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Elliptic genera of Berglund-Hübsch Landau-Ginzburg orbifolds

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We match the elliptic genus of a Berglund-Hübsch Landau-Ginzburg orbifold with the supertrace of $y^{J[0]}q^{L[0]}$ on a vertex algebra $V_{\mathbf{1},\mathbf{1}}$. We show that it is a weak Jacobi form and the elliptic genus of one theory is equal (up to a sign) to the elliptic genus of its mirror.

1. Introduction

Mirror symmetry was originally formulated as a correspondence between the $N = (2, 2)$ superconformal field theories constructed for a Calabi-Yau n-fold X and for its mirror partner $X^\vee$. On the level of cohomology groups, there is a 90-degree rotation of the Hodge diamond, i.e. $h^{p,q}(X, \mathbf{C}) = h^{n-p,q}(X^\vee, \mathbf{C})$. Batyrev's construction of Calabi-Yau hypersurfaces in Gorenstein Fano toric varieties associated to a pair of reflexive polytopes ([B]) is a prolific source of examples of mirror Calabi-Yau varieties. This construction was later generalized by Borisov to Calabi-Yau complete intersections in Gorenstein Fano toric varieties ([B1]), and further by Batyrev and Borisov to the mirror duality of reflexive Gorenstein cones ([BB1]). They proved that the string-theoretic Hodge numbers of (singular) Calabi-Yau varieties arising from their constructions satisfy the expected mirror duality ([BB2]).

Around the same time, physicists Berglund and Hübsch proposed a way to construct mirror pairs of $(2, 2)$-superconformal field theories in the formalism of orbifold Landau-Ginzburg theories ([BH]). They considered a non-degenerate invertible polynomial potential W whose transpose $W^\vee$ is again a non-degenerate invertible potential. They claimed that there exists a suitable group H such that the Landau-Ginzburg orbifolds W and $W^\vee/H$ form a mirror pair. Recently, Krawitz found a general construction of the dual group $G^\vee$ for any subgroup G of diagonal symmetries of W, and proved an "LG-to-LG" mirror symmetry theorem for the pair $(W/G, W^\vee/G^\vee)$ at the level of double-graded state spaces ([K]).

Under a certain CY condition, the polynomials W, $W^\vee$ define Calabi-Yau hypersurfaces X_W, $X_{W^\vee}$ in (usually different) weighted projective spaces.

For suitable groups G, the Calabi-Yau orbifolds X_W/G, $X_{W^\vee}/G^\vee$ are expected to be a mirror pair. Indeed, this "CY-to-CY" mirror symmetry theorem was proved by Chiodo and Ruan ([CR]) at the level of double-graded dimensions of the state spaces by first establishing a cohomological LG/CY correspondence and then invoking Krawitz's theorem.

Elliptic genera are invariants of oriented (or almost complex) manifolds that take values in the ring of modular forms. In the present article, we consider a two-variable elliptic genus associated to an $N = (2, 2)$ superconformal field theory in the physics literature. When the field theory comes from a smooth Calabi-Yau manifold M, one has a mathematical formula for the genus in terms of the holomorphic Euler characteristic of the double-graded bundle

$$y^{-\frac{\dim M}{2}} \otimes_{n\geq 1} (\wedge_{-yq^{n-1}} T_M^* \otimes \wedge_{-y^{-1}q^n} T_M \otimes S_{q^n} T_M^* \otimes S_{q^n} T_M).$$

When the manifold is Calabi-Yau, this Euler characteristic is a weak Jacobi form of weight 0 and index $\frac{\dim M}{2}$ ([BL]), and mirror Calabi-Yau varieties must have the same elliptic genera.

In [BHe], Berglund and Henningson computed the elliptic genus of an arbitrary $N = (2, 2)$ Landau-Ginzburg orbifold using physical argument and showed that the elliptic genera of mirror Berglund-Hübsch LG-orbifolds are equal by comparing a particular limit of the genera. The main result of this paper is to give a mathematical proof of this statement using vertex algebras. We will match their formula with the supertrace of an operator on a vertex algebra, and then derive the duality by exploring the mirror-symmetric nature of this vertex algebra.

The idea of interpreting the elliptic genus as the trace of an operator on a vertex algebra first appeared in [BL]. The starting point is that the elliptic genus of a smooth variety M is equal to $y^{-\frac{\dim M}{2}}$ times the supertrace of the operator $y^{J[0]}q^{L[0]}$ on the cohomology of the chiral de Rham complex [1] of M. This interpretation allowed Borisov and Libgober to extend the notion of elliptic genus to singular varieties, for which the chiral de Rham complexes have been constructed, e.g. Calabi-Yau hypersurfaces in Gorenstein Fano toric varieties ([B2]). Using the full force of the machinery developed in [B2], they proved that the elliptic genera of mirror Calabi-Yau hypersurfaces in Gorenstein Fano toric varieties are equal up to a sign.

Recently, a vertex algebra approach to Berglund-Hübsch-Krawitz mirror symmetry was developed by Borisov in [B3]. Similar to the toric case, a

[1] The chiral de Rham complex is a sheaf of vertex superalgebras introduced by Malikov, Schechtman, and Vaintrob.

vertex algebra $V_{\mathbf{1},\mathbf{1}}$ was constructed from the combinatorial data of a non-degenerate invertible polynomial potential W and a subgroup G of diagonal symmetries of W. The vertex algebra $V_{\mathbf{1},\mathbf{1}}$ contains the A and B rings of the Landau-Ginzburg orbifolds $(W/G, W^\vee/G^\vee)$ as subspaces. Using the properties of $V_{\mathbf{1},\mathbf{1}}$, Borisov reproved Krawitz's result and moreover showed a ring isomorphism between the A ring of W/G and the B ring of $W^\vee/G^\vee$.

The paper is organized as follows: Section 2 recalls the combinatorial data of the Berglund-Hübsch-Krawitz construction following [B3] and [CR]. The vertex algebra $V_{\mathbf{1},\mathbf{1}}$ is defined in Section 3. In Section 4, we match the Berglund-Henningson formula of the elliptic genus of W/G with the supertrace of an operator $y^{J[0]}q^{L[0]}$ on $V_{\mathbf{1},\mathbf{1}}$, and show that it is a weak Jacobi form. Finally we prove that the elliptic genera of W/G and $W^\vee/G^\vee$ coincide up to a sign.

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2. The Berglund-Hübsch-Krawitz mirror symmetry construction

We will use notations from [B3] and [CR]. Consider a non-degenerate polynomial potential

$$W(x_1, \dots, x_d) = \sum_{i=1}^{d} \prod_{j=1}^{d} x_j^{a_{ij}} \tag{2.1}$$

with invertible exponent matrix $A = (a_{ij})$. Suppose the variables x_j can be assigned positive rational degrees q_j such that W is homogeneous of degree 1, i.e. we have

$$\sum_j a_{ij} q_j = 1 \tag{2.2}$$

for all i. Since A is invertible, the rational degrees q_j are uniquely determined by W. Non-degeneracy means that the hypersurface $W = 0$ in $\mathbb{C}^d$ is smooth away from the origin. In fact, Kreuzer and Skarke ([KS]) classified all non-degenerate potentials; they are sums of decoupled invertible potentials of the following types

$$W_{\text{Fermat}} = x^a \tag{2.3}$$

$$W_{\text{loop}} = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}x_1 \tag{2.4}$$

$$W_{\text{chain}} = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}. \tag{2.5}$$

Decoupled means that the set of variables $\{x_1, \ldots, x_d\}$ is partitioned into a disjoint union of subsets, and the variables in each subset contribute a polynomial of one of the above types.

