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Zeros of Dirichlet L-functions over function fields

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Random matrix theory has successfully modeled many systems in physics and mathematics, and often analysis in one area guides development in the other. Hughes and Rudnick computed one-level density statistics for low-lying zeros of the family of primitive Dirichlet L-functions of fixed prime conductor Q, as $Q \to \infty$, and verified the unitary symmetry predicted by the random matrix theory. We compute one- and two-level statistics of the analogous family of Dirichlet L-functions over $\mathbb{F}_q(T)$. Whereas the Hughes–Rudnick results were restricted by the support of the Fourier transform of their test function, our test function is periodic and our results are only restricted by a decay condition on its Fourier coefficients. Our statements are more general and also include error terms. In concluding, we discuss an $\mathbb{F}_q(T)$-analog of Montgomery's Hypothesis on the distribution of primes in arithmetic progressions, which Fiorilli and Miller show would remove the restriction on the Hughes–Rudnick results.

1. Introduction

1.1. Background

In the 1970s, Montgomery and Dyson conjectured that local statistics of critical zeros of the Riemann zeta function — in the limit of large height — should match those of angles of eigenvalues of matrices in the Gaussian Unitary Ensemble (GUE), which Wigner, Dyson and others (see [12] for a historical overview) had already used with great success in modeling the energy levels of heavy nuclei. Their ideas, exemplified by the Pair Correlation Conjecture in [24], began a long history of investigation into connections between number theory, physics and random matrix theory. Odlyzko [25, 26] checked various statistics of critical zeros of the Riemann zeta function high up on the critical line numerically, including pair correlation, and found extraordinary agreement with GUE predictions. Katz and Sarnak extended this philosophy to families of L-functions in [19, 20]. They proposed that zeros of L-functions in suitable "families" would have similar statistics to each other, and that the statistics of a given family, in the limit of large analytic conductor, would match those of eigenangles of matrices in some classical compact group under Haar measure, in the limit of large dimension. Thus, families of L-functions would correspond to one of three basic symmetry types: unitary, symplectic, or orthogonal. For a recent discussion about a working definition of families of L–functions, as well as how to determine the underlying symmetry see [7, 31, 32].

Originally, the Katz–Sarnak Conjectures were investigated in the so-called local regime near the central point $s = 1/2$; that is, in intervals around $s = 1/2$ shrinking as the conductor grows, so that the number of zeros it contains is roughly constant (see, among others, [15, 16, 18, 22, 29, 39]). In this regime the conjectures are very difficult, and most results are limited to test functions whose Fourier transforms have severely restricted support. Moreover, it is necessary to average over a "family", as one L-function cannot have sufficiently many normalized zeros near the central point. This is in sharp contrast to other statistics such as the n-level correlation; these statistics study zeros high up on the critical line, and one L-function has sufficiently many zeros far from the central point to permit an averaging. For example, Rudnick and Sarnak [28] computed the n-level correlation for the zeroes of not just the Riemann zeta function but any cuspidal automorphic form for a restricted class of test functions, proving their expression agrees with the n-level correlation for the eigenvalues of random unitary matrices.

Our own work extends [16], in which Hughes and Rudnick compute the mth centered moment of the one-level density of the family of primitive Dirichlet L-functions of fixed conductor Q as $Q \to \infty$, for test functions ϕ such that $\mathrm{supp}(\hat{\phi}) \subset (-2/m, +2/m)$. This family should have unitary symmetry, which the authors verified for suitably restricted test functions. We consider analogous questions in the function field case. There has been significant progress in this area of late (see [13, 30] among others); we briefly comment on one particular example which illuminates the contributions that function field results can have to random matrix theory and mathematical physics.

In his thesis, Rubinstein [29] showed the n-level density of quadratic Dirichlet L-functions agrees with the random matrix theory predictions for support in $(-1/n, 1/n)$. In the course of his investigations, he analyzed the combinatorial expansions for the n-level densities of the classical compact groups, although he only needed the results for restricted test functions due to the limitations on the number theory calculations. Gao [14] doubled the support in his thesis, but do to the complexity of the combinatorics was only able to show the two computed quantities agreed for $n \leq 3$.

Levinson and Miller [21] devised a new approach which allowed them to show agreement for $n \leq 7$; unlike the *ad-hoc* method of Gao, they developed a canonical formulation of the quantities and reduced the general case to a (still open) combinatorial identity involving Fourier transforms. The work of Entin, Roditty–Gershon and Rudnick [10] bypasses these calculations using known results in the function field case to deduce that the combinatorics must equal.

Katz and Sarnak suggested that a possible motivation for their conjectures is the analogy between number fields and global function fields. The Riemann zeta function and Dirichlet L-functions can be considered L-functions "over $\mathbb{Q}$"; they possess analogues "over $\mathbb{F}_q(T)$," which occur as factors of numerators of zeta functions of projective curves over $\mathbb{F}_q$. As proved by Deligne [6], the zeros of the latter have a spectral interpretation, as reciprocals of eigenvalues of the Frobenius endomorphism acting on ℓ-adic cohomology. Katz and Sarnak [19] proved agreement with GUE n-level correlation *unconditionally* for the family of isomorphism classes of curves of genus g over $\mathbb{F}_q$, in the limit as both $g, q \to \infty$. Their main tool was Deligne's result that the Frobenius conjugacy classes become equidistributed in the family's monodromy group as $q \to \infty$.

Recently, there has been interest in the "opposite limit," where q is held fixed and $g \to \infty$. In [13], the authors considered zeros of zeta functions of hyperelliptic curves of genus g over $\mathbb{F}_q$. Instead of looking at zeros in the *local* regime, they look at zeros in (1) *global* and (2) *mesoscopic* regimes: that is, in intervals $\mathcal{I}$ around 0 such that either (1) $|\mathcal{I}|$ is fixed, or (2) $|\mathcal{I}| \to 0$ but $g|\mathcal{I}| \to \infty$. In both regimes, they show that the zeros become equidistributed in $\mathcal{I}$ as $g \to \infty$, and the normalized fluctuations in the number of the zeros are Gaussian. Xiong [38] extended their work to families of ℓ-fold covers of $\mathbb{P}^1(\mathbb{F}_q)$, for prime ℓ such that $q \equiv 1 \pmod{\ell}$, again obtaining Gaussian behavior. For other works in this direction see [1–3].

1.2. Outline

We study the $\mathbb{F}_q(T)$-analog of the Dirichlet L-function family of [16]. Specifically, we compute one- and two-level statistics of its zeros in the global regime, then show how they imply statistics in the local regime. In this introduction, we only state the results in the global regime; we save for later sections the full statements of the local results, as these require significantly more notation to state. Whereas the Hughes–Rudnick results were restricted by the support of $\hat{\phi}$, our global test function ψ is periodic and our results are only restricted by a decay condition on the Fourier coefficients $\hat{\psi}(n)$.

In what follows, let $Q \in \mathbb{F}_q[T]$. Let $\mathcal{F}_Q$ be the family of primitive Dirichlet characters $\chi : \mathbb{F}_q[T] \to \mathbb{C}$ of modulus Q, and let $\mathcal{F}_Q^{\text{even}}$ be the subfamily of even characters in $\mathcal{F}_Q$.

1.2.1. One-level statistics. A *(one-dimensional) test function of period 1* is a holomorphic Fourier series $\psi(s) = \sum_{n\in\mathbb{Z}} \hat{\psi}(n)e(ns)$. The average or expectation of a function $F : \mathcal{F}_Q \to \mathbb{C}$ is

$$\mathbb{E}F \;=\; \frac{1}{\#\mathcal{F}_Q} \sum_{\chi\in\mathcal{F}_Q} F(\chi); \tag{1.1}$$

this sum is well defined as there are only finitely many $\chi \in \mathcal{F}_Q$. With these definitions, we set

$$F_{1,\chi}(\psi) \;:=\; \frac{1}{d-1} \sum_{-\frac{T_q}{2}\le\gamma_\chi<\frac{T_q}{2}} \psi\left(\frac{\gamma_\chi}{T_q}\right), \tag{1.2}$$

where above, γ_χ runs through the ordinates of the zeros $1/2 + i\gamma_\chi$ of $L(s,\chi)$.

