection. Since $\zeta_s : Z_s \to \rho_Z^{-1}(s)$ is an isomorphism and $q_{Z,s} \circ \zeta_s = 1_{Z_s}$, it follows that $\zeta_s^*\big(\mathcal{F}_1|_{\rho_Z^{-1}(s)}\big)$ is a subsheaf of $E|_{Z_s}$. In particular it is scheme theoretically supported on D_E, and equation (2.10) yields an isomorphism

$$\zeta_E^* \mathcal{F}_1 \otimes_X \mathcal{O}_{Z_s} \simeq \zeta_s^*\big(\mathcal{F}_1|_{\rho_Z^{-1}(s)}\big). \tag{2.11}$$

Now let $f = \zeta_E^* \varphi : \zeta_E^* \mathcal{F}_1 \to \zeta_E^* \mathcal{F} \simeq E$ and let $F = \mathrm{Im}(f) \subset E$. Since $\zeta_s^*\big(\mathcal{F}_1|_{\rho_Z^{-1}(s)}\big)$ is a subsheaf of $E|_{Z_s}$, equation (2.11) implies that

$$F|_{Z_s} \simeq \zeta_E^*(\mathcal{F}_1|_{\rho_Z^{-1}(s)})$$

is also a subsheaf of $E|_{Z_s}$. By construction this is the first term in the Harder-Narasimhan filtration of $E|_{Z_s}$. Since E is vertical, Lemma 2.6 implies that

$\chi(E|_{Z_s}) = 0$. Thefore, as $E|_{Z_s}$ is not semistable by assumption, one must have $\chi(F|_{Z_s}) > 0$. Then Lemma 2.7 implies that $c_1(L)\cdot\alpha_F > 0$, where α_F is the horizontal part of $\mathrm{ch}_2(F)$ as in loc.cit.

Let ω_B be the dualizing sheaf of B. Applying the above construction to $L = \omega_B^{-m}$, for sufficiently large $m \geq 1$, one is then led to a contradiction with Lemma 2.8 since E is assumed adiabatically ω-slope semistable. □

Lemma 2.11 admits the following converse.

Lemma 2.12. *Suppose $\omega = t\Theta - sK_B$ with $s, t \in \mathbb{R}$, $s > t > 0$. Let E be an ω-semistable vertical pure dimension two sheaf on X with topological invariants*

$$\mathrm{ch}_1(E) = p^*C, \qquad \mathrm{ch}_2(E) = kf$$

where C is a nonzero effective divisor class on B and $k \in (1/2)\mathbb{Z}$. Suppose E is generically semistable. Then E is adiabatically ω-semistable.

Proof. Let $0 \neq F \subset E$ be a proper pure dimension two subsheaf of E such that $G = E/F$ is also pure dimension two. Let H be a sufficiently generic very ample divisor on B as in Definition 2.5 and $Z = p^{-1}(H)$. Lemma 2.6 shows that $\mathcal{T}or_1^X(G, \mathcal{O}_Z) = 0$, hence there is an exact sequence

$$0 \to F|_Z \to E|_Z \to G|_Z \to 0.$$

Let $\mathrm{ch}_1(F) = p^*C_F$ and $\mathrm{ch}_2(F) = \sigma_*(\alpha_F) + k_F f$ with C_F, α_F divisor classes on B, C_F nonzero, effective, and $k_F \in (1/2)\mathbb{Z}$. Since $E|_Z$ is semistable by assumption, and $\chi(E|_Z) = 0$, it follows that

$$\chi(F|_Z) = Z\cdot \mathrm{ch}_2(F) = H\cdot\alpha_F \leq 0.$$

In particular, for H in the linear system $|-K_B|$,

$$K_B\cdot\alpha_F \geq 0. \tag{2.12}$$

Let $\omega' = t'\Theta - sp^*K_B$ with $0 < t' \leq t$. Then

$$\mu_{\omega'}(F) = -\frac{s}{t'(s-t'/2)}\frac{K_B\cdot\alpha_F}{|K_B\cdot C_F|} + \frac{1}{s-t'/2}\frac{k_F + K_B\cdot\alpha_F}{|K_B\cdot C_F|}$$

and

$$\mu_{\omega'}(E) = \frac{1}{s-t'/2}\frac{k}{|K_B\cdot C|}.$$

Since $\mu_\omega(F) \leq \mu_\omega(E)$, one finds

$$\frac{k_F + K_B \cdot \alpha_F}{|K_B \cdot C_F|} \leq \frac{k}{|K_B \cdot C|} + \frac{s}{t}\frac{K_B \cdot \alpha_F}{|K_B \cdot C_F|}.$$

Using (2.12), this implies that

$$\mu_{\omega'}(F) - \mu_{\omega'}(E) \leq \frac{s(t'-t)}{tt'(s-t'/2)}\frac{K_B \cdot \alpha_F}{|K_B \cdot C_F|} \leq 0$$

for any $0 < t' < t$. Moreover equality holds for some $0 < t' < t$ if and only if $K_B \cdot \alpha_F = 0$. If this is the case, ω-stability implies that

$$\nu_\omega(F) \leq \nu_\omega(E)$$

which is equivalent to

$$\frac{\chi(F)}{|K_B \cdot C_F|} \leq \frac{\chi(E)}{|K_B \cdot C|}.$$

This implies that $\nu_{\omega'}(F) \leq \nu_{\omega'}(E)$. Therefore E is ω'-semistable. □

Let $\mathcal{M}_\omega(X,\gamma)$ denote the moduli stack of ω-semistable pure dimension two sheaves E with topological invariants $\gamma = (C,k,n) \in H_2(B,\mathbb{Z}) \oplus (1/2)\mathbb{Z} \oplus \mathbb{Z}$. Let $\mathcal{M}^{\mathsf{ad}}_\omega(X,\gamma)$ denote the substack of adiabatically semistable objects. To conclude this section it will be shown that $\mathcal{M}^{\mathsf{ad}}_\omega(X,\gamma)$ is an open substack of $\mathcal{M}_\omega(X,\gamma)$ for any discrete invariants γ and for any Kähler class $\omega = t\Theta - sp^*K_B,\ s,t,\in \mathbb{R},\ s > t > 0$. For any $0 < t' < t < s$ let $\omega_{t'} = t'\Theta - sp^*K_B$. Then one has:

Lemma 2.13. *Suppose E is a vertical (ω_t,β)-semistable sheaf with discrete invariants $\gamma = (C,k,n)$, $C \neq 0$, which is not $\omega_{t'}$-semistable for some $0 < t' < t$. Then E is not $\omega_{t''}$-semistable for any $0 < t'' < t'$.*

Proof. Let $F \subset E$ be a destabilizing proper non-zero subsheaf with respect to $\omega_{t'}$-stability. This means that

$$\mu_{\omega_{t'}}(F) \geq \mu_{\omega_{t'}}(E), \tag{2.13}$$

and, if equality holds, $\nu_{\omega_{t'}}(F) > \nu_{\omega_{t'}}(E)$. At the same time, note that $\mu_{\omega_t}(F) \leq \mu_{\omega_t}(E)$. As in the proof of Lemma 2.12, let

$$\mathrm{ch}_1(F) = p^*(C_F), \qquad \mathrm{ch}_2(E') = \sigma_*(\alpha_F) + k_F f$$

where C_F is a nonzero effective curve class on B. Then the same computation as in loc.cit. shows that

$$\mu_{\omega_{t'}}(F) - \mu_{\omega_{t'}}(E) \leq \frac{s(t'-t)}{tt'(s-t'/2)} \frac{K_B \cdot \alpha_F}{|K_B \cdot C_F|}.$$

Therefore inequality (2.13) implies that $K_B \cdot \alpha_F \leq 0$. Moreover, if $K_B \cdot \alpha_F = 0$, equality must hold in (2.13).

Now suppose E is $\omega_{t''}$-semistable for some $0 < t'' < t'$. Then

$$\mu_{\omega_{t''}}(F) \leq \mu_{\omega_{t''}}(E). \tag{2.14}$$

and, if equality holds, $\nu_{\omega_{t''}}(F) < \nu_{\omega_{t''}}(E)$. However inequality (2.13) yields

$$\frac{k_F + K_B \cdot \alpha_F}{|K_B \cdot C_F|} \geq \frac{k}{|K_B \cdot C|} + \frac{s}{t'} \frac{K_B \cdot \alpha_F}{|K_B \cdot C_F|}.$$

Therefore

$$\mu_{\omega_{t''}}(F) - \mu_{\omega_{t''}}(E) \geq \frac{s(t''-t')}{t't''(s-t''/2)} \frac{K_B \cdot \alpha_F}{|K_B \cdot C_F|} \geq 0.$$

This implies that $K_B \cdot \alpha_F = 0$, hence equality must hold in (2.13) and (2.14). However, in this case, $\nu_{\omega_{t'}(F)} > \nu_{\omega_{t'}}(E)$, which is equivalent to

$$\frac{\chi(F)}{|K_B \cdot C_F|} > \frac{\chi(E)}{|K_B \cdot C_F|}.$$

This further implies $\nu_{\omega_{t''}(F)} > \nu_{\omega_{t''}}(E)$, leading to a contradiction. □

In order to formulate the last result of this section, let $M_\omega(X, \gamma)$ denote the coarse moduli scheme parameterizing S-equivalence classes of ω-semistable sheaves on X. Note that according to [1, Ex.8.7], $M_\omega(X, \gamma)$ is a good coarse moduli space for the moduli stack $\mathcal{M}_\omega(X, \gamma)$. This means that there is a morphism $\varrho : \mathcal{M}_\omega(X, \gamma) \to M_\omega(X, \gamma)$ satisfying the properties listed in [1, Thm. 4.16]. Let $M^{\mathsf{ad}}_\omega(X, \gamma) \subset M_\omega(X, \gamma)$ be the scheme theoretic image $\varrho(\mathcal{M}^{\mathsf{ad}}_\omega(X, \gamma))$.

Lemma 2.14. *For any Kähler class $\omega = t\Theta - sp^*K_B$ with $s > t > 0$, and for any discrete invariants γ, the subscheme $M^{\mathsf{ad}}_\omega(X, \gamma)$ is open in $M_\omega(X, \gamma)$, and the substack $\mathcal{M}^{\mathsf{ad}}_\omega(X, \gamma)$ is open in $\mathcal{M}_\omega(X, \gamma)$.*

Proof. For any $0 < t' < t$, let $\mathcal{N}_{t'}(\gamma)$ be the substack of ω-semistable vertical sheaves which are not $\omega_{t'}$-semistable. Note that this is an closed substack of $\mathcal{M}_\omega(X,\gamma)$ since $\omega_{t'}$-semistability is an open condition in flat families. Lemma 2.13 shows that $\mathcal{N}_{t'}(\gamma) \subset \mathcal{N}_{t''}(\gamma)$ for any $0 < t'' < t' < t$. According to [1, Thm 4.16.(i)], the morphism ϱ is universally closed. Therefore the scheme theoretic image $\varrho(\mathcal{N}_{t'}(\gamma))$ is a closed subscheme $N_{t'}(\gamma) \subset M_\omega(X,\gamma)$. Moreover and [1, Thm.4.16.(iii)] implies that $N_{t'}(\gamma) \subseteq N_{t''}(\gamma)$ for any $0 < t'' < t' < t$. Since $M_\omega(X,\gamma)$ is noetherian, it follows that the union $N_\omega(\gamma) = \cup_{0<t'<t} N_{t'}(\gamma)$ must be a closed subscheme of $M_\omega(X,\gamma)$. Therefore its inverse image $\mathcal{N}_\omega(\gamma) = \varrho^{-1}(N_\omega(\gamma))$ is a closed substack of $\mathcal{M}_\omega(X,\gamma)$. To conclude the proof note that $\mathcal{M}^{\mathsf{ad}}_\omega(X,\gamma)$ is the complement of $\mathcal{N}_\omega(\gamma)$ according to Lemma 2.13. □

3. Fourier-Mukai transform and stability

This section contains the detailed proof of Theorem 1.1. Since the proof is fairly long and complicated, it will be divided into subsections. The first subsection reviews the basic proeprties of the relative Fourier-Mukai transform on elliptic fibrations.

3.1. Basics of Fourier-Mukai transform

The main references for this section will be [10–12] and the review article [2]. Let X be a smooth generic elliptic Weierstrass model over a smooth Fano variety B. In particular all singular elliptic fibers are either nodal of cuspidal. In this subsection X will be assumed of dimension $n \in \{2,3\}$ and not necessarily Calabi-Yau. Let $\hat{X}$ be the Altman-Kleiman compactification of the degree zero relative Jacobian of X and $\hat{p} : \hat{X} \to B$ the natural projection. This is a fine relative moduli space for rank one degree zero torsion free sheaves on the fibers of p, hence there is a (non-unique) universal rank one torsion free sheaf $\mathcal{P}$ on $\widehat{X} \times_B X$. There is also a canonical morphism $\theta : X \to \hat{X}$ mapping a closed point $x \in X$ to $\mathcal{I}_x \otimes \mathcal{O}_{X_{p(x)}}(\sigma(p(x)))$, where $\mathcal{I}_x \subset \mathcal{O}_{X_{p(x)}}$ is the ideal sheaf of $\{x\} \subset X_{p(x)}$, and $\sigma : B \to X$ is the canonical section. Under the current assumptions θ is an isomorphism. Hence $\hat{p} : \hat{X} \to B$ is a smooth Weierstrass model with a canonical section $\hat{\sigma} : B \to \hat{X}$.

Note that $\mathcal{P}$ is flat over $\hat{X}$ and also flat over X according to [12, Lemma 8.4]. Extending $\mathcal{P}$ by zero to $\widehat{X} \times X$, let

$$\mathcal{Q} = \mathrm{R}\mathcal{H}om_{\widehat{X}\times X}(\mathcal{P}, \pi_X^*\omega_X)[n-1] \tag{3.1}$$

where $\pi_X : \widehat{X} \times X \to X$ is the canonical projection and ω_X is the dualizing sheaf of X. Then [12, Lemma 8.4] proves that $\mathcal{Q}$ is a sheaf on $\widehat{X} \times X$ which is flat over both $\widehat{X}$ and X. Moreover, $\mathcal{Q}$ is pure and scheme theoretically supported on $\widehat{X} \times_B X$.

