COMMUNICATIONS IN
NUMBER THEORY AND PHYSICS
Volume 8, Number 03, 369–402, 2014

On an extension of the universal monodromy representation for $\mathbb{P}^1\backslash\{0,1,\infty\}$

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The ideas behind the TSUCHIYA–KANIE representations of braid groups on spaces of N-point correlation functions are emulated to represent the modular group $PSL(2,\mathbb{Z})$ on a space of degenerate 3-point correlation functions. This extends the CHEN series map giving the universal monodromy representation of $\mathbb{P}^1 \setminus \{0,1,\infty\}$ to an injective 1-cocycle of $PSL(2,\mathbb{Z})$ into power series with complex coefficients in two non-commuting variables, twisted by an action of S_3. The definition of the 1-cocycle is effected by parallel transport of flat sections of the bundle, also with an S_3 twisting, along paths in $\mathbb{P}^1 \setminus \{0,1,\infty\}$ which are explicitly associated with elements of $PSL(2,\mathbb{Z})$. Injectivity is proven using a DE RHAM-type theorem due to CHEN. The resulting action of $PSL(2,\mathbb{Z})$ on the polylogarithm generating function is shown to yield a family of proofs of the analytic continuation and functional equation of the RIEMANN zeta function.

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Introduction

In the work [18] in which they gave a rigorous mathematical foundation to groundbreaking developments brought forward by V. G. Knizhnik and A. B. Zamolodchikov, A. Tsuchiya and Y. Kanie produced explicit monodromy representations of the braid group on N strands, B_N, on spaces of N-point correlation functions. A key idea facilitating these representations is the lifting of the braid groups, which act as the fundamental groups of the quotients of the configuration spaces

$$\mathrm{Config}_N := \{(z_0, \ldots, z_{N-1}) \in \mathbb{C}^N | z_i \neq z_j \ \text{ if } \ i \neq j\}$$

by the action of the symmetric groups S_N, to spaces of paths on Config_N itself. Thereby, one obtains monodromy representations of B_N on any suitable space of functions from Config_N into some S_N-module, via analytic continuation twisted by the S_N action coming from the surjection $B_N \to S_N$.

In this paper, a similar circle of ideas is used to represent the modular group $PSL(2,\mathbb{Z})$ on a space of degenerate 3-point correlation functions. In particular, we give an explicit lifting of $PSL(2,\mathbb{Z})$ as the fundamental group of the quotient stack $\left[S_3 \backslash (\mathbb{P}^1 \backslash \{0,1,\infty\})\right]$, to homotopy classes of paths in $\mathbb{P}^1 \backslash \{0,1,\infty\}$. Next, interpreting the degenerate 3-point correlation functions as flat sections of the universal unipotent bundle with connection on $\mathbb{P}^1 \backslash \{0,1,\infty\}$, parallel transport along the elements of the modular group, twisted by the S_3 action on $\mathbb{P}^1 \backslash \{0,1,\infty\}$, gives rise to an injective 1-cocycle of $PSL(2,\mathbb{Z})$ on an S_3-module comprising formal non-commuting power series in two variables over $\mathbb{C}$, which extends the monodromy representation of the fundamental group of $\mathbb{P}^1 \backslash \{0,1,\infty\}$.

The $PSL(2,\mathbb{Z})$ action is then shown to yield a family of proofs of the analytic continuation and functional equation of the RIEMANN zeta function.

Here are some details and relevant background: consider the curve in Config_3 (with notation as above) cut out by the equations $z_0 = 0$ and $z_1 = 1$. This is $\mathbb{P}^1 \setminus \{0,1,\infty\}$, isomorphic via the cross-ratio to the moduli space $\mathfrak{M}_{0,4}$ of genus 0 curves with four marked points. Owing to work of BELYI, DELIGNE, DRINFEL'D, IHARA and many others, the arithmetic significance of $\mathbb{P}^1 \setminus \{0,1,\infty\}$ is well known. Now via the covering map $\chi : \mathrm{Config}_3 \to \mathbb{P}^1 \setminus \{0,1,\infty\}$ the structures of [18] degenerate to $\mathbb{P}^1 \setminus \{0,1,\infty\}$. Among these is the degenerate version of the KNIZHNIK–ZAMOLODCHIKOV (KZ) equations

$$\kappa \frac{\partial}{\partial z_i} \psi(z_0, \ldots, z_{N-1}) = \sum_{j \neq i, 0 \leq j \leq N-1} \frac{\Omega_{ij}}{z_i - z_j} \psi(z_0, \ldots, z_{N-1}) \quad i = 0, \ldots, N-1$$

(where the Ω_{ij} are operators on tensor products of duals of highest weight modules over a certain affine LIE algebra — see [18] or [6] for the details), which takes the form

$$\kappa \frac{d}{dz} \psi(z) = \left[\frac{\Omega_{20}}{z} + \frac{\Omega_{21}}{z-1} \right] \psi(z). \tag{1}$$

As is shown in Proposition 4.2.1 of [6], this equation is auxilliary to solving the KZ system in the case $N = 3$. As the equation for flat sections of the universal unipotent bundle with connection on $\mathbb{P}^1 \setminus \{0,1,\infty\}$, this equation has geometric significance, in addition to the above-mentioned arithmetic significance, which this paper aims to explore.

As is well known the monodromy representation corresponding to the universal prounipotent bundle $\mathfrak{U}$ with connection ∇ on $\mathbb{P}^1 \setminus \{0,1,\infty\}$ (cf. Section 1 of [12]) may be described by means of the CHEN series map on homotopy classes of paths $[\gamma] \in \pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}, c)$:

$$[\gamma] \mapsto \sum_w \int_\gamma \omega_{i_1} \cdots \omega_{i_k} X_{i_1} \cdots X_{i_k}, \tag{2}$$

where the sum is taken over all words in the non-commuting formal variables X_0 and X_1 (including the empty word, for which the corresponding integral is 1), c is any (possibly tangential) basepoint, and if z denotes the usual parameter on $\mathbb{P}^1 \setminus \{0,1,\infty\}$, $\omega_0 = \frac{dz}{z}$ while $\omega_1 = \frac{dz}{1-z}$ (see Proposition 11 in [7]).

The CHEN iterated integrals appearing here are defined in the general case of a piecewise smooth path $\tilde{\gamma}$ on a smooth manifold M as integrals over time-ordered simplices (pulling back via the path) as follows:

$$\int_{\tilde{\gamma}} \alpha_1 \cdots \alpha_r := \int_{\Delta^r} (\tilde{\gamma}^* \alpha_1)(t_1) \wedge \cdots \wedge (\tilde{\gamma}^* \alpha_r)(t_r),$$

where Δ^r denotes the r-fold time-ordered simplex $\{0 \leq t_1 \leq \cdots \leq t_r \leq 1\}$ and the α_j are 1-forms on M. For further details see the classic reference [3]. The CHEN series in (2) can thus be interpreted as a time-ordered exponential.

When c is the tangential basepoint $\overrightarrow{01}$ and γ is a path from c to z which does not cut the real axis unless z is real, the series which results is called the polylogarithm-generating series, and is denoted $\mathrm{Li}(z, X_0, X_1)$ or $\mathrm{Li}(z)$ for short. Here, the integrals which appear are regularized in the usual way — cf. [11]. For such γ, the coefficients of the terms of the form of $X_1 X_0^{n_1} X_1 X_0^{n_1} \cdots X_1 X_0^{n_r}$ are the multiple polylogarithm functions.

Because the bundle $\mathcal{U}$ is given by

$$\mathcal{U} = \lim_{\leftarrow_N} \left[(\mathbb{C}\!<\!X_0, X_1\!> / (X_0, X_1)^{N+1}) \otimes \mathcal{O}_{\mathbb{P}^1 \setminus \{0,1,\infty\}} \right]$$

and ∇ is the formal KZ connection

$$\nabla = d - \left(\frac{dz}{z} X_0 + \frac{dz}{1-z} X_1 \right),$$

where we set $\Omega_{2i} = \frac{X_i}{\kappa}$ for $i = 0, 1$ in (1), one verifies without difficulty that $\mathrm{Li}(z)$ is a flat section of $(\mathcal{U}, \nabla)$. An excellent presentation of general KZ equations and related mathematical structures appearing in conformal field theory may be found in [6].

Here we prove:

Theorem A. *The monodromy representation*

$$F_\bullet : \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, \overrightarrow{01}) \to \mathbb{C}\!\ll\! X_0, X_1 \!\gg^\times$$

admits an extension to an injective 1-cocycle

$$F_\bullet : PSL(2, \mathbb{Z}) \to \mathbb{C}\!\ll\! X_0, X_1 \!\gg^\times_\Lambda .$$

(See 1.15 and 1.17 below.)

$\mathbb{C}\langle\langle X_0, X_1\rangle\rangle$ denotes the algebra of power series with complex coefficients in X_0 and X_1, and $\mathbb{C}\langle\langle X_0, X_1\rangle\rangle_\Lambda^\times$ denotes invertible power series with an action of $PSL(2,\mathbb{Z})$ which factors through $\Lambda \simeq S_3$ via the usual surjection $\mathcal{A}$ of (3) below. The S_3 action on power series is induced by the action of the group of automorphisms Λ of $\mathbb{P}^1 \setminus \{0,1,\infty\}$ on the connection ∇, and was given in (25) of [16]. It is also described in Section 1.1 below.

The existence of this extension is facilitated by the following short exact sequence:

(3)
$$1 \to \Gamma(2)/\{\pm 1\} \simeq \pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}, c) \to PSL(2,\mathbb{Z}) \overset{\mathcal{A}}{\to} SL(2,\mathbb{Z}/2\mathbb{Z}) \to 1.$$

Where the flat section $\mathrm{Li}(z)$ is concerned, we prove in Proposition 1.15 below that the parallel transport action of $PSL(2,\mathbb{Z})$ twisted by S_3 amounts to multiplying the section by a power series. This power series gives the extension of the monodromy representation to a 1-cocycle on $PSL(2,\mathbb{Z})$, and is given by the formula:

$$\alpha \mapsto \sum_w \int_\alpha \omega_{i_1} \cdots \omega_{i_k} Y_{i_1} \cdots Y_{i_k} \Big|_{Y_{i_j} = \overline{\alpha} X_{i_j}},$$

with sum and ω_{i_j} notation as above, writing $\overline{\alpha}$ for the reduction of $\alpha \in PSL(2,\mathbb{Z})$ to $SL(2,\mathbb{Z}/2\mathbb{Z}) \simeq S_3$, and $\overline{\alpha} X_{i_j}$ for the action of $\overline{\alpha} \in S_3$ on X_{i_j}.

The proof of the injectivity rests on Chen's π_1 DeRham Theorem (cf. Theorem 10 of [7]).

