# Algebraic cycles and local quantum cohomology

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We review the Hodge theory of some classic examples from mirror symmetry, with an emphasis on what is intrinsic to the A-model. In particular, we illustrate the construction of a quantum $\mathbb{Z}$-local system on the cohomology of $K_{\mathbb{P}^2}$ and suggest how this should be related to the higher algebraic cycles studied in [10].

This note concerns three types of polarized variations of mixed Hodge structure (PVMHS) which arise in mirror symmetry:
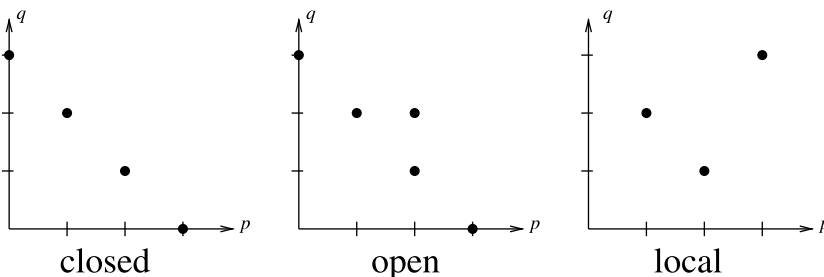
In each case, at the large complex structure boundary point one obtains a limiting mixed Hodge structure (LMHS) of Hodge–Tate type. It follows that replacing $W_\bullet$ by the relative weight filtration $M_\bullet$ produces a new PVMHS of the form
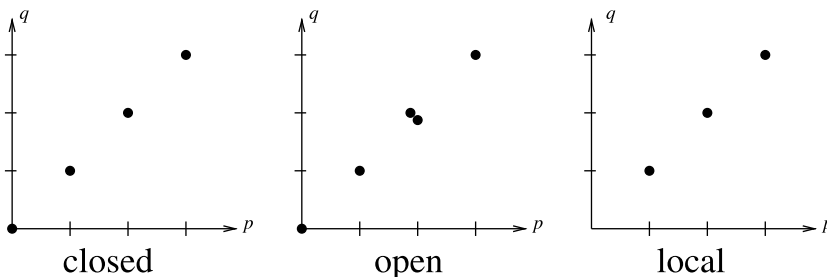
simultaneously in the A and B models. In particular, the $F^p \cap M_{2p}$ subspaces identify with $H^{3-p,3-p}$ in quantum cohomology.

Let Δ^* denote the punctured unit disk and write $\mathcal{O}_{\Delta^*} =: \mathcal{O}$, $\Omega^1_{\Delta^*} =: \Omega^1$. A PVMHS $(\mathbb{V}, \mathcal{V}, \mathcal{F}^\bullet, W_\bullet, \nabla, Q)$ over Δ^* comprises

- a $\mathbb{Z}$-local system $\mathbb{V}$ on Δ^*,

- the holomorphic vector bundle $\mathcal{V}$ with sheaf of sections $\mathbb{V} \otimes \mathcal{O}$,

- a decreasing filtration by holomorphic subbundles $\mathcal{F}^j \subset \mathcal{V}$,

- an increasing filtration by sublocal systems $W_i \subset \mathbb{V}_\mathbb{Q} := \mathbb{V} \otimes \mathbb{Q}$,

- a flat connection $\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega^1$ with $\nabla(\mathcal{F}^\bullet) \subset \mathcal{F}^{\bullet-1}$ and $\nabla(\mathbb{V}) = 0$ and

- bilinear forms $Q_i : \left(\mathrm{Gr}^W_i \mathbb{V}\right)^{\otimes 2} \to \mathbb{Z}$,

such that each $(\mathrm{Gr}^W_i \mathbb{V}_s, \mathrm{Gr}^W_i \mathcal{F}^\bullet_s, Q_i)$ $(s \in \Delta^*)$ yields a polarized Hodge structure. The PVMHS considered here, as well as all PVMHS arising from geometry, are *admissible* — i.e., have well-defined LMHS at 0.

In the above pictures, the number of bullets in position (p, q) signifies the dimension of the summand in the Deligne bigrading on $\mathcal{V}$ defined pointwise by

$$I^{p,q}(\mathcal{V}_s) := F^p \cap W_{p+q} \cap \left(\overline{F^q} + \sum_{j \geq 0} \left\{ \overline{F^{q-j-1}} \cap W_{p+q-j-2} \right\} \right).$$

This bigrading is uniquely determined by the properties

(1) $\oplus_{p \geq j} \oplus_q I^{p,q}(\mathcal{V}_s) = \mathcal{F}^j_s$,

(2) $\oplus_{p+q \leq i} I^{p,q}(\mathcal{V}_s) = (W_i)_s \otimes \mathbb{C}$,

(3) $\overline{I^{b,a}(\mathcal{V}_s)} \equiv I^{a,b}(\mathcal{V}_s)$ modulo $\oplus_{p<a} \oplus_{q<b} I^{p,q}(\mathcal{V}_s)$.

In passing to the limit, heuristically one may visualize the bullets in each line $p + q = i$ moving up and down in such a way that the end result remains symmetric about this line.

Notation: Set $\ell(s) := \frac{\log(s)}{2\pi i}$. We shall often write $\mathcal{V}$ (instead of the six-tuple) for a PVMHS.

1. Closed string

Beginning on the B-model side, recall how the LMHS construction works for a pure $(\mathbb{Z}\text{-})$VHS $\mathcal{V}$ of weight 3 over Δ^* with unimodular polarization

Q. The weight filtration is the trivial one $W_3 = \mathcal{V} \supset W_2 = \{0\}$. Denote the (unipotent part of the) monodromy operator by T, with nilpotent logarithm

$$N := \log(T) : \mathbb{V}_{\mathbb{Q}} \to \mathbb{V}_{\mathbb{Q}}.$$

There exists an unique filtration

$$M_{-1} = \{0\} \subset M_0 \subset M_1 \subset \cdots \subset M_6 = \mathbb{V}_{\mathbb{Q}}$$

satisfying $N(M_\alpha) \subset M_{\alpha-2}$ and $N^\ell : \mathrm{Gr}_{3+\ell}^M \overset{\cong}{\to} \mathrm{Gr}_{3-\ell}^M$. Untwisting the local system by

$$\tilde{\mathbb{V}} := e^{-\ell(s)N}\mathbb{V},$$

we obtain the canonical extension

$$\mathcal{V}_e := \tilde{\mathbb{V}} \otimes \mathcal{O}_\Delta.$$

Let $\{\gamma_i\}$ be a multivalued basis of $\mathbb{V}$ generating the steps of the integral filtration $M_m^{\mathbb{Z}} := \mathbb{V} \cap M_m$, and set $\tilde{\gamma}_i := e^{-\ell(s)N}\gamma_i \in \Gamma(\Delta, \tilde{\mathbb{V}})$.

Definition 1.1. The LMHS of $\mathcal{V}$, denoted informally V_{lim}, is given by the data $V_{\mathrm{lim}}^{\mathbb{Z}} := \mathbb{Z}\langle\{\tilde{\gamma}_i(0)\}\rangle$, $\mathcal{F}_{\mathrm{lim}}^\bullet := \mathcal{F}_e^p(0)$ and (monodromy weight) filtration $M_\bullet$ on $V_{\mathrm{lim}} := \mathcal{V}_e(0)$.