Consider the group $\text{Aut}(W)$ of diagonal automorphisms

$$\gamma : x_j \mapsto \gamma_j x_j \tag{2.6}$$

that preserve the potential W, that is

$$\text{Aut}(W) = \left\{ \gamma = (\gamma_j) : \prod_j \gamma_j^{a_{ij}} = 1 \text{ for all } i \right\} \tag{2.7}$$

Since the matrix A is invertible, each γ_j is a root of unity. If we write

$$\gamma_j = \exp(2\pi i p_j) \tag{2.8}$$

for some rational number p_j (determined up to an integer), then the defining relation of $\text{Aut}(W)$ translates to

$$\sum_{j=1}^{d} a_{ij} p_j \in \mathbb{Z}. \tag{2.9}$$

This identifies the group $\text{Aut}(W)$ with d-tuple of rational numbers $p = (p_j)$ defined up to $\mathbb{Z}^d$ such that $Ap \in \mathbb{Z}^d$. Let ρ_i be the i-th column of A^{-1}, then the group $\text{Aut}(W)$ is generated by the ρ_i-s, and we have $q = \sum_i \rho_i$ where $q = (q_j)$ is the vector encoding the rational degrees of x_j. The corresponding scaling operator

$$J_W : x_j \mapsto \exp(2\pi i q_j) x_j \tag{2.10}$$

is called the exponential grading operator. Other than the subgroup of $\text{Aut}(W)$ generated by J_W, we are also interested in the subgroup $SL_W =$

$SL_d \cap \mathrm{Aut}(W)$ defined as follows:

$$SL_W = \left\{ \gamma \in \mathrm{Aut}(W) : \prod_j \gamma_j = 1 \right\}. \tag{2.11}$$

This corresponds to the condition that $\sum_j p_j \in \mathbb{Z}$ for $p = (p_j) \in \mathrm{Aut}(W)$.

We impose the condition that the subgroup $\langle J_W \rangle$ generated by the exponential grading operator lies in SL_W. This translates to the generalized Calabi-Yau condition ([B3, Remark 2.1.1]): [2]

$$\sum_{j=1}^{d} q_j = k \in \mathbb{Z}_{>0}. \tag{2.12}$$

Let G be a subgroup of $\mathrm{Aut}(W)$ that contains J_W and is contained in SL_W, that is $\langle J_W \rangle \subset G \subset SL_W$. To describe the mirror of the Landau-Ginzburg orbifold X_W/G, we need the notion of dual potential and dual group.

The dual potential $W^\vee$ is obtained by transposing the exponent matrix A, i.e.

$$W^\vee = \sum_{i=1}^{d} \prod_{j=1}^{d} x_j^{a_{ij}} \tag{2.13}$$

It follows from the classification of [KS] that if W is non-degenerate, then $W^\vee$ is also non-degenerate. In fact, one observes that the dual potential of each type in (2.3)-(2.5) is a potential of the same type. We also consider the group $\mathrm{Aut}(W^\vee)$ of diagonal automorphisms that preserve $W^\vee$. If identified with (row) vectors $\bar{p}$ in $\mathbb{Q}^d/\mathbb{Z}^d$ such that $\bar{p}A \in \mathbb{Z}^d$, $\mathrm{Aut}(W^\vee)$ is generated by the rows of A^{-1}. Similarly, we define the dual exponential grading operator $J_{W^\vee}$ and the subgroup $SL_{W^\vee}$. Given G such that $\langle J_W \rangle \subset G \subset SL_W$, there is a natural way of defining a dual group $G^\vee$ such that $\langle J_{W^\vee} \rangle \subset G^\vee \subset SL_{W^\vee}$ ([K]). We will describe this duality in the language of dual lattices ([B3]).

[2] $k = 1$ is the Calabi-Yau condition in [CR]. In this case, $W = 0$ defines a Calabi-Yau orbifold X_W in the weighted projective space $\mathbb{P}(mq_1, \ldots, mq_d)$ where m is the smallest positive integer such that $mq_j \in \mathbb{Z}$ for all j. When $k > 1$, this does not define a Calabi-Yau hypersurface in the weighted projective space. Nevertheless, in Borisov's reformulation of the Berglund-Hübsch-Krawitz construction, k is similar to the index of a reflexive Gorenstein cone in the Batyrev-Borisov construction. In good cases, reflexive Gorenstein cones of index k give rise to Calabi-Yau complete intersections of k hypersurfaces. See [B3] and [BB1] for details.

Let M_0 and N_0 be free abelian groups with bases $\{u_i\}, i = 1, \ldots, d$ and $\{v_j\}, j = 1, \ldots, d$. Define a non-degenerate integral pairing on these lattices by putting

$$u_i \cdot v_j = a_{ij}, \tag{2.14}$$

where a_{ij} are the exponents in the polynomial potential W. Because the pairing is integral, we have

$$M_0 \subset N_0^\vee, \qquad N_0 \subset M_0^\vee$$

where $N_0^\vee$ and $M_0^\vee$ are the dual lattices of N_0 and M_0. It was shown in [B3] that $\mathrm{Aut}(W)$ is naturally isomorphic to $M_0^\vee/N_0$. Indeed, given $(p_j) \in \mathbb{Q}^d$, form $v = \sum_j p_j v_j$, then (2.9) is equivalent to $u_i \cdot v \in \mathbb{Z}$ for all i which implies $v \in M_0^\vee$. Moreover, integer-valued (p_j) corresponds to $v \in N_0$. The image of J_W under this isomorphism can be represented by

$$\deg^\vee = \sum_{j=1}^{d} q_j v_j \in M_0^\vee, \tag{2.15}$$

then

$$u_i \cdot \deg^\vee = 1 \tag{2.16}$$

for all i. Similarly, the group $\mathrm{Aut}(W^\vee)$ is naturally isomorphic to $N_0^\vee/M_0$, and $J_{W^\vee}$ is represented by $\deg \in N_0^\vee$ such that

$$\deg \cdot v_j = 1 \tag{2.17}$$

for all j.

Each subgroup $G \subset \mathrm{Aut}(W)$ determines a suplattice $N \supset N_0$ such that $G \cong N/N_0$. The dual group $G^\vee \subset \mathrm{Aut}(W^\vee)$ is defined to be M/M_0 where M is the dual lattice of N. In particular, the dual of $\langle J_W \rangle$ is $SL_{W^\vee}$; the dual of SL_W is $\langle J_{W^\vee} \rangle$. Indeed, for the entries of $(p_j) \in \mathrm{Aut}(W)$ to add up to an integer (the condition of (p_j) being in SL_W) is equivalent to $(\deg \cdot \sum_j p_j v_j)$ being an integer. It is now clear that if G sits between $\langle J_W \rangle$ and SL_W, then $G^\vee$ sits between $\langle J_{W^\vee} \rangle$ and $SL_{W^\vee}$; the corresponding lattices then satisfy $\deg \in M$ and $\deg^\vee \in N$.

