Communications in
Number Theory and Physics
Volume 10, Number 1, 87–131, 2016

A relative basis for mixed Tate motives over the projective line minus three points

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In a previous work, the author built two families of distinguished algebraic cycles in Bloch-Kriz cubical cycle complex over the projective line minus three points.

The goal of this paper is to show how these cycles induce well-defined elements in the H^0 of the bar construction of the cycle complex and thus generate comodules over this H^0, that is a mixed Tate motives over the projective line minus three points.

In addition, it is shown that out of the two families only one is needed at the bar construction level. As a consequence, the author obtains that one of the family gives a basis of the tannakian Lie coalgebra of mixed Tate motives over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ relatively to the tannakian Lie coalgebra of mixed Tate motives over $\mathrm{Spec}(\mathbb{Q})$. This in turns provides a new formula for Goncharov motivic coproduct, which should really be thought as a coaction. This new presentation is explicitly controlled by the structure coefficients of Ihara's action by special derivation on the free Lie algebra on two generators.

2010 Mathematics Subject Classification: 14C25 (19E15 18D50 05C05).

1. Introduction

1.1. Multiple polylogarithms and mixed Tate motives

For a tuple of integers $(k_1, \dots, k_m)$, the multiple polylogarithm is defined by:

$$Li_{k_1,\dots,k_m}(z) = \sum_{n_1 > \cdots > n_m} \frac{z^{n_1}}{n_1^{k_1} \cdots n_m^{k_m}} \qquad (z \in \mathbb{C},\ |z| < 1).$$

This is one of the one variable version of multiple polylogarithms in many variables defined by Goncharov in [10].

When $k_1 \geqslant 2$, the series converges as z goes to 1 and one recovers the multiple zeta value

$$\zeta(k_1, \dots k_m) = Li_{k_1,\dots,k_m}(1) = \sum_{n_1 > \cdots > n_m} \frac{1}{n_1^{k_1} \cdots n_m^{k_m}}.$$

The case $m = 1$ recovers the classical *polylogarithm* and the value of Riemann zeta function at k_1: $\zeta(k_1)$.

The values of multiple polylogarithms are important in geometric as they naturally appear as periods, in the Hodge or motivic sense, of moduli spaces of curves in genus 0 ([3]); as periods of the fundamental groups of $\mathbb{P}^1$ minus a finite set of points ([5]). In number theory, Zagier's conjecture [28] predicts that regulators of number fields are linear combinations of polylogarithms at special points.

Bloch and Kriz in [1] constructed algebraic avatars of classical polylogarithms. However this was part of a larger work proposing in 1994 a tannakian category $MTM(F)$ of mixed Tate motives over a number field F. Their construction begins with the cubical complex computing higher Chow groups which in the case of $\operatorname{Spec}(F)$ is commutative differential graded algebra $\mathcal{N}_F$. The H^0 of the bar construction $B(\mathcal{N}_F)$ is a Hopf algebra and $MTM(F)$ is defined as

$$MTM(F) = \text{category of comodule over } \mathrm{H}^0(\mathcal{N}_F).$$

Spitzweck in [25] (as presented in [16]) proved that this construction agrees with Voevodsky definition of motives [26] and Levine's approach to mixed Tate motive [15]. More recently, M. Levine generalized this approach in [17] to any quasi projective variety X over the spectrum of a field $\mathbb{K}$ such that

- the motive of X is mixed Tate in Cisinski and Déglise category $DM(\mathbb{K})$,

- the motive of X satisfies Beilinson-Soulé vanishing property.

In order to do so, Levine used the complex $\mathcal{N}_X^{qf,\bullet}$ of quasi-finite cycles over X (Definition 3.11) instead of the original Bloch-Kriz complex. This modification has better functoriality properties and allows a simpler definition of the product structure. Let Spec($\mathbb{Q}$) be the ground field, and Let x be a $\mathbb{Q}$-point of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. $G_{\mathbb{P}^1\setminus\{0,1,\infty\}}$ and $G_{\mathbb{Q}}$ denote the Spectrum of $\mathrm{H}^0(B(\mathcal{N}^{qf,\bullet}_{\mathbb{P}^1\setminus\{0,1,\infty\}}))$ and $\mathrm{H}^0(B(\mathcal{N}^{qf,\bullet}_{\mathbb{Q}}))$ respectively. M. Levine's work shows in particular that for $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$:

Theorem 1.1 ([17][Section 6.6 and Corollary 6.6.2]). *There is a tannakian category of mixed Tate motives over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$:*

$$MTM(\mathbb{P}^1 \setminus \{0, 1, \infty\}) = \textit{category of comodule over } \mathrm{H}^0(\mathcal{N}^{qf,\bullet}_{\mathbb{P}^1\setminus\{0,1,\infty\}}).$$

Moreover there is a split exact sequence:

$$(1) \quad 1 \longrightarrow \pi_1^{mot}(\mathbb{P}^1 \setminus \{0, 1, \infty\}, x) \longrightarrow G_{\mathbb{P}^1\setminus\{0,1,\infty\}} \underset{x^*}{\overset{p^*}{\rightleftarrows}} G_{\mathbb{Q}} \longrightarrow 1$$

where p is the structural morphism $p : \mathbb{P}^1 \setminus \{0, 1, \infty\} \longrightarrow \mathrm{Spec}(\mathbb{Q})$. In the above exact sequence $\pi_1^{mot}(\mathbb{P}^1 \setminus \{0, 1, \infty\}, x)$ denotes Deligne and Goncharov motivic fundamental group [5].

The exact sequence (1) is the motivic avatar of the short exact sequence for etale fundamental groups. M. Levine however did not produce any specific motives. In particular, M. Levine did not produce any specific element in $\mathrm{H}^0(B(\mathcal{N}^{qf,\bullet}_{\mathbb{P}^1\setminus\{0,1,\infty\}}))$; a natural motive being then the comodule cogenerated by such an element.

1.2. Distinguished algebraic cycles over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$

In order to describe explicitly some elements in $\mathrm{H}^0(B(\mathcal{N}^{qf,\bullet}_{\mathbb{P}^1\setminus\{0,1,\infty\}}))$, a first possible step is to produce a family of degree 1 elements in $\mathcal{N}^{qf,\bullet}_{\mathbb{P}^1\setminus\{0,1,\infty\}}$ having a decomposable boundary inside the family. More explicitly, the differential of such an element is a linear combination of products of other elements inside the family.

In [23] the author produces such a family. Together with two explicit degree 1 weight 1 algebraic cycles generating $\mathrm{H}^1(\mathcal{N}^{qf,\bullet}_{\mathbb{P}^1\setminus\{0,1,\infty\}})$, the author obtains:

Theorem. *For any Lyndon word W in the letters $\{0,1\}$ of length $p \geqslant 2$, there exists a cycle $\mathcal{L}_W^0$ in $\mathcal{N}^{qf,1}_{\mathbb{P}^1\setminus\{0,1,\infty\}}(p)$; i.e. a cycle of codimension p in $\mathbb{P}^1 \setminus \{0,1,\infty\} \times \mathbb{A}^{2p-1} \times \mathbb{A}^p$ dominant and quasi-finite over $\mathbb{P}^1 \setminus \{0,1,\infty\} \times \mathbb{A}^{2p-1}$. The cycle $\mathcal{L}_W^0$ satisfies :*

- *$\mathcal{L}_W^0$ has a decomposable boundary,*
- *$\mathcal{L}_W^0$ admits an equidimensional extension to $\mathbb{A}^1$ with empty fiber at 0.*

A similar statement holds for 1 in place of 0.

The above result relies on

- The dual of the action of the free Lie algebra on two generators on itself by Ihara special derivations in order to "guess" the differential of cycles $\mathcal{L}_W^0$.
- The pull-back by the multiplication $\mathbb{A}^1 \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1$ in order to build the cycles $\mathcal{L}_W^0$ from their boundaries.

The free Lie algebra on two generators $\mathrm{Lie}(X_0, X_1)$ is the Lie algebra associated to $\pi_1^{mot}(\mathbb{P}^1 \setminus \{0,1,\infty\}, x)$ from the exact sequence (1) and hence appears naturally in the construction. However its graded dual Q_{geom} is more closely related to $\mathrm{H}^0(B(\mathcal{N}^{qf,\bullet}_{\mathbb{P}^1\setminus\{0,1,\infty\}}))$ and is more natural in our context. It appears in the sequence dual to (1) :

$$0 \longrightarrow Q_{\mathbb{Q}} \longrightarrow Q_{\mathbb{P}^1\setminus\{0,1,\infty\}} \longrightarrow Q_{geom} \longrightarrow 0$$

where $Q_{\mathbb{P}^1\setminus\{0,1,\infty\}}$ and $Q_{\mathbb{Q}}$ denote respectively the set of indecomposable elements of $\mathrm{H}^0(B(\mathcal{N}^{qf,\bullet}_{\mathbb{P}^1\setminus\{0,1,\infty\}}))$ and $\mathrm{H}^0(B(\mathcal{N}^{qf,\bullet}_{\mathrm{Spec}(\mathbb{Q})}))$.

1.3. Main results

In this paper, using the unit of the adjunction between bar and cobar construction in the commutative/coLie case, we lift the above algebraic cycles to elements in $Q_{\mathbb{P}^1\setminus\{0,1,\infty\}}$ viewed as a subspace of $\mathrm{H}^0(B(\mathcal{N}^{qf,\bullet}_{\mathbb{P}^1\setminus\{0,1,\infty\}}))$ by the mean of Hain's projector $p_{\text{Ш}}$ (see [12] or Section 2.3). Let

$$\pi_1 : B(\mathcal{N}^{qf,\bullet}_{\mathbb{P}^1\setminus\{0,1,\infty\}}) \longrightarrow \mathcal{N}^{qf,\bullet}_X$$

be the projection onto the tensor degree 1 part of the bar construction. We obtain:

Theorem (Theorem 3.16). *For any Lyndon word W of length $p \geqslant 2$ there exists an element $\mathcal{L}_W^B$, in the bar construction $B(\mathcal{N}^{qf,\bullet}_{\mathbb{P}^1\setminus\{0,1,\infty\}})$ satisfying:*

- *One has $\pi_1(\mathcal{L}_W^B) = \mathcal{L}_W^0$.*
- *It is in the image of $p_{Ш}$; that is in $Q_{\mathbb{P}^1\setminus\{0,1,\infty\}}$;*
- *It is of degree 0 and map to 0 under the bar differential; it induces a class in $\mathrm{H}^0(B_{\mathbb{P}^1\setminus\{0,1,\infty\}})$ and in $\mathrm{H}^0(Q_{\mathbb{P}^1\setminus\{0,1,\infty\}})$.*
- *Its cobracket in $Q_{\mathbb{P}^1\setminus\{0,1,\infty\}}$ is given by the differential of $\mathcal{L}_W^0$.*

A similar statement holds for cycles $\mathcal{L}_W^1$ and constant cycles $\mathcal{L}_W^0(1)$ induced by the fiber at 1 of $\mathcal{L}_W^0$ (after extension to $\mathbb{A}^1$).

Then we show that the elements $\mathcal{L}_W^B$ and $\mathcal{L}_W^{1,B}$ are related:

Theorem (Theorem 4.5). *For any Lyndon word W of length $p \geqslant 2$ the following relation holds in $\mathrm{H}^0(Q_X) = Q_{\mathrm{H}^0(B_X)}$*

$$\mathcal{L}_W^B - \mathcal{L}_W^{1,B} = \mathcal{L}_W^B(1). \tag{2}$$

The proof relies on the relation between the situation on $\mathbb{P}^1 \setminus \{0,1,\infty\}$ and on $\mathbb{A}^1$ where the cohomology of $\mathcal{N}^{qf,\bullet}_{\mathbb{A}^1}$ is given by constant cycles because of $\mathbb{A}^1$-homotopy invariance of higher Chow groups. As a corollary one obtains a description of the cobracket of $\mathcal{L}_W^B$ in terms of the structure coefficients of Ihara action by special derivation. This makes explicit the relation between the dual of Ihara action (or bracket) and Goncharov motivic coproduct which here, as in Brown [2], is in fact a coaction. In a group setting, Goncharov coproduct corresponds really to the action of $G_{\mathbb{Q}}$ on $\pi_1^{mot}(\mathbb{P}^1 \setminus \{0,1,\infty\}, x)$ induced by the short exact sequence (1).

We conclude the paper by showing at Theorem 4.9 that the family of elements $\mathcal{L}_W^B$ induces a basis of Q_{geom}; that is a basis of $Q_{\mathbb{P}^1\setminus\{0,1,\infty\}}$ relatively to $Q_{\mathbb{Q}}$.

Our methods are structural and geometric by opposition to Gangl Goncharov and Levin approach [7] toward lifting cycles to bar elements using the combinatorics of “rooted polygons”.

The paper is organized as follow

- In the next section, Section 2, we begin by a short review of differential graded (dg) vector spaces. Then we present the bar cobar adjunction in the case of associative algebras and coalgebras and in the case of commutative algebras and Lie coalgebras.

- In Section 3, we present the action of $\mathrm{Lie}(X_0, X_1)$ on itself by Ihara's special derivations and the corresponding Lie coalgebra. From there we recall the results from [23] constructing the cycles $\mathcal{L}_W^{\varepsilon}$. We conclude this section by lifting the cycles to elements in the bar construction.
- In section 4, we prove that bar elements $\mathcal{L}_W^B$ and $\mathcal{L}_W^{1,B}$ are equal up to the constant (over $\mathbb{P}^1 \setminus \{0,1,\infty\}$) bar element $\mathcal{L}_W^B(1)$. From there we make explicit the relation with Ihara's coaction and prove that the elements $\mathcal{L}_W^B$ provide a basis for Q_{geom} graded dual of the Lie algebra associated to $\pi_1^{mot}(\mathbb{P}^1 \setminus \{0,1,\infty\}, x)$.