Now consider the commutative diagram

$$\begin{array}{ccc} \widehat{X} \times_B X & \xrightarrow{\rho} & X \\ \downarrow{\scriptstyle \hat{\rho}} & \searrow^{q} & \downarrow{\scriptstyle p} \\ \widehat{X} & \xrightarrow{\hat{p}} & B. \end{array} \tag{3.2}$$

and define the Fourier-Mukai functors $\Phi : D^b(\widehat{X}) \to D^b(X)$,

$$\Phi(\widehat{E}) = R\rho_*(L\hat{\rho}^*(\widehat{E}) \overset{L}{\otimes} \mathcal{P}) \tag{3.3}$$

and $\widehat{\Phi} : D^b(X) \to D^b(\widehat{X})$,

$$\widehat{\Phi}(E) = R\hat{\rho}_*(L\rho^*(E) \overset{L}{\otimes} \mathcal{Q}). \tag{3.4}$$

Theorem [12, Thm 1.2] proves the following relations:

$$\widehat{\Phi} \circ \Phi \simeq \mathrm{Id}_{D^b(\widehat{X})}[-1], \qquad \Phi \circ \widehat{\Phi} \simeq \mathrm{Id}_{D^b(X)}[-1]. \tag{3.5}$$

For any object E in $D^b(X)$ let $\widehat{\Phi}^i(E)$ denote the i-th cohomology sheaf of $\widehat{\Phi}(E)$. Since $\mathcal{Q}$ is flat over X, the base change theorem implies that $\widehat{\Phi}^i(E)$ is nonzero only for $i \in \{0, 1\}$. A sheaf E on X is called $\widehat{\Phi} - WIT_i$ if $\widehat{\Phi}^j(\widehat{E}) = 0$ for all $j \neq i$. The same applies to sheaves on $\widehat{X}$ with respect to the inverse functor Φ.

For any closed point $\hat{x} \in \widehat{X}$ let $\iota_{\hat{x}} : \hat{x} \times X \hookrightarrow \widehat{X} \times X$ denote the canonical embedding and $\mathcal{P}_{\hat{x}} = \iota_{\hat{x}}^* \mathcal{P}$, $\mathcal{Q} = \iota_{\hat{x}}^* \mathcal{Q}$. Note that $\mathcal{P}_{\hat{x}}$ is isomorphic to the extension by zero of a rank one torsion free sheaf on the elliptic fiber $X_{\hat{p}(\hat{x})}$. Since $\mathcal{P}, \mathcal{Q}$ are flat over $\widehat{X}$, [11, Lemma 3.1.1] implies that

$$\mathrm{L}_k \iota_{\hat{x}}^* \mathcal{P} = 0, \qquad \mathrm{L}_k \iota_{\hat{x}}^* \mathcal{Q} = 0$$

for all $k > 0$. Then, using [24, Prop. III.8.8], relation (3.1) yields the isomorphism

$$\mathcal{Q}_{\hat{x}} \simeq \mathrm{R}\mathcal{H}om_X(\mathcal{P}_{\hat{x}}, \omega_X)[n-1] \tag{3.6}$$

in $D^b(X)$. This implies that $\mathcal{Q}_{\hat{x}}$ is a pure dimension one sheaf on X with scheme theoretic support on $X_{\hat{p}(\hat{x})}$. Taking a further derived dual, one also has

$$\mathcal{P}_{\hat{x}} \simeq \mathrm{R}\mathcal{H}om_X(\mathcal{Q}_{\hat{x}}, \omega_X)[n-1] \tag{3.7}$$

Analogous results hold for the fibers of ρ.

Next note the following lemma, which is a simple consequence of the definitions.

Lemma 3.1. *(i) For any closed point $\hat{x} \in \widehat{X}$ the skyscraper sheaf $\mathcal{O}_{\hat{x}}$ is $\Phi - WIT_0$ and*

$$\Phi^0(\mathcal{O}_{\hat{x}}) \simeq \mathcal{P}_{\hat{x}}.$$

(ii) For any closed point $x \in X$ the $\mathcal{O}_{\widehat{X}}$-module $\mathcal{P}_x$ is $\Phi - WIT_1$ and

$$\Phi^1(\mathcal{P}_x) \simeq \mathcal{O}_x.$$

(iii) Analogous results hold for closed points $x \in X$ relative to $\widehat{\Phi}$.

Further results needed in the following include [12, Lemma 9.2] and [12, Lemma 9.3] which will be reproduced below for convenience.

Lemma 3.2. *Let $\widehat{E}$ be a sheaf on $\widehat{X}$. Then $\Phi^i(\widehat{E})$ is $\widehat{\Phi} - WIT_{1-i}$ for $i \in \{0,1\}$ and there is a short exact sequence*

$$0 \to \widehat{\Phi}^1(\Phi^0(\widehat{E})) \to \widehat{E} \to \widehat{\Phi}^0(\Phi^1(\widehat{E})) \to 0.$$

An analogous statement holds of sheaves E on X with Φ and $\widehat{\Phi}$ reversed.

Lemma 3.3. *A sheaf $\widehat{F}$ on $\widehat{X}$ is $\Phi - WIT_0$ if and only if $\mathrm{Hom}_{\widehat{X}}(\widehat{F}, \mathcal{Q}_x) = 0$ for all $x \in X$.*

Now suppose X is a Calabi-Yau threefold. Choosing the normalization of [2] let $\mathcal{P}$ be given by

$$\mathcal{P} = \mathcal{I}_\Delta \otimes \rho^*\mathcal{O}_X(\Theta) \otimes \hat{\rho}^*\mathcal{O}_{\widehat{X}}(\widehat{\Theta}) \otimes q^*\omega_B^{-1}$$

where $\mathcal{I}_\Delta$ is the ideal sheaf of the diagonal $\Delta \subset X \times_B \widehat{X} \simeq X \times_B X$, ω_B is the dualizing sheaf of B, and $\Theta \subset X$, $\widehat{\Theta} \subset \widehat{X}$ are the canonical sections. This particular choice for $\mathcal{P}$ will be used throughout the remaining part of the paper. Then note that equations (17) and (18) in [2, Sect 5.3] yield the following formulas for the Chern characters of the Fourier-Mukai transform of vertical sheaves.

Let $\widehat{F}$ be a pure dimension one sheaf on $\widehat{X}$ with

$$\mathrm{ch}_2(\widehat{F}) = \hat{\sigma}_*(C) + m\hat{f}, \qquad \mathrm{ch}_3(\widehat{F}) = l\mathrm{ch}_3(\mathcal{O}_{\hat{x}}), \qquad m, l \in \mathbb{Z}.$$

Then

$$\begin{aligned} &\mathrm{ch}_0(\Phi(\widehat{F})) = 0, \qquad \mathrm{ch}_1(\Phi(\widehat{F})) = p^*C, \\ &\mathrm{ch}_2(\Phi(\widehat{F})) = \big(l + K_B \cdot C/2\big) f \qquad \mathrm{ch}_3(\Phi(\widehat{F})) = -m\mathrm{ch}_3(\mathcal{O}_x). \end{aligned} \tag{3.8}$$

with $x \in X$ an arbitrary closed point. Conversely, let E be a vertical pure dimension two sheaf on X with

$$\mathrm{ch}_1(E) = p^*C, \qquad \mathrm{ch}_2(E) = kf, \qquad \mathrm{ch}_3(E) = -n\mathrm{ch}_3(\mathcal{O}_x)$$

where C is an effective curve class on B and $k \in (1/2)\mathbb{Z}$, $n \in \mathbb{Z}$, $k \equiv K_B \cdot C/2$ mod $\mathbb{Z}$. Then

$$\begin{aligned} &\mathrm{ch}_0(\widehat{\Phi}(E)) = 0, \qquad \mathrm{ch}_1(\widehat{\Phi}(E)) = 0, \qquad \mathrm{ch}_2(\widehat{\Phi}(E)) = -\hat{\sigma}_*C - n\hat{f} \\ &\mathrm{ch}_3(\widehat{\Phi}(E)) = \big(-k + K_B \cdot C/2\big)\mathrm{ch}_3(\mathcal{O}_{\hat{x}}). \end{aligned} \tag{3.9}$$

3.2. From sheaves on X to sheaves on $\widehat{X}$

Lemma 3.4. *Let E be a vertical pure dimension two sheaf on X. Let $\widehat{U} \subset \widehat{X}$ be an arbitrary affine open subset. Then $\widehat{\Phi}(E)|_{\widehat{U}}$ is quasi-isomorphic to a three term complex of coherent locally free $\mathcal{O}_{\widehat{U}}$-modules*

$$0 \to W_{-1} \xrightarrow{\phi_0} W_0 \xrightarrow{\phi_1} W_1 \to 0$$

where the degree of W_i is i for $-1 \leq i \leq 1$ and ϕ_0 is injective.

Proof. Since E is pure dimension two, it has a locally free resolution

$$V_{-2} \to V_{-1} \to V_0$$

on X, where V_{-i} is in degree $-i$ for $0 \leq i \leq 2$. Since ρ is flat and $\mathcal{Q}$ is flat over X, $L\rho^*(E)\overset{L}{\otimes}\mathcal{Q}$ is isomorphic to the complex

$$\rho^*V_{-2} \otimes \mathcal{Q} \to \rho^*V_{-1} \otimes \mathcal{Q} \to \rho^*V_0 \otimes \mathcal{Q}$$

in $D^b(\widehat{X} \times_B X)$. Let $\mathcal{V}$ denote the above complex and note that each term of this complex is flat over $\widehat{X}$.

Given any affine open subscheme $\widehat{U} \subset \widehat{X}$ let $\hat{\rho}_{\widehat{U}}$ denote the restriction of $\hat{\rho}$ to $\hat{\rho}^{-1}(\widehat{U})$. Then $\widehat{\Phi}(E)|_{\widehat{U}}$ is given by $\mathrm{R}\hat{\rho}_{\widehat{U}*}(\mathcal{V}|_{\hat{\rho}^{-1}(\widehat{U})})$. According to [23, Thm. 6.10.5] and [23, Remark 6.10.6], or [45, Sect 5, page 46], $\mathrm{R}\hat{\rho}_{\widehat{U}*}(\mathcal{V}|_{\hat{\rho}^{-1}(\widehat{U})})$ is quasi-isomorphic to a finite complex $W_\bullet$ of locally free $\mathcal{O}_{\widehat{U}}$-modules. Moreover, for any point $\hat{x} \in \widehat{U}$, the cohomology group $H^i(W_\bullet|_{\hat{x}})$ is isomorphic to the hypercohomology group $\mathbb{H}^i(\mathcal{V}|_{\hat{\rho}^{-1}(\hat{x})})$ for all values of i.

Next note that $W_\bullet$ can be truncated to a three term locally free complex of amplitude $[-1,\ 1]$. By general properties of the Fourier-Mukai transform, $\widehat{\Phi}(E)|_{\widehat{U}}$ has nontrivial cohomology sheaves only in degrees $0, 1$ hence one can truncate W to a locally free complex

$$\cdots \to W_{-1} \to W_0 \to W_1 \to 0$$

where W_i is in degree i for all $i \leq 1$. Recall that the cokernel of an injective morphism $f_i : W_i \to W_{i-1}$ of locally free sheaves is locally free if and only if f_i is injective on fibers. Then the claim will follow if one shows that

$$\mathbb{H}^{-i}(\mathcal{V}|_{\hat{\rho}^{-1}(\hat{x})}) = 0$$

for all $i \geq 2$. In order to prove this, note that the cohomology sheaf $\mathcal{H}^{-i}(\mathcal{V}|_{\hat{\rho}^{-1}(\hat{x})})$ is isomorphic to the local Tor sheaf $\mathcal{T}or_i^X(E, \mathcal{Q}_{\hat{x}})$. Then relation (3.6) yields isomorphisms

$$\mathcal{H}^{-i}(\mathcal{V}|_{\hat{\rho}^{-1}(\hat{x})}) \simeq \mathcal{E}xt_X^{2-i}(\mathcal{P}_{\hat{x}}, E) \tag{3.10}$$

for all $i \in \mathbb{Z}$. In particular $\mathcal{H}^{-i}(\mathcal{V}|_{\hat{\rho}^{-1}(\hat{x})}) = 0$, $i \geq 3$ for degree reasons, and $\mathcal{H}^{-2}(\mathcal{V}|_{\hat{\rho}^{-1}(\hat{x})}) = 0$ since E is pure dimension two. Then the required vanishing result follows from the hypercohomology spectral sequence since the remaining cohomology sheaves of $\mathcal{V}|_{\hat{\rho}^{-1}(\hat{x})}$ are set theoretically supported in dimension one. In conclusion $\widehat{\Phi}(E)|_U$ is quasi-isomorphic to a complex of the form

$$0 \to W_{-1} \xrightarrow{\phi_0} W_0 \xrightarrow{\phi_1} W_1 \to 0$$

where ϕ_0 is injective. □

Lemma 3.5. *Let H be a smooth projective curve and Z be a smooth Weiestrass model over H with at most nodal fibers. Let F be a coherent sheaf on Z with set theoretic support on a reduced fiber Z_b, for $b \in H$ a closed point. Suppose $\chi(F) = 0$ and F is stable with respect to an arbitrary polarization ω_Z. Then F is the extension by zero of a rank one torsion free sheaf G on Z_b with $\chi(G) = 0$.*

Proof. According to [19, Thm 1.1], any stable torsion free sheaf G on Z_b with $\chi(G)=0$ must have rank one. Therefore it suffices to prove that F is scheme theoretically supported on Z_b.

Since F is stable, it must be pure of dimension one. Hence it is scheme theoretically supported on a nonreduced divisor kZ_b on Z for some $k\in\mathbb{Z}$, $k>0$. Consider the morphism $F\xrightarrow{\zeta}F\otimes_Z \mathcal{O}_Z(Z_b)$, where $\zeta\in\mathcal{O}_Z(Z_b)$ is a defining section. Note that $\mathcal{O}_Z(Z_b)\simeq\mathcal{O}_Z(Z_{b'})$ for any point $b'\in H\setminus\{b\}$. Pick any such point b' and let $\zeta'\in\mathcal{O}_Z(Z_b)$ be its defining section. Obviously ζ' is nonzero on $Z\setminus Z_{b'}$, hence its yields an isomorphism $F\otimes_H\mathcal{O}_Z(Z_b)\simeq F$. Since F is assumed stable it follows that $F\xrightarrow{\zeta}F\otimes_H\mathcal{O}_Z(Z_b)$ must be either identically zero or an isomorphism. However note that $F\xrightarrow{\zeta^k}F\otimes_Z\mathcal{O}_Z(kZ_b)$ must be identically zero since F is scheme theoretically supported on kZ_b. Therefore $F\xrightarrow{\zeta}F\otimes_Z\mathcal{O}_Z(Z_b)$ cannot be an isomorphism, which implies that it must be identically zero. In conclusion F is scheme theoretically supported on the reduced fiber Z_b, hence it must be isomorphic to the extension by zero of a stable sheaf G on Z_b. □

Let E be a vertical pure dimension two sheaf on X scheme theoretically supported on a divisor

$$D_E=\sum_{i=1}^{k}\ell_i p^{-1}(C_i) \tag{3.11}$$

for some reduced irreducible effective divisors C_i in B. Using the notation introduced above Definition 2.5, let H be a very ample divisor in B corresponding to a closed point $s\in S_{E,\mathsf{tr}}\cap S_{E,\mathsf{pure}}\subset|H|$. Therefore $Z=p^{-1}(H)$ is a smooth elliptic surface with finitely many nodal fibers which intersects each component $p^{-1}(C_i)$ transversely along a finite collection of elliptic fibers.