Assume that V_{lim} is *Hodge–Tate*, i.e., $\mathrm{Gr}_{2j}^M \cong \mathbb{Z}(-j)^{\oplus d_j}$ for $j = 0, 1, 2, 3$ and $\{0\}$ otherwise. (For example, the large complex structure LMHS for H^3 of the quintic mirror is of this type, whereas that for the Fermat quintic pencil[1] is *not*.) In the rank 4 setting, where we must have all $d_j = 1$, we may pick (for each j) a holomorphic section $e_j \in \Gamma(\Delta, \mathcal{F}_e^j \cap M_{2j}^{\mathbb{C}})$ mapping to the image of $\gamma_j \in \Gamma(\mathfrak{H}, M_{2j}^{\mathbb{Z}})$ in $\Gamma(\Delta^*, \mathrm{Gr}_{2j}^M \mathcal{V})$ hence generating the latter. (Here $\mathfrak{H}$ denotes the upper half-plane.) Write $e = \{e_3, e_2, e_1, e_0\}$ and $\gamma = \{\gamma_3, \gamma_2, \gamma_1, \gamma_0\}$ for the two bases.

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[1]The $t = 0$ LMHS $H_{\mathrm{lim}}^3(Y_t)$ of the pencil $Y_t := \left\{t\sum_{i=0}^4 X_i^5 = \prod_{i=0}^4 X_i\right\} \subset \mathbb{P}^4$ has nonzero Gr_3^W (cf. the analysis in [13, Section IV]).

To make things explicit, we have (for some $a, b \in \mathbb{Z}$ and $e, f \in \mathbb{Q}$)

$$(1.1) \quad [Q]_\gamma = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = [Q]_e \quad \text{and} \quad [N]_\gamma = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ e & b & 0 & 0 \\ f & e & -a & 0 \end{pmatrix}$$

(cf. [12]), in which we shall demand that $|a| = 1$. Replacing the local coordinate s by $q := e^{2\pi\sqrt{-1}\tau}$, where $\tau := Q(\gamma_1, e_3)$, and making full use of the bilinear relations (e.g., $Q(\mathcal{F}^1, \mathcal{F}^3) = 0 = Q(\mathcal{F}^2, \mathcal{F}^2)$), the limiting period matrix becomes (cf. [op. cit.])

$$(1.2) \qquad \tilde{\gamma}(0)[\mathbf{1}]_{e(0)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{f}{2} & e & 1 & 0 \\ \alpha_0 & \frac{f}{2} & 0 & 1 \end{pmatrix}.$$

Example 1.2. For the mirror quintic family, we have (cf. [op. cit.], where the computation is based on [1]) $a = -1$, $b = 5$, $e = \frac{11}{2}$, $f = -\frac{25}{6}$ and $\alpha_0 = \frac{25i}{\pi^3}\zeta(3) =: C$.

Following Deligne [9], the $e_j(q)|_{\Delta^*}$ provide the Hodge(-Tate) basis of a PVMHS $(\mathbb{V}, \mathcal{V}, \mathcal{F}^\bullet, M_\bullet, \nabla)$ on Δ^*, denoted $\mathcal{V}_{\text{rel}}$ for short. For the connection, we have

$$[\nabla]_e = d + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -Y(q) & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \otimes \frac{dq}{(2\pi\sqrt{-1})q},$$

where $Y(q)$ defines the Yukawa coupling. In the event that $\mathcal{V}$ comes from $H^3(X)$, and Φ denotes the Gromov–Witten prepotential of the mirror X° (composed with the inverse mirror map), according to mirror symmetry we have $Y = \Phi''' := \frac{d^3\Phi}{d\tau^3}$.

Example 1.3. The mirror quintic VHS arises from H^3 of X_ξ, which is a smooth compactification of

$$\left\{ 1 - \xi \left(\sum_{i=1}^{4} x_i + \frac{1}{\prod_{i=1}^{4} x_i} \right) = 0 \right\} \subset (\mathbb{C}^*)^{\times 4}.$$

Taking $s := \xi^5$, we obtain τ and q as above, and

$$\Phi(q) = \frac{5}{6}\tau^3 + \Phi_h(q),$$

where the holomorphic part

$$\Phi_h(q) = \frac{1}{(2\pi i)^3} \sum_{d \geq 1} N_d q^d.$$

From [1, 12, 23], we have the mixed Hodge basis

$$e_0 = \gamma_0,$$

$$e_1 = \gamma_1 - \tau\gamma_0,$$

$$e_2 = \gamma_2 - \left(5\tau + \frac{11}{2} + \Phi_h''\right)\gamma_1 + \left(\frac{5}{2}\tau^2 + \frac{25}{12} + \tau\Phi_h'' - \Phi_h'\right)\gamma_0,$$

$$e_3 = \gamma_3 + \tau\gamma_2 - \left(\frac{5}{2}\tau^2 + \frac{11}{2}\tau - \frac{25}{12} + \Phi_h'\right)\gamma_1$$

$$+ \left(\frac{5}{6}\tau^3 + \frac{25}{12}\tau - C + \tau\Phi_h' - 2\Phi_h\right)\gamma_0.$$

Here e_3 can also be viewed as the class of a holomorphic 3-form in the original VHS, whose LMHS is reflected by the presence of C. The mirror X° is the Fermat quintic.

Turning to the A-model, we need to define an integral structure, Hodge and weight filtrations on

$$H^{\mathrm{even}}(X^\circ) = H^{3,3} \oplus H^{2,2} \oplus H^{1,1} \oplus H^{0,0},$$

which will lead to VHS, LMHS and VMHS isomorphic to those on $H^3(X)$. These variations will be defined over a small disk $0 < |q| < \epsilon$. For constructing them, the general idea is to use the family of algebraic structures on H^{even} parametrized by $\tau[H] \in H^{1,1}(X^\circ)$, known as the *(small) quantum cohomology*. (Here $[H]$ the the class of a hyperplane section and $\tau = \ell(q)$, and we are working in the rank 4 setting.)

For the filtrations, we set

$$F^a H^{\mathrm{even}} = \oplus_{i \leq 3-a} H^{i,i}, \qquad M_b H^{\mathrm{even}} = \oplus_{j \geq 3-\frac{b}{2}} H^{j,j},$$

so that $\mathcal{F}^{3-k} \cap M_{6-2k} = H^{i,i}(X^\circ, \mathbb{C})$ as a subspace of H^{even}. This is where the "naive" fundamental classes of coherent sheaves or algebraic cycles of

codimension i lie. In contrast, the integral local system will be generated by *quantum-deformed* fundamental classes of algebraic cycles on X°. Alternately, we can regard the flat structure as given by the solution to a quantum differential equation

$$\nabla = d + E \otimes \frac{dq}{(2\pi\sqrt{-1})q},$$

which gives the integral structure up to a constant. (Note that d differentiates with respect to $\oplus_i H^{i,i}(X^\circ, \mathbb{C})$; E is an $\mathcal{O}(\Delta^*)$-linear operator we shall illustrate in the example below.) Since E kills M-graded pieces, we get a natural identification between Gr_{2i}^M of this "integral structure" and $H^{i,i}(X^\circ, \mathbb{Z})$.

Example 1.4. For X° the Fermat quintic, we have Hodge basis

$$[X^\circ] = e_3, \quad [H] = e_2, \quad -[L] = e_1, \quad [p] = e_0,$$

where H is a hyperplane section, L a line and p a point. The minus sign on $[L]$ ensures that the form

$$Q(\alpha, \beta) := (-1)^{\frac{\deg(\alpha)}{2}} \int_{X^\circ} \alpha \cup \beta$$

has matrix $[Q]_e$ as above, which is necessary for equality of *polarized* VHS.