Theorem 1.1. *Suppose Q is irreducible of degree $d \geq 2$. Let ψ be a test function of period 1 such that*

$$C(\psi) = \sum_{n\in\mathbb{Z}} |\hat{\psi}(n)|q^{|n|/2} \tag{1.3}$$

converges. Then

$$\mathbb{E}F_{1,\chi}(\psi) \;=\; \hat{\psi}(0) - \frac{1}{(d-1)(q-1)} \sum_{n\in\mathbb{Z}} \frac{\hat{\psi}(n)}{q^{|n|/2}} + O\left(\frac{C(\psi)}{dq^d}\right). \tag{1.4}$$

The variance of a function $F : \mathcal{F}_Q \to \mathbb{C}$ is

$$\text{Var } F \;:=\; \mathbb{E}|F - \mathbb{E}F|^2 \;=\; \mathbb{E}|F|^2 - |\mathbb{E}F|^2.$$

Theorem 1.2. *Suppose Q is irreducible of degree $d \geq 2$. Let ψ be a test function of period 1 such that $C(\psi)$ converges, where $C(\psi)$ is defined in Theorem 1.1. Then*

$$\text{Var } F_{1,\chi}(\psi) = \frac{1}{(d-1)^2} \sum_{n\in\mathbb{Z}} |n||\hat{\psi}(n)|^2 + O\left(\frac{C(\psi)^2}{d^2q^d}\right). \tag{1.5}$$

1.2.2. Two-level statistics. A *two-dimensional test function of period* 1 is a bivariate Fourier series $\psi(s_1, s_2) = \psi_1(s_1)\psi_2(s_2) = \sum_{n_1,n_2\in\mathbb{Z}} \hat{\psi}_1(n_1)\hat{\psi}_2(n_2)e(n_1s_1 + n_2s_2)$. We set

$$F_{2,\chi}(\psi) = \frac{1}{(d-1)^2} \sum_{\substack{-\frac{T_q}{2}\le\gamma_{\chi,1},\gamma_{\chi,2}<\frac{T_q}{2} \\ \gamma_{\chi,2}\neq\gamma_{\chi,1}}} \psi\left(\frac{\gamma_{\chi,1}}{T_q}, \frac{\gamma_{\chi,2}}{T_q}\right). \tag{1.6}$$

Theorem 1.3. *Suppose Q is irreducible of degree $d \geq 2$. Let ψ be a two-dimensional test function of period 1 such that*

$$C(\psi) = C(\psi_1)C(\psi_2) \tag{1.7}$$

converges, where $C(\psi)$ is defined for one-dimensional test functions ϕ of period 1 *in Theorem 1.1. Let $\psi_{\text{diag}}(s) = \psi(s, s)$. Then*

(1.8)
$$\mathbb{E}F_{2,\chi}(\psi) = -\mathbb{E}F_{1,\chi}(\psi_{\text{diag}}) + \hat{\psi}(0,0) + \frac{1}{(d-1)^2}\sum_{n\in\mathbb{Z}} |n|\hat{\psi}(n,-n) + \frac{C_{2,\Gamma}(\psi)}{q-1} + O\left(\frac{C(\psi_1)+C(\psi_2)}{dq^d}\right) + O\left(\frac{C(\psi)}{d^2q^d}\right),$$

where
(1.9)
$$C_{2,\Gamma}(\psi) = -\frac{1}{d-1}\sum_{n\in\mathbb{Z}} \frac{\hat{\psi}(0,n)+\hat{\psi}(n,0)}{q^{|n|/2}} + \frac{1}{(d-1)^2}\sum_{n_1,n_2\in\mathbb{Z}} \frac{\hat{\psi}(n_1,n_2)}{q^{(|n_1|+|n_2|)/2}}.$$

1.2.3. Complements. In Section 5, we explain how to pass from global to local. Our global statements above imply local statements analogous to those of [16], but are more general and also include error terms. In Section 6, we discuss an $\mathbb{F}_q(T)$-analog of Montgomery's Hypothesis, which Fiorilli and Miller [11] show would remove the restriction on the Hughes–Rudnick results.

1.2.4. Connections. We end this introduction by briefly describing the connections this work emphasizes between number theory and physics. For us, the most important bridge between these two areas is the one provided by random matrix theory, which since the work of Wigner [35–37], Dyson [8, 9] and others has highlighted commonalities between the two topics (see the recent work of Conrey et al. [4, 5] for a related, relevant example). One of the most important issues in both fields is to obtain results in the greatest possible generality. In physics, this can be seen in the attempts to shoot neutrons of arbitrary energy into a heavy nucleus. The corresponding problem in number theory (where zeros of L-functions play an analogous role to those of energy levels, and the support of the Fourier transform corresponds to the energy level of the neutron) is to understand sums of the Fourier transform of the test function for arbitrary support. In physics, we obtain a greater understanding of the internal structure from incident neutrons of varying energies; however, in practice we can only work with neutrons whose energies are in a specified band. Similar insights occur in number theory; in the ideal situation where we could take test functions whose Fourier transforms have arbitrarily large support, this would correspond to a test function that is a Dirac delta spike, and hence yield perfect information at a point.

While both number theory and physics are quite far from being able establish to the above, the hope is that insights and formulas in one area can help drive progress in the other. In particular, in this paper we study a function field problem (these are typically easier than number theory problems) and investigate global statistics that can be computed in larger regimes than other number theory quantities; we hope the resulting formulas will be of interest and use to researchers in both disciplines.

2. Dirichlet L-function preliminaries

We always assume q is a prime power below. We write $\sum', \prod'$ to denote a sum or product restricted to monic polynomials in $\mathbb{F}_q[T]$, and $\sum_P, \prod_P$ to denote a sum or product over irreducibles in $\mathbb{F}_q[T]$. If $f \in \mathbb{F}_q[T]$, then $|f|$ equals 0 if $f = 0$ and $q^{\deg f}$ if $f \neq 0$.

Fix a nonconstant modulus $Q \in \mathbb{F}_q[T]$ of degree d, and consider Dirichlet characters $\chi : \mathbb{F}_q[T] \to \mathbb{C}$ of modulus Q. To each nontrivial character χ, one associates the L-function

$$L(s,\chi) := {\sum_f}' \frac{\chi(f)}{|f|^s} = \sum_{n=0}^{d-1} {\sum_{\deg f = n}}' \chi(f) q^{-ns}. \tag{2.1}$$

We briefly review its properties, following Chapter 4 of [27]. It possesses the Euler product

$$L(s,\chi) = {\prod_P}' \frac{1}{1-\chi(P)|P|^{-s}}. \tag{2.2}$$

Taking logarithmic derivatives of both sides gives

$$\frac{L'}{L}(s,\chi) = -(\log q)\sum_{n=0}^{\infty} c_\chi(n) q^{-ns}, \tag{2.3}$$

where $c_\chi(n) = \sum'_{\deg f = n} \Lambda(f)\chi(f)$ and

$$\Lambda(f) = \begin{cases} \deg P & f = P^\nu \text{ for some irreducible monic } P \text{ and } \nu \in \mathbb{Z}_+ \\ 0 & \text{otherwise} \end{cases} \tag{2.4}$$

is the von Mangoldt function over $\mathbb{F}_q[T]$.

Since we wish to emphasize the analogy between these L-functions and number-field Dirichlet L-functions, we prefer to consider their zeros in the variable s rather than q^{-s}. The Riemann Hypothesis, proved for these L-functions by Weil [34], implies that the critical zeros of $L(s,\chi)$ live on the line $\Re s = 1/2$ and thus are *vertically periodic* with period $2\pi/\log q$. Moreover, the Riemann Hypothesis implies that $c_\chi(n) \ll dq^{n/2}$ for all $\chi \neq \chi_0$ (see [27]).