The Berglund-Hübsch-Krawitz mirror symmetry asserts that the Landau-Ginzburg orbifolds W/G and $W^\vee/G^\vee$ are mirror of each other for $\langle J_W \rangle \subset G \subset SL_W$.

3. The vertex algebra $V_{1,1}$

In this section, we define a vertex algebra associated to the above combinatorial data, and state some results about it. We mostly follow the exposition of [B2, B3].

Fix dual lattices M and N such that $M_0 \subset M \subset N_0^\vee$, $N_0 \subset N \subset M_0^\vee$, $\deg \in M$, and $\deg^\vee \in N$. We define a vertex superalgebra $\mathrm{Fock}_{M\oplus N}$ which is the tensor product of the lattice vertex algebra associated to $M \oplus N$ and a vertex superalgebra generated by $2d$ fermions.

Let

$$L = M \oplus N.$$

The non-degenerate pairing between M and N extends to a non-degenerate bilinear form on L where the only non-zero pairing is between an element of M and an element of N. L is thus an even lattice, though not positive-definite. Consider the 2-cocycle

$$c : L \times L \to \{\pm 1\} \tag{3.1}$$

defined by

$$c((m,n),(m_1,n_1)) = (-1)^{m\cdot n_1}. \tag{3.2}$$

Let V_L be the lattice vertex algebra associated to L and this cocycle. We use A, B to distinguish modes coming from elements of N and M. That is, we denote

$$m \cdot B(z) = \sum_{k\in\mathbb{Z}} m \cdot B[k] z^{-k-1}, \qquad n \cdot A(z) = \sum_{k\in\mathbb{Z}} n \cdot A[k] z^{-k-1} \tag{3.3}$$

which have OPE:

$$\begin{gathered} m \cdot B(z)\, m_1 \cdot B(w) \sim n \cdot A(z)\, n_1 \cdot A(w) \sim 0, \\ m \cdot B(z)\, n \cdot A(w) \sim \frac{m \cdot n}{(z-w)^2}. \end{gathered} \tag{3.4}$$

As a vector space, V_L is isomorphic to the direct sum of infinitely many polynomial algebras indexed by elements of M and N in infinitely many variables

$$\oplus_{m\in M, n\in N}\, \mathbb{C}[B[-1], B[-2], \ldots, A[-1], A[-2], \ldots]\, |m,n\rangle$$

Here, each $B[-k], k \geq 1$ does not stand for one mode, but d linearly independent ones corresponding to a basis of M; the same is true for $A[-k]$.

Each $|m,n\rangle$ is annihilated by $m_1 \cdot B[k]$, $n_1 \cdot A[k]$ for $k>0$ and

$$m_1 \cdot B[0]\,|m,n\rangle = m_1 \cdot n\,|m,n\rangle, \qquad n_1 \cdot A[0]\,|m,n\rangle = n_1 \cdot m|m,n\rangle. \tag{3.5}$$

We denote the vertex operators of $|m,n\rangle$ by $e^{\int m\cdot B(z)+n\cdot A(z)}$. By definition, it acts on an arbitrary element of V_L as follows

$$\begin{aligned} & e^{\int m\cdot B(z)+n\cdot A(z)} \prod A[\cdots] \prod B[\cdots]|m_1,n_1\rangle \\ &= (-1)^{m\cdot n_1} \exp\left(\sum_{k<0}(m\cdot B[k]+n\cdot A[k])\frac{z^{-k}}{-k}\right) \\ &\quad \exp\left(\sum_{k>0}(m\cdot B[k]+n\cdot A[k])\frac{z^{-k}}{-k}\right) z^{m\cdot n_1+n\cdot m_1} \\ &\quad \prod A[\cdots]\prod B[\cdots]|m+m_1,n+n_1\rangle. \end{aligned}$$

Let Λ_L be the vertex super-algebra generated by the fermionic fields:

$$m\cdot\Phi(z) = \sum_{k\in\mathbb{Z}} m\cdot\Phi[k]z^{-k-1}, \qquad n\cdot\Psi(z) = \sum_{k\in\mathbb{Z}} n\cdot\Psi[k]z^{-k}$$

with OPE:

$$m\cdot\Phi(z)\ n\cdot\Psi(w) \sim \frac{m\cdot n}{z-w}.$$

As a vector space, Λ_L is isomorphic to the exterior algebra

$$\wedge^{\cdot}\left(\oplus_{k<0}\Phi[k]\oplus\oplus_{k\le 0}\Psi[k]\right).$$

Define a vertex super-algebra

$$\mathrm{Fock}_{M\oplus N} = V_L\otimes\Lambda_L.$$

Consider the following two bosonic fields in $\mathrm{Fock}_{M\oplus N}$ with normal ordering implicit:

$$J(z) = -\sum_{i=1}^{d} m_i\cdot\Phi(z)\,n_i\cdot\Psi(z) - \deg\cdot B(z) + \deg^{\vee}\cdot A(z), \tag{3.6}$$

$$\begin{aligned} L(z) &= \sum_{i=1}^{d} m_i\cdot B(z)\,n_i\cdot A(z) \\ &\quad -\sum_{i=1}^{d} m_i\cdot\Phi(z)\partial_z\,n_i\cdot\Psi(z) - \partial_z\deg\cdot B(z), \end{aligned} \tag{3.7}$$

where $\{m_i\}$, $\{n_i\}$ are dual bases in M and N. Write

$$J(z) = \sum_{k\in\mathbb{Z}} J[k]z^{-k-1}, \qquad L(z) = \sum_{k\in\mathbb{Z}} L[k]z^{-k-2}.$$

The eigenvalues of $J[0]$ and $L[0]$ equip $\mathrm{Fock}_{M\oplus N}$ with a double grading. Explicitly, given an element

$$\prod A[\cdots]\prod B[\cdots]\prod \Phi[\cdots]\prod \Psi[\cdots]|m,n\rangle \in \mathrm{Fock}_{M\oplus N},$$

$J[0]$ counts the number of occurrences of Ψ minus the number of occurrences of Φ, plus $(\deg^\vee\cdot m - \deg\cdot n)$, while $L[0]$ counts the opposite of the sum of indices in [ ], plus $m\cdot n + \deg\cdot n$.

Switching the roles of M and N, we obtain

$$\begin{aligned}J^*(z) &= -\sum_{i=1}^{d} n_i\cdot\Psi(z)\, m_i\cdot\Phi(z) - \deg^\vee\cdot A(z) + \deg\cdot B(z)\\ &= -J(z)\end{aligned} \tag{3.8}$$

$$\begin{aligned}L^*(z) &= \sum_{i=1}^{d} n_i\cdot A(z)\, m_i\cdot B(z)\\ &\quad -\sum_{i=1}^{d} n_i\cdot\Psi(z)\partial_z\, m_i\cdot\Phi(z) - \partial_z\deg^\vee\cdot A(z)\\ &= L(z) - \partial_z J(z).\end{aligned} \tag{3.9}$$

In particular

$$J^*[0] = -J[0], \qquad L^*[0] = L[0] + J[0]$$

This is important to us because we will use $(J[0], L[0])$ to compute the elliptic genus of W/G and use $(J^*[0], L^*[0])$ to compute the elliptic genus of $W^\vee/G^\vee$.