Extending the monodromy representation to $PSL(2,\mathbb{Z})$ yields an additional symmetry on $\mathrm{Li}(z)$ which can be used to prove the analytic continuation of Riemann's zeta function $\zeta(s)$. This allows us to draw parallels between the classical theta function technique used to prove the analytic continuation and functional equation of $\zeta(s)$, and Riemann's original contour integral approach. As Hecke noticed in [9], the following two facts comprise the essence of the theta function proof:

T0. The Jacobi theta function $\theta(\tau, z)$ is modular in τ in the usual sense, with respect to the congruence subgroup $\Gamma(2)$ of $PSL(2,\mathbb{Z})$. (This explains the existence of the Fourier series expansion for $\theta(\tau, z)$.)
T1. $\theta(\tau, z)$ satisfies an additional symmetry property with respect to the involutive generator σ of $PSL(2,\mathbb{Z})$ (given by $\sigma : \tau \mapsto -1/\tau$ in the action

on H), namely the functional equation of $\theta(\tau, z)$, which is regarded as an additional modularity property in τ.

Here, we show that RIEMANN'S contour integral expression for $\zeta(s)$ fits into the context of a family of integral expressions, each of which may be used to prove the analytic continuation and functional equation for $\zeta(s)$. Taken together, these proofs result from the following facts:

P0. The monodromy of the polylogarithm generating function $\mathrm{Li}(z)$ may be calculated (as for example in [15]) by directly performing the analytic continuation along the paths of the fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. The equations which result may be thought of as transformation rules for $\mathrm{Li}(z)$ with respect to elements of $\Gamma(2)/\{\pm 1\} \simeq \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, c)$ (where c is any basepoint — possibly tangential).
P1. $\mathrm{Li}(z)$ satisfies an additional symmetry property with respect to $\sigma \in PSL(2, \mathbb{Z})$, namely a functional equation involving the DRINFEL'D associator.

Properties P0 and P1 represent particular instances of the above-mentioned action of the modular group on sections of $(\mathcal{U}, \nabla)$. P0 encapsulates monodromy data which are essential to the proofs, while P1 facilitates the analytic continuation in that it gives rise to the EULER connection formulae (see Proposition 5 in [16]). The latter allow us to avoid non-integrable monodromy terms by shifting monodromy of the integrands from $0 \in \mathbb{C}$ to ∞ — for the details see Section 2.2.

1. The extension of the monodromy representation

1.1. Explicit lifting of $PSL(2, \mathbb{Z})$ to classes of paths in $\mathbb{P}^1 \setminus \{0, 1, \infty\}$

Suppose that $X = \overline{X} \setminus S$ is a smooth curve over $\mathbb{C}$ where S is some finite set of points. In [5], DELIGNE introduced a notion of fundamental group of X based at any given omitted point $a \in S$, in the direction of some specified tangent vector to $\overline{X}$ at a. Classically, as in [8] such fundamental groups with *tangential basepoint* may be defined as follows: If $\vec{v_j} \in T_{a_j}$ is a tangent vector at $a_j \in S$ for $j = 0, 1$, set

$$P_{\vec{v_0}, \vec{v_1}} := \{\gamma : [0, 1] \to \overline{X} | \gamma'(0) = \vec{v_0}, \gamma'(1) = -\vec{v_1}, \gamma((0, 1)) \subset X\}.$$

Definition 1.1. The fundamental path space $\pi_1(X, \vec{v_0}, \vec{v_1})$ is the set of path components of $P_{\vec{v_0}, \vec{v_1}}$.
When $\vec{v_1} = \vec{v_0}$, this is the fundamental group denoted $\pi_1(X, \vec{v_0})$.

This naive description is sufficient for the use of the paper. For our purposes, $\overline{X} = \mathbb{P}^1_{\mathbb{C}}$, $S = \{0, 1, \infty\}$, and $\overrightarrow{ab}$ will denote the tangent vector of unit length over $\overline{X}$ at $a \in S$, pointing in the direction of $b \in S$ for any $b \neq a$.

Definition 1.2. Any fundamental path space of the form of

$$\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, \overrightarrow{a_0b_0}, \overrightarrow{a_1b_1}),$$

where $a_j, b_j \in \{0, 1, \infty\}$ and $a_j \neq b_j$ for $j = 0, 1$ will be called a *real-based fundamental path space* of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, and the tangential basepoints $\overrightarrow{a_jb_j}$ will be referred to as real tangential basepoints.

Fix a real tangential basepoint $\overrightarrow{ab}$. Then form the set

$$G_{\overrightarrow{ab}} := \cup \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, \overrightarrow{ab}, \overrightarrow{a_0b_0}),$$

where the union is taken over all $a_0, b_0 \in \{0, 1, \infty\}$ with $a_0 \neq b_0$. The utility of restricting attention to the real tangential basepoints lies in the fact that they admit an action of $SL(2, \mathbb{Z}/2\mathbb{Z})$ (see below). Using this action, $G_{\overrightarrow{ab}}$ will be endowed with a group structure, by means of which it can be identified with $PSL(2, \mathbb{Z})$.

Now as is described in [2], the symmetries of the classical λ function effecting the covering of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ by H are captured by the classical anharmonic group Λ, to which $SL(2, \mathbb{Z}/2\mathbb{Z})$ is isomorphic. Λ is given explicitly as the following group of linear fractional automorphisms of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$:

$$\Lambda = \left\{ \lambda \mapsto \lambda, \lambda \mapsto 1 - \lambda, \lambda \mapsto \frac{\lambda}{\lambda - 1}, \lambda \mapsto \frac{1}{\lambda}, \lambda \mapsto \frac{\lambda - 1}{\lambda}, \lambda \mapsto \frac{1}{1 - \lambda} \right\}.$$

It is evident from the topology that Λ is exactly the group of *all* such linear fractional automorphisms of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

Note that these transformations necessarily permute the real tangential basepoints, as is also immediate from the above explicit description. In fact, the elements of Λ are characterized by the corresponding permutations of the symbols 0, 1 and ∞ so that also $\Lambda \simeq S_3$.

Once and for all fix isomorphisms

$$SL(2, \mathbb{Z}/2\mathbb{Z}) \simeq \Lambda \simeq S_3 \simeq \langle \overline{\sigma}, \overline{\rho} | \overline{\sigma}^2 = \overline{\rho}^2 = 1; \overline{\sigma\rho\sigma} = \overline{\rho\sigma\rho} \rangle \tag{4}$$

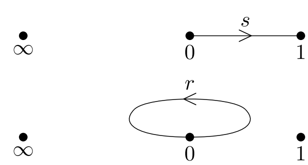
by identifying the respective generators

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \leftrightarrow (\lambda \mapsto 1-\lambda) \leftrightarrow (01) \leftrightarrow \overline{\sigma}$$

and

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \leftrightarrow \left(\lambda \mapsto \frac{\lambda}{\lambda-1}\right) \leftrightarrow (1\infty) \leftrightarrow \overline{\rho}.$$

Now suppose given the real tangential basepoint $\overrightarrow{ab} = \overrightarrow{01}$. Then let s denote the homotopy class of paths in $\mathbb{P}^1 \setminus \{0,1,\infty\}$ represented by the tangential path $[0,1]$ and let r be the homotopy class of paths represented by the loop from $\overrightarrow{01}$ to $\overrightarrow{0\infty}$ in the upper half-plane, as pictured below.



The use of tangential basepoints prevents homotopies which would otherwise occur — in particular, the homotopy classes can detect an *upper half-plane* owing to the rigidity of the real line with respect to a choice of a pair of real tangential basepoints. In this way, one sees that r is well defined as a homotopy class of paths which differs from the class of a similar loop in the lower half-plane.

The group structure on $G_{\overrightarrow{ab}}$ is facilitated by the distinct presentations of $SL(2,\mathbb{Z}/2\mathbb{Z})$ coming from (4): Firstly, we define the surjection $[\,\cdot\,]_{ab}$ of $G_{\overrightarrow{ab}}$ onto $SL(2,\mathbb{Z}/2\mathbb{Z}) \simeq S_3$ by sending a given homotopy class t in $G_{\overrightarrow{ab}}$ with endpoint $\overrightarrow{a_t b_t}$, to the permutation $[t]_{ab}$ of $\{0,1,\infty\}$ sending a to a_t and b to b_t. Next, we exploit the fact that the fractional linear automorphisms Λ are also isomorphic to $SL(2,\mathbb{Z}/2\mathbb{Z})$ to define an action of this group on $G_{\overrightarrow{ab}}$: Any $\overline{\alpha} \in \Lambda$ is a self-mapping of $\mathbb{P}^1 \setminus \{0,1,\infty\}$ and as such sends any homotopy class u of paths between real tangential basepoints, to some other such homotopy class of paths. We denote the latter by $\overline{\alpha} * u$.

Sythesizing these definitions, we have a map of $G_{\overrightarrow{ab}} \times G_{\overrightarrow{ab}}$ into $G_{\overrightarrow{ab}}$ given by

$$(t,u) \mapsto [t]_{ab} * u.$$

Remark 1.3. When $\overrightarrow{ab} = \overrightarrow{01}$ we write $[\,\cdot\,]$ for $[\,\cdot\,]_{01}$. Then notice that, viewed as linear fractional transformations of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$,

$$[r] : z \mapsto \frac{z}{z-1},$$

while

$$[s] : z \mapsto 1 - z.$$

Furthermore, for any $t \in G_{\overrightarrow{01}}$ with endpoint $\overrightarrow{a_1 b_1}$, one checks by direct computation that $[t] * r$ may be represented by a loop in the upper or lower half-plane (according to the corresponding permutation $[t]$ being even or odd, respectively), beginning at $\overrightarrow{a_1 b_1}$ and ending at $\overrightarrow{a_1 c_1}$ where $c_1 \neq b_1$, while $[t] * s$ may be represented by a straight line segment beginning at $\overrightarrow{a_1 b_1}$ and ending at $\overrightarrow{b_1 a_1}$.

Using this action, we define a concatenation procedure for homotopy classes of paths in $G_{\overrightarrow{01}}$ according to the following inductive prescription: if η is a homotopy class of paths formed from the concatenation procedure applied successively to classes in $\{r, s\}$, and ν is either r or s, let $\eta\nu$ be the homotopy class of η followed by $[\eta] * \nu$. Since $[\eta]$ sends ν to a homotopy class of paths originating at the endpoint of the paths in η, it follows that $\eta\nu \in G_{\overrightarrow{01}}$.

The construction may be repeated for any choice of real tangential basepoint $\overrightarrow{ab}$. In cases other than $\overrightarrow{ab} = \overrightarrow{01}$ write r_{ab} and s_{ab} for the corresponding generators. To be precise, r_{ab} is a loop based at $\overrightarrow{ab}$ of the form of r as above, which is in the upper half-plane for $\overrightarrow{ab} = \overrightarrow{\infty 0}$ and $\overrightarrow{ab} = \overrightarrow{1\infty}$ but in the lower half-plane when $\overrightarrow{ab}$ is $\overrightarrow{10}$, $\overrightarrow{0\infty}$, or $\overrightarrow{\infty 1}$; while s_{ab} is a straight line segment from $\overrightarrow{ab}$ to $\overrightarrow{ba}$.