2. Bar and cobar adjunctions

In this section we recall how the bar/cobar constructions give a pair of adjoint functors in the two following cases:

$$B : \left\{ \begin{array}{c} \text{diff. gr. ass.} \\ \text{algebras} \end{array} \right\} \rightleftharpoons \left\{ \begin{array}{c} \text{diff. gr. coass.} \\ \text{coalgebras} \end{array} \right\} : \Omega$$

and

$$B_{com} : \left\{ \begin{array}{c} \text{diff. gr. com.} \\ \text{ass. algebras} \end{array} \right\} \rightleftharpoons \left\{ \begin{array}{c} \text{diff. gr. coLie.} \\ \text{coalgebras} \end{array} \right\} : \Omega_{coL}.$$

A differential graded commutative algebra A is also an associative algebra and we will recall how the two constructions are related in this case.

The material developed here is well known and can be found in Ginzburg-Kapranov [9] work even if their use of graded duals replaces coalgebra structures by algebra structures. The presentation used here is closer to the Kozul duality as developed by Jones and Getzler in [8]. We follow here the signs conventions and the formalism presented by Loday and Vallette in [18]. The associative case is directly taken from [18, Chap. 2] in a cohomological version. More about the commutative/coLie adjunction can be found in [8, 9, 19, 22].

2.1. Notation and convention

Koszul sign rule. The objects are all objects of the category of (sign) *graded* $\mathbb{Q}$*-vector spaces*. The degree of an homogeneous element v in V is denoted by $|v|$ or $|v|_V$ if we want to emphasis where v is. The symmetric structure is given by the switching map

$$\tau : V \otimes W \longrightarrow W \otimes V\,, \qquad \tau(v \otimes w) = (-1)^{|v||w|} w \otimes v.$$

For any maps $f : V \to V'$ and $g : W \to W'$ of graded spaces, the tensor product

$$f \otimes g : V \otimes W \longrightarrow V' \otimes W'$$

is defined by

$$(f \otimes g)(v \otimes w) = (-1)^{|g||v|} f(v) \otimes g(w).$$

A *differential graded* (dg) vector space is a graded vector space equipped with a *differential* d_V (or simply d) ; that is a degree 1 linear map satisfying $d_V^2 = 0$. For V and W two dg vector spaces the differential on $V \otimes W$ is defined by

$$d_{V\otimes W} = \mathrm{id}_V \otimes d_W + d_V \otimes \mathrm{id}_W .$$

Definition 2.1. Let $V = \oplus_n V^n$ and $W = \oplus_n W^n$ be two graded vector spaces. A *morphism of degree* r, say $f : V \longrightarrow W$, is a collection of morphisms $f_n : V^n \longrightarrow W^{n+r}$. Let $\mathrm{Hom}(V,W)_r$ be the vector space of morphisms of degree r.

Let V and W be two dg vector spaces. Then, the graded vector space $\mathrm{Hom}(V,W) = \oplus \mathrm{Hom}(V,W)_r$ turns into a a dg vector space with differential given by :

$$d_{\mathrm{Hom}}(f) = d_W \circ f - (-1)^r f \circ d_V$$

for any homogeneous element f of degree r. A *dg morphism* $f : V \longrightarrow W$ is a morphism satisfying $d_{\mathrm{Hom}}(f) = 0$.

The dual of a graded vector space $V = \oplus_n V^n$ is defined by

$$V^* = \oplus_n \mathrm{Hom}_{Vect}(V^{-n}, \mathbb{Q}) = \mathrm{Hom}(V, N)$$

where the dg vector space N is defined by $N = \mathbb{Q}$ concentrated in degree 0 with 0 differential. One has an obvious notion of cohomology on dg vector space.

Definition 2.2 ((de)suspension). Let $S = s\mathbb{Q}$ be the 1 dimensional dg vector space concentrated in degree 1 (that is $d_S = 0$) generated by s.

The dual of S is a one dimensional dg vector space denoted by S^{-1} and generated by a degree -1 element s^{-1} dual to s.

Let V, d_V be a dg vector space. Its *suspension* (sV, d_{sV}), is the dg vector space $S \otimes V$. Its desuspension $(s^{-1}V, d_{s^{-1}V})$ is $S^{-1} \otimes V$.

There is a canonical identification $V^{n-1} \simeq (sV)^n$ given by

$$i_s : V \longrightarrow sV \qquad v \longmapsto (-1)^{|v|_V} s \otimes v;$$

under this identification $d_{sV} = -d_V$.

Associative dg algebra. A *differential graded associative* algebra (A, d_A) abbreviated into dga algebra is a dg vector space equipped with a unital associative product μ_A of degree 0 commuting with the differential :

$$d_A \circ \mu_A = \mu_A \circ d_{A \otimes A}$$

and satisfying the usual commutative diagrams for an associative algebra, all the maps involved being maps of dg vector spaces.

The above equality is nothing but Leibniz rule. The unit 1_A belongs to A^0. On elements, one writes $a \cdot_A b$ or simply $a \cdot b$ instead of $\mu_A(a \otimes b)$.

Definition 2.3. The dga A is *connected* if $A^0 = 1_A \mathbb{Q}$.

Definition 2.4. The *tensor algebra* over a dg vector space V is defined by

$$T(V) = \bigoplus_{n \geqslant 0} V^{\otimes n}$$

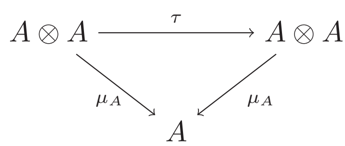
and equipped with the differential induced on each $V^{\otimes n}$ by d_V and with the concatenation product given by

$$[a_1|\cdots|a_n] \otimes [a_{n+1}|\cdots a_{n+m}] \longmapsto [a_1|\cdots|a_n|a_{n+1}|\cdots a_{n+m}]$$

where the "bar" notation $[a_1|\cdots|a_n]$ stands for $a_1 \otimes \cdots \otimes a_n$ in $V^{\otimes n}$.

Note that the degree of $[a_1|\cdots|a_n]$ is $|a_1|_V + \cdots |a_n|_V$ and that $T(V)$ admits a natural augmentation given by $\varepsilon([a_1|\cdots|a_n]) = 0$ for $n > 0$ and the convention $V^{\otimes 0} = \mathbb{Q}$. The concatenation product is associative.

Commutative, symmetric and antisymmetric algebras. A commutative dga algebra (A, d_A, μ_A) or cdga algebra is a dga algebra such that the multiplication commutes with the switching map:



On homogeneous elements, this reads as

$$a \cdot b = (-1)^{|a||b|} b \cdot a.$$

Let V be a dg vector space and $n \geqslant 1$ an integer. The symmetric group $\mathbb{S}_n$ acts on $V^{\otimes n}$ in two natural ways : the symmetric action ρ_S and the antisymmetric action ρ_Λ (both graded).

For i in $\{1, \ldots, n-1\}$, let τ_i be the permutation exchanging i and $i+1$. It is enough to define both actions for the τ_i:

$$\rho_S(\tau_i) = \underbrace{\mathrm{id} \otimes \cdots \mathrm{id}}_{i-1 \text{ factors}} \otimes \tau \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}$$

and

$$\rho_\Lambda(\tau_i) = \underbrace{\mathrm{id} \otimes \cdots \mathrm{id}}_{i-1 \text{ factors}} \otimes (-\tau) \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id} \ .$$

where τ is the usual switching map. Both actions involved signs. The graded signature $\varepsilon^{gr}(\sigma) \in \{\pm 1\}$ of a permutation σ is defined by

$$\rho_S(\sigma)(v_1 \otimes \cdots \otimes v_n) = \varepsilon^{gr}(\sigma) v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}.$$

Then, one has

$$\rho_\Lambda(\sigma)(v_1 \otimes \cdots \otimes v_n) = \varepsilon(\sigma)\varepsilon^{gr}(\sigma) v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$$

where $\varepsilon(\sigma)$ is the usual signature. Let $p_{S,n}$ be the projector defined on $V^{\otimes n}$ by

$$p_{S,n} = \frac{1}{n!} \left(\sum_{\sigma \in \mathbb{S}_n} \rho_S(\sigma) \right).$$

Definition 2.5. The (graded) symmetric algebra $S^{gr}(V)$ over V is defined as the quotient of $T(V)$ by the two side ideal generated by $(\mathrm{id} - \tau)(a \otimes b)$.

One can write

$$S^{gr}(V) = \bigoplus_{n\geqslant 0} S^{gr,n}(V).$$

One also has the isomorphism $S^{gr,n}(V) = p_{S,n}(V^{\otimes n})$. $S^{gr}(V)$ is the free commutative algebra over V. We may write simply $V \odot V$ for $S^{gr,2}(V)$.

Definition 2.6. The (graded) antisymmetric algebra $\Lambda^{gr}(V)$ V is defined as the quotient of $T(V)$ by the two side ideal generated by $(\mathrm{id} + \tau)(a \otimes b)$.

One can write

$$\Lambda^{gr}(V) = \bigoplus_{n\geqslant 0} \Lambda^{gr,n}(V).$$

We may write $V \wedge V$ for $\Lambda^{gr,2}(V)$.

As in the symmetric case, $\Lambda^{gr}(V)$ is also the image of $V^{\otimes n}$ by the projector $p_{\Lambda,n} = 1/(n!)(\sum_\sigma \rho_\Lambda(\sigma))$ and $\Lambda(V)$ is the free antisymmetric algebra over V.

Associative coalgebra. A *differential graded associative* coalgebra (C, d_C) abbreviated into dga coalgebra is a dg vector space equipped with a counital coassociative coproduct Δ_C of degree 0 commuting with the differential :

$$d_{C\otimes C} \circ \Delta_C = \Delta_C \circ d_C$$

and satisfying the usual commutative diagrams for an associative coalgebra, all the maps involved being maps of dg vector spaces.

The iterated coproduct $\Delta^n : C \longrightarrow C^{\otimes(n+1)}$ is

$$\Delta^n = (\Delta \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id})\Delta^{n-1} \qquad \text{and} \qquad \Delta^1 = \Delta.$$

This definition is independent of the place of the Δ factor (here in first position) because of the associativity of the coproduct. We will use Sweedler's notation:

$$\Delta(x) = \sum x_{(1)} \otimes x_{(2)}\,, \qquad (\Delta \otimes \mathrm{id})\Delta(x) = \sum x_{(1)} \otimes x_{(2)} \otimes x_{(3)}$$

and

$$\Delta^n(x) = \sum x_{(1)} \otimes \cdots \otimes x_{(n+1)}.$$

A coaugmentation on C is a morphism of dga coalgebra $u : \mathbb{Q} \to C$. In this case, C is canonically isomorphic to $\ker(\varepsilon) \oplus \mathbb{Q}u(1)$. Let $\bar{C} = \ker(\varepsilon)$ be the kernel of the counit.

When C is coaugmented, the *reduced coproduct* is defined by $\bar{\Delta} = \Delta - 1 \otimes \mathrm{id} - \mathrm{id} \otimes 1$. It is associative and there is an iterated reduced coproduct $\bar{\Delta}^n$ for which we also use Sweedler's notation.

Definition 2.7. *C is conilpotent when it is coaugmented and when, for any x in C, one has $\bar{\Delta}^n(x)$ vanishes for n large enough.*

A *cofree* dga coalgebra over the dg vector space is by definition a *conilpotent* dga coalgebra $F^c(V)$ equipped with a linear map of degree 0 $p : F^c(V) \to V$ commuting with the differential such that $p(1) = 0$. It factors any morphism of dg vector space $\phi : C \longrightarrow V$ where C is a conilpotent dga coalgebra with $\phi(1) = 0$.

Definition 2.8. The *tensor coalgebra* over V is defined by

$$T^c(V) = \bigoplus_{n \geqslant 0} V^{\otimes n}$$

and equipped with the differential induced on each $V^{\otimes n}$ by d_V and with the deconcatenation coproduct given by

$$[a_1|\cdots|a_n] \longmapsto \sum_{i=0}^{n+1} [a_1|\cdots|a_i] \otimes [a_{i+1}|\cdots a_n].$$

The deconcatenation coproduct is associative. The natural projection $\pi_V : T^c(V) \longrightarrow \mathbb{Q} = V^{\otimes 0}$ onto the tensor degree 0 part is a counit for $T^c(V)$ while the inclusion $\mathbb{Q} = V^{\otimes 0} \longrightarrow T^c(V)$ gives the coaugmentation. The tensor coalgebra $T^c(V)$ is the cofree counital dga coalgebra over V.

dg Lie algebra. We review here the definition of Lie algebra and Lie coalgebra in the dg formalism. For any dg vector space V, let ξ be the cyclic permutation of $V \otimes V \otimes V$ defined by

$$\xi = (\mathrm{id} \otimes \tau)(\tau \otimes \mathrm{id}).$$

It corresponds to the cycle sending 1 to 3, 3 to 2 and 2 to 1.

Definition 2.9. A dg Lie algebra L is a dg vector space equipped with a degree 0 map of dg vector spaces $c : L \otimes L \longrightarrow L$ (c stands for "*crochet*") satisfying

$$c \circ \tau = -c \qquad \text{and} \qquad c \circ (c \otimes \mathrm{id}) \circ (\mathrm{id} + \xi + \xi^2) = 0.$$

On elements, we will use a bracket notation $[x, y]$ instead of $c(x \otimes y)$.

In the above definition, the first relation is the usual antisymmetry of the bracket which gives in the dg context:

$$[x, y] = (-1)^{|x||y|}[y, x].$$

The second relation is the Jacobi relation:

$$[[x, y], z] + (-1)^{|x|(|y|+|z|)}[[y, z], x] + (-1)^{|z|(|y|+|x|)}[[z, x], y] = 0.$$

One remarks that $(c \otimes \mathrm{id}) \circ \xi = \tau \circ (\mathrm{id} \otimes c)$ and that $(c \otimes \mathrm{id}) \circ \xi^2 = ((c \circ \tau) \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau)$. Using this and the antisymmetry relation, one can rewrite the Jacobi relation as a Leibniz relation:

$$c \circ (c \otimes \mathrm{id}) = c \circ (\mathrm{id} \otimes c) + c \circ (c \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau).$$

The definition of a dg Lie coalgebra is dual to the definition of a Lie algebra.