Denote by $\triangle$ the set (u_i) and by $\triangle^\vee$ the set (v_j). Define the cones K_M in M and K_N in N by

$$K_M := M\cap\sum_i \mathbb{Q}_{\geq 0}u_i, \qquad K_N := N\cap\sum_j \mathbb{Q}_{\geq 0}v_j.$$

Since $u_i\cdot\deg^\vee = 1$ for all i and $\deg^\vee\in N$, (u_i) are the primitive generators of the rays in K_M, so are (v_j) primitive generators of the rays in K_N. Consider the following operators

$$D_{1,0} = \mathrm{Res}_{z=0} \sum_{m\in\triangle} m \cdot \Phi(z) e^{\int m\cdot B(z)},$$

$$D_{0,1} = \mathrm{Res}_{z=0} \sum_{n\in\triangle^\vee} n \cdot \Psi(z) e^{\int n\cdot A(z)},$$

$$D_{1,1} = D_{1,0} + D_{0,1}.$$

They all commute with $J[0]$ and $L[0]$. It is straightforward to check that $D_{1,0}$ and $D_{0,1}$ are both differentials, and they anticommute, hence $D_{1,1}$ is also a differential. Introduce a bi-grading on $\mathrm{Fock}_{M\oplus N}$ by the eigenvalues of

$$\deg^\vee \cdot A[0] \quad \text{and} \quad \deg \cdot B[0].$$

Then $D_{1,0}$ and $D_{0,1}$ change the $(\deg^\vee \cdot A[0], \deg \cdot B[0])$-grading by $(1,0)$ and $(0,1)$ respectively, hence $(\mathrm{Fock}_{M\oplus N}, D_{1,0}, D_{0,1})$ form a double complex.

Define the vertex super-algebra $V_{1,1}$ as the cohomology of $\mathrm{Fock}_{M\oplus N}$ with respect to the total differential $D_{1,1}$. The $(J[0], L[0])$-grading on $\mathrm{Fock}_{M\oplus N}$ descends to a $(J[0], L[0])$-grading on the cohomology of $\mathrm{Fock}_{M\oplus N}$ with respect to the operators $D_{1,0}$, $D_{0,1}$, and $D_{1,1}$.

We need the following results from [B2, B3].

Proposition 3.1. *[B3, Theorem 5.2.3] The cohomology of $\mathrm{Fock}_{M\oplus N}$ with respect to $D_{1,1}$ is equal to the cohomology of $\mathrm{Fock}_{M\oplus K_N}$ with respect to $D_{1,1}$.*

Proposition 3.2. *[B3, Theorem 6.2.1] For fixed eigenvalues of $J[0]$ and $L[0]$, the corresponding eigenspace in $V_{1,1}$ is finite-dimensional.*

We are interested in computing the supertrace of the operator $y^{J[0]}q^{L[0]}$ on $V_{1,1}$. Supertrace means that we subtract the dimension of the odd part from the dimension of the even part of the corresponding eigenspaces. By the previous proposition, this gives a well-defined double series in y and q. To compute this invariant, we first describe the cohomology of $\mathrm{Fock}_{M\oplus K_N}$ with respect to $D_{0,1}$.

Proposition 3.3. *[B2, Proposition 9.3] Denote by $(v_j^\vee)$ the dual basis of (v_j) in $N_0^\vee$. For each i, define*

$$b^i(z) = e^{\int v_i^\vee \cdot B(z)},$$

$$\phi^i(z) = (v_i^\vee \cdot \Phi(z)) e^{\int v_i^\vee \cdot B(z)}, \qquad \psi_i(z) = (v_i \cdot \Psi(z)) e^{-\int v_i^\vee \cdot B(z)}$$

$$a_i(z) =: (v_i \cdot A(z)) e^{-\int v_i^\vee \cdot B(z)} : + : (v_i^\vee \cdot \Phi(z))(v_i \cdot \Psi(z)) e^{-\int v_i^\vee \cdot B(z)} :$$

These fields generate a vertex subalgebra $\mathcal{VA}_{K_N,N_0^\vee}$ *inside* $Fock_{N_0^\vee\oplus 0}$. *Consider all elements from* $\mathcal{VA}_{K_N,N_0^\vee}$ *whose* $A[0]$ *eigenvalues lie in* M. *Denote the resulting algebra by* $\mathcal{VA}_{K_N,M}$.

Let $Box(K_N)$ *be the set of all elements* $n \in K_N$ *such that* $n - v_j \notin K_N$ *for all* j. *Equivalently,* $Box(K_N) = \{\sum_j p_j v_j \in K_N : 0 \le p_j < 1\}$. *For every* $n \in Box(K_N)$, *consider the following set of elements of* $Fock_{M\oplus n}$. *For every* $v = \prod A[\cdots]\prod B[\cdots]\prod \Phi[\cdots]\prod \Psi[\cdots]|m,0\rangle$ *that lies in* $\mathcal{VA}_{K_N,M} \subset Fock_{M\oplus 0}$, *consider* $v' = \prod A[\cdots]\prod B[\cdots]\prod \Phi[\cdots]\prod \Psi[\cdots]|m,n\rangle$ *which is obtained by applying the same modes of* A, B, Φ, *and* Ψ *to* $|m,n\rangle$ *instead of* $|m,0\rangle$. *We denote this space by* $\mathcal{VA}^{(n)}_{K_N,M}$.

Then the cohomology of $Fock_{M\oplus K_N}$ *with respect to* $D_{0,1}$ *is equal to*

$$Fock_{M\oplus K_N}/D_{0,1} = \oplus_{n\in Box(K_N)}\mathcal{VA}^{(n)}_{K_N,M}. \tag{3.10}$$

Basically, the differential $D_{0,1}$ preserves $\mathrm{Fock}_{M\oplus(n+\sum_j \mathbb{N}v_j)}$ for each $n \in Box(K_N)$. The cohomology $\mathrm{Fock}_{M\oplus(n+\sum_j \mathbb{N}v_j)}$ with respect to $D_{0,1}$ for various n all look like the cohomology of $\mathrm{Fock}_{M\oplus\sum_j \mathbb{N}v_j}$ with respect to $D_{0,1}$.