Throughout write $\cdot$ for concatenation of (homotopy classes of) paths.

Definition 1.4. The mapping

$$S_{ab} : G_{\overrightarrow{ab}} \times G_{\overrightarrow{ab}} \to G_{\overrightarrow{ab}}$$

with

$$S_{ab}(\eta, \mu) = \eta\mu := \eta \cdot ([\eta]_{ab} * \mu)$$

for any $\eta, \mu \in G_{\overrightarrow{ab}}$, will be referred to as $SL(2, \mathbb{Z}/2\mathbb{Z})$ concatenation of tangential paths in $G_{\overrightarrow{ab}}$.

One checks that for any $\eta, \mu \in G_{\overrightarrow{ab}}$,

$$[\eta\mu] = [\eta] \circ [\mu]. \tag{5}$$

Using this fact, one readily proves the associativity of successive application of S_{ab}: i.e., for any η, μ and ν in $G_{\overrightarrow{ab}}$,

$$S_{ab}(\eta, S_{ab}(\mu, \nu)) = S_{ab}(S_{ab}(\eta, \mu), \nu).$$

Because of the associativity, for any $n \geq 1$, the $SL(2, \mathbb{Z}/2\mathbb{Z})$ concatenation $\nu_1 \cdots \nu_n$ of elements $\nu_j \in \{r, s\}$ is uniquely determined. It is given by

$$\nu_1 \cdot ([\nu_1] * \nu_2) \cdot ([\nu_1 \cdot [\nu_1] * \nu_2] * \nu_3) \cdot \ldots \\ \cdot ([\nu_1 \cdot [\nu_1] * \nu_2 \cdot \ldots \cdot [\ldots[[\nu_1 \cdot [\nu_1] * \nu_2] * \nu_3] \ldots] * \nu_{n-1}] * \nu_n),$$

where $\cdot$ again denotes concatenation of (homotopy classes of) paths. Applying (5) iteratively, one sees that for any $m \leq n$,

$$[\nu_1 \cdots \nu_m] = [\nu_1] \circ \cdots \circ [\nu_m],$$

so $\nu_1 \cdots \nu_n$ may be rewritten

$$\nu_1 \cdot ([\nu_1] * \nu_2) \cdot (([\nu_1] \circ [\nu_2]) * \nu_3) \cdots (([\nu_1] \circ \cdots \circ [\nu_{n-1}]) * \nu_n).$$

Now it is possible to show that for any real tangential basepoint $\overrightarrow{ab}$, $G_{\overrightarrow{ab}}$ may be endowed with a group structure with multiplication given by S_{ab}. To simplify the notation, consider the case of $\overrightarrow{ab} = \overrightarrow{01}$. Begin by observing that the class e of the trivial path acts as the identity. Also, s is its own inverse, since $[s] * s$ is the homotopy class of paths represented by the tangential path $[1, 0]$, which is inverse to $[0, 1]$. The inverse of r is the homotopy class q of paths represented by the loop from $\overrightarrow{01}$ to $\overrightarrow{0\infty}$ in the *lower* half-plane — one checks easily that $rq = qr = e$. We write $q = r^{-1}$. Of course $[r] = [r^{-1}]$.

With the group structure induced in this way, it is easy to prove that

$$G_{\overrightarrow{ab}} \simeq \langle r_{ab}, s_{ab} \rangle / (s_{ab}^2, (s_{ab} r_{ab})^3),$$

where $\langle r_{ab}, s_{ab} \rangle = F_2$ denotes the free group on the two generators r_{ab} and s_{ab}.

Now it is a well-known fact that

$$PSL(2,\mathbb{Z}) = \langle \rho, \sigma \rangle / (\sigma^2, (\rho \circ \sigma)^3),$$

where $\langle \rho, \sigma \rangle = F_2$, the free group on two generators. (For example, consult [13], in which the Bruhat decomposition is given, by means of which one can write down generators and relations for $SL(2,\mathbb{R})$.) Viewing $PSL(2,\mathbb{Z})$ as a group of linear fractional transformations of H, generators may be given by

$$\rho : \tau \mapsto 1 + \tau$$

and

$$\sigma : \tau \mapsto -\frac{1}{\tau}.$$

It then follows that for any real tangential basepoint $\overrightarrow{ab}$,

$$G_{\overrightarrow{ab}} \simeq PSL(2,\mathbb{Z}). \tag{6}$$

Since we now have

$$\pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}, \overrightarrow{ab}) \lhd G_{\overrightarrow{ab}},$$

the isomorphism of (6) gives the isomorphism of the fundamental group with the congruence subgroup $\Gamma(2)/\{\pm 1\}$ on the level of the generators ρ and σ.

Notational remark 1.5. The multiplication in $PSL(2,\mathbb{Z})$ is written in the functional order, whereas concatenation of paths in $G_{\overrightarrow{ab}}$ occurs in the order in which the paths are written.

Remark 1.6. Denote the isomorphism of (6) by

$$\Psi_{ab} : G_{\overrightarrow{ab}} \xrightarrow{\simeq} PSL(2,\mathbb{Z}),$$

writing $\Psi := \Psi_{01}$ in the special case of $\overrightarrow{ab} = \overrightarrow{01}$.

We know that for any given $u \in G_{\overrightarrow{ab}}$, $\Psi_{ab}(u)$ is a transformation of the upper half-plane which sends the lift of $\overrightarrow{ab}$ under the covering map $\lambda : H \to \mathbb{P}^1 \setminus \{0,1,\infty\}$ in some fixed fundamental domain for $\mathbb{P}^1 \setminus \{0,1,\infty\}$, to some lift of the endpoint of u under λ.

Finally, we remark that with notation as above,

$$[\Psi_{ab}^{-1}(\cdot)]_{ab} : PSL(2,\mathbb{Z}) \to SL(2,\mathbb{Z}/2\mathbb{Z})$$

is the usual projection (i.e., $\mathcal{A}$ of (3)).

Subsequently write $\mathcal{A}(\upsilon) = \overline{\upsilon}$ for any $\upsilon \in PSL(2,\mathbb{Z})$, and suppress the mapping Ψ_{ab} from the notation (i.e., implicitly identify elements of $PSL(2,\mathbb{Z})$ with those of $G_{\overrightarrow{ab}}$).

1.2. Extending the monodromy representation

1.2.1. The universal prounipotent bundle with connection on $\mathbb{P}^1 \setminus \{0,1,\infty\}$. For definitions and properties of CHEN iterated integrals, the reader is referred to [7] or [11], and for general facts related to bundles with connections on curves (and parallel transport), to [4] or [10].

Concretely, the universal prounipotent bundle with connection (cf. [12]) on $\mathbb{P}^1 \setminus \{0,1,\infty\}$ is constructed as follows: with X_0 and X_1 formal non-commuting variables as above and $I = (X_0, X_1)$ the augmentation ideal, let

$$U_n := \mathbb{C}\langle X_0, X_1\rangle / I^{n+1},$$

i.e. the algebra comprising linear combinations of words in the X_j of length less than or equal to n. The inverse limit of the U_n is the power series algebra in the non-commuting variables

$$U := \lim_{\leftarrow} U_n = \mathbb{C}\ll X_0, X_1 \gg .$$

Now we set $\mathcal{U}_n := U_n \otimes \mathcal{O}_{\mathbb{P}^1\setminus\{0,1,\infty\}}$ and $\mathcal{U} := \lim_{\leftarrow} \mathcal{U}_n$. With the ω_j defined as above for $j = 0, 1$, and $|w|$ denoting the length of the word w in the X_j, a compatible family of connections on the $\mathcal{U}_n$ can be defined, giving rise to a connection on $\mathcal{U}$: Indeed, let

$$\sum_{|w|\le n} f_w w \in \mathcal{U}_n$$

be arbitrary, and set

$$\nabla_n \left(\sum_{|w|\le n} f_w w \right) = \sum_{|w|\le n} df_w w - pr_n \sum_{|w|\le n} f_w \sum_{i=1}^{m} \omega_i w X_i,$$

where pr_n is the projection to $\mathcal{U}_n$ — i.e., the augmented words $[wX_i]$ having length greater than n are disregarded. One checks readily that ∇_n is a connection on $\mathcal{U}_n$, which is unipotent (that is to say, a successive extension of trivial bundles $(\mathcal{O}^r_{\mathbb{P}^1\setminus\{0,1,\infty\}}, d)$; for a similar computation see [12]). Moreover, for $k > 0$ the (suitably interpreted) restriction of the connection on $\mathcal{U}_{n+k}$ to $\mathcal{U}_n$ evidently agrees with ∇_n. Hence $(\mathcal{U}, \nabla)$ is the inverse limit of unipotent connections on X.

∇ is identical to the formal KZ equation

$$dG(z, X_0, X_1) = \left(\frac{dz}{z}X_0 + \frac{dz}{1-z}X_1\right) G(z, X_0, X_1).$$

A fundamental solution to this equation asymptotic to $\exp(X_0 \log z)$ as z approaches 0 is the polylogarithm-generating function $\mathrm{Li}(z, X_0, X_1)$, given by the Chen series

$$\mathrm{Li}(z, X_0, X_1) := \sum_w \int_{[\overrightarrow{01}, z]} \omega_{i_1} \cdots \omega_{i_k} X_{i_1} \cdots X_{i_k},$$

where $[\overrightarrow{01}, z]$ denotes a tangential path from $\overrightarrow{01}$ to z which winds around neither 1 nor ∞ in $\mathbb{P}^1 \setminus \{0, 1, \infty\}$; and other notation is as in the introduction.

1.2.2. The reduced action on sections of $\mathcal{U}$. The Λ action on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ by linear fractional transformations lifts to the (global) sections of $\mathcal{O}_{\mathbb{P}^1\setminus\{0,1,\infty\}}$ in the obvious way. This produces an action on sections of $\mathcal{U}$ once a suitable action of Λ on the formal variables X_0 and X_1 is defined. The latter was determined by Okuda and Ueno in Section 3 of [16], in which formal algebraic arguments and the theory of differential equations were used to compute the Λ action on the fundamental solutions to the KZ equation with specific asymptotics at 0, 1 and ∞, respectively, generalizing a calculation of Drinfel'd. The action on X_0 and X_1 arises from a simple substitution action on the KZ equation:

Example 1.7. Consider the element $\overline{\sigma} : z \mapsto 1 - z$ of Λ. Making this substitution in the KZ equation yields

$$-\frac{d}{dz}G(1-z, X_0, X_1) = \left(\frac{X_0}{1-z} + \frac{X_1}{z}\right) G(1-z, X_0, X_1)$$

— i.e.

$$\frac{d}{dz}\tilde{G}(z, X_0, X_1) = \left(\frac{-X_1}{z} + \frac{-X_0}{1-z}\right) \tilde{G}(z, X_0, X_1). \tag{7}$$

This equation is identical to the original KZ equation but for the interchanging of $X_0 \leftrightarrow -X_1$. Therefore we define the action of $\overline{\sigma}$ on the pair (X_0, X_1) of formal non-commuting variables, as the involution $(X_0, X_1) \mapsto (-X_1, -X_0)$.