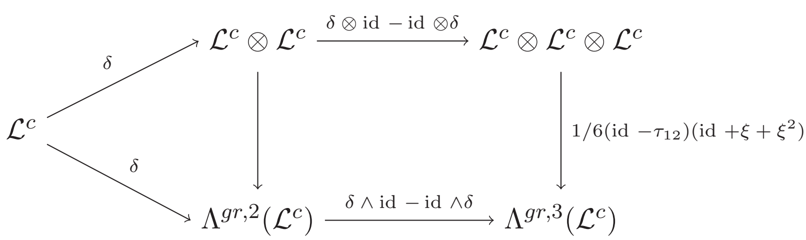
Definition 2.10. A dg Lie coalgebra $\mathcal{L}^c$ is a dg vector space equipped with a degree 0 map of dg vector spaces $\delta : \mathcal{L}^c \longrightarrow \mathcal{L}^c \otimes \mathcal{L}^c$ satisfying

$$\tau \circ \delta = -\delta \qquad \text{and} \qquad (\mathrm{id} + \xi + \xi^2) \circ (\delta \otimes \mathrm{id}) \circ \delta = 0$$

The first condition shows that δ induces a map (again denoted by δ)

$$\delta : \mathcal{L}^c \longrightarrow \mathcal{L}^c \wedge \mathcal{L}^c.$$

Let $\tau_{12} : \mathcal{L}^{c\otimes 3} \longrightarrow \mathcal{L}^{c\otimes 3}$ be the permutation exchanging the two first factors. The second condition shows that the following diagram is commutative



and that the composition going through the bottom line is 0.

2.2. Bar/cobar adjunction: associative case

Bar construction. In this subsection, we recall briefly the bar/cobar construction and how they give a pair of adjoint functor in the associative case.

$$B : \left\{ \begin{array}{c} \text{diff. gr. ass.} \\ \text{algebras} \end{array} \right\} \rightleftharpoons \left\{ \begin{array}{c} \text{diff. gr. coass.} \\ \text{coalgebras} \end{array} \right\} : \Omega.$$

Let $(A, d_A, \mu_A, \varepsilon_A)$ be an augmented dga algebra and $\bar{A} = \ker(\varepsilon_A)$ its augmentation ideal. The bar construction of A is obtained by twisting the differential of the dga free coalgebra $T^c(s^{-1}\bar{A})$.

The differential d_A makes $\bar{A}$ and thus $s^{-1}\bar{A}$ into a dg vector vector space. Let D_1 denote the induced differential on $T^c(s^{-1}\bar{A})$ which in tensor degree n is:

$$\sum_{i=1}^{n} \mathrm{id}^{\,i-1} \otimes d_{s^{-1}\bar{A}} \otimes \mathrm{id}^{\,n-i}.$$

$S^{-1} = s^{-1}\mathbb{Q}$ admits an associative product-like map of degree $+1$ defined by:

$$\Pi_s : s^{-1}\mathbb{Q} \otimes s^{-1}\mathbb{Q} \longrightarrow s^{-1}\mathbb{Q} \qquad \Pi_s(s^{-1} \otimes s^{-1}) = s^{-1}.$$

The map Π_s and the restriction $\mu_{\bar{A}}$ of the multiplication μ_A to $\bar{A}$ induce the following map:

$$\begin{aligned} f : s^{-1}\mathbb{Q} \otimes \bar{A} \otimes s^{-1}\mathbb{Q} \otimes \bar{A} &\xrightarrow{\mathrm{id} \otimes \tau \otimes \mathrm{id}} s^{-1}\mathbb{Q} \otimes s^{-1}\mathbb{Q} \otimes \bar{A} \otimes \bar{A} \\ &\xrightarrow{\Pi_s \otimes \mu_{\bar{A}}} s^{-1}\mathbb{Q} \otimes \bar{A}. \end{aligned}$$

This map induces a degree 1 map $D_2 : T^c(s^{-1}\bar{A}) \longrightarrow T^c(s^{-1}\bar{A})$ which satisfies $D_2^2 = 0$ because of the associativity of μ_A

One checks that the degree 1 morphisms D_1 and D_2 commute (in the graded sense):

$$D_1 \circ D_2 + D_2 \circ D_1 = 0$$

The coproduct on $T^c(s^{-1}\bar{A})$ is given by the deconcatenation coproduct. From these definitions, one obtains (see [18][Section 2.2.1]) the following.

Lemma 2.11. *The complex $B(A) = (T^c(s^{-1}\bar{A}), d_B)$ with $d_B = D_1 + D_2$ and endowed with the deconcatenation coproduct Δ is a conilpotent dga coalgebra.*

We recall below the explicit formulas related to the bar construction $B(A)$:

- An homogeneous element $\mathbf{a}$ of tensor degree n is denoted by
$$[s^{-1}a_1|\cdots|s^{-1}a_n]$$
or when the context is clear enough not to forget the shifting simply by $[a_1|\cdots|a_n]$. Its degree is given by:
$$\deg_B(\mathbf{a}) = \sum_{i=1}^{n} \deg_{s^{-1}\bar{A}}(s^{-1}a_i) = \sum_{i=1}^{n}(\deg_A(a_i) - 1)$$
- the coproduct is given by:
$$\Delta(\mathbf{a}) = \sum_{i=1}^{n}[s^{-1}a_1|\cdots|s^{-1}a_i] \otimes [s^{-1}a_{i+1}|\cdots|s^{-1}a_n].$$
- Let $\eta_{\mathbf{a}}(i)$ or simply $\eta(i)$ denote the "partial degree" of $\mathbf{a}$:
$$\eta_{\mathbf{a}}(i) = \sum_{k=1}^{i} \deg_{s^{-1}\bar{A}}(s^{-1}a_k) = \sum_{k=1}^{i}(\deg_A(a_k) - 1).$$
- The differential D_1 and D_2 are explicitly given by the formulas:
$$D_1(\mathbf{a}) = -\sum_{i=1}^{n}(-1)^{\eta(i-1)}[s^{-1}a_1|\cdots|s^{-1}d_A(a_i)|\cdots|s^{-1}a_n]$$
and
$$D_2(\mathbf{a}) = -\sum_{i=1}^{n}(-1)^{\eta(i)}[s^{-1}a_1|\cdots|s^{-1}\mu_A(a_i, a_{i+1})|\cdots|s^{-1}a_n].$$

The global minus sign in D_1 appears because the differential of the dg vector space $s^{-1}\bar{A}$ is given by $d_{s^{-1}\bar{A}}(s^{-1}a) = -s^{-1}d_A(a)$. The other signs are due to the Kozul sign rules taking care of the shifting.

Remark 2.12. This construction can be seen as a simplicial total complex associated to the complex A (as in [1]). Here, the augmentation makes it possible to use directly $\bar{A}$ without referring to the tensor coalgebra over A and without the need of killing the degeneracies. However the simplicial presentation usually masks the need of working with the shifted complex which is important for sign issues.

The bar construction $B(A)$ also admits a product ш which shuffles the tensor factors. However, this extra structure becomes more interesting when A is graded commutative and it will described at Section 2.3.

The bar construction is a quasi-isomorphism invariant as shown in [18] (Proposition 2.2.4) and the construction provides a functor:

$$B : \{\text{aug. dga algebra}\} \longrightarrow \{\text{coaug. dga coalgebra}\}.$$

Cobar construction. The cobar construction follows similar but dual lines. Let $(C, d_C, \Delta_C, \varepsilon_C)$ be a coaugmented dga coalgebra decomposed as $C = \bar{C} \oplus \mathbb{Q}$. Consider $T(s\bar{C})$ the free algebra over $s\bar{C}$ (with concatenation product). The differential on C induces a differential d_1 on $T(s\bar{C})$. $S = s\mathbb{Q}$ comes with a coproduct-like degree 1 map dual to Π_s:

$$\Delta_s : s\mathbb{Q} \longrightarrow s\mathbb{Q} \otimes s\mathbb{Q}, \qquad \Delta_s(s) = -s \otimes s.$$

The map Δ_s and the restriction of the reduced coproduct $\bar{\Delta}_c$ to $\bar{C}$ induce the following map:

$$g : s\bar{C} \xrightarrow{\Delta_s \otimes \bar{\Delta}_C} s\mathbb{Q} \otimes s\mathbb{Q} \otimes \bar{C} \otimes \bar{C} \xrightarrow{\text{id} \otimes \tau \otimes \text{id}} s\mathbb{Q} \otimes \bar{C} \otimes s\mathbb{Q} \otimes \bar{C}.$$

It induces a degree 1 map d_2 on $T(s\bar{C})$ satisfying $d_2^2 = 0$ because of the coassociativity of Δ_c. The two degree 1 maps d_1 and d_2 commute (in the graded sense):

$$d_1 \circ d_2 + d_2 \circ d_1 = 0.$$

Lemma 2.13. *The complex $\Omega(C) = (T(s\bar{C}), d_\Omega)$ with differential $d_\Omega = d_1 + d_2$ and endowed with the concatenation product is an augmented dga algebra called the cobar construction of C.*

Note that the cobar construction is not in general a quasi-isomorphisms invariant. The reader may look at [18, Section 2.4] for more details.

Adjunction. The two functors bar and cobar induces an adjunction which is described as follows:

Theorem 2.14 ([18, Theorem 2.2.9 and Corollary 2.3.4]). *For every augmented dga algebra A and every conilpotent dga coalgebra C there exists* *a natural bijection*

$$\mathrm{Hom}_{\text{dga alg}}(\Omega(C), A) \simeq \mathrm{Tw}(C, A) \simeq \mathrm{Hom}_{\text{dga coalg}}(C, B(A)).$$

The unit $\upsilon : C \to B \circ \Omega(C)$ and the counit $\epsilon : \Omega \circ B(A) \to A$ are quasi-isomorphisms of dga coalgebras and algebras respectively.

2.3. Bar/cobar adjunction: commutative/coLie case

In this section we recall the bar/cobar adjunction in the commutative dga algebra and dg Lie coalgebra case giving a pair of functors:

$$B_{com} : \left\{ \begin{array}{c} \text{diff. gr. com.} \\ \text{ass. algebras} \end{array} \right\} \rightleftharpoons \left\{ \begin{array}{c} \text{diff. gr. Lie.} \\ \text{coalgebras} \end{array} \right\} : \Omega_{coL}.$$

The cobar construction in the coLie case is a little more delicate. We will concentrate on this construction. The bar construction in the commutative case, will be presented as the set of indecomposable elements of the associative bar construction. A direct construction can be found in [22]. Other descriptions were given in [8, 9].

Cobar construction for Lie coalgebras. The construction follows the lines of the dg associative coalgebra case. However the lack of associativity and the use of the symmetric algebra must be taken into account.

First we need a notion of conilpotency for a Lie coalgebra $(\mathcal{L}^c, \delta, d)$. As δ is not associative, one can not directly use an iterated coproduct. One introduces trivalent trees controlling this lack of associativity.

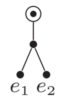
A rooted trivalent tree, or simply a tree, is a planar tree (at each internal vertex a cyclic ordering of the incident edges is given) where vertices have valency 1 (external vertices) or 3 (internal vertices) together with a distinguished external vertex (the root); other external vertices are called leaves. The leaves are numbered from left to right beginning at 1. The trees are drawn with the root (with number 0) at the top.

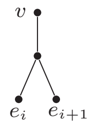
Let $(\mathcal{L}^c, \delta, d)$ be a dg Lie coalgebra. Recall that δ is a dg morphism $\delta : \mathcal{L}^c \longrightarrow \mathcal{L}^c \otimes \mathcal{L}^c$.

Definition 2.15. Let T be a tree with n leaves as above and let $\{e_1, \ldots, e_n\}$ be the set of its leaves (e_i is the i-th leaf). T induces a morphism

$$\delta_T : \mathcal{L}^c \longrightarrow \mathcal{L}^{c \otimes n}$$

as follows:

- if T has $n = 1$ leaf, $\delta_T = \mathrm{id}_{\mathcal{L}^c}$;
- if T has $n = 2$ leaves, then $T =$  and $\delta_T = \delta$;
- if T has $n \geqslant 3$ leaves, then there exists at least one leaf e_i in a strict subtree of the form

  $T_0 =$ 

  where v is an internal vertex of T. Let T' be the tree $T \setminus T_0$, where the subtree T_0 of T has been removed and replace by a leaf which is in position i by construction. The morphism δ_T is defined by

$$\delta_T = (\mathrm{id}^{\otimes(i-1)} \otimes \delta \otimes \mathrm{id}^{\otimes(n-i)}) \circ \delta_{T'}.$$

This definition does not depend on the choice of the subtree T_0. By analogy with the associative case, we define:

Definition 2.16. A Lie coalgebra $(\mathcal{L}^c, \delta, d)$ is *conilpotent* if for any $x \in \mathcal{L}^c$ there exists n big enough such that for any tree T with k leaves, $k \geqslant n$, $\delta_T(x) = 0$.

We now fix a conilpotent dg Lie coalgebra $(\mathcal{L}^c, \delta, d_{\mathcal{L}^c})$. Its cobar construction is given by twisting the differential of the free commutative dga $S^{gr}(s\mathcal{L}^c)$.

The differential $d_{\mathcal{L}^c}$ induces a differential $d_{s\mathcal{L}^c}$ on $s\mathcal{L}^c$ and thus on $(s\mathcal{L}^c)^{\otimes n}$ given by

$$\sum_{i=1}^{n} id^{\otimes(i-1)} \otimes d_{s\mathcal{L}^c} \otimes \mathrm{id}^{\otimes(n-i)} : (s\mathcal{L}^c)^{\otimes n} \longrightarrow (s\mathcal{L}^c)^{\otimes n}.$$

This differential goes down to a differential on n-th symmetric power of $s\mathcal{L}^c$:

$$D_1 : S^{gr,n}(s\mathcal{L}^c) \longrightarrow S^{gr,n}(s\mathcal{L}^c).$$

Using the map Δ_s and the cobracket $\delta_{\mathcal{L}^c}$, one has a morphism:

$$g_L : s\mathcal{L}^c \xrightarrow{\Delta_s \otimes \delta} s\mathbb{Q} \otimes s\mathbb{Q} \otimes \mathcal{L}^c \otimes \mathcal{L}^c \xrightarrow{\mathrm{id} \otimes \tau \otimes id} s\mathbb{Q} \otimes \mathcal{L}^c \otimes s\mathbb{Q} \otimes \mathcal{L}^c$$

which induces a degree 1 map $g_L : s\mathcal{L}^c \longrightarrow S^{2,gr}(s\mathcal{L}^c)$ because of the relation $\tau \circ \delta = -\delta$ and the shift in the degree.