For the quantum deformed classes, we invert the relations of Example 1.3 to obtain

$$[X^\circ]_\mathcal{Q} = \gamma_3 = [X^\circ] - \tau[H] + \left(\frac{5}{2}\tau^2 + \frac{25}{12} + \tau\Phi_h'' - \Phi_h' \right) [L]$$

$$+ \left(-\frac{5}{6}\tau^3 - \frac{25}{12}\tau + C - \tau\Phi_h' + 2\Phi_h \right) [p],$$

$$[H]_\mathcal{Q} = \gamma_2 = [H] - \left(5\tau + \frac{11}{2} + \Phi_h'' \right) [L] + \left(\frac{5}{2}\tau^2 + \frac{11}{2}\tau - \frac{25}{12} + \Phi_h' \right) [p],$$

$$[L]_\mathcal{Q} = -\gamma_1 = [L] - \tau[p],$$

$$[p]_\mathcal{Q} = \gamma_0 = [p].$$

These are solutions to the above differential equation with E given by the (small) quantum product $[H]*$ defined by

$$[H] * [X^\circ] = [H], \quad [H] * [H] = \Phi'''[L], \quad [H] * [L] = [p], \text{ and } [H] * [p] = 0.$$

(Note that this is consistent with cup product, in the sense that $[H] \cup [H] = 5[L] = \Phi'''(0)[L]$.) The resulting variations of HS on $H^{\text{even}}(X^\circ)$ and $H^3(X)$ match by construction.

The natural question at this point is: *how much of this "common $\mathbb{Z}$-VHS" is intrinsic to the A-model, and not just the B-model?* Clearly the issue lies not in the Hodge and monodromy weight filtrations (given by the grading of H^{even} by degree), or the polarizing form Q, or the ∇-flat complex local system (given by the quantum product), but in the integral structure on the latter. Another way to think of this (cf. [9]) is that we must determine the "constant of integration" of the VHS, or equivalently the LMHS (1.2).

Naively, one could try to find a basis δ of the local system with integral $[Q]_\delta$ and integral monodromy matrices (which are computable in principle by analytic continuation). Unfortunately the result may not be unique, even after identifying bases related by a rational symplectic matrix. In the above example, one could have

$$\delta_3 = \frac{\gamma_3}{\sqrt{5}} + \frac{\gamma_2}{\sqrt{5}}, \quad \delta_2 = \frac{\gamma_2}{\sqrt{5}} - \frac{3\gamma_1}{\sqrt{5}} - \frac{3\gamma_0}{\sqrt{5}}, \quad \delta_1 = \sqrt{5}\xi_1, \quad \delta_0 = \sqrt{5}\gamma_0,$$

which produces the (distinct) quintic *twin* mirror $\mathbb{Z}$-VHS. Indeed, in [11] this phenomenon is responsible for the bifurcation of each $\mathbb{R}$-VHS into finitely many distinct $\mathbb{Z}$-VHS.

Instead, what is needed is a direct construction of an integral structure on quantum cohomology, which has only recently been realized by Iritani [15, 16] and Katzarkov–Kontsevich–Pantev [18]. We illustrate how this works in the setting where X° is a smooth CY 3-fold, and $\dim H^{\text{even}}(X^\circ) = 4$. A map σ from H^{even} to multivalued ∇-flat sections (in a neighborhood of $q = 0$), defined in terms of Gromov–Witten theory, has been known for some time (cf. [7, Sections 8.5.3, 10.2.2]). If $\alpha_i \in H^{2(3-i)}(X^\circ)$ ($i = 0, 1, 2, 3$) denote a Q-symplectic basis with $\alpha_2 = [H]$, this boils down to first setting

$$\tilde{\sigma}(\alpha_0) := \alpha_0, \quad \tilde{\sigma}(\alpha_1) := \alpha_1, \quad \tilde{\sigma}(\alpha_2) := \alpha_2 + \Phi_h''\alpha_1 + \Phi_h'\alpha_0,$$
$$\tilde{\sigma}(\alpha_3) := \alpha_3 + \Phi_h'\alpha_1 + 2\Phi_h\alpha_0$$

and then

$$\sigma(\alpha) := \tilde{\sigma}\left(e^{-\tau[H]} \cup \alpha\right) := \sum_{k \geq 0} \frac{(-1)^k}{k!}\tilde{\sigma}\left([H]^k \cup \alpha\right).$$

(In our running example, we obviously have in mind $\alpha_3 = [X^\circ]$, $\alpha_2 = [H]$, $\alpha_1 = -[L]$, and $\alpha_0 = [p]$.) These are ∇-flat sections with monodromy

$$(1.3) \qquad T(\sigma(\alpha)) = \sigma\left(e^{-[H]} \cup \alpha\right).$$

We also set $\sigma_\infty(\alpha) := \tilde{\sigma}(\alpha)|_{q=0}$.

The key new ingredient introduced by the authors [15, 18] is a characteristic class defined using the Γ-function, and which in our setting specializes to

$$(1.4) \qquad \hat{\Gamma}(X^\circ) := \exp\left(\sum_{k \geq 2} \frac{(-1)^k(k-1)!}{(2\pi i)^k}\zeta(k)ch_k(TX^\circ)\right) \in H^{\text{even}}(X^\circ).$$

Using it, we may assign a flat section

$$(1.5) \qquad \gamma(\xi) := \sigma\left(\hat{\Gamma}(X^\circ) \cup ch(\xi)\right)$$

to each $\xi \in K_0^{\text{num}}(X^\circ)$, which defines a $\mathbb{Z}$-local system. (Similarly, we can define $\tilde{\gamma}(\xi)$, $\gamma_\infty(\xi)$ by applying $\tilde{\sigma}$, σ_∞.) A strong indication that $\hat{\Gamma}$ gives the right "correction" is Iritani's result (cf. [15, Proposition 2.10]) that the Mukai pairing

$$\langle \xi, \xi' \rangle := \int_{X^\circ} ch(\xi^\vee \otimes \xi') \cup Td(X^\circ) \ = \ Q(\gamma(\xi), \gamma(\xi')).$$

Moreover, since $ch(\mathcal{O}(-1)) = e^{-[H]}$, (1.3) implies that

$$T(\gamma(\xi)) = \gamma(\mathcal{O}(-1) \otimes \xi)$$

— an elementary example of how a categorical autoequivalence of $D^b(X^\circ)$ corresponds to monodromy. The autoequivalences corresponding to monodromies arising away from $q = 0$ have been explicitly identified in [5].

Example 1.5. Once more we take X° to be the Fermat quintic, which has total Chern class $c(X^\circ) = 1 + 50[L] - 200[p]$ and Todd class $Td(X^\circ) = 1 + \frac{25}{6}[L]$. A Mukai-symplectic basis of $K_0^{\text{num}}(X^\circ)$ is

$$\xi_3 := \mathcal{O}_{X^\circ}, \quad \xi_2 := \mathcal{O}_H - 3\mathcal{O}_L - 8\mathcal{O}_p, \quad \xi_1 := -\mathcal{O}_L - \mathcal{O}_p \equiv -\mathcal{O}_L(1),$$
$$\xi_0 := \mathcal{O}_p;$$

this in fact (referring to Example 1.2 and (1.1)) satisfies $[\mathcal{O}(-1)\otimes]_\xi = \exp([N]_\gamma)$. (Note that taking $\xi_2 = \mathcal{O}_H$ and $\xi_1 = \mathcal{O}_L$ does *not* yield a

symplectic basis.) From

$$ch(\xi_3) = [X^\circ], \quad ch(\xi_2) = [H] - \frac{11}{2}[L] - \frac{25}{6}[p], \quad ch(\xi_1) = -[L],$$
$$ch(\xi_0) = [p]$$

and $\hat{\Gamma}(X^\circ) = [X^\circ] + \frac{25}{12}[L] + C[p]$, a straightforward computation gives that

$$\gamma(\xi_i) = \gamma_i \quad (i = 0, 1, 2, 3),$$

with the $\{\gamma_i\}$ exactly as in Example 1.4. Moreover, the $\{\gamma_\infty(\xi_i)\}$ recover the LMHS matrix (1.2) (with e, f, α_0 as in Example 1.2), including the crucial constant C which visibly comes from $\hat{\Gamma}$.