This example may be imitated for each element of Λ, and as in (25) of [16] it is convenient to summarize all transformations of (X_0, X_1) which arise in this way. The associated linear fractional transformations of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ are also tabulated:

Elt. of $SL(2, \mathbb{Z}/2\mathbb{Z})$	Lin. frac. tr.	Action on (X_0, X_1)
$1:$	$z \mapsto z$	$(X_0, X_1) \mapsto (X_0, X_1)$,
$\overline{\sigma}:$	$z \mapsto 1 - z$	$(X_0, X_1) \mapsto (-X_1, -X_0)$,
$\overline{\rho}:$	$z \mapsto \frac{z}{z-1}$	$(X_0, X_1) \mapsto (X_0, X_0 - X_1)$,
$\overline{\sigma} \circ \overline{\rho}:$	$z \mapsto \frac{1}{1-z}$	$(X_0, X_1) \mapsto (-X_1, X_0 - X_1)$,
$\overline{\rho} \circ \overline{\sigma}:$	$z \mapsto \frac{z-1}{z}$	$(X_0, X_1) \mapsto (X_1 - X_0, -X_0)$,
$\overline{\rho} \circ \overline{\sigma} \circ \overline{\rho} = \overline{\sigma} \circ \overline{\rho} \circ \overline{\sigma}:$	$z \mapsto \frac{1}{z}$	$(X_0, X_1) \mapsto (X_1 - X_0, X_1)$.

Now one can state the

Definition 1.8. For every $\overline{v} \in SL(2, \mathbb{Z}/2\mathbb{Z})$ and every global section $L(z, X_0, X_1)$ of $\mathcal{U}$, set

$$L^{\overline{v}}(z, X_0, X_1) := L(\overline{v}(z), \overline{v}X_0, \overline{v}X_1)$$

and refer to this as the $SL(2, \mathbb{Z}/2\mathbb{Z})$-action on global sections of $\mathcal{U}$.

Example 1.9. We compute $\mathrm{Li}^{\overline{\sigma}}(z, X_0, X_1)$: By construction,

$$\mathrm{Li}^{\overline{\sigma}}(z, \overline{\sigma}X_0, \overline{\sigma}X_1) = \mathrm{Li}(1 - z, X_0, X_1)$$

is a fundamental solution to (7). Formally, $\mathrm{Li}(z, -X_1, -X_0)$ is also. Recall from Section 1.2.1 that

$$\mathrm{Li}(z, X_0, X_1) \exp(-X_0 \log z) \to 1$$

as $z \to 0$. Hence

$$\mathrm{Li}(z, -X_1, -X_0) \exp(X_1 \log z) \to 1 \tag{8}$$

as $z \to 0$. Now recall from [1]

$$\lim_{z \to 1} \mathrm{Li}(z, X_0, X_1) \exp(X_1 \log(1 - z)) = \Phi_{\mathrm{KZ}}(X_0, X_1),$$

where $\Phi_{\mathrm{KZ}}(X_0, X_1)$ denotes the DRINFEL'D associator[1], or equivalently,

$$\lim_{z \to 0} \mathrm{Li}(1-z, X_0, X_1) \exp(X_1 \log z) = \Phi_{\mathrm{KZ}}(X_0, X_1).$$

But then $\Phi_{\mathrm{KZ}}(X_0, X_1)\mathrm{Li}(z, -X_1, -X_0)$ and $\mathrm{Li}(1-z, X_0, X_1)$ share the same asymptotics near zero and both are solutions to the KZ equation. By uniqueness of such solutions, then

$$\mathrm{Li}^{\overline{\sigma}}(z, X_0, X_1) = \mathrm{Li}(1-z, -X_1, -X_0) = \Phi_{\mathrm{KZ}}(-X_1, -X_0)\mathrm{Li}(z, X_0, X_1).$$

We remark that by the symmetry in the above computation, it is evident that $\Phi_{\mathrm{KZ}}(X_0, X_1)^{-1} = \Phi_{\mathrm{KZ}}(-X_1, -X_0)$, a fact which will be used often in what follows.

With the notation of 1.8, the computations of Propostion 2 of [16], (which run in the same vein as 1.9), may be summarized by

Proposition 1.10.

$$\begin{aligned}
\mathrm{Li}^{\overline{\sigma}}(z, X_0, X_1) &= \Phi_{\mathrm{KZ}}(-X_1, -X_0)\mathrm{Li}(z, X_0, X_1),\\
\mathrm{Li}^{\overline{\rho}}(z, X_0, X_1) &= \exp(\mp X_0 i\pi)\mathrm{Li}(z, X_0, X_1),\\
\mathrm{Li}^{\overline{\sigma}\circ\overline{\rho}}(z, X_0, X_1) &= \exp(\pm X_1 i\pi)\Phi_{\mathrm{KZ}}(-X_1, -X_0)\mathrm{Li}(z, X_0, X_1),\\
\mathrm{Li}^{\overline{\rho}\circ\overline{\sigma}}(z, X_0, X_1) &= \Phi_{\mathrm{KZ}}(X_0, X_0 - X_1)^{-1} \exp(\mp X_0 i\pi)\mathrm{Li}(z, X_0, X_1)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Li}^{\overline{\rho}\circ\overline{\sigma}\circ\overline{\rho}}(z, X_0, X_1) &= \exp(\pm(X_0 - X_1)i\pi)\Phi_{\mathrm{KZ}}(X_1 - X_0, -X_0)\\
&\quad \times \exp(\mp X_0 i\pi)\mathrm{Li}(z, X_0, X_1)\\
&= \mathrm{Li}^{\overline{\sigma}\circ\overline{\rho}\circ\overline{\sigma}}(z, X_0, X_1)\\
&= \Phi_{\mathrm{KZ}}(X_1 - X_0, X_1) \exp(\pm X_1 i\pi)\Phi_{\mathrm{KZ}}(-X_1, -X_0)\\
&\quad \times \mathrm{Li}(z, X_0, X_1),
\end{aligned}$$

where the ambiguity in sign is according to z being in the upper or lower half plane respectively.

The ambiguity in sign will be resolved in lifting the action to $PSL(2, \mathbb{Z})$.

We remark that the equality $\mathrm{Li}^{\overline{\rho}\circ\overline{\sigma}\circ\overline{\rho}} = \mathrm{Li}^{\overline{\sigma}\circ\overline{\rho}\circ\overline{\sigma}}$ follows from the well-definedness of the Λ action and is a means of using the braid relation $\overline{\rho} \circ$

[1] This expression can be taken as the definition of Φ_{KZ}, but this formal power series can also be given more explicitly. See [14].

$\overline{\sigma}\circ\overline{\rho}=\overline{\sigma}\circ\overline{\rho}\circ\overline{\sigma}$ to establish the (highly non-trivial) hexagonal relations of Drinfel'd, to wit

$$\Phi_{\mathrm{KZ}}(X_1-X_0,X_1)\exp(\pm X_1 i\pi)\Phi_{\mathrm{KZ}}(-X_1,-X_0)$$
$$=\exp(\pm(X_0-X_1)i\pi)\Phi_{\mathrm{KZ}}(X_1-X_0,-X_0)\exp(\mp X_0 i\pi).$$

1.2.3. Lifting the action on sections of $\mathcal{U}$ to $PSL(2,\mathbb{Z})$. The action of $SL(2,\mathbb{Z}/2\mathbb{Z})$ on the formal variables X_0 and X_1 as given in the table in Section 1.2.2 extends by linearity to polynomials in the X_j with complex coefficients, and thereby to the quotients

$$\mathbb{C}\langle X_0,X_1\rangle/I^{N+1}$$

(where $I=(X_0,X_1)$ denotes the augmentation ideal); and hence to the inverse limit $\mathbb{C}\ll X_0,X_1\gg$. More precisely, we have:

Definition 1.11. The action of $\overline{\alpha}\in SL(2,\mathbb{Z}/2\mathbb{Z})$ on a formal power series $F(X_0,X_1)\in\mathbb{C}\ll X_0,X_1\gg$ is given by

$$F(X_0,X_1)^{\overline{\alpha}}:=F(\overline{\alpha}X_0,\overline{\alpha}X_1).$$

A given element $\alpha\in PSL(2,\mathbb{Z})$ then acts on power series by reduction to $SL(2,\mathbb{Z}/2\mathbb{Z})$. In this case we replace $\overline{\alpha}$ by α in the notation for the above action — i.e., we set

$$F(X_0,X_1)^{\alpha}:=F(X_0,X_1)^{\overline{\alpha}}. \tag{9}$$

Now let V_a be some open neighbourhood of $a\in\{0,1,\infty\}$ in $\mathbb{P}^1$ for which $(V_a\setminus\{a\})\cap\{0,1,\infty\}$ is empty. Then set $U_a:=V_a\setminus\{a\}$. This is an open set in $\mathbb{P}^1\setminus\{0,1,\infty\}$. Suppose that $L_a(z,X_0,X_1)$ is a section of $(\mathcal{U},\nabla)$ defined over U_a — i.e., $L_a(z,X_0,X_1)\in\Gamma(U_a,\mathcal{U})$. As above, let $\overline{\upsilon}$ denote the image of $\upsilon\in PSL(2,\mathbb{Z})$ under the usual projection map to $SL(2,\mathbb{Z}/2\mathbb{Z})$. By identifying the elements of $PSL(2,\mathbb{Z})$ with those of $G_{\overrightarrow{ab}}$ as in Section 1.1, the $SL(2,\mathbb{Z}/2\mathbb{Z})$-action on section of $\mathcal{U}$ as in Definition 1.8 can be lifted to an action of $PSL(2,\mathbb{Z})$ as follows:

Definition 1.12. Fixing a choice of basepoint $\overrightarrow{ab}$, then given $\upsilon\in PSL(2,\mathbb{Z})$, analytically continue $L_a(z,\overline{\upsilon}X_0,\overline{\upsilon}X_1)$ along any path of the corresponding homotopy class of paths $\Psi^{-1}_{\overrightarrow{ab}}(\upsilon)$ in $G_{\overrightarrow{ab}}$. This is an action of $PSL(2,\mathbb{Z})$ for which the image of $L_a(z,X_0,X_1)$ will be denoted by

$$L_a^{\upsilon}(z,X_0,X_1).$$

We proceed to compute this action for the distinguished section $\mathrm{Li}(z, X_0, X_1) \in \Gamma(U_0, \mathfrak{U})$. Begin by setting

$$\omega(X_0, X_1) := \frac{dz}{z} X_0 + \frac{dz}{1-z} X_1$$

and write

$$\omega(X_0, X_1)^\alpha = \omega(\overline{\alpha} X_0, \overline{\alpha} X_1).$$

The following results will prove to be essential. The proofs are elementary.