The relation $(\mathrm{id} + \xi + \xi^2) \circ (\delta \otimes \mathrm{id}) \circ \delta = 0$, combined with the shift in the degree and Δ_s, shows that g_L induces a differential D_2 on $S^{gr}(s\mathcal{L}^c)$ given by the formula:

$$D_2|_{S^{gr,n}(s\mathcal{L}^c)} = \sum_{i=1}^{n} \mathrm{id}^{\otimes(i-1)} \otimes g_L \otimes \mathrm{id}^{\otimes(n-i)}.$$

This is the classical duality between Jacobi identity and $D_2^2 = 0$ for a classical Lie coalgebra (that is with a dg structure concentrated in degree 0).

The differential D_1 and D_2 commute (in the graded sens), that is

$$D_1 \circ D_2 + D_2 \circ D_1 = 0$$

and one obtains the following:

Lemma 2.17. *The complex $\Omega_{coL}(\mathcal{L}^c) = (S^{gr}(s\mathcal{L}^c), d_{\Omega,coL})$ with differential $d_{\Omega,coL} = D_1 + D_2$ and endowed with the symmetric concatenation product is an augmented commutative dga algebra called the cobar (coLie) construction of $\mathcal{L}^c$.*

Bar construction for commutative dga algebras. Let $(A, d_A, \mu_A, \varepsilon_A)$ be an augmented commutative dga algebra and $\bar{A} = \ker(\varepsilon_A)$. One can consider its bar construction $B(A)$ as associative algebra. One defines on the coalgebra $B(A)$ an associative product ш by the formula

$$\begin{aligned}[x_1|\cdots|x_n] ш [x_{n+1}|\cdots|x_{n+m}] &= \sum_{\sigma\in \mathrm{sh}(n,m)} \rho_S(\sigma)([x_1|\cdots|x_n|x_{n+1}|\cdots|x_{n+m}]) \\ &= \sum_{\sigma\in \mathrm{sh}(n,m)} \varepsilon^{gr}(\sigma)([x_{\sigma^{-1}(1)}|\cdots|x_{\sigma^{-1}(n+m)}])\end{aligned}$$

where $\mathrm{sh}(n,m)$ denotes the subset $\mathbb{S}_{n+m}$ preserving the order of the ordered sets $\{1,\ldots,n\}$ and $\{n+1,\ldots,n+m\}$. For grading reasons, and thus for signs issues, it is important to note that, in the above formula, the x_i's are elements of $s^{-1}A$.

A direct computation shows that ш turns $B(A)$ into an augmented commutative dga Hopf algebra.

The augmentation of $B(A)$ is the projection onto the tensor degree 0 part. Let $\overline{B(A)}$ be the kernel of the augmentation of $B(A)$ and

$$Q_B(A) = \overline{B(A)}/\left(\left(\overline{B(A)}\right)^2\right.$$

be the set of its indecomposable elements. Ree's theorem [18, Theorem 1.3.9] (originally in [20]) shows the following

Lemma 2.18. *The differential d_B induces a differential d_Q on $Q_B(A)$. The reduced coproduct $\bar{\Delta}$ induces a cobracket $\delta_Q = 1/2(\bar{\Delta} - \tau\bar{\Delta})$ on $Q_B(A)$ making it into a conilpotent Lie dg coalgebra.*

The complex $(Q_B(A), d_Q)$ endowed with the cobracket δ_Q is the commutative bar construction *of A and denoted $B_{com}(A)$.*

Working with the set of indecomposable elements as a quotient may be complicated. In particular, some structure, say for example extra filtrations, may not behave well by taking a quotient. For this purpose, R. Hain, dealing with Hodge structure problems, gave in [12] a splitting

$$i_Q : Q_B(A) \longrightarrow \overline{B(A)}$$

of the projection $p_Q : \overline{B(A)} \longrightarrow Q_B(A)$ commuting with the differential. The projector of $\overline{B(A)}$ given by the composition $p_{ш} = i_Q \circ p_Q$ can be expressed using the following explicit formula given in [12]:

$$p_{ш}([a_1|\cdots|a_n]) = \sum_{i=1}^{n} \frac{(-1)^{i-1}}{i} ш \circ (\bar{\Delta})^{i-1}([a_1|\cdots|a_n])$$

where the associative product ш has been extended to $B(A)^{\otimes n}$ for all $n \geqslant 2$ and where $\bar{\Delta}^{(0)} = \mathrm{id}$.

Adjunction. As in the case of associative algebras and coalgebras, the functors Ω_{coL} and B_{com} are adjoint.

Theorem 2.19. *For any augmented commutative dga algebra A and any conilpotent dg Lie coalgebra $\mathcal{L}^c$ there exists a natural bijection*

$$\mathrm{Hom}_{\text{com dga alg}}(\Omega_{coL}(\mathcal{L}^c), A) \simeq \mathrm{Hom}_{\text{conil dg coLie}}(\mathcal{L}^c, B_{com}(A)).$$

The unit $\upsilon : \mathcal{L}^c \to B_{com} \circ \Omega_{coL}(\mathcal{L}^c)$ and the counit $\epsilon : \Omega_{coL} \circ B_{com}(A) \to A$ are quasi-isomorphisms of conilpotent dg Lie coalgebras and commutative dga algebras respectively.

Note that the following diagram of dg vector spaces is commutative:

$$\begin{array}{ccc}
\mathcal{L}^c & \longrightarrow & \Omega_{coL}(\mathcal{L}^c) = (S^{gr}(s\mathcal{L}^c), d_{\Omega,coL}) \\
& \searrow^{\upsilon} & \uparrow \pi_1 \\
& & B_{com}(\Omega_{coL}(\mathcal{L}^c)) = Q_B\,(S^{gr}(s\mathcal{L}^c), d_{\Omega,coL})
\end{array}$$

where π_1 is the projection onto the tensor degree 1 part restricted to the set of indecomposable elements; that is its restriction to the image of $p_{ш}$.

2.4. An explicit map

We present here an explicit description for the unit $\upsilon : \mathcal{L}^c \longrightarrow B_{com} \circ \Omega_{coL}(\mathcal{L}^c)$ with image in $T^c(S^{gr}(s\mathcal{L}^c))$ using the projector $p_{ш}$ onto the indecomposable elements.

Definition 2.20. Let n be a positive integer.

- The Catalan number $C(n-1)$ gives the number of rooted trivalent trees with n leaves.
- Let T be such a rooted trivalent tree with n leaves. Define $\tilde{\delta}_T$ by
$$\tilde{\delta}_T = \frac{1}{2^n}\delta_T$$
where the map $\delta_T : \mathcal{L}^c \longrightarrow \mathcal{L}^{c\otimes n}$ was introduced at Definition 2.15.
- The morphism $\tilde{\delta}_n$ is up to a normalizing coefficient the sums for all trees T with n leaves of the morphism $\tilde{\delta}_T$:
$$\tilde{\delta}_n = \sum_T \frac{1}{nC(n-1)}\tilde{\delta}_T.$$
where the sum runs through all trivalent trees with n leaves.

Note that using the identification $\mathcal{L}^c \simeq s^{-1}\mathbb{Q} \otimes s\mathbb{Q} \otimes \mathcal{L}^c$ given by $x \mapsto s^{-1} \otimes s \otimes x$, the morphism $\tilde{\delta}_n$ induces a morphism

$$\mathcal{L}^c \longrightarrow \left(s^{-1} \otimes s \otimes \mathcal{L}^c\right)^{\otimes n}$$

again denoted by $\tilde{\delta}_n$.

Lemma 2.21. *The composition*

$$\phi_{\mathcal{L}^c} = p_{Ш} \circ \left(\sum_{n \geqslant 1} \tilde{\delta_n} \right)$$

gives a morphism

$$\phi_{\mathcal{L}^c} : \mathcal{L}^c \longrightarrow B_{com} \circ \Omega_{coL}(\mathcal{L}^c) = Q_B\left(S^{gr}(s\mathcal{L}^c)\right)$$

which is equal to the unit of the adjunction:

$$\phi_{\mathcal{L}^c} = \upsilon.$$

In the above identification $\Omega_{coL}(\mathcal{L}^c) = S^{gr}(s\mathcal{L}^c)$, the differential on $S^{gr}(s\mathcal{L}^c)$ is the bar differential $d_{\Omega,coL}$.

The idea for this formula arises by mimicking the associative case where the unit morphism being a coalgebra morphisms has to be compatible with the iterated reduced coproduct. This formula can be derived as a consequence of the work of Getzler and Jones [8] or of the work of Sinha and Walter [22].

3. Families of bar elements

In [23], the author defined a family of algebraic cycles $\mathcal{L}_W^{\varepsilon}$ indexed by couple (W, ε) where W is a Lyndon word and ε is in $\{0,1\}$.

One of the idea underlying the construction of the cycles was to follow explicitly a 1-minimal model construction described in [6] with the hope to use the relation between 1-minimal model and bar construction in order to obtain explicit motives over $\mathbb{P}^1 \setminus \{0,1,\infty\}$ in the sense of Bloch and Kriz [1].

The construction of the family of cycles provides in fact a differential system for these cycles related to the action of the free Lie algebra $\mathrm{Lie}(X_0, X_1)$ on itself by Ihara's special derivations. In this section we will associate bar elements to the previously defined algebraic cycles using the unit of the bar/cobar adjunction in the commutative algebra/Lie coalgebra case.

Before dealing with the algebraic cycles situation, we need to recall the combinatorial situation from [23] and its relation with Ihara action. This is needed to relate the Lie coalgebra situation (dual to Ihara action) with the differential system for algebraic cycles.

3.1. Lie algebra, special derivations and Lyndon words

We present here the Lyndon brackets basis for the free Lie algebra $\mathrm{Lie}(X_0, X_1)$ and its action on itself by special derivations. The Lie bracket of $\mathrm{Lie}(X_0, X_1)$ is denoted by $[\,,\,]$ as usual.

A Lyndon word in 0 and 1 is a word in 0 and 1 strictly smaller than any of its nonempty proper right factors for the lexicographic order with $0 < 1$ (for more details, see [21]). All words considered in this work are Lyndon words in the two letters 0 and 1.

The standard factorization $[W]$ of a Lyndon word W is defined inductively by $[0] = X_0$, $[1] = X_1$ and otherwise by $[W] = [[U], [V]]$ with $W = UV$, U and V nontrivial and such that V is minimal.

Example 3.1. Lyndon words in letters $0 < 1$ in lexicographic order are up to weight 4:

$$0 < 0001 < 001 < 0011 < 01 < 011 < 0111 < 1$$

Their standard factorization is given in weight 1 and 2 by

$$[0] = X_0, \quad [1] = X_1, \quad [01] = [X_0, X_1],$$

and in weight 3 by

$$[001] = [X_0, [X_0, X_1]] \qquad \text{and} \qquad [011] = [[X_0, X_1], X_1].$$

In weight 4, one has

$$[0001] = [X_0, [X_0, [X_0, X_1]]], \quad \text{and} \quad [0111] = [[[X_0, X_1], X_1], X_1]$$

and

$$[0011] = [X_0, [[X_0, X_1], X_1]]$$

The sets of Lyndon brackets $\{[W]\}$, that is Lyndon words in standard factorization, form a basis of $\mathrm{Lie}(X_0, X_1)$ ([21, Theorem 5.1]). This basis can then be used to write the Lie bracket:

Definition 3.2. For any Lyndon word W, the coefficients $\alpha_{U,V}^W$ (with $U < V$ Lyndon words) are defined by:

$$[[U],[V]] = \sum_{\substack{W \text{ Lyndon} \\ \text{words}}} \alpha_{U,V}^W [W]$$

with $U < V$ Lyndon words. The α's are the structure coefficients of the Lie algebra $\mathrm{Lie}(X_0, X_1)$.

A derivation of $\mathrm{Lie}(X_0, X_1)$ is a linear endomorphism satisfying

$$D([f,g]) = [D(f), g] + [f, D(g)] \qquad \forall f, g \in \mathrm{Lie}(X_0, X_1).$$

Definition 3.3 (Special derivation, [13, 14]). For any f in $\mathrm{Lie}(X_0, X_1)$ we define a derivation D_f by:

$$D_f(X_0) = 0, \qquad D_f(X_1) = [X_1, f].$$

Ihara bracket on $\mathrm{Lie}(X_0, X_1)$ is given by

$$\{f, g\} = [f, g] + D_f(g) - D_g(f).$$

Ihara bracket is simply the bracket of derivations

$$[D_1, D_2]_{Der} = D_1 \circ D_2 - D_2 \circ D_1$$

restricted to special derivations :

$$[D_f, D_g]_{Der} = D_h, \qquad \text{with } h = \{f, g\}.$$

Let L_1 and L_x be two copies of the vector space $\mathrm{Lie}(X_0, X_1)$. The subscript x denotes a formal variable but it can be thought as a point x in $\mathbb{A}^1$. L_x is endowed with the free bracket $[\,,\,]$ of $\mathrm{Lie}[X_0, X_1]$ while L_1 is endowed with Ihara bracket $\{\,,\,\}$. The Lie algebra L_1 acts on L_x by special derivations ; which act on X_1 hence the subscript. If f is an element of $\mathrm{Lie}(X_0, X_1)$, we write $f(1)$ its image in L_1 and $f(x)$ its image in L_x.

Definition 3.4 ([11]). The semi-direct sum $L_{1;x}$ of L_x by L_1 is as a vector space the direct sum

$$L_{1;x} = L_x \oplus L_1$$

with bracket $\{\,,\,\}_{1;x}$ given by $[\,,\,]$ on L_x, by $\{\,,\,\}$ on L_1 and by

$$\{g(1), f(x)\}_{1;x} = -\{f(x), g(1)\}_{1;x} = D_g(f)(x) \qquad \forall f, g \in \mathrm{Lie}(X_0, X_1)$$

on cross-terms.