Remark 1.6. The toric-hypersurface CY 3-fold families from which B-model VHS's are often produced are intrinsically defined over $\mathbb{Q}$. Moreover, by virtue of its toric nature, the large complex structure limit may be regarded as a $\mathbb{Q}$-semistable degeneration. The general conjectural framework surrounding the limiting motive (cf. [12, (III.B.5)]) therefore predicts that the class $\alpha_0 \in \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(-3), \mathbb{Q}(0)) \cong \mathbb{C}/\mathbb{Q}$ arising in the corresponding LMHS is always a rational multiple of the constant $\frac{\zeta(3)}{(2\pi i)^3}$, motivating its appearance in (1.4).[2]

The "non-toric" degenerations at the conifold and Gepner points, on the other hand, produce singular fibers whose desingularization may introduce an algebraic extension of $\mathbb{Q}$, leading to an arithmetically richer LMHS. One should try to use mirror symmetry to get at this, perhaps beginning with

PROBLEM 1.7. Adapt the (A-model) $\hat{\Gamma}$-integral structure on FJRW theory introduced in [5] to the explicit computation of the periods of (B-model) LMHS at the Gepner point ($s = \infty$).

Remark 1.8. The periods of an LMHS are the entries of a matrix (such as (1.2)) relating rational (or integral) and Hodge bases, which are chosen compatibly with the monodromy weight filtration.[3] Of greatest interest in Problem 1.7 (at least, when the monodromy about the Gepner point is not finite) is the explicit computation of nontorsion extension classes between

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[2]Note that we are interested in the arithmetic of locally complete CY families; taking irrational "slices" of such to force an extension both misses the point and will not affect α_0.

[3]For instance, one can take the $\{e_i(0)\}$ to give a basis of the Deligne bigrading of $(F_{\mathrm{lim}}^\bullet, M_\bullet)$.

the $\{\mathrm{Gr}_k^M\}$ — possibly in terms of Gamma- and zeta-values, hypergeometric special values and the like. While Section 4 (also see Corollary 1.4) of [op. cit.] does not carry this out, it appears to contain the technical background for doing so.

See Section 4 for another source of algebraic extensions.

2. Local string

This section is based on a simple example studied by [8, 10, 14, 21]. Once and for all we set

(2.1)
$$
Y_\xi := \left\{ (x, y; u, v) \in (\mathbb{C}^*)^2 \times \mathbb{C}^2 \, \bigg| \, 1 - \xi \left(x + y + \frac{1}{xy} \right) + u^2 + v^2 = 0 \right\},
$$

the so-called *Hori–Vafa mirror* of $Y^\circ = K_{\mathbb{P}^2}$. The canonical holomorphic $(3, 0)$ form on Y_ξ is given by

$$
\eta_\xi = 2\sqrt{-1}\mathrm{Res}_{Y_\xi} \left(\frac{\frac{dx}{x} \wedge \frac{dy}{y} \wedge du \wedge dv}{1 - \xi(x + y + \frac{1}{xy}) + u^2 + v^2} \right).
$$

The 3-cycles are spanned in homology by (a) a real 3-torus $\mathbb{T}^3$ and (b) circle-bundles over membranes in $(\mathbb{C}^*)^2$ bounding 1-cycles on the thrice-punctured elliptic curve $W_\xi^* := \left\{ (x, y) \in (\mathbb{C}^*)^2 \, \big| \, 1 - \xi(x + y + \frac{1}{xy}) = 0 \right\}$. The circle is pinched to a point over the 1-cycles.
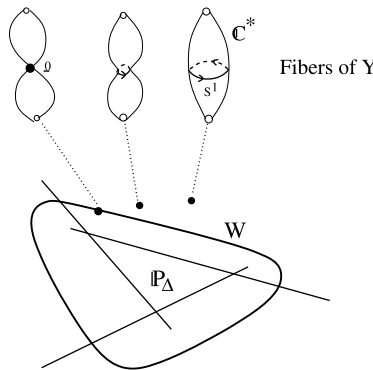
We write W_ξ for the complete elliptic curve, $\tilde{\omega}_\xi := \frac{1}{2\pi i}\mathrm{Res}_{W_\xi} \left(\frac{\frac{dx}{x} \wedge \frac{dy}{y}}{1 - \xi(x + y + \frac{1}{xy})} \right)$ for the canonical holomorphic 1-form, and φ_0, φ_1 for 1-cycles spanning H_1

$(W_\xi, \mathbb{Z})$ with periods $\pi_i := \int_{\varphi_i} \tilde{\omega}_\xi$. In particular, we let φ_0 be the vanishing cycle and $\omega_\xi := \tilde{\omega}_\xi / \pi_0$ the normalization of the 1-form so that $\int_{\varphi_0} \omega_\xi \equiv 1$.

Denoting the membrane construction (b) by $\mathcal{M}$, we have the short exact sequence

$$0 \longrightarrow \mathbb{Z}\langle \mathbb{T}^3 \rangle \longrightarrow H_3(Y) \longrightarrow \ker\left\{ H_1(W^*) \to H_1((\mathbb{C}^*)^2) \right\}(1) \longrightarrow 0$$

$$\overset{\mathcal{M}}{} \qquad \overset{\cong}{\downarrow}$$

$$H_1(W)(1)$$

(cf. [DK, Section 5]).[4] Its dual

$$0 \longleftarrow \mathbb{Z}(-3) \longleftarrow H^3(Y) \overset{\mu}{\longleftarrow} H^1(W)(-1) \longleftarrow 0$$

yields an extension class

$$\varepsilon \in \mathrm{Ext}^1_{\mathrm{MHS}}\left(\mathbb{Z}(-2), H^1(W)\right) \cong \mathrm{Hom}\left(H_1(W), \mathbb{C}/\mathbb{Z}(2)\right).$$

Miraculously, this is the image of a higher cycle $\Xi \in K_2^{\mathrm{alg}}(W)$ by a generalized Abel–Jacobi map [10], and the periods of η may be described by

$$\frac{1}{2\pi\sqrt{-1}} \int_{\mathcal{M}(\gamma)} \eta \underset{\mathbb{Z}(2)}{\equiv} \langle AJ(\Xi), \gamma \rangle_W, \quad \frac{1}{(2\pi\sqrt{-1})^3} \int_{\mathbb{T}^3} \eta = 1.$$

Normalizing the local coordinate $s := \xi^3$ to q where

$$\ell(q) := \tau := \frac{\pi_1}{\pi_0} = \int_{\varphi_1} \omega_\xi,$$

we remark that $s \mapsto q$ gives the mirror map for the family W of elliptic curves. Similarly, if we set

$$\ell(Q) := \mathcal{T} := \frac{1}{(2\pi\sqrt{-1})^3} \int_{\mathcal{M}(3\varphi_0)} \eta,$$

then $s \mapsto Q$ is the local mirror map for Y. The initial VMHS $\mathcal{V}$ is that on $H^3(Y)$, with integral basis[5] $\gamma = \{\gamma_3, \gamma_2, \gamma_1\}$ where

$$\gamma_3 := \mathbb{T}^\vee, \quad \gamma_2 = \mathcal{M}(3\varphi_0)^\vee, \quad \gamma_1 = \mathcal{M}(3\varphi_1)^\vee.$$

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[4]The isomorphism is valid only rationally, but can be made integral by replacing $H_1(W, \mathbb{Z})$ by $\mathbb{Z}\langle 3\varphi_0, \varphi_1 \rangle$, which is done tacitly below.

[5]We will ignore for now the fact that γ_1 is really $\frac{1}{3}$ of an integral class; it is a more convenient choice for our purposes than $\mathcal{M}(\varphi_1)^\vee$.

