communications in
number theory and physics
Volume 8, Number 3, 541–588, 2014

Invariants of spectral curves and intersection theory of moduli spaces of complex curves

B. Eynard

To any spectral curve $\mathcal{S}$, we associate a topological class $\hat{\Lambda}(\mathcal{S})$ in a moduli space $\mathcal{M}^{\mathfrak{b}}_{g,n}$ of "$\mathfrak{b}$–colored" stable Riemann surfaces of given topology (genus g, n boundaries), whose integral coincides with the topological recursion invariants $W_{g,n}(\mathcal{S})$ of the spectral curve $\mathcal{S}$. This formula can be viewed as a generalization of the ELSV formula (whose spectral curve is the Lambert function and the associated class is the Hodge class), or Mariño–Vafa formula (whose spectral curve is the mirror curve of the framed vertex, and the associated class is the product of three Hodge classes), but for an arbitrary spectral curve. In other words, to a B-model (i.e., a spectral curve) we systematically associate a mirror A-model (integral in a moduli space of "colored" Riemann surfaces). We find that the mirror map, i.e., the relationship between the A-model moduli and B-model moduli, is realized by the Laplace transform.

1. Introduction

In the past few years, many developments have unearthed a deep and fascinating relationship between integrable systems, algebraic geometry, combinatorics, enumerative geometry and random matrices, and much is yet to be understood.

In particular, an integrable system can be encoded by its "spectral curve" (i.e., the locus of eigenvalues of a Lax operator), which is a plane analytical curve embedded in $\mathbb{C}\times\mathbb{C}$, for example given by its equation, like $y=\sin(\sqrt{x})$, or $\mathrm{e}^x = y\,\mathrm{e}^{-y}$.

In [13] (led by recent developments in random matrix theory), it was proposed to define a sequence of "invariants" $W_n^{(g)}$, $g=0,1,2,\dots,$ $n=0,1,2,\dots$ associated with a spectral curve. For many specific examples of spectral curves, those invariants had an enumerative geometry interpretation, as "counting" surfaces in a moduli space of Riemann surfaces.

For example:

– For the spectral curve $\mathbf{y} = \sqrt{\mathbf{x}}$, the $W_n^{(g)}$'s are the generating functions of Witten–Kontsevich intersection numbers on the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of genus g with n marked points [20]. Writing $z_i = \sqrt{x_i}$ we have

$$W_n^{(g)} = (-2)^{2g-2+n} \sum_{d_1+\dots+d_n=3g-3+n} \left\langle \prod_{i=1}^n \tau_{d_i} \right\rangle_{\mathcal{M}_{g,n}} \prod_{i=1}^n \frac{(2d_i+1)!!\, dz_i}{z_i^{2d_i+2}}.$$

– For the spectral curve $\mathbf{y} = \frac{\mathbf{1}}{\mathbf{2}\pi} \sin(\mathbf{2}\pi\sqrt{\mathbf{x}})$, the $W_n^{(g)}$'s are the Laplace transforms of Weil–Petersson volumes of the moduli spaces $\mathcal{M}_{g,n}$ (see [14, 26]). Writing $z_i = \sqrt{x_i}$ we have

$$\begin{aligned} W_n^{(g)} &= \prod_{i=1}^n \int_0^\infty L_i dL_i\, \mathrm{e}^{-z_i L_i}\ \mathrm{Vol}(\mathcal{M}_{g,n}(L_1,\dots,L_n)) \\ &= (-2)^{2g-2+n} \sum_{d_0+d_1+\dots+d_n=3g-3+n} \frac{1}{d_0!} \left\langle (2\pi^2\kappa_1)^{d_0} \prod_{i=1}^n \tau_{d_i} \right\rangle_{\mathcal{M}_{g,n}} \\ &\quad \times \prod_{i=1}^n \frac{(2d_i+1)!!\ dz_i}{z_i^{2d_i+2}} \\ &= (-2)^{2g-2+n} \sum_{d_1,\dots,d_n} \left\langle \mathrm{e}^{2\pi^2\kappa_1} \prod_{i=1}^n \tau_{d_i} \right\rangle_{\mathcal{M}_{g,n}} \prod_{i=1}^n \frac{(2d_i+1)!!\ dz_i}{z_i^{2d_i+2}}. \end{aligned}$$

– For the spectral curve $\mathrm{e}^{\mathbf{x}} = \mathbf{y}\,\mathrm{e}^{-\mathbf{y}}$, it was proposed by Bouchard and Mariño [6] that the $W_n^{(g)}$'s are the generating functions of simple Hurwitz numbers $\mathcal{H}_{g,\mu}$, which was then proved in [4, 12]. And simple Hurwitz numbers can themselves be translated into integrals of the Hodge class $\Lambda(1)$ in the moduli space of curves, through ELSV formula [9]:

$$\begin{aligned} W_n^{(g)} &= (-1)^n \sum_{\mu\,,\,\ell(\mu)=n} \mathcal{H}_{g,\mu} \prod_{i=1}^n \mu_i\, \mathrm{e}^{-\mu_i\, x_i}\, dx_i \\ &= (-1)^n \sum_{\mu_1,\dots,\mu_n} \left\langle \Lambda(1) \prod_{i=1}^n \frac{\mu_i}{1-\mu_i\psi_i} \right\rangle_{\mathcal{M}_{g,n}} \prod_{i=1}^n \frac{\mu_i^{\mu_i}}{\mu_i!}\, \mathrm{e}^{-\mu_i\, x_i}\, dx_i\,. \end{aligned}$$

– More generally, it was initially proposed by Mariño [23] and refined by Bouchard–Klemm–Mariño–Pasquetti (BKMP) [5], that if we choose the spectral curve to be the mirror curve of a toric Calabi–Yau 3–fold $\mathfrak{X}$, then the

$W_n^{(g)}$'s should be generating functions for the Gromov–Witten invariants of $\mathfrak{X}$, i.e., they enumerate maps from a Riemann surface of genus g into $\mathfrak{X}$ with n boundaries into a Lagrangian submanifold of $\mathfrak{X}$. This BKMP conjecture was proved to low genus for many spaces $\mathfrak{X}$, and to all genus only for the case $\mathfrak{X} = \mathbb{C}^3$ in [7, 32].

Then, if a spectral curve is not among the list of "known examples" (the list above is not exhaustive, there are also more known examples associated with matrix models [13], combinatorics of maps and combinatorics of two-dimensional (2D) and 3D partitions, counting Grothendieck's dessins d'enfants [25], ...), the natural question is:

– do the $W_n^{(g)}$'s of an arbitrary spectral curve have a meaning in terms of enumerative geometry, counting the "volume of some moduli space of surfaces"?

In [11] we proposed a partial answer for all spectral curves having only one branchpoint. The goal of the present paper is to extend that to an arbitrary number of branchpoints. We shall thus define a compact moduli space $\overline{\mathcal{M}}^{\mathfrak{b}}_{g,n}$ of "colored" Riemann surfaces of genus g with n marked points, and some (cohomology class of) differential forms $\hat\Lambda(\mathcal{S})$ and $\hat B(z, 1/\psi)$ on it, so that $W_n^{(g)}$ is indeed an integral on that moduli-space.

We shall prove in Theorem 4.1 that:

$$\boxed{W_n^{(g)}(\mathcal{S}; z_1, \ldots, z_n) = 2^{\dim \mathcal{M}^{\mathfrak{b}}_{g,n}} \int_{\overline{\mathcal{M}}^{\mathfrak{b}}_{g,n}} \hat\Lambda(\mathcal{S}) \prod_{i=1}^{n} \hat B(z_i, 1/\psi_i),} \tag{1.1}$$

where notations are explained below.

We shall see that the term $\hat B(z_i, 1/\psi_i)$ or more precisely its Laplace transform $\int \mathrm{e}^{-\mu_i x(z_i)} \hat B(z_i, 1/\psi_i)$ is the analog of $\frac{\mu_i}{1+\mu_i\psi_i}$ in the ELSV formula, and that it indeed reduces to it if we choose the spectral curve to be the Lambert function. In some sense our formula generalizes the ELSV formula.

The moduli space $\overline{\mathcal{M}}^{\mathfrak{b}}_{g,n}$, defined in Section 4.3, is a "combinatorial" moduli space, whose strata are labelled by $\mathfrak{b}$– colored dual graphs of nodal surfaces. In other words, labelled by degenerations of Riemann surfaces, together with a color.

2. Definition and notations: spectral curves and their invariants

The goal of this paper is to show that invariants of an arbitrary spectral curve can be written in terms of intersection numbers, so we first recall the definition of symplectic invariants and their descendants.

Definition 2.1 (Spectral curve). A spectral curve $\mathcal{S} = (\mathcal{C}, x, y, B)$, is the data of:

- a Riemann surface $\mathcal{C}$ (not necessarily compact nor connected);
- two analytical function $x : \mathcal{C} \to \mathbb{C}$, $y : \mathcal{C} \to \mathbb{C}$;
- a Bergman kernel B, i.e., a symmetric 2nd kind bilinear meromorphic differential, having a double pole on the diagonal and no other pole, and normalized (in any local coordinate z) as:

$$B(z_1, z_2) \underset{z_2 \to z_1}{\sim} \frac{dz_1 \otimes dz_2}{(z_1 - z_2)^2} + \text{analytical}. \tag{2.1}$$

Moreover, the spectral curve $\mathcal{S}$ is called regular if the meromorphic form dx has a finite number of zeroes on $\mathcal{C}$, and they are simple zeroes, and dy does not vanish at the zeroes of dx. In other words, locally near a branchpoint a, y behaves like a square root of x:

$$y(z) \underset{z \to a}{\sim} y(a) + y'(a)\,\sqrt{x(z) - x(a)} + O(x(z) - x(a)), \qquad y'(a) \neq 0. \tag{2.2}$$

From now on, all spectral curves considered shall always be chosen to be regular[1].

In [13], it was proposed how to associate with a regular spectral curve, an infinite sequence of symmetric meromorphic n-forms, and a sequence of complex numbers $F_g(\mathcal{S})$. The definition is given by a recursion, often called "topological recursion", which we recall:

Definition 2.2 (Invariants $W_n^{(g)}(\mathcal{S})$). Let $\mathcal{S} = (\mathcal{C}, x, y, B)$ be a regular spectral curve. Let $a_1, \ldots, a_{\mathfrak{b}}$ be its branchpoints (zeroes of dx in $\mathcal{C}$).

[1] A generalized definition of symplectic invariants for non-regular spectral curves was also introduced in [30], but for simplicity, we consider only regular spectral curves here.

We define

$$W_1^{(0)}(\mathcal{S};z) = y(z)\,dx(z), \tag{2.3}$$

$$W_2^{(0)}(\mathcal{S};z_1,z_2) = B(z_1,z_2), \tag{2.4}$$

and for $2g-2+(n+1)>0$:

(2.5)

$$W_{n+1}^{(g)}(\mathcal{S};z_1,\dots,z_n,z_{n+1}) = \sum_{i=1}^{\mathfrak{b}} \operatorname*{Res}_{z\to a_i} K(z_{n+1},z)\Big[W_{n+2}^{(g-1)}(z,\bar z,z_1,\dots,z_n) + \sum_{h=0}^{g} \sum_{I\sqcup J=\{z_1,\dots,z_n\}}' W_{1+\#I}^{(h)}(z,I)\,W_{1+\#J}^{(g-h)}(z,J)\Big],$$

Where the prime in $\sum_h \sum'_{I\sqcup J}$ means that we exclude from the sum the terms $(h=0, I=\emptyset)$ and $(h=g, J=\emptyset)$, and where $\bar z$ means the other branch of the square-root in (2.2) near a branchpoint a_i, i.e., if z is in the vicinity of a_i, $\bar z \neq z$ is the other point in the vicinity of a_i such that

$$x(\bar z) = x(z), \tag{2.6}$$

and thus $y(\bar z) \sim y(a) - y'(a)\sqrt{x(z)-x(a)}$. The recursion kernel $K(z_{n+1},z)$ is defined as

$$K(z_{n+1},z) = \frac{\int_{z'=\bar z}^{z} B(z_{n+1},z')}{2(y(z)-y(\bar z))\,dx(z)} \tag{2.7}$$

K is a 1-form in z_{n+1} defined on $\mathcal{C}$ with a simple pole at $z_{n+1}=z$ and at $z_{n+1}=\bar z$, and in z it is the inverse of a 1-form, defined only locally near branchpoints, and it has a simple pole at $z=a_i$.

We also define for $g\geq 2$:

(2.8)

$$F_g(\mathcal{S}) = W_0^{(g)}(\mathcal{S}) = \frac{1}{2-2g}\sum_{i=1}^{\mathfrak{b}} \operatorname*{Res}_{z\to a_i} W_1^{(g)}(\mathcal{S};z)\left(\int_{z'=a_i}^{z} y(z')dx(z')\right).$$

With this definition, $F_g(\mathcal{S})\in\mathbb{C}$ is a complex number associated with $\mathcal{S}$, sometimes called the gth symplectic invariant of $\mathcal{S}$, and $W_n^{(g)}(\mathcal{S};z_1,\dots,z_n)$ is a symmetric multilinear differential $\in T^*(\mathcal{C})\otimes\cdots\otimes T^*(\mathcal{C})$, sometimes called the nth descendant of F_g. Very often we denote $F_g = W_0^{(g)}$. If $2-2g-n<0$,

$W_n^{(g)}$ is called stable, and otherwise unstable, the only unstable cases are $F_0, F_1, W_1^{(0)}, W_2^{(0)}$. For $2-2g-n<0$, $W_n^{(g)}$ has poles only at branchpoints (when some z_k tends to a branchpoint a_i), without residues, and the degrees of the poles are $\leq 6g+2n-4$.

It is also possible to define F_0 and F_1, see [13], but we shall not use them here.

Those invariants F_g and $W_n^{(g)}$'s have many fascinating properties, in particular related to integrability, to modular functions, and to special geometry, and we refer the reader to [13, 15].

3. Intersection numbers

Our goal now is to relate those $W_n^{(g)}$'s to intersection numbers in moduli spaces of curves, so let us first introduce basic concepts.

3.1. Definitions

Let $\mathcal{M}_{g,n}$ be the moduli space of complex curves of genus g with n marked points. It is a complex orbifold (manifold quotiented by a group of symmetries), of dimension

$$\dim \mathcal{M}_{g,n} = d_{g,n} = 3g-3+n. \tag{3.1}$$

Each element $(\Sigma, p_1, \dots, p_n) \in \mathcal{M}_{g,n}$ is a smooth complex curve Σ of genus g with n smooth distinct marked points $p_1, \dots, p_n$. $\mathcal{M}_{g,n}$ is not compact because the limit of a family of smooth curves may be non-smooth, some cycles may shrink, or some marked points may collapse in the limit. The Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of $\mathcal{M}_{g,n}$ also contains stable nodal curves of genus g with n marked points (a nodal curve is a set of smooth curves glued at nodal points, and thus nodal points are equivalent to pairs of marked points, and stability means that each punctured component curve has an Euler characteristics <0), see figure 1. $\overline{\mathcal{M}}_{g,n}$ is then a compact space.

Let $\mathcal{L}_i$ be the cotangent bundle at the marked point p_i, i.e., the bundle over $\overline{\mathcal{M}}_{g,n}$ whose fiber is the cotangent space $T^*(p_i)$ of Σ at p_i. It is customary to denote its first Chern class:

$$\psi_i = \psi(p_i) = c_1(\mathcal{L}_i). \tag{3.2}$$

ψ_i is (the cohomology equivalence class modulo exact forms, of) a 2-form on $\overline{\mathcal{M}}_{g,n}$. Since $\dim_{\mathbb{R}} \overline{\mathcal{M}}_{g,n} = 2\dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,n} = 6g-6+2n$, it makes sense to

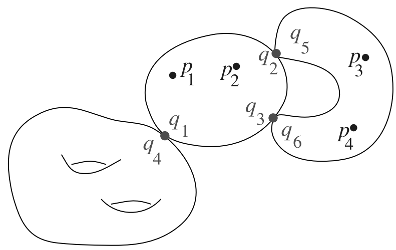


Figure 1: A stable curve in $\overline{\mathcal{M}}_{g,n}$ can be smooth or nodal. Here we have an example in $\overline{\mathcal{M}}_{3,4}$ of a stable curve of genus $g = 3$, with $n = 4$ marked points $p_1, \ldots, p_4$, and made of three components, glued by three nodal points. Each nodal point is a pair of marked points (q_i, q_j). Each component is a smooth Riemann surface of some genus g_i, and with n_i marked or nodal points. Stability means that for each component $\chi_i = 2 - 2g_i - n_i < 0$. Here, one component has genus 2 and 1 nodal point q_4 so $\chi = -3$, another component is a sphere with two marked points p_1, p_2 and three nodal points q_1, q_2, q_3, i.e., $\chi = -3$, and the last component is a sphere with two marked points p_3, p_4 and two nodal points q_5, q_6 so $\chi = -2$. The total Euler characteristics is $\chi = -3 - 3 - 2 = -8$ which indeed corresponds to $2 - 2g - n$ for a Riemann surface of genus $g = 3$ with $n = 4$ marked points.

compute the integral of the exterior product of $3g - 3 + n$ 2-forms, i.e., to compute the "intersection number"

$$\left\langle \psi_1^{d_1} \ldots \psi_n^{d_n} \right\rangle_{g,n} = \int_{[\overline{\mathcal{M}}_{g,n}]^{\mathrm{vir}}} \psi_1^{d_1} \ldots \psi_n^{d_n} \tag{3.3}$$

on the Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of $\mathcal{M}_{g,n}$ (or more precisely, on a virtual cycle $[]^{\mathrm{vir}}$ of $\overline{\mathcal{M}}_{g,n}$, taking careful account of the non-smooth curves at the boundary of $\mathcal{M}_{g,n}$), provided that

$$\sum_i d_i = d_{g,n} = 3g - 3 + n. \tag{3.4}$$

If this equality is not satisfied, we define $\left\langle \psi_1^{d_1} \ldots \psi_n^{d_n} \right\rangle_{g,n} = 0$.