The union of $\{[W](x), [W](1)\}$ for all Lyndon words gives a basis of $L_{1;x}$ while a basis of $L_{1;x} \wedge L_{1;x}$ is given by the union of the following families

$$\begin{aligned} &[U](x) \wedge [V](x) \text{ for any Lyndon word } U < V \\ &[U](x) \wedge [V](1) \text{ for any Lyndon word } U \neq V \\ &[U](1) \wedge [V](1) \text{ for any Lyndon word } U < V. \end{aligned}$$

Definition 3.5. The structure coefficients $\alpha_{U,V}^W$, $\beta_{U,V}^W$ and $\gamma_{U,V}^W$ of $L_{1;x}$ are given for any Lyndon words W by the family of relations

$$\begin{aligned} \{[U](x), [V](x)\}_{1;x} &= \sum_{W \in Lyn} \alpha_{U,V}^W [W](x) \quad \text{for any Lyndon word } U < V \\ \{[U](x), [V](1)\}_{1;x} &= \sum_{W \in Lyn} \beta_{U,V}^W [W](x) \quad \text{for any Lyndon word } U \neq V \\ \{[U](1), [V](1)\}_{1;x} &= \sum_{W \in Lyn} \gamma_{U,V}^W [W](1) \quad \text{for any Lyndon word } U < V. \end{aligned} \tag{3}$$

All coefficients above are integers.

Because $\{\,,\,\}_{1;x}$ restricted to L_x is the usual bracket on $\mathrm{Lie}(X_0, X_1)$, the $\alpha_{U,V}^W$ are the α's of Definition 3.2 ; Similarly the γ's are the structure coefficients of Ihara's bracket.

Special derivations acts on X_1 and D_{X_0} is simply bracketing with X_0. This and the above remark show:

Lemma 3.6 ([23, Lemma 4.18]). *Let W be a Lyndon word of length greater than or equal to 2. Then the following holds for any Lyndon words U, V :*

- $\beta_{0,V}^W = 0,$
- $\beta_{V,0}^W = \alpha_{0,V}^W$

- $\beta_{U,1} = 0$,
- $\beta^W_{1,U} = \alpha^W_{U,1}$.
- $\gamma^W_{U,V} = \alpha^W_{U,V} + \beta^W_{U,V} - \beta^W_{V,U}$.

In particular, $\beta^W_{0,0} = \beta^W_{1,1} = 0$. *We also have for* $W = 0$ *and* $W = 1$*:*

$$\alpha^W_{U,V} = \beta^W_{U,V} = \gamma^W_{U,V} = 0.$$

3.2. The dual setting : a coaction and a Lie coalgebra

The Lie algebra $Lie(X_0, X_1)$ is graded by the number of letters appearing inside a bracket. Hence there is an induced grading on $L_{1;x}$. Taking the graded dual of the $L_{1;x}$, we obtain a Lie coalgebra $\mathcal{T}^{coL}_{1;x}$.

Definition 3.7. The elements of the dual basis of the Lyndon bracket basis $[W](x)$ of L_x are denoted by $T_{W^*}(x)$. Similarly, $T_{W^*}(1)$ denotes, for a Lyndon word W the corresponding element in the basis dual to the basis of L_1 given by the $[W](1)$'s.

For a in $\{1, x\}$, the elements $T_{W^*}(a)$ can be represented by linear combinations of rooted trivalent trees with leaves decorated by 0 and 1 and root decorated by a (cf. [23, Section 4.3]). This remark explains the notation "T" which stands for "trees" (cf. [23]).

A basis of $\mathcal{T}^{coL}_{1;x} \wedge \mathcal{T}^{coL}_{1;x}$ is given by the union of the following families:

$$\begin{array}{l} T_{U^*}(x) \wedge T_{V^*}(x) \text{ for any Lyndon word } U < V \\ T_{U^*}(x) \wedge T_{V^*}(1) \text{ for any Lyndon word } U \neq V \\ T_{U^*}(1) \wedge T_{V^*}(1) \text{ for any Lyndon word } U < V. \end{array}$$

By duality between $L_{1;x}$ and $\mathcal{T}^{coL}_{1;x}$ one has the following:

Proposition 3.8. *The bracket* $\{,\}_{1;x}$ *on* $L_{1,x}$ *induces a cobracket on* $\mathcal{T}^{coL}_{1;x}$

$$\delta_{1;x} : \mathcal{T}^{coL}_{1;x} \longrightarrow \mathcal{T}^{coL}_{1;x} \wedge \mathcal{T}^{coL}_{1;x}.$$

In terms of the above basis one gets

$$\text{(ED-T)}\qquad \delta_{1;x}(T_{W^*}(x)) = \sum_{U<V} \alpha^W_{U,V} T_{U^*}(x) \wedge T_{V^*}(x) + \sum_{U,V} \beta^W_{U,V} T_{U^*}(x) \wedge T_{V^*}(1)$$

and

$$(4)\qquad \delta_{1;x}(T_{W^*}(1)) = \sum_{U<V} \gamma^W_{U,V} T_{U^*}(1) \wedge T_{V^*}(1)$$

where U and V are Lyndon words. The coefficients $\alpha^W_{U,V}$, $\beta^W_{U,V}$ and $\gamma^W_{U,V}$ are those defined in Equation (3)*.*

Note that one has

$$\delta_{1;x}(T_{0^*}(x)) = \delta_{1;x}(T_{0^*}(1)) = 0 \qquad \text{and} \qquad \delta_{1;x}(T_{1^*}(x)) = \delta_{1;x}(T_{1^*}(1)) = 0.$$

In weight 2 one has

$$\delta_{1;x}(T_{01}(x)) = T_{0^*}(x) \wedge T_{1^*}(x) + T_{1^*}(x) \wedge T_{0^*}(1).$$

Because of geometric constraints, one cannot use directly the combinatorics of the cobracket $\delta_{1;x}$ in this basis to define a family of algebraic cycles over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. One defines for any Lyndon word W

$$T^0_{W^*} = T_{W^*}(x) \qquad \text{and} \qquad T^1_{W^*} = T_{W^*}(x) - T_{W^*}(1).$$

The definition of $T^0_{W^*}$ can be thought as the difference $T^0_{W^*} = T_{W^*}(x) - T_{W^*}(0)$ where the element $T_{W^*}(0)$ is equal to 0. The elements $T^0_{W^*}$ and $T^1_{W^*}$ form a basis of $\mathcal{T}^{coL}_{1;x}$ when W runs through the set of Lyndon words.

Lemma 3.9 ([23, Lemma 4.32]). *In this basis the cobracket $\delta_{1;x}$ is given by*

$$\text{(ED-}T^0\text{)}\qquad \delta_{1;x}(T^0_{W^*}) = \sum_{U<V} a^W_{U,V} T^0_{U^*} \wedge T^0_{V^*} + \sum_{U,V} b^W_{U,V} T^1_{U^*} \wedge T^0_{V^*},$$

and

$$\text{(ED-}T^1\text{)}\qquad \delta_{1;x}(T^1_{W^*}) = \sum_{U<V} {a'}^W_{U,V} T^1_{U^*} \wedge T^1_{V^*} + \sum_{U,V} {b'}^W_{U,V} T^1_{U^*} \wedge T^0_{V^*}$$

where the coefficients a's, b's a''s and b''s are given by

$$
\begin{aligned}
&a^W_{U,V} = \alpha^W_{U,V} + \beta^W_{U,V} - \beta^W_{V,U} && \text{for } U < V \\
&b^W_{U,V} = \beta^W_{V,U} && \text{for any } U, V
\end{aligned}
\tag{5}
$$

and

$$
\begin{aligned}
&{a'}^W_{U,V} = -a^W_{U,V} && \text{for } U < V, \\
&{b'}^W_{U,V} = a^W_{U,V} + b^W_{U,V} && \text{for } U < V, \\
&{b'}^W_{V,U} = -a^W_{U,V} + b^W_{V,U} && \text{for } U < V, \\
&{b'}^W_{U,U} = b^W_{U,U} && \text{for any } U.
\end{aligned}
\tag{6}
$$

$>$From the explicit description of the coaction, Lemma 3.6 gives explicitly some of the coefficients α's and β's. This translates as

Lemma 3.10 ([23, Lemma 4.33]).

- *If W is the Lyndon word 0 or 1, then :*

$$a^0_{U,V} = b^0_{U,V} = {a'}^0_{U,V} = {b'}^0_{U,V} = 0, \quad a^1_{U,V} = b^1_{U,V} = {a'}^1_{U,V} = {b'}^1_{U,V} = 0$$

for any Lyndon words U and V.

- *For any Lyndon word W, U and V of length at least 2, one has*

$$a^W_{0,V} = {a'}^W_{0,V} = 0, \quad {b'}^W_{U,0} = {b'}^W_{U,0} = 0$$

and

$$a^W_{U,1} = {a'}^W_{U,1} = 0, \quad b^W_{1,V} = {b'}^W_{1,V} = 0.$$

Moreover for W a Lyndon word,

$$a^W_{U,V} = b^W_{U,V} = {a'}^W_{U,V} = {b'}^W_{U,V} = 0$$

as soon as the length of U plus the length of V is not equal to the length of W.

>From the definition of $a_{U,V}^{W}$ and Lemma 3.6 one sees that $a_{U,V}^{W} = \gamma_{U,V}^{W}$. This and Equations (ED-T^0) and (ED-T^1) shows that

$$\delta_{1;x}(T_{W^*}(1)) = \delta_{1;x}(T^0_{W^*} - T^1_{W^*}) = \sum_{U<V} a_{U,V}^{W}(T^0_{U^*} - T^1_{U^*}) \wedge (T^0_{V^*} - T^1_{V^*}) \tag{7}$$

$$= \sum_{U<V} a_{U,V}^{W}(T^0_{U^*}(1)) \wedge (T^0_{V^*}(1)) \tag{8}$$

3.3. A differential system for cycles

In this subsection we review the cubical complex of quasi-finite cycles over X computing higher Chow groups of X. This complex has a natural cdga structure. M. Levine, in [17], proved that the H^0 of its bar construction is the tannakian Hopf algebra of mixed Tate motive over X.

The ground field is Spec($\mathbb{Q}$). The projective line minus three points $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ is simply denoted by X. A generic smooth quasi-projective variety will be denoted by Y.

We define $\square^1$ to be $\square^1 = \mathbb{P}^1 \setminus \{1\}$ and $\square^n$ to be $(\square^1)^n$. The standard projective coordinates on $\square^n$ is $[U_i : V_i]$ on the i-th factor; and $u_i = U_i/V_i$ is the corresponding affine coordinate. A face F of codimension p of $\square^n$ is given by $u_{i_k} = \varepsilon_k$ for $k = 1, \ldots, p$ and ε_k in $\{0, \infty\}$. Such a face is isomorphic to $\square^{n-p}$. For $\varepsilon = 0, \infty$ and i in $\{1, \ldots, n\}$, let s_i^{ε} denote the insertion morphism of a codimension 1 face

$$s_i^{\varepsilon} : \square^{n-1} \longrightarrow \square^n$$

given by the identification

$$\square^{n-1} \simeq \square^{i-1} \times \{\varepsilon\} \times \square^{n-i}.$$

Definition 3.11 ([17, Example 4.1.6]). Let Y be an irreducible smooth variety.

- Let $\mathcal{Z}^p_{q.f.}(Y, n)$ denote the free abelian group generated by irreducible closed subvarieties

  $$Z \subset Y \times \square^n \times (\mathbb{P}^1 \setminus \{1\})^p$$

  such that the restriction of the projection on $Y \times \square^n$,

  $$p_1 : Z \longrightarrow Y \times \square^n,$$

  is dominant and quasi finite (that is of pure relative dimension 0).

- We say that elements of $\mathcal{Z}^p_{q.f.}(Y,n)$ are *quasi-finite*.

Intersection with codimension 1 faces give morphisms

$$\partial_i^\varepsilon = (s_i^\varepsilon)^* : \mathcal{Z}^p_{q.f.}(Y,n) \longrightarrow \mathcal{Z}^p_{q.f.}(Y,n-1).$$

The symmetric group $\mathfrak{S}_p$ acts on $\mathcal{Z}^p_{q.f.}(Y,n)$ by permutation of the factors of $(\mathbb{P}^1 \setminus \{1\})^p$. Let $Sym^p_{\mathbb{P}^1\setminus\{1\}}$ denotes the projector corresponding to the *symmetric* representation.

The symmetric group $\mathfrak{S}_n$ acts on $\mathcal{Z}^p_{q.f.}(Y,n)$ by permutation of the factor $\square^1$, and $(\mathbb{Z}/2\mathbb{Z})^n$ acts on $\mathcal{Z}^p_{q.f.}(Y,n)$ by $u_i \mapsto 1/u_i$ on the $\square^1$. The sign representation of $\mathfrak{S}_n$ extends to a sign representation

$$G_n = (\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathfrak{S}_n \longrightarrow \{1,-1\}.$$

Let $\mathrm{Alt}_n \in \mathbb{Q}[G_n]$ be the corresponding projector.

Definition 3.12. Let $\mathcal{N}_Y^{qf,\,k}(p)$ denote

$$\mathcal{N}_Y^{qf,\,k}(p) = Sym^p_{\mathbb{P}^1\setminus\{1\}} \circ \mathrm{Alt}_{2p-k}\left(\mathcal{Z}^p_{q.f.}(Y,2p-k)\otimes\mathbb{Q}\right).$$

- The intersection with codimension 1 faces of $\square^{2p-k}$ induces a differential

$$\partial_Y = \sum_{i=1}^{2p-k}(-1)^{i-1}(\partial_i^0 - \partial_i^\infty)$$

  of degree 1.

- The complex of quasi finite cycles is defined by

$$\mathcal{N}_Y^{qf,\bullet} = \mathbb{Q} \oplus \bigoplus_{p\geqslant 1}\mathcal{N}_Y^{qf,\bullet}(p).$$

Concatenation of factors $\square^n$ and of factors $\mathbb{P}^1 \setminus \{1\}$ followed by the pull-back by the diagonal $\Delta_Y : Y \to Y \times Y$ induces a product structure to $\mathcal{N}_Y^{qf,\bullet}$. This product is graded commutative and $\mathcal{N}_Y^{qf,\bullet}$ is a cdga ([17, Section 4.2]).