4. Elliptic genera of Berglund-Hübsch orbifolds

We aim to derive a formula for the supertrace of $y^{J[0]}q^{L[0]}$ on the cohomology of $\mathrm{Fock}_{M\oplus K_N}$ with respect to $D_{0,1}$. By (3.10), it is sufficient to compute the supertrace of $y^{J[0]}q^{L[0]}$ on each summand $\mathcal{VA}^{(n)}_{K_N,M}$. For $n=0$, $\mathcal{VA}_{K_N,M}$ consists of those from $\mathcal{VA}_{K_N,N_0^\vee}$ whose $A[0]$-eigenvalue lies in M. The vertex superalgebra $\mathcal{VA}_{K_N,N_0^\vee}$ is generated by the $2d$ bosonic fields $b^i(z)$, $a_i(z)$, and $2d$ fermionic fields $\phi^i(z)$, and $\psi_i(z)$ with the following OPE

$$a_i(z)b^j(w) \sim \frac{\delta_{ij}}{z-w}, \qquad \phi^i(z)\psi_j(w) \sim \frac{\delta_{ij}}{z-w}$$

(all other OPEs vanish). These fields (by the state-field correspondence) admit the following $(J[0], L[0])$-eigenvalues:

	$J[0]$	$L[0]$
$a_i(z)$	$-\deg^\vee \cdot v_i^\vee$	1
$b^i(z)$	$\deg^\vee \cdot v_i^\vee$	0
$\phi^i(z)$	$\deg^\vee \cdot v_i^\vee - 1$	1
$\psi_i(z)$	$-\deg^\vee \cdot v_i^\vee + 1$	0

Note that $\deg^\vee = \sum_{j=1}^d q_j v_j$ (see (2.15)), hence $\deg^\vee \cdot v_i^\vee = q_i$. The previous table becomes

	$J[0]$	$L[0]$
$a_i(z)$	$-q_i$	1
$b^i(z)$	q_i	0
$\phi^i(z)$	$q_i - 1$	1
$\psi_i(z)$	$-q_i + 1$	0

Now, the supertrace of $y^{J[0]}q^{L[0]}$ on $\mathcal{VA}_{K_N,N_0^\vee}$ can be computed as follows:

$$\text{(4.1)} \quad \mathrm{ST}_{\mathcal{VA}_{K_N,N_0^\vee}} y^{J[0]}q^{L[0]} = \prod_{i=1}^d \frac{\prod_{k\geq 0}(1-y^{-q_i+1}q^k)\prod_{k\geq 1}(1-y^{q_i-1}q^k)}{\prod_{k\geq 0}(1-y^{q_i}q^k)\prod_{k\geq 1}(1-y^{-q_i}q^k)}$$

The infinite products on the numerator come from the modes of $\phi^i(z)$ and $\psi_i(z)$; the products on the denominator come from the modes of $a_i(z)$ and $b^i(z)$. This expression involves rational powers of y and q. To extract the supertrace of $y^{J[0]}q^{L[0]}$ on the subalgebra $\mathcal{VA}_{K_N,M} \subset \mathcal{VA}_{K_N,N_0^\vee}$ from (4.1), we need to insert certain roots of 1 to eliminate the terms contributed by those in $\mathcal{VA}_{K_N,N_0^\vee}$ whose $A[0]$-eigenvalue lies outside M.

Recall the finite abelian group $G = N/N_0$. As a set, G is isomorphic to $\mathrm{Box}(K_N)$. For any $n_1 \in N$, we define

$$\text{(4.2)} \qquad \theta_j(n_1) = v_j^\vee \cdot n_1.$$

Consider the group algebra $\mathbb{C}[N_0^\vee] = \mathbb{C}[x_1^{\pm 1}, \dots, x_d^{\pm 1}]$. The variables (x_j) correspond to the basis $(v_j^\vee)$ of $N_0^\vee$. The group G acts on $\mathbb{C}[N_0^\vee]$ as follows: for any $n_1 \in N/N_0$, $m \in N_0^\vee$, we have

$$n_1.[m] = \exp(2\pi i(n_1 \cdot m))\,[m].$$

Then the G-invariant of $\mathbb{C}[N_0^\vee]$ is $\mathbb{C}[M]$, i.e. $\mathbb{C}[N_0^\vee]^G = \mathbb{C}[M]$. There is an "averaging over G" operation from $\mathbb{C}[N_0^\vee]$ to $\mathbb{C}[M]$ that we can use to obtain the supertrace of $y^{J[0]}q^{L[0]}$ on $\mathcal{VA}_{K_N,M}$, that is to insert $\frac{1}{|G|}\sum_{n_1\in G}\exp(2\pi i\, m \cdot n_1)$ in front of the term contributed by element

$$\prod A[\cdots]\prod B[\cdots]\prod \Phi[\cdots]\prod \Psi[\cdots]|m,0\rangle \in \mathcal{VA}_{K_N,N_0^\vee}.$$

Hence, we have

$$\begin{aligned}\text{(4.3)}\qquad &\mathrm{ST}_{\mathcal{VA}_{K_N,M}} y^{J[0]}q^{L[0]}\\ &= \frac{1}{|G|}\sum_{n_1\in G}\prod_{j=1}^{d}\frac{\prod_{k\geq 0}(1-y^{-q_j+1}q^k e^{-2\pi i\theta_j(n_1)})}{\prod_{k\geq 0}(1-y^{q_j}q^k e^{2\pi i\theta_j(n_1)})}\\ &\qquad\frac{\prod_{k\geq 1}(1-y^{q_j-1}q^k e^{2\pi i\theta_j(n_1)})}{\prod_{k\geq 1}(1-y^{-q_j}q^k e^{-2\pi i\theta_j(n_1)})}\end{aligned}$$

In general for $n\in \mathrm{Box}(K_N)$, the supertrace of $y^{J[0]}q^{L[0]}$ on $\mathcal{VA}^{(n)}_{K_N,M}$ is given by

$$\begin{aligned}\text{(4.4)}\qquad &\mathrm{ST}_{\mathcal{VA}^{(n)}_{K_N,M}} y^{J[0]}q^{L[0]}\\ &= (y^{-1}q)^{\deg\cdot n}\frac{1}{|G|}\sum_{n_1\in G}\prod_{j=1}^{d}\frac{\prod_{k\geq 0}(1-y^{-q_j+1}q^{k-\theta_j(n)} e^{-2\pi i\theta_j(n_1)})}{\prod_{k\geq 0}(1-y^{q_j}q^{k+\theta_j(n)} e^{2\pi i\theta_j(n_1)})}\\ &\qquad\frac{\prod_{k\geq 1}(1-y^{q_j-1}q^{k+\theta_j(n)} e^{2\pi i\theta_j(n_1)})}{\prod_{k\geq 1}(1-y^{-q_j}q^{k-\theta_j(n)} e^{-2\pi i\theta_j(n_1)})}\end{aligned}$$

The above is understood as a Laurent series in y, q with rational powers and non-negative powers of q. Indeed, each $\theta_j(n)$ lies in $[0,1)$. The powers of q that appear on the denominator are all non-negative, hence when the reciprocal of the denominator terms are expressed as a power series, only non-negative powers of q appear. On the numerator, the only term that could have a negative power of q is when $k=0$ in the first infinite product. However, we have an extra term $q^{\deg\cdot n}$ in the front, and the fact that $\deg\cdot n=\sum_j\theta_j(n)$ takes care of it. This double series converges absolutely when $|q|<|y^{q_j}q^{\theta_j(n)}|<1$ for all j. In fact, we can write it in terms of the theta function. Let

$$\text{(4.5)}\quad \Theta(\nu,\tau)=iq^{\frac{1}{8}}e^{-i\pi\nu}(1-e^{i2\pi\nu})\prod_{n=1}^{\infty}(1-q^n)(1-q^n e^{i2\pi\nu})(1-q^n e^{-i2\pi\nu})$$

be Jacobi's theta function where $q=e^{i2\pi\tau}$. It is a holomorphic function for $\nu\in\mathbb{C}$, $\tau\in H$ where H is the upper-half plane. If we fix $\tau\in H$, then $\Theta(\nu,\tau)$, as a function of ν, has single zeroes at all the lattice points in $\mathbb{Z}\tau+\mathbb{Z}$. Multiplying both the numerator and the denominator of (4.4) by