More interesting characteristic classes and intersection numbers are defined as follows. Let (we follow the notations of [18], and refer the reader to it for details)

$$\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$$

be the forgetful morphism (which forgets the last marked point), and let $\sigma_1, \dots, \sigma_n$ be the canonical sections of π, and $D_1, \dots, D_n$ be the corresponding divisors in $\overline{\mathcal{M}}_{g,n+1}$. Let ω_π be the relative dualizing sheaf. We consider the following tautological classes on $\overline{\mathcal{M}}_{g,n}$:

- The ψ_i classes (which are 2-forms), already introduced above:

$$\psi_i = c_1(\sigma_i^*(\omega_\pi))$$

It is customary to use Witten's notation:

$$\psi_i^{d_i} = \tau_{d_i}. \tag{3.5}$$

- The Mumford κ_k classes [1, 27]:

$$\kappa_k = \pi_*(c_1(\omega_\pi(\sum_i D_i))^{k+1}).$$

κ_k is a $2k$-form. κ_0 is the Euler class, and in $\overline{\mathcal{M}}_{g,n}$, we have

$$\kappa_0 = -\chi_{g,n} = 2g - 2 + n.$$

κ_1 is in the same cohomology class as the Weil–Petersson form: $2\pi^2\kappa_1 \equiv \sum_i dl_i \wedge d\theta_i$ in the Fenchel–Nielsen coordinates (l_i, θ_i) in Teichmüller space [31].

In some sense, κ classes are the remnants of the ψ classes of (clusters of) forgotten points. There is the formula [1]:

$$\pi_* \psi_1^{d_1} \cdots \psi_n^{d_n}\, \psi_{n+1}^{k+1} = \psi_1^{d_1} \cdots \psi_n^{d_n}\, \kappa_k \tag{3.6}$$

$$\pi_*\pi_* \psi_1^{d_1} \cdots \psi_n^{d_n}\, \psi_{n+1}^{k+1}\, \psi_{n+2}^{k'+1} = \psi_1^{d_1} \cdots \psi_n^{d_n}\, (\kappa_k\, \kappa_{k'} + \kappa_{k+k'}) \tag{3.7}$$

and so on ...

- The Hodge class $\Lambda(\alpha) = 1 + \sum_{k=1}^{g} (-1)^k\, \alpha^{-k}\, c_k(\mathbb{E})$ where $c_k(\mathbb{E})$ is the kth Chern class of the Hodge bundle $\mathbb{E} = \pi_*(\omega_\pi)$. Mumford's formula [16, 27] says that

(3.8)
$$\Lambda_{\text{Hodge}}(\alpha) = \mathrm{e}^{\sum_{k\geq 1} \frac{B_{2k}\, \alpha^{1-2k}}{2k(2k-1)} \left(\kappa_{2k-1} - \sum_i \psi_i^{2k-1} + \frac{1}{2}\sum_\delta \sum_j (-1)^j\; l_{\delta *}\psi^j\, \psi'^{2k-2-j}\right)},$$

where $\mathcal{B}_k$ is the kth Bernoulli number, δ a boundary divisor (i.e., a cycle which can be pinched so that the pinched curve is a stable nodal curve, i.e., replacing the pinched cycle by a pair of marked points, all components have a strictly negative Euler characteristics), and $l_{\delta *}$ is the natural inclusion into the moduli spaces of each connected component. In other words $\sum_\delta l_{\delta *}$ adds a nodal point in all possible stable ways.

In fact, all tautological classes in $\overline{\mathcal{M}}_{g,n}$ can be expressed in terms of ψ-classes or their pull back or push forward from some $\overline{\mathcal{M}}_{h,m}$ [3]. Faber's conjecture [16] (partly proved in [26] and [18]) proposes an efficient method to compute intersection numbers of ψ, κ and Hodge classes.

3.2. Reminder one branch-point

If $\mathcal{S}_a = (\mathcal{C}_a, x, y, B)$ is a spectral curve with only one branchpoint a, the following theorem was proved in [11]:

Theorem 3.1 (one branchpoint).

(3.9)
$$\begin{aligned} W_n^{(g)}(\mathcal{S}_a; z_1, \dots, z_n) &= 2^{d_{g,n}} \sum_{d_1,\dots,d_n} \prod_{i=1}^n d\xi_{a,d_i}(z_i) \left\langle \psi_1^{d_1} \cdots \psi_n^{d_n}\ \hat\Lambda(\mathcal{S}_a) \right\rangle_{\mathcal{M}_{g,n}} \\ &= 2^{d_{g,n}} \left\langle \prod_{i=1}^n \hat B_a(z; 1/\psi_i)\ \hat\Lambda(\mathcal{S}_a) \right\rangle_{\mathcal{M}_{g,n}} \end{aligned}$$

or alternatively after Laplace transform:

(3.10)
$$\begin{aligned} &\int_{\gamma_a^n} W_n^{(g)}(\mathcal{S}_a; z_1, \dots, z_n) \prod_{i=1}^n \mathrm{e}^{-\mu_i\, x(z_i)} \\ &\quad = 2^{d_{g,n}} \prod_{i=1}^n \frac{2\sqrt{\pi}\,\mathrm{e}^{-\mu_i x(a)}}{\sqrt{\mu_i}} \left\langle \prod_{i=1}^n \left(\frac{\mu_i}{1+\mu_i\psi_i} - \check B_{a,a}(\mu_i, 1/\psi_i) \right) \hat\Lambda(\mathcal{S}_a) \right\rangle_{\mathcal{M}_{g,n}}, \end{aligned}$$

where

$$\hat\Lambda(\mathcal{S}_a) = \mathrm{e}^{\sum_k \hat t_{a,k}\kappa_k + \frac{1}{2}\sum_{\delta\in[\partial\mathcal{M}_{g,n}]} l_{\delta *}\ \check B_{a,a}(1/\psi, 1/\psi')} \tag{3.11}$$

with the times $\hat t_{a,k}$ defined by the Laplace transform of ydx

$$\mathrm{e}^{-\sum_k \hat t_{a,k} u^{-k}} = \frac{u^{3/2}\,\mathrm{e}^{ux(a)}}{2\,\sqrt{\pi}} \int_{\gamma_a} ydx\,\mathrm{e}^{-ux} \tag{3.12}$$

with $\mathrm{Re} u > 0$ *and* γ_a *the steepest descent contour going through* a*, i.e., such that*

$$x(\gamma_a) - x(a) = \mathbb{R}_+, \tag{3.13}$$

and $\hat{B}$ *and* $\check{B}$ *are defined by Laplace transforms of the Bergman kernel*

$$\hat{B}_a(z;u) = -\frac{\sqrt{u}\,\mathrm{e}^{ux(a)}}{\sqrt{\pi}} \int_{z'\in\gamma_a} B(z,z')\ \mathrm{e}^{-ux(z')} = \sum_d u^{-d}\, d\xi_{a,d}(z), \tag{3.14}$$

(3.15)

$$\begin{aligned}\check{B}_{a,a}(u,v) &= \frac{uv}{u+v} + \frac{\sqrt{uv}\,\mathrm{e}^{(u+v)x(a)}}{2\,\pi} \int_{z,z'\in\gamma_a} B(z,z')\ \mathrm{e}^{-ux(z)}\,\mathrm{e}^{-vx(z')},\\ &= \sum_{k,l} \check{B}_{a,k;a,l}\, u^{-k}\, v^{-l}.\end{aligned}$$

$[\partial\mathcal{M}_{g,n}]$ *is the set of boundary divisors of* $\mathcal{M}_{g,n}$*, and* $l_{\delta *}$ *is the natural inclusion of a boundary of* $\mathcal{M}_{g,n}$ *into* $\mathcal{M}_{g-1,n+2} \cup \cup_{h,m}^{\text{stable}} \mathcal{M}_{h,m} \times \mathcal{M}_{g-h,n-m}$*, and* ψ*,* ψ' *represent the* ψ *classes associated with the two marked points in* $\mathcal{M}_{g-1,n+2} \cup \cup_{h,m} \mathcal{M}_{h,m} \times \mathcal{M}_{g-h,n-m}$*, associated with the nodal point in* $\partial\mathcal{M}_{g,n}$*.*

Proof. This theorem was proved in [11], using the known result for the Kontsevich integral's spectral curve, and the deformation theory in [13] (special geometry) satisfied by symplectic invariants $W_n^{(g)}$'s. This allowed to view any spectral curve with one branch point, as a deformation of the Kontsevich's spectral curve, and thus prove this theorem. □

Remark 3.1 (ELSV–like). Notice that formula (3.10) is very reminiscent of the ELSV formula [9].

Remark 3.2 (The Laplace transform). here our definition of the Laplace transform is by integration along a contour $\gamma_a \subset \mathcal{C}$, where $x(z) - x(a) > 0$. Since a is a simple branchpoint, i.e., $x(z)$ behaves locally like $x(a) + \frac{x''(a)}{2}(z-a) + O(z-a)^3$, the image of γ_a in the x-plane contains two copies of $x(a) + \mathbb{R}_+$, with opposite orientation. This implies that Laplace transforms of integer powers of x vanish, and only Laplace transform of the skew invariant part, i.e., half-integer powers of x contribute to the Laplace transform. This explains that the large u asymptotic expansion of our Laplace transforms contains only integer powers of u^{-1} and no half-integer powers.

Another remark is that in fact we are interested only in the large u asymptotic expansion of the Laplace transform, which is controlled by the Taylor expansion near the branchpoint a. In other words, the path γ_a can also be chosen as any segment $\subset x^{-1}(x(a)+\mathbb{R}_+)$ containing the branchpoint a, it would give the same asymptotic series expansion at large u.

In fact the Laplace tranform here is to be taken in the sense of formal series of u^{-1}, and is just a convenient way of encoding the coefficients $\hat{t}_{a,d}$ or $\check{B}_{a,k;b,l}$.

3.2.1. Example: topological vertex. A very important example of application of Theorem 3.1 is the topological vertex.

The spectral curve of the topological vertex is given by

$$\mathcal{S} = (\mathbb{C}\backslash]-\infty,0]\cup[1,\infty[, x, y, B)$$

$$\begin{cases} x(z) = -f\ln z - \ln(1-z), \\ y(z) = -\ln z, \\ B(z_1,z_2) = \dfrac{dz_1\otimes dz_2}{(z_1-z_2)^2}. \end{cases} \tag{3.16}$$

It is more often described by observing that $X = \mathrm{e}^{-x}$ and $Y = \mathrm{e}^{-y}$ are related by the following algebraic equation (known as the mirror curve of $\mathbb{C}^3$ with framing f):

$$X = Y^f\,(1-Y). \tag{3.17}$$

The only branchpoint is at $a = f/(1+f)$, and $\gamma_a =]0,1[$, so that the Laplace transform of ydx is the Euler Beta function (we first integrate by parts):

$$\begin{aligned} \int_{\gamma_a} \mathrm{e}^{-xu}\, ydx &= \frac{1}{u}\int_{\gamma_a} \mathrm{e}^{-xu}\, dy = \frac{1}{u}\int_0^1 z^{fu}\,(1-z)^u\,\frac{dz}{z} \\ &= \frac{1}{u}\,\frac{\Gamma(u+1)\Gamma(fu)}{\Gamma((f+1)u+1)} \\ &= \frac{1}{(f+1)u}\,\frac{\Gamma(u)\Gamma(fu)}{\Gamma((f+1)u)}. \end{aligned} \tag{3.18}$$

Using the Stirling large u expansion of the Γ function, we thus have from (3.12)

(3.19)
$$e^{\hat{t}_0} = \sqrt{2f(f+1)},$$
(3.20)
$$g(u) = \sum_{k\geq 1} \hat{t}_k u^{-k} = \sum_k \frac{\mathcal{B}_{2k}}{2k(2k-1)} u^{1-2k} \left(1 + f^{1-2k} + (-f-1)^{1-2k}\right),$$

where $\mathcal{B}_k$ is the kth Bernoulli number. Similarly we find (see Appendix B)

$$\check{B}(u,v) = uv\, \frac{1 - e^{-g(u)}\, e^{-g(v)}}{u+v}. \tag{3.21}$$

Using a few combinatorial identities (see [11]), and thanks to Mumford's formula (3.8) (which writes Hodge classes in terms of ψ and κ classes, see [27]), we find that the spectral curve's class of the vertex is the product of three Hodge classes (see [11]):

$$\hat{\Lambda}_a(\mathcal{S}) \prod_{i=1}^n e^{-g(1/\psi_i)} = \Lambda_{\text{Hodge}}(1)\Lambda_{\text{Hodge}}(f)\Lambda_{\text{Hodge}}(-f-1). \tag{3.22}$$

Notice also that from (3.21) we have

$$\int_{\gamma_a} e^{-\mu x(z)}\, \hat{B}(z;1/\psi) = \frac{2\sqrt{\pi} e^{-\mu x(a)}}{\sqrt{\mu}}\, e^{-g(\mu)}\, \frac{\mu}{1+\mu\,\psi}\, e^{-g(1/\psi)}, \tag{3.23}$$

i.e., formula (3.9) reads in that case (after Laplace transform):

$$\begin{aligned} &\int_{\gamma_a^n} W_n^{(g)}(\mathcal{S}; z_1,\dots,z_n) \prod_i e^{-\mu_i x(z_i)} \\ &\quad = 2^{d_{g,n}}\, e^{\hat{t}_0(2g-2+n)} \int_{[\overline{\mathcal{M}}_{g,n}]^{\text{vir}}} \Lambda_{\text{Hodge}}(1)\Lambda_{\text{Hodge}}(f)\Lambda_{\text{Hodge}}(-f-1) \\ &\quad\quad \times \prod_{i=1}^n \frac{1}{1+\mu_i\,\psi_i} \prod_{i=1}^n \frac{\Gamma(\mu_i)\Gamma(f\mu_i)}{\Gamma((f+1)\mu_i)} \left(\frac{(f+1)^{f+1}}{f^f}\right)^{\mu_i} \frac{2\sqrt{2\pi f}}{\sqrt{f+1}}, \end{aligned} \tag{3.24}$$

whose right-hand side is the famous Mariño–Vafa formula [24].

Using the result of [7, 32], i.e., that the symplectic invariants $W_n^{(g)}$ of the vertex are the Gromov–Witten invariants of $\mathbb{C}^3$, we see that (3.9) translates into the Mariño–Vafa formula [24], and thus, $W_n^{(g)}$'s are generating functions of Gromov–Witten invariants of $\mathbb{C}^3$.

The large f limit of that identity, is the ELSV formula [9] combined with Bouchard–Mariño formula [6].

4. Spectral curve with several branchpoints

Let $\mathcal{S} = (\mathcal{C}, x, y, B)$ be a spectral curve, let $\{a_1, a_2, \dots, a_{\mathfrak{b}}\}$ be the set of its branchpoints. We first need to set up notations.

4.1. Local description of the spectral curve near branchpoints

For each branchpoint a_i we define the steepest descent path γ_{a_i}, as a connected arc on $\mathcal{C}$ passing through a_i such that

$$x(\gamma_{a_i}) - x(a_i) = \mathbb{R}_+. \tag{4.1}$$

In a vicinity of a_i we define the local coordinate

$$\zeta_{a_i}(z) = \sqrt{x(z) - x(a_i)}. \tag{4.2}$$

Remark 4.1. For our purposes, it is sufficient that γ_{a_i} is defined only in a vicinity of a_i, but for many examples, it is actually a well-defined path in $\mathcal{C}$.