Thanks to [27, Chapter IV and VI], the cohomology of $\mathcal{N}_Y^{qf,\bullet}$ agrees with higher Chow groups of Y tensored with $\mathbb{Q}$ (one can also see [17, Lemma 4.2.1]).

In [23], the author defined two weight 1 degree 1 cycles $\mathcal{L}_0^1$ and $\mathcal{L}_1^0$ in $\mathcal{N}_X^{qf,1}$ as

$$\mathcal{L}_0^1 = Sym^1_{\mathbb{P}^1\setminus\{1\}} \circ \mathrm{Alt}_1(Z_0) \quad \text{and } \mathcal{L}_1^0 = Sym^1_{\mathbb{P}^1\setminus\{1\}} \circ \mathrm{Alt}_1(Z_1)$$

where Z_0 and Z_1 are the irreducible varieties defined respectively by:

$$Z_0 \subset X \times \square^1 \times (\mathbb{P}^1 \setminus \{1\}) : (U-V)(A-B)(U-xV) + x(1-x)UVB = 0$$

and

$$Z_1 : (U-V)(A-B)(U-(1-x)V) + x(1-x)UVB = 0$$

where x denotes the standard affine coordinate on $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, $[U : V]$ the projective coordinates on $\square^1$ and $[A : B]$ that on $\mathbb{P}^1 \setminus \{1\}$.

Starting with these two cycles, the author built in [23] two families of degree 1 elements in $\mathcal{N}_X^{qf,\bullet}$ whose differential are given by the cobracket in $\mathcal{T}_{1;x}^{coL}$.

Let j be the inclusion $\mathbb{P}^1 \setminus \{0, 1, \infty\} = X \hookrightarrow \mathbb{A}^1$. The differential on $\mathcal{N}_X^{qf,\bullet}$ is simply denoted by ∂.

Theorem 3.13 ([23]). *For any Lyndon word of length $p \geqslant 2$, there exist two cycles $\mathcal{L}_W^0$ and $\mathcal{L}_W^1$ in $\mathcal{N}_X^{qf,1}(p)$ such that:*

- *There exist cycles $\overline{\mathcal{L}_W^0}$, $\overline{\mathcal{L}_W^1}$ in $\mathcal{N}_{\mathbb{A}^1}^{qf,1}(p)$ such that*

$$\mathcal{L}_W^0 = j^*(\overline{\mathcal{L}_W^0}) \qquad \textit{and} \qquad \mathcal{L}_W^1 = j^*(\overline{\mathcal{L}_W^1}).$$

- *The restriction of $\overline{\mathcal{L}_W^0}$ (resp. $\overline{\mathcal{L}^1}$) to the fiber $t = 0$ (resp. $t = 1$) is empty.*
- *The cycle $\mathcal{L}_W^0$ and $\mathcal{L}_W^1$ satisfy the following differential equations in $\mathcal{N}_X^{qf,\bullet}$:*

$$\text{(ED-}\mathcal{L}^0\text{)} \qquad \partial(\mathcal{L}_W^0) = -\left(\sum_{U<V} a_{U,V}^W \mathcal{L}_U^0 \mathcal{L}_V^0 + \sum_{U,V} b_{U,V}^W \mathcal{L}_U^1 \mathcal{L}_V^0 \right)$$

and

$$\text{(ED-}\mathcal{L}^1\text{)} \qquad \partial(\mathcal{L}_W^1) = -\left(\sum_{U<V} a'^W_{U,V} \mathcal{L}_U^1 \mathcal{L}_V^1 + \sum_{U,V} b'^W_{U,V} \mathcal{L}_U^1 \mathcal{L}_V^0 \right)$$

where coefficients a's, b's, a′'s and b′'s are the ones of equations (ED-T^0) *and Lemma 3.9.*

One will write generically a cycle in the above families as $\mathcal{L}_W^\varepsilon$ with ε in $\{0,1\}$ when working over $X = \mathbb{P}^1 \setminus \{0,1,\infty\}$ and $\overline{\mathcal{L}_W^\varepsilon}$ when working over $\mathbb{A}^1$.

The above equations differ from the cobracket in $\mathcal{T}_{1;x}^{coL}$ given at equations (ED-T^0) and (ED-T^1) by a global minus sign. This is due to a shift in the degree. Hence the above cycles $\mathcal{L}_W^\varepsilon$ differ from the ones defined in [23] by a global minus sign.

Remark 3.14. The extension $\overline{\mathcal{L}_W^\varepsilon}$ of the cycle to $\mathcal{N}_{\mathbb{A}^1}^{qf,\bullet}$ satisfies the same differential equations as $\mathcal{L}_W^\varepsilon$ by considering the Zariski closure of the product in the R.H.S. of (ED-$\mathcal{L}^0$) and (ED-$\mathcal{L}^1$). However, this Zariski closure is *not* decomposable: terms of the form $\overline{\mathcal{L}_0^1\mathcal{L}_V^0}$ are not products in $\mathcal{N}_{\mathbb{A}^1}^{qf,\bullet}$ because $\overline{\mathcal{L}_0^1}$ is not in $\mathcal{N}_{\mathbb{A}^1}^{qf,1}$; it is not quasi-finite over 0 (cf proof of Proposition 6.3 in [23]).

Despite the above remark, Theorem 3.13 and the proof of Theorem 5.8 in [23] give two other but related families of cycles with decomposable boundary in $\mathcal{N}_{\mathbb{A}^1}^{qf,1}$. They are described below.

Let W be a Lyndon word of length greater than 2. We define $\overline{\mathcal{L}_W^{0-1}}$ to be the difference

$$\overline{\mathcal{L}_W^{0-1}} = \overline{\mathcal{L}_W^0} - \overline{\mathcal{L}_W^1}.$$

The geometric situation relates $\mathbb{P}^1 \setminus \{0,1,\infty\}$, $\mathbb{A}^1$ and the point $\{1\}$ as follows:

$$\begin{array}{ccc} X = \mathbb{P}^1 \setminus \{0,1,\infty\} & \xrightarrow{\quad j \quad} & \mathbb{A}^1 \\ & & p_1 \big\downarrow \big\uparrow i_1 \\ & & \{1\} \end{array}$$

where j is the open inclusion, p_1 is the projection onto $\{1\}$ and i_1 the closed inclusion (or the 1-section). We define the constant cycle $\overline{\mathcal{L}_W^0(1)}$ as

$$\overline{\mathcal{L}_W^0(1)} = p_1^* \circ i_1^*(\overline{\mathcal{L}_W^0}) = p_1^*(\overline{\mathcal{L}_W^0}|_{x=1})$$

where $\overline{\mathcal{L}_W^0}|_{x=1}$ denotes the fiber at 1 of the cycle $\overline{\mathcal{L}_W^0}$. Its restriction to X is denoted by $\mathcal{L}_W^0(1)$.

Lemma 3.15. *For any Lyndon word of length $p \geqslant 2$ the cycle $\overline{\mathcal{L}_W^{0-1}}$ satisfies*

$$\text{(ED-}\overline{\mathcal{L}^{0-1}}\text{)} \quad \partial\left(\overline{\mathcal{L}_W^0} - \overline{\mathcal{L}_W^1}\right) = -\left(\sum_{0<U<V<1} a_{U,V}^W \left(\overline{\mathcal{L}_U^0} - \overline{\mathcal{L}_U^1}\right)\left(\overline{\mathcal{L}_V^0} - \overline{\mathcal{L}_V^1}\right)\right).$$

The differential of $\overline{\mathcal{L}_W^0(1)}$ is given by

$$\partial\left(\overline{\mathcal{L}_W^0(1)}\right) = -\left(\sum_{0<U<V<1} a_{U,V}^W \overline{\mathcal{L}_U^0(1)}\,\overline{\mathcal{L}_V^0(1)}\right) \tag{9}$$

The above equation also holds for $\mathcal{L}_W^0(1)$ and $i_1^*(\overline{\mathcal{L}_W^0})$.

Proof. The combinatoric being the same as in $\mathcal{T}_{1;x}^{coL}$, the first part follows from Equation (7). The second part is a consequence of Lemma 3.10 because products of the form $\overline{\mathcal{L}_U^1 \mathcal{L}_V^0}$ have an empty fiber at 1. □

The rest of this section shows that each cycle attached to one of the above family of "differential system" gives rise to an element in the corresponding bar constructions.

Let B_X, $B_{\mathbb{A}^1}$ and $B_{\{1\}}$ denote the bar construction over $\mathcal{N}_X^{qf,\bullet}$, $\mathcal{N}_{\mathbb{A}^1}^{qf,\bullet}$ and $\mathcal{N}_{\mathrm{Spec}(\mathbb{Q})}^{qf,\bullet}$ respectively. Let Q_X, $Q_{\mathbb{A}^1}$ and $Q_{\{1\}}$ be the corresponding set of indecomposable elements. By an abuse of notation, we will write d_B, Δ, ш and δ_Q the natural operation in the corresponding spaces. When required by the context, we will precise the "base" space by the subscript X, $\mathbb{A}^1$ and $\{1\}$ respectively.

Note that the geometric relation between $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, $\mathbb{A}^1$ and $\{1\}$ gives rise to morphisms of cdga between the corresponding cycles algebras:

$$\begin{array}{ccc} \mathcal{N}_X^{qf,\bullet} & \text{———} & \mathcal{N}_{\mathbb{A}^1}^{qf,\bullet} \\ & & p_1^* \uparrow \;\; \downarrow i_1^* \\ & & \mathcal{N}_{\mathrm{Spec}(\mathbb{Q})}^{qf,\bullet} \end{array}$$

which then induce morphisms between bar construction and sets of indecomposable elements. These morphisms are also denoted j^*, p_1^* and i_1^*.

Theorem 3.16 (bar elements). *For any Lyndon word W of length p there exists an element $\mathcal{L}_W^B$, in the bar construction B_X satisfying:*

- *Its image under the projection onto the tensor degree* 1 *part* $\pi_1 : B_X \longrightarrow \mathcal{N}_X^{qf,\bullet}$ *is* $\pi_1(\mathcal{L}_W^B) = \mathcal{L}_W^0$.
- *It is in the image of the projector* $p_{\text{ш}}$*; hence it is in* Q_X.
- *It is of bar degree* 0 *and its image under* d_B *is* 0*. Thus it induces a class in* $\mathrm{H}^0(B_X)$ *and in* $\mathrm{H}^0(Q_X) = Q_{\mathrm{H}^0(B_X)}$.
- *Its image under* δ_Q *is given by the differential equations* (ED-$\mathcal{L}^0$) *without the minus sign*

$$\delta_Q(\mathcal{L}_W^0) = \sum_{U<V} a_{U,V}^W \mathcal{L}_U^0 \mathcal{L}_V^0 + \sum_{U,V} b_{U,V}^W \mathcal{L}_U^1 \mathcal{L}_V^0.$$

A similar statement holds for $\mathcal{L}_W^1$, $\overline{\mathcal{L}_W^{0-1}}$ *and* $\mathcal{L}_W^0(1)$ *with Equation* (ED-$\mathcal{L}^0$) *replaced by equation* (ED-$\mathcal{L}^1$), (ED-$\overline{\mathcal{L}^{0-1}}$) *and* (9) *respectively.*

Proof. The main point is the relation between $\mathcal{T}_{1;x}^{coL}$ and the above family of algebraic cycles and to use the unit of the adjunction cobar/bar.

As in Section 2.3, $\Omega_{coL}(\mathcal{T}_{1;x}^{coL}) = S^{gr}(s\mathcal{T}_{1;x}^{coL})$ denotes the cobar construction over the Lie coalgebra $\mathcal{T}_{1;x}^{coL}$ concentrated purely in degree 0 (hence with 0 as differential). Let

$$\psi : \Omega_{coL}(\mathcal{T}_{1;x}^{coL}) \longrightarrow \mathcal{N}_X^{qf,\bullet}$$

be the morphism of cdga induced by

$$sT_{W^*}^0 \longmapsto \mathcal{L}_W^0, \qquad sT_{W^*}^1 \longmapsto \mathcal{L}_W^1$$

for any Lyndon word W of length $p \geqslant 2$ together with

$$\psi(sT_{0^*}^1) = \mathcal{L}_0^1, \quad \psi(sT_{1^*}^0) = \mathcal{L}_1^0 \text{ and } \psi(sT_{0^*}^0) = \psi(sT_{1^*}^1) = 0$$

where the prefix s denotes the suspension.

The morphism ψ is compatible with the differential because of equations (ED-$\mathcal{L}^0$) and(ED-$\mathcal{L}^1$) and Lemma 3.10. The overall minus sign difference between equations (ED-$\mathcal{L}^0$) and (ED-T^0) (and similarly for Equation (ED-$\mathcal{L}^1$)) makes it possible to define ψ without sign (cf Section 2.3).

It induces a morphism on the bar construction (for the associative case)

$$\psi_B : B(\Omega_{coL}(\mathcal{T}_{1;x}^{coL})) \longrightarrow B_X = B(\mathcal{N}_X^{qf,\bullet})$$

compatible with projection on tensor degree n part (for any n) and with the projector $p_{\text{ш}}$ onto the indecomposable elements.

Hence we obtain the following commutative diagram (of vector space)

$$\begin{array}{ccccc}
 & & \Omega_{coL}(\mathcal{T}^{coL}_{1;x}) & \xrightarrow{\psi} & \mathcal{N}_X^{qf,\bullet} \\
 & \nearrow & \uparrow{\scriptstyle \pi_1} & & \uparrow{\scriptstyle \pi_1} \\
 & & B(\Omega_{coL}(\mathcal{T}^{coL}_{1;x})) & \xrightarrow{\psi_B} & B_X \\
 & & \uparrow & & \uparrow \\
\mathcal{T}^{coL}_{1;x} & \xrightarrow{\upsilon} & B_{com}(\Omega_{coL}(\mathcal{L}^c)) & \xrightarrow{\psi_Q} & Q_X
\end{array}$$

where the morphisms in the bottom line are morphisms of dg Lie algebras and where υ is the unit of the bar/cobar adjunction.