Lemma 1.13. *For any $\alpha \in PSL(2, \mathbb{Z})$,*

$$\overline{\alpha}^* \omega(X_0, X_1) := \left[\overline{\alpha}^* \left(\frac{dz}{z}\right)\right] X_0 + \left[\overline{\alpha}^* \left(\frac{dz}{1-z}\right)\right] X_1 = \omega(X_0, X_1)^{\overline{\alpha}^{-1}}.$$

Lemma 1.14. *For any $\alpha, \beta \in PSL(2, \mathbb{Z})$,*

$$\overline{\alpha}^* [\overline{\beta}^* \omega(X_0, X_1)] = \omega(X_0, X_1)^{(\overline{\alpha} \circ \overline{\beta})^{-1}} = (\overline{\alpha} \circ \overline{\beta})^* \omega(X_0, X_1).$$

Now let $\int_\alpha \tilde{\omega}^n$ denote the n-fold Chen iterated integral of the form $\tilde{\omega}$ along α — i.e., an iterated integral in which $\tilde{\omega}$ is repeated n times. Also write $\int_\alpha \tilde{\omega}^0 = 1$. Then we have

Proposition 1.15. *For any $\alpha \in PSL(2, \mathbb{Z})$,*

$$\mathrm{Li}^\alpha(z, X_0, X_1) = F_\alpha(X_0, X_1) \mathrm{Li}(z, X_0, X_1),$$

where $F_\alpha(X_0, X_1)$ is a formal power series given by the Chen *series*

$$F_\alpha(X_0, X_1) := \sum_{n \geq 0} \int_\alpha \omega(\overline{\alpha} X_0, \overline{\alpha} X_1)^n.$$

Implicitly here, $PSL(2, \mathbb{Z})$ is identified with $G_{\overrightarrow{01}}$.

Proof. When the Chen iterated integrals are suitably interpreted — regularizing $\frac{dz}{z}$ at $z = 0$ and $\frac{dz}{1-z}$ at $z = 1$ in the usual way (cf. [11]) — for $z \notin (-\infty, 0) \cup (1, \infty)$ the polylogarithm-generating function may be expressed as

$$\mathrm{Li}(z, X_0, X_1) = \sum_{n \geq 0} \int_{[\overrightarrow{01}, z]} \omega(X_0, X_1)^n$$

with notation as above.

Now consider $\alpha \in G_{\overrightarrow{01}}$. Denote the endpoint thereof by $\overrightarrow{cd}$, and a path from $\overrightarrow{cd}$ to $\overline{\alpha}z$ which does not cross the real axis by $[\overrightarrow{cd}, \overline{\alpha}z]$. Then composing paths in the order as written (i.e., not the functional order), the analytic continuation of $\mathrm{Li}(z, X_0, X_1)$ along α is given by

$$\sum_{n\geq 0} \int_{\alpha\cdot[\overrightarrow{cd},\overline{\alpha}(z)]} \omega(X_0, X_1)^n.$$

Consider a typical integral which appears here. Using the coproduct formula for iterated integrals (since $\omega(X_0, X_1)$ is a 1-form),

$$\int_{\alpha\cdot[\overrightarrow{cd},\overline{\alpha}z]} \omega(X_0, X_1)^n = \sum_{k=0}^{n} \int_{\alpha} \omega(X_0, X_1)^k \cdot \int_{[\overrightarrow{cd},\overline{\alpha}z]} \omega(X_0, X_1)^{n-k}.$$

Now

$$\begin{aligned}\int_{[\overrightarrow{cd},\overline{\alpha}z]} \omega(X_0, X_1)^{n-k} &= \int_{[\overrightarrow{01},z]} \overline{\alpha}^* \omega(X_0, X_1)^{n-k} \\ &= \int_{[\overrightarrow{01},z]} \omega(\overline{\alpha}^{-1} X_0, \overline{\alpha}^{-1} X_1)^{n-k}\end{aligned}$$

by the Lemma 1.13. Hence, replacing X_j by $\overline{\alpha}X_j$ for $j = 0, 1$, $\mathrm{Li}^{\alpha}(z, X_0, X_1)$ is the same as

$$\begin{aligned}&\sum_{n\geq 0}\sum_{k=0}^{n} \int_{\alpha} \omega(\overline{\alpha}X_0, \overline{\alpha}X_1)^k \cdot \int_{[\overrightarrow{01},z]} \omega(\overline{\alpha}^{-1} \circ \overline{\alpha}X_0, \overline{\alpha}^{-1} \circ \overline{\alpha}X_1)^{n-k} \\ &\quad = \left[\sum_{n\geq 0} \int_{\alpha} \omega(\overline{\alpha}X_0, \overline{\alpha}X_1)^n\right] \cdot \mathrm{Li}(z, X_0, X_1).\end{aligned}$$

□

Now we can prove the fundamental

Corollary 1.16.

$$\begin{aligned}\mathrm{Li}^{\rho}(z, X_0, X_1) &= \exp(i\pi X_0)\mathrm{Li}(z, X_0, X_1), \\ \mathrm{Li}^{\sigma}(z, X_0, X_1) &= \Phi_{\mathrm{KZ}}(X_0, X_1)^{\sigma}\mathrm{Li}(z, X_0, X_1).\end{aligned}$$

Proof. Recall that $\Psi(r) = \rho$ and $\Psi(s) = \sigma$ in the notation of Section 1.1. Also, $\overline{\rho}(X_0, X_1) = (X_0, X_0 - X_1)$, while $\overline{\sigma}(X_0, X_1) = (-X_1, -X_0)$.

One computes

$$\int_r \left(\frac{dz}{z}X_0 + \frac{dz}{1-z}X_1\right)^n = \frac{X_0^n}{n!}\left(\int_r \frac{dz}{z}\right)^n = \frac{(i\pi X_0)^n}{n!}$$

by repeatedly using the shuffle product for iterated integrals and the fact that the integrals in which $\frac{dz}{1-z}$ occur vanish along r. Thence

$$\begin{aligned} F_\rho(X_0, X_1) &= \sum_{n=0}^{\infty} \int_r \left(\frac{dz}{z}X_0 + \frac{dz}{1-z}(X_0 - X_1)\right)^n \\ &= \sum_{n=0}^{\infty} \frac{(i\pi X_0)^n}{n!} \\ &= \exp(i\pi X_0). \end{aligned}$$

It is well known that

$$\Phi_{\mathrm{KZ}}(X_0, X_1) = \sum_{n=0}^{\infty} \int_{[0,1]} \left(\frac{dz}{z}X_0 + \frac{dz}{1-z}X_1\right)^n,$$

in which expression we understand the integrals to be regularized at 0 and 1 as before, and the shuffle regularization of iterated integrals is applied to those terms which otherwise diverge (see 3.4 of [1], for example). From Proposition 1.15 the second assertion of the corollary follows. □

The power series which arise here do not look too different from those which result in the case of the $SL(2, \mathbb{Z}/2\mathbb{Z})$ action as in Section 1.2.2. Where σ is concerned, the reason for this is that $\overline{\sigma}$ has a unique lift to $PSL(2, \mathbb{Z})$. On the other hand, unlike that of $\overline{\rho}$ the action of ρ is not involutive.

Theorem 1.17. *$F_\bullet(X_0, X_1)$ is an injective 1-cocycle for $PSL(2,\mathbb{Z})$ in the multiplicative group of formal power series in the non-commuting variables X_0 and X_1 equipped with the action of $PSL(2,\mathbb{Z})$ factoring through that of $SL(2,\mathbb{Z}/2\mathbb{Z})$. Specifically, for any $\upsilon, \upsilon' \in PSL(2,\mathbb{Z})$,*

$$F_\upsilon(X_0, X_1)^{\upsilon'} F_{\upsilon'}(X_0, X_1) = F_{\upsilon' \circ \upsilon}(X_0, X_1).$$

Proof. Consider such arbitrary υ and $\upsilon' \in PSL(2,\mathbb{Z})$ and identify them with some choices of paths in the corresponding homotopy classes of $G_{\overrightarrow{ab}}$. Also, interpret $\int_\upsilon \omega^0$ as 1 so that

$$F_{\upsilon'\circ\upsilon}(X_0, X_1) = \sum_{n=0}^{\infty} \int_{\upsilon\upsilon'} \left(\frac{dz}{z}Y_0 + \frac{dz}{1-z}Y_1\right)^n \Bigg|_{Y_j = \overline{\upsilon}'\circ\overline{\upsilon}X_j:\, j=0,1}.$$

Now recall from Section 1.1 that $\upsilon\upsilon' = \upsilon \cdot [\upsilon] * \upsilon'$, write $Y_j = \overline{\upsilon}' \circ \overline{\upsilon}X_j$ for $j = 0, 1$ as above, and use the coproduct formula for iterated integrals to compute

$$\begin{aligned}
&\int_{\upsilon\upsilon'} \left(\frac{dz}{z}Y_0 + \frac{dz}{1-z}Y_1\right)^n \\
&\quad = \sum_{k=0}^{n} \int_\upsilon \omega(Y_0, Y_1)^k \int_{[\upsilon]*\upsilon'} \omega(\overline{\upsilon}' \circ \overline{\upsilon}X_0, \overline{\upsilon}' \circ \overline{\upsilon}X_1)^{n-k} \\
&\quad = \sum_{k=0}^{n} \int_\upsilon \omega(Y_0, Y_1)^k \int_{\upsilon'} \overline{\upsilon}^* \omega(\overline{\upsilon}' \circ \overline{\upsilon}X_0, \overline{\upsilon}' \circ \overline{\upsilon}X_1)^{n-k} \\
&\quad = \sum_{k=0}^{n} \int_\upsilon \omega(Y_0, Y_1)^k \int_{\upsilon'} \omega((\overline{\upsilon}' \circ \overline{\upsilon}) \circ \overline{\upsilon}^{-1}X_0, (\overline{\upsilon}' \circ \overline{\upsilon}) \circ \overline{\upsilon}^{-1}X_1)^{n-k}
\end{aligned}$$

using Lemmas 1.13 and 1.14. Hence

$$\begin{aligned}
F_{\upsilon'\circ\upsilon}(X_0, X_1) &= \sum_{n\geq 0} \sum_{k=0}^{n} \int_\upsilon \omega(\overline{\upsilon}' \circ \overline{\upsilon}X_0, \overline{\upsilon}' \circ \overline{\upsilon}X_1)^k \int_{\upsilon'} \omega(\overline{\upsilon'}X_0, \overline{\upsilon'}X_1)^{n-k} \\
&= F_\upsilon(X_0, X_1)^{\upsilon'} F_{\upsilon'}(X_0, X_1).
\end{aligned}$$