$\prod_{n=1}^{\infty}(1-q^n)$, we obtain

$$
\begin{aligned}
(4.6)\qquad & \mathrm{ST}_{\mathcal{VA}^{(n)}_{K_N,M}} y^{J[0]} q^{L[0]} \\
&= q^{\deg\cdot n}\frac{1}{|G|}\sum_{n_1\in G}\prod_{j=1}^{d} y^{-\theta_j(n)}\frac{e^{i\pi\{(1-q_j)z-\theta_j(n)\tau-\theta_j(n_1)\}}}{e^{i\pi\{q_j z+\theta_j(n)\tau+\theta_j(n_1)\}}} \\
&\qquad \frac{\Theta((1-q_j)z-\theta_j(n)\tau-\theta_j(n_1),\tau)}{\Theta(q_j z+\theta_j(n)\tau+\theta_j(n_1),\tau)} \\
&= q^{\deg\cdot n}\frac{1}{|G|}\sum_{n_1\in G} e^{i\pi z(d-2\sum_j q_j)} e^{-i2\pi\tau\sum_j\theta_j(n)} e^{-i2\pi\sum_j\theta_j(n_1)} \\
&\qquad \prod_{j=1}^{d} e^{-i2\pi z\theta_j(n)}\frac{\Theta(\cdots)}{\Theta(\cdots)}
\end{aligned}
$$

where $y=e^{i2\pi z}$, $q=e^{i2\pi\tau}$. Note that $q^{\deg\cdot n}$ cancels with $e^{-i2\pi\tau\sum_j\theta_j(n)}$. Moreover, $\sum_j\theta_j(n_1)=\deg\cdot n_1\in\mathbb{Z}$ because $\deg\in M$, $n_1\in N$, and M and N are dual lattices, hence $e^{-i2\pi\sum_j\theta_j(n_1)}=1$. We have

$$
\begin{aligned}
(4.7)\qquad & \mathrm{ST}_{\mathcal{VA}^{(n)}_{K_N,M}} y^{J[0]} q^{L[0]} \\
&= y^{\frac{1}{2}(d-2\sum_j q_j)}\frac{1}{|G|} \\
&\qquad \sum_{n_1\in G}\prod_{j=1}^{d} e^{-i2\pi z\theta_j(n)}\frac{\Theta((1-q_j)z-\theta_j(n)\tau-\theta_j(n_1),\tau)}{\Theta(q_j z+\theta_j(n)\tau+\theta_j(n_1),\tau)}.
\end{aligned}
$$

Note that

$$
(4.8)\qquad q_j = v_j^{\vee}\cdot\deg^{\vee} = \theta_j(\deg^{\vee})
$$

for $\deg^{\vee}\in N$. The number

$$
(4.9)\qquad \hat{c} = d-2\sum_j q_j = d-2\,\deg\cdot\deg^{\vee}
$$

is the central charge of the $N=2$ structure on $V_{1,1}$ ([B3]).

Theorem 4.1. *The elliptic genus of the Berglund-Hübsch Landau-Ginzburg orbifold W/G calculated in [BHe] is equal to $y^{-\frac{1}{2}\hat{c}}\mathit{SuperTrace}_{V_{1,1}} y^{J[0]} q^{L[0]}$.*

Proof. Consider the double complex $(\mathrm{Fock}_{M\oplus K_N}, D_{1,0}, D_{0,1})$ with bi-grading $(\deg^{\vee}\cdot A[0], \deg\cdot B[0])$. It lies in the upper half plane because $\deg\cdot B[0]$ has non-negative eigenvalues on K_N. The vertex algebra $V_{1,1}$ is the cohomology of the total complex. Consider the filtration of the total complex such that the E^0 terms of the associated spectral sequence is the cohomology of $\mathrm{Fock}_{M\oplus K_N}$ with respect to the (vertical) differential $D_{0,1}$. The filtration is bounded below and exhaustive, so the spectral sequence converges to the cohomology of the total complex. The cohomology of $\mathrm{Fock}_{M\oplus K_N}$ with respect to $D_{0,1}$ is described in Proposition 3.3. The $\deg\cdot B[0]$-grading on $\mathrm{Fock}_{M\oplus K_N}/D_{0,1}$ is bounded by d. Indeed, each summand $\mathcal{VA}^{(n)}_{K_N,M}$ in (3.10) has the $\deg\cdot B[0]$-grading equal to $\deg\cdot n$ which is $< d$ because n lies in the $\mathrm{Box}(K_N)$. Hence, the spectral sequence degenerates after finitely many steps. Since the differentials of the spectral sequence change parity and commute with $J[0]$ and $L[0]$, they have no effect on the supertrace. We have

$$\mathrm{SuperTrace}_{V_{1,1}} y^{J[0]}q^{L[0]} = \mathrm{SuperTrace}_{\mathrm{Fock}_{M\oplus K_N}/D_{0,1}} y^{J[0]}q^{L[0]}.$$

Finally, we sum up (4.7) over $n \in \mathrm{Box}(K_N) \cong G$. When multiplied with $y^{-\frac{1}{2}\hat{c}}$, it matches with the formulae (2.6), (2.7), and (2.14) of [BHe]. □

We denote the elliptic genus of W/G by

$$\begin{aligned}(4.10)\qquad & Ell(W/G, z, \tau)\\ &= \frac{1}{|G|}\sum_{n,n_1\in G}\prod_{j=1}^{d} e^{-i2\pi z\theta_j(n)}\frac{\Theta((1-q_j)z-\theta_j(n)\tau-\theta_j(n_1),\tau)}{\Theta(q_jz+\theta_j(n)\tau+\theta_j(n_1),\tau)}.\end{aligned}$$

Our next goal is to show that this is a weak Jacobi form. First, we establish the holomorphicity.

Theorem 4.2. *$Ell(W/G, z, \tau)$ is a holomorphic function of two variables for all $z \in \mathbb{C}$, $\tau \in H$.*

Proof. We will show explicitly with appeal to the classification of non-degenerate potentials that the zeroes of the theta functions on the denominator of (4.7) cancel with (some of) the zeroes of the theta functions on the numerator. Then it follows that the double series (4.4) converge absolutely to holomorphic functions for all $y \in \mathbb{C}^*$, $|q| < 1$.