We consider the completed L-function (a good reference is Chapter 7 of [33]). Suppose χ is primitive. Then the completed L-function associated with χ is

$$\mathcal{L}(s,\chi) = \frac{1}{1-\lambda_\infty(\chi)q^{-s}} L(s,\chi) \tag{2.5}$$

where $\lambda_\infty(\chi)$ equals 1 if χ is even, meaning $\mathbb{F}_q^\times \subset \ker\chi$, and 0 if χ is odd. The functional equation of $\mathcal{L}(s,\chi)$ is

$$\mathcal{L}(s,\chi) = \epsilon(\chi)(q^{d(\chi)})^{1/2-s}\mathcal{L}(1-s,\bar{\chi}), \tag{2.6}$$

where $d(\chi) = d - 1 - \lambda_\infty(\chi)$ is the degree of $L(s,\chi)$ seen as a polynomial in the variable q^{-s} and $\epsilon(\chi) \in S^1$ is some root number. Translating (2.5)–(2.6) into statements about the logarithmic derivatives gives

$$\frac{\mathcal{L}'}{\mathcal{L}}(s,\chi) = \frac{L'}{L}(s,\chi)\frac{\lambda_\infty(\chi)\log q}{\lambda_\infty(\chi) - q^s} \tag{2.7}$$

and

$$\frac{\mathcal{L}'}{\mathcal{L}}(s,\chi) = -d(\chi)\log q - \frac{\mathcal{L}'}{\mathcal{L}}(1-s,\bar{\chi}). \tag{2.8}$$

Therefore, using the fact that $\lambda_\infty(\bar{\chi}) = \lambda_\infty(\chi)$, we find

(2.9)
$$\begin{aligned} &-\frac{L'}{L}(1-s,\bar{\chi}) \\ &\quad= d(\chi)\log q + \frac{L'}{L}(s,\chi) + \lambda_\infty(\chi)\left(\frac{1}{\lambda_\infty(\chi)-q^s} + \frac{1}{\lambda_\infty(\chi)-q^{1-s}}\right)\log q \\ &\quad= d(\chi)\log q + \frac{L'}{L}(s,\chi) + \lambda_\infty(\chi)\left(\frac{1}{1-q^s} + \frac{1}{1-q^{1-s}}\right)\log q. \end{aligned}$$

The following formula is essentially Lemma 2.2 of [13]. We rederive it in appendix A in a way that facilitates comparison with classical Explicit Formulas, such as that of [28]. To state the result, abbreviate

$$T_q = \frac{2\pi}{\log q}, \tag{2.10}$$

which is the correct rescaling for zeros near the central point.

Proposition 2.1. *Let $Q \in \mathbb{F}_q[T]$ be of degree $d \geq 2$, and let χ be a non-trivial Dirichlet character of modulus Q. Let ψ be a test function of period 1. Then*

(2.11)
$$\begin{aligned} F_{1,\chi}(\psi) &:= \frac{1}{d-1} \sum_{-\frac{T_q}{2} \leq \gamma_\chi < \frac{T_q}{2}} \psi\left(\frac{\gamma_\chi}{T_q}\right) \\ &= \hat{\psi}(0) - \frac{\lambda_\infty(\chi)}{d-1} \sum_{n\in\mathbb{Z}} \frac{\hat{\psi}(n)}{q^{|n|/2}} \frac{1}{d-1} \sum_{n=0}^{\infty} \frac{c_\chi(n)\hat{\psi}(n) + c_{\bar{\chi}}(n)\hat{\psi}(-n)}{q^{n/2}}, \end{aligned}$$

where $\lambda_\infty(\chi)$ equals 1 if χ is even and 0 otherwise, and $c_\chi(n) = \sum'_{\deg f = n} \Lambda(f)\chi(f)$.

3. The one-level global regime

The computations in this section are closely based on those in [16].

3.1. Expectation

Proof of 1.1. By the Explicit Formula

$$\mathbb{E}F_{1,\chi}(\psi) = \hat{\psi}(0) - \frac{1}{d-1} \cdot \frac{\#\mathcal{F}_Q^{\text{even}}}{\#\mathcal{F}_Q} \sum_{n \in \mathbb{Z}} \frac{\hat{\psi}(n)}{q^{|n|/2}} + \mathbb{E}(F_{1,\chi}(\psi)^{\text{osc}}), \tag{3.1}$$

where

$$F_{1,\chi}(\psi)^{\text{osc}} = -\frac{1}{d-1} \sum_{n=0}^{\infty} \frac{c_\chi(n)\hat{\psi}(n) + c_{\bar{\chi}}(n)\hat{\psi}(-n)}{q^{n/2}}. \tag{3.2}$$

Since Q is monic and irreducible, the only imprimitive character modulo Q is the principal character χ_0. In this case, there are $(|Q|-1)/(q-1)$ even characters including χ_0, so we know $\#\mathcal{F}_Q^{\text{even}}/\#\mathcal{F}_Q$ is roughly $1/(q-1)$. It remains to estimate $\mathbb{E}(F_{1,\chi}(\psi)^{\text{osc}})$.

By Schur orthogonality,

$$\mathbb{E}\chi(f) = \begin{cases} 0 & f \equiv 0 \pmod{Q}, \\ 1 & f \equiv 1 \pmod{Q}, \\ -1/\#\mathcal{F}_Q & \text{otherwise}, \end{cases} \tag{3.3}$$

for all $f \in \mathbb{F}_q[T]$, and similarly with $\mathbb{E}\bar{\chi}(f)$. Therefore

$$\mathbb{E}(F_{1,\chi}(\psi)^{\text{osc}}) = -\frac{1}{d-1} \sum_{n=0}^{\infty} \left(\sideset{}{'}\sum_{\substack{\deg f = n \\ f \equiv 1 \pmod{Q}}} \frac{1}{\#\mathcal{F}_Q} \sideset{}{'}\sum_{\substack{\deg f = n \\ f \not\equiv 0,1 \pmod{Q}}} \right) \times \Lambda(f) \frac{\hat{\psi}(n) + \hat{\psi}(-n)}{q^{n/2}}. \tag{3.4}$$

To estimate the contribution of the first term in the big parenthesized expression above we make use of the function-field analog of the Brun–Titchmarsh Theorem (see [17]), which states that

$$\sideset{}{'}\sum_{\substack{\deg f = n \\ f \equiv 1 \pmod{Q}}} \Lambda(f) \ \leq \ C q^{n-d} \tag{3.5}$$

for some $C > 0$ independent of Q, n. Thus the contribution from this term is

$$(3.6)\quad -\frac{1}{(d-1)}\sum_{n=0}^{\infty}\ \sideset{}{'}\sum_{\substack{\deg f=n\\ f\equiv 1 \pmod Q}} \Lambda(f)\frac{\hat{\psi}(n)+\hat{\psi}(-n)}{q^{n/2}} \ll \frac{1}{dq^d}\sum_{n\in\mathbb{Z}}\hat{\psi}(n)q^{|n|/2}.$$

On the other hand, by the Prime Number Theorem in this setting (see [27]) we have

$$(3.7)\qquad \sideset{}{'}\sum_{\deg f=n}\Lambda(f) = q^n + O(q^{n/2}),$$

where the implied constant is independent of q. Thus the second term of the big parenthesized expression in (3.4) contributes with the same order, completing the proof. □

Corollary 3.1. *Suppose Q is irreducible. Let ψ be a test function of period 1 such that*

$$(3.8)\qquad \hat{\psi}(n) \ll \frac{1}{|n|^{1+\epsilon}q^{|n|/2}}$$

for some $\epsilon > 0$. Then

$$(3.9)\qquad \mathbb{E}F_{1,\chi}(\psi) = \hat{\psi}(0) - \frac{1}{(d-1)(q-1)}\sum_{n\in\mathbb{Z}}\frac{\hat{\psi}(n)}{q^{|n|/2}} + O\left(\frac{1}{d}\right),$$

where $F_{1,\chi}$ is defined in (1.2).