4.1.1. Coefficients $\hat{B}_{a_i,k;a_j,l}$. We expand the Bergman kernel in the vicinity of branchpoints as follows:

$$B(z,z') \underset{\substack{z'\to a_j\\ z\to a_i}}{\sim} \left(\frac{\delta_{i,j}}{(\zeta_{a_i}(z) - \zeta_{a_j}(z'))^2} + \sum_{d,d'\geq 0} B_{a_i,d;a_j,d'} \zeta_{a_i}(z)^d\, \zeta_{a_j}(z')^{d'} \right) \times d\zeta_{a_i}(z) \otimes d\zeta_{a_j}(z') \tag{4.3}$$

and then we define

$$\hat{B}_{a_i,k;a_j,k'} = (2k-1)!!\,(2l-1)!!\; 2^{-k-l-1}\; B_{a_i,2k;a_j,2k'}. \tag{4.4}$$

It is useful to notice that the generating function of these last quantities can also be defined through Laplace transform, we define:

$$\check{B}_{a_i,a_j}(u,v) = \sum_{k,k'\geq 0} \hat{B}_{a_i,k;a_j,k'} u^{-k}\, v^{-l}, \tag{4.5}$$

which is given by the Laplace transform of the Bergman kernel

$$\check{B}_{a_i,a_j}(u,v) = \delta_{i,j}\,\frac{uv}{u+v} + \frac{\sqrt{uv}\,\mathrm{e}^{ux(a_i)+vx(a_j)}}{2\pi}\int_{z\in\gamma_{a_i}}\int_{z'\in\gamma_{a_j}} \times\left(B(z,z') - \delta_{i,j}\,\frac{d\zeta_{a_i}(z)\,d\zeta_{a_j}(z')}{(\zeta_{a_i}(z)-\zeta_{a_j}(z'))^2}\right)\mathrm{e}^{-ux(z)}\,\mathrm{e}^{-vx(z')} \tag{4.6}$$

where the double integral is conveniently regularized when $i=j$, so that $\check{B}_{a_i,a_j}(u,v)$ is a power series of u^{-1} and v^{-1}. The proof that the regularizing term is $\frac{uv}{u+v}$ is for example in Appendix B or in [11], and is similar to Givental formalism [2, 17].

Remark 4.2. Again, as in Remark 3.2, we notice that the Laplace transform is to be taken in the formal sense, i.e., formal series of u^{-1} and v^{-1}, as a convenient way of storing the Taylor expansion information near the branchpoints.

4.1.2. Basis of differential forms $d\xi_{a_i,d}(z)$. We define the set of meromorphic 1-forms $d\xi_{a_i,d}(z)$ as follows:

$$d\xi_{a_i,d}(z) = -\,(2d-1)!!\,2^{-d}\,\operatorname*{Res}_{z'\to a_i} B(z,z')\,\zeta_{a_i}(z')^{-2d-1}. \tag{4.7}$$

It is a meromorphic 1-form defined on $\mathcal{C}$, with a pole only at $z=a_i$, of degree $2d+2$.

Namely, near $z\to a_j$ it behaves like

$$d\xi_{a_i,d}(z) \underset{z\to a_j}{\sim} -\,\delta_{i,j}\,\frac{(2d+1)!!\,d\zeta_{a_i}(z)}{2^d\,\zeta_{a_i}(z)^{2d+2}} - \frac{(2d-1)!!}{2^d}\sum_k B_{a_i,2d;a_j,k}\,\zeta_{a_j}(z)^k\,d\zeta_{a_j}(z). \tag{4.8}$$

These differential forms will play an important role because they give the behavior of the Bergman kernel B near a branchpoint:

$$B(z,z') - B(\bar{z},z') \underset{z\to a_i}{\sim} -2\sum_{d\geq 0}\frac{2^d}{(2d-1)!!}\,\zeta_{a_i}(z)^{2d}\,d\zeta_{a_i}(z)\otimes d\xi_{a_i,d}(z'). \tag{4.9}$$

$\xi_{a_j,0}(z)$ plays a special role, notice that it has a simple pole at $z=a_j$ and no other pole:

$$\xi_{a_j,0}(z) \underset{z\to a_j}{\sim} \frac{1}{\zeta_{a_j}(z)} + \text{analytical}. \tag{4.10}$$

4.1.3. Laplace transform $f_{i,j}(u)$. Knowing $\xi_{a_j,0}(z)$, it is useful to define its Laplace transform along γ_{a_i} as

$$f_{i,j}(u) = \frac{\sqrt{u}}{2\sqrt{\pi}}\,\mathrm{e}^{ux(a_i)} \int_{\gamma_{a_i}} \mathrm{e}^{-u\,x}\ \xi_{a_j,0}\,dx = \frac{1}{2\sqrt{\pi\,u}}\,\mathrm{e}^{ux(a_i)} \int_{\gamma_{a_i}} \mathrm{e}^{-u\,x}\ d\xi_{a_j,0} = \delta_{i,j} - \sum_{k\geq 0} \frac{\hat{B}_{a_j,0;a_i,k}}{u^{k+1}}. \tag{4.11}$$

In Appendix B, we show that

Lemma 4.1 (proved in Appendix B). *If $\mathcal{C}$ is a compact Riemann surface and dx is a meromorphic form on $\mathcal{C}$ and B is the fundamental form of the second kind normalized on $\mathcal{A}$-cycles, we have*

$$\check{B}_{a_i,a_j}(u,v) = \frac{uv}{u+v}\left(\delta_{i,j} - \sum_{k=1}^{\mathfrak{b}} f_{i,k}(u)\,f_{j,k}(v)\right) \tag{4.12}$$

so that all we need to compute is in fact $f_{i,j}(u)$.

4.1.4. Half-Laplace transform. We also define

(4.13)
$$\hat{B}_{a_i}(z;u) = -\frac{\sqrt{u}\,\mathrm{e}^{u\,x(a_i)}}{\sqrt{\pi}} \int_{z'\in\gamma_{a_i}},\ B(z,z')\,\mathrm{e}^{-u\,x(z')} = \sum_d u^{-d}\,d\xi_{a_i,d}(z).$$

If we do a second Laplace transform we have

$$\frac{\sqrt{v}\,\mathrm{e}^{v\,x(a_j)}}{2\sqrt{\pi}} \int_{z\in\gamma_{a_j}} \hat{B}_{a_i}(z;u)\,\mathrm{e}^{-vx(z)} = \delta_{i,j}\,\frac{uv}{u+v} - \check{B}_{a_i,a_j}(v,u) = \frac{uv}{u+v}\sum_k f_{i,k}(v)\,f_{j,k}(u). \tag{4.14}$$

4.1.5. The times $\hat{t}_{a_i,k}$. Finally we define the times $\hat{t}_{a_i,k}$ at branchpoint a_i in terms of the local behavior of $y(z)$ by the Laplace transform of ydx along γ_{a_i}

$$e{-}\hat{t}_{a_i,0}\,\mathrm{e}^{-g_{a_i}(u)} = \frac{u^{3/2}\,\mathrm{e}^{ux(a_i)}}{2\sqrt{\pi}} \int_{z\in\gamma_{a_i}} \mathrm{e}^{-ux(z)}\,y(z)\,dx(z) = \frac{\sqrt{u}\,\mathrm{e}^{ux(a_i)}}{2\sqrt{\pi}} \int_{z\in\gamma_{a_i}} \mathrm{e}^{-ux(z)}\,dy(z). \tag{4.15}$$

The times $\hat{t}_{a_i,k}$ are the coefficients of the expansion of $g(u)$ at large u:

$$g_{a_i}(u) = \sum_{k\geq 1} \hat{t}_{a_i,k} u^{-k}. \tag{4.16}$$

Notice that the time $\hat{t}_{a_i,0}$ is given by

$$\mathrm{e}^{-\hat{t}_{a_i,0}} = \frac{1}{4} \lim_{z\to a_i} \frac{y(z)-y(\bar{z})}{\zeta_{a_i}(z)} = \frac{y'(a_i)}{\sqrt{2\,x''(a_i)}}. \tag{4.17}$$

4.2. Structure of invariants

Since the definition of $W_n^{(g)}$ involves only residues at branchpoints, this means that for each (g,n), $W_n^{(g)}$ is a polynomial of the coefficients of B and y in a Taylor expansion near the branchpoints, in other words, $W_n^{(g)}$ is a polynomial in the times $\hat{t}_{a_i,k}$ and $B_{a_i,k;a_j,l}$.

Since K is linear in B, and $W_2^{(0)} = B$ is also linear in B, one easily finds by recursion that $W_n^{(g)}$ is polynomial in the coefficients $B_{a_i,d;a_j,d'}$, of degree $3g-3+2n = d_{g,n}+n$.

When we compute residues at $z\to a_i$, we may replace $B(z,z')$, or more precisely the combination $B(z,z')-B(\bar{z},z')$ (indeed the Bergmann kernels always enter the residue computation with this combination, this due to the fact that $K(z_0,z)=K(z_0,\bar{z})$) by

$$B(z,z')-B(\bar{z},z') = -2\sum_d \zeta_{a_i}(z)^{2d}\,d\zeta_{a_i}(z)\;\frac{2^d}{(2d-1)!!}\,d\xi_{a_i,d}(z'), \tag{4.18}$$

where

$$d\xi_{a_i,d}(z) = -\frac{(2d-1)!!}{2^d} \operatorname*{Res}_{z'\to a_i} B(z,z')\;\zeta_{a_i}(z)^{-2d-1} \tag{4.19}$$

and thus one clearly sees by recursion that there exists some coefficients $A_n^{(g)}$ such that

(4.20)
$$W_n^{(g)}(\mathcal{S};z_1,\dots,z_n) = \sum_{i_1,\dots,i_n}\sum_{d_1,\dots,d_n} A_n^{(g)}(\mathcal{S};i_1,d_1;\dots;i_n,d_n)\prod_{k=1}^n d\xi_{a_{i_k},d_k}(z_k)$$

and the coefficients $A_n^{(g)}$ are polynomials of the $\hat{B}_{a_i,d;a_j,d'}$ of degree $3g-3+n=d_{g,n}$. $A_n^{(g)}$ are also polynomials in the $\hat{t}_{a_i,k}$ because the residues also picks terms in the Taylor expansion of the denominator of K, i.e., the Taylor expansion of $y(z)dx(z)$ near branchpoints, i.e., the coefficients $\hat{t}_{a_i,k}$.

So we have:

Proposition 4.1. *For any spectral curve $\mathcal{S}$ with branchpoints $a_1,\dots,a_{\mathfrak{b}}$, there exist some coefficients $A_n^{(g)}(\mathcal{S};i_1,d_1;\dots;i_n,d_n)$ such that*

(4.21)
$$W_n^{(g)}(\mathcal{S};z_1,\dots,z_n) = \sum_{i_1,\dots,i_n}\sum_{d_1,\dots,d_n} A_n^{(g)}(\mathcal{S};i_1,d_1;\dots;i_n,d_n)\prod_{k=1}^n d\xi_{a_{i_k},d_k}(z_k)$$

- *The coefficients $A_n^{(g)}$ are non-vanishing only if $\sum_k d_k \leq 3g-3+n$.*
- *The coefficients $A_n^{(g)}$ are polynomials of the $\hat{B}_{a_i,d;a_j,d'}$ of degree $3g-3+n=d_{g,n}$.*
- *The coefficients $A_n^{(g)}$ are polynomials in the $\hat{t}_{a_i,k}$, of degree at most $d_{g,n}$ (where $\hat{t}_{a_i,k}$ is weighted with degree k).*

Our goal now, is to show that the coefficients $A_n^{(g)}$ can be written as intersection numbers in some moduli space.

4.3. Definition of an appropriate moduli space

Definition 4.1. We define the "combinatorial moduli space"

$$\overline{\mathcal{M}}^{\mathfrak{b}}_{g,n} = \{(\Sigma;p_1,\dots,p_n;\sigma)\} \tag{4.22}$$

where $(\Sigma;p_1,\dots,p_n)\in\overline{\mathcal{M}}_{g,n}$ is a nodal curve of genus g with n smooth marked points, and $\sigma:\Sigma\setminus\mathfrak{K}\{\text{nodal points}\}\to\mathfrak{K}_{\mathfrak{b}}$ is a continuous stable map from Σ to the complete graph $\mathfrak{K}_{\mathfrak{b}}$ with $\mathfrak{b}$ vertices labeled by branchpoints

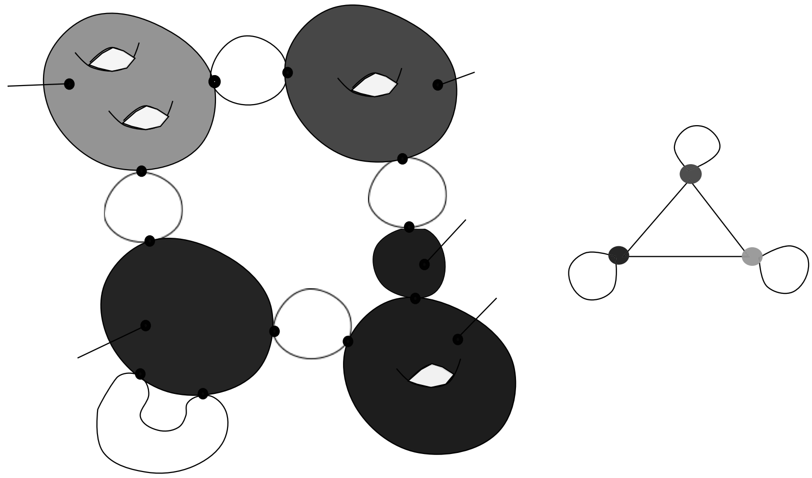


Figure 2: $\overline{\mathcal{M}}^{\mathfrak{b}}_{g,n}$ is the moduli space of nodal curves Σ with some stable "color" map $\sigma : \Sigma \to \mathfrak{K}_{\mathfrak{b}}$, where $\mathfrak{K}_{\mathfrak{b}}$ is the complete graph with $\mathfrak{b}$ vertices labeled by the branchpoints. Stability means that σ must map any stable component with $\chi < 0$ to a vertex of $\mathfrak{K}_{\mathfrak{b}}$, and every unstable ($\chi \geq 0$) component to an edge of $\mathfrak{K}_{\mathfrak{b}}$.

modulo continuous deformations. Stability means that σ must not be constant on an unstable $\chi \geq 0$ component, and must be constant on stable components with $\chi < 0$.

This moduli space is compact and contains $\mathfrak{b}$ copies of $\overline{\mathcal{M}}_{g,n}$.

Notice that if $(\Sigma; p_1, \dots, p_n)$ is a smooth curve ($\in \mathcal{M}_{g,n}$), i.e., with no nodal points then σ must be constant, and if $(\Sigma; p_1, \dots, p_n)$ is a nodal curve with several components, then σ must be piecewise constant, i.e., is constant on each stable component.

Definition 4.2. Let $\{a_1, a_2, \dots, a_{\mathfrak{b}}\}$ be a set of $\mathfrak{b}$ elements (later on, these will be the branchpoints of some spectral curve). For every $i = 1, \dots, \mathfrak{b}$, let $\hat{\Lambda}_{a_i} = f_i(\psi_1, \dots, \psi_n, \kappa_0, \kappa_1, \kappa_2, \dots)$ be a tautological class on $\overline{\mathcal{M}}_{g,n}$, which is a combination of ψ and κ classes, defined independently of g and n (as for instance (3.11)). Let $C_{a_i,d;a_j,d'}$ be an arbitrary sequence of complex numbers indexed by $i, j \in [1, \dots, \mathfrak{b}]$ and $d, d' \in \mathbb{N}$. We define the topological class $\hat{\Lambda}$ on $\overline{\mathcal{M}}^{\mathfrak{b}}_{g,n}$ as follows on each stratum of $\overline{\mathcal{M}}^{\mathfrak{b}}_{g,n}$:

A stratum of $\overline{\mathcal{M}}^{\mathfrak{b}}_{g,n}$, is given by the dual graph G of a nodal curve $\Sigma \in \overline{\mathcal{M}}_{g,n}$, and a map $\sigma : G \to \mathfrak{K}_{\mathfrak{b}}$. The stratum consists of $(\Sigma; p_1, \dots, p_n; \sigma)$

with k stable components $\cup_{m=1}^k(\Sigma_m,\vec{p}_m,\vec{q}_m)$ where σ is constant on Σ_m, taking the value σ_m, and where $\cup_m\vec{p}_m=\{p_1,\dots,p_n\}$ is a partition of the marked points, and pairs (q_i,q_j) are the nodal points attached to an unstable component (see figure 1). We define the topological class on the stratum as

(4.23)
$$\hat\Lambda=\prod_{m=1}^k\hat\Lambda_{a_{\sigma_m}}\prod_{\langle q_i,q_j\rangle=\text{nodal point}}\left(\sum_{d,d'}C_{a_{\sigma(q_i)},d;a_{\sigma(q_j)},d'}\,l_*\,\psi(q_i)^d\psi(q_j)^{d'}\right),$$

where l_* is the natural inclusion into $\mathcal{M}_{g_m,n_m}\times\mathcal{M}_{g_{m'},n_{m'}}$ (if $q_i\in\Sigma_m$ and $q_j\in\Sigma_{m'}$).