Any non-degenerate potential W is a sum of decoupled potentials of three types: Fermat, loop, and chain. It is sufficient to prove the holomorphicity for W of each type. If $W = x^a$, $a > 2$ is of the Fermat type, then $q = 1/a$ and $\theta(n) \in (1/a)\mathbb{Z}$ for all $n \in G = \mathbb{Z}_a$. The zeroes of $\Theta(qz + \theta(n)\tau +$

$\theta(n_1), \tau)$ correspond to those z, τ such that

$$qz + \theta(n)\tau + \theta(n_1) \in \mathbb{Z}\tau + \mathbb{Z}.$$

When this is true, multiplying with $a-1$, we get

$$(1-q)z + (a-1)\theta(n)\tau + (a-1)\theta(n_1) \in \mathbb{Z}(a-1)\tau + \mathbb{Z}(a-1).$$

Since $a\theta(n), a\theta(n_1) \in \mathbb{Z}$, it follows that $(1-q)z - \theta(n)\tau - \theta(n_1) \in \mathbb{Z}\tau + \mathbb{Z}$, hence they are also zeroes of the numerator.

If $W = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{k-1}^{a_{k-1}}x_k + x_k^{a_k}x_1$ is of the loop type, then we have

$$\begin{aligned} a_iq_i + q_{i+1} &= 1 \quad \text{for } 1 \le i \le k-1, \\ a_kq_k + q_1 &= 1 \\ a_i\theta_i(n) + \theta_{i+1}(n) &\in \mathbb{Z} \quad \text{for } 1 \le i \le k-1, n \in \mathrm{Box}(K_N) \\ a_k\theta_k(n) + \theta_1(n) &\in \mathbb{Z} \quad \text{for } n \in \mathrm{Box}(K_N). \end{aligned}$$

The same arguments apply as in the Fermat case: the zeroes of $\Theta(q_jz + \theta_j(n)\tau + \theta_j(n_1), \tau)$ cancel with the zeroes of $\Theta((1-q_{j+1})z - \theta_{j+1}(n)\tau - \theta_{j+1}(n_1), \tau)$ for $1 \le j \le k-1$, and the zeroes of $\Theta(q_kz + \theta_k(n)\tau + \theta_k(n_1), \tau)$ cancel with the zeroes of $\Theta((1-q_1)z - \theta_1(n)\tau - \theta_1(n_1), \tau)$.

If $W = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{k-1}^{a_{k-1}}x_k + x_k^{a_k}$ is of the chain type, then

$$\begin{aligned} a_iq_i + q_{i+1} &= 1 \quad \text{for } 1 \le i \le k-1, \\ a_kq_k &= 1 \\ a_i\theta_i(n) + \theta_{i+1}(n) &\in \mathbb{Z} \quad \text{for } 1 \le i \le k-1, n \in \mathrm{Box}(K_N) \\ a_k\theta_k(n) &\in \mathbb{Z} \quad \text{for } n \in \mathrm{Box}(K_N). \end{aligned}$$

The previous argument fails to cancel the zeroes of $\Theta(q_kz + \theta_k(n)\tau + \theta_k(n_1), \tau)$. Instead, we will "distribute" its zeroes to each term of the numerator. Here is the mechanism of how this works, isolated. Suppose we have a ratio of theta functions

$$\frac{1}{\Theta(\frac{k}{ml}z + \alpha_3\tau + \beta_3, \tau)} \frac{\Theta(\frac{k}{m}z + \alpha_1\tau + \beta_1, \tau)}{\Theta(\frac{1}{m}z + \alpha_2\tau + \beta_2, \tau)} \tag{4.11}$$

where m, k, l are integers, α_i, β_i are rational numbers, $(m, k) = 1$, and $m\alpha_2 \in \mathbb{Z}$, $m\beta_2 \in \mathbb{Z}$. Moreover, $k\alpha_2 \equiv \alpha_1 \pmod{\mathbb{Z}}$, $k\beta_2 \equiv \beta_1 \pmod{\mathbb{Z}}$, $l\alpha_3 \equiv \alpha_1$ (mod

$\mathbb{Z}$), $l\beta_3 \equiv \beta_1 \pmod{\mathbb{Z}}$. The zeroes of the first theta function on the denominator lie on the lines

$$\frac{k}{ml}z + \alpha_3\tau + \beta_3 = p\tau + q, \quad p, q \in \mathbb{Z}$$

which is equivalent to

$$\frac{k}{m}z + l\alpha_3\tau + l\beta_3 = pl\tau + ql, \quad p, q \in \mathbb{Z}. \tag{4.12}$$

By assumption, this family of lines belong to the set of lines containing the zeroes of the numerator. Similarly, the second theta function on the denominator has zeroes on the lines

$$\frac{k}{m}z + k\alpha_2\tau + k\beta_2 = p'k\tau + q'k, \quad p', q' \in \mathbb{Z}. \tag{4.13}$$

These lines again coincide with some of the lines containing the zeroes of the numerator. If the two families of lines (4.12) and (4.13) have no intersection, then the zeroes of the denominator are all cancelled by the zeroes from the numerator, the ratio is therefore holomorphic. Otherwise, we have $\gcd(k,l)|(k\alpha_2 - l\alpha_3)$, $\gcd(k,l)|(k\beta_2 - l\beta_3)$. The lines in (4.13) that are not "eliminated" by the lines from the numerator are those with (p', q') such that

$$l|(p'k - k\alpha_2 + l\alpha_3), \qquad l|(q'k - k\beta_2 + l\beta_3). \tag{4.14}$$

Fix such a pair (p', q'), then every other such pair (p'', q'') satisfies that

$$p'' - p',\ q'' - q' \in \frac{l}{\gcd(k,l)}\mathbb{Z}.$$

Hence, we have the following lines remaining

$$\frac{1}{m}z + \alpha_2\tau + \beta_2 = (p' + \frac{l}{\gcd(k,l)}s)\tau + (q' + \frac{l}{\gcd(k,l)}t), \quad s, t \in \mathbb{Z}$$

or

$$\frac{\gcd(k,l)}{ml}z + \frac{\gcd(k,l)}{l}(\alpha_2 - p')\tau + \frac{\gcd(k,l)}{l}(\beta_2 - q') = s\tau + t, \quad s, t \in \mathbb{Z}.$$

Set

$$m^{\text{new}} = \frac{ml}{\gcd(k,l)}, \qquad \alpha_2^{\text{new}} = \frac{\gcd(k,l)}{l}(\alpha_2 - p'),$$
$$\beta_2^{\text{new}} = \frac{\gcd(k,l)}{l}(\beta_2 - q'),$$

then it is clear that $m^{\text{new}}\alpha_2^{\text{new}}, m^{\text{new}}\beta_2^{\text{new}} \in \mathbb{Z}$. The ratio (4.11) has the same poles as

$$\frac{1}{\Theta(\frac{1}{m^{\text{new}}}z + \alpha_2^{\text{new}}\tau + \beta_2^{\text{new}}, \tau)}. \tag{4.15}$$

Finally, consider $\Theta((1-\frac{k}{ml})z - \alpha_3\tau - \beta_3, \tau)$. We have

$$1 - \frac{k}{ml} = \frac{\frac{ml}{\gcd(k,l)} - \frac{k}{\gcd(k,l)}}{\frac{ml}{\gcd(k,l)}} =: \frac{k^{\text{new}}}{m^{\text{new}}}, \qquad (k^{\text{new}}, m^{\text{new}}) = 1. \tag{4.16}$$

Moreover,

$$k^{\text{new}}\alpha_2^{\text{new}} = m(\alpha_2 - p') - \frac{k(\alpha_2 - p')}{l} \equiv -\alpha_3 (\bmod \mathbb{Z}) \tag{4.17}$$

because $m\alpha_2 \in \mathbb{Z}$ by assumption and (4.14). Similarly, we have

$$k^{\text{new}}\beta_2^{\text{new}} = m(\beta_2 - q') - \frac{k(\beta_2 - q')}{l} \equiv -\beta_3 (\bmod \mathbb{Z}) \tag{4.18}$$

All of (4.15)–(4.18) will be the beginning of another round of the same arguments.