3.2. Variance

Proof of 1.2. Let $C_{1,\Gamma}(\psi) = (d-1)^{-1}\sum_{n\in\mathbb{Z}}\hat{\psi}(n)q^{-|n|/2}$. Then

$$(3.10)\qquad F_{1,\chi}(\psi) - \mathbb{E}F_{1,\chi}(\psi) = (F_{1,\chi}(\psi)^{\mathrm{osc}} - \mathbb{E}(F_{1,\chi}(\psi))^{\mathrm{osc}}) + C_{1,\Gamma}(\psi)\left(\lambda_\infty(\chi) - \frac{1}{q-1}\right),$$

from which

$$\begin{aligned}\text{Var } F_{1,\chi}(\psi) &= \text{Var } (F_{1,\chi}(\psi)^{\text{osc}}) + 2\Re\mathbb{E}((F_{1,\chi}(\psi)^{\text{osc}} - \mathbb{E}(F_{1,\chi}(\psi))^{\text{osc}}) \\ &\quad \times C_{1,\Gamma}(\psi)\lambda_\infty(\chi)) + O\left(\frac{1}{d^2}\right) \\ &= \text{Var } (F_{1,\chi}(\psi)^{\text{osc}}) + O\left(\frac{1}{d}\mathbb{E}(F_{1,\chi}(\psi)^{\text{osc}})\right). \end{aligned} \tag{3.11}$$

Next, $\text{Var } (F_{1,\chi}(\psi)^{\text{osc}}) = \mathbb{E}|F_{1,\chi}(\psi)^{\text{osc}}|^2 - |\mathbb{E}(F_{1,\chi}(\psi)^{\text{osc}})|^2$ where

$$\begin{aligned}|F_{1,\chi}(\psi)^{\text{osc}}|^2 = \frac{1}{(d-1)^2} \sum_{n_1,n_2=0}^{\infty} \sideset{}{'}\sum_{\substack{\deg f_1 = n_1 \\ \deg f_2 = n_2}} \frac{\Lambda(f_1)\Lambda(f_2)}{q^{(n_1+n_2)/2}} \\ \Big(\chi(f_1)\bar{\chi}(f_2)\hat{\psi}(n_1)\bar{\hat{\psi}}(n_2) + \chi(f_1)\chi(f_2)\hat{\psi}(n_1)\bar{\hat{\psi}}(-n_2) \\ + \bar{\chi}(f_1)\bar{\chi}(f_2)\hat{\psi}(-n_1)\bar{\hat{\psi}}(n_2) + \bar{\chi}(f_1)\chi(f_2)\hat{\psi}(-n_1)\bar{\hat{\psi}}(-n_2)\Big).\end{aligned} \tag{3.12}$$

Again by Schur orthogonality,

$$\mathbb{E}\left(\chi(f_1)\bar{\chi}(f_2)\right) = \begin{cases} 0 & f_1 \equiv 0 \text{ or } f_2 \equiv 0 \pmod{Q}, \\ 1 & f_1 \equiv f_2 \not\equiv 0 \pmod{Q}, \\ -1/\#\mathcal{F}_Q & \text{otherwise,} \end{cases} \tag{3.13}$$

$$\mathbb{E}\left(\chi(f_1)\chi(f_2)\right) = \begin{cases} 0 & f_1 \equiv 0 \text{ or } f_2 \equiv 0 \pmod{Q}, \\ 1 & f_1 f_2 \equiv 1 \pmod{Q}, \\ -1/\#\mathcal{F}_Q & \text{otherwise.} \end{cases} \tag{3.14}$$

Therefore

$$\begin{aligned}\mathbb{E}|F_{1,\chi}(\psi)^{\text{osc}}|^2 &= \frac{1}{(d-1)^2} \sum_{n_1,n_2=0}^{\infty} \frac{1}{q^{(n_1+n_2)/2}} \\ &\quad \times \Big(C_1(n_1,n_2;Q)\left(\hat{\psi}(n_1)\bar{\hat{\psi}}(n_2) + \hat{\psi}(-n_1)\bar{\hat{\psi}}(-n_2)\right) \\ &\quad + C_2(n_1,n_2;Q)\left(\hat{\psi}(n_1)\bar{\hat{\psi}}(-n_2) + \hat{\psi}(-n_1)\bar{\hat{\psi}}(n_2)\right)\Big) \\ &\quad + O\left(\frac{1}{\#\mathcal{F}_Q}\left(\frac{1}{d-1}\sum_{n=0}^{\infty}\sideset{}{'}\sum_{\deg f = n} \Lambda(f)\frac{\hat{\psi}(n)}{q^{n/2}}\right)^2\right),\end{aligned} \tag{3.15}$$

where

$$C_1(n_1, n_2; Q) = \sideset{}{'}\sum_{\substack{\deg f_1 = n_1 \\ \deg f_2 = n_2 \\ f_1 \equiv f_2 \not\equiv 0 \pmod Q}} \Lambda(f_1)\Lambda(f_2), \tag{3.16}$$

$$C_2(n_1, n_2; Q) = \sideset{}{'}\sum_{\substack{\deg f_1 = n_1 \\ \deg f_2 = n_2 \\ f_1 f_2 \equiv 1 \pmod Q}} \Lambda(f_1)\Lambda(f_2), \tag{3.17}$$

and the big-O term of (3.15) is $O(C(\psi)^2/(d^2 q^d))$.

As before, we use the Hsu–Brun–Titchmarsh Theorem to bound the contribution of the C_2 sum:

$$\begin{aligned} &\frac{1}{(d-1)^2} \sum_{n_1, n_2 = 0}^{\infty} C_2(n_1, n_2; Q) \frac{\hat{\psi}(n_1)\bar{\hat{\psi}}(-n_2) + \hat{\psi}(-n_1)\bar{\hat{\psi}}(n_2)}{q^{(n_1+n_2)/2}} \\ &\ll \frac{1}{(d-1)^2 q^d} \sum_{n_1, n_2 = 0}^{\infty} \left(\hat{\psi}(n_1)\bar{\hat{\psi}}(-n_2) + \hat{\psi}(-n_1)\bar{\hat{\psi}}(n_2)\right) q^{(n_1+n_2)/2} \\ &\ll \frac{C(\psi)^2}{d^2 q^d}. \end{aligned} \tag{3.18}$$

Using the same theorem, we break the contribution of the C_1 sum into a main diagonal term and an off-diagonal term that depends on Q; the latter contributes with the same order as the C_2 sum:

$$\begin{aligned} &\frac{1}{(d-1)^2} \sum_{n_1, n_2 = 0}^{\infty} C_1(n_1, n_2; Q) \frac{\hat{\psi}(n_1)\bar{\hat{\psi}}(n_2) + \hat{\psi}(-n_1)\bar{\hat{\psi}}(-n_2)}{q^{(n_1+n_2)/2}} \\ &= \frac{1}{(d-1)^2} \sum_{n=0}^{\infty} \sideset{}{'}\sum_{\deg f = n} \Lambda(f)^2 \frac{|\hat{\psi}(n)|^2 + |\hat{\psi}(-n)|^2}{q^n} + O\left(\frac{C(\psi)^2}{d^2 q^d}\right). \end{aligned} \tag{3.19}$$

By writing

$$\sideset{}{'}\sum_{\deg f = n} \Lambda(f)^2 = n \sideset{}{'}\sum_{\deg f = n} \Lambda(f) = n(q^n + O(q^{n/2})) \tag{3.20}$$

we conclude the proof. □

Corollary 3.2. *Suppose Q is irreducible. Let ψ be a test function of period 1 such that (3.8) holds. Then*

$$\text{Var } F_{1,\chi}(\psi) = \frac{1}{(d-1)^2} \sum_{n \in \mathbb{Z}} |n||\hat{\psi}(n)|^2 + O\left(\frac{1}{d^2}\right), \tag{3.21}$$

where $F_{1,\chi}$ is defined in (1.2).

4. The two-level global regime

Since the computations rapidly become laborious, we do not derive an unaveraged two-level explicit formula, but instead compute the expectation of $F_{2,\chi}(\psi)$ directly. We do not compute the two-level higher moments for the same reason.