Then we have our main theorem:

Theorem 4.1. *For any spectral curve $\mathcal{S}=(\mathcal{C},x,y,B)$ with $\mathfrak{b}$ branchpoints $\{a_1,a_2,\dots,a_{\mathfrak{b}}\}$, we consider $\mathcal{S}_{a_i}=(\mathcal{C}_i,x,y,\overset{\circ}{B})$, where $\mathcal{C}_i\subset\mathcal{C}$ is a vicinity of the branchpoint a_i, x and y are the restrictions of x and y to $\mathcal{C}_i$, and $\overset{\circ}{B}$ can be any arbitrary Bergman kernel chosen in $\mathcal{C}_i$ (it does not need to be the restriction of B to $\mathcal{C}_i\times\mathcal{C}_i$).*

Then, for any $n\geq 0$ and g such that $2g-2+n>0$, the symplectic invariant of $\mathcal{S}$ is an integral of the class $\hat\Lambda(\mathcal{S})$ on $\overline{\mathcal{M}}^{\mathfrak{b}}_{g,n}$:

(4.24)
$$W_n^{(g)}(\mathcal{S};z_1,\dots,z_n)=2^{d_{g,n}}\int_{\overline{\mathcal{M}}^{\mathfrak{b}}_{g,n}}\hat\Lambda(\mathcal{S})\prod_{i=1}^n\hat B_{a_{\sigma(p_i)}}(z_i;1/\psi(p_i))$$

or in Laplace transform

(4.25)
$$\begin{aligned}&\int_{z_i\in\gamma_{a_{k_i}}}W_n^{(g)}(\mathcal{S};z_1,\dots,z_n)\prod_{i=1}^n\mathrm{e}^{-\mu_i x(z_i)}\\&\quad=2^{d_{g,n}}\prod_{i=1}^n\frac{2\sqrt{\pi}\,\mathrm{e}^{-\mu_i\,x(a_{k_i})}}{\sqrt{\mu_i}}\int_{\overline{\mathcal{M}}^{\mathfrak{b}}_{g,n}}\hat\Lambda(\mathcal{S})\\&\qquad\times\prod_{i=1}^n\left(\frac{\mu_i\,\delta_{k_i,\sigma(p_i)}}{1+\mu_i\psi(p_i)}-\check B_{a_{k_i},a_{\sigma(p_i)}}(\mu_i,1/\psi(p_i))\right).\end{aligned}$$

or using Lemma 4.1

$$(4.26)\quad \boxed{\begin{aligned}&\int_{z_i\in\gamma_{a_{k_i}}} W_n^{(g)}(\mathcal{S};z_1,\dots,z_n)\,\prod_{i=1}^n \mathrm{e}^{-\mu_i x(z_i)}\\ &= 2^{d_{g,n}}\,\prod_{i=1}^n (2\sqrt{\pi\mu_i}\,\mathrm{e}^{-\mu_i\,x(a_{k_i})})\\ &\quad\times \int_{\overline{\mathcal{M}}_{g,n}^{\mathfrak{b}}} \hat\Lambda(\mathcal{S})\,\prod_{i=1}^n \frac{\sum_{r=1}^{\mathfrak{b}} f_{k_i,r}(\mu_i) f_{\sigma(p_i),r}(1/\psi(p_i))}{1+\mu_i\psi(p_i)}.\end{aligned}}$$

In particular for $n=0$:

$$(4.27)\quad \boxed{F_g(\mathcal{S}) = 2^{d_{g,0}}\,\int_{\overline{\mathcal{M}}_{g,0}^{\mathfrak{b}}} \hat\Lambda(\mathcal{S}),}$$

where $\hat\Lambda(\mathcal{S})$ is defined as in Definition 4.2, with $\hat\Lambda_{a_i}$ defined in Theorem 3.1: if $(\Sigma,p_1,\dots,p_n,\sigma)\in\mathcal{M}_{g,n}^{\mathfrak{b}}$ is made of k components $\Sigma=\cup_{m=1}^k\Sigma_m$, with $\vec{p}_m=\{p_1,\dots,p_n\}\cap\Sigma_m$ and (q_i,q_j) are the nodal points, we define

$$(4.28)\quad \begin{aligned}\hat\Lambda(\mathcal{S}) &= \prod_{m=1}^k \hat\Lambda_{a_m} \prod_{\langle q_i,q_j\rangle=\text{nodal point}} \Big(\sum_{d,d'}(\hat B_{a_{\sigma(q_i)},d;a_{\sigma(q_j)},d'}\\ &\qquad - \delta_{\sigma(q_i),\sigma(q_j)}\hat{\mathring{B}}_{a_{\sigma(q_i)},d;a_{\sigma(q_i)},d'})\, l_*\psi(q_i)^d\,\psi(q_j)^{d'}\Big)\\ &= \prod_{m=1}^k \hat\Lambda_{a_m} \prod_{\langle q_i,q_j\rangle=\text{nodal point}} \Big(\check B_{a_{\sigma(q_i)},a_{\sigma(q_j)}}(1/\psi(q_i),1/\psi(q_j))\\ &\qquad - \delta_{\sigma(q_i),\sigma(q_j)}\,\check{\mathring{B}}_{a_{\sigma(q_i)},a_{\sigma(q_i)}}(1/\psi(q_i),1/\psi(q_j))\Big),\end{aligned}$$

where $\hat\Lambda_{a_i}$ is the class defined in Theorem 3.1 or in [11]:

$$(4.29)\quad \hat\Lambda_{a_i} = \mathrm{e}^{\sum_k \hat t_{a_i,k}\,\kappa^k + \frac12\sum_{\delta\in\partial\mathcal{M}_{g,n}} l_{\delta*}\,\check{\mathring{B}}_{a_i,a_i}(1/\psi,1/\psi')}$$

with the times $\hat t_{a_i,k}$ defined by the Laplace transform of ydx along γ_{a_i}

$$(4.30)\quad \mathrm{e}^{-\sum_k \hat t_{a_i,k} u^{-k}} = \frac{u\sqrt{u}\ \mathrm{e}^{ux(a_i)}}{2\sqrt{\pi}}\ \int_{z\in\gamma_{a_i}} \mathrm{e}^{-ux(z)}\ y(z)dx(z),$$

and where the $\hat{B}_{a_i,d;a_j,d'}$ *are defined by the Laplace transform of the Bergman kernel*

$$\begin{aligned}(4.31)\qquad \check{B}_{a_i,a_j}(u,v) &= \delta_{i,j}\,\frac{uv}{u+v} + \frac{\sqrt{uv}\;\mathrm{e}^{u\,x(a_i)+vx(a_j)}}{2\pi} \\ &\quad\times \int_{z\in\gamma_{a_i}}\int_{z'\in\gamma_{a_j}} B(z,z')\;\mathrm{e}^{-u\,x(z)}\;\mathrm{e}^{-v\,x(z')} \\ &= \sum_{d,d'} \hat{B}_{a_i,d;a_j,d'}\;u^{-d}\;v^{-d'} \\ &= \frac{uv}{u+v}\left(\delta_{i,j} - \sum_{r=1}^{\mathfrak{b}} f_{i,r}(u)\,f_{j,r}(v)\right)\end{aligned}$$

and

$$\begin{aligned}(4.32)\qquad \hat{B}_{a_i}(z;u) &= -\frac{\sqrt{u}\;\mathrm{e}^{u\,x(a_i)}}{\sqrt{\pi}}\int_{z'\in\gamma_{a_j}} B(z,z')\;\mathrm{e}^{-u\,x(z)} \\ &= \sum_d u^{-d}\;d\xi_{a_i,d}(z),\end{aligned}$$

and

$$(4.33)\qquad f_{i,j}(u) = \frac{\sqrt{u}\,\mathrm{e}^{ux(a_i)}}{2\sqrt{\pi}}\int_{z\in\gamma_{a_i}} \mathrm{e}^{-ux(z)}\,\xi_{a_j,0}(z)\,dx(z).$$

Proof. The proof almost follows the definition of $W_n^{(g)}$'s. It can also be viewed as a simple application of the method of Kostov and Orantin [21, 22, 29]. We give the full proof in Appendix A. □

Remark, the theorem also holds for $n=0$, i.e., the F_g's, this is due to the fact that the diagrammatic decomposition of [11, 21, 22, 29] works also for $n=0$.

4.4. How to use the formula

Example, assume that there are 2 branch points a_1, a_2, and let us compute $W_3^{(0)}$ and $W_4^{(0)}$.

• For $W_3^{(0)}$, consider a stable curve Σ of genus 0, with 3 marked points (p_1,p_2,p_3). The only possibility is that Σ is a sphere with 3 marked points, and thus it is a smooth surface, and σ must be constant. There are two possibilities $\sigma = 1$ or $\sigma = 2$. In other words $\mathcal{M}_{0,3}^2 \sim \mathcal{M}_{0,3} \cup \mathcal{M}_{0,3}$. We thus have

(4.34)
$$W_3^{(0)}(\mathcal{S}, z_1, z_2, z_3) = \left\langle \hat{\Lambda}_1 \ \hat{B}_{a_1}(z_1, 1/\psi_1)\, \hat{B}_{a_1}(z_2, 1/\psi_2)\, \hat{B}_{a_1}(z_3, 1/\psi_3) \right\rangle_{0,3} + \left\langle \hat{\Lambda}_2 \ \hat{B}_{a_2}(z_1, 1/\psi_1)\, \hat{B}_{a_2}(z_2, 1/\psi_2)\, \hat{B}_{a_2}(z_3, 1/\psi_3) \right\rangle_{0,3}.$$

Moreover, we have $\hat{B}_{a_i}(z, 1/\psi) = \sum_d \psi^d \ d\xi_{a_i,d}(z)$, and since $\dim \mathcal{M}_{0,3} = d_{0,3} = 0$, only the term $d = 0$ may contribute, i.e.,

$$W_3^{(0)}(\mathcal{S}, z_1, z_2, z_3) = \left\langle \hat{\Lambda}_1 \right\rangle_{0,3} d\xi_{a_1,0}(z_1)\, d\xi_{a_1,0}(z_2)\, d\xi_{a_1,0}(z_3) + \left\langle \hat{\Lambda}_2 \right\rangle_{0,3} d\xi_{a_2,0}(z_1)\, d\xi_{a_2,0}(z_2)\, d\xi_{a_2,0}(z_3). \tag{4.35}$$

Then, we have

$$\hat{\Lambda}_1 = \mathrm{e}^{\sum_k \hat{t}_{a_1,k}\kappa_k} \ \mathrm{e}^{\frac{1}{2} \sum_{\delta \in \partial \mathcal{M}_{0,3}} \sum_{d,d'} l_{\delta *} \hat{\mathring{B}}_{1\,a_1,d;a_1,d'} \psi^d \, \psi'^{d'}} \tag{4.36}$$

and since $\partial \mathcal{M}_{0,3} = \emptyset$, and since $d_{0,3} = 0$, only the term κ_0 may contribute, and thus

$$\left\langle \hat{\Lambda}_1 \right\rangle_{0,3} = \mathrm{e}^{\hat{t}_{a_1,0}} \ \langle 1 \rangle_{0,3} = \mathrm{e}^{\hat{t}_{a_1,0}}, \quad \left\langle \hat{\Lambda}_2 \right\rangle_{0,3} = \mathrm{e}^{\hat{t}_{a_2,0}} \ \langle 1 \rangle_{0,3} = \mathrm{e}^{\hat{t}_{a_2,0}}, \tag{4.37}$$

and finally, since $\langle 1 \rangle_{0,3} = 1$:

$$W_3^{(0)}(\mathcal{S}, z_1, z_2, z_3) = \mathrm{e}^{\hat{t}_{a_1,0}} \, d\xi_{a_1,0}(z_1)\, d\xi_{a_1,0}(z_2)\, d\xi_{a_1,0}(z_3) + \mathrm{e}^{\hat{t}_{a_2,0}} \, d\xi_{a_2,0}(z_1)\, d\xi_{a_2,0}(z_2)\, d\xi_{a_2,0}(z_3). \tag{4.38}$$

• For $W_4^{(0)}$, consider a stable curve Σ of genus 0, with four marked points (p_1, p_2, p_3, p_4). If Σ is smooth, then a continuous map $\sigma : \Sigma \to \{1, 2\}$ must be constant, it takes either the value $\sigma = 1$ or $\sigma = 2$. If Σ is not smooth, then it is a nodal curve, let us call (q_1, q_2) the nodal point. The only stable possibility is that Σ has two components $\Sigma = \Sigma_1 \cup \Sigma_2$, which are both spheres with three marked points, and thus $q_1 \in \Sigma_1$, $q_2 \in \Sigma_2$, and two of the points p_1, p_2, p_3, p_4 belong to Σ_1 and the two others belong to Σ_2. Then a continuous map $\sigma : \Sigma \to \{1, 2\}$ must be constant on Σ_1 and on Σ_2, for instance it may take the value $\sigma = 1$ on Σ_1 and $\sigma = 2$ on Σ_2. This shows that

$$\mathcal{M}^2_{0,4} \sim \mathcal{M}_{0,4} \cup \mathcal{M}_{0,4} \cup \overbrace{\mathcal{M}_{0,3} \times \mathcal{M}_{0,3}}^{12\,\text{times}}, \tag{4.39}$$

where the 12 times correspond to the 12 possibilities to have two point with color 1 and two point with color 2 among the four marked points p_1, p_2, p_3, p_4.

We thus have

$$\begin{aligned} &\frac{1}{2} W_4^{(0)}(\mathcal{S}, z_1, \dots, z_4) \\ &= \left\langle \prod_{i=1}^4 \hat{B}_{a_1}(z_i, 1/\psi(p_i))\hat{\Lambda}_1 \right\rangle_{0,4} + \left\langle \prod_{i=1}^4 \hat{B}_{a_2}(z_i, 1/\psi(p_i))\hat{\Lambda}_2 \right\rangle_{0,4} \\ &\quad + \frac{1}{2} \sum_{i,j\in\{1,2\}^2} \sum_{d,d'} C_{a_i,d;a_j,d'} \\ &\quad \times \left\langle \hat{B}_{a_i}(z_1, 1/\psi(p_1))\hat{B}_{a_i}(z_2, 1/\psi(p_2))\psi(q_1)^d\hat{\Lambda}_i \right\rangle_{0,3} \\ &\quad \times \left\langle \hat{B}_{a_j}(z_3, 1/\psi(p_3))\, \hat{B}_{a_j}(z_4, 1/\psi(p_4))\ \psi(q_2)^{d'}\ \hat{\Lambda}_j \right\rangle_{0,3} \\ &\quad + \text{symmetrize on}\,(z_1, z_2, z_3, z_4) \end{aligned} \tag{4.40}$$

where all intersection numbers in the right-hand side are now usual intersection numbers in the corresponding $\overline{\mathcal{M}}_{g,n}$.