From the exact sequence we can read off the weight filtration

$$W_6 = \mathcal{V} \supset W_5 = W_4 = W_3 = \langle \gamma_2, \gamma_1 \rangle = \mathrm{im}\{\mu\} \supset W_2 = \{0\},$$

and Hodge filtration (except for $\mathcal{F}^3 = \langle \eta \rangle$). The extension data are recorded by $\mathcal{T} = \langle AJ(\Xi), 3\varphi_0 \rangle$ and $\Phi := \langle AJ(\Xi), 3\varphi_1 \rangle$.

The monodromy logarithm

$$[N]_\gamma = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ \frac{1}{2} & -1 & 0 \end{pmatrix}$$

leads to a relative weight filtration $M_\bullet$. The resulting $\mathcal{V}_{\mathrm{rel}}$ has Hodge–Tate basis

$$e_3 := \frac{\eta}{(2\pi\sqrt{-1})^3} = \gamma_3 + \mathcal{T}\gamma_2 + \Phi\gamma_1 \in \mathcal{F}^3 \cap M_6,$$

$$e_2 := \mu(\omega) = \gamma_2 + \tau\gamma_1 \in \mathcal{F}^2 \cap M_4,$$

$$e_1 \ = \gamma_1 \in \mathcal{F}^1 \cap M_2.$$

From transversality

$$\gamma_2 + \frac{d\Phi}{d\mathcal{T}}\gamma_1 = \nabla_{\partial_\mathcal{T}} e_3 \in \mathcal{F}^2,$$

we deduce that $\frac{d\Phi}{d\mathcal{T}} = \tau$, which may also be derived from the fact that logarithmic derivatives of the extension classes give periods[6] of $\tilde{\omega}_\xi$ [op. cit.]:

$$\frac{d\Phi}{d\mathcal{T}} = \frac{s \cdot d\Phi/ds}{s \cdot d\mathcal{T}/ds} = \frac{\pi_1}{\pi_0} = \tau.$$

This equality has the important consequence

$$\Phi'' := \frac{d^2\Phi}{d\mathcal{T}^2} = \frac{d\tau}{d\mathcal{T}} = \frac{\delta_s(\pi_1/\pi_0)}{\delta_s\mathcal{T}} = \frac{2\pi\sqrt{-1}(\pi_0\delta_s\pi_1 - \pi_1\delta_s\pi_0)}{\pi_0^3} = \frac{\mathcal{Y}}{\pi_0^3},$$

where $\mathcal{Y}$ is the (suitably normalized) Yukawa coupling for the family $\{W_\xi\}$ of elliptic curves. Noting as well that $\nabla_{\partial_\mathcal{T}} e_2 = \frac{d\tau}{d\mathcal{T}} e_1$, we conclude that

$$[\nabla]_e = d + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \Phi'' & 0 \end{pmatrix} \otimes \frac{dQ}{(2\pi\sqrt{-1})Q},$$

where $e = \{e_3, e_2, e_1\}$.

---

[6]That is, we have $\delta_s\mathcal{T} = \frac{1}{2\pi i}\pi_0$, $\delta_s\Phi = \frac{1}{2\pi i}\pi_1$.

Turning to the A-model, we shall seek a quantum interpretation of ∇. Before doing so, we remark that by [14] and [10], under the local mirror map Φ may be identified as the local Gromov–Witten prepotential

$$(2.2) \qquad \Phi \equiv \frac{1}{2}\mathcal{T}^2 - \frac{1}{(2\pi\sqrt{-1})^2}\sum_d 3dN_d Q^d.$$

modulo lower-order terms in $\mathcal{T}$.[7] Differentiating (2.2) twice, we have

$$1 - \sum_d 3d^3 N_d Q^d = \frac{\mathcal{Y}}{\pi_0^3},$$

in which the right-hand side has a pole where the family W degenerates. Directly computing $\langle AJ(\Xi), \varphi_0\rangle$ at this singular elliptic curve gives $\Im(\mathcal{T}_0) = \frac{27\sqrt{3}}{8\pi^2}L(\chi_{-3}, 2)$ [10], and hence $Q_0 = |e^{2\pi\sqrt{-1}\mathcal{T}_0}| = e^{-2\pi\Im(\mathcal{T}_0)}$ for the radius of convergence. This ties the asymptotic growth rate

$$\limsup_{d\to\infty} |N_d|^{\frac{1}{d}} = e^{2\pi\Im(\mathcal{T}_0)}$$

of the local Gromov–Witten numbers directly to the Beilinson regulator of an algebraic cycle.

For the quantum interpretation, we consider the dual VMHS $\mathcal{V}^\vee$ on $H_3(Y)$ under the pairing $H^3(Y) \times H_3(Y) \to H_0(Y) = \mathbb{Z}$. The dual integral (flat) basis is of course

$$\gamma_1^\vee = \mathbb{T}^3, \quad \gamma_2^\vee = \mathcal{M}(3\varphi_0), \quad \gamma_1^\vee = \mathcal{M}(3\varphi_1),$$

and in the dual Hodge basis $e^\vee = \{e_3^\vee, e_2^\vee, e_1^\vee\}$ we have

$$(2.3) \qquad [\nabla]_{e^\vee} = d - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \Phi'' \\ 0 & 0 & 0 \end{pmatrix} \otimes \frac{dQ}{2\pi\sqrt{-1}Q}.$$

Now recalling that $Y^\circ = K_{\mathbb{P}^2}$, Hosono [14] proposed a homological mirror map

$$\mathrm{mir}: K_0^c(Y^\circ) \to H_3(Y, \mathbb{Z})$$

---

[7]A different form of this result is already present in Section 6.2 of [8], about which we shall say more in the next section.

from coherent sheaves with compact support to homology classes of Lagrangian 3-cycles, given explicitly by

$$(2.4) \qquad \mathcal{O}_p \mapsto \gamma_3^\vee, \quad \mathcal{O}_{\mathbb{P}^1}(-1) \mapsto \gamma_2^\vee, \quad \mathcal{O}_{\mathbb{P}^2}(-2) \mapsto \gamma_1^\vee.$$

(The sheaves are all supported on the zero-section $\mathbb{P}^2 \subset Y^\circ$.) Making the identifications $e_3^\vee = [p]$, $e_2^\vee = [\mathbb{P}^1]$, $e_1^\vee = [\mathbb{P}^2]$ under $\overline{\mathrm{mir}} : H_{\mathrm{even}}(Y^\circ) \overset{\cong}{\to} H_3$ (Y), we impose as before an integral structure on the A-model side by means of the quantum deformed classes

$$([p] =) [p]_\mathcal{Q} := \gamma_3^\vee, \quad [\mathbb{P}^1]_\mathcal{Q} := \gamma_2^\vee, \quad [\mathbb{P}^2]_\mathcal{Q} := \gamma_1^\vee.$$

Together with the filtrations $W_{-6} = W_{-5} = W_{-4} = \langle [p] \rangle \subset W_{-3} = H_{\mathrm{even}}$, and $\langle [p] \rangle = \mathcal{F}^{-3} \cap M_{-6}$, $\langle [\mathbb{P}^1] \rangle = \mathcal{F}^{-2} \cap M_{-4}$, $\langle [\mathbb{P}^2] \rangle = \mathcal{F}^{-1} \cap M_{-2}$, this determines the A-model (relative) variation matching that on the B-model.

Finally, consider the formal quantum product

$$(2.5) \qquad \begin{array}{llll} e_1^\vee * e_3^\vee = 0, & e_1^\vee * e_2^\vee = -3e_3^\vee, & e_1^\vee * e_1^\vee = -3\Phi'' e_2^\vee, \\ e_2^\vee * e_3^\vee = 0, & e_3^\vee * e_3^\vee = 0, & e_2^\vee * e_2^\vee = 0, \end{array}$$

where we continue to identify classes under $\overline{\mathrm{mir}}$. This is compatible with the ordinary cup product in the sense that

$$e_1^\vee \cup e_3^\vee = [\mathbb{P}^2] \cup [p] = 0,$$
$$e_1^\vee \cup e_2^\vee = [\mathbb{P}^2] \cup [\mathbb{P}^1] = (\mathbb{P}^2 \cdot \mathbb{P}^1)_{Y^\circ}[p] = -3[p] = -3e_3^\vee, \quad \text{and}$$
$$e_1^\vee \cup e_1^\vee = [\mathbb{P}^2] \cup [\mathbb{P}^2] = -3[\mathbb{P}^1] = -3e_2^\vee,$$

the last of which contains the leading term of $-3\Phi'' = -3 + \cdots$.