Proof of 1.3. Let

$$\psi^*(s_1, s_2) = \frac{1}{(d-1)^2} \psi\left(\frac{s_1}{T_q}, \frac{s_2}{T_q}\right). \tag{4.1}$$

Let ℓ_c be defined as in the proof of Proposition 2.1. For $j = 1, 2$, let $c_j = 1/2 + \epsilon_j$, where $0 < \epsilon_1 < \epsilon_2 < 1/2$. Writing $\int_{\mathcal{C}_j} = \int_{\ell_{c_j}} - \int_{\ell_{1-c_j}}$, Cauchy's Theorem implies

(4.2)
$$\begin{aligned} F_{2,\chi}(\psi) = -F_{1,\chi}(\psi_{\text{diag}}) + F_{2,\chi}(\psi; \epsilon_1, \epsilon_2) - F_{2,\chi}(\psi; \epsilon_1, -\epsilon_2) - F_{2,\chi}(\psi; -\epsilon_1, \epsilon_2) \\ + F_{2,\chi}(\psi; -\epsilon_1, -\epsilon_2) + O(\max(\epsilon_1, \epsilon_2)), \end{aligned}$$

where

$$\begin{aligned} F_{2,\chi}(\psi; \epsilon_1, \epsilon_2) = \frac{1}{(2\pi)^2} \iint_{A_q} \frac{L'}{L}(1/2 + \epsilon_1 + it_1, \chi) \\ \frac{L'}{L}(1/2 + \epsilon_2 + it_2, \chi) \psi^*((t_j - i\epsilon_j)_{j=1,2})\, dt_1\, dt_2 \end{aligned} \tag{4.3}$$

and $A_q = [-T_q/2, +T_q/2]^2$.

Again, we employ the functional equation to replace those terms of (4.2) that have $-\epsilon_1$ or $-\epsilon_2$ as a parameter. First, define

(4.4)

$$F^{2,2}_{2,\chi}(+1) = \frac{1}{(2\pi)^2}\iint_{A_q}\bigg(\frac{L'}{L}(1/2+\epsilon_1+it_1,\chi)\\ \frac{L'}{L}(1/2+\epsilon_2-it_2,\bar{\chi})\psi^*(t_1-i\epsilon_1,t_2+i\epsilon_2)+\frac{L'}{L}(1/2+\epsilon_2+it_2,\chi)\\ \times\frac{L'}{L}(1/2+\epsilon_1-it_1,\bar{\chi})\psi^*(t_1+i\epsilon_1,t_2-i\epsilon_2)\bigg)\,dt_1\,dt_2,$$

(4.5)

$$F^{2,2}_{2,\chi}(-1) = \frac{1}{(2\pi)^2}\iint_{A_q}\left(\prod_{j=1,2}\frac{L'}{L}(1/2+\epsilon_j-it_j,\chi)\right)\\ \times\psi^*(t_1+i\epsilon_1,t_2+i\epsilon_2)dt_1\,dt_2,$$

(4.6)

$$F^{3,3}_{2,\chi} = \frac{1}{(2\pi)^2}\iint_{A_q} G_\chi(1/2+it_1)G_\chi(1/2+it_2)\psi^*(t_1,t_2)\,dt_1\,dt_2,$$

where

$$G_\chi(s) = \lambda_\infty(\chi)\left(-1+\frac{1}{1-q^s}+\frac{1}{1-q^{1-s}}\right)\log q. \tag{4.7}$$

Also define

$$F^{1,2}_{2,\chi}(\delta) = \frac{1}{(2\pi)^2}\iint_{A_q}((d-1)\log q)\\ \times\bigg(\frac{L'}{L}(1/2+\epsilon_1+i\delta t_1,\chi)\psi^*(t_1-i\delta\epsilon_1,t_1-i\epsilon_2)\\ +\frac{L'}{L}(1/2+\epsilon_2+i\delta t_2,\chi)\psi^*(t_1-i\epsilon_1,t_1-i\delta\epsilon_2)\bigg)\,dt_1\,dt_2, \tag{4.8}$$

$$F^{1,3}_{2,\chi} = \frac{1}{(2\pi)^2}\iint_{A_q}((d-1)\log q)\,(G_\chi(1/2+it_1)\\ +G_\chi(1/2+it_2))\,\psi^*(t_1,t_2)\,dt_1\,dt_2. \tag{4.9}$$

$$(4.10)\quad F_{2,\chi}^{2,3}(\delta) = \frac{1}{(2\pi)^2}\iint_{A_q}\bigg(\frac{L'}{L}(1/2+\epsilon_1+i\delta t_1,\chi)G_\chi(1/2+\epsilon_2+it_2) \\ \times\ \psi^*(t_1-i\delta\epsilon_1,t_2-i\epsilon_2)+\frac{L'}{L}(1/2+\epsilon_2+i\delta t_2,\chi) \\ \times\ G_\chi(1/2+\epsilon_1+it_1)\psi^*(t_1-i\epsilon_1,t_2-i\delta\epsilon_2)\bigg)\,dt_1,dt_2.$$

It is straightforward to check that

$$(4.11)\quad -(F_{2,\chi}(\psi;\epsilon_1,-\epsilon_2)+F_{2,\chi}(\psi;-\epsilon_1,\epsilon_2)) = F_{2,\chi}^{1,2}(+1)+F_{2,\chi}^{2,2}(+1)+F_{2,\chi}^{2,3}(+1)$$

and

$$(4.12)\quad F_{2,\chi}(\psi;-\epsilon_1,-\epsilon_2) = \frac{1}{(2\pi)^2}\iint_{A_q}((d-1)\log q)^2\psi^*(t_1,t_2)\,dt_1\,dt_2 \\ + F_{2,\bar\chi}^{2,2}(-1)+F_{2,\chi}^{3,3}+F_{2,\bar\chi}^{1,2}(-1)+F_{2,\chi}^{1,3}+F_{2,\bar\chi}^{2,3}(-1) \\ = \hat\psi(0,0)+F_{2,\bar\chi}^{2,2}(-1)+F_{2,\chi}^{3,3}+F_{2,\bar\chi}^{1,2}(-1) \\ + F_{2,\chi}^{1,3}+F_{2,\bar\chi}^{2,3}(-1).$$

In what follows, we estimate each of the individual contributions. We will implicitly substitute the appropriate Dirichlet series and send $\epsilon_1,\epsilon_2\to 0$ in all of the L'/L terms.

Diagonal contributions. By an argument similar to that in the proof of Proposition 2.1,

$$(4.13)\quad F_{2,\chi}^{3,3} = \frac{\lambda_\infty(\chi)}{(d-1)^2}\sum_{n_1,n_2\in\mathbb{Z}}\frac{\hat\psi(n_1,n_2)}{q^{(|n_1|+|n_2|)/2}}.$$

The analysis of the expectation of the two $F_{2,\chi}^{2,2}$ terms is reminiscent of that of $\mathbb{E}|F_{1,\chi}(\psi)^{\rm osc}|^2$ in the proof of Theorem 1.2. With $C_1(n_1,n_2,Q)$ and $C_2(n_1,n_2,Q)$ defined as in (3.16) and (3.17), respectively,

$$(4.14)\quad \mathbb{E}(F_{2,\chi}^{2,2}(+1)+F_{2,\chi}^{2,2}(-1)) \\ = \frac{1}{(d-1)^2}\sum_{n_1,n_2=0}^{\infty}\frac{1}{q^{(n_1+n_2)/2}}\Big(C_1(n_1,n_2;Q)\Big(\hat\psi(n_1,-n_2)+\hat\psi(-n_1,n_2)\Big) \\ + C_2(n_1,n_2;Q)\hat\psi(-n_1,-n_2)\Big)$$

$$= \frac{1}{(d-1)^2} \sum_{n=0}^{\infty} \sideset{}{'}\sum_{\deg f = n} \Lambda(f)^2 \frac{\hat{\psi}(n,-n) + \hat{\psi}(-n,n)}{q^n} + O\left(\frac{C(\psi)}{d^2 q^d}\right)$$
$$= \frac{1}{(d-1)^2} \sum_{n\in\mathbb{Z}} |n| \hat{\psi}(n,-n) + O\left(\frac{C(\psi)}{d^2 q^d}\right).$$