Since $\mathcal{M}_{0,3}$ is a point ($\dim \mathcal{M}_{0,3} = d_{0,3} = 0$), we must have $d = d' = 0$ in the last term, and we may replace $\hat{B}_{a_i}(z, 1/\psi) = \sum_d \psi^d\, d\xi_{a_i,d}(z)$ by $d\xi_{a_i,0}(z)$. And since $d_{0,4} = 1$, we may replace $\hat{B}_{a_i}(z, 1/\psi) = \sum_d \psi^d\, d\xi_{a_i,d}(z)$ by $d\xi_{a_i,0}(z) + \psi\, d\xi_{a_i,1}(z)$ in the $\langle\rangle_{0,4}$ terms. More explicitly, that gives

$$\begin{aligned} &\frac{1}{2} W_4^{(0)}(\mathcal{S}, z_1, \dots, z_4) \\ &= \left\langle \prod_{i=1}^4 (d\xi_{a_1,0}(z_i) + \psi(p_i)\, d\xi_{a_1,1}(z_i))\ \hat{\Lambda}_1 \right\rangle_{0,4} \\ &\quad + \left\langle \prod_{i=1}^4 (d\xi_{a_2,0}(z_i) + \psi(p_i)\, d\xi_{a_2,1}(z_i))\ \hat{\Lambda}_2 \right\rangle_{0,4} \\ &\quad + \frac{1}{2} \sum_{i,j\in\{1,2\}^2} C_{a_i,0;a_j,0}\ \left\langle \hat{\Lambda}_i \right\rangle_{0,3} d\xi_{a_i,0}(z_1)\, d\xi_{a_i,0}(z_2) \\ &\quad \times \left\langle \hat{\Lambda}_j \right\rangle_{0,3} d\xi_{a_j,0}(z_3)\, d\xi_{a_j,0}(z_4) + \text{symmetrize on}\,(z_1, z_2, z_3, z_4) \end{aligned} \tag{4.41}$$

Again, since $d_{0,3}=0$, we may keep only the κ_0 term in the computation of $\langle\hat\Lambda_i\rangle_{0,3}$, i.e.,

$$\left\langle \hat\Lambda_i \right\rangle_{0,3} = \left\langle \mathrm{e}^{\hat t_{a_i,0}\kappa_0} \right\rangle_{0,3} = \mathrm{e}^{\hat t_{a_i,0}}. \tag{4.42}$$

Since $d_{0,4}=1$, we need to keep κ_0 and κ_1, and since $\partial\mathcal{M}_{0,4}\sim\overbrace{\mathcal{M}_{0,3}\times\mathcal{M}_{0,3}}^{6\,\text{times}}$, we have

$$\begin{aligned}\left\langle \hat\Lambda_i \right\rangle_{0,4} &= \left\langle \mathrm{e}^{\hat t_{a_i,0}\kappa_0}\,(1+\hat t_{a_i,1}\kappa_1) \right\rangle_{0,4} + \frac{6}{2}\hat{\mathring{B}}_{a_i,0;a_i,0}\left\langle \hat\Lambda_i \right\rangle_{0,3}\left\langle \hat\Lambda_i \right\rangle_{0,3}\\ &= \hat t_{a_i,1}\left\langle \mathrm{e}^{\hat t_{a_i,0}\kappa_0}\,\kappa_1 \right\rangle_{0,4} + 3\hat{\mathring{B}}_{a_i,0;a_i,0}\ \mathrm{e}^{\hat t_{a_i,0}}\ \mathrm{e}^{\hat t_{a_i,0}}\\ &= \hat t_{a_i,1}\,\mathrm{e}^{2\hat t_{a_i,0}}\ \langle\kappa_1\rangle_{0,4} + 3\hat{\mathring{B}}_{a_i,0;a_i,0}\ \mathrm{e}^{\hat t_{a_i,0}}\ \mathrm{e}^{\hat t_{a_i,0}}\\ &= \mathrm{e}^{2\hat t_{a_i,0}}\ \left(\hat t_{a_i,1} + 3\hat{\mathring{B}}_{a_i,0;a_i,0}\right),\end{aligned}\tag{4.43}$$

where we used that $\langle\kappa_1\rangle_{0,4}=1$. Similarly, since $\dim\psi=1$, we have

$$\left\langle \psi\hat\Lambda_i \right\rangle_{0,4} = \left\langle \psi\,\mathrm{e}^{\hat t_{a_i,0}\kappa_0} \right\rangle_{0,4} = \mathrm{e}^{2\hat t_{a_i,0}}\ \langle\psi\rangle_{0,4} = \mathrm{e}^{2\hat t_{a_i,0}}\ \langle\tau_1\rangle_{0,4} = \mathrm{e}^{2\hat t_{a_i,0}}, \tag{4.44}$$

where we used that $\langle\tau_1\rangle_{0,4}=1$. We thus have

$$\begin{aligned}&\frac{1}{2}W_4^{(0)}(\mathcal{S},z_1,\dots,z_4)\\ &\quad= \mathrm{e}^{2\hat t_{a_1,0}}\ \left(\hat t_{a_1,1} + 3\,\hat{\mathring{B}}_{a_1,0;a_1,0}\right)\,d\xi_{a_1,0}(z_1)\,d\xi_{a_1,0}(z_2)\,d\xi_{a_1,0}(z_3)\,d\xi_{a_1,0}(z_4)\\ &\qquad+\mathrm{e}^{2\hat t_{a_1,0}}\left(d\xi_{a_1,1}(z_1)\,d\xi_{a_1,0}(z_2)\,d\xi_{a_1,0}(z_3)\,d\xi_{a_1,0}(z_4)+\text{sym. on}\,(z_1,z_2,z_3,z_4)\right)\\ &\qquad+\mathrm{e}^{2\hat t_{a_2,0}}\ \left(\hat t_{a_2,1} + 3\,\hat{\mathring{B}}_{a_2,0;a_2,0}\right)\,d\xi_{a_2,0}(z_1)\,d\xi_{a_2,0}(z_2)\,d\xi_{a_2,0}(z_3)\,d\xi_{a_2,0}(z_4)\\ &\qquad+\mathrm{e}^{2\hat t_{a_2,0}}\left(d\xi_{a_2,1}(z_1)\,d\xi_{a_2,0}(z_2)\,d\xi_{a_2,0}(z_3)\,d\xi_{a_2,0}(z_4)+\text{sym. on}\,(z_1,z_2,z_3,z_4)\right)\\ &\qquad+\frac{1}{2}\sum_{i,j\in\{1,2\}^2} C_{a_i,0;a_j,0}\ \mathrm{e}^{\hat t_{a_i,0}+\hat t_{a_j,0}}\ \Big(\,d\xi_{a_i,0}(z_1)\,d\xi_{a_i,0}(z_2)\ d\xi_{a_j,0}(z_3)\,d\xi_{a_j,0}(z_4)\\ &\qquad+\text{sym. on}\,(z_1,z_2,z_3,z_4)\Big)\end{aligned}\tag{4.45}$$

$$
\begin{aligned}
&= \mathrm{e}^{2\hat{t}_{a_1,0}} \left(\hat{t}_{a_1,1} + 3\,\hat{\mathring{B}}_{a_1,0;a_1,0}\right) d\xi_{a_1,0}(z_1)\, d\xi_{a_1,0}(z_2)\, d\xi_{a_1,0}(z_3)\, d\xi_{a_1,0}(z_4) \\
&\quad + \mathrm{e}^{2\hat{t}_{a_1,0}} \Big(d\xi_{a_1,1}(z_1)\, d\xi_{a_1,0}(z_2)\, d\xi_{a_1,0}(z_3)\, d\xi_{a_1,0}(z_4) \\
&\quad + \text{sym. on}\,(z_1,z_2,z_3,z_4)\Big) \\
&\quad + \mathrm{e}^{2\hat{t}_{a_2,0}} \left(\hat{t}_{a_2,1} + 3\,\hat{\mathring{B}}_{a_2,0;a_2,0}\right) d\xi_{a_2,0}(z_1)\, d\xi_{a_2,0}(z_2)\, d\xi_{a_2,0}(z_3)\, d\xi_{a_2,0}(z_4) \\
&\quad + \mathrm{e}^{2\hat{t}_{a_2,0}} \Big(d\xi_{a_2,1}(z_1)\, d\xi_{a_2,0}(z_2)\, d\xi_{a_2,0}(z_3)\, d\xi_{a_2,0}(z_4) \\
&\quad + \text{sym. on}\,(z_1,z_2,z_3,z_4)\Big) \\
&\quad + \frac{1}{2}\, C_{a_1,0;a_1,0}\ \mathrm{e}^{2\hat{t}_{a_1,0}} \Big(6\ d\xi_{a_1,0}(z_1)\, d\xi_{a_1,0}(z_2)\, d\xi_{a_1,0}(z_3)\, d\xi_{a_1,0}(z_4)\Big) \\
&\quad + \frac{1}{2}\, C_{a_2,0;a_2,0}\ \mathrm{e}^{2\hat{t}_{a_2,0}} \Big(6\ d\xi_{a_2,0}(z_1)\, d\xi_{a_2,0}(z_2)\, d\xi_{a_2,0}(z_3)\, d\xi_{a_2,0}(z_4)\Big) \\
&\quad + C_{a_1,0;a_2,0}\ \mathrm{e}^{\hat{t}_{a_1,0}+\hat{t}_{a_2,0}} \Big(d\xi_{a_1,0}(z_1)\, d\xi_{a_1,0}(z_2)\ \ d\xi_{a_2,0}(z_3)\, d\xi_{a_2,0}(z_4) \\
&\quad + \text{sym. on}\,(z_1,z_2,z_3,z_4)\Big)
\end{aligned}
$$

and then, remember that

$$
C_{a_i,0;a_j,0} = \hat{B}_{a_i,0;a_j,0} - \delta_{i,j}\ \hat{\mathring{B}}_{a_i,0;a_j,0} \tag{4.46}
$$

i.e., finally

(4.47)

$$
\begin{aligned}
&\frac{1}{2} W_4^{(0)}(\mathcal{S}, z_1, \ldots, z_4) \\
&\quad = \mathrm{e}^{2\hat{t}_{a_1,0}} \left(\hat{t}_{a_1,1} + 3\hat{B}_{a_1,0;a_1,0}\right) d\xi_{a_1,0}(z_1) d\xi_{a_1,0}(z_2) d\xi_{a_1,0}(z_3) d\xi_{a_1,0}(z_4) \\
&\qquad + \mathrm{e}^{2\hat{t}_{a_1,0}} \Big(d\xi_{a_1,1}(z_1) d\xi_{a_1,0}(z_2) d\xi_{a_1,0}(z_3) d\xi_{a_1,0}(z_4) \\
&\qquad + \text{symmetrizeon}(z_1,z_2,z_3,z_4)\Big) \\
&\qquad + \mathrm{e}^{2\hat{t}_{a_2,0}} \left(\hat{t}_{a_2,1} + 3\hat{B}_{a_2,0;a_2,0}\right) d\xi_{a_2,0}(z_1) d\xi_{a_2,0}(z_2) d\xi_{a_2,0}(z_3) d\xi_{a_2,0}(z_4) \\
&\qquad + \mathrm{e}^{2\hat{t}_{a_2,0}} \Big(d\xi_{a_2,1}(z_1) d\xi_{a_2,0}(z_2) d\xi_{a_2,0}(z_3) d\xi_{a_2,0}(z_4) \\
&\qquad + \text{symmetrizeon}(z_1,z_2,z_3,z_4)\Big)
\end{aligned}
$$

$$+ \hat{B}_{a_1,0;a_2,0} e^{\hat{t}_{a_1,0}+\hat{t}_{a_2,0}} \Big(d\xi_{a_1,0}(z_1) d\xi_{a_1,0}(z_2) d\xi_{a_2,0}(z_3) d\xi_{a_2,0}(z_4) + \text{symmetrizeon}(z_1,z_2,z_3,z_4) \Big).$$

It is easy to verify that this coincides with the direct computation of residues in the very definition of $W_4^{(0)}$.

4.5. Examples

In this section, we consider some classical examples of spectral curves with two branch points, and we compute the corresponding Laplace transforms.

4.5.1. Ising model and the Airy function. The spectral curve of the Ising model coupled to gravity, i.e., the $(4,3)$ minimal model, is (see [15]):

$$\mathcal{S} = (\mathbb{C}, x, y, B) \tag{4.48}$$
$$\begin{cases} x(z) = z^3 - 3z \\ y(z) = z^4 - 4z^2 + 2 \\ B(z_1,z_2) = \dfrac{dz_1 \otimes dz_2}{(z_1-z_2)^2}. \end{cases}$$

That curve has two branch points:

$$a_+ = 1, \quad a_- = -1. \tag{4.49}$$

Instead of choosing the colors $\sigma \in \{1,2\}$, we choose to name the colors $\sigma \in \{+,-\}$.

Notice that we have the symmetry

$$x(-z) = -x(z), \quad y(-z) = y(z), \tag{4.50}$$

and thus all that we compute for a_+ can easily be transposed to a_-.

We have:

$$\zeta_+(z) = \sqrt{x(z)+2} = (z-1)\sqrt{z+2}. \tag{4.51}$$

We easily find

$$\xi_{a_+,0}(z) = \frac{1}{\sqrt{3}} \frac{1}{z-1} = \frac{1}{\zeta_+} + \frac{3}{2} \sum_k \frac{(-1)^k \zeta_+^k}{3^{3/2(k+1)}} \frac{\Gamma(3k/2)}{(k+1)!\,\Gamma(k/2)}, \tag{4.52}$$

i.e.,

$$B_{+,0;+,k} = \frac{(-1)^k}{2\; 3^{3k/2+2}}\, \frac{\Gamma(3/2+3k/2)}{(k+2)!\,\Gamma(1/2+k/2)}, \tag{4.53}$$

$$\hat{B}_{+,0;+,k} = \frac{1}{2^{3(k+1)}\; 3^{3k+2}}\, \frac{(6k+1)!!}{(2k+2)!} = \frac{1}{2^{4(k+1)}\; 3^{3k+2}}\, \frac{(6k+1)!!}{(k+1)!\;(2k+1)!!}, \tag{4.54}$$

i.e.,

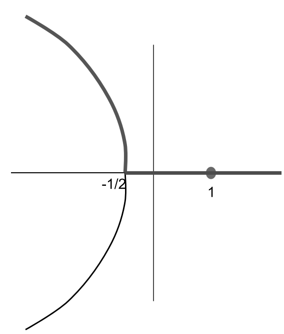
$$f_{+,+}(u) = 1 - \sum_k \frac{1}{2^{4k}\, 3^{3k-1}}\, \frac{(6k-5)!!}{k!\,(2k-1)!!}\, u^{-k}. \tag{4.55}$$

Since $f_{+,+}$ can also be obtained from the Laplace transform, we write

(4.56)

$$\begin{aligned} f_{+,+}(u^{3/2}/3) &= \frac{(u^{3/2}/3)^{1/2}\, e^{-\frac{2}{3}u^{3/2}}}{2\sqrt{\pi}} \int e^{-u^{3/2}/3(z^3-3z)}\, \frac{1}{\sqrt{3}\,(z-1)}\; 3(z^2-1)\, dz \\ &= \frac{u^{3/4}\, e^{-\frac{2}{3}u^{3/2}}}{2\sqrt{\pi}} \int e^{-u^{3/2}/3(z^3-3z)}\, (z+1)\, dz \\ &= \frac{u^{1/4}\, e^{-\frac{2}{3}u^{3/2}}}{2\sqrt{\pi}} \int e^{-(z^3/3-uz)}\, (z/\sqrt{u}+1)\, dz \\ &= \frac{u^{1/4}\, e^{-\frac{2}{3}u^{3/2}}}{2\sqrt{\pi}} \left(Ai(u) + Ai'(u)/\sqrt{u}\right), \end{aligned}$$

where Ai is an "Airy function" (solution of $Ai'' = xAi$), with integration path the following contour where $x(z)+2>0$:



i.e.,

$$f_{+,+}(u) = \frac{(3u)^{1/6}\,\mathrm{e}^{-2u}}{2\sqrt{\pi}}\,\Big(Ai((3u)^{2/3}) + Ai'((3u)^{2/3})/(3u)^{1/3}\Big)\,. \tag{4.57}$$

Similarly, we have

$$\xi_{-,0}(z) = \frac{1}{\sqrt{3}\,(z+1)} = \sum_{k\geq 0} \frac{(-1)^k}{2\,3^{(3k-1)/2}}\,\frac{\Gamma(3k/2)}{k!\,\Gamma(k/2)}\,\zeta_+(z)^k, \tag{4.58}$$

i.e.,

$$B_{-,0;+,k} = \frac{(-1)^k}{2\,3^{(3k+2)/2}}\,\frac{\Gamma(3k/2+3/2)}{k!\,\Gamma(k/2+1/2)} \tag{4.59}$$

and

$$\hat{B}_{-,0;+,k} = \frac{1}{2^{3(k+1)}\,3^{3k+1}}\,\frac{(6k+1)!!}{(2k)!} = \frac{1}{2^{4k+3}\,3^{3k+1}}\,\frac{(6k+1)!!}{k!\,(2k-1)!!}. \tag{4.60}$$

This gives

$$f_{+,-}(u) = -\sum_k \frac{1}{u^{k+1}}\,\frac{1}{2^{4k+3}\,3^{3k+1}}\,\frac{(6k+1)!!}{k!\,(2k-1)!!}, \tag{4.61}$$

which can also be computed by Laplace transform:

(4.62)

$$\begin{aligned} f_{+,-}(u^{3/2}/3) &= \frac{(u^{3/2}/3)^{1/2}\,\mathrm{e}^{-\frac{2}{3}u^{3/2}}}{2\sqrt{\pi}} \int \mathrm{e}^{-u^{3/2}/3(z^3-3z)}\,\frac{1}{\sqrt{3}\,(z+1)}\,3(z^2-1)\,dz \\ &= \frac{u^{3/4}\,\mathrm{e}^{-\frac{2}{3}u^{3/2}}}{2\sqrt{\pi}} \int \mathrm{e}^{-u^{3/2}/3(z^3-3z)}\,(z-1)\,dz \\ &= \frac{u^{1/4}\,\mathrm{e}^{-\frac{2}{3}u^{3/2}}}{2\sqrt{\pi}} \int \mathrm{e}^{-(z^3/3-uz)}\,(z/\sqrt{u}-1)\,dz \\ &= \frac{u^{1/4}\,\mathrm{e}^{-\frac{2}{3}u^{3/2}}}{2\sqrt{\pi}}\,\big(-Ai(u) + Ai'(u)/\sqrt{u}\big)\,, \end{aligned}$$

where Ai is the Airy function, i.e.,

$$f_{+,-}(u) = \frac{(3u)^{1/6}\,\mathrm{e}^{-2u}}{2\sqrt{\pi}}\left(-Ai((3u)^{2/3}) + Ai'((3u)^{2/3})/(3u)^{1/3}\right). \tag{4.63}$$