Finally we prove the injectivity: consider any $\alpha \in PSL(2, \mathbb{Z})$ for which $F_\alpha(X_0, X_1) = 1$. We show that such α is necessarily trivial. First observe that because $\overline{\alpha}$ is invertible, also $F_\alpha(\overline{\alpha}X_0, \overline{\alpha}X_1) = 1$. But then

$$\int_\alpha \omega(X_0, X_1)^n = 0$$

for each $n \geq 1$, since each such integral expression is homogeneous of degree n in the X_j. Even further, the coefficients of the monomials $X_{i_1} \cdots X_{i_n}$ here are all of the form of

$$\int_\alpha \omega_{i_1} \cdots \omega_{i_n}$$

where i_j is either 0 or 1. Consequently, all such integrals are necessarily zero. But then by Chen's π_1 De Rham Theorem (given for this case in Theorem 10 in [7]), necessarily α is trivial, as was to be shown. □

Let Γ denote an arbitrary fixed subgroup of $PSL(2,\mathbb{Z})$ and define

$$\mathcal{F}_\Gamma := \{F_\alpha(X_0,X_1) \in \mathbb{C} \ll X_0, X_1 \gg | \alpha \in \Gamma\}.$$

Lemma 1.18. *The elements of $\mathcal{F}_{PSL(2,\mathbb{Z})}$ are group-like.*

Proof. $\Phi_{\mathrm{KZ}}(X_0,X_1)$ is group-like by construction. $e^{i\pi X_0}$ is group-like since X_0 is primitive. In fact, replacing (X_0,X_1) in each of these formal series by any pair of primitive elements of $\mathbb{C}\langle X_0,X_1\rangle$, the resulting series are also group-like. Now the images of X_0 and X_1 under the action of the elements of $SL(2,\mathbb{Z}/2\mathbb{Z})$ are all primitive. Consequently, each $F_\alpha(X_0,X_1)$ is a product of group-like elements, making it group-like too, since the Lie exponentials form a group. □

Endow $\mathcal{F}_\Gamma$ with a multiplication $\circledast$ coming from the $SL(2,\mathbb{Z}/2\mathbb{Z})$ action — i.e., set

$$F_\beta(X_0,X_1) \circledast F_\alpha(X_0,X_1) := F_\alpha(X_0,X_1)^\beta F_\beta(X_0,X_1) = F_{\beta\circ\alpha}(X_0,X_1).$$

This is well defined by 1.17. Also from 1.17 one obtains

Theorem 1.19. *$(\mathcal{F}_\Gamma,\circledast)$ is a group which is isomorphic to $(\Gamma,\circ)$.*

1.3. Monodromy of polylogarithms

Identifying any $g \in \pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\},\overrightarrow{01})$ with $\gamma \in PSL(2,\mathbb{Z})$ via (3), as made explicit in Section 1.1, the monodromy of $\mathrm{Li}(z,X_0,X_1)$ about g is equal to $\mathrm{Li}^\gamma(z,X_0,X_1)$, since $\overline{\gamma}$ is trivial. As a consequence of 1.15 and 1.17, determining the monodromy is now an easy calculation. For example, about the generators of the fundamental group, as first proven in [15] by means of direct methods, we have

Proposition 1.20. *The monodromy of $\mathrm{Li}(z,X_0,X_1)$ around the loop r^2 about 0 in $\mathbb{P}^1 \setminus \{0,1,\infty\}$ based at $\overrightarrow{01}$ is given by*

$$\mathrm{Li}^{\rho^2}(z,X_0,X_1) = \exp(2i\pi X_0)\mathrm{Li}(z,X_0,X_1),$$

while that about the loop sr^2s about 1 in $\mathbb{P}^1 \setminus \{0,1,\infty\}$ is given by

$$\mathrm{Li}^{\sigma\rho^2\sigma}(z,X_0,X_1) = \Phi_{\mathrm{KZ}}(X_0,X_1)^\sigma \exp(-2\pi i X_1)\Phi_{\mathrm{KZ}}(X_0,X_1)\mathrm{Li}(z,X_0,X_1).$$

Proof. Recall that the loop r^2 about 0 in $\mathbb{P}^1 \setminus \{0,1,\infty\}$ based at $\overrightarrow{01}$ corresponds to ρ^2 in $PSL(2,\mathbb{Z})$. Now

$$\begin{aligned}\mathrm{Li}^{\rho^2}(z,X_0,X_1) &= F_\rho(X_0,X_1)^\rho F_\rho(X_0,X_1)\mathrm{Li}(z,X_0,X_1)\\ &= \exp(\pi i X_0)^\rho \exp(\pi i X_0)\mathrm{Li}(z,X_0,X_1)\\ &= \exp(2\pi i X_0)\mathrm{Li}(z,X_0,X_1),\end{aligned}$$

since $\overline{\rho}(X_0) = X_0$.

The loop about 1 in $\mathbb{P}^1 \setminus \{0,1,\infty\}$ based at $\overrightarrow{01}$ is sr^2s in former notation, corresponding to $\sigma\rho^2\sigma$ in $PSL(2,\mathbb{Z})$.

$$\begin{aligned}&\mathrm{Li}^{\sigma\rho^2\sigma}(z,X_0,X_1)\\ &= F_\sigma(X_0,X_1)^{\sigma\rho^2} F_\rho(X_0,X_1)^{\sigma\rho} F_\rho(X_0,X_1)^\sigma F_\sigma(X_0,X_1)\mathrm{Li}(z,X_0,X_1)\\ &= \Phi_{\mathrm{KZ}}(X_0,X_1)\exp(-\pi i X_1)\exp(-\pi i X_1)\Phi_{\mathrm{KZ}}(X_1,X_0)^\sigma \mathrm{Li}(z,X_0,X_1)\end{aligned}$$

from the definition of the respective actions of $\overline{\sigma}$ and $\overline{\rho}$ on X_0, X_1.

□

2. Application: proving the analytic continuation and functional equation of $\zeta(s)$

As outlined in the Introduction, data encoded in the modular action on $\mathrm{Li}(z,X_0,X_1)$ may be used to give a family of proofs of the analytic continuation and functional equation of the Riemann zeta function. In particular, the analytic continuation may be affected (as in Section 2.2 below) using functional relations known as the Euler connection formulae. As shown in Proposition 5 of [16], these arise from equating coefficients of the respective sides of

$$\mathrm{Li}(1-z,-X_1,-X_0) = \Phi_{\mathrm{KZ}}(-X_1,-X_0)\mathrm{Li}(z,X_0,X_1).$$

This equation giving $\mathrm{Li}^{\overline{\sigma}}(z,X_0,X_1)$ is necessarily the same as

$$\mathrm{Li}^\sigma(z,X_0,X_1) = \Phi_{\mathrm{KZ}}(-X_1,-X_1)\mathrm{Li}(z,X_0,X_1)$$

since σ and the reduction thereof are both involutions.

Monodromy data of polylogarithm functions is used to prove the functional equation itself in Section 2.3. As shown in Proposition 1.20 above, such information is given by an equation which is an easy consequence of the modular action computed in Corollary 1.16.

The proofs are modifications of RIEMANN'S contour integral method, each based on a member of an infinite family of integral expressions for $\zeta(s)$ given in Section 2.1.

2.1. Families of integral expressions for $\zeta(s)$

In previous work, [11], the author developed a theory of complex iterated integral generalizing the usual notion of iterated integral as in the work of CHEN. In particular, on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, if $F(z)$ denotes a function with $F(0) = 0$ having TAYLOR series expansion on the unit disc for which the nth coefficient is $O(n^k)$ for some $k \geq 0$, then we define

$$L[F](s, z) = \int_{[0,z]} F(t) \left(\frac{dt}{t}\right)^s := \int_0^z \frac{(\log z - \log t)^{s-1}}{\Gamma(s)} F(t) \frac{dt}{t}, \tag{10}$$

where the usual regularization of the logarithm at zero is understood, in which case the integral can be shown to converge on $\mathfrak{Re}\ s > k + 1$. In what follows, we shall write $L[F](s) := L[F](s, 1)$ and say that the functions $F(z)$ satisfying the conditions given above are k-BIEBERBACH.

This complex iterated integral turns out to coincide under the change of variables $x = -\log t$ with the fractional integral as defined by RIEMANN and LIOUVILLE; for which the *additive iterativity* property

$$L[F](s) = L\left[\int_{[0,z]} F(t) \left(\frac{dt}{t}\right)^w\right](s - w) \tag{11}$$

holds for those w for which all relevant integrals converge. (w should have $\mathfrak{Re}\ w > k + 1$ and $\mathfrak{Re}\ (s - w) > k + 1$.) Although this much is classical, the iterated integral perspective lends itself to powerful generalization and has various number theoretic consequences (see [11]), among them the non-classical *multiplicative iterativity* property

$$\int_{[0,1]} \sum_{n=1}^{\infty} a_n t^n \left(\frac{dt}{t}\right)^s = \int_{[0,1]} \sum_{n=1}^{\infty} a_n t^{n^k} \left(\frac{dt}{t}\right)^{s/k}$$

for positive integer k.

Each of these respective iterativity properties gives rise to an infinite family of integral expressions for the RIEMANN zeta function $\zeta(s)$:

In $\mathbb{P}^1 \setminus \{0,1,\infty\}$ coordinates ABEL'S integral for $\zeta(s)$ becomes

$$\zeta(s) = \int_{[0,1]} \frac{t}{1-t}\left(\frac{dt}{t}\right)^s = L\left[\frac{t}{1-t}\right](s) \tag{12}$$

and hence by (11) may be expressed by any one of the family of integrals

$$\zeta(s) = \int_{[0,1]} \mathrm{Li}_\mu(t)\left(\frac{dt}{t}\right)^{s-\mu} \tag{13}$$

for integer μ, with $\mathrm{Li}_\mu(z)$ denoting the usual polylogarithm function

$$\mathrm{Li}_\mu(z) := \int_{[0,z]} \frac{t}{1-t}\left(\frac{dt}{t}\right)^\mu$$

when $\mu > 0$. On the other hand, by multiplicative iterativity and use of ABEL'S integral, for positive integer k

$$\zeta(s) = \int_{[0,1]} \sum_{n=1}^{\infty} t^{n^k}\left(\frac{dt}{t}\right)^{s/k}.$$

Observe that the case of $k = 2$ corresponds to the theta function integral used in RIEMANN'S second proof of the functional equation of $\zeta(s)$, under the change of variables $t = e^{-\pi u}$.

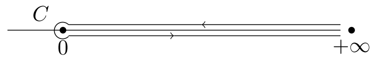
Since ABEL'S integral (12) (which forms the basis of RIEMANN'S first proof of the functional equation) belongs to both families of integrals it is interesting to exhibit a proof of the functional equation making use of an integral which is a member of the additive family of integrals but not of the multiplicative family. To this we next proceed.

