

# Umbral moonshine

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We describe surprising relationships between automorphic forms of various kinds, imaginary quadratic number fields and a certain system of six finite groups that are parameterized naturally by the divisors of 12. The Mathieu group correspondence recently discovered by Eguchi–Ooguri–Tachikawa is recovered as a special case. We introduce a notion of extremal Jacobi form and prove that it characterizes the Jacobi forms arising by establishing a connection to critical values of Dirichlet series attached to modular forms of weight 2. These extremal Jacobi forms are closely related to certain vector-valued mock modular forms studied recently by Dabholkar–Murthy–Zagier in connection with the physics of quantum black holes in string theory. In a manner similar to monstrous moonshine the automorphic forms we identify constitute evidence for the existence of infinite-dimensional graded modules for the six groups in our system. We formulate an Umbral moonshine conjecture that is in direct analogy with the monstrous moonshine conjecture of Conway–Norton. Curiously, we find a number of Ramanujan’s mock theta functions appearing as McKay–Thompson series. A new feature not apparent in the monstrous case is a property which allows us to predict the fields of definition of certain homogeneous submodules for the groups involved. For four of the groups in our system we find analogues of both the classical McKay correspondence and McKay’s monstrous Dynkin diagram observation manifesting simultaneously and compatibly.

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## 1. Introduction

The term *monstrous moonshine* was coined by Conway in order to describe the unexpected and mysterious connections between the representation theory of the largest sporadic group — the *Fischer–Griess monster*,  $\mathbb{M}$  — and modular functions that stemmed from McKay’s observation that  $196883 + 1 = 196884$ , where the summands on the left are degrees of irreducible representations of  $\mathbb{M}$  and the number on the right is the coefficient of  $q$  in the Fourier expansion of the *elliptic modular invariant* [1]

$$(1.1) \quad J(\tau) = \sum_{m \geq -1} a(m)q^m = q^{-1} + 196884q + 21493760q^2 + 864299970q^3 + \cdots$$

Thompson expanded upon McKay’s observation in [2] and conjectured the existence of an infinite-dimensional monster module

$$(1.2) \quad V = \bigoplus_{m \geq -1} V_m$$

with  $\dim V_m = a(m)$  for all  $m$ . He also proposed [3] to consider the series, now known as *McKay–Thompson series*, given by

$$(1.3) \quad T_g(\tau) = \sum_{m \geq -1} \operatorname{tr}_{V_m}(g) q^m$$

for  $g \in \mathbb{M}$ , and detailed explorations [1] by Conway–Norton led to the astonishing *moonshine conjecture*.

For each  $g \in \mathbb{M}$  the function  $T_g$  is a principal modulus for some genus zero group  $\Gamma_g$ .

(A discrete group  $\Gamma < PSL_2(\mathbb{R})$  is said to have *genus zero* if the Riemann surface  $\Gamma \backslash \mathbb{H}$  is isomorphic to the Riemann sphere minus finitely many points, and a holomorphic function  $f$  on  $\mathbb{H}$  is called a *principal modulus* for a genus zero group  $\Gamma$  if it generates the field of  $\Gamma$ -invariant functions on  $\mathbb{H}$ .)

Thompson's conjecture was verified by Atkin, Fong and Smith (cf. [4, 5]). A more constructive verification was obtained by Frenkel–Lepowsky–Meurman [6, 7] with the explicit construction of a monster module  $V = V^\natural$  with graded dimension given by the Fourier expansion (1.1) of the elliptic modular invariant. They used *vertex operators* — structures originating in the dual resonance theory of particle physics and finding contemporaneous application [8, 9] to affine Lie algebras — to recover the non-associative *Griess algebra* structure (developed in the first proof [10] of the existence of the monster) from a subspace of  $V^\natural$ . Borcherds found a way to attach vertex operators to every element of  $V^\natural$  and determined the precise sense in which these operators could be given a commutative associative composition law, and thus arrived at the notion of *vertex algebra* [11], an axiomatization of the operator product expansion of chiral *conformal field theory* (CFT). The closely related notion of *vertex operator algebra* (VOA) was subsequently introduced by Frenkel–Lepowsky–Meurman [12] and they established that the monster is precisely the group of automorphisms of a VOA structure on  $V^\natural$ ; the Frenkel–Lepowsky–Meurman construction of  $V^\natural$  would ultimately prove to furnish the first example of an *orbifold conformal field theory*.

Borcherds introduced the notion of *generalized Kac–Moody algebra* in [13] and by using the VOA structure on  $V^\natural$  was able to construct a particular example — the monster Lie algebra — and use the corresponding equivariant denominator identities to arrive at a proof [14] of the Conway–Norton moonshine conjectures. Thus by 1992 monstrous moonshine had already become a phenomenon encompassing elements of finite group theory, modular forms, vertex algebras and generalized Kac–Moody algebras, as well as aspects of conformal field theory and string theory.