Kernels

The kernel is thus closely related to the Airy kernel

(4.64)

$$\begin{aligned}&\frac{uv}{u+v} - \check{B}_{+,+}(u,v)\\ &= uv\frac{(9uv)^{1/6}\,\mathrm{e}^{-2(u+v)}}{2\pi}\\ &\quad \frac{Ai((3u)^{2/3})Ai((3v)^{2/3}) + Ai'((3u)^{2/3})/(3u)^{1/3}Ai'((3v)^{2/3})/(3v)^{1/3}}{u+v}.\end{aligned}$$

- The times are computed by:

$$\begin{aligned}\mathrm{e}^{-g_+(u^{3/2}/3)} &= \frac{(u^{3/2}/3)^{1/2}}{2\sqrt{\pi}}\int \mathrm{e}^{-u^{3/2}/3(z^3-3z+2)}\,4(z^3-2z)\,dz\\ &= \frac{2\,u^{3/4}\,\mathrm{e}^{-\frac{2}{3}u^{3/2}}}{\sqrt{3\pi}}\int \mathrm{e}^{-(z^3/3-uz)}\,(z^3/u^2-2z/u)\,dz\\ &= \frac{2\,u^{-5/4}\,\mathrm{e}^{-\frac{2}{3}u^{3/2}}}{\sqrt{3\pi}}\int \mathrm{e}^{-(z^3/3-uz)}\,(z^3-2uz)\,dz\\ &= \frac{2\,u^{-5/4}\,\mathrm{e}^{-\frac{2}{3}u^{3/2}}}{\sqrt{3\pi}}\,\left(Ai'''(u) - 2uAi'(u)\right)\\ &= \frac{2\,u^{-5/4}\,\mathrm{e}^{-\frac{2}{3}u^{3/2}}}{\sqrt{3\pi}}\,\left(Ai(u) - uAi'(u)\right),\end{aligned} \tag{4.65}$$

i.e.,

$$\mathrm{e}^{-g_+(u)} = \frac{2\,(3u)^{-5/6}\,\mathrm{e}^{-2u}}{\sqrt{3\pi}}\,\left(Ai((3u)^{2/3}) - (3u)^{2/3}\,Ai'((3u)^{2/3})\right). \tag{4.66}$$

In other words, the spectral curve class of the Ising model, is the class.

$$\mathrm{e}^{\frac{55}{12^2}\kappa_1 + \frac{3.55}{12^3}\kappa_2 + \frac{8855}{12^5}\kappa_3 + \cdots}, \tag{4.67}$$

$$\mathrm{e}^{\hat{t}_{+,0}} = -\frac{\sqrt{3}}{2}. \tag{4.68}$$

4.5.2. Example Gromov–Witten theory of P^1 and the Hankel function. In [28], Norbury and Scott claim that the Gromov–Witten invariants of $\mathbb{P}^1$

$$\mathcal{W}_n^{(g)}(x_1,\dots,x_n) = \sum_\mu \mathcal{N}_{g,\mu}(\mathbb{P}^1)\ \prod_{i=1}^n \mathrm{e}^{-\mu_i\, x_i}\ \mu_i\, dx_i \tag{4.69}$$

are computed by the invariants $W_n^{(g)}$'s of the following spectral curve:

$$\mathcal{S} = (\mathbb{C}^* \setminus \mathbb{R}_+, x, y, B),$$

$$\begin{cases} x(z) = z + 1/z, \\ y(z) = \ln z, \\ B(z_1,z_2) = \dfrac{dz_1 \otimes dz_2}{(z_1-z_2)^2}. \end{cases} \tag{4.70}$$

That curve has two branchpoints:

$$a_+ = 1, \quad a_- = -1. \tag{4.71}$$

Again, the curve has the symmetry:

$$x(-z) = -x(z), \quad dy(-z) = dy(z), \tag{4.72}$$

so that all what we compute for a_+ can easily be transposed to a_-.

We have:

$$\xi_{a_+,0}(z) = \frac{1}{z-1} = \frac{1}{\sqrt{x-2}} - \frac{1}{2} + \sum_k \frac{(2k-1)!!\ (-1)^k}{2^{3k+3}\,(k+1)!}\ (x-2)^{k+1/2}, \tag{4.73}$$

i.e.,

$$\begin{aligned} B_{a_+,0;a_+,2k-2} &= (-1)^k\ \frac{(2k-1)!!}{2^{3k}\ k!}, \\ \hat{B}_{a_+,0;a_+,k-1} &= (-1)^k\ \frac{(2k-1)!!\ (2k-3)!!}{2^{4k}\ k!}. \end{aligned} \tag{4.74}$$

Similarly we have

$$\xi_{a_-,0}(z) = \frac{1}{z+1} = \frac{1}{2} - \sum_k \frac{(2k-1)!!\ (-1)^k}{2^{3k+2}\ k!}\ (x-2)^{k+1/2}, \tag{4.75}$$

i.e.,

(4.76)
$$B_{a_-,0;a_+,2k} = (-1)^k \,\frac{(2k+1)!!}{2^{3k+2}\;k!}, \quad \hat{B}_{a_-,0;a_+,k} = (-1)^k \,\frac{(2k+1)!!\;(2k-1)!!}{2^{4k+3}\;k!}.$$

Therefore we have

(4.77)
$$f_{+,+}(u) = 1 - \sum_{k\geq 0} \frac{\hat{B}_{a_+,,0;a_+,k}}{u^{k+1}} = 1 - \sum_{k\geq 1} (-1)^k \,\frac{(2k-1)!!\;(2k-3)!!}{2^{4k}\;k!}\,\frac{1}{u^k},$$

(4.78)
$$f_{+,-}(u) = -\sum_k (-1)^k \,\frac{(2k+1)!!\;(2k-1)!!}{2^{4k+3}\;k!}\,\frac{1}{u^{k+1}} = -2\,u\,f'_{+,+}(u).$$

Again, $f_{+,+}$ and $f_{+,-}$ can be computed by Laplace transforms as integrals and are found equal to Hankel functions:

(4.79)
$$\begin{aligned} f_{+,+}(u) &= \frac{u^{1/2}\,\mathrm{e}^{2\,u}}{2\sqrt{\pi}} \int_0^\infty \mathrm{e}^{-u(z+1/z)}\,\frac{1}{(z-1)}\,(1-1/z^2)\,dz \\ &= \frac{u^{1/2}\,\mathrm{e}^{2\,u}}{2\sqrt{\pi}} \int_0^\infty \mathrm{e}^{-u(z+1/z)}\,\frac{z+1}{z^2}\,dz \\ &= \frac{u^{1/2}\,\mathrm{e}^{2\,u}}{2\sqrt{\pi}} \int_0^\infty \mathrm{e}^{-u(z+1/z)}\,(1+1/z)\,\frac{dz}{z} \\ &= \frac{u^{1/2}\,\mathrm{e}^{2\,u}}{4\sqrt{\pi}} \int_0^\infty \mathrm{e}^{-u(z+1/z)}\,(2+z+1/z)\,\frac{dz}{z} \\ &= -\frac{u^{1/2}\,\mathrm{e}^{4\,u}}{4\sqrt{\pi}}\,\frac{\partial}{\partial u} \int_0^\infty \mathrm{e}^{-u(z+1/z-2)}\,\frac{dz}{z} \\ &= -\frac{u^{1/2}\,\mathrm{e}^{4\,u}}{4\sqrt{\pi}}\,\frac{\partial}{\partial u}\,H_0(2iu) \\ &= \frac{u^{1/2}\,\mathrm{e}^{2\,u}}{2\sqrt{\pi}}\,\pi\,(-iH_0(2iu)+H_1(2iu)), \end{aligned}$$

where H_0 is the Hankel function.

Applying the formula of appendix B, we have

(4.80)
$$\begin{aligned} \hat{B}_{a_+,a_+}(u,v) &= \frac{uv}{u+v}\,(1 - f_{+,+}(u)f_{+,+}(v) - f_{+,-}(u)f_{-,+}(v)) \\ &= \frac{uv}{u+v}\,\left(1 - f_{+,+}(u)f_{+,+}(v) - 4\,uv\,f'_{+,-}(u)f'_{-,+}(v)\right). \end{aligned}$$

Then, we compute the times $\hat{t}_{a_\pm,k}$ from

(4.81)
$$\begin{aligned}
e^{-g_+(u)} &= 2\,e^{u\,x(a_+)}\,\sqrt{u/\pi}\,\int_{\gamma_{a_+}} dy(z)\;e^{-u\,x(z)} \\
&= 2\,e^{2u}\,\sqrt{u/\pi}\,\int_0^\infty \frac{dz}{z}\;e^{-u(z+1/z)} \\
&= 4\sqrt{u/\pi}\,\int_{-\infty}^\infty d\phi\;e^{-4u\sinh^2\phi} \qquad \text{change of variable } z=e^{2\phi} \\
&= 4\sqrt{u/\pi}\,\int_{-\infty}^\infty \frac{ds}{\sqrt{1+s^2}}\;e^{-4us^2} \qquad \text{change of variable } s=\sinh\phi \\
&= \frac{2}{\sqrt{\pi}}\int_{-\infty}^\infty \frac{ds}{\sqrt{1+\frac{s^2}{4u}}}e^{-s^2} \qquad \text{change of variable } s\to s/2\sqrt{u} \\
&= \frac{2}{\sqrt{\pi}}\,\sum_k \left(\begin{array}{c}-1/2\\ k\end{array}\right)(4u)^{-k}\int_{-\infty}^\infty ds\,s^{2k}\;e^{-s^2} \\
&= 2\sum_k \left(\begin{array}{c}-1/2\\ k\end{array}\right)(4u)^{-k}\;\frac{(2k-1)!!}{2^k} \\
&= 2\sum_k \frac{(2k-3)!!}{2^k\,k!}\,(4u)^{-k}\;\frac{(-1)^k\;(2k-1)!!}{2^k} \\
&= 2\sum_k \frac{\hat{B}_{a_1,0;a_1,k-1}}{u^k} \\
&= -2\,f_{+,+}(u).
\end{aligned}$$

The expansion of $g_+(u)$ is

(4.82)
$$\begin{aligned}
e^{-g_+(u)} &= -2\,f_{+,+}(u) \\
&= 2\,e^{-\frac{1}{16u}+\frac{1}{64u^2}-\frac{25}{3\;2^{10}\;u^3}+\dots} \\
&= 2\,e^{-\sum_k \hat{t}_k u^{-k}},
\end{aligned}$$

where the $\hat{t}_k$'s satisfy the recursion:

(4.83)
$$-4\,k\,\hat{t}_k = k\,(k-1)\,\hat{t}_{k-1} + \sum_{j=2}^{k-1}(j-1)(k-j)\,\hat{t}_{j-1}\,\hat{t}_{k-j}$$
$$\hat{t}_1 = \frac{-1}{16},\quad \hat{t}_2 = \frac{1}{64},\quad \hat{t}_3 = \frac{-25}{3\;2^{10}},\quad \dots.$$

We have

$$\hat{B}_{a_+,a_+}(u,v) = -4\frac{uv}{u+v}\left(1+\frac{4}{uv}g'_+(u)g'_+(v)\right)\,\mathrm{e}^{-g_+(u)}\,\mathrm{e}^{-g_+(v)}, \tag{4.84}$$

$$\hat{B}_{a_+,a_+}(\mu,1/\psi) = -4\frac{\mu\,\mathrm{e}^{-g(\mu)}}{1+\mu\,\psi}\left(1+\frac{4\,g'_+(\mu)}{\mu}\sum_k k\,\hat{t}_k\psi^{k+2}\right)\,\mathrm{e}^{-\sum_k \hat{t}_k\psi^k}. \tag{4.85}$$

4.6. Resolved conifold

The spectral curve of resolved conifold, is (see [5]):

$$\mathcal{S} = (\mathcal{C}, x, y, B),$$
$$\begin{cases} x(z) = -f\ln z - \ln\frac{1-z}{1-z/Q}, \\ y(z) = -\ln z, \\ B(z_1,z_2) = \frac{dz_1\otimes dz_2}{(z_1-z_2)^2}. \end{cases} \tag{4.86}$$

It is customary to denote

$$X(z) = \mathrm{e}^{-x(z)}, \quad Y(z) = \mathrm{e}^{-y(z)} \tag{4.87}$$

so that the equation of the spectral curve is

$$X = Y^f\,\frac{1-Y}{1-Y/Q}. \tag{4.88}$$

4.6.1. Determination of $\boldsymbol{\xi_{\pm,0}}$. We have

$$x'(z) = -f\,\frac{(z-a_+)(z-a_-)}{z(z-1)(z-Q)} \tag{4.89}$$

and we find:

$$\xi_{a_+,0}(z) = \frac{1}{\sqrt{x''(a_+)/2}} \frac{1}{z-a_+} \tag{4.90}$$

(4.91)

$$\begin{aligned}
&f_{++}(u)\\
&= -\frac{f}{Q} \frac{\sqrt{u}\, \mathrm{e}^{ux(a_+)}}{\sqrt{2\pi\, x''(a_+)}} \int (z-a_-) z^{uf-1}(1-z)^{u-1}(1-z/Q)^{-u-1}\, dz\\
&= -\frac{f}{Q} \frac{\sqrt{u}\, \mathrm{e}^{ux(a_+)}}{\sqrt{2\pi\, x''(a_+)}} \sum_k Q^{-k} \frac{\Gamma(u+k+1)}{k!\,\Gamma(u+1)} \int (z-a_-) z^{k+uf-1}(1-z)^{u-1}\, dz\\
&= -\frac{f}{Q} \frac{\sqrt{u}\, \mathrm{e}^{ux(a_+)}}{\sqrt{2\pi\, x''(a_+)}} \sum_k Q^{-k} \frac{\Gamma(u+k+1)}{k!\,\Gamma(u+1)}\\
&\quad \times \left(\frac{\Gamma(k+uf)\,\Gamma(u)}{\Gamma(k+uf+u)} - a_- \frac{\Gamma(k+1+uf)\,\Gamma(u)}{\Gamma(k+1+uf+u)} \right)\\
&= -\frac{f}{Q} \frac{\mathrm{e}^{ux(a_+)}}{\sqrt{2\pi u\, x''(a_+)}} \sum_k Q^{-k} \frac{\Gamma(u+k+1)}{k!}\\
&\quad \times \left(\frac{\Gamma(k+uf)}{\Gamma(k+uf+u)} - a_- \frac{\Gamma(k+1+uf)}{\Gamma(k+1+uf+u)} \right)\\
&= -\frac{f}{Q} \frac{\mathrm{e}^{ux(a_+)}}{\sqrt{2\pi u\, x''(a_+)}} \sum_k Q^{-k} \frac{\Gamma(u+k+1)}{k!}\\
&\quad \times \frac{\Gamma(k+uf)}{\Gamma(k+1+uf+u)} \left(u + (1-a_-)(k+uf)\right).
\end{aligned}$$

Case $f=-1$:

(4.92)

$$\begin{aligned}
f_{++}(u) = \frac{1}{Q} &\frac{(a_+(1-a_+/Q)/(1-a_+))^u (Q(Q-1))^{1/4} (\sqrt{Q}+\sqrt{Q-1})}{\sqrt{\pi u}}\\
&\sum_k Q^{-k}\ \frac{\Gamma(u+k+1)\Gamma(k-u)}{k!^2} \left(u + (1-a_-)(k-u)\right),
\end{aligned}$$

(4.93)

$$\begin{aligned}
f_{+-}(u) = -&\frac{Q^{u+uf-1} f}{a_-} \frac{\sqrt{u}\, \mathrm{e}^{ux(a_-)}}{\sqrt{2\pi\, x''(a_+)}}\\
&\times \int (z-a_-)\ z^{-uf} (1-z/Q)^{u-1} (1-z)^{-u-1}\, dz.
\end{aligned}$$

The times are given by

$$
\begin{aligned}
(4.94)\qquad \mathrm{e}^{-g(u)} &= \frac{\sqrt{u}\,\mathrm{e}^{ux(a)}}{2\sqrt{\pi}}\int_0^1 \frac{z^{fu}\,(1-z)^u}{(1-z/Q)^u}\,\frac{dz}{z}\\
&= \frac{\sqrt{u}\,\mathrm{e}^{ux(a)}}{2\sqrt{\pi}}\sum_k Q^{-k}\,\frac{\Gamma(u+k)}{k!\,\Gamma(u)}\int_0^1 z^{fu+k}\,(1-z)^u\,\frac{dz}{z}\\
&= \frac{\sqrt{u}\,\mathrm{e}^{ux(a)}}{2\sqrt{\pi}}\sum_k Q^{-k}\,\frac{\Gamma(u+k)}{k!\,\Gamma(u)}\,\frac{\Gamma(fu+k)\Gamma(u+1)}{\Gamma((f+1)u+k+1)}\\
&= \frac{u\sqrt{u}\,\mathrm{e}^{ux(a)}}{2\sqrt{\pi}}\sum_k Q^{-k}\,\frac{\Gamma(u+k)\,\Gamma(fu+k)}{k!\,\Gamma((f+1)u+k+1)}.
\end{aligned}
$$

This is a hypergeometric function

$$
(4.95)\qquad \mathrm{e}^{-g(u)} = \frac{\sqrt{u}\,\mathrm{e}^{ux(a)}}{2\sqrt{\pi}}\,\frac{\Gamma(fu)\Gamma(u+1)}{\Gamma((f+1)u+1)}\,{}_2F_1(u,fu,(f+1)u+1;Q).
$$

5. Conclusion

In this paper, we have shown that the invariants of any spectral curve, have an interpretation in terms of intersection numbers in a moduli space

$$
(5.1)\qquad \mathcal{M}_{g,n}^{\mathfrak{b}}
$$

of maps of "colored" Riemann surfaces of genus g with n marked points into a discrete set $\{a_1,\dots,a_{\mathfrak{b}}\}$. This result generalizes the idea that Kontsevich–Witten intersection numbers compute the Gromov–Witten theory of a point, here instead we have the Gromov–Witten theory of a discrete set of points.