2.2. Analytic continuation at integer parameter $\mu > 1$

Consider the dilogarithm integral

$$\zeta(s) = \int_{[0,1]} \mathrm{Li}_2(z)\left(\frac{dz}{z}\right)^{s-2} = \int_0^\infty \mathrm{Li}_2(e^{-x})\frac{x^{s-2}}{\Gamma(s-2)}\frac{dx}{x},$$

(where we take $x = -\log z$ to obtain the last integral). For the analytic continuation of this integral to complex values of s for which $\mathfrak{Re}\, s \leq 2$ the standard (HANKEL) contour C (as pictured below) is not suitable with this integrand, because the dilogarithm monodromy term arising from moving about 0 is $2\pi i x$, which does not have finite Mellin transform.



Instead we use Euler's dilogarithm inversion formula, which is a functional equation of the dilogarithm effecting a change between $z = 0$ and $z = 1$. This shifts the monodromy of $\mathrm{Li}_2(e^{-x})$ to $x = +\infty$, so that Hankel's contour may be used.

Now write $\mathrm{Li}_{2,1\times(n-2)}(z) := \mathrm{Li}_{211\ldots1}(z)$ with the index 1 repeated $n-2$ times; i.e., the multiple polylogarithm

$$\mathrm{Li}_{211\ldots1}(z) = \int_{[0,z]} \frac{dt}{1-t}\,\frac{dt}{t}\,\frac{dt}{1-t}\,\frac{dt}{1-t}\cdots\frac{dt}{1-t}$$

in which the form $\frac{dt}{1-t}$ occurs a total of $n-1$ times. Using a generalized version of the dilogarithm inversion formula, we find in general:

Theorem 2.1. *For each integer $m \geq 2$,*

$$\zeta(s) = -\frac{(s-m)\Gamma(m)\Gamma(1-s)}{2\pi i}\int_C \frac{\mathrm{Li}_{2,1\times(m-2)}(1-e^{-x})}{x^m}(-x)^s\frac{dx}{x}$$

for all complex $s \neq 1$ satisfying $\mathfrak{Re}\ s < m$.

For each fixed m, together with (13) this result gives the analytic continuation of $\zeta(s)$ to all values other than $\{1\} \cup \{\mathfrak{Re}\ s = m\}$.

Proof. Throughout let m denote an integer with $m \geq 2$. Then

$$\begin{aligned}\zeta(s) &= \int_0^1 \mathrm{Li}_m(e^{-x})\frac{x^{s-m}}{\Gamma(s-m)}\frac{dx}{x} + \int_1^\infty \mathrm{Li}_m(e^{-x})\frac{x^{s-m}}{\Gamma(s-m)}\frac{dx}{x}\\ &= \int_0^1 \left[\frac{\mathrm{Li}_m(e^{-x})}{x} - \frac{\zeta(m)}{x}\right]\frac{x^{s-m}}{\Gamma(s-m)}dx + \frac{\zeta(m)}{(s-m)\Gamma(s-m)}\\ &\quad + \int_1^\infty \mathrm{Li}_m(e^{-x})\frac{x^{s-m}}{\Gamma(s-m)}\frac{dx}{x}.\end{aligned}$$

This last expression holds also for $m-1 < \mathfrak{Re}\ s < m$ by analytic continuation, since $\mathrm{Li}_m(e^0) = \zeta(m)$. But on this vertical strip,

$$\frac{1}{s-m} = -\int_1^\infty x^{s-m-1}dx.$$

Consequently, provided that $m-1 < \mathfrak{Re}\ s < m$, we can write

$$\zeta(s) = \int_0^\infty \left(\frac{\mathrm{Li}_m(e^{-x}) - \zeta(m)}{x} \right) \frac{x^{s-m}}{\Gamma(s-m)} dx. \tag{14}$$

(This much is patterned on a similar analytic continuation in [17].)

Now from (45) in [16] using exponential coordinates and $\mathrm{Li}_1(z) = -\log(1-z)$, EULER'S connection formula takes the form of

$$\begin{aligned}\mathrm{Li}_m(e^{-x}) - \zeta(m) = -\mathrm{Li}_{2,1\times(m-2)}(1-e^{-x}) - \frac{x^{m-1}}{(m-1)!}(-\log(1-e^{-x})) \\ -\frac{x^{m-2}}{(m-2)!}\mathrm{Li}_2(e^{-x}) - \cdots - \frac{x^2}{2!}\mathrm{Li}_{m-2}(e^{-x}) - x\mathrm{Li}_{m-1}(e^{-x}).\end{aligned}$$

In substituting this expression into (14) we notice immediately that all other polylogarithm integral expressions as in (13) with $\mu = 1, \ldots, m-1$ appear. These expressions are also valid for $m-1 < \mathfrak{Re}\ s < m$, so in each case, the resulting expression may be replaced by some multiple of $\zeta(s)$. We show this explicitly when $m = 2$: Then,

$$\zeta(s) = \int_0^\infty \left(\frac{\log(1-e^{-x})}{x} - \frac{\mathrm{Li}_2(1-e^{-x})}{x^2} \right) \frac{x^{s-1}}{\Gamma(s-2)} dx \tag{15}$$

whenever $1 < \mathfrak{Re}\ s < 2$. Here, for $\mathfrak{Re}\ s > 1$ we have

$$\zeta(s) = \int_{[0,1]} \mathrm{Li}_1(z) \left(\frac{dz}{z}\right)^{s-1} = -\int_0^\infty \log(1-e^{-x}) \frac{x^{s-1}}{\Gamma(s-1)} \frac{dx}{x}.$$

From $(s-2)\Gamma(s-2) = \Gamma(s-1)$ it then follows that

$$\int_0^\infty \frac{\log(1-e^{-x})}{x} \frac{x^{s-1}}{\Gamma(s-2)} dx = -(s-2)\zeta(s), \tag{16}$$

for $1 < \mathfrak{Re}\ s < 2$. Then using (16) in (15),

$$(s-1)\zeta(s) = -\int_0^\infty \frac{\mathrm{Li}_2(1-e^{-x})}{x^2} \frac{x^{s-1}}{\Gamma(s-2)} dx.$$

Most generally, repeated use of the functional equation

$$\Gamma(r+1) = r\Gamma(r) \tag{17}$$

together with (13) shows that for each integer k with $1 \le k \le m-1$,

$$-\int_0^\infty \frac{x^{m-k}\mathrm{Li}_k(e^{-x})}{(m-k)!x}\frac{x^{s-m}}{\Gamma(s-m)}dx = -\frac{(s-m)\dots(s-k-1)}{(m-k)!}\zeta(s)$$
$$= -\binom{s-k-1}{m-k}\zeta(s).$$

Adding the negative of such expressions to both sides of our equation for $\zeta(s)$ (found by substitution of the EULER connection formula into (14)) and using a simple inductive argument to add up the terms of the coefficient, (adding first $1+(s-m)$ to obtain $s-m+1$ then taking this as a common factor in summing with the next term and so on), the left side becomes

$$\left[1+\binom{s-m}{1}+\dots+\binom{s-2}{m-1}\right]\zeta(s) = \binom{s-1}{m-1}\zeta(s)$$

while the right side is given by

$$-\int_0^\infty \mathrm{Li}_{2,1\times(m-2)}(1-e^{-x})\frac{x^{s-m}}{\Gamma(s-m)}\frac{dx}{x}.$$

By equating coefficients of the first equation of 1.20, one sees that $\mathrm{Li}_{2,1\times(m-2)}(z)$ has no monodromy about $z=0$. Thus, $\mathrm{Li}_{2,1\times(m-2)}(1-e^{-x})$ has no monodromy about $x=0$. Now with C as above, consider

$$I(s) := \int_C \frac{\mathrm{Li}_{2,1\times(m-2)}(1-e^{-x})}{x^m}(-x)^{s-1}dx.$$

Here the branch cut for the logarithm is taken along the negative real axis, so that for the portion of C above the real axis from $x=+\infty$ to $x=0$,

$$(-x)^s = e^{s(\log x - i\pi)}$$

and along the part of C below the real axis back from 0 to $+\infty$,

$$(-x)^s = e^{s(\log x + i\pi)}.$$

Now along the arc, say with $|x|=\varepsilon$, which is the piece of C around $x=0$, the integrand is bounded by

$$M\varepsilon\left|x^{(\mathfrak{Re}\, s)-m}\right|e^{2\pi\varepsilon}$$

for some constant $M>0$ because $\mathrm{Li}_{2,1\times(m-2)}(1-e^{-x})$ vanishes at $x=0$ at least to the same order as does x. Now since $\mathfrak{Re}\, s > m-1$, the integral about $|x| = \varepsilon$ approaches 0 as ε becomes very small. (The integral is of the order of $\varepsilon^{\mathfrak{Re}\, s-m+1}$.)

Consequently, in the limit as ε approaches 0, we have

$$\begin{aligned} I(s) \to -e^{-i\pi s} \int_\infty^0 \frac{\mathrm{Li}_{2,1\times(m-2)}(1-e^{-x})}{x^m} x^s \frac{dx}{x} \\ - e^{i\pi s} \int_0^\infty \frac{\mathrm{Li}_{2,1\times(m-2)}(1-e^{-x})}{x^m} x^s \frac{dx}{x} \\ = (e^{i\pi s} - e^{-i\pi s}) \binom{s-1}{m-1} \Gamma(s-m)\zeta(s). \end{aligned}$$

Now $2i\sin(\pi s) = e^{i\pi s} - e^{-i\pi s}$ and we may repeatedly use (17) to see that

$$\binom{s-1}{m-1}\Gamma(s-m) = \frac{\Gamma(s)}{(s-m)\cdot(m-1)!} = \frac{\Gamma(s)}{(s-m)\Gamma(m)}.$$

Moreover,

$$\frac{\pi}{\sin \pi s} = \Gamma(s)\Gamma(1-s),$$

so that

$$\begin{aligned} (e^{i\pi s} - e^{-i\pi s}) \binom{s-1}{m-1}\Gamma(s-m) &= 2i\sin(\pi s)\frac{\Gamma(s)}{(s-m)\Gamma(m)} \\ &= \frac{2i\pi}{(s-m)\Gamma(m)\Gamma(1-s)}. \end{aligned}$$

Hence

$$\zeta(s) = \frac{(s-m)\Gamma(m)\Gamma(1-s)}{2\pi i} \int_C \frac{\mathrm{Li}_{2,1\times(m-2)}(1-e^{-x})}{x^m}(-x)^{s-1}dx. \tag{18}$$

This expression has been proven for $m > \mathfrak{Re}\ s > m-1$, but converges for all s having $\mathfrak{Re}\ s < m$ with the possible exception of the poles $s = 1, \ldots, m-1$ of $\Gamma(1-s)$ in this region of the plane. (When $\mathfrak{Re}\ s \geq m$, the integrand exhibits unsuitable behavior at infinity.) However, for $s = 2, \ldots, m-1$ the integral vanishes by the usual argument: there is no monodromy about 0, so the integrals above and below the real line differ by a factor of -1 and approach the same absolute value in the limit as the contour approaches the real line; while near zero, one computes by L'HÔPITAL'S Rule that

$$\lim_{x\to 0}\frac{\mathrm{Li}_{2,1\times(m-2)}(1-e^{-x})}{x^m}=\lim_{x\to 0}\frac{(-1)^{m-1}}{m!x}.$$

This leaves only the simple pole at $s=1$, for which we may compute the residue using (18). Firstly,

$$\mathrm{Res}_{s=1}\Gamma(1-s)=-1,$$

so

$$\begin{aligned}\mathrm{Res}_{s=1}\zeta(s)&=\lim_{s=1}(s-1)\Gamma(1-s)\frac{2\pi i}{(m-1)!(m-1)}\frac{(s-m)\Gamma(m)}{2\pi i}\\&=(-1)\frac{(1-m)(m-1)!}{(m-1)!(m-1)}=1\end{aligned}$$

as is well known.