Next note that $\widehat{Z}=\hat{p}^{-1}(H)\subset\widehat{X}$ is a smooth elliptic surface isomorphic to Z over H. Moreover $\widehat{Z}\times_H Z=(\widehat{X}\times_B H)\times_H Z=\widehat{X}\times_H Z$ is the inverse image $\rho^{-1}(Z)$ under the projection $\rho:\widehat{X}\times_B X\to X$. In particular $\widehat{Z}\times_H Z$ is a closed subscheme of $\widehat{X}\times_B X$ and $\rho^*\mathcal{O}_Z\simeq\mathcal{O}_{\widehat{Z}\times_H Z}$. Let

$$\jmath:\widehat{Z}\times_H Z\to\widehat{X}\times_B\widehat{X} \tag{3.12}$$

denote the canonical closed embedding and let $\widehat{\Phi}_Z:D^b(Z)\to D^b(\widehat{Z})$ be the Fourier-Mukai functor with kernel $\mathrm{L}\jmath^*\mathcal{Q}$.

Lemma 3.6. *Suppose E is a nonzero vertical pure dimension two sheaf with scheme theoretic support* (3.11) *and let $Z=p^{-1}(H)\subset X$ be a vertical* *divisor as above. Then there is an isomorphism*

$$\widehat{\Phi}^1(E) \otimes_X \mathcal{O}_{\widehat{Z}} \simeq \widehat{\Phi}^1_Z(E \otimes_X \mathcal{O}_Z) \tag{3.13}$$

Proof. Since $\rho : \widehat{X} \times_B X \to X$ is flat, there is an exact sequence

$$0 \to \rho^*\mathcal{O}_X(-Z) \to \rho^*\mathcal{O}_X \to \rho^*\mathcal{O}_Z \to 0$$

where $\rho^*\mathcal{O}_X \simeq \mathcal{O}_{\widehat{X}\times_B X}$ and $\rho^*\mathcal{O}_Z \simeq \mathcal{O}_{\widehat{Z}\times_H Z}$. Hence this is a two term locally free resolution of $\mathcal{O}_{\widehat{Z}\times_H Z}$. Since $\mathcal{Q}$ is flat over X, this sequence will remain exact when one takes a tensor product with $\mathcal{Q}$. Therefore $L\jmath^*\mathcal{Q}$ is quasi-isomorphic to $\mathcal{Q}|_{\widehat{Z}\times_H Z}$.

Since the Fourier-Mukai transform is compatible with base change there is an isomorphism

$$\widehat{\Phi}_Z(\mathrm{L}\iota_Z^*(E)) \simeq L\iota_{\widehat{Z}}^*\widehat{\Phi}(E)$$

in $D^b(\widehat{Z})$, where $\iota_{\widehat{Z}} : \widehat{Z} \to \widehat{X}$ is the natural closed embedding. However, Lemma 2.6 yields an isomorphism $\mathrm{L}\iota_Z^*E \simeq E \otimes_X \mathcal{O}_Z$ in $D^b(Z)$, hence one obtains

$$\widehat{\Phi}_Z(E \otimes_X \mathcal{O}_Z) \simeq L\iota_{\widehat{Z}}^*\widehat{\Phi}(E).$$

Since $\widehat{\Phi}(E)$ has cohomology only in degrees 0 and 1, the base change theorem [23, Thm. 7.7.5] implies that

$$\widehat{\Phi}^1_Z(E \otimes_X \mathcal{O}_Z) \simeq \iota_{\widehat{Z}}^*\widehat{\Phi}^1(E) \simeq \widehat{\Phi}^1(E) \otimes_{\widehat{X}} \mathcal{O}_{\widehat{Z}}. \tag{3.14}$$

□

Lemma 3.7. *Let E be a nonzero generically semistable vertical pure dimension two sheaf on X as in Definition 2.5. Then $\widehat{\Phi}^0(E) = 0$ and $\widehat{\Phi}^1(E)$ is a pure dimension one sheaf on $\widehat{X}$.*

Proof. Using the notation of Lemma 3.4 it suffices the prove that the complex $W_\bullet$ is exact in degree 0 for any open affine subset $\widehat{U} \subset X$. Under the current assumptions the scheme theoretic support of E is of the form (3.11). Note that the first Chern character of E is of the form

$$\mathrm{ch}_1(E) = \sum_{i=1}^k r_i p^*(C_i) \tag{3.15}$$

for some integers $r_i \in \mathbb{Z}$, $r_i \geq 1$, $1 \leq i \leq k$.

Let H be a smooth very ample divisor on B satisfying the genericity conditions in Definition 2.5. In particular H intersects each C_i transversely at

$n_i \geq 1$ finitely many smooth points $b_{i,j}$ on C_i, where $1 \leq j \leq n_i$. The inverse image $Z = p^{-1}(H)$ is a smooth Weierstrass model over H with at most nodal fibers. Let $F = E \otimes_X \mathcal{O}_Z$. By assumption, F is an $\omega|_Z$- semistable sheaf on Z set theoretically supported on a finite union of elliptic fibers. According to Lemma 2.6, there is an exact sequence

$$0 \to E(-Z) \to E \to F \to 0$$

of sheaves on X which yields $\chi(F) = 0$ via the Riemann-Roch theorem. Moreover, the above sequence also implies that

$$\mathrm{ch}_1(F) = \iota_Z^* \mathrm{ch}_1(E)$$

as a sheaf on Z. Using a Jordan Hölder filtration and Lemma 3.1, it is straightforward to check that F is $\widehat{\Phi}_Z - WIT_1$ and $\widehat{\Phi}^1_Z(F)$ is a zero dimensional sheaf of length

$$\chi(\widehat{\Phi}^1_Z(F)) = \sum_{i=1}^{k} r_i n_i.$$

This holds for any very ample divisor H in B satisfying the genericity conditions in Definition 2.5. Then Lemma 3.6 implies that the set theoretical support of $\widehat{\Phi}^1(E)$ is at most one dimensional. If it had dimension two or higher, the restriction of $\widehat{\Phi}^1(E)$ to a generic $\widehat{Z}$ would be supported in dimension at least one since any effective divisor on $\widehat{X}$ intersects $\widehat{Z}$ along a nonempty curve.

Let $T \subset \widehat{\Phi}^1(E)$ be the maximal zero dimensional subsheaf, and let $\widehat{\Phi}^1(E)' = \widehat{\Phi}^1(E)/T$, which is a sheaf of pure dimension one. Obviously, the set theoretic support of $\widehat{\Phi}^1(E)'$ intersects $\widehat{Z}$ at finitely many closed points, hence $\mathcal{T}or_1^{\widehat{X}}(\mathcal{O}_{\widehat{Z}}, \widehat{\Phi}^1(E)')$ is a zero dimensional sheaf. Then, using the locally free resolution

$$0 \to \mathcal{O}_{\widehat{X}}(-\widehat{Z}) \to \mathcal{O}_{\widehat{X}} \to \mathcal{O}_{\widehat{Z}} \to 0,$$

it follows that $\mathcal{T}or_1^{\widehat{X}}(\mathcal{O}_{\widehat{Z}}, \widehat{\Phi}^1(E)') = 0$ since $\mathcal{O}_{\widehat{X}}(-\widehat{Z}) \otimes_{\widehat{X}} \widehat{\Phi}^1(E)'$ is pure of dimension one. As the higher local tor sheaves are obviously zero, one obtains a quasi-isomorphism

$$L\iota_{\widehat{Z}}^* \widehat{\Phi}^1(E)' \simeq \mathcal{O}_{\widehat{Z}} \otimes_{\widehat{X}} \widehat{\Phi}^1(E)'.$$

Moreover, since T depends only on E one can choose H sufficiently generic such that $\widehat{Z}$ does not intersect the support of T. Then Lemma 3.6 yields an isomorphism

$$\mathcal{O}_{\widehat{Z}} \otimes_{\widehat{X}} \widehat{\Phi}^1(E)' \simeq \mathcal{O}_{\widehat{Z}} \otimes_{\widehat{X}} \widehat{\Phi}^1(E) \simeq \widehat{\Phi}^1_Z(F).$$

Since $\mathcal{T}or_i^{\widehat{X}}(\mathcal{O}_{\widehat{Z}}, \widehat{\Phi}^1(E)') = 0$ for all $i \geq 1$, the Riemann-Roch theorem yields

$$\chi(\mathcal{O}_{\widehat{Z}} \otimes_{\widehat{X}} \widehat{\Phi}^1(E)') = \mathrm{ch}_2(\widehat{\Phi}^1(E)') \cdot \widehat{Z}.$$

Therefore for any H satisfying the required genericity assumptions there is an identity

$$\mathrm{ch}_2(\widehat{\Phi}^1(E)) \cdot \widehat{Z} = \chi(\widehat{\Phi}^1_Z(F)) = \sum_{i=1}^{k} n_i r_i.$$

However equations (3.9) imply that

$$\mathrm{ch}_2(\widehat{\Phi}(E)) = -\sum_{i=1}^{k} r_i \hat{\sigma}_*(C_i) - nf$$

where $\mathrm{ch}_3(E) = -n\mathrm{ch}_3(\mathcal{O}_x)$. Therefore

$$\mathrm{ch}_2(\widehat{\Phi}^1(E)) \cdot \widehat{Z} - \mathrm{ch}_2(\widehat{\Phi}^0(E)) \cdot \widehat{Z} = \sum_{i=1}^{k} n_i r_i.$$

In conclusion

$$\mathrm{ch}_2(\widehat{\Phi}^0(E)) \cdot \widehat{Z} = 0$$

for any very ample class H in B. This implies that $\mathrm{ch}_2(\widehat{\Phi}^0(E)) \in \mathbb{Q}\langle f \rangle$. However, equations (3.9) imply that

$$\mathrm{ch}_i(\widehat{\Phi}^0(E)) = 0$$

for $i \in \{0, 1\}$ since $\widehat{\Phi}^1(E)$ has one dimensional support. Therefore $\widehat{\Phi}^0(E)$ is set theoretically supported on a finite union of elliptic fibers.

Now recall that $\widehat{\Phi}^0(E)$ is $\Phi - WIT_1$ and there is an injective morphism

$$\Phi^1(\widehat{\Phi}^0(E)) \hookrightarrow E$$

according to Lemma 3.2. Since $\widehat{\Phi}^0(E)$ is $\Phi - WIT_1$ and set theoretically supported on a finite union of fibers, equations (3.9) imply that $\Phi^1(\widehat{\Phi}^0(E))$ will be also supported on a finite union of elliptic fibers. Since E is pure of dimension two, it follows that $\Phi^1(\widehat{\Phi}^0(E)) = 0$, which further implies that

$\widehat{\Phi}^0(E) = 0$ since $\widehat{\Phi}^0(E)$ is $\Phi - WIT_1$. This implies that for any open subset of $\widehat{X}$ the complex $W_\bullet$ constructed in Lemma 3.4 is a locally free resolution of $\widehat{\Phi}^1(E)$. Therefore $\widehat{\Phi}^1(E)$ must be a pure dimension one sheaf on $\widehat{X}$. $\square$

For the remaining part of this section set

$$\omega = t\Theta - sp^*K_B, \qquad \hat{\omega} = \widehat{\Theta} - s\hat{p}^*K_B, \tag{3.16}$$

where $s, t \in \mathbb{R}$, $s > t > 0$ and $s > 1$.

Let E be a vertical ω-semistable sheaf on X with topological invariants

$$\mathrm{ch}_1(E) = p^*C, \qquad \mathrm{ch}_2(E) = kf, \qquad \mathrm{ch}_3(E) = -n\mathrm{ch}_3(\mathcal{O}_x) \tag{3.17}$$

where $0 \neq C \in H_2(B, \mathbb{Z})$ is an effective curve class, $k \in (1/2)\mathbb{Z}$, $n \in \mathbb{Z}$. Suppose E is also generically semistable. Then E is $\widehat{\Phi} - WIT_1$ according to Lemma 3.7 and $\widehat{F} = \widehat{\Phi}^1(E)$ is a pure dimension one sheaf on $\widehat{X}$ with topological invariants

$$\mathrm{ch}_2(\widehat{F}) = \hat{\sigma}_*(C) + n\hat{f}, \qquad \chi(\widehat{F}) = k - \frac{K_B \cdot C}{2}. \tag{3.18}$$

Remark 3.8. Note that Lemma 3.7 and Corollary 2.2.ii. imply that $n \geq 0$ for any sheaf E as above since $\mathrm{ch}_2(\widehat{F})$ must be effective.

The next goal is to show that $\widehat{F}$ is $\hat{\omega}$-semistable for sufficiently large s provided that $\chi = k - K_B \cdot C/2 \geq 1$. This will be carried out in several steps. For fixed C, k, n as above with $C \neq 0$ effective, $\chi \geq 1$, $n \geq 0$, let

$$\begin{aligned} \mathcal{S}(C, k, n) = \{(C', l, m) \in \mathrm{Pic}(B) \times \mathbb{Z} \times \mathbb{Z} \,|\, & C', C - C' \text{ effective}, \\ & l \geq 0,\ |K_B \cdot C|l - |K_B \cdot C'|\chi \leq 0,\ 0 \leq m \leq n\}. \end{aligned}$$

Note that $|K_B \cdot C'| \leq |K_B \cdot C|$ for any $(C', l, m) \in \mathcal{S}(C, k, n)$, hence the second defining inequality of $\mathcal{S}(C, k, n)$ yields

$$0 \leq l \leq \chi.$$

Therefore $\mathcal{S}(C, k, n)$ is a finite set. Moreover,

$$|nl - m\chi| \leq n\chi \tag{3.19}$$

for any $(C', l, m) \in \mathcal{S}(C, k, n)$.

Lemma 3.9. *Suppose E, $\widehat{F}$ are as above. Let $\widehat{G} \subset \widehat{F}$ be a nonzero subsheaf such that $\widehat{F}/\widehat{G}$ is a nonzero pure dimension one sheaf on $\widehat{X}$. Let*

$$\mathrm{ch}_2(\widehat{G}) = \hat{\sigma}_*(C_{\widehat{G}}) + m\hat{f} \tag{3.20}$$

with $m \in \mathbb{Z}$. Suppose $\widehat{G}$ is $\hat{\omega}$-semistable and $\mu_{\hat{\omega}}(\widehat{G}) > \mu_{\hat{\omega}}(\widehat{F})$ for some $s > 1$. Then $(C_{\widehat{G}}, \chi(\widehat{G}), m) \in \mathcal{S}(C, k, n)$.