The computation also shows that to every spectral curve $\mathcal{S}$ is associated a certain class $\hat\Lambda(\mathcal{S})$ in that moduli space, which in some sense generalizes the Hodge class defined through the Mumford formula.

Descendents correspond to insertion of ψ classes, and the formula in Laplace transform, looks very similar to ELSV or Mariño–Vafa formula:

(5.2)

$$
\left(\mathrm{Laplace}_{k_1,\dots,k_n}\,W_n^{(g)}\right)(\mu_i) \propto \left\langle \hat\Lambda(\mathcal{S})\,\prod_{i=1}^n \frac{\sum_r \mu_i f_{k_i,r}(\mu_i)\,f_{\sigma(p_i),r}(1/\psi_i)}{1+\mu_i\,\psi_i}\right\rangle_{\mathcal{M}_{g,n}^{\mathfrak{b}}}
$$

Mirror symmetry

The definition of topological recursion and invariants $W_n^{(g)}$ starts with a spectral curve, i.e., a B-model geometry, and the moduli of the spectral curve are given by a 1-form ydx, and a Bergman kernel B.

On the other side, we have the Gromov–Witten theory of a set of points, i.e., an A-model geometry, and the coefficients (the moduli) are the coefficients of $\hat{\Lambda}$, and the coefficients of $f_{i,j}(u)$, and these are obtained by Laplace transform of the B-model moduli:

	B model	A model
moduli	$ydx\,,\ B$	$\hat{t}_{a_\sigma,k}\,,\ \check{B}_{a_\sigma,d;a_{\sigma'},d'}$

with

$$(\text{Laplace}_\sigma ydx)\,(\mu) \propto \mathrm{e}^{-\sum_k \hat{t}_{a_\sigma,k}\,\mu^{-k}}, \tag{5.3}$$

$$\big(\text{Laplace}_{\sigma,\sigma'} B\big)\,(\mu,\nu) \propto \sum_{k,l} \check{B}_{a_\sigma,d;a_{\sigma'},d'}\,\mu^{-k}\,\nu^{-l}. \tag{5.4}$$

In other words, the mirror map, which expresses the A-moduli in terms of the B-moduli, is simply the Laplace transform.

BKMP conjecture ?

BKMP conjecture [5, 23] claims that if we choose $\mathcal{S} = \hat{\mathfrak{X}}$ to be the mirror curve of a toric Calabi–Yau 3-fold $\mathfrak{X}$, then the $W_n^{(g)}$'s are the generating function of Gromov–Witten invariants of stable maps $f : \Sigma_{g,n} \to \mathfrak{X}$ with their boundaries on a Lagrangian submanifold $L \subset \mathfrak{X}$ (to which is associated the coordinate x of $\mathcal{S}$), i.e., the Gromov–Witten theory of

$$\mathcal{M}(\mathfrak{X}, L)_{g,n}.$$

According to what we have just found, this would mean that the Gromov–Witten theory of $\mathcal{M}(\mathfrak{X}, L)_{g,n}$ actually reduces to the the Gromov–Witten theory of a set of points $\{a_1, \ldots, a_{\mathfrak{b}}\}$, where the a_i's are the invariant points of the torus symmetry of $\mathfrak{X}$.

This would not be too surprising, since toric symmetry implies localization on invariant loci, and in particular near invariant points a_i. This well-known fact has given rise to the famous "topological vertex" theory [8, 19].

However, BKMP conjecture is yet to be proved (it was proved so far to all genus g only for $\mathfrak{X} = \mathbb{C}^3$), and our theorem here, does not give directly a proof…

Acknowledgments

I would like to thank G. Borot, A. Brini, C. Kozcaz, M. Mariño, M. Mulase for useful and fruitful discussions on this subject. This work is partly supported by the ANR project GranMa "Grandes Matrices Aléatoires" ANR-08-BLAN-0311-01, by the European Science Foundation through the Misgam program, by the Quebec government with the FQRNT, and the CERN for its support and hospitality.

Appendix F

A. Proof of Theorem 4.1

A.1. Graphical representation

In [10, 13], it was observed that the definition of $W_n^{(g)}$ can be written diagramatically:

$$W_n^{(g)}(\mathcal{S}; z_1, \ldots, z_n) = \sum_{G \in \mathcal{G}^{\mathfrak{b}}_{g,n}(z_1,\ldots,z_n)} w(G), \tag{A.1}$$

where $\mathcal{G}^{\mathfrak{b}}_{g,n}(z_1, \ldots, z_n)$ is the set of graphs with n external legs, g loops, constructed as follows:

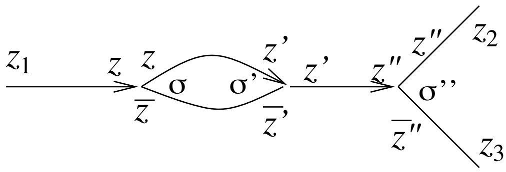
Definition A.1. a graph $G \in \mathcal{G}^{\mathfrak{b}}_{g,n}(z_1, \ldots, z_n)$ if and only if:

• G is a graph with $2g - 2 + n$ trivalent vertices, n labeled external legs (each ending on a 1-valent vertex)

• G has $2g - 2 + n$ arrowed edges forming an **oriented tree**, and $n + g - 1$ unoriented edges. The left and the right branch at each vertex of the tree are distinguished.

• Each oriented edge ends on a trivalent vertex, and all trivalent vertex sit at the end of an oriented edge.

• Each external leg but one is an unoriented edge. One external leg is at the beginning of an oriented edge, it is the root of the tree of oriented edges it has the label z_1. The other external legs have labels $z_2, \ldots, z_n$.



$$w(G)=\mathop{\mathrm{Res}}_{z\to a_i}\mathop{\mathrm{Res}}_{z'\to a_j}\mathop{\mathrm{Res}}_{z''\to a_k}K(z_1,z)K(z,z')K(z',z'')B(\bar z,\bar z')B(z'',z_2)B(\bar z'',z_3)$$

Figure 3: Example of a graph in $G\in\mathcal{G}_{1,3}^{\mathfrak{b}}(z_1,z_2,z_3)$. G has three external legs, one loop, three tri–valent vertices, three oriented edges and three unoriented edges. The residue in z'' is computed first, then z' then z.

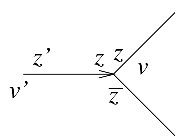
• G has g internal unoriented edges, each such internal unoriented edge can connect two points only if they are on the same branch of the tree (i.e., if one is the descendent of the other).

• Each trivalent vertex v carries an index $\sigma(v)\in\{1,2,\dots,\mathfrak{b}\}$.

To associate a weight $w(G)$ to a graph G, we label each trivalent vertex v by a spectral variable (i.e. $\in\mathcal{C}$) z_v, and the 1-valent vertices (i.e., the root and the $n-1$ leaves of the trees are labeled by $z_1,\dots,z_n$. This induces a labeling of edges $e=(z_{e+},z_{e-})$ of the graph in the following way:

let v be a tri-valent vertex with one oriented edge e with labels (z_{e_+},z_{e_-}) arriving on it, and two edges (oriented or not) going out of the vertex, the left child edge $e_{\text{left}}=(z_{e_{\text{left}}+},z_{e_{\text{left}}-})$ and the right child edge $e_{\text{right}}=(z_{e_{\text{right}}+},z_{e_{\text{right}}-})$. Then we have:

$$z_{e_-}=z_v,\quad z_{e_{\text{left}}+}=z_v,\quad z_{e_{\text{right}}+}=\bar z_v \tag{A.2}$$



– if an endpoint of an edge is a 1-valent vertex, it receives the corresponding variable z_i, in particular the root receives z_1.

Then the weight $w(G)$ is

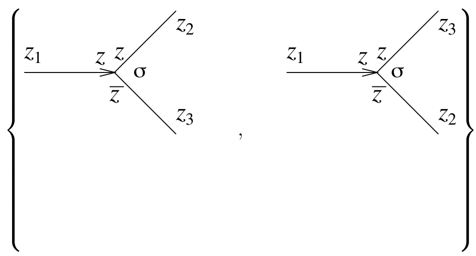
$$w(G)=:\prod_{v=\text{vertices}}:\mathop{\mathrm{Res}}_{z_v\to a_{\sigma(v)}}\prod_{e=\text{oriented edges}}K(z_{e+},z_{e-})\prod_{e=\text{unoriented edges}}B(z_{e+},z_{e-}), \tag{A.3}$$

where : $\prod$: means that we compute the residues in a "time ordered manner", i.e., the reverse order of the arrows along the tree, i.e., "leaves first, root last".

Remark A.1. the definition gives a special role to z_1, but it was shown in [13] that $W_n^{(g)}(\mathcal{S}; z_1, \ldots, z_n)$ is symmetric in all its variables, i.e., the result of the sum of weights is independent of which z_i is chosen as the root.

Example $\mathcal{G}_{0,3}^{\mathfrak{b}}(z_1, z_2, z_3)$ contains $2\mathfrak{b}$ graphs. Each graph $G \in \mathcal{G}_{0,3}^{\mathfrak{b}}$ (z_1, z_2, z_3) has one tri-valent vertex, one oriented edge and two unoriented edges. The oriented edge goes from the root z_1 to the vertex (label z), and the two unoriented edges go from the vertex to the leaves z_2 and z_3. z_2 is either the right or the left edge (and z_3 is the other). Moreover, the vertex has an index $\sigma \in \{1, 2, \ldots, \mathfrak{b}\}$.

$\mathcal{G}_{0,3}^{\mathfrak{b}}(z_1, z_2, z_3) =$



Eventually we have

$$W_3^{(0)}(\mathcal{S}; z_1, z_2, z_3) = \sum_{\sigma} \operatorname*{Res}_{z \to a_\sigma} K(z_1, z) B(z, z_2) B(\bar{z}, z_3) + \sum_{\sigma} \operatorname*{Res}_{z \to a_\sigma} K(z_1, z) B(z, z_3) B(\bar{z}, z_2). \tag{A.4}$$

A.2. Structure of weights

Since the weight $w(G)$ of a graph is evaluated by taking residues at branch-points, it has the same structure discussed in Section 4.2, i.e.,

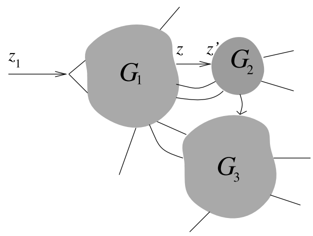
$$w(G(\sigma_1, z_1; \ldots, \sigma_n, z_n)) = \sum_{d_1, \ldots, d_n} A(G(\sigma_1, z_1; \ldots, \sigma_n, z_n); d_1, \ldots, d_n) \times \prod_{j=1}^{n} d\xi_{a_{\sigma_j}, d_j}(z_j) \tag{A.5}$$

and the coefficients $A(G; d_1, \ldots, d_n)$ is a polynomial in the $\hat{B}_{a_\sigma, k; a_{\sigma'}, l}$'s in the $\hat{t}_{a_\sigma, k}$'s of total degree at most $3g - 3 + n$ (where $\hat{B}_{a_i, k; a_j, l}$ is counted with degree 1, and $\hat{t}_{a_i,k}$ is counted with degree k), and it vanishes if $\sum_j d_j > 3g-3+n$.

A.3. Cutting graphs into clusters of constant index

Let $G = G(i_1, z_1; \ldots, i_n, z_n) \in \mathcal{G}^{\mathfrak{b}}_{g,n}(z_1, \ldots, z_n)$ be a graph.

Let us cut all lines that separate vertices of different index, that gives a set of subgraphs with constant index in each subgraph:



The computation of the weight $w(G)$ involves computing residues at $z \to a_\sigma$ and $z' \to a_{\sigma'}$, i.e., taking integrals along small circles centered around a_σ and $a_{\sigma'}$.

If $\sigma = \sigma'$, the order of computing the two residues (z or z' first ?) matters, because exchanging the order means moving one circle across the other, and since $K(z,z')$ and $B(z,z')$ have poles when $z = z'$, the exchange of order produces a residue at $z = z'$ (and possibly at $z' = \bar{z}$ for $K(z,z')$). But if, as we assume here, $\sigma \neq \sigma'$, the circles have different centers, and exchanging the order does not matter.

So, let us compute first all residues in a subgraph G_1, and let (z,z') be one of the external legs of graph G_1, i.e., it is a line separating vertices of different indexes σ and σ'. If (z,z') is an oriented line from z to z', let us write

$$K(z,z') = \mathop{\mathrm{Res}}_{z''\to z} \mathop{\mathrm{Res}}_{z'''\to z'} B(z,z'') \ \ln E(z'',z''') \ K(z''',z') \tag{A.6}$$

and If (z,z') is an unoriented line between z and z', let us write

$$B(z,z') = \mathop{\mathrm{Res}}_{z''\to z} \mathop{\mathrm{Res}}_{z'''\to z'} B(z,z'') \ \ln E(z'',z''') \ B(z''',z'), \tag{A.7}$$

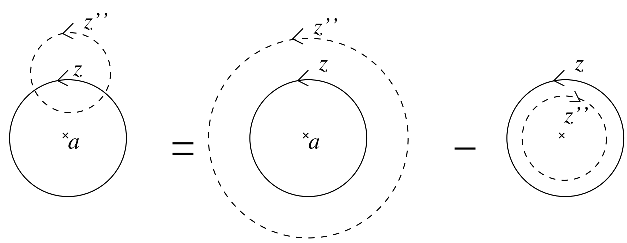
where $E(z'',z''')$ is the prime form, i.e., its second derivative with respect to z'' and z''' is $B(z'',z''')$:

$$d_{z''}\, d_{z'''} \ \ln E(z'',z''') = B(z'',z'''). \tag{A.8}$$

The computation of $w(G)$ involves computing a residue of the type:

$$\operatorname*{Res}_{z\to a_\sigma}\operatorname*{Res}_{z''\to z} f(z)\,B(z,z'')\;\ln E(z',z'''), \tag{A.9}$$

where $f(z)$ may have poles at a_σ and possibly at other spectral variables associated with adjacent vertices in the graph G. This last expression means that we should first compute the residue in z'' then in z. This means that z is integrated around a small circle around a_σ, and z'' is integrated around a small circle around z. The circle of z'' can be deformed into two circles around a_σ, one exterior to the circle of z, and one interior:



In other words we have:

$$\operatorname*{Res}_{z\to a_\sigma}\operatorname*{Res}_{z''\to z} = \operatorname*{Res}_{z''\to a_\sigma}\operatorname*{Res}_{z\to a_\sigma} - \operatorname*{Res}_{z\to a_\sigma}\operatorname*{Res}_{z''\to a_\sigma}. \tag{A.10}$$

Since the integrand has no pole at $z''=a_\sigma$, the last residue vanishes, and therefore we may write:

$$\operatorname*{Res}_{z\to a_\sigma}\operatorname*{Res}_{z''\to z} = \operatorname*{Res}_{z''\to a_\sigma}\operatorname*{Res}_{z\to a_\sigma}, \tag{A.11}$$

i.e., now we compute the z residue first.