The analytic continuation for $\zeta(s)$ to $\mathfrak{Re}\ s<m$ is achieved by (18). □

2.3. Dilogarithm proof of the functional equation

In principle, we may now imitate RIEMANN'S contour integral proof using each of the integrals of (18). For each $m\geq 2$ this would be done using the monodromy of $\mathrm{Li}_{2,1\times(m-2)}(z)$, as may be calculated using 1.20 (by equating the coefficients of terms of the respective power series). The interesting aspect of the computation is that the monodromy terms coming from considering $\mathrm{Li}_{2,1\times(m-2)}(1-e^{-x})$ around ∞ have themselves monodromy about $2\pi in$ for integer n, and this monodromy of the monodromy is what contributes terms that add up to $\zeta(1-s)$ multiplied by some factor.

Here is a sketch of some details in the case that $m=2$:

As in RIEMANN'S original calculation, consider $\mathbb{C}$ with a logarithmic branch cut along the negative real axis (viewed as a model for the punctured surface $X_{\log}\xrightarrow{\sim}\mathbb{C}\backslash 2\pi i\mathbb{Z}$ on which the logarithm function is single valued). Assuming that $\mathfrak{Re}\ s<0$ we lift the contour C, negatively oriented, to a closed path E_N in this space which excludes those points (of imaginary part with absolute value bounded by $2\pi(N+1)$ for some large positive integer N) where the integrand in (18) is singular — i.e., we exclude the positive real axis and the points $2\pi in$ for integer n with $|n|<N+1$. We can take this path to consist of translates of C by $2\pi in$ for such integers n, (where we abuse notation and denote the negatively oriented path also by C), joined by

straight line segments forming a closed path which includes a portion of the vertical line segment with real part $-R$ for some large positive real number R. Then the functional equation can be recovered from the statement of CAUCHY'S Theorem when $\mathfrak{Re}\ s < 0$:

$$\frac{(s-2)\Gamma(1-s)}{2\pi i}\int_{E_N}\frac{\mathrm{Li}_2(1-e^{-x})}{x^2}(-x)^s\frac{dx}{x}=0.$$

Computing this integral in the limit as C shrinks closer to the real axis and extends to $+\infty$ along the real axis, and N nears ∞, under the assumption that $\mathfrak{Re}\ s < -2$, yields the functional equation.

In performing this calculation, a term $-\zeta(s)$ arises from the integral along the contour straddling the positive real axis.

Next, if the path C extends to $+R$ and lies a distance of $\varepsilon > 0$ to either side of the real axis, consider

$$\frac{(s-2)\Gamma(1-s)}{2\pi i}\int_{R+i\varepsilon}^{R+2\pi i-i\varepsilon}\frac{\mathrm{Li}_2(1-e^{-x})}{x^2}(-x)^s\frac{dx}{x}.$$

Passage along this line segment produces a monodromy term from the dilogarithm, of

$$\frac{2\pi i\log(1-e^{-x})}{x^3}(-x)^s, \tag{19}$$

which must be taken into account along all subsequent paths.

In computing the dilogarithm integral along the translate $C + 2\pi i$ we obtain

$$\frac{(s-2)\Gamma(1-s)}{2\pi i}\int_{\gamma_1}\frac{2\pi i\log(1-e^{-x})}{x^2}(-x)^s\frac{dx}{x},$$

where γ_k will denote the (negatively oriented) loop of radius δ about $2\pi ik$ for integer k, along with

$$(s-2)\Gamma(1-s)(2\pi i)\int_{2\pi i+\delta+i\varepsilon}^{R+2\pi i+i\varepsilon}(-x)^{s-2}\frac{dx}{x},$$

the term arising from the monodromy of $\log(1-e^{-x})$ about $2\pi i$. As before, this last integrand must be considered along all subsequent subpaths of E_N. But notice that along the remaining translates of C, this monodromy term (from passage around $2\pi i$) is 0 (again in the limit as $\varepsilon \to 0$) by CAUCHY'S Theorem.

Now let $D_{R,n}$ denote the rectangular path comprising straight line segments between successively: $R+2n\pi i, R+(2N+1)\pi i, -R+(2N+1)\pi i, -R-(2N+1)\pi i, R-(2N+1)\pi i$ and $R-2n\pi i$ for non-negative integer $n \leq N$. Continuing along E_N we find thus that the integrals which are yet to be computed add to

$$\begin{aligned}
&\frac{(s-2)\Gamma(1-s)}{2\pi i}\int_{D_{R,0}} \frac{\mathrm{Li}_2(1-e^{-x})}{x^2}(-x)^s\frac{dx}{x} \\
&\quad+\sum_{n=1}^{N}\frac{(s-2)\Gamma(1-s)}{2\pi i}\int_{\gamma_n}\frac{n2\pi i\log(1-e^{-x})}{x^2}(-x)^s\frac{dx}{x} \\
&\quad+\frac{(s-2)\Gamma(1-s)}{2\pi i}\sum_{n=1}^{N}\left\{\int_{2n\pi i+\delta+i\varepsilon}^{R+2n\pi i+i\varepsilon} n(2\pi i)^2(-x)^{s-2}\frac{dx}{x}\right. \\
&\quad\left.+\int_{D_{R,n}}\sum_{m=1}^{n} m(2\pi i)^2(-x)^{s-2}\frac{dx}{x}\right\}
\end{aligned}$$

along with terms with integrand of the form of (19) integrated along the outside contour, and certain similar terms for suitable negative values of n.

Now along the portion of E_N proceeding from the point $-R+(2N+1)\pi i$, further monodromy terms arise from the dilogarithm terms, in this instance the negatives of terms of the form of (19). All such terms themselves exhibit monodromy each time the path traverses a segment of length $2\pi i$ along the line segment from $-R+2N\pi i$ to $-R-2N\pi i$ since images of such segments trace out circles of radius e^R about $z=1$ under the mapping $x \mapsto 1-e^{-x}=z$.

These new dilogarithm monodromy terms cancel out those from the first vertical portion of the path, so that all such terms add to zero by when the point $-R$ is reached along $C^{\delta,\varepsilon}_{R,N}$. Below the real axis, negative terms accumulate so that once one reaches the point $-R-(2N+1)\pi i$, the remaining dilogarithm monodromy terms add to

$$-2\pi i N\log(1-e^{-x}).$$

On the other hand, since the number of logarithmic terms as one moves along $-R+i\alpha$ (for real decreasing α) decreases from N to $N-1$ to $N-2$ and so on, the sum of the logarithmic monodromy terms number successively $N(N+1)/2; N(N+1)/2+N$; then $N(N+1)/2+N+(N-1)$ and so on, until at the point $-R$, there are $N(N+1)$ such terms. Thereafter, the increasing number of negative logarithmic terms decrease the total number of these second monodromy terms. Eventually, at $-R-(2N+1)\pi i$, the end of the vertical line, the terms which remain sum to

$$(2\pi i)^2[1+2+\cdots+N].$$

By the same argument as before, the integral coming from the terms $(2\pi i)^2$ $[1+2+\cdots+N-1]$ is zero around $C-2N\pi i$, but because of the monodromy of the log term about $-2N\pi i$, the integral

$$\frac{(s-2)\Gamma(1-s)}{2\pi i}\int_{R-2N\pi i-i\varepsilon}^{-2N\pi i+\delta-i\varepsilon} N(2\pi i)^2(-x)^{s-2}\frac{dx}{x}$$

does need to be taken into account. Continuing back to the starting point of E_N, similar terms add to

$$\frac{(s-2)\Gamma(1-s)}{2\pi i}\sum_{n=1}^{N}\int_{R-2n\pi i-i\varepsilon}^{-2n\pi i+\delta-i\varepsilon} n(2\pi i)^2(-x)^{s-2}\frac{dx}{x}.$$

This expression, along with its counterpart from the part of E_N with positive imaginary part, is readily computed in the limit as δ and ε approach 0 while R tends to ∞: indeed, using CAUCHY'S Theorem applied to rectangular contours respectively below and above the real axis, we find that

$$\lim_{R\to\infty;\delta,\varepsilon\to 0}\int_{R-2n\pi i-i\varepsilon}^{-2n\pi i+\delta-i\varepsilon} n(2\pi i)^2(-x)^{s-2}\frac{dx}{x}=\frac{(2\pi)^s(i)^s n^{s-1}}{s-2}, \tag{20}$$

whereas

$$\lim_{R\to\infty;\delta,\varepsilon\to 0}\int_{2n\pi i+\delta+i\varepsilon}^{R+2n\pi i+i\varepsilon} n(2\pi i)^2(-x)^{s-2}\frac{dx}{x}=-\frac{(2\pi)^s(-i)^s n^{s-1}}{s-2}. \tag{21}$$

Adding all such terms of the integral along E_N then gives

$$\begin{aligned}&\frac{(s-2)\Gamma(1-s)}{2\pi i}\,\frac{(2\pi)^s[i^s-(-i)^s]}{s-2}\sum_{n=1}^{\infty}n^{s-1}\\ &=\Gamma(1-s)2^s\pi^{s-1}\frac{e^{i\frac{\pi}{2}s}-e^{-i\frac{\pi}{2}s}}{2i}\zeta(1-s)\\ &=\Gamma(1-s)2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\zeta(1-s).\end{aligned}$$

Since this term added to $-\zeta(s)$ gives 0, the functional equation follows.

Estimates of the usual kind suffice to show that in the limit, the remaining terms approach 0.

Acknowledgments

The author is glad of the chance to express his thanks to Minhyong Kim for his unwavering encouragement and patient explanations, as well as for sharing his insights. He would also like to thank the referees for their kind comments and very helpful suggestions.

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RECEIVED MARCH 12, 2013