Proof. Given $E, \widehat{F}, \widehat{G}$ as in Lemma 3.9, note that $\mu_{\hat{\omega}}(\widehat{G}) > \mu_{\hat{\omega}}(\widehat{F}) > 0$. Since $\widehat{G}$ is assumed $\hat{\omega}$-semistable for some $s > 1$, Lemma 3.3 implies that $\widehat{G}$ is $\Phi - WIT_0$. Since E is $\widehat{\Phi} - WIT_1$ and $\widehat{F} = \widehat{\Phi}^1(E)$, Lemma 3.3 implies that $\Phi^0(\widehat{G})$ is a subsheaf of E. Moreover equations (3.8) yield

$$\mathrm{ch}_1(\Phi^0(\widehat{G})) = p^* C_{\widehat{G}}, \qquad \mathrm{ch}_2(\Phi^0(\widehat{G})) = (\chi(\widehat{G}) + K_B \cdot C_{\widehat{G}}/2) f,$$
$$\mathrm{ch}_3(\Phi^0(\widehat{G})) = -m\mathrm{ch}_3(\mathcal{O}_x).$$

Since $\widehat{G}$ is $\Phi - WIT_0$ and $\Phi^0(\widehat{G})$ is a nonzero subsheaf of E one must have $C_{\widehat{G}} \neq 0$. Otherwise $\Phi^0(\widehat{G})$ would be a nonzero sheaf supported on a finite union of elliptic fibers, leading to a contradiction since E is purely two dimensional. Moreover, Corollary 2.2 implies that $C_{\widehat{G}}$ is effective and $m \geq 0$. Since $\mathrm{ch}_2(\widehat{F}/\widehat{G})$ must be an effective curve class, Corollary 2.2 also implies that $C = C_{\widehat{G}} + C'$ where C' is an effective curve class on B and $n - m \geq 0$.

Since E is ω-semistable, one also has $\mu_\omega(\Phi^0(\widehat{G})) \leq \mu_\omega(E)$, which is equivalent to

$$\chi(\widehat{G})|K_B \cdot C| - \chi(\widehat{F})|K_B \cdot C_{\widehat{G}}| \leq 0.$$

In conclusion, $(C_{\widehat{G}}, \chi(\widehat{G}), m) \in \mathcal{S}(C, k, n)$. □

Now consider the subset

$$\mathcal{S}'(C, k, n) = \{(C', l, m) \in \mathcal{S}(C, k, n) \,|\, |K_B \cdot C| l - |K_B \cdot C'| \chi \leq -1\}$$
$$\subset \mathcal{S}(C, k, n).$$

For any $s \in \mathbb{R}$, $s > 0$, let $f_s : \mathcal{S}'(C, k, n) \to \mathbb{R}$ be the function

$$f_s(C', l, m) = (s - 1)(|K_B \cdot C| l - |K_B \cdot C'| \chi) + (nl - m\chi).$$

Then the following is a straightforward consequence of inequality (3.19).

Lemma 3.10. *For fixed C, k, n as above there exists $s_1 \in \mathbb{R}$, $s_1 > 1$ depending only on (C, k, n) such that for any $s > s_1$ one has $f_s(C', l, m) < 0$ for all $(C', l, m) \in \mathcal{S}'(C, k, n)$.*

Lemma 3.11. *For fixed (C,k,n) as above let $s_1 > 1$ be as in Lemma 3.10. Then for any $s > s_1$ the Fourier-Mukai transform $\widehat{F} = \Phi^1(E)$ of any ω-semistable and generically semistable sheaf E with topological invariants* (3.17) *is $\hat{\omega}$-semistable.*

Proof. Suppose $s > s_1$. The goal is to show that no destabilizing subsheaf $\widehat{G} \subset \widehat{F}$ as in Lemma 3.9 can exist for any pair $(E, \widehat{F})$. Suppose $\widehat{G} \subset \widehat{F}$ is such a subsheaf for some pair $(E, \widehat{F})$. Note that $\mu_{\hat{\omega}}(\widehat{G}) > \mu_{\hat{\omega}}(\widehat{F})$ is equivalent to

$$(s-1)\delta_1 + \delta_2 > 0, \tag{3.21}$$

where

$$\delta_1 = \chi(\widehat{G})|K_B \cdot C| - \chi(\widehat{F})|K_B \cdot C_{\widehat{G}}|, \qquad \delta_2 = n\chi(\widehat{G}) - m\chi(\widehat{F}).$$

According to Lemma 3.9, $(C_{\widehat{G}}, \chi(\widehat{G}), m) \in \mathcal{S}(C,k,n)$. In particular $\delta_1 \leq 0$. Since $\delta_1 \in \mathbb{Z}$, there are two cases.

(*i*) $\delta_1 \leq -1$. Then according to Lemma 3.10

$$(s-1)\delta_1 + \delta_2 = f_s(C_{\widehat{G}}, \chi(\widehat{G}), m) < 0,$$

contradicting (3.21).

(*ii*) $\delta_1 = 0$. Solving for $\chi(\widehat{G})$, δ_2 reduces to

$$\delta_2 = \frac{\chi(\widehat{F})}{|K_B \cdot C|}\big(n|K_B \cdot C_{\widehat{G}}| - m|K_B \cdot C|\big).$$

In this case $\mu_\omega(\Phi^0(\widehat{G})) = \mu_\omega(E)$, hence one must have

$$\nu_\omega(\Phi^0(\widehat{G})) \leq \nu_\omega(E)$$

since E is ω-semistable. This is equivalent to $\delta_2 \leq 0$, leading again to a contradiction. □

3.3. From sheaves on $\widehat{X}$ to sheaves on X

Again, consider Kähler classes of the form (3.16) on X, $\widehat{X}$ respectively. Suppose $\widehat{F}$ is a pure dimension one sheaf on $\widehat{X}$ and let L be a very ample line bundle on B. Using the same notation as in Definition 2.5, let $S_{\mathsf{sm}} \subset |L|$ be the nonempty open subset parametrizing smooth irreducible divisors $H \in |L|$ such that $Z = p^{-1}(H)$ is also smooth. Since $\widehat{F}$ is scheme theoretically supported on a closed subscheme of $\widehat{X}$ of pure dimension one, there exists a nonempty open subset $S_{\widehat{F}} \subset S_{\mathsf{sm}}$ such that:

• the set theoretic intersection between $Z_s = p^{-1}(H_s)$ and the support of $\widehat{F}$ consists of finitely many closed points, and

• H_s intersects the discriminant $\Delta \subset B$ transversely at finitely many points in the smooth locus of Δ.

for any closed point $s \in S_{\widehat{F}}$. If the above conditions are satisfied, $\widehat{Z}_s = \hat{p}^{-1}(H_s)$ is a smooth Weierstrass model over H_s with at most finitely many nodal fibers.

Lemma 3.12. *Let $\widehat{F}$ be an $\hat{\omega}$-semistable pure dimension one sheaf on $\widehat{X}$ with*

$$\mathrm{ch}_2(\widehat{F}) = \hat{\sigma}(C) + n\hat{f}, \qquad \chi(\widehat{F}) > 0, \tag{3.22}$$

where $C \neq 0$. Then the following hold.

(i) $\widehat{F}$ is $\Phi - WIT_0$ and $\Phi^0(\widehat{F})$ is a vertical pure dimension two sheaf on $\widehat{X}$.

(ii) Let L be a very ample line bundle on B, let H be a divisor in B corresponding to a closed point in $S_{\widehat{F}} \subset |L|$, and $Z = p^{-1}(H) \subset X$. Then $\Phi^0(\widehat{F}) \otimes_X \mathcal{O}_Z$ is a semistable pure dimension one sheaf on Z.

Proof. (i) Since $\widehat{F}$ is $\hat{\omega}$-semistable, condition (3.22) implies that $\mathrm{Hom}_{\widehat{X}}(\widehat{F}, \mathcal{Q}_x) = 0$ for any closed point $x \in X$. Therefore Lemma 3.3 implies that $\widehat{F}$ is $\Phi - WIT_0$. Moreover, equations (3.8) imply that $\Phi^0(\widehat{F})$ is a vertical two dimensional sheaf. The proof of purity is completely analogous to the proof of Lemma 3.4.i.

(ii) Under the current assumptions $\widehat{F}|_Z = \widehat{F} \otimes_{\widehat{X}} \mathcal{O}_{\widehat{Z}}$ is a zero dimensional sheaf on $\widehat{Z}$. Using the same notation as in Lemma 3.6, let $\Phi_{\widehat{Z}} : D^b(\widehat{Z}) \to D^b(Z)$ denote the Fourier-Mukai functor with kernel $\mathcal{P}|_{\widehat{Z}\times_H Z}$. Then it is straightforward to show that $\Phi^0_{\widehat{Z}}(\widehat{F} \otimes_{\widehat{X}} \mathcal{O}_{\widehat{Z}})$ is a semistable sheaf on Z of pure dimension one set theoretically supported on a finite union of elliptic fibers. Moreover, by analogy with Lemma 3.6, there is an isomorphism

$$\Phi^0_Z(\widehat{F} \otimes_{\widehat{X}} \mathcal{O}_{\widehat{Z}}) \simeq \Phi^0(\widehat{F}) \otimes_X \mathcal{O}_Z$$

□

Next let $\widehat{F}$ be an $\hat{\omega}$-semistable pure dimension one sheaf on $\widehat{X}$ as in Lemma 3.12 with

$$\mathrm{ch}_2(\widehat{F}) = \hat{\sigma}_*(C) + n\hat{f}, \qquad \chi(\widehat{F}) = k - \frac{K_B \cdot C}{2} \geq 1, \tag{3.23}$$

where C is a nonzero divisor class on B and $n \in \mathbb{Z}$, $k \in (1/2)\mathbb{Z}$. Note that Corollary 2.2 implies that C must be effective and $n \geq 0$. According to Lemma 3.12, $\widehat{F}$ is $\Phi - WIT_0$ and $E = \Phi^0(\widehat{F})$ is a vertical pure dimension two sheaf on X with topological invariants

$$\mathrm{ch}_1(E) = p^*C, \qquad \mathrm{ch}_2(E) = kf, \qquad \mathrm{ch}_3(E) = -n\mathrm{ch}_3(\mathcal{O}_x)$$

where $\mathcal{O}_x$ is the structure sheaf of an arbitrary closed point $x \in X$. In the remaining part of this section it will be shown that E is ω-semistable for sufficiently small $t > 0$. This will be carried out in several stages.

First suppose $E = \Phi^0(\widehat{F}) \twoheadrightarrow G$ is a nonzero pure dimension two quotient such that $\mu_\omega(G) \leq \mu_\omega(E)$ and G is not isomorphic to E. Then G will have topological invariants

$$\mathrm{ch}_1(G) = p^*C_G, \qquad \mathrm{ch}_2(G) = \sigma_*(\alpha_G) + cf, \qquad \mathrm{ch}_3(G) = -m\mathrm{ch}_3(\mathcal{O}_x),$$

where C_G is a nonzero effective divisor class on B, α_G is an arbitrary divisor class on B, and $c, m \in (1/2)\mathbb{Z}$, $c \equiv K_B \cdot C_G/2 \bmod \mathbb{Z}$, $m \equiv C_G \cdot \alpha_G/2 \bmod \mathbb{Z}$. Since G is a quotient of E, not isomorphic to E, the curve class $C - C_G$ is effective, nonzero. Therefore

$$|K_B \cdot C_G| < |K_B \cdot C|. \tag{3.24}$$

Lemma 3.13. *Under the above assumptions α_G is an effective divisor class on B and*

$$\frac{c - |K_B \cdot \alpha_G|}{|K_B \cdot C_G|} \leq \frac{k}{|K_B \cdot C|}. \tag{3.25}$$

Moreover equality holds in (3.25) *if and only if $\alpha_G = 0$.*

Proof. Note that

$$\mu_\omega(E) = \frac{1}{(s - t/2)} \frac{k}{|K_B \cdot C|}, \qquad \mu_\omega(G) = \frac{-(s/t - 1)(\alpha_G \cdot K_B) + c}{(s - t/2)|K_B \cdot C_G|}.$$

Given any very ample linear system Π on B, Lemma 3.12 shows that $E|_Z$ is semistable for any sufficiently generic very ample divisor $H \in \Pi$, where

$Z = p^{-1}(H)$. Moreover using Lemma 2.6, one has

$$\chi(E|_Z) = 0, \qquad \chi(G|_Z) = H \cdot \alpha_G.$$

Therefore $H \cdot \alpha_G \geq 0$ for any very ample divisor H on B. This implies that α_G must be an effective divisor class on B, in particular $\alpha_G \cdot K_B \leq 0$. Then

$$\mu_\omega(G) = \frac{(s/t-1)|K_B \cdot \alpha_G| + c}{(s-t/2)|K_B \cdot C_G|},$$

and inequality (3.25) follows from the slope inequality $\mu_\omega(G) \leq \mu_\omega(E)$. □

Lemma 3.14. *There exists a constant A depending on (C, k, n) and s, but not t, such that*

$$|c - |K_B \cdot \alpha_G|| < A$$

for all quotients $E = \Phi^0(\widehat{F}) \twoheadrightarrow G$ as above and for all $\hat{\omega}$-semistable sheaves $\widehat{F}$ with topological invariants (3.23).

Proof. Recall that the set of isomorphism classes of $\hat{\omega}$-semistable sheaves with fixed topological invariants is bounded [28, Thm. 3.3.7]. Since the Fourier-Mukai transform preserves families of sheaves [11, Prop. 6.13.], this implies that the family of sheaves $E = \Phi^0(\widehat{F})$ is also bounded and depends on (C, k, n), and s, but not t. Moreover, [28, Lemma 1.7.6] implies that the same holds for the family $E_B = \sigma^* E$.

Let $\eta_0 = -K_B$, which is very ample on B. Then the set of Hilbert polynomials $\mathcal{P} = \{P_{\eta_0, E_B}\}$ is finite and indepedent of t. Let $P \in \mathcal{P}$ be fixed. Obviously, the set of isomorphism classes $\{[E_B]\}_P$ of sheaves E_B with fixed $P_{\eta_0, E_B} = P$ is also bounded and independent of t.