Off-diagonal contributions. Similarly to the argument in the proof of Proposition 2.1,

$$F^{1,3}_{2,\chi} = -\frac{\lambda_\infty(\chi)}{d-1} \sum_{n\in\mathbb{Z}} \frac{\hat{\psi}(0,n) + \hat{\psi}(n,0)}{q^{|n|/2}}. \tag{4.15}$$

Also, similarly to the proof of Theorem 1.1,

$$\begin{aligned} &\mathbb{E}(F^{1,2}_{2,\chi}(+1) + F^{1,2}_{2,\chi}(-1)) \\ &= -\frac{1}{d-1} \sum_{n=0}^{\infty} \left(\sideset{}{'}\sum_{\substack{\deg f = n \\ f\equiv 1 \pmod Q}} - \frac{1}{\#\mathcal{F}_Q} \sideset{}{'}\sum_{\substack{\deg f = n \\ f\not\equiv 0,1 \pmod Q}} \right) \frac{\Lambda(f)}{q^{n/2}} \\ &(\hat{\psi}(n,0) + \hat{\psi}(-n,0) + \hat{\psi}(0,n) + \hat{\psi}(0,-n)) \\ &\ll \frac{C(\psi_1) + C(\psi_2)}{dq^d} \end{aligned} \tag{4.16}$$

and

$$\mathbb{E}(F^{2,3}_{2,\chi}(+1) + F^{2,3}_{2,\chi}(-1)) \ll \frac{C(\psi)}{d^2 q^d}. \tag{4.17}$$

□

5. The local regime

5.1. n-level density

An *n-dimensional test function of rapid decay* is a smooth function $\phi(s_1, \ldots, s_n) = \phi_1(s_1) \ldots \phi_n(s_n)$, defined in a region $U \subset \mathbb{C}^n$ containing $\mathbb{R}^n$, such that $\phi(s) \ll (1+|s|)^{-(1+\delta)}$ for some $\delta > 0$. The (homogeneous) periodization of

ϕ, scaled by a parameter N, is

$$\phi_N(s) = \sum_{\nu\in\mathbb{Z}^n} \phi\left(N(s+\nu_1),\ldots,N(s+\nu_n)\right). \tag{5.1}$$

Let U be an $N\times N$ unitary matrix with eigenangles $\theta_1,\ldots,\theta_N$. Then the *n-level density* of the θ_j with respect to ϕ is

$$W_{n,U}(\phi) := \sum_{\substack{1\le j_1,\ldots,j_n\le N\\ j_k \text{ distinct}}} \phi_N\left(\frac{\theta_{j_1}}{2\pi},\ldots,\frac{\theta_{j_n}}{2\pi}\right). \tag{5.2}$$

Let $L(s,\chi)$ be a Dirichlet L-function over $\mathbb{F}_q(T)$. Recall that $L(s,\chi)$ has $d+O(1)$ zeros of the form $1/2+\gamma_{\chi,j}$ in an interval of periodicity $[1/2 - iT_q/2, 1/2+iT_q/2)$. Therefore, by analogy, we define the *n-level density* of $L(s,\chi)$ with respect to ϕ to be

$$W_{n,\chi}(\phi) \;=\; \sum_{\substack{-\frac{T_q}{2}\le\gamma_{j_1},\ldots,\gamma_{\chi,j_n}<\frac{T_q}{2}\\ j_k \text{ distinct}}} \phi_{d-1}\left(\frac{\gamma_{\chi,1}}{T_q},\ldots,\frac{\gamma_{\chi,n}}{T_q}\right). \tag{5.3}$$

Remark 5.1. To facilitate comparison with [16], we compare the above to the situation over number fields in the one-level case. Let $L(s)$ be a Selberg-class L-function with analytic conductor $c>0$. Write $1/2+i\gamma_j$ to denote its jth critical zero above the real line, ordered by height. Then the one-level density of $L(s)$ with respect to ϕ is

$$W_1(\phi) := \lim_{T\to\infty} \sum_{0\le\gamma_j<T} \phi\left(\gamma_j \frac{\log c}{2\pi}\right). \tag{5.4}$$

Above, $(\log c)/2\pi$ normalizes the average consecutive spacing between ordinates of zeros near the central point to be 1 in the limit $T\to\infty$.

5.2. Unitary predictions

Let $\mathrm{U}(N)$ be the group of $N\times N$ unitary matrices under Haar measure. If F is a function on $\mathrm{U}(N)$, then the expectation of F is

$$\mathbb{E}F(U) \;=\; \int_{\mathrm{U}(N)} F(U)\,dU. \tag{5.5}$$

In [16], the authors prove that if ϕ is an *even* one-dimensional test function of rapid decay, then

$$\mathbb{E}W_U(\phi) \;\to\; \hat{\phi}(0), \tag{5.6}$$
$$\mathbb{E}(W_{1,U}(\phi) - \mathbb{E}W_{1,U}(\phi))^2 \;\to\; \sigma(\phi)^2 \tag{5.7}$$

as $N \to \infty$, where

$$\sigma(\phi)^2 \;=\; \int_{-\infty}^{+\infty} \min(1, |t|)\hat{\phi}(t)^2\, dt. \tag{5.8}$$

If $\hat{\phi}$ is supported in the interval $[-2/m, +2/m]$, then they also obtain

$$\mathbb{E}(W_{1,U}(\phi) - \mathbb{E}W_{1,U}(\phi))^m \;\to\; \begin{cases} 0 & m \text{ odd,} \\ \dfrac{m!}{2^{m/2}(m/2)!}\sigma(\phi)^m & m \text{ even.} \end{cases} \tag{5.9}$$

By [19], if $\phi = \phi_1 \cdot \phi_2$ is an even two-dimensional test function of rapid decay, then

$$\mathbb{E}W_{2,U}(\phi) \;\to\; -\hat{\phi}_{\text{diag}}(0) + \hat{\phi}(0,0) + \int_{-\infty}^{+\infty} |t|\hat{\phi}_1(t)\hat{\phi}_2(t)\, dt, \tag{5.10}$$

where $\phi_{\text{diag}}(s) = \phi(s, s)$.

5.3. From global to local

Set $\psi = \phi_{d-1}$ in the results from Sections 3 and 4. Recall that $\psi(s_1, \ldots, s_n)$ is a Fourier series, whereas the Fourier *transform* of ϕ is

$$\hat{\phi}(s_1, \ldots, s_n) \;=\; \int_{-\infty}^{+\infty} \phi(t_1, \ldots, t_n) e(-(s_1 t_1 + \cdots + s_n t_n))\, dt_1 \cdots dt_n. \tag{5.11}$$

We have

$$\hat{\psi}(\nu_1, \ldots, \nu_n) \;=\; \frac{1}{(d-1)^n}\hat{\phi}\left(\frac{\nu_1}{d-1}, \ldots, \frac{\nu_n}{d-1}\right) \tag{5.12}$$

and

$$F_{n,\chi}(\psi) \;=\; \frac{1}{(d-1)^n} W_{n,\chi}(\phi) \tag{5.13}$$

for all n. Thus we obtain the following local-regime results.