Now, let us see how this applies to prove the holomorphicity of $Ell(W/G, z, \tau)$ for W of the chain type. We need to examine

$$\cdots\cdots \frac{\Theta((1-q_{k-1})z - \theta_{k-1}(n)\tau - \theta_{k-1}(n_1), \tau)}{\Theta(q_{k-1}z + \theta_{k-1}(n)\tau + \theta_{k-1}(n_1), \tau)} \frac{\Theta((1-q_k)z - \theta_k(n)\tau - \theta_k(n_1), \tau)}{\Theta(q_k z + \theta_k(n)\tau + \theta_k(n_1), \tau)}.$$

Recall that $q_k = \frac{1}{a_k}$ and $a_k\theta_k(n), a_k\theta_k(n_1) \in \mathbb{Z}$. It is clear that the ratio of the last theta function on the numerator and the last two theta functions on the denominator satisfy the assumptions of the above discussion. We apply the above arguments finitely many times from right to left, in the end, we are able to cancel all zeroes of the denominator. $\square$

Theorem 4.3. *$Ell(W/G, z, \tau)$ is a weak Jacobi form of weight 0 and index $\frac{\hat{c}}{2}$.*

Weak here means that it obeys the transformation laws of the Jacobi forms, however at the cusp we require that only non-negative powers of q appear ([EZ]). Also, when $\hat{c}$ is odd, the definition of the Jacobi form is modified to allow a character.

Proof. The condition at the cusp holds because (4.4) has no negative powers of q. It is now enough to verify the following modular properties of $Ell(W/G, z, \tau)$:

$$Ell(W/G, z, \tau+1) = Ell(W/G, z, \tau) \tag{4.19}$$

$$Ell(W/G, z+1, \tau) = (-1)^{\hat{c}} Ell(W/G, z, \tau) \tag{4.20}$$

$$Ell(W/G, z+\tau, \tau) = (-1)^{\hat{c}} e^{-i\pi\hat{c}(\tau+2z)} Ell(W/G, z, \tau) \tag{4.21}$$

$$Ell\left(W/G, \frac{z}{\tau}, -\frac{1}{\tau}\right) = e^{\frac{i\pi\hat{c}z^2}{\tau}} Ell(W/G, z, \tau) \tag{4.22}$$

We need the following identities of the theta function:

$$\Theta(\nu, \tau+1) = \Theta(\nu, \tau) \tag{4.23}$$

$$\Theta(\nu+1, \tau) = -\Theta(\nu, \tau) \tag{4.24}$$

$$\Theta(\nu+\tau, \tau) = -e^{-i2\pi\nu - i\pi\tau}\Theta(\nu, \tau) \tag{4.25}$$

$$\Theta\left(\frac{\nu}{\tau}, -\frac{1}{\tau}\right) = -i\sqrt{\frac{\tau}{i}} e^{\frac{i\pi\nu^2}{\tau}}\Theta(\nu, \tau). \tag{4.26}$$

(4.19) follows from (4.23) and the change of variable $nn_1 \to n_1$ in (4.10). (4.20) follows from (4.24), $e^{-i2\pi\sum_j \theta_j(n)} = 1$ (because $\sum_j \theta_j(n) = \deg \cdot n \in \mathbb{Z}$), and the change of variable $\deg^\vee n_1 \to n_1$ (note that $q_j = \theta_j(\deg^\vee)$ and $\deg^\vee \in G$). Also, $\sum_j q_j \in \mathbb{Z}$, hence $\hat{c} = d \pmod 2$. (4.21) follows from (4.25), $\sum_j \theta_j(n_1) \in \mathbb{Z}$, and the change of variable $\deg^\vee\, n \to n$. (4.22) follows from (4.26) and change of variables $n \to n_1^{-1}$, $n_1 \to n$. □

Remark 4.4. It was shown in [BHe] that the elliptic genus $Ell(W/G)$ satisfy modular transformation properties with respect to $(z, \tau) \to (z, \tau + 1)$, $(z, \tau) \to (\frac{z}{\tau}, -\frac{1}{\tau})$ (same as here), and $(z, \tau) \to (z + L, \tau)$, $(z, \tau) \to (z + L\tau, \tau)$, where L is the smallest integer such that $g^L = \text{id}$ for all $g \in G$. The modularity here appears to be stronger. The reason is that we made the assumption $\langle J_W \rangle \subset G$, i.e. $\deg^\vee \in N$, so that we can combine $q_j = \theta_j(\deg^\vee)$ with $\theta_j(n)$ and do a change of variable. We also assumed that $G \subset SL_W$, or equivalently $\deg \in M$, then $\sum_j \theta_j(n) \in \mathbb{Z}$ for all $n \in G$. This enables us to reduce terms $\prod_j e^{i2\pi\theta_j(n)}$ to 1.

We now prove that the elliptic genera of mirror Berglund-Hübsch Landau-Ginzburg orbifolds are equal up to a sign.

Theorem 4.5. *$Ell(W/G, z, \tau) = (-1)^{\hat{c}} Ell(W^\vee/G^\vee, z, \tau)$.*

Proof. The Fock space $\mathrm{Fock}_{M\oplus N}$ and the differential $D_{1,1}$ are both symmetric with respect to the switching of M and N. The discrepancy of the cocycle (3.1) and its counterpart with the role of M and N switched can be resolved by multiplying $|m, n\rangle$ by $(-1)^{m\cdot n}$. However, to obtain the elliptic genus of the dual theory $W^\vee/G^\vee$ from the double-graded superdimension of $V_{1,1}$, we need to consider a different bi-grading $(J^*[0], L^*[0]) = (-J[0], L[0] + J[0])$. Then by Theorem 4.3, we have

$$\begin{aligned} Ell(W^\vee/G^\vee, y, q) &= y^{-\frac{\hat{c}}{2}} \mathrm{SuperTrace}_{V_{1,1}} y^{J^*[0]} q^{L^*[0]} \\ &= y^{-\frac{\hat{c}}{2}} \mathrm{SuperTrace}_{V_{1,1}} (y^{-1}q)^{J[0]} q^{L[0]} \\ &= y^{-\frac{\hat{c}}{2}} (y^{-1}q)^{\frac{\hat{c}}{2}} Ell(W/G, y^{-1}q, q) \\ &= y^{-\hat{c}} q^{\frac{\hat{c}}{2}} Ell(W/G, y^{-1}q, q). \end{aligned}$$

It remains to use the following transformation property of Ell:

$$Ell(W/G, -z+\tau, \tau) = (-1)^{\hat{c}} e^{-i\pi\hat{c}(\tau-2z)} Ell(W/G, z, \tau).$$

□

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