The bar element $\mathcal{L}^B_W$ is then defined by

$$\mathcal{L}^B_W = \psi_Q \circ \upsilon(T^0_{W^*}).$$

Similarly we define $\mathcal{L}^{1,B}_W$ and $\overline{\mathcal{L}^{0-1,B}_W}$.

In order to define $\mathcal{L}^B_W(1)$, one considers only the sub Lie coalgebra $\mathcal{T}^{coL}_1$ of $\mathcal{T}^{coL}_{1;x}$ and the morphism

$$\psi : sT_{W^*}(1) \longmapsto \mathcal{L}^0_W(1)$$

when W has length $p \geqslant 2$ and sending $sT_{0^*}(1)$ and $sT_{1^*}(1)$ to zero. □

Over $\mathbb{A}^1$ a similar statement holds:

Proposition 3.17. *For any Lyndon word W of length $p \geqslant 2$, there exists an element $\overline{\mathcal{L}^{0-1,B}_W}$ in the bar construction $B_{\mathbb{A}^1}$ satisfying*

- $\pi_1(\overline{\mathcal{L}^{0-1,B}_W}) = \overline{\mathcal{L}^{0-1}_W} = \overline{\mathcal{L}^0_W} - \overline{\mathcal{L}^1_W}$.
- *It is in the image of the projector $p_{ш}$; hence in $Q_{\mathbb{A}^1}$.*
- *It is of bar degree* 0 *and its image under d_B is* 0*; hence it gives a class in* $\mathrm{H}^0(B_X)$ *and in* $\mathrm{H}^0(Q_X)$.
- *Its image under δ_Q is given by Equation* (ED-$\overline{\mathcal{L}^{0-1}}$) *without the minus sign.*
- $j^*(\overline{\mathcal{L}^{0-1,B}_W}) = \mathcal{L}^B_W - \mathcal{L}^{1,B}_W$.

A similar statement holds for $\overline{\mathcal{L}^0_W(1)}$ *and* $i_1^*(\overline{\mathcal{L}^0_W}) = i_1^*(\overline{\mathcal{L}^0_W(1)})$ *with equation* (ED-$\overline{\mathcal{L}^{0-1}}$) *replaced by* (9)*. The corresponding bar elements are denoted*

$$\overline{\mathcal{L}^B_W(1)} \qquad \textit{and} \qquad \mathcal{L}^B_{W,x=1}$$

respectively. We have the appropriate compatibilities with p_1^* *and* j^*.

Proof. The proof goes as in Theorem 3.16 above but using only the sub Lie coalgebra $\mathcal{T}_1^{coL}$ and equations (ED-$\overline{\mathcal{L}^{0-1}}$) and (9). The coefficients $a^W_{U,V}$ appearing in the differential equation for the cycles are equal to coefficients $\gamma^W_{U,V}$ giving the cobracket of the element $T_{W^*}(1)$ (cf. Lemma 3.6).

The relations between bar elements over $\{1\}$, $\mathbb{A}^1$ and $X = \mathbb{P}^1 \setminus \{0,1,\infty\}$ follow because i_1^*, p_1^* and j^* are morphisms of cdga algebra. □

Lemma 3.18. *In* $B_{\mathbb{Q}}$ *the following relation holds :*

$$i_1^*(\overline{\mathcal{L}_W^{0-1,B}}) = i_1^*(\overline{\mathcal{L}^B_W(1)}) = \mathcal{L}^B_{W,x=1}.$$

Proof. It follows from equations (7) and (8) which holds for the cycle on $\mathbb{A}^1$ and because in the cycle setting one has in $\mathcal{N}_{\mathbb{Q}}^{qf,\bullet}$:

$$i_1^*(\mathcal{L}_W^{0-1}) = i_1^*(\overline{\mathcal{L}^0_W(1)}) = \mathcal{L}^0_W|_{x=1}$$

for any Lyndon word W. □

4. A relative basis for mixed Tate motive over $\mathbb{P}^1 \setminus \{0,1,\infty\}$

4.1. Relations between bar elements

A motivation for introducing cycles $\mathcal{L}^1_W$ in [23] was the idea of a correspondence

$$\mathcal{L}^0_W - \mathcal{L}^0_W(1) \leftrightarrow \mathcal{L}^1_W$$

In this section, we prove that this relation is an equality in the H^0 of the bar construction modulo shuffle products; that is

$$\mathcal{L}^B_W - \mathcal{L}^B_W(1) = \mathcal{L}^{1,B}_W \qquad \text{in } \mathrm{H}^0(Q_X).$$

A key point in order to build cycles $\mathcal{L}^0_W$ and $\mathcal{L}^1_W$ in [23] is a pull-back by the multiplication. More precisely, the usual multiplication $\mathbb{A}^1 \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1$

composed with the isomorphism $\mathbb{A}^1 \times \square^1 \simeq \mathbb{A}^1 \times \mathbb{A}^1$ gives a multiplication

$$m_0 : \mathbb{A}^1 \times \square^1 \longrightarrow \mathbb{A}^1$$

sending (t, u) to $\frac{t}{1-u}$. Twisting m_0 by $\theta : t \mapsto 1 - t$ gives a "twisted multiplication"

$$m_1 = (\theta \times \mathrm{id}\,) \circ m_0 \circ \theta : \mathbb{A}^1 \times \square^1 \longrightarrow \mathbb{A}^1.$$

Proposition 4.1 ([23]). *For $\varepsilon = 0, 1$ the morphism m_ε induces a linear morphism*

$$m_\varepsilon^* : \mathcal{N}_{\mathbb{A}^1}^{qf,\,k} \longrightarrow \mathcal{N}_{\mathbb{A}^1}^{qf,\,k-1}$$

giving a homotopy

$$\partial_{\mathbb{A}^1} \circ m_\varepsilon^* + m_\varepsilon^* \circ \partial_{\mathbb{A}^1} = \mathrm{id}\, - p_\varepsilon^* \circ i_\varepsilon^*$$

where $p_\varepsilon : \mathbb{A}^1 \longrightarrow \{\varepsilon\}$ is the projection onto the point $\{\varepsilon\}$ and i_ε its inclusion in $\mathbb{A}^1$.

From this homotopy property, one derives the following relation between m_0^* and m_1^*.

Lemma 4.2. *One has:*

(10)
$$m_1^* = m_0^* - p_1^* \circ i_1^* \circ m_0^* - \partial_{\mathbb{A}^1} \circ m_1^* \circ m_0^* + m_1^* \circ m_0^* \circ \partial_{\mathbb{A}^1} + m_1^* \circ p_0^* \circ i_0^*$$

and a similar expression for m_1^*.

In particular, when $b \in \mathcal{N}_{\mathbb{A}^1}^{qf,\,k}$ satisfies $\partial_{\mathbb{A}^1}(b) = 0$ and $i_0(b) = 0$, one has:

$$m_1^*(b) = m_0^*(b) - p_1^* \circ i_1^*(m_0^*(b)) + \partial_{\mathbb{A}^1}(m_0^* \circ m_1^*(b))$$

Proof. Let b be in $\mathcal{N}_{\mathbb{A}^1}^{qf,\,k}$. We treat only Equation 10. Using the homotopy property for m_0^*, one writes

$$b = \partial_{\mathbb{A}^1} \circ m_0^*(b) + m_0^* \circ \partial_{\mathbb{A}^1} + p_0^* \circ i_0^*(b).$$

Computing $m_1^*(b)$, the homotopy property

$$m_1^* \circ \partial_{\mathbb{A}^1}(m_0(b)) = m_0^*(b) - p_1^* \circ i_1^*(m_0(b)) - \partial_{\mathbb{A}^1} \circ m_1^*(m_0^*(b)),$$

gives the desired formula. □

Writing $\overline{A_W}$ the $\mathbb{A}^1$-extension of the right hand side of Equation (ED-$\mathcal{L}^0$), cycles $\overline{\mathcal{L}^0_W}$ are obtained in [23] as

$$\overline{\mathcal{L}^0_W} = m_0^*(\overline{A_W})$$

and similarly for $\mathcal{L}^1_W$.

An explicit computation in low weight [23, 24] shows that

$$\overline{\mathcal{L}^0_{01}} = -m_0^*(\overline{\mathcal{L}^1_0}\,\overline{\mathcal{L}^0_1}) \quad \text{and} \quad \overline{\mathcal{L}^1_{01}} = -m_1^*(\overline{\mathcal{L}^1_0}\,\overline{\mathcal{L}^0_1}).$$

Using Lemma 4.2, one gets

$$\overline{\mathcal{L}^1_{01}} = \overline{\mathcal{L}^0_{01}} - \overline{\mathcal{L}^0_{01}(1)} - \partial_{\mathbb{A}^1}(m_1^* \circ m_0^*(\overline{\mathcal{L}^0_0}\,\overline{\mathcal{L}^0_1})). \tag{11}$$

Remark 4.3. In [24, Example 5.5], the author gave a parametrized equidimensional cycle over $\mathbb{A}^1$, C_{01} relating $\mathcal{L}^0_{01}$ and $\mathcal{L}^1_{01}$ defined in terms of equidimensional cycles. Up to a global minus sign and a reparametrization, the cycle C_{01} agrees with the expression

$$-m_1^* \circ m_0^*(\overline{\mathcal{L}^0_0}\,\overline{\mathcal{L}^0_1}).$$

Thus, $\overline{\mathcal{L}^0_{01}} - \overline{\mathcal{L}^0_{01}(1)}$ and $\overline{\mathcal{L}^1_{01}}$ differs only by a boundary. The differential of $\overline{\mathcal{L}^0_{01}(1)}$ is zero and one can compute explicitly the corresponding bar elements:

$$\overline{\mathcal{L}^{0-1,B}_{01}} = [\mathcal{L}^0_{01}] - [\mathcal{L}^1_{01}], \quad \mathcal{L}^B_{01}(1) = [\mathcal{L}^0_{01}(1)].$$

Lemma 4.4. *In $B(\mathcal{N}^{qf,\bullet}_{\mathbb{A}^1})$, one has the following relation*

$$\overline{\mathcal{L}^{0-1,B}_{01}} - \mathcal{L}^B_{01}(1) = d_B([m_1^*(\overline{\mathcal{L}^0_{01}})]). \tag{12}$$

Thus in $\mathrm{H}^0(B_{\mathbb{A}^1})$ as in $\mathrm{H}^0(Q_{\mathbb{A}^1})$ one has the equality

$$\overline{\mathcal{L}^{0-1,B}_{01}} - \mathcal{L}^B_{01}(1) = 0.$$

Taking the restriction to $\mathbb{P}^1 \setminus \{0,1,\infty\}$, one obtains in $\mathrm{H}^0(Q_X)$

$$\mathcal{L}^B_{01} - \mathcal{L}^{1,B}_{01} = j^*(\overline{\mathcal{L}^{0-1,B}_{01}}) = \mathcal{L}^B_{01}(1). \tag{13}$$

For W a Lyndon word of length $p \geqslant 2$, the explicit comparison between $\mathcal{L}_W^B$ and $\mathcal{L}_W^{1,B}$ is in general much more complicated as

$$\partial_{\mathbb{A}^1}(\overline{\mathcal{L}_W^0} - \overline{\mathcal{L}_W^1}) = -\sum_{0<U<V<1} a_{U,V}^W(\overline{\mathcal{L}_U^0} - \overline{\mathcal{L}_U^1})(\overline{\mathcal{L}_V^0} - \overline{\mathcal{L}_V^1}) \neq 0.$$

However, working at the bar construction level in $\mathrm{H}^0(Q_{\mathbb{A}^1})$ allows to use an induction argument.

Theorem 4.5. *For any Lyndon word W of length $p \geqslant 2$ the following relation holds*

$$\overline{\mathcal{L}_W^{0-1,B}} = \overline{\mathcal{L}_W^B(1)} \qquad \text{in} \quad \mathrm{H}^0(Q_{\mathbb{A}^1}) = Q_{\mathrm{H}^0(B_{\mathbb{A}^1})}. \tag{14}$$

The restriction to $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ gives in $\mathrm{H}^0(Q_X) = Q_{\mathrm{H}^0(B_X)}$

$$\mathcal{L}_W^B - \mathcal{L}_W^{1,B} = j^*(\overline{\mathcal{L}_W^{0-1,B}}) = \mathcal{L}_W^B(1). \tag{15}$$

Proof. From Lemma 4.4 above it is true for $p = 2$ as there is then only one Lyndon word to consider $W = 01$.

Now we assume that the theorem is true for all Lyndon words of length k with $2 \leqslant k \leqslant p - 1$. Let W be a Lyndon word of length p.

From Proposition 3.17, one has in $Q_{\mathbb{A}^1}$ and in particular in $\mathrm{H}^0(Q_{\mathbb{A}^1})$:

$$\delta_Q(\overline{\mathcal{L}_W^{0-1,B}}) = \sum_{0<U<V<1} a_{U,V}^W(\overline{\mathcal{L}_U^{0-1,B}}) \wedge (\overline{\mathcal{L}_V^{0-1,B}})$$

and

$$\delta_Q(\overline{\mathcal{L}_W^B(1)}) = \sum_{0<U<V<1} a_{U,V}^W(\overline{\mathcal{L}_U^B(1)}) \wedge (\overline{\mathcal{L}_V^B(1)}).$$

Using the induction hypothesis, one has in $\mathrm{H}^0(Q_{\mathbb{A}^1})$

$$\delta_Q(\overline{\mathcal{L}_W^{0-1,B}}) = \sum_{0<U<V<1} a_{U,V}^W(\overline{\mathcal{L}_U^B(1)}) \wedge (\overline{\mathcal{L}_V^B(1)})$$

and thus

$$\delta_Q\left(\overline{\mathcal{L}_W^{0-1,B}} - \overline{\mathcal{L}_W^B(1)}\right) = 0 \qquad \text{in} \quad \mathrm{H}^0(Q_{\mathbb{A}^1}).$$

Let C_W be the class of $\overline{\mathcal{L}_W^{0-1,B}} - \overline{\mathcal{L}_W^B(1)}$ in $\mathrm{H}^0(Q_{\mathbb{A}^1})$ and sC_W its image in

$$\Omega_{coL}(\mathrm{H}^0(Q_{\mathbb{A}^1})) = S^{gr}(s\mathbb{Q} \otimes \mathrm{H}^0(Q_{\mathbb{A}^1})).$$

where the above equality is in terms of commutative algebras.