PROPOSITION 2.1. *With the product (2.5), (2.3) may be rewritten*

$$\nabla = d + \left(\frac{1}{3} e_1^\vee * \right) \otimes \frac{dQ}{2\pi \sqrt{-1} Q}$$

in terms of the quantum product with the zero-section $\mathbb{P}^2 \subset K_{\mathbb{P}^2}$.

This motivates the following

PROBLEM 2.2. Develop a general theory of quantum cohomology for the local setting recovering $W_\bullet$ and ∇ on $H_{\mathrm{even}}(Y^\circ)$ as above. In particular, this

will entail the construction of dual admissible PVMHS on H^{even} and H_{even} for the total space of the canonical line bundle over a toric Fano orbifold.

We will obtain a solution for our running example in the next section. Remark that the required ∇ is defined in passing in [17, Section 2.5], which we shall comment in below (see Remark 3.2).

The Abel–Jacobi maps from [10] touched on above may be viewed as maps from $K_2^{\text{alg}}(W) = K_2(\text{Coh}(W))$ to $(\mathbb{C}/\mathbb{Z}(2)$-valued) functionals on (classes of) Lagrangian 1-cycles on W. Noting that W° is also an elliptic curve, we propose

PROBLEM 2.3. Derive (in general) a homological mirror to AJ. This would produce a "symplectic regulator" map from $K_2(\text{Fuk}(W^\circ))$ to functionals on coherent sheaves on W°. The functional mirroring the AJ class in our example would send $\mathcal{O}_p \mapsto \frac{(2\pi\sqrt{-1})^2}{3}\mathcal{T}$ and $\mathcal{O}_{W^\circ} \mapsto \frac{(2\pi\sqrt{-1})^2}{3}\Phi$.

The motivation for such a quantum AJ map is clear: it would bring Beilinson's conjectures directly to bear upon the arithmetic of Gromov–Witten (GW) invariants, in the context of the A-model VHS on quantum cohomology. A first step might be to construct, in our example, a mirror in $K_2(\text{Fuk}(W^\circ))$ to the toric symbol $\{x, y\} \in K_2^{\text{alg}}(W)$ (i.e., the higher cycle), by representing $K_2^{\text{alg}}(W)$ using the Quillen category of $\text{Coh}(W)$ and applying homological mirror symmetry for elliptic curves.

3. Closed to local

We begin by summarizing a computation from [8]. The setting is a 2-parameter family X_{ξ_1,ξ_2} of $h^{2,1} = 2$ CY 3-folds over a product $(\Delta^*)^2$ of punctured disks, with $\hat{\eta} \in \Omega^3(X)$. The mirror ($h^{1,1} = 2$) CY has an elliptic fibration

$$X^\circ \xrightarrow{\bar{\rho}} \mathbb{P}^2$$

with

- zero-section $D_2 \cong \mathbb{P}^2$,
- a line $C_2 \cong \mathbb{P}^1 \subset D_2$ with preimage $D_1 = \bar{\rho}^{-1}(C_2)$ and
- a fiber $C_1 = \bar{\rho}^{-1}(p)$.

We will use the bases

$$\begin{cases} J_1 = [D_2] + 3[D_1], & J_2 = [D_1] & \text{for } H^{1,1}(X^\circ), \\ C_1, & C_2 & \text{for } H^{2,2}(X^\circ), \end{cases}$$

which are dual under cup product. The period vector for $\hat{\eta}$ takes the form

$$\left(\Pi_0, \tau_1 \Pi_0, \tau_2 \Pi_0, \partial_{\tau_1}\tilde{\Phi}, \partial_{\tau_2}\tilde{\Phi}, 2\tilde{\Phi} - \delta_{\tau_1}\tilde{\Phi} - \delta_{\tau_2}\tilde{\Phi} \right),$$

where Π_0 is the "holomorphic period" and

$$\tilde{\Phi} := \tfrac{3}{2}\tau_1^3 + \tfrac{3}{2}\tau_1^2\tau_2 + \tfrac{1}{2}\tau_1\tau_2^2 + \left\{ \tfrac{17}{4}\tau_1 + \tfrac{3}{2}\tau_2 + C \right\} + \tfrac{1}{(2\pi\sqrt{-1})^3}\sum_{d_1,d_2}\tilde{N}_{d_1,d_2}q_1^{d_1}q_2^{d_2}$$

is the prepotential.[8] Here, $q_j = e^{2\pi\sqrt{-1}\tau_j}$ $(j = 1, 2)$ are the disk-coordinates[9] and $\tilde{N}_{d_1,d_2}$ is the GW invariant of the class $d_1[C_1] + d_2[C_2]$ on X°; the Kähler class is simply $\tau_1 J_1 + \tau_2 J_2$.

Now we take $\tau_1 \to i\infty$ $(q_1 \to 0)$ considered as the "large volume limit" for the fibers of $\bar{\rho}$. For the purposes of GW theory on the A-model, in this limit X° is equivalent to the total space of $\mathcal{N}_{D_2/X^\circ} \cong K_{\mathbb{P}^2}$, i.e., Y° in the last section (with the map $\rho : Y^\circ \twoheadrightarrow \mathbb{P}^2$). On the B-model, which we shall henceforth ignore, the periods remaining finite are Π_0, $\tau_2 \Pi_0$, and

$$(3.1) \quad (\partial_{\tau_1} - 3\partial_{\tau_2})\tilde{\Phi} = \frac{1}{2}\tau_2^2 - \frac{1}{4} + \frac{1}{(2\pi\sqrt{-1})^2}\sum_{d_1,d_2}\tilde{N}_{d_1,d_2}(d_1 - 3d_2)q_1^{d_1}q_2^{d_2}.$$

Indeed, actually taking the limit of (3.1) (and writing $\mathcal{T} := \tau_2$, $Q := e^{2\pi\sqrt{-1}\mathcal{T}}$, $N_d := \tilde{N}_{0,d}$) defines the local prepotential

$$\Phi_{\text{loc}} := \frac{1}{2}\mathcal{T}^2 - \frac{1}{4} - \frac{1}{(2\pi\sqrt{-1})^2}\sum_d 3dN_d Q^d$$

in agreement with (2.2).[10]

The next step is to consider the limit of the quantum products of classes in $H^{\text{even}}(X^\circ)$ which come from $H_c^{\text{even}}(Y^\circ)(\cong H_{\text{even}}(Y^\circ))$, namely $[p]$, $[C_2]$,

---

[8]This is the usual GW prepotential *plus* the bracketed lower-order correction terms.

[9]That is, the mirror map sends $(\xi_1, \xi_2) \mapsto (q_1, q_2)$ [op. cit., Section 6.1] and we may replace the former by the latter as coordinates on Δ^2.