The computation of the z residue, gives exactly the weight $w(G_1)$, and since G_1 has all its vertices of color σ, the weight $w(G_1)$ is the same as the one computed from the spectral curve $\mathcal{S}_{a_\sigma}$, i.e.,

$$w(G_1) = \overset{\circ}{w}(G_1), \tag{A.12}$$

and it takes the form (A.5)

$$\overset{\circ}{w}(G_1) = \sum_{d_1,\dots,d_n} \overset{\circ}{A}(G_1;d_1,\dots,d_n)\prod_{j=1}^{n} d\xi_{a_\sigma,d_j}(z_j). \tag{A.13}$$

and this has to be done for all subgraphs $G_1, G_2, \dots$ of G.

Now it remains to perform the integral over variables z'', z''' associated with lines with non-constant indices in G, and which now correspond to external legs of G_1, G_2.

Therefore we now have to compute:

$$C_{a_\sigma,d;a_{\sigma'},d'} = \mathop{\mathrm{Res}}_{z''\to a_\sigma} \mathop{\mathrm{Res}}_{z'''\to a_{\sigma'}} d\xi_{a_\sigma,d}(z'')\ \ln E(z'',z''')\ d\xi_{a_{\sigma'},d'}(z'''). \tag{A.14}$$

For this purpose, we Taylor expand $\ln E(z'',z''')$ in the vicinity of a_σ and $a_{\sigma'}$ using (4.3) and (A.8):

$$\ln E(z'',z''') \sim \sum_{k,l} \frac{B_{a_\sigma,k;a_{\sigma'},d'}}{(k+1)(l+1)}\, \zeta_{a_\sigma}(z'')^{k+1}\, \zeta_{a_{\sigma'}}(z''')^{l+1} \tag{A.15}$$

and

$$d\xi_{a_\sigma,d}(z'') \sim -\frac{(2d+1)!!}{2^d}\ \frac{d\zeta_{a_\sigma}(z'')}{\zeta_{a_\sigma}(z'')^{2d+2}} + \text{analytical near } a_\sigma \tag{A.16}$$

we have

$$C_{a_\sigma,d;a_{\sigma'},d'} = \frac{(2d-1)!!}{2^d}\ \frac{(2d'-1)!!}{2^{d'}}\ B_{a_\sigma,2d;a_{\sigma'},2d'} = 2\,\hat{B}_{a_\sigma,d;a_{\sigma'},d'}. \tag{A.17}$$

Eventually, this shows that the weight of a graph G can be obtained from the weights of its subgraphs:

$$\begin{aligned} A(G, d_{\text{external legs of } G}) = &\sum_{(d_v,d_{v'})=\text{cut edges }(v,v')} \prod 2\,\hat{B}_{a_{\sigma(v)},d_v;a_{\sigma(v')},d_{v'}} \\ &\times \prod_{G_i=\text{subgraphs}} \mathring{A}(G_i, d_{\text{external legs of } G_i}). \end{aligned} \tag{A.18}$$

Note that since the number of internal edges of a graph is $d_{g,n} = 3g-3+n$, we have

$$d_{g,n}(G) = \#\text{cut edges} + \sum_i d_{g_i,n_i}(G_i) \tag{A.19}$$

and therefore the powers of 2 match:

(A.20)
$$2^{-d_{g,n}} A(G, d_{\text{external legs of } G}) = \sum_{(d_v,d_{v'})=\text{cut edges}} \prod_{(v,v')} \hat{B}_{a_{\sigma(v)},d_v;a_{\sigma(v')},d_{v'}} \times \prod_{G_i=\text{subgraphs}} 2^{-d_{g_i,n_i}} \mathring{A}(G_i, d_{\text{external legs of } G_i}).$$

Now it remains to perform the sum over all graphs G, which is equivalent to a sum over all cut edges and for given cut edges, the sum over all subgraphs G_i. Since every graph and subgraph contain at least one trivalent vertex, and the number of trivalent vertices is $2g_i - 2 + n_i$, we have $2g_i - 2 + n_i > 0$ for each subgraph G_i, and thus the sum contains only stable terms. It is clear that the set of possible (cutting edges+ stable subgraphs), is in bijection with the nodal degenerations of a surface of genus g with n marked points.

From Theorem 3.1, the sum over all possible G_i's with given g_i, n_i and external legs with given degrees d_k is

$$2^{-d_{g_i,n_i}} \sum_{G_i} \mathring{A}(G_i, \{d_k\}) = \left\langle \hat{\Lambda}_{\sigma(G_i)} \prod_k \psi_k^{d_k} \right\rangle_{g_i,n_i}. \tag{A.21}$$

Eventually, substituting into (A.20), gives

(A.22)
$$2^{-d_{g,n}} A(G, d_{\text{external legs of } G}) = \sum_{\text{nodal degenerations}} \prod_{\text{nodal points}(q,q')} \sum_{d_q,d_{q'}} \prod_{(q,q')} \times \hat{B}_{a_{\sigma(q)},d_q;a_{\sigma(q')},d_{q'}} \prod \left\langle \hat{\Lambda}_{\sigma(G_i)} \prod_{k=1}^{n_i} \psi_k^{d_k} \right\rangle_{g_i,n_i}$$

and we recognize that the right-hand side is exactly what we have defined as

$$\int_{\mathcal{M}_{g,n}^{\flat}} \hat{\Lambda} \prod_{i=1}^{n} \psi(p_i)^{d_i} \tag{A.23}$$

i.e., we have proved the theorem, at least in the case where $\mathring{B} = B$ on each $\mathcal{C}_{a_i}$.

The derivation above, relied on exchanging the order of residues when $\sigma \neq \sigma'$ thanks to the fact that small circles around a_σ and $a_{\sigma'}$ do not intersect.

The exchange of order of residues can also work when $\sigma = \sigma'$ provided that we can move one circle through the other, which is possible if the integrand has no pole when circles cross each other. In particular, let us write:

$$B(z,z') - \overset{\circ}{B}(z,z') = \operatorname*{Res}_{z''\to z} \operatorname*{Res}_{z'''\to z'} \overset{\circ}{B}(z,z'') \, \ln \frac{E(z'',z''')}{\overset{\circ}{E}(z'',z''')} \, \overset{\circ}{B}(z''',z'). \tag{A.24}$$

The procedure above can be repeated, the only difference is that now we also allow to cut edges with the same color on both vertices. The coefficients $C_{a_\sigma,d;a_\sigma,d'}$ are then given by

$$C_{a_\sigma,d;a_\sigma,d'} = \operatorname*{Res}_{z''\to a_\sigma} \operatorname*{Res}_{z'''\to a_\sigma} d\xi_{a_\sigma,d}(z'') \, \ln \frac{E(z'',z''')}{\overset{\circ}{E}(z'',z''')} \, d\xi_{a_{\sigma'},d'}(z''') \tag{A.25}$$

i.e.,

$$C_{a_\sigma,d;a_{\sigma'},d'} = 2\, \check{B}_{a_\sigma,d;a_{\sigma'},d'} - 2\, \delta_{\sigma,\sigma'} \, \check{\overset{\circ}{B}}_{a_\sigma,d;a_\sigma,d'}. \tag{A.26}$$

This completes the proof of Theorem 4.1 □.

B. Case dx = meromorphic

Let $\mathcal{C}$ be a compact Riemann surface of some genus $\mathfrak{g}$, and let $\mathcal{A}_i, \mathcal{B}_j$ be a symplectic basis of non-contractible cycles on $\mathcal{C}$ such that the Bergman kernel is normalized on $\mathcal{A}$-cycles:

$$\oint_{z_2\in\mathcal{A}_i} B(z_1,z_2) = 0. \tag{B.1}$$

Let

$$dS_{z_1,z_2}(z) = \int_{z'=z_2}^{z_1} B(z,z'). \tag{B.2}$$

In that case we have

$$\xi_{a_i,0}(z) = -\frac{dS_{z,o}(a_i)}{d\zeta_{a_i}(a_i)} \qquad \text{and} \qquad d\xi_{a_i,0}(z) = -\frac{B(z,a_i)}{d\zeta_{a_i}(a_i)}. \tag{B.3}$$

Since dx is meromorphic, we see that $-d\xi_{a_i,d}/dx$ is a meromorphic function on $\mathcal{C}$, with a pole of degree $2d+3$ at a_i, and simple poles at a_j, $j \neq i$, and which behaves near a_i as $(2d+1)!!\, d^{-2-1}\, \zeta_{a_i}(z)^{-2d-3} - \hat{B}_{a_i,d;a_i,0}\, \zeta_{a_i}(z)^{-1} + O(1)$, and therefore, after substracting the simple poles at $z = a_j$, we see that this quantity has all the properties of $\xi_{a_i,d+1}(z)$, and thus (we use here that $\mathcal{C}$ is compact)

$$\xi_{a_i,d+1}(z) = -\frac{d\xi_{a_i,d}(z)}{dx(z)} - \sum_{j=1}^{\mathfrak{b}} \hat{B}_{a_i,d;a_j,0}\, \xi_{a_j,0}(z). \tag{B.4}$$

Taking the Laplace transform on contour γ_k we thus have

$$f_{a_k,a_i,d+1}(u) = -\, u f_{a_k,a_i,d}(u) - \sum_{j=1}^{\mathfrak{b}} \hat{B}_{a_i,d;a_j,0}\, f_{k,j}(u), \tag{B.5}$$

where

$$\begin{aligned} f_{a_k,a_i,d}(u) &= \frac{\sqrt{u}}{2\sqrt{\pi}} \int_{z\in\gamma_{a_k}} \mathrm{e}^{-u(x(z)-x(a_k))} dx(z)\, \xi_{a_i,d}(z) \\ &= \delta_{i,k}\, (-1)^d\, u^d - \sum_{d'} \hat{B}_{a_i,d;a_k,d'}\, u^{-d'-1}, \end{aligned} \tag{B.6}$$

$$f_{i,j}(u) = \frac{\sqrt{u}\, \mathrm{e}^{ux(a_i)}}{2\sqrt{\pi}} \int_{z\in\gamma_{a_i}} \mathrm{e}^{-ux(z)}\, \xi_{a_j,0}(z)\, dx(z). \tag{B.7}$$

Then

$$\check{B}_{a_k,a_i}(u,v) = \sum_{d,l} \hat{B}_{a_i,d;a_k,l}\, v^{-d}\, u^{-l} = \delta_{i,k}\, \frac{uv}{u+v} - u \sum_d v^{-d}\, f_{a_k,a_i,d}(u), \tag{B.8}$$

where the first equality is definition in Equation (4.5), and the second follows from (B.6). Applying (B.5) to (B.8) we get

$$\begin{aligned} \sum_d v^{-d}\, f_{a_k,a_i,d}(u) &= \sum_d v^{-d-1}\, f_{a_k,a_i,d+1}(u) \\ &= -v^{-1} \sum_d v^{-d}\, u f_{a_k,a_i,d}(u) \\ &\quad - \sum_{j=1}^{\mathfrak{b}} v^{-1} \sum_d v^{-d} \hat{B}_{a_i,d;a_j,0} f_{k,j}(u), \end{aligned}$$

which implies

$$(1 + uv^{-1}) \sum_d v^{-d} f_{a_k,a_i,d}(u) = -\sum_{j=1}^{\mathfrak{b}} v^{-1} \sum_d v^{-d} \hat{B}_{a_i,d;a_j,0} f_{k,j}(u). \\ = \sum_{j=1}^{\mathfrak{b}} f_{i,j}(v) f_{k,j}(u). \tag{B.9}$$

Inserting again in (B.8) we get

$$\check{B}_{a_k,a_i}(u,v) = \frac{uv}{u+v} \left(\delta_{i,k} - \sum_{j=1}^{\mathfrak{b}} f_{k,j}(u)\, f_{i,j}(v) \right). \tag{B.10}$$

References

[1] E Arbarello and M Cornalba, *Combinatorial and algebro-geometric cohomology classes on the moduli space of curves*, J. Algebr. Geom., **5** (1996), 705–709

[2] Y. Zhang and B. Dubrovin, *Normal forms of hierarchies of integrable pdes, frobenius manifolds and gromov-witten invariants*

[3] A. Bertram, R. Cavalieri and G. Todorov, *Evaluating tautological classes using only Hurwitz numbers*. page 11, August 2006

[4] G. Borot, B. Eynard, M. Mulase and B. Safnuk, *A matrix model for simple Hurwitz numbers, and topological recursion*

[5] V. Bouchard, A. Klemm, M. Mariño and S. Pasquetti, *Remodeling the B-Model*, Commun. Math. Phys., **287**(1) (2008), 117–178

[6] V. Bouchard and M. Marino, *Hurwitz numbers, matrix models and enumerative geometry*. page 21, September 2007

[7] C. Lin, *Bouchard–Klemm–Marino–Pasquetti conjecture for* $\mathbb{C}^3$. `arXiv:0910.3739`, 2009

[8] D.-E. Diaconescu and B. Florea, *Localization and Gluing of Topological Amplitudes*. Commun. Math. Phys., **257**(1) (2005), 119–149

[9] T. Ekedahl, S. Lando, M. Shapiro and A. Vainshtein, *On Hurwitz numbers and Hodge integrals*. C. R. Acad. Sci. I — Math., **328**(12) (1999), 1175–1180

[10] B Eynard, *Topological expansion for the 1-Hermitian matrix model correlation functions*, J. High Energy Phys., **11** (2004), 031–031

[11] B. Eynard, *Intersection numbers of spectral curves.* page 53, April 2011

[12] B. Eynard, M. Mulase, and B. Safnuk, *The Laplace transform of the cut-and-join equation and the Bouchard–Marino conjecture on Hurwitz numbers*

[13] B. Eynard and N. Orantin, *Invariants of algebraic curves and topological expansion.* February 2007

[14] B. Eynard and N. Orantin, *Weil–Petersson volume of moduli spaces, Mirzakhani's recursion and matrix models.* page 9, May 2007

[15] B. Eynard and N. Orantin, Algebraic methods in random matrices and enumerative geometry. page 139, November 2008

[16] C. Faber and R. Pandharipande, *Hodge integrals and Gromov–Witten theory*. `arXiv:math/9810173`, 1998

[17] A. Givental, *Gromov–Witten invariants and quantization of quadratic Hamiltonians*, Mosc. Math. J. **1**(4) (2001), 551–568

[18] H. Xu, K. Liu, A proof of the Faber intersection number conjecture. December 2009

[19] M. Kontsevich, *Enumeration of rational curves via torus actions*, in 'The Moduli space of curves', eds. R. Dijkgraaf, C. Faber and G. Van der Geer, Progress in Mathematics **129**, 335–368, Birkhauser, Boston, MA, 1995. `e-print hep-th/9405035`.

[20] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix airy function*, Commun. Math. Phys., **147**(1) (1992), 1–23

[21] I. Kostov and N. Orantin, *CFT and topological recursion.* J. High Energy Phys., **2010**(11) (2010)

[22] I.K. Kostov, *Conformal field theory techniques in Random matrix models*. page 25, July 1999

[23] M. Mariño, *Open string amplitudes and large order behavior in topological string theory*, J. High Energy Phys., **03** (2008), 060–060

[24] M. Marino and C. Vafa, *Framed knots at large N*. page 26, August 2001

[25] M. Mulase and M. Penkava, *Topological recursion for the Poincare polynomial of the combinatorial moduli space of curves*. page 16, September

2010

[26] M. Mulase and B. Safnuk, *Mirzakhani's recursion relations, Virasoro constraints and the KdV hierarchy.* page 21, January 2006

[27] D. Mumford, *Towards an enumerative geometry of the moduli space of curves*, pages 271–328. Birkhauser, 1983

[28] P. Norbury and N. Scott, *Gromov–Witten invariants of* $\mathbb{P}^1$ *and Eynard-Orantin invariants.* page 30, June 2011

[29] N. Orantin, *Symplectic invariants, Virasoro constraints and Givental decomposition.* `arXiv:0808.0635`, 2008

[30] A. Prats Ferrer, *New recursive residue formulas for the topological expansion of the Cauchy matrix model*, J. High Energy Phys., **10** (2010), 42

[31] S. Wolpert, *On the homology of the Moduli space of stable curves*, Ann. Math., **118**(3) (1983), 491–523

[32] J. Zhou, *Local mirror symmetry for one-legged topological vertex.* `arXiv:0910.4320`, 2009

Institut de Physique Théorique, CE Saclay,
F-91191 Gif-sur-Yvette Cedex
France
Département de Mathématiques
Université of Genève
2-4, rue du Lièvre, Case postale 64, 1211 Genève 4
Switzerland
E-mail address: bertrand.eynard@cea.fr

Received January 9, 2014