Given a quotient $E \twoheadrightarrow G$, note that $G_B = \sigma^* G$ is also a quotient of E_B, and there is an exact sequence of $\mathcal{O}_B$-modules

$$0 \to T_G \to G_B \to G'_B \to 0$$

where T_G is the maximal zero dimensional subsheaf of G_B and G'_B has pure dimension one. Since G is pure of dimension two and has vertical support Lemma 2.6 yields an exact sequence

$$0 \to G(-\Theta) \to G \to \sigma_* G_B \to 0.$$

Using the above exact sequence and the Grothedieck-Riemann-Roch theorem for the embedding $\Theta \hookrightarrow X$, one obtains

$$\mathrm{ch}_1(G_B) = C_G, \qquad \mathrm{ch}_2(G_B) = (c - |K_B \cdot \alpha_G|)\,\mathrm{ch}_2(\mathcal{O}_b) \tag{3.26}$$

with $b \in B$ an arbitrary closed point. Since T_G is zero dimensional, $\mu_{\eta_0}(G'_B) \leq \mu_{\eta_0}(G_B)$. Then inequality (3.25) yields

$$\mu_{\eta_0}(G'_B) \leq \frac{c - |K_B \cdot \alpha_G|}{|K_B \cdot C_G|} \leq \frac{k}{|K_B \cdot C|}. \tag{3.27}$$

For fixed $P = P_{\eta_0, E_B} \in \mathcal{P}$, let $\mathcal{Q}_P$ denote the set of isomorphism classes of pure dimension one sheaves F on B such that

(a) there exists an epimorphism $E_B \twoheadrightarrow F$, for some $E = \Phi^0(\widehat{F})$ as above with $P_{\eta_0, E_B} = P$, and

(b) $\mu_{\eta_0}(F) \leq k/|K_B \cdot C|$.

Then Grothendieck's lemma [28, Lemma 1.7.9] implies that $\mathcal{Q}_P$ is bounded and depends only on P and the bounded family $\{[E_B]\}_P$. In particular it is independent of t. This implies that the set $\{P_{\eta_0, G'_B}\}_P$ of Hilbert polynomials of all quotients $E_B \twoheadrightarrow G'_B$ where $P_{\eta_0, E_B} = P$ is finite and $|\{P_{\eta_0, G'_B}\}_P|$ is bounded above by a constant depending on P and the bounded family $\{[E_B]\}_P$, but not on t. Since the whole family $\{[E_B]\} = \cup_P \{[E_B]\}_P$ is bounded and depends only on (C, k, n) and s, it follows that there exists a constant A_1 depending on (C, k, n) and s, but not t, such that

$$|\chi(G'_B)| < A_1$$

for all pure dimension one quotients $E_B \twoheadrightarrow G'_B$, for all $E = \Phi^0(\widehat{F})$ as above.

To conclude the proof, note that $\chi(G_B) = \chi(T_G) + \chi(G'_B) \geq \chi(G'_B)$ since T_G is zero dimensional. On the other hand, using equation (3.26) and the Riemann-Roch theorem,

$$\chi(G_B) = c - |K_B \cdot \alpha_G| + |K_B \cdot C_G|/2.$$

Therefore, using inequality (3.24),

$$c - |K_B \cdot \alpha_G| > -A_1 - |K_B \cdot C_G|/2 > -A_1 - |K_B \cdot C|/2.$$

At the same time inequalities (3.24), (3.25) yield

$$c - |K_B \cdot \alpha_G| \leq \frac{|K_B \cdot C_G|}{|K_B \cdot C|} k < |k|.$$

Therefore the claim follows. □

Lemma 3.15. *There exists a constant $t_1 \in \mathbb{R}$, $0 < t_1 < s$, depending on (C, k, n) and s, such that for all $0 < t < t_1$ and for any $\hat{\omega}$-semistable sheaf $\widehat{F}$ with topological invariants (3.23) any pure dimension two quotient $E \twoheadrightarrow G$ with $\mu_\omega(G) \leq \mu_\omega(E)$ is vertical, where $E = \Phi^0(\widehat{F})$.*

Proof. For any $0 < t < s$ set $\omega_t = t\Theta - sp^*K_B$. The topological invariants (C, k, n) and $s > 0$ are fixed in the following.

Suppose the opposite statement holds. Given any $0 < t_1 < s$, there exist $0 < t < t_1$, a sheaf $\widehat{F}$ as in Lemma 3.15 and a nonzero quotient $E \twoheadrightarrow G$, not isomorphic to E, such that $\mu_{\omega_t}(G) \leq \mu_{\omega_t}(E)$ and G is not vertical. It will be shown below that this leads to a contradiction.

Note that G has topological invariants

$$\mathrm{ch}_1(G) = p^*C_G, \qquad \mathrm{ch}_2(G) = \sigma_*(\alpha_G) + cf, \qquad \mathrm{ch}_3(G) = -m\mathrm{ch}_3(\mathcal{O}_x)$$

and G is vertical if and only if $\alpha_G = 0$. Suppose $\alpha_G \neq 0$. Lemma 3.13 shows that α_G is effective, hence

$$\mu_\omega(G) = \frac{s}{t(s - t/2)} \frac{|K_B \cdot \alpha_G|}{|K_B \cdot C_G|} + \delta, \qquad \delta = \frac{1}{s - t/2} \frac{c - |K_B \cdot \alpha_G|}{|K_B \cdot C_G|}. \tag{3.28}$$

According to Lemma 3.14, there is a constant A depending on (C, k, n) and s, but not t, such that

$$|c - |K_B \cdot \alpha_G|| < A$$

for any quotient $E \twoheadrightarrow G$ as above. Moreover since $-K_B$ is very ample, the set

$$\{|\beta \cdot K_B| \,|\, 0 \neq \beta \in \mathrm{Pic}(B) \text{ effective}\} \subset \mathbb{Z}_{>0}$$

is bounded from below. Let $M \in \mathbb{Z}_{>0}$ denote its minimum and note that $|K_B \cdot C_G| \geq M$, $|K_B \cdot \alpha_G| \geq M$ since C_G, α_G are effective, nonzero.

Suppose $0 < t < 2$. Then $0 < s - 1 < s - t/2$, hence

$$|\delta| < \frac{1}{s-1} \frac{A}{M}.$$

Using inequality (3.24),

$$\frac{s}{t(s - t/2)} \frac{|K_B \cdot \alpha_G|}{|K_B \cdot C_G|} > \frac{s}{t(s - t/2)} \frac{M}{|K_B \cdot C|}.$$

Moreover, the map

$$f : (0,\ s) \to \mathbb{R}, \qquad f(t) = \frac{s}{t(s - t/2)}$$

is a decreasing function of t on the interval $0 < t < s$ for fixed $s > 0$, and $\lim_{t\to 0} f(t) = +\infty$. Therefore there exists a constant $0 < t_1 < \min\{s, 2\}$ depending on (C, k, n) and s such that for any $0 < t < t_1$,

$$\mu_\omega(G) > \frac{1}{(s-1)} \frac{|k|}{|K_B \cdot C|} + 1$$

for all quotients G as in Lemma 3.15 with $\alpha_G \neq 0$. In order to conclude the proof note that under the current assumptions

$$|\mu_\omega(E)| = \frac{|k|}{(s-t/2)|K_B \cdot C|} \leq \frac{1}{(s-1)} \frac{|k|}{|K_B \cdot C|}$$

for any $0 < t < t_1$, leading to a contradiction. □

Lemma 3.16. *Let $s > s_1$ be fixed, where $s_1 > 1$ is a constant as in Lemma 3.10 and $0 < t < t_1$ where t_1 is a constant as in Lemma 3.15 for fixed (C, k, n) and s. Then the Fourier-Mukai transform $E = \Phi^0(\widehat{F})$ of any $\hat{\omega}$-semistable sheaf $\widehat{F}$ with topological invariants* (3.23) *is ω-semistable for all $0 < t < t_1$.*

Proof. Recall that under the current assumptions

$$\chi(\widehat{F}) = k - \frac{K_B \cdot C}{2} \geq 1.$$

Let $E \twoheadrightarrow G$ be a nonzero pure dimension two quotient of E such that G is ω-semistable and destabilizes E. This means either

$$\mu_\omega(G) < \mu_\omega(E) = \frac{1}{(s-t/2)} \frac{k}{|K_B \cdot C|}$$

or $\mu_\omega(G) = \mu_\omega(E)$ and

$$\nu_\omega(G) < \nu_\omega(E).$$

According to Lemma 3.15, G must be vertical i.e.

$$\mathrm{ch}_1(G) = p^*C_G, \qquad \mathrm{ch}_2(G) = cf, \qquad \mathrm{ch}_3(G) = -m\mathrm{ch}_3(\mathcal{O}_x)$$

where C_G is a nonzero effective divisor class on B and $c \in (1/2)\mathbb{Z}$, $m \in \mathbb{Z}$, $c \equiv K_B \cdot C_G/2 \bmod \mathbb{Z}$. Therefore

$$\mu_\omega(G) = \frac{c}{(s-t/2)|K_B \cdot C_G|}.$$

At the same time E is generically semistable according to Lemma 3.12. Hence, given any very ample linear system Π on B, $E|_Z$ is semistable for any sufficiently generic very ample divisor $H \in \Pi$, where $Z = p^{-1}(H)$. Moreover Lemma 2.6 yields $\chi(E|_Z) = \chi(G|_Z) = 0$. This implies that G must be generically semistable as well. Then Lemma 3.7 implies that G is $\widehat{\Phi} - WIT_1$ and $\widehat{\Phi}^1(G)$ is pure dimension one. Furthermore the epimorphism $E \twoheadrightarrow G$ yields an epimorphism $\widehat{F} \twoheadrightarrow \widehat{\Phi}^1(G)$. Therefore

$$\mu_{\hat{\omega}}(\widehat{F}) \leq \mu_{\hat{\omega}}(\widehat{\Phi}^1(G)) \tag{3.29}$$

since $\widehat{F}$ is $\hat{\omega}$-semistable. The topological invariants of $\widehat{\Phi}^1(G)$ are

$$\mathrm{ch}_2(\widehat{\Phi}^1(G)) = \hat{\sigma}_*(C_G) + m\hat{f}, \qquad \chi(\widehat{\Phi}^1(G)) = c - K_B \cdot C_G/2.$$

Note that Corollary 2.2 implies that $0 \leq m \leq n$. Moreover, $\chi(\widehat{\Phi}^1(G)) > 0$ since $\chi(\widehat{F}) > 0$ under the current assumptions. At the same time, the slope inequality $\mu_\omega(G) \leq \mu_\omega(E)$ is equivalent to $\delta_1 \leq 0$, where

$$\delta_1 = \chi(\widehat{\Phi}^1(G))|K_B \cdot C| - \chi(\widehat{F})|K_B \cdot C_G|.$$

In conclusion $(C_G, \chi(\widehat{\Phi}^1(G)), m) \in \mathcal{S}(C, k, n)$, where $\mathcal{S}(C, k, n)$ is the finite set defined above Lemma 3.9.

The slope inequality (3.29) is equivalent to

$$(s-1)\delta_1 + \delta_2 \geq 0,$$

where

$$\delta_2 = n\chi(\widehat{\Phi}^1(G)) - m\chi(\widehat{F}).$$

Since $\delta_1 \in \mathbb{Z}$, one has to distinguish two cases.

(i) $\delta_1 \leq -1$. In this case

$$(s-1)\delta_1 + \delta_2 = f_s(C_G, \chi(\widehat{\Phi}^1(G)), m) < 0$$

where $f_s : \mathcal{S}'(C, k, n) \to \mathbb{R}$ is the function defined above Lemma 3.10. Obviously, this leads to a contradiction.

(*ii*) Suppose $\delta_1 = 0$. This implies

$$\delta_2 = \frac{\chi(\widehat{F})}{|K_B \cdot C|}\big(n|K_B \cdot C_G| - m|K_B \cdot C|\big).$$

However in this case $\mu_\omega(G) = \mu_\omega(E)$, hence one must have

$$\nu_\omega(G) < \nu_\omega(E),$$

which is equivalent to $n|K_B \cdot C_G| - m|K_B \cdot C| < 0$. Since $\chi(\widehat{F}) > 0$, this implies $\delta_2 < 0$, leading again to a contradiction. □

3.4. Proof of Theorem 1.1

This subsection concludes the proof of Theorem 1.1. Let $\hat{\gamma} \in H_2(B, \mathbb{Z}) \oplus \mathbb{Z} \oplus \mathbb{Z}$ be fixed topological invariants with $\hat{\gamma}_1$ an effective curve class on B, and $\hat{\gamma}_3 > 0$. Let $\gamma = \phi(\hat{\gamma})$. Let $s_1(\hat{\gamma}) > 1$ be a constant as in Lemma 3.10. For any $s \in \mathbb{R}$, $s > s_1(\hat{\gamma})$, let $t_1(\hat{\gamma}, s) \in \mathbb{R}$, $0 < t_1(\hat{\gamma}, s) < s$ be a constant as in Lemma 3.15. Let

$$\omega = t\Theta - sp^*K_B, \qquad \hat{\omega} = \widehat{\Theta} - s\hat{p}^*K_B.$$

Lemmas 3.12 and 3.16 prove that any $\hat{\omega}$-semistable sheaf $\widehat{F}$ with topological invariants $\hat{\gamma}$ is $\Phi - WIT_0$ and $\Phi^0(\widehat{F})$ is an ω-semistable vertical pure dimension two sheaf E on X with invariants γ. Moreover E is also generically semistable. Conversely, Lemmas 3.7 and 3.11 prove that any ω-semistable and generically semistable vertical sheaf E with topological invariants γ is $\widehat{\Phi} - WIT_1$ and $\widehat{\Phi}^1(E)$ is an $\hat{\omega}$-semistable sheaf on $\widehat{X}$ with invariants $\hat{\gamma}$. Furthermore, Lemmas 2.11 and 2.12 prove that generic semistability is equivalent to adiabatic semistability for ω-semistable sheaves.

In order to conclude the proof of Theorem 1.1.i. note that the Fourier-Mukai transform preserves flat families of sheaves [11, Prop. 6.13.].