[10]In fact, by a computation in [14], $\Phi = \Phi_{\text{loc}} - \frac{1}{2}\mathcal{T} + \frac{1}{2}$.

and

$$[D_2] = J_1 - 3J_2.$$

In general, the only interesting products (not given by the cup product) are

$$J_j * J_k = \sum_\ell \left(\partial_{\tau_j} \partial_{\tau_k} \partial_{\tau_\ell} \tilde{\Phi} \right) [C_\ell].$$

So (using (3.1)) we have

$$[D_2] * [D_2] = (\partial_{\tau_1} - 3\partial_{\tau_2})^2 \left(\partial_{\tau_1} \tilde{\Phi}[C_1] + \partial_{\tau_2} \tilde{\Phi}[C_2] \right)$$

$$= -3[C_2] + \sum_{d_1,d_2} \tilde{N}_{d_1,d_2}(d_1 - 3d_2)^2(d_1[C_1] + d_2[C_2])q_1^{d_1}q_2^{d_2},$$

whereupon taking the limit

$$\lim_{q_1 \to 0}[D_2] * [D_2] = \left\{ -3 + \sum_d N_d(-3d)^2 dQ^d \right\} [C_2]$$

$$= -3 \left\{ 1 - \sum_d 3d^3 N_d Q^d \right\} [C_2]$$

gives $[\mathbb{P}^2] * [\mathbb{P}^2] = -3\Phi''[\mathbb{P}^1]$, which is exactly what we wanted.

This makes a case for the general principle that the "local restriction" of the quantum product in a closed CY should remain finite under an appropriate large volume limit. Beyond establishing this, a solution to Problem 2.2 would have to show the result is consistent with a formula of the shape[11]

$$(3.2) \qquad \alpha *_{\text{loc}} \beta := \sum_k \sum_d \langle \alpha, \beta, \phi^k \rangle_{0,3,d}^{\text{loc}} \phi_k e^{\langle d, \mathcal{T} \rangle}$$

for $\alpha, \beta \in H_{\text{even}} \cong H_c^{\text{even}}$, $\mathcal{T} \in H^2$, $d \in H_2$, and ϕ^k resp. ϕ_k dual bases of H^{even} resp. H_{even}.

The resulting local quantum cohomology would then provide a direct A-model approach to "most" of the variation of mixed Hodge structure (the $\{I^{p,q}\}$ and ∇-flat structure), leaving only the

---

[11] For $Y^\circ \cong K_{\mathbb{P}} \to \mathbb{P}$ with $\mathbb{P}$ a toric Fano surface, negativity of $K_{\mathbb{P}}$ allows us to express the local invariants as closed invariants $\langle \alpha, \beta, \phi^k \rangle_{0,3,\iota_*(d)}$ for $\overline{Y^\circ} := \mathbb{P}(\mathcal{O} \oplus K_{\mathbb{P}}) \overset{\iota}{\supset} Y^\circ$; cf. [6, Section 9].

PROBLEM 3.1. Extend Iritani's construction of an integral structure on ∇-flat sections of $H_{\mathrm{even}}(Y^\circ)$ to the local CY setting (with $Y^\circ = K_{\mathbb{P}_\Delta}$).

This is easily accomplished in our running example by "taking LMHS along $q_1 = 0$" of the $\mathbb{Z}$-VHS over $(\Delta^*)^2$ (common to both the A- and B-models). More precisely, if T_1 denotes the monodromy about $q_1 = 0$, with logarithm N_1, then the limiting variation of MHS takes the form
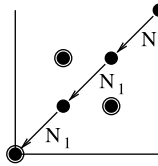
where the circled bullets denote $\ker(N_1) = \ker(T_1 - \mathrm{id})$. For our purposes, it will suffice to compute the limit of the T_1-invariant "cycles" in the $\hat{\Gamma}$-integral structure on the closed A-model VHS $H^{\mathrm{even}}(X^\circ)$.

Indeed, together with the Clemens–Schmid sequence, the assumption that "Y° is the A-model limit of X°" implies that

$$0 \to H_3(Y)(-2) \longrightarrow \lim_{q_1 \to 0} H^3(X)(1) \xrightarrow{N_1} \lim_{q_1 \to 0} H^3(X) \longrightarrow H^3(Y) \to 0$$

$$\Big\| \qquad\qquad \Big\| \qquad\qquad \Big\| \qquad\qquad \Big\|$$

$$0 \to H_{\mathrm{even}}(Y^\circ)(-2) \longrightarrow \lim_{q_1 \to 0} H^{\mathrm{even}}(X^\circ)(1) \xrightarrow{N_1} \lim_{q_1 \to 0} H^{\mathrm{even}}(X^\circ) \longrightarrow H^{\mathrm{even}}(Y^\circ) \to 0$$

is an exact sequence of VMHS (in q_2). Iritani's procedure necessarily gives integral ∇-flat sections $\{\hat{\gamma}_i\}_{i=1}^6$ in $H^{\mathrm{even}}(X^\circ)$, with $\{\hat{\gamma}_4, \hat{\gamma}_5, \hat{\gamma}_6\} \subset \mathrm{im}(N_1)$ and $\{\hat{\gamma}_1^\vee, \hat{\gamma}_2^\vee, \hat{\gamma}_3^\vee\} \subset \ker(N_1)$, such that[12]

$$[X^\circ]\left(= \frac{\hat{\eta}}{\Pi_0}\right) = \hat{\gamma}_1 + \tau_2 \hat{\gamma}_2 + \left\{(\partial_{\tau_1} - 3\partial_{\tau_2})\tilde{\Phi}\right\}\hat{\gamma}_3 + \sum_{j=4}^6 \hat{\pi}_j(\underline{\tau})\hat{\gamma}_j.$$

---

[12] Here the $\{\pi_j(\underline{\tau})\}$ are defined by the equation, and the $\{\hat{\gamma}_i\}$ exist by virtue of the agreement between $\gamma(K_0^{\mathrm{num}}(X^\circ))$ (cf. (1.5)) and the B-model $\mathbb{Z}$-structure. (The authors do not know explicit $\xi_i \in K_0^{\mathrm{num}}(X^\circ)$ with $\gamma(\xi_i) = \hat{\gamma}_i$; whereas by [15] these must exist, the argument here uses only what we know from the B-model, and so does not require them.)

Taking the limit whilst killing $\mathrm{im}(N_1)$, then making the change of basis[13] $\{\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3\} =: \{\gamma_1 + \frac{1}{2}\gamma_3, \gamma_2 - \frac{1}{2}\gamma_3, \gamma_3\}$, recovers

$$e_3 = \gamma_1 + \mathcal{T}\gamma_2 + \Phi\gamma_3$$

in $H^{\mathrm{even}}(Y^\circ)$.

Of course, in analogy to (3.2), it would be better to solve Problem 3.1 in a manner intrinsic to the local A-model. That is, there should be a direct construction as in (1.5) assigning flat sections of $H_{\mathrm{even}}(Y^\circ)$ to classes in $K_0^c(Y^\circ)$, and "compatible with monodromy". In our example, (2.4) has this compatibility, since $\otimes\mathcal{O}_{Y^\circ}(-J_2)$ on the coherent sheaves and monodromy about $q = 0$ on the cycles have the same matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Remark 3.2. As mentioned above, Iritani [15, Section 2.5] briefly describes an extension of the Dubrovin connection and $\hat{\Gamma}$–$\mathbb{Z}$-structure which should provide such a "direct" solution to Problem 3.1. Whether this produces an *admissible* $\mathbb{Z}$-PVMHS, or (more precisely) matches the VMHS on the middle cohomology of the Hori–Vafa mirror Y, remains to be verified. It would, for example, suffice to exhibit a degenerating family of CY n-folds X_{ξ_1,ξ_2} over Δ^2 with isomorphisms (a) $H^n(X_{0,\xi}) \cong H_n(Y_\xi)(-n)$ and (b) $H_{\mathrm{even}}(Y^\circ)(-n) \cong \ker(N_1) \subseteq \lim q_1 \to 0 H^{\mathrm{even}}(X^\circ)$ of VMHS. Here (a) is a geometric/Hodge-theoretic calculation on the B-model side, while (b) is a compatibility check for the A-model $\mathbb{Z}$-structures in the compact and noncompact cases.