As $\delta_Q(C_W)=0$, $d_{\Omega,coL}(sC_W)=0$ and sC_W gives a class in

$$\mathrm{H}^1(\Omega_{coL}(\mathrm{H}^0(Q_{\mathbb{A}^1}))) \simeq \mathrm{H}^1(\mathcal{N}_{\mathbb{A}^1}^{qf,\bullet});$$

where the above isomorphism is given by Bloch and Kriz in [1, Corollary 2.31] after a choice of a 1-minimal model in the sense of Sullivan. Using the comparison between $\mathrm{H}^1(\mathcal{N}_{\mathbb{A}^1}^{qf,\bullet})$ and the higher Chow groups, this class can be represented by $p_1^*(C)$ in $\mathcal{N}_{\mathbb{A}^1}^{qf,1}$ with C a cycle in $\mathcal{N}_{\mathbb{Q}}^{qf,1}$.

The cycle $p_1^*(C)$ satisfies $\partial_{\mathbb{A}^1}(p_1^*(C))=0$ and $[p_1^*(C)]$ gives a degree 0 bar element C^B in $B_{\mathbb{A}^1}$ whose bar differential and reduced coproduct are equal to 0.

>From this, one gets a class $\tilde{C}_W = C_W - C^B$ in $\mathrm{H}^0(Q_{\mathbb{A}^1})$. Its image $s\tilde{C}_W$ in $\Omega_{coL}(\mathrm{H}^0(Q_{\mathbb{A}^1}))$ also gives a class in

$$\mathrm{H}^1(\Omega_{coL}(\mathrm{H}^0(Q_{\mathbb{A}^1})))$$

which is 0 by construction.

As the differential $d_{\Omega,coL}$ is zero on the degree 0 part of $\Omega_{coL}(\mathrm{H}^0(Q_{\mathbb{A}^1}))$, one obtains that $s\tilde{C}_W=0$ in $s\mathbb{Q}\otimes \mathrm{H}^0(Q_{\mathbb{A}^1})$. Thus $\tilde{C}_W$ is zero in $\mathrm{H}^0(Q_{\mathbb{A}^1}) = Q_{\mathrm{H}^0(B_{\mathbb{A}^1})}$. The above discussion shows that:

$$0=\tilde{C}_W = C_W - C^B = C_W - [p_1^*(C)]$$

So far one has obtained that in $B_{\mathbb{A}^1}$:

$$\overline{\mathcal{L}_W^{0-1,B}} - \overline{\mathcal{L}_W^B(1)} - [p_1^*(C)] = d_B(b) \qquad \text{modulo ш products} \tag{16}$$

with b in the degree -1 part of $B_{\mathbb{A}^1} = B(\mathcal{N}_{\mathbb{A}^1}^{qf,\bullet})$.

Because taking the fiber at 1 commutes with products and differential, one gets modulo shuffles

$$i_1^*(\overline{\mathcal{L}_W^{0-1,B}}) - i_1^*(\overline{\mathcal{L}_W^B(1)}) - [C] = d_B(i_1^*(b)).$$

Lemma 3.18 insures that $i_1^*(\overline{\mathcal{L}_W^{0-1,B}}) - i_1^*(\overline{\mathcal{L}_W^B(1)}) = 0$. Thus one has

$$-[C] = d_B(i_1^*(b)) + \text{ shuffle products.}$$

This shows that $[p^*(C)]$ is zero in $\mathrm{H}^0(B_{\mathbb{A}^1})$ modulo shuffles. Hence Equation (16) can be written has

$$\overline{\mathcal{L}_W^{0-1,B}} - \overline{\mathcal{L}_W^B(1)} = 0 \qquad \text{in} \quad Q_{\mathrm{H}^0(B_{\mathbb{A}^1})} = \mathrm{H}^0(Q_{\mathbb{A}^1})$$

Finally, taking the restriction to $\mathbb{P}^1 \setminus \{0,1,\infty\}$, one has $\mathcal{L}^B_W - \mathcal{L}^{1,B}_W = j^*(\overline{\mathcal{L}^{0-1,B}_W})$ which concludes the proof.

□

The main consequence of Equation (15) in the previous theorem is that in $Q_{\mathrm{H}^0(B_X)}$ one can replace the bar avatar of the geometric differential system (ED-$\mathcal{L}^0$) by a bar avatar of the differential system (ED-T) coming from Ihara action by special derivations.

Corollary 4.6. *In $Q_{\mathrm{H}^0(B_X)}$, the following holds for any (non-empty) Lyndon word W:*

$$\text{(ED-}Q_X\text{)} \qquad \delta_Q(\mathcal{L}^B_W) = \sum_{U<V} \alpha^W_{U,V} \mathcal{L}^B_U \wedge \mathcal{L}^B_V + \sum_{U,V} \beta^W_{U,V} \mathcal{L}^B_U \wedge \mathcal{L}^B_V(1).$$

Proof. Let W be a Lyndon word. The statement holds when W has length equal 1 and one can assume that W has length greater or equal to 2. One begins with the formula giving $\delta_Q(\mathcal{L}^B_W)$ from Theorem 3.16:

$$\delta_Q(\mathcal{L}^B_W) = \sum_{U<V} a^W_{U,V} \mathcal{L}^B_U \wedge \mathcal{L}^B_V + \sum_{U,V} b^W_{U,V} \mathcal{L}^B_U \wedge \mathcal{L}^{1,B}_V.$$

Then using the relations given by Equation (15), one has

$$\delta_Q(\mathcal{L}^B_W) = \sum_{U<V} a^W_{U,V} \mathcal{L}^B_U \wedge \mathcal{L}^B_V + \sum_{U,V} b^W_{U,V} \mathcal{L}^B_U \wedge \left(\mathcal{L}^B_V - \mathcal{L}^B_V(1)\right).$$

Expanding terms as $\mathcal{L}^B_U \wedge \left(\mathcal{L}^B_V - \mathcal{L}^B_V(1)\right)$, we conclude the proof using the expression of coefficients a's and b' in terms of α's and β's given at Lemma 3.9.

□

4.2. A basis for the geometric Lie coalgebra

This section shows that the image of the family of bar elements $\mathcal{L}^B_W$ in Deligne-Goncharov motivic fundamental Lie coalgebra is a basis of this coLie coalgebra. Hence the family $\mathcal{L}^B_W$ induced a basis of the tannakian coLie coalgebra of mixed Tate motives over $\mathbb{P}^1 \setminus \{0,1,\infty\}$ relatively to the one for mixed Tate motives over $\mathbb{Q}$.

For $X = \mathbb{P}^1 \setminus \{0,1,\infty\}$, M. Levine in [17][Theorem 5.3.2 and beginning of the section 6.6] shows that one can identify the Tannakian group associated with MTM(X) with the spectrum of $\mathrm{H}^0(B_X)$:

$$G_{\mathrm{MTM}(X)} \simeq \mathrm{Spec}(\mathrm{H}^0(B_X)).$$

Then, he uses a relative bar-construction in order to relate $G_{MTM(X)}$ to the motivic fundamental group $\pi_1^{mot}(X,x)$ of Goncharov and Deligne (see [4] and [5]).

Let x be a $\mathbb{Q}$-point of $X = \mathbb{P}^1 \setminus \{0,1,\infty\}$.

Theorem 4.7 ([17][Corollary 6.6.2]). *There is a split exact sequence:*

$$1 \longrightarrow \pi_1^{mot}(X,x) \longrightarrow \mathrm{Spec}(\mathrm{H}^0(B(\mathcal{N}_X))) \overset{p^*}{\longrightarrow} \mathrm{Spec}(\mathrm{H}^0(B(\mathcal{N}_{\mathbb{Q}}))) \longrightarrow 1$$

$$x^* : \mathrm{Spec}(\mathrm{H}^0(B(\mathcal{N}_{\mathbb{Q}}))) \longrightarrow \mathrm{Spec}(\mathrm{H}^0(B(\mathcal{N}_X)))$$

where p is the structural morphism $p : \mathbb{P}^1 \setminus \{0,1,\infty\} \longrightarrow \mathrm{Spec}(\mathbb{Q})$.

Theorem 4.7 can be reformulate in terms of Lie coalgebras, looking at indecomposable elements of the respective Hopf algebras.

Proposition 4.8. *There is a split exact sequence of Lie coalgebras:*

$$0 \longrightarrow Q_{\mathrm{H}^0(B_{\mathbb{Q}})} \overset{\tilde{p}}{\longrightarrow} Q_{\mathrm{H}^0(B_X)} \overset{\phi}{\longrightarrow} Q_{geom} \longrightarrow 0$$

$$\tilde{x} : Q_{\mathrm{H}^0(B_X)} \longrightarrow Q_{\mathrm{H}^0(B_{\mathbb{Q}})}$$

where Q_{geom} is the set of indecomposable elements of $\mathcal{O}(\pi_1^{mot}X, x)$ and is isomorphic as Lie coalgebra to the graded dual of the Lie algebra associated to $\pi_1^{mot}(X,x)$. Hence Q_{geom} is isomorphic as Lie coalgebra to the graded dual of the free Lie algebra on two generators $\mathrm{Lie}(X_0, X_1)$.

Considering the family of bar elements $\mathcal{L}_W^B$ for all Lyndon words W in this short exact sequence of Lie coalgebra, ones gets:

Theorem 4.9. *The family $\phi(\mathcal{L}_W^B)$ for any Lyndon words W is a basis of the Lie coalgebra Q_{geom}. Hence the family $\mathcal{L}_W^B$ is a basis of $Q_{\mathrm{H}^0(B_X)}$ relatively to $Q_{\mathrm{H}^0(B_{\mathbb{Q}})}$.*

Proof. The above short exact sequence being a sequence of Lie coalgebra, one has:

$$\begin{array}{ccccccccc}
0 \longrightarrow & Q_{\mathrm{H}^0(B_{\mathbb{Q}})} \wedge Q_{\mathrm{H}^0(B_{\mathbb{Q}})} & \xrightarrow{\tilde{p}\wedge\tilde{p}} & Q_{\mathrm{H}^0(B_X)} \wedge Q_{\mathrm{H}^0(B_X)} & \xrightarrow{\phi\wedge\phi} & Q_{geom} \wedge Q_{geom} & \longrightarrow 0 \\
 & \uparrow{\scriptstyle \delta_{Q,\mathbb{Q}}} & & \uparrow{\scriptstyle \delta_{Q,X}} & & \uparrow{\scriptstyle \delta_{geom}} & \\
0 \longrightarrow & Q_{\mathrm{H}^0(B_{\mathbb{Q}})} & \xrightarrow{\tilde{p}} & Q_{\mathrm{H}^0(B_X)} & \xrightarrow{\phi} & Q_{geom} & \longrightarrow 0.
\end{array}$$

As $\delta_{Q,X}(\mathcal{L}_0^B) = \delta_{Q,X}(\mathcal{L}_1^B) = 0$, weight reasons show that $\phi(\mathcal{L}_0^B)$ and $\phi(\mathcal{L}_1^B)$ are dual to the weight 1 generators of $\mathrm{Lie}(X_0, X_1)$.

In order to show that the family $\phi(\mathcal{L}_W^B)$ is a basis of Q_{geom}, it is enough to show that the elements $\phi(\mathcal{L}_W^B)$ satisfy:

$$\delta_{geom}(\phi(\mathcal{L}_W^B)) = \sum_{U<V} \alpha_{U,V}^W \phi(\mathcal{L}_U^B) \wedge \phi(\mathcal{L}_V^B)$$

because δ_{geom} is dual to the bracket $[\,,\,]$ of $\mathrm{Lie}(X_0, X_1)$.

As ϕ commutes with the cobracket, it is enough to compute $(\phi \wedge \phi) \circ \delta_X(\mathcal{L}_W^B)$:

$$\begin{aligned}
\delta_{geom}(\phi(\mathcal{L}_W^B)) =& (\phi \wedge \phi) \circ \delta_X(\mathcal{L}_W^B) \\
=& (\phi \wedge \phi) \left(\sum_{U<V} \alpha_{U,V}^W \mathcal{L}_U^B \wedge \mathcal{L}_V^B + \sum_{U,V} \beta_{U,V}^W \mathcal{L}_U^B \wedge \mathcal{L}_V^B(1) \right) \\
=& \sum_{U<V} \alpha_{U,V}^W \phi(\mathcal{L}_U^B) \wedge \phi(\mathcal{L}_V^B) + \sum_{U,V} \beta_{U,V}^W \phi(\mathcal{L}_U^B) \wedge \phi(\mathcal{L}_V^B(1))
\end{aligned}$$

By construction $\phi(\mathcal{L}_V^B(1))$ is zero. Thus one obtains the expected formula for $\delta_{geom}(\phi(\mathcal{L}_W^B))$. □

Note that δ_X gives the coaction of $Q_{\mathrm{H}^0(B_{\mathbb{Q}})}$ on Q_{geom} described in [2] in relation with Goncharov motivic coproduct Δ^{mot}. In this context, Equation (ED-Q_X)

$$\delta_Q(\mathcal{L}_W^B) = \sum_{U<V} \alpha_{U,V}^W \mathcal{L}_U^B \wedge \mathcal{L}_V^B + \sum_{U,V} \beta_{U,V}^W \mathcal{L}_U^B \wedge \mathcal{L}_V^B(1)$$

is nothing but another expression for Goncharov motivic cobracket $\frac{1}{2}(\Delta^{mot} - \tau \circ \Delta^{mot})$. This new expression has the advantage that it is stable under the generating family $\mathcal{L}_W^B$.

Acknowledgments

None of this would have been possible without M. Levine patience, his explanations and inputs. The author is grateful to the MPIM in Bonn for providing ideal working conditions and support during his stay there as a guest where this work has been accomplished.

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Received January 29, 2015