For the second statement, note that the substack $\mathcal{M}^{\mathsf{ad}}_{\underset{\sim}{\omega}}(X, \gamma)$ is open in $\mathcal{M}_\omega(X, \gamma)$ according to Lemma 2.14. Moreover, let $M_{\hat{\omega}}(\widehat{X}, \hat{\gamma})$, $M_\omega(X, \gamma)$ be the coarse moduli schemes parameterizing S-equivalence classes of semistable sheaves. As noted above Lemma 2.14, according to [1, Ex. 8.7], the coarse moduli schemes are good moduli coarse moduli spaces for the moduli stacks

$\mathcal{M}_{\hat{\omega}}(\widehat{X}, \hat{\gamma})$, $\mathcal{M}_\omega(X,\gamma)$. Using [1, Thm 4.16], this yields a commutative diagram

$$\begin{array}{ccc} \mathcal{M}_{\hat{\omega}}(\widehat{X}, \hat{\gamma}) & \xrightarrow{\varphi} & \mathcal{M}_\omega(X,\gamma) \\ \downarrow{\scriptstyle \hat{\varrho}} & & \downarrow{\scriptstyle \varrho} \\ M_{\hat{\omega}}(\widehat{X}, \hat{\gamma}) & \xrightarrow{f} & M_\omega(X,\gamma) \end{array}$$

where φ factors through the natural embedding $\mathcal{M}^{\mathsf{ad}}_\omega(X,\gamma) \subset \mathcal{M}_\omega(X,\gamma)$. In the above diagram f is a morphism of schemes, and the vertical morphisms are those constructed in [1, Thm 4.16]. Since both coarse moduli spaces are projective, it follows that f is proper. At the same time, according to Lemma 2.14, the scheme theoretic image $M^{\mathsf{ad}}_\omega(X,\gamma) = \varrho(\mathcal{M}^{\mathsf{ad}}_\omega(X,\gamma))$ is open in $M_\omega(X,\gamma)$. Since f is proper and $M_{\hat{\omega}}(\widehat{X},\hat{\gamma})$ is projective, it follows that $M^{\mathsf{ad}}_\omega(X,\gamma)$ is open and closed in $M_\omega(X,\gamma)$. Therefore $\mathcal{M}^{\mathsf{ad}}_\omega(X,\gamma)$ is open and closed in $\mathcal{M}_\omega(X,\gamma)$.

□

4. Vertical sheaves on elliptic $K3$ pencils

Using the notation in Section 1.2, let X be a smooth generic Weierstrass model over the Hirzebruch surface $B = \mathbb{F}_a$, $0 \le a \le 1$. Let $\pi : X \to \mathbb{P}^1$ be the natural projection to $\mathbb{P}^1$. Note that all fibers of π are reduced irreducible elliptic $K3$-surfaces in Weierstrass form. For sufficiently generic X, the generic $K3$ fiber is a smooth Weierstrass model and the singular fibers will be Weierstrass models with finitely many isolated type I_1 and I_2 fibers. In particular all singular K3 fibers are reduced, irreducible with isolated simple nodal singularities. This will be assumed throughout this section.

Let $\Xi \in \mathrm{Pic}(B) \simeq H_2(B,\mathbb{Z})$ denote the fiber class of the Hirzebruch surface and note that the $K3$ fiber class is $D = p^*\Xi \in \mathrm{Pic}(X) \simeq H_4(X,\mathbb{Z})$. Let $\omega = t\Theta - sp^*K_B$ be a Kähler class on X with $t, s \in \mathbb{R}$, $0 < t < s$. In order to simply the notation the pushforward $\sigma_* C \in H_2(X,\mathbb{Z})$ of a curve class on B will be denoted by C. The distinction will be clear from the context.

First note the following simple fact.

Lemma 4.1. *Let E be a nonzero pure dimension two sheaf set theoretically supported on a finite union of $K3$ fibers. Then*

$$\mathrm{ch}_1(E) = rD, \qquad \mathrm{ch}_2(E) = m\Xi + lf \tag{4.1}$$

for some $r, m, l \in \mathbb{Z}$, $r \ge 1$.

Proof. Let $C_0 \in H_2(B,\mathbb{Z})$ be a section class with $C_0^2 = -a$ for $B = \mathbb{F}_a$, $0 \leq a \leq 1$. Let $D_0 = p^*C_0 \in H_4(X,\mathbb{Z})$. According to Lemma 2.1

$$H_4(X,\mathbb{Z})/_{\text{torsion}} \simeq \mathbb{Z}\langle D_0, D, \Theta\rangle, \qquad H_2(X,\mathbb{Z})/_{\text{torsion}} \simeq \mathbb{Z}\langle C_0, \Xi, f\rangle.$$

Obviously $\mathrm{ch}_i(E)\cdot D = 0$ in the intersection ring of X for $1 \leq i \leq 3$ since under the current assumptions $E \simeq E \otimes_X \mathcal{O}_X(-D)$ and $D^2 = 0$. Then the claim follows easily from Lemma 2.1 and the following relations in the intersection ring of X:

$$\begin{aligned} &D_0 \cdot D = f, \qquad D\cdot D = 0, \qquad \Theta \cdot D = \Xi \\ &C_0 \cdot D = 1, \qquad \Xi \cdot D = 0, \qquad f \cdot D = 0. \end{aligned} \tag{4.2}$$

□

Now suppose $\iota : S \hookrightarrow X$ is a singular K3 fiber. Under the current genericity assumptions S is an elliptic surface over $\mathbb{P}^1$ in Weierstrass form with finitely many type I_1 and I_2 fibers. Therefore S will have finitely many isolated simple nodes and the singular locus of S is disjoint from the canonical section of the Weierstrass model. Let $\rho : \widetilde{S} \to S$ be a smooth crepant resolution of singularities and let $\psi = \iota \circ \rho : \widetilde{S} \to X$. Let $\widetilde{\Xi} = \psi^*\Theta$, $\tilde{f} = \psi^* D_0$ be induced divisor classes on $\widetilde{S}$. Let also $\epsilon_1, \ldots, \epsilon_k$ denote the exceptional (-2) curve classes on $\widetilde{S}$ and note that

$$\begin{aligned} &\widetilde{\Xi}^2 = -2, \qquad \widetilde{\Xi}\cdot \tilde{f} = 1, \qquad \tilde{f}^2 = 0, \\ &\epsilon_i \cdot \widetilde{\Xi} = \epsilon_i \cdot \tilde{f} = 0, \qquad 1 \leq i \leq k, \end{aligned} \tag{4.3}$$

in the intersection ring of $\widetilde{S}$, and

$$\psi_*\epsilon_i = 0, \qquad 1 \leq i \leq k.$$

Then note the following.

Lemma 4.2. (*i*) *Let $\iota : S \hookrightarrow X$ be a smooth K3 fiber of X and F a torsion free sheaf on S such that*

$$\mathrm{ch}_1(\iota_*F) = rD, \qquad \mathrm{ch}_2(\iota_*F) = m\Xi + lf, \qquad \mathrm{ch}_3(\iota_*F) = -n\mathrm{ch}_3(\mathcal{O}_x)$$

for some $l, m, n, r \in \mathbb{Z}$, $r \geq 1$. Then

$$\mathrm{ch}_0(F) = r, \qquad \mathrm{ch}_1(F) = m\Xi + lf + \beta, \qquad \mathrm{ch}_2(F) = -n\mathrm{ch}_2(\mathcal{O}_s)$$

for a curve class $\beta \in H_2(S,\mathbb{Q})$ such that $\beta\cdot\Xi = \beta \cdot f = 0$ in the intersection ring of S and $\iota_*\beta = 0$.

(ii) Let $\iota : S \hookrightarrow$ be a singular K3 fiber of X and $\widetilde{F}$ a torsion free sheaf on the resolution $\widetilde{S}$ such that

$$\mathrm{ch}_1(\psi_*\widetilde{F}) = rD, \qquad \mathrm{ch}_2(\psi_*\widetilde{F}) = m\Xi + lf, \qquad \mathrm{ch}_3(\psi_*\widetilde{F}) = -n\mathrm{ch}_3(\mathcal{O}_x)$$

for some $l, m, n, r \in \mathbb{Z}$, $r \geq 1$. Then

$$\mathrm{ch}_0(\widetilde{F}) = r, \quad \mathrm{ch}_1(F) = m\widetilde{\Xi} + l\tilde{f} + \tilde{\beta} + \sum_{i=1}^{k} p_i\epsilon_i, \quad \mathrm{ch}_2(\widetilde{F}) = -\tilde{n}\mathrm{ch}_2(\mathcal{O}_s)$$

for some $p_i \in \mathbb{Q}$, $1 \leq i \leq k$, $\tilde{n} \in \mathbb{Z}$, $\tilde{n} \leq n$, and a curve class $\tilde{\beta} \in H_2(\widetilde{S}, \mathbb{Q})$ such that

$$\tilde{\beta} \cdot \widetilde{\Xi} = \tilde{\beta} \cdot \tilde{f} = \tilde{\beta} \cdot \epsilon_i = 0, \qquad 1 \leq i \leq k$$

in the intersection ring of $\widetilde{S}$ and $\psi_*\tilde{\beta} = 0$.

Proof. For (i) note that the Grothendieck-Riemann-Roch theorem yields

$$\mathrm{ch}_0(F) = r, \qquad \iota_*\mathrm{ch}_1(F) = m\Xi + lf, \qquad \mathrm{ch}_2(F) = -n\mathrm{ch}_3(\mathcal{O}_s) \tag{4.4}$$

with $s \in S$ a closed point. Then the push pull formula yields

$$\mathrm{ch}_1(F) \cdot \Xi = l - 2m, \qquad \mathrm{ch}_1(F) \cdot f = m$$

in the intersection ring of S. Therefore

$$\mathrm{ch}_1(F) = m\Xi + lf + \beta$$

where $\beta \in H_2(S, \mathbb{Z})$ is orthogonal to Ξ, f. Moreover the second equation in (4.4) implies $\iota_*\beta = 0$.

(ii) Since $\psi = \iota \circ \rho$ and $\rho : \widetilde{S} \to S$ is an isomorphism onto the smooth open part of S, $R^1\psi_*\widetilde{F}$ is a zero dimensional sheaf supported at the nodes of S. Then the Grothendieck-Riemann-Roch theorem gives

$$\mathrm{ch}_0(\widetilde{F}) = r, \qquad \psi_*\mathrm{ch}_1(\widetilde{F}) = m\Xi + lf,$$
$$\psi_*\mathrm{ch}_2(\widetilde{F}) = -n\mathrm{ch}_3(\mathcal{O}_x) + \mathrm{ch}_3(R^1\psi_*\widetilde{F}).$$

The remaining part of the proof is analogous to (i). □

For any pure dimension two sheaf E with scheme theoretic support on a reduced nodal K3 fiber $S \subset X$, let $\widetilde{F}_E = \psi^*E/\mathrm{torsion}$. Note that given an ample class ω on X, the real divisor class $\tilde{\omega}_\lambda = \lambda\psi^*\omega - \sum_{i=1}^k \epsilon_i$ is ample on $\widetilde{S}$ for sufficiently large $\lambda \in \mathbb{R}$, $\lambda > 0$. Then the following result is similar to [29, Lemma 2.1].

Lemma 4.3. *Let $\iota : S \to X$ be a reduced nodal $K3$ fiber. Let E be a nonzero ω-slope stable pure dimension two sheaf on X set theoretically supported on S. Then E is scheme theoretically supported on S and $\widetilde{F}_E$ is $\tilde{\omega}_\lambda$-slope stable for sufficiently large $\lambda > 0$.*

Proof. Proving that E is scheme theoretically supported on S is completely analogous to Lemma 3.5. The details will be omitted. For the second statement, by construction there is an exact sequence

$$0 \to \mathcal{T} \to \psi^*E \to \widetilde{F}_E \to 0$$

where $\mathcal{T}$ is set theoretically supported on the exceptional locus of ρ. This yields a second sequence

$$0 \to \psi_*\mathcal{T} \to \psi_*\psi^*E \xrightarrow{f} \psi_*\widetilde{F}_E \to R^1\psi_*\mathcal{T} \to \cdots$$

where $\psi_*\mathcal{T}, R^1\psi_*\mathcal{T}$ are set theoretically supported on the singular locus $S^{\mathsf{sing}} \subset S$, which consists of fintely many points. Moreover there is a natural morphism $g : E \to \psi_*\psi^*E$ which is an isomorphism on the smooth locus $S \setminus S^{\mathsf{sing}}$. The morphism $f \circ g : E \to \psi_*\widetilde{F}_E$ is also an isomorphism on $S \setminus S^{\mathsf{sing}}$, hence it must be injective since E is purely two dimensional. In conclusion there is an exact sequence

$$0 \to E \to \psi_*\widetilde{F}_E \xrightarrow{f} T \to 0 \tag{4.5}$$

with T zero dimensional. This implies that $\mu_\omega(E) = \mu_\omega(\psi_*\widetilde{F}_E)$.

If $r = 1$, $\widetilde{F}_E$ is a rank one torsion free sheaf which is slope stable for any polarization of $\widetilde{S}$. Recall that slope stability is defined with respect to saturated nonzero test subsheaves as in [28, Sect. 1.6].

Let $r \geq 2$ and suppose $\widetilde{G} \subset \widetilde{F}_E$ is a nonzero proper saturated subsheaf of rank $1 \leq r' \leq r-1$. Then $\psi_*\widetilde{G}$ is a subsheaf of $\psi_*\widetilde{F}_E$. Let $I \subset T$, $G \subset \psi_*\widetilde{G}$ be the image and respectively the kernel of $f|_{\psi_*\widetilde{G}}$ in the exact sequence (4.5). Then I is zero dimensional and G is a subsheaf of E. This implies that $\mu_\omega(G) = \mu_\omega(\psi_*\widetilde{G})$, hence $\mu_\omega(\psi_*\widetilde{G}) < \mu_\omega(E) = \mu_\omega(\psi_*\widetilde{F}_E)$ since E is ω-stable by assumption. Therefore

$$(r'\mathrm{ch}_1(\widetilde{F}_E) - r\mathrm{ch}_1(\widetilde{G})) \cdot \psi^*\omega > 0. \tag{4.6}$$

Let $\lambda_0 > 0$ be fixed such that $\tilde{\omega}_0 = \lambda_0\psi^*\omega - \sum_{i=1}^k \epsilon_i$ is ample on $\widetilde{S}$. The subsheaves $\widetilde{G} \subset \widetilde{F}_E$ are of two types:

a) $(r'\mathrm{ch}_1(\widetilde{F}) - r\mathrm{ch}_1(\widetilde{G})) \cdot \omega_0 > 0$. Then, using inequality (4.6),

$$(\omega_0 + \lambda\psi^*\omega) \cdot (r'\mathrm{ch}_1(\widetilde{F}_E) - r\mathrm{ch}_1(\widetilde{G}) > 0 \tag{4.7}$$

for any $\lambda > 0$.

b) $(r'\mathrm{ch}_1(\widetilde{F}_E) - r\mathrm{ch}_1(\widetilde{G})) \cdot \omega_0 \leq 0$. According to Grothendieck's Lemma [28, Lemma 1.7.9] the family of such subsheaves is bounded for fixed $\widetilde{F}_E$ and ω_0. Therefore there exists a constant $c_1 > 0$ depending on $\widetilde{F}_E$, ω_0 such that

$$(r'\mathrm{ch}_1(\widetilde{F}_E) - r\mathrm{ch}_1(\widetilde{G})) \cdot \psi^*\omega > c_1$$

for any subsheaf $\widetilde{G}$ of type (b). Furthermore there is a second constant $c_2 > 0$ depending on $\widetilde{F}$, ω_0 such that

$$(r'\mathrm{ch}_1(\widetilde{F}_E) - r\mathrm{ch}_1(\widetilde{G})) \cdot \omega_0 > -c_2$$

for any such subsheaf. This implies that there exists a sufficiently large $\lambda > 0$ such that inequality (4.7) holds for all subsheaves of type (b) as well. In conclusion $\widetilde{F}_E$ is $(\omega_0 + \lambda\psi^*\omega)$-slope stable. □

Now recall that the discriminant of a rank $r \geq 1$ torsion free sheaf F on a smooth projective surface S is defined (up to normalization) by

$$\Delta(F) = n + \frac{1}{2r}\mathrm{ch}_1(F)^2$$

where $\mathrm{ch}_2(F) = -n\mathrm{ch}_2(\mathcal{O}_s)$, with $s \in S$ and arbitrary closed point. For any vertical pure dimension two sheaf E with

$$\mathrm{ch}_1(E) = rD, \qquad \mathrm{ch}_2(E) = m\Xi + lf, \qquad \mathrm{ch}_3(E) = -n\mathrm{ch}_3(\mathcal{O}_x)$$

let

$$\delta(E) = n - \frac{1}{r}m(m-l). \tag{4.8}$$

Then note the following.