Apparently, either solution to Problem 3.1 still leaves us a long way from the "holy grail" of Problem 2.3.

4. Open string

Problem 2.3 is probably intractable without major theoretical developments. However, its rough analog in the *relative* situation studied by Morrison and Walcher [22] appears to be more accessible. In particular, there is nothing mysterious about the mirror of the (usual, not higher) algebraic cycle — it is just a Lagrangian.

---

[13]Note that Clemens–Schmid is only *rationally* exact, and that N_1, N_2 are only defined over $\mathbb{Q}$.

The B-model in the example we consider (following [op. cit.]) comprises:

- $X =$ a double-cover of the mirror quintic family, with holomorphic form $\omega \in \Omega^3(X)$;

- $Z \in CH^2(X)_{\text{hom}}$ a family of algebraic 1-cycles (for analogy to Section 2, think "$K_0(\text{Coh}(X))$"); and

- $\langle AJ_X^2(Z), \omega \rangle =$ the resulting "truncated normal function", solving

- the inhomogeneous Picard–Fuchs equation $D_{\text{PF}}^\omega \langle AJ_X^2(Z), \omega \rangle =: g$.

On the A-model side these data mirror to:

- $X^\circ =$ the Fermat quintic;

- $Z^\circ \cong \mathbb{RP}^3$ the real quintic, viewed as a Lagrangian 3-cycle (think "$K_0(\text{Fuk}(X))$"); and

- the open Gromov–Witten generating function whose coefficients count holomorphic disks bounding on Z°,

which (under the mirror map) solves the same PF equation.

As in the closed and local stories, GW numbers are therefore obtained as power-series coefficients of a Hodge-theoretic function, with (in this latter role) the Yukawa coupling replaced by the truncated normal function.

PROBLEM 4.1. Work out (in analogy with Sections 1 and 2) the $[\nabla]_e$ story. This will require the *full* normal function (not considered in [op. cit.]), which means computing also $\langle AJ_X^2(Z), \nabla_{\partial_\tau} \omega^{3,0} \rangle$.

Since the B-model VMHS is an extension of the constant variation $\mathbb{Z}(-2)$ by the pure VHS $H^3(X)$, the extension class is defined over $\mathbb{R}$ hence given completely by $\langle AJ_X^2(Z), \omega^{3,0} \rangle$ and $\langle AJ_X^2(Z), \nabla_{\partial_\tau} \omega^{3,0} \rangle$. The extension arises geometrically from the residue exact sequence

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H^3(X) & \longrightarrow & H^3(X \setminus |Z|) & \longrightarrow & \ker\left(H^4_{|Z|}(X) \to H^4(X)\right) & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & H^3(X) & \longrightarrow & \mathbb{E} & \longrightarrow & \mathbb{Q}(-2) & \longrightarrow & 0
\end{array}
$$

Completely missing, however, is an approach to the following.

PROBLEM 4.2. Can one produce the extension class from the pair X°, Z° from the standpoint of quantum cohomology and the A-model VMHS?

To illustrate its difficulty, a naive attempt to mirror the exact sequence approach, viz.

$$0 \to \frac{H^{\mathrm{even}}(X^\circ)}{H^6_{Z^\circ}(X^\circ)} \to H^{\mathrm{even}}(X^\circ \backslash Z^\circ) \to \ker\left(H^3_{Z^\circ}(X^\circ) \to H^3(X^\circ)\right) \to 0,$$

fails due to the vanishing of the third term. The result of [op. cit.], however, that the "truncated" extension class is given by the open GW generating function, gives one reason to believe the problem has interesting content.

Remark 4.3. We briefly note another interesting phenomenon that arises in the open setting, related to Remark 1.6. Even on a family of CY 3-folds defined over $\mathbb{Q}$, algebraic cycles often force an algebraic extension $L/\mathbb{Q}$ upon us, as in the case of the van Geemen lines on the mirror quintic family studied by Laporte and Walcher [19]. The resulting limits of truncated normal functions can then often be expressed in terms of the Borel regulator on $K_3^{\mathrm{ind}}(L)$ (see [13] for the theoretical reason). This makes the open setting ideal terrain for exploring generalizations of the A-model $\hat{\Gamma}$-construction where the B-model LMHS does not correspond to a $\mathbb{Q}$-rational limiting motive.

4.1. Local to open

Recent work of Chan, Lau, Leung, Tseng and Wu [3, 4, 20] has brought to light an interesting relation between the (local) mirror map and certain open Gromov–Witten invariants for a toric Calabi–Yau manifold Y°. The first three authors conjecture in [3] that the SYZ mirror construction (applied to Y°) inverts the mirror map given by a normalized integral basis of single-log-divergent periods of the Hori–Vafa mirror Y. With the integrality hypothesis dropped, the conjecture is established in [4] for $Y^\circ = K_Z$ with Z a compact toric Fano variety; it is known integrally for toric surfaces [20] and a handful of other examples [3], including $K_{\mathbb{P}^2}$.

We briefly describe the case $Y^\circ = K_{\mathbb{P}^2}$ in the notation of Section 2. Let ℓ be the class of a line $L\,(\cong \mathbb{P}^1)$ contained in the zero section $D\,(\cong \mathbb{P}^2) \subset Y^\circ$, and $\mathcal{T}[\rho^{-1}(L)] \in H^2(Y^\circ, \mathbb{C})$ be the Kähler class with corresponding Kähler parameter $Q = e^{2\pi i \mathcal{T}}$. Take β_0 to denote the class of a holomorphic disk with $\beta_0 \cdot D = 1$ bounding on a Lagrangian torus (see [3, Section 4.2.3] for the precise definition). Then the SYZ construction in [3] produces the noncompact

Calabi–Yau in $(\mathbb{C}^*)^2 \times \mathbb{C}^2$ given by

$$(4.1) \qquad\qquad UV = c(Q) + X + Y + \frac{Q}{XY},$$

where $c(Q) = 1 + \sum_{k \geq 1} n_{\beta_0 + k\ell} Q^k$ is an open Gromov–Witten generating series. An easy change of coordinates exhibits (4.1) as the Hori–Vafa manifold Y_ξ of (2.1), with $\xi = -\frac{Q^{\frac{1}{3}}}{c(Q)}$; taking the cube gives

$$(4.2) \qquad\qquad s(Q) = -\frac{Q}{c(Q)^3}.$$

The observation of [op. cit.] is that (4.2) inverts the local mirror map

$$Q(s) = e^{2\pi\sqrt{-1}\mathcal{T}(s)} = \exp\left(\frac{1}{(2\pi\sqrt{-1})^2} \int_{\mathcal{M}(3\varphi_0)} \eta \right)$$

in Section 2. So just as for Φ, we have an enumerative interpretation for $\mathcal{T}$, and one can use the computation[14]

$$\mathcal{T}(s) = \ell(s) + \frac{1}{2} + \frac{1}{2\pi\sqrt{-1}} \sum_{k \geq 1} (-1)^k \frac{\binom{3k}{k,k,k}}{k} s^k$$

in [3] or [10] to compute $c(Q) = 1 - 2Q + 5Q^2 - 32Q^3 + \cdots$.

We conclude with one final

PROBLEM 4.4. Can one use the formulae in Section 5 of [10] for the integral periods of Hori–Vafa mirrors, to establish integrality in [4]?

Remark 4.5. While this paper was under review, Problem 4.4 was answered affirmatively for $Y^\circ = K_Z$ a threefold (cf. the footnote in [2, Section 1]). We expect that the relevant computations in [10, Section 5] generalize to dimensions $n > 3$ as well; the more interesting question is whether similar constructions exist for more general semi-projective toric Calabi–Yau varieties (not of the form K_Z).

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[14]up to the sign and term $\frac{1}{2}$ which are required for consistency with Section 2 and [14].

Acknowledgments

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