Corollary 5.2. *Suppose Q is irreducible. Let ϕ be a one-dimensional test function of rapid decay. Let*

$$C(\phi; d) \ := \ \sum_{\nu\in\mathbb{Z}} \left|\hat{\phi}\left(\frac{\nu}{d-1}\right)\right| q^{|\nu|/2}. \tag{5.14}$$

(1) *If $C(\phi; d) \ll dq^d$ as $d \to \infty$, then*

$$\mathbb{E}W_{1,\chi}(\phi) \ = \ \hat{\phi}(0) - \frac{1}{(d-1)(q-1)} \sum_{\nu\in\mathbb{Z}} \frac{1}{q^{|\nu|/2}} \hat{\phi}\left(\frac{\nu}{d-1}\right) + O\left(\frac{C(\phi)}{dq^d}\right). \tag{5.15}$$

(2) *If $C(\phi; d) \ll dq^{d/2}$ as $d \to \infty$, then*

$$\mathrm{Var}\ W_{1,\chi}(\phi) \ = \ \frac{1}{(d-1)^2} \sum_{\nu\in\mathbb{Z}} |\nu| \left|\hat{\phi}\left(\frac{\nu}{d-1}\right)\right|^2 + O\left(\frac{C(\phi)^2}{d^2q^d}\right). \tag{5.16}$$

Corollary 5.3. *Suppose Q is irreducible. Let ϕ be a two-dimensional test function of rapid decay such that $C(\phi_1; d), C(\phi_2; d) \ll dq^{d/2}$, where $C(\phi; d)$ is defined for one-dimensional test functions ϕ of rapid decay in Corollary 5.2. Let $C(\phi) = C(\phi_1)C(\phi_2)$ and $\phi_{\mathrm{diag}}(s) = \phi(s, s)$. Then*

$$\begin{aligned} \mathbb{E}W_{2,\chi}(\phi) \ &= \ -\mathbb{E}W_{1,\chi}(\phi_{\mathrm{diag}}) + \hat{\phi}(0,0) + \frac{1}{(d-1)^2} \sum_{\nu\in\mathbb{Z}} |\nu| \hat{\phi}\left(\frac{n}{d-1}, -\frac{n}{d-1}\right) \\ &\quad + \frac{C_{2,\Gamma}(\phi; d)}{q-1} + O\left(\frac{C(\phi_1; d) + C(\phi_2; d)}{dq^d}\right) + O\left(\frac{C(\phi; d)}{d^2q^d}\right), \end{aligned} \tag{5.17}$$

where

$$\begin{aligned} C_{2,\Gamma}(\phi; d) \ &= \ -\frac{1}{d-1} \sum_{\nu\in\mathbb{Z}} \frac{1}{q^{|\nu|/2}} \left(\hat{\phi}\left(0, \frac{\nu}{d-1}\right) + \hat{\phi}\left(\frac{\nu}{d-1}, 0\right)\right) \\ &\quad + \frac{1}{(d-1)^2} \sum_{\nu_1,\nu_2\in\mathbb{Z}} \frac{1}{q^{(|\nu_1|+|\nu_2|)/2}} \hat{\phi}\left(\frac{\nu_1}{d-1}, \frac{\nu_2}{d-1}\right). \end{aligned} \tag{5.18}$$

In Corollary 5.2, if $\mathrm{supp}\hat{\phi} \subset [-2, +2]$, then the hypothesis of (1) is satisfied, and if $\mathrm{supp}\hat{\phi} \subset [-1, +1]$, then the hypothesis of (2) is satisfied. In both cases, our results match the unitary predictions of [16] in the limit. More precisely our results: (1) implies the $\mathbb{F}_q(T)$-analog of their Theorem 3.1, and (2) implies the $\mathbb{F}_q(T)$-analog of their Theorem 3.4. Similarly, in Corollary

5.3, if $\mathrm{supp}\hat{\phi}_1, \mathrm{supp}\hat{\phi}_2 \subset [-1, +1]$, then our result matches the prediction of [19].

6. Montgomery's hypothesis

Returning to the classical setting, Fiorilli and Miller [11] showed how to relate certain conjectures about prime distribution to improvements in the available support for $\hat{\phi}$ in the local-regime density results of Hughes–Rudnick. One that generalizes to our setting is a weakened version of Montgomery's Hypothesis, itself originally stated in [23].

Let $Q \in \mathbb{Z}_+$. Let Λ be the classicial von Mangoldt function, and let $\Psi(X) = \sum_{n \leq X} \Lambda(n)$, the Chebyshev function. For all $a \in \mathbb{Z}_+$ coprime to Q, let

$$\Psi(X) = \sum_{\substack{n \leq X \\ n \equiv a \pmod{Q}}} \Lambda(n). \tag{6.1}$$

Conjecture 6.1 (θ-Montgomery). *Let Φ be the classical Euler totient function. Then for all $Q \geq 3$, there exists $\theta \in (0, 1/2]$ such that*

$$\Psi(X; Q, 1) - \frac{\Psi(X)}{\Phi(Q)} \ll_\epsilon \frac{X^{1/2+\epsilon}}{Q^\theta} \tag{6.2}$$

for all $X \geq Q$.

Although we do not expect the conjecture to hold for $\theta = 1/2$, it is likely to hold for any arbitrarily smaller value. Theorem 1.16 of [11] implies the following.

Theorem 6.2 (Fiorilli–Miller). *Let $Q \geq 3$ be prime, and let $\mathcal{F}_Q$ be the family of primitive Dirichlet L-functions of conductor Q. Let ϕ be a one-dimensional test function of rapid decay such that $\hat{\phi}$ is compactly support. If Conjecture 6.1 holds for θ, then*

$$\frac{1}{\#\mathcal{F}_Q} \sum_{\chi \in \mathcal{F}_Q} \sum_{\gamma_\chi} \phi\left(\gamma_\chi \frac{\log Q}{2\pi}\right) = \hat{\phi}(0) + \textit{gamma-factor term} + O(Q^{-\theta+\epsilon}). \tag{6.3}$$

That is, θ-Montgomery implies that the one-level density of $\mathcal{F}_Q$ tends to that of the unitary group for *all* ϕ such that $\hat{\phi}$ has compact support, and the error term improves exponentially with θ.

We return to the function-field setting. Let Λ_q be the von Mangoldt function for $\mathbb{F}_q(T)$, and let $\Psi_q(n) = \sum'_{\deg f = n} \Lambda_q(f)$. (Note how this differs from the most naïve analog of the Chebyshev function.) For all non-constant $Q \in \mathbb{F}_q[T]$ and $f \in \mathbb{F}_q[T]$ coprime to Q, let

$$\Psi_q(n; Q, f) = \sum_{\substack{\deg g = n \\ g \equiv f \pmod{Q}}}' \Lambda_q(g). \tag{6.4}$$

Let $\mathcal{F}_Q$ resume its definition from Section 4. Recall from the proof of Theorem 1.1 that if ψ is a one-dimensional test function of period 1 and Q is irreducible of degree ≥ 2, then $\mathbb{E}F_{1,\chi}(\psi) = \hat{\psi}(0) +$ gamma-factor term $+$ $\mathbb{E}(F_{1,\chi}(\psi)^{\text{osc}})$, where

$$\mathbb{E}(F_{1,\chi}(\psi)^{\text{osc}}) = -\frac{1}{d-1} \sum_{n=0}^{\infty} \left(\Psi_q(n; Q, 1) - \frac{\Psi_q(n)}{\#\mathcal{F}_Q} \right) \frac{\hat{\psi}(n) + \hat{\psi}(-n)}{q^{n/2}}. \tag{6.5}$$

Thinking of q^n as the correct analog of the X variable in Conjecture 6.1, we are led to the following conjecture.

Conjecture 6.3 (θ-Montgomery for $\mathbb{F}_q(T)$). *Let $\Phi_q(f) = \#(\mathbb{F}_q[T]/f)^\times$ be the Euler totient function for $\mathbb{F}_q[T]$. Then for all $Q \in \mathbb{F}_q[T]$ of degree $d \geq 2$, there exists $\theta \in (0, 1/2]$ such that*

$$\Psi_q(n; Q, 1) - \frac{\Psi_q(n)}{\Phi_q(Q)} \ll_\epsilon q^{n(1/2+\epsilon) - d\theta} \tag{6.6}$$

for all $n \geq d$.

We remark that if Montgomery's Hypothesis is translated from the language of primes to the language of zeros, guided by the duality that exists between primes and zeros of L-functions, then we obtain a conjecture that relates to $F_{1,\chi}$ more naturally. For all $\chi \in \mathcal{F}_Q$, let $\{\gamma_\chi, j\}_{j=1}^{d(\chi)}$ be the ordinates of the zeros of $L(s, \chi)$. We propose the following.