Lemma 4.4. *Let E be an ω-slope semistable pure dimension two sheaf on X with topological invariants*

$$\mathrm{ch}_1(E) = rD, \qquad \mathrm{ch}_2(E) = m\Xi + lf, \qquad \mathrm{ch}_2(E) = -n\mathrm{ch}_3(\mathcal{O}_x)$$

where $r, l, m, n \in \mathbb{Z}$, $r \geq 1$, and $x \in X$ is an arbitrary closed point. Suppose E is scheme theoretically supported on a reduced $K3$ fiber $\iota : S \hookrightarrow X$. Then $\delta(E) \geq 0$.

Proof. Obviously, $E = \iota_* F$ for a torsion free sheaf on S.

Suppose first that S is smooth. Then F is $\omega|_S$-slope semistable. According to Lemma 4.2.i,

$$\mathrm{ch}_1(F) = m\Xi + lf + \beta$$

where $\beta \in H_2(S, \mathbb{Q})$ is a curve class such that $\beta \cdot \Xi = \beta \cdot f = 0$. At the same time $\omega|_S = t\Xi + 2sf$, hence $\beta \cdot \omega|_S = 0$. Then $\beta^2 \leq 0$ according to the Hodge index theorem. Since F is $\omega|_S$-slope semistable, it satisfies the Bogomolov inequality, $\Delta(F) \geq 0$, where

$$\Delta(F) = n - \frac{1}{r} m(m-l) + \frac{\beta^2}{2r} = \delta(E) + \frac{\beta^2}{2r}.$$

Since $\beta^2 \leq 0$, this implies the claim.

Next let S be a singular $K3$ fiber. Suppose first that E is ω-slope stable. Then it is scheme theoretically supported on S. Let $\widetilde{F}_E$ be the corresponding torsion free sheaf on $\widetilde{S}$. Lemma 4.3 shows that $\widetilde{F}_E$ is stable for a suitable ample class $\tilde{\omega}$ on $\widetilde{S}$, hence $\Delta(\widetilde{F}_E) \geq 0$. Moreover as shown in the proof of Lemma 4.3, there is an exact sequence

$$0 \to E \to \psi_* \widetilde{F}_E \to T \to 0$$

with T zero dimensional. Setting

$$\mathrm{ch}_3(E) = -n\mathrm{ch}_3(\mathcal{O}_x), \qquad \mathrm{ch}_3(\psi_* \widetilde{F}_E) = -n'\mathrm{ch}_3(\mathcal{O}_x)$$

this implies $n \geq n'$. Furthermore, according to Lemma 4.2.ii,

$$\mathrm{ch}_0(\widetilde{F}_E) = r, \qquad \mathrm{ch}_1(\widetilde{F}_E) = m\widetilde{\Xi} + l\tilde{f} + \tilde{\beta} + \sum_{i=1}^{k} p_i \epsilon_i,$$
$$\mathrm{ch}_2(\widetilde{F}_E) = -\tilde{n}\mathrm{ch}_2(\mathcal{O}_s)$$

with $p_i \in \mathbb{Q}$, $\tilde{n} \in \mathbb{Z}$, $\tilde{n} \leq n'$, and $\tilde{\beta} \in H_2(\widetilde{S}, \mathbb{Q})$ a curve class orthogonal to $\widetilde{\Xi}, \tilde{f}, \epsilon_i$ for all $1 \leq i \leq k$. In particular $\tilde{\beta} \cdot \tilde{\omega} = 0$. Then

$$\Delta(\widetilde{F}_E) = \tilde{n} - \frac{1}{r} m(m-l) + \frac{1}{2r}\left(\tilde{\beta}^2 - 2\sum_{i=1}^{n} p_i^2\right)$$

Since $\tilde{\beta} \cdot \tilde{\omega} = 0$, the Hodge index theorem shows that $\tilde{\beta}^2 \leq 0$. Since $\tilde{n} \leq n' \leq n$, this implies the claim.

To finish the proof, suppose E is strictly ω-slope semistable. According to [28, Thm 1.6.7.ii], there is a Jordan-Hölder filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_j = E$$

for slope semistability with $j \geq 2$. Each succesive quotient E_i/E_{i-1}, $1 \leq i \leq j$, is ω-slope polystable, hence scheme theoretically supported on S. Therefore $\delta(E_i/E_{i-1}) \geq 0$ for all $1 \leq i \leq j$. Then the claim follows by a recursive application of Lemma 4.5 below. □

Lemma 4.5. *Let*

$$0 \to E_1 \to E \to E_2 \to 0$$

be an extension of nonzero pure dimension two sheaves such that E_1, E_2 are ω-slope semistable and set theoretically supported on finite unions of K3 fibers. Suppose that $\mu_\omega(E_1) = \mu_\omega(E_2)$. Then

$$\delta(E) \geq \delta(E_1) + \delta(E_2)$$

Proof. Let

$$\mathrm{ch}_1(E_i) = r_i D, \qquad \mathrm{ch}_2(E_i) = m_i \Xi + l_i l$$

for $1 \leq i \leq 2$, where $r_1, r_2 \geq 1$. Then

$$\delta(E) - \delta(E_1) - \delta(E_2) = d$$

where

$$d = \frac{(r_1 m_2 - r_2 m_1)}{r_1 r_2 (r_1 + r_2)} \left[(r_1 m_2 - r_2 m_1) - (r_1 l_2 - r_2 l_1)\right].$$

Let S be a generic smooth K3 fiber and $\Xi, f \in H_2(S, \mathbb{Z})$ the section, respectively fiber class. Then

$$d = -\frac{\alpha^2}{2 r_1 r_2 (r_1 + r_2)}$$

where

$$\alpha = r_1(m_2\Xi + l_2 f) - r_2(m_1\Xi + l_1 f).$$

The slope equality $\mu_\omega(E_1) = \mu_\omega(E_2)$ is equivalent to $\alpha \cdot \omega|_S = 0$. Since S is smooth, the Hodge index theorem shows that $\alpha^2 \leq 0$. This proves the claim. □

Lemma 4.6. *Let E be an ω-slope semistable sheaf on X with topological invariants*

$$\mathrm{ch}_1(E) = rD, \qquad \mathrm{ch}_2(E) = m\Xi + lf, \qquad \mathrm{ch}_2(E) = -n\mathrm{ch}_3(\mathcal{O}_x)$$

where $r, l, m, n \in \mathbb{Z}$, $r \geq 1$, and $x \in X$ is an arbitrary closed point. Suppose E is scheme theoretically supported on a reduced $K3$ fiber $\iota : S \hookrightarrow X$ and there is an extension

$$0 \to E_1 \to E \to E_2 \to 0$$

with E_1, E_2 nonzero pure dimension two sheaves with $\mathrm{ch}_1(E_i) = r_i D$, $r_i \in \mathbb{Z}$, $r_i \geq 1$, $1 \leq i \leq 2$. Moreover suppose

$$\mu_\omega(E_1) = \mu_\omega(E_2) \quad \text{and} \quad \frac{1}{r_1}\mathrm{ch}_2(E_1) - \frac{1}{r_2}\mathrm{ch}_2(E_2) \neq 0. \tag{4.9}$$

Then

$$\frac{t}{s} \geq \frac{2}{1 + r^3\delta(E)}. \tag{4.10}$$

Proof. As in the proof of Lemma 4.5, let $\iota' : S' \hookrightarrow X$ be a smooth generic K3 fiber and $\Xi, f \in H_2(S', \mathbb{Z})$ the section and fiber class respectively. Note that

$$\mathrm{ch}_2(E_i) = \iota'_* \alpha_i$$

for $\alpha_i = m_i\Xi + l_i f \in H_2(S', \mathbb{Z})$, $1 \leq i \leq 2$ and

$$\delta(E) - \delta(E_1) - \delta(E_2) = -\frac{\alpha^2}{2rr_1r_2}$$

where $\alpha = r_1\alpha_2 - r_2\alpha_1$. Then Lemma 4.4 implies that

$$-\frac{\alpha^2}{2rr_1r_2} \leq \delta(E).$$

For simplicity let $\alpha = a\Xi + bf$, $a, b \in \mathbb{Z}$. The slope equality in (4.9) implies that $\alpha \cdot \omega|_{S'} = 0$, which yields

$$b = 2a\left(1 - \frac{s}{t}\right).$$

Therefore

$$-\alpha^2 = 2a^2\left(\frac{2s}{t} - 1\right).$$

Next note that $a \neq 0$; if $a = 0$, one has $b = 0$ as well, hence $\alpha = 0$, contradicting the second condition in (4.9). Therefore $a^2 \geq 1$ since $a \in \mathbb{Z}$. Moreover, $\delta(E) \geq 0$ according to Lemma 4.4, and $1 \leq r_1, r_2 \leq r$. This implies inequality (4.10). □

Lemma 4.7. *Let E be an ω-slope stable pure dimension two sheaf on X with topological invariants*

$$\mathrm{ch}_1(E) = rD, \qquad \mathrm{ch}_2(E) = lf, \qquad \mathrm{ch}_3(E) = -n\mathrm{ch}_3(\mathcal{O}_x),$$

$l, n, r \in \mathbb{Z}$, $r \geq 1$. *Suppose there exists $t' \in \mathbb{R}$, $0 < t' < t$ such that E is not ω'-slope semistable, where $\omega' = t'\Theta - sp^*K_B$. Then*

$$\frac{t}{s} > \frac{2}{1 + r^3\delta(E)}. \tag{4.11}$$

Proof. Any sheaf E with $\mathrm{ch}_1(E) = rD$ must be set theoretically supported on a finite union of $K3$ fibers of X. Since E is ω-slope stable, it must be scheme theoretically supported on a reduced irreducible fiber $\iota : S \hookrightarrow X$.

Let $\mathcal{Q}_E(t', t)$ denote the family of sheaves E' such that E' is a nonzero pure dimension two quotient of E, not isomorphic to E, and $\mu_{\omega'}(E') < \mu_{\omega'}(E)$. According to Grothendieck's lemma [28, 1.7.9], $\mathcal{Q}_E(t', t)$ is bounded. Any quotient E' of E is also scheme theoretically supported on S and has invariants of the form

$$\mathrm{ch}_1(E') = r'D, \qquad \mathrm{ch}_2(E') = m'\Xi + l'f, \qquad \mathrm{ch}_3(E') = -n'\mathrm{ch}_3(\mathcal{O}_x), \tag{4.12}$$

$l', m', n', r' \in \mathbb{Z}$, $r' \geq 1$. Since the family $\mathcal{Q}_E(t', t)$ is bounded, the set of numerical invariants (r', m', l', n') of all sheaves in this family is finite.

For any $t'' \in \mathbb{R}$, $t' \leq t'' \leq t$ set $\omega'' = t''\Theta - sp^*K_B$. For any $\gamma' = (l', m', n', r') \in \mathbb{Z}^4$, $r' \geq 1$ let $\eta_{\gamma'} : [t',\ t] \to \mathbb{R}$ be the linear function

$$\eta_{\gamma'}(t'') = \frac{2m'}{r'}s - \left(\frac{2m' - l'}{r'} + \frac{l}{r}\right)t''.$$

Then note that for any sheaf E' with invariants (4.12) one has

$$\mu_{\omega''}(E') - \mu_{\omega''}(E) = \frac{\eta_{\gamma'}(t'')}{t''(2s - t'')}.$$

Since E is ω-slope stable and not ω'-slope semistable, one has

$$\eta_{\gamma'}(t') < 0, \qquad \eta_{\gamma'}(t) > 0$$

for any sheaf E' in the family $\mathcal{Q}_E(t', t)$. Therefore $\eta_{\gamma'}$ is an increasing linear function of t'' for any such sheaf . In particular there exists exactly one point $t' < t(\gamma') < t$ such that $\eta_{\gamma'}(t(\gamma')) = 0$. The set of all $t(\gamma')$ associated to E' in $\mathcal{Q}_E(t', t)$ is finite. Let t_0 be its maximal element and $\omega_0 = t_0\Theta - sp^*K_B$. Then it will be shown below that E is strictly ω_0-slope semistable.

Given the choice of t_0, one has $\eta_{\gamma'}(t_0) \geq 0$ for any quotient $E \twoheadrightarrow E'$ in $\mathcal{Q}_E(t', t)$. Moreover, there exists E'_0 in $\mathcal{Q}_E(t', t)$ such that $\eta_{\gamma'}(t_0) = 0$. Clearly, E'_0 cannot be isomorphic to E since $\mu_{\omega'}(E_0) < \mu_{\omega'}(E)$. Hence the kernel $E''_0 = \mathrm{Ker}(E \twoheadrightarrow E'_0)$ is nontrivial. This implies that $\mathrm{ch}_1(E'_0) = r'_0 D$, $\mathrm{ch}_1(E''_0) = r''_0 D$ with $r'_0, r''_0 \geq 1$.

Given a quotient $E \twoheadrightarrow E'$ not in $\mathcal{Q}_E(t', t)$, one has

$$\eta_{\gamma'}(t') \geq 0, \qquad \eta_{\gamma'}(t) > 0.$$

Since $\eta_{\gamma'}$ is linear this implies that $\eta_{\gamma'}(t_0) > 0$, hence E' cannot destabilize E with respect to ω_0.

In conclusion E is indeed ω_0-slope semistable and there is an exact sequence

$$0 \to E''_0 \to E \to E'_0 \to 0$$

such that $\mu_{\omega_0}(E''_0) = \mu_{\omega_0}(E'_0)$ and $r'_0, r''_0 \geq 1$. Moreover, since E is ω-slope stable one must have

$$\frac{1}{r'_0}\mathrm{ch}_2(E'_0) - \frac{1}{r''_0}\mathrm{ch}_2(E''_0) \neq 0.$$

Then Lemma 4.6 implies that $t_0/s \geq 2/(1 + r^3\delta(E))$. $\square$

4.1. Proof of Proposition 1.2

Let $(n, r) \in \mathbb{Z} \times \mathbb{Z}$ be fixed integers, $n \geq 0$, $r \geq 1$. For any $j \in \mathbb{Z}$, $1 \leq j \leq r$, let

$$\Gamma_j(n,r) = \Bigg\{ ((n_1, r_1), \ldots (n_j, r_j)) \in (\mathbb{Z} \times \mathbb{Z})^{\times j} \ \Bigg| \ n_i \geq 0,\ r_i \geq 1,\ 1 \leq i \leq j,\ \sum_{i=1}^{j} r_i = r,\ \sum_{i=1}^{j} n_i = n \Bigg\}.$$

Then let $\Gamma(n, r) = \cup_{j=1}^{r} \Gamma_j(n, r)$. Clearly $\Gamma(n, r)$ is a finite set. Let $t \in \mathbb{R}$, $t > 0$ be such that $t/s \in \mathbb{R} \setminus \mathbb{Q}$ and

$$\frac{t}{s} < \frac{2}{1 + r_i^3 n_i}, \qquad 1 \leq i \leq j, \tag{4.13}$$

for any element $\big((n_i, r_i)\big)_{1 \leq i \leq j} \in \Gamma_j(n, r)$, and for all $1 \leq j \leq r$.

Let E be an ω-semistable sheaf on X with topological invariants

$$\mathrm{ch}_1(E) = rD, \qquad \mathrm{ch}_2(E) = lf, \qquad \mathrm{ch}_3(E) = -n\mathrm{ch}_3(\mathcal{O}_x),$$

$l, n, r \in \mathbb{Z}$, $r \geq 1$. Then E is ω-slope semistable. Let

$$0 = E_0 \subset E_1 \subset \cdots \subset E_j = E \tag{4.14}$$

be a Jordan-Hölder filtration of E with respect to ω-slope stability. Let

$$\begin{aligned} &\mathrm{ch}_1(E_i/E_{i-1}) = r_i D, \qquad \mathrm{ch}_2(E_i/E_{i-1}) = m_i \Xi + l_i f, \\ &\mathrm{ch}_3(E_i/E_{i-1}) = -n_i \mathrm{ch}_3(\mathcal{O}_x) \end{aligned} \tag{4.15}$$

be the topological invariants of the i-th successive quotient, where $r_i, l_i, n_i \in \mathbb{Z}$, $r_i \geq 1$. Since $t/s \in \mathbb{R} \setminus \mathbb{Q}$ a simple computation shows that

$$m_i = 0, \qquad \frac{l_i}{r_i} = \frac{l}{r} \tag{4.16}$$

for each $1 \leq i \leq j$. Obviously,

$$\sum_{i=1}^{j} r_i = r, \qquad \sum_{i=1}^{j} n_i = n$$

Moreover, Lemma 4.4 shows that $\delta(E_i/E_{i-1}) = n_i \geq 0$ for each $1 \leq i \leq j$. Since t/s satisfies inequalities (4.13), Lemma 4.6 implies that each E_i/E_{i-1} is adiabatically ω-slope semistable. According to Lemma 2.11, this implies that each E_i/E_{i-1} is generically semistable as in Definition 2.5. Let H be a very ample divisor in B satisfying the genericity conditions in loc. cit. for E as well as for each successive quotient E_i/E_{i-1}. In particular $Z = p^{-1}(H)$ is a smooth elliptic surface which intersects the set theoretic support of E along a finite union of elliptic fibers. Then Lemma 2.6 implies that the filtration (4.14) restricts to a filtration of $E|_Z$ with successive quotients $(E_i/E_{i-1})|_Z$, $1 \leq i \leq j$, and $\chi((E_i/E_{i-1}))|_Z = 0$ for all $1 \leq i \leq j$. Since each E_i/E_{i-1} is generically semistable, $(E_i/E_{i-1})|_Z$ is a zero slope semistable pure dimension one sheaf on Z. Hence $E|_Z$ is also semistable, which means that E is generically semistable. Finally, Lemma 2.12 implies that E is adiabatically ω-semistable. □

References

[1] J. Alper, *Good moduli spaces for Artin stacks*, `arXiv:0804.2242`.

[2] B. Andreas and D. Hernández Ruipérez, *Fourier Mukai transforms and applications to string theory*, RACSAM. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. **99**(1):29–77, 2005.

[3] B. Andreas, D. H. Ruipérez, and D. S. Gómez, *Stable sheaves over K3 fibrations*, Internat. J. Math. **21**(1):25–46, 2010.

[4] C. Bartocci, U. Bruzzo, and D. Hernández Ruipérez, *A Fourier-Mukai transform for stable bundles on K3 surfaces*, J. Reine Angew. Math. **486**:1–16, 1997.

[5] C. Bartocci, U. Bruzzo, and D. Hernández Ruipérez, *Existence of μ-stable vector bundles on K3 surfaces and the Fourier-Mukai transform*, In Algebraic geometry (Catania, 1993/Barcelona, 1994), volume 200 of Lecture Notes in Pure and Appl. Math., pages 245–257. Dekker, New York, 1998.

[6] C. Bartocci, U. Bruzzo, and D. Hernández Ruipérez, *Fourier-Mukai and Nahm transforms in geometry and mathematical physics*, volume 276 of Progress in Mathematics, Birkhäuser Boston, Inc., Boston, MA, 2009.

[7] C. Bartocci, U. Bruzzo, D. Hernández Ruipérez, and J. M. Muñoz Porras, *Relatively stable bundles over elliptic fibrations*, Math. Nachr. **238**:23–36, 2002.

[8] C. Bartocci, U. Bruzzo, D. H. Ruipérez, and J. M. Muñoz Porras, *Mirror symmetry on $K3$ surfaces via Fourier-Mukai transform*, Comm. Math. Phys. **195**(1):79–93, 1998.

[9] M. Bernardara and G. Hein, *The Euclid-Fourier-Mukai algorithm for elliptic surfaces*, Asian J. Math. **18**(2):345–364, 2014.

[10] T. Bridgeland, *Fourier-Mukai transforms for elliptic surfaces*, J. Reine Angew. Math. **498**:115–133, 1998.

[11] T. Bridgeland, *Fourier-Mukai transforms for surfaces and moduli spaces of stable sheaves*, 1998. PhD thesis available from `www.maths.ed.ac.uk/pg/thesis/bridgeland.pdf`.

[12] T. Bridgeland and A. Maciocia, *Fourier-Mukai transforms for $K3$ and elliptic fibrations*, J. Algebraic Geom. **11**(4):629–657, 2002.

[13] A. Căldăraru, *Derived categories of twisted sheaves on elliptic threefolds*, J. Reine Angew. Math. **544**:161–179, 2002.

[14] A. Căldăraru, *Fiberwise stable bundles on elliptic threefolds with relative Picard number one*, C. R. Math. Acad. Sci. Paris, **334**(6):469–472, 2002.

[15] W.-y. Chuang and J. Lo, *Stability and Fourier-Mukai transforms on higher dimensional elliptic fibrations*, preprint, `arXiv:1307.1845`.

[16] F. Denef and G. W. Moore, *Split states, entropy enigmas, holes and halos*, JHEP 1111:129, 2011.

[17] R. Friedman, *Vector bundles and* SO(3)*-invariants for elliptic surfaces*, J. Amer. Math. Soc. **8**(1):29–139, 1995.

[18] R. Friedman, J. Morgan, and E. Witten, *Vector bundles and* F *theory*, Comm. Math. Phys. **187**(3):679–743, 1997.

[19] R. Friedman, J. W. Morgan, and E. Witten, *Vector bundles over elliptic fibrations*, J. Algebraic Geom. **8**(2):279–401, 1999.

[20] D. Gaiotto, A. Strominger, and X. Yin, *The M5-brane elliptic genus: modularity and BPS states*, J. High Energy Phys. (8):070, 18 pp. (electronic), 2007.

[21] A. Gholampour and A. Sheshmani, *Donaldson-Thomas Invariants of 2-Dimensional sheaves inside threefolds and modular forms*, `arXiv:1309.0050`, 2013.

[22] A. Gholampour and A. Sheshmani, *Generalized Donaldson-Thomas Invariants of 2-Dimensional sheaves on local P^2*, to appear in Adv. Theor. Math. Physics, arXiv:1309.0056, 2013.

[23] A. Grothendieck, *Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. II*, Inst. Hautes Études Sci. Publ. Math. (17):91, 1963.

[24] R. Hartshorne, *Residues and duality*, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20. Springer-Verlag, Berlin, 1966.

[25] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, No. 52, Springer-Verlag, New York, 1977.

[26] S. Hosono, M.-H. Saito, and A. Takahashi, *Relative Lefschetz action and BPS state counting*, Internat. Math. Res. Notices (15):783–816, 2001.

[27] K. Hulek and R. Kloosterman, *Calculating the Mordell-Weil rank of elliptic threefolds and the cohomology of singular hypersurfaces*, Ann. Inst. Fourier (Grenoble) **61**(3):1133–1179, 2011.

[28] D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves*, Aspects of Mathematics, E31. Friedr. Vieweg & Sohn, Braunschweig, 1997.

[29] A. Ishii, *On the moduli of reflexive sheaves on a surface with rational double points*, Math. Ann. **294**(1):125–150, 1992.

[30] M. Jardim and A. Maciocia, *A Fourier-Mukai approach to spectral data for instantons*, J. Reine Angew. Math. **563**:221–235, 2003.

[31] D. Joyce and Y. Song, *A theory of generalized Donaldson-Thomas invariants*, Mem. Amer. Math. Soc. **217**(1020):iv+199, 2012.

[32] S. Katz, *Genus zero Gopakumar-Vafa invariants of contractible curves*, J. Differential Geom. **79**(2):185–195, 2008.

[33] A. Klemm, J. Manschot, and T. Wotschke, *Quantum geometry of elliptic Calabi-Yau manifolds*, Commun. Number Theory Phys. **6**(4):849–917, 2012.

[34] M. Kontsevich and Y. Soibelman, *Stability structures, Donaldson-Thomas invariants and cluster transformations*, arXiv:0811.2435.

[35] J. Lo, *T-structures on elliptic fibrations*, preprint, arXiv:1509.03216, to appear in Kyoto J. Math.

[36] J. Lo, *On some moduli of complexes on K3 surfaces*, arXiv:1203.1558, 2012.

[37] J. Lo, *Stability and Fourier-Mukai transforms on elliptic fibrations*, Adv. Math. **255**:86–118, 2014.

[38] A. Maciocia and C. Meachan, *Rank 1 Bridgeland stable moduli spaces on a principally polarized abelian surface*, Int. Math. Res. Not. IMRN (9):2054–2077, 2013.

[39] D. Maulik and R. Pandharipande, *Gromov-Witten theory and Noether-Lefschetz theory*, In A celebration of algebraic geometry, volume 18 of Clay Math. Proc., pages 469–507, Amer. Math. Soc., Providence, RI, 2013.

[40] J. A. Minahan, D. Nemeschansky, C. Vafa, and N. P. Warner, *E-strings and $N=4$ topological Yang-Mills theories*, Nuclear Phys. B **527**(3):581–623, 1998.

[41] H. Minamide, S. Yanagida, and K. Yoshioka, *Fourier-Mukai transforms and the wall-crossing behavior for Bridgeland's stability conditions*, June 2011.

[42] H. Minamide, S. Yanagida, and K. Yoshioka, *Some moduli spaces of Bridgeland's stability conditions*, Int. Math. Res. Not. IMRN (19):5264–5327, 2014.

[43] S. Mukai, *Semi-homogeneous vector bundles on an Abelian variety*, J. Math. Kyoto Univ. **18**(2):239–272, 1978.

[44] S. Mukai, *Fourier functor and its application to the moduli of bundles on an abelian variety*, In Algebraic geometry, Sendai, 1985, volume 10 of Adv. Stud. Pure Math., pages 515–550. North-Holland, Amsterdam, 1987.

[45] D. Mumford, *Abelian varieties*, Tata Institute of Fundamental Research Studies in Mathematics, No. 5, Published for the Tata Institute of Fundamental Research, Bombay; Oxford University Press, London, 1970.

[46] K. G. O'Grady, *The weight-two Hodge structure of moduli spaces of sheaves on a $K3$ surface*, J. Algebraic Geom. **6**(4):599–644, 1997.

[47] R. Pandharipande and R. P. Thomas, *Curve counting via stable pairs in the derived category*, Invent. Math. **178**(2):407–447, 2009.

[48] S. Rose and N. Yui, *Elliptic Calabi-Yau threefolds over a del Pezzo surface*, preprint, `arXiv:1312.0911`.

[49] D. H. Ruipérez and J. M. Muñoz Porras, *Stable sheaves on elliptic fibrations*, J. Geom. Phys. **43**(2-3):163–183, 2002.

[50] C. T. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety. I*, Inst. Hautes Études Sci. Publ. Math. (79):47–129, 1994.

[51] R. P. Thomas, *A holomorphic Casson invariant for Calabi-Yau 3-folds, and bundles on* $K3$ *fibrations*, J. Differential Geom. **54**(2):367–438, 2000.

[52] Y. Toda, *Stability conditions and curve counting invariants on Calabi-Yau 3-folds*, Kyoto J. Math. **52**(1):1–50, 2012.

[53] Y. Toda, *Generalized Donaldson-Thomas invariants on the local projective plane*, May 2014.

[54] S. Yanagida and K. Yoshioka, *Semi-homogeneous sheaves, Fourier-Mukai transforms and moduli of stable sheaves on abelian surfaces*, J. Reine Angew. Math. **684**:31–86, 2013.

[55] K. Yoshioka, *Some notes on the moduli of stable sheaves on elliptic surfaces*, Nagoya Math. J. **154**:73–102, 1999.

[56] K. Yoshioka, *Moduli spaces of stable sheaves on abelian surfaces*, Math. Ann. **321**(4):817–884, 2001.

[57] K. Yoshioka, *Stability and the Fourier-Mukai transform. I*, Math. Z. **245**(4):657–665, 2003.

[58] K. Yoshioka, *Fourier-Mukai transform on abelian surfaces*, Math. Ann. **345**(3):493–524, 2009.

[59] K. Yoshioka, *Stability and the Fourier-Mukai transform. II*, Compos. Math. **145**(1):112–142, 2009.

[60] K. Yoshioka, *Perverse coherent sheaves and Fourier-Mukai transforms on surfaces, I*, Kyoto J. Math. **53**(2):261–344, 2013.

NHETC Rutgers University
126 Frelinghuysen Rd., Piscataway NJ 08854, USA
E-mail address: duiliu@physics.rutgers.edu

Received October 1, 2015