Conjecture 6.4. *Let $Q \in \mathbb{F}_q[T]$ be of degree $d \geq 2$. Then there exist $\theta_1, \theta_2 \in (0, 1)$ such that for all $n \in \mathbb{Z}_+$,*

$$\sum_{\chi \in \mathcal{F}_Q} \sum_j q^{in\gamma_{\chi,j}} \ll_{\theta_1, \theta_2} d(\chi)^{(1-\theta_1)} q^{d(1-\theta_2)} \tag{6.7}$$

as $d \to \infty$.

Proposition 6.5. *Let $Q \in \mathbb{F}_q[T]$ be irreducible of degree $d \geq 2$. Let ψ be a one-dimensional test function of period* 1.

(1) *Suppose Conjecture 6.3 holds for some θ. If $C_\epsilon(\psi) = \sum_{n\in\mathbb{Z}} \hat{\psi}(n) q^{|n|\epsilon}$ converges for all $\epsilon > 0$ small enough, then*

$$\mathbb{E}(F_{1,\chi}(\psi)^{\mathrm{osc}}) \ll_\epsilon \frac{C_\epsilon(\psi)}{dq^{d\theta}}. \tag{6.8}$$

(2) *Suppose Conjecture 6.4 holds for some θ_1, θ_2. Then*

$$\mathbb{E}(F_{1,\chi}(\psi)^{\mathrm{osc}}) \ll_{\theta_1,\theta_2} \frac{\theta_2}{d^{\theta_1-1}(\#\mathcal{F}_Q)^{\theta_2}}. \tag{6.9}$$

Proof. (1) is immediate. (2) follows from the Erdős–Turán Inequality, which, together with Conjecture 6.4, implies that for all $[a,b] \subset [-T_q/2, +T_q/2]$ and $N \in \mathbb{Z}_+$,

(6.10)

$$\begin{aligned}\left| \frac{\#\{(\chi, j) : \gamma_{\chi,j} \in [a,b]\}}{(d-1)\#\mathcal{F}_Q} - \frac{b-a}{T_q} \right| &\ll \frac{1}{N} + \sum_{n=1}^{N} \frac{1}{n} \left| \frac{1}{(d-1)\#\mathcal{F}_Q} \sum_\chi \sum_j q^{in\gamma_{\chi,j}} \right| \\ &= \frac{1}{N} + \frac{1}{d^{\theta_1}(\#\mathcal{F}_Q)^{\theta_2}} \sum_{n=1}^{N} \frac{1}{n}.\end{aligned}$$

The supremum of the expression on the left over all a, b is an upper bound for $\mathbb{E}(F_{1,\chi}(\psi)^{\mathrm{osc}})$, as we can approximate ψ arbitrarily well by linear combinations of indicator functions. Choosing $N = \lfloor d^{\theta_1}(\#\mathcal{F}_Q)^{\theta_2} \rfloor$ completes the proof. □

Either of the two possibilities suggested by Proposition 6.5 is considerably stronger than Theorem 1.1. They imply the following results in the local regime.

Corollary 6.6. *Let $Q \in \mathbb{F}_q[T]$ be irreducible of degree $d \geq 2$. Let ϕ be a one-dimensional test function of rapid decay such that $\hat{\phi}$ has compact support.*

(1) *Suppose Conjecture 6.3 holds for some θ. Then the hypotheses in Corollary 5.2 can be lifted and the error term can be sharpened to $O_\epsilon(q^{-d(\theta-\epsilon)})$.*

(2) *Suppose Conjecture 6.4 holds for some θ_1, θ_2. Then the hypotheses in Corollary 5.2 can be lifted and the error term can be sharpened to $O_{\theta_1,\theta_2}(d_2^{\theta} d^{1-\theta_1}(\#\mathcal{F}_Q)^{-\theta_2})$.*

Appendix A. Proof of the explicit formula

Proof of 2.1. Let

$$\psi^*(s) = \frac{1}{d-1}\psi\left(\frac{s}{T_q}\right). \tag{A.1}$$

For all real c, let ℓ_c be the segment from $c - iT_q/2$ to $c + iT_q/2$ in the complex plane. Let $0 < \epsilon < 1/4$ and $c = 1/2 + \epsilon$. Using Cauchy's Theorem,

(A.2)
$$\begin{aligned}\sum_{-\frac{T_q}{2} \le \gamma_\chi < \frac{T_q}{2}} \psi^*(\gamma_\chi) &= \frac{1}{2\pi i}\left(\int_{\ell_c} - \int_{\ell_{1-c}}\right)\frac{L'}{L}(s,\chi)\psi^*(-i(s-1/2))\,ds + O(\epsilon)\\ &= F_{1,\chi}(\psi;\epsilon) - F_{1,\chi}(\psi;-\epsilon) + O(\epsilon),\end{aligned}$$

where

$$F_{1,\chi}(\psi;\epsilon) = \frac{1}{2\pi}\int_{-T_q/2}^{+T_q/2}\frac{L'}{L}(1/2+\epsilon+it,\chi)\psi^*(t-i\epsilon)\,dt. \tag{A.3}$$

To deal with $F_{1,\chi}(\psi;-\epsilon)$ we substitute the formula (2.9). Distributing the integral among the resulting three terms, and sending $\epsilon \to 0$ in the first and last, we find

(A.4)
$$\begin{aligned}-F_{1,\chi}(\psi;-\epsilon) = \frac{d-1}{T_q}&\int_{-T_q/2}^{+T_q/2}\psi^*(t)\,dt\\ &+ \frac{1}{2\pi}\int_{-T_q/2}^{+T_q/2}\frac{L'}{L}(1/2+\epsilon-it,\bar{\chi})\psi^*(t+i\epsilon)\,dt + \frac{\lambda_\infty(\chi)}{T_q}\\ &\times\int_{-T_q/2}^{+T_q/2}\left(-1 + \frac{1}{1-q^{1/2-it}} + \frac{1}{1-q^{1/2+it}}\right)\psi^*(t)\,dt.\end{aligned}$$

The first term of the right-hand side is $\hat{\psi}(0) = \int_{-1/2}^{+1/2} \psi(t)\, dt$, whereas the last term equals

$$\text{(A.5)} \qquad \frac{\lambda_\infty(\chi)}{(d-1)T_q} \int_{-T_q/2}^{+T_q/2} \left(-1 + \frac{1}{1-q^{1/2-it}} + \frac{1}{1-q^{1/2+it}} \right) \psi\left(\frac{t}{T_q}\right) dt$$
$$= \frac{\lambda_\infty(\chi)}{(d-1)T_q} \left(-1 - \frac{q^{-1/2+it}}{1-q^{1/2-it}} - \frac{q^{-1/2-it}}{1-q^{1/2+it}} \right) \psi\left(\frac{t}{T_q}\right) dt$$
$$= -\frac{\lambda_\infty(\chi)}{d-1} \sum_{n\in\mathbb{Z}} \frac{\hat{\psi}(n)}{q^{|n|/2}}.$$

Finally,

$$\text{(A.6)} \qquad F_{1,\chi}(\psi;\epsilon) = \frac{1}{(d-1)T_q} \sum_{n=0}^{\infty} \int_{-T_q/2}^{+T_q/2} \frac{c_\chi(n)}{q^{n(1/2+\epsilon+it)}} \psi\left(\frac{t-i\epsilon}{T_q}\right) dt,$$

and similarly with the middle term of (A.4), where our use of the Dirichlet series in the region $\Re s > 1/2$ is justified by the bound $c_\chi(n) \ll dq^{n/2}$ from the Riemann Hypothesis. Interchanging the sum with the integral, and sending $\epsilon \to 0$, we arrive at the desired result. □

Interestingly, the middle term on the right-hand side of (2.11) corresponds to the gamma-factor term in the classical Explicit Formula, but is visually much simpler. This is because the $\mathbb{F}_q(T)$-analog of the Riemann zeta function and of the gamma function are itself simpler objects.

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