Recently Eguchi–Ooguri–Tachikawa have presented evidence [15] for a new kind of moonshine involving the elliptic genus of  $K3$  surfaces (the elliptic genus is a topological invariant and therefore independent of the choice of  $K3$  surface) and the *largest Mathieu group*  $M_{24}$  (cf. Section 3.1). This connection between  $K3$  surfaces and  $M_{24}$  becomes apparent only after decomposing the elliptic genus into characters of the  $N = 4$  superconformal algebra (cf. [16–18]). This decomposition process (cf. Section 2.4) reveals the presence of a

mock modular form of weight  $1/2$  (cf. Section 2.1) satisfying

$$(1.4) \quad \begin{aligned} H^{(2)}(\tau) &= \sum_{n=0}^{\infty} c_1^{(2)}(n-1/8)q^{n-1/8} \\ &= 2q^{-1/8}(-1 + 45q + 231q^2 + 770q^3 + 2277q^4 + \dots) \end{aligned}$$

and one recognizes here the dimensions of several irreducible representations of  $M_{24}$  (cf. Table B.1).

One is soon led to follow the path forged by Thompson in the case of the monster: to suspect the existence of a graded infinite-dimensional  $M_{24}$ -module

$$(1.5) \quad K^{(2)} = \bigoplus_{n=0}^{\infty} K_{n-1/8}^{(2)}$$

with  $\dim K_{n-1/8}^{(2)} = c^{(2)}(n-1/8)$  for  $n \geq 1$ , and to study the analogues  $H_g^{(2)}$  of the monstrous McKay–Thompson series obtained by replacing  $c^{(2)}(n-1/8) = \dim K_{n-1/8}^{(2)}$  with  $\text{tr}_{K_{n-1/8}^{(2)}}(g)$  in (1.4). This idea has been implemented successfully in [19–22] and provides strong evidence for the existence of such an  $M_{24}$ -module  $K^{(2)}$ . A proof of the existence of  $K^{(2)}$  has now been established in [23] although no explicit construction is yet known. In particular, there is as yet no known analogue of the VOA structure which conjecturally characterizes [12] the monster module  $V^\natural$ .

The strong evidence in support of the  $M_{24}$  analogue of Thompson’s conjecture invites us to consider the  $M_{24}$  analogue of the Conway–Norton moonshine conjectures — this will justify the use of the term moonshine in the  $M_{24}$  setting — except that it is not immediately obvious what the analogue should be. Whilst the McKay–Thompson series  $H_g^{(2)}$  is a mock modular form of weight  $1/2$  on some  $\Gamma_g < SL_2(\mathbb{Z})$  for every  $g$  in  $M_{24}$  [22], it is not the case that  $\Gamma_g$  is a genus zero group for every  $g$ , and even if it were, there is no obvious sense in which one mock modular form of weight  $1/2$  on some group can “generate” all the others, and thus no obvious analogue of the principal modulus property.

A solution to this problem — the formulation of the moonshine conjecture for  $M_{24}$  — was found in [24] (see also Section 5.2) via an extension of the program that was initiated in [25]; the antecedents of which include Rademacher’s pioneering work [26] on the elliptic modular invariant  $J(\tau)$ , quantum gravity in three dimensions [27–29], the anti-de sitter (AdS)/CFT correspondence in physics [30–32], and the application of *Rademacher sums*

to these and other settings in string theory [33–39] (and in particular [40]). To explain the formulation of the moonshine conjecture for  $M_{24}$  we recall that in [25] a Rademacher sum  $R_\Gamma(\tau)$  is defined for each discrete group  $\Gamma < PSL_2(\mathbb{R})$  commensurable with the modular group in such a way as to naturally generalize Rademacher’s Poincaré series-like expression for the elliptic modular invariant derived in [26]. It is then shown in [25] that a holomorphic function on the upper-half plane (with invariance group commensurable with  $PSL_2(\mathbb{Z})$ ) is the principal modulus for its invariance group if and only if it coincides with the Rademacher sum attached to this group. Thus the genus zero property of monstrous moonshine may be reformulated as follows.

For each  $g$  in  $\mathbb{M}$  we have  $T_g = R_{\Gamma_g}$  where  $\Gamma_g$  is the invariance group of  $T_g$ .

Write  $R_\Gamma^{(2)}$  to indicate a weight  $1/2$  generalization of the (weight 0) Rademacher sum construction  $R_\Gamma$  studied in [25]. (Note that a choice of multiplier system on  $\Gamma$  is also required.) Then the following natural  $M_{24}$ -analogue of the Conway–Norton moonshine conjecture comes into view.

For each  $g$  in  $M_{24}$  we have  $H_g^{(2)} = R_{\Gamma_g}^{(2)}$  where  $\Gamma_g$  is the invariance group of  $H_g^{(2)}$ .

This statement is confirmed in [24] for the functions  $H_g^{(2)}$  that are, at this point, conjecturally attached to  $M_{24}$  via the conjectural  $M_{24}$ -module  $K^{(2)}$ .

Through these results we come to envisage the possibility that both monstrous moonshine and the  $M_{24}$  observation of Eguchi–Ooguri–Tachikawa will eventually be understood as aspects of one underlying structure which will include finite groups, various kinds of automorphic forms, extended algebras and string theory, and will quite possibly be formulated in terms of the AdS/CFT correspondence in the context of some higher-dimensional string or gravitational theory.

In fact, we can expect this *moonshine structure* to encompass more groups beyond the monster and  $M_{24}$ : in this paper we identify the  $M_{24}$  observation as one of a family of correspondences between finite groups and (vector-valued) mock modular forms, with each member in the family admitting a natural analogue of the Conway–Norton moonshine conjecture according to the philosophy of [24, 25] (see also [41]). For each  $\ell$  in  $\Lambda = \{2, 3, 4, 5, 7, 13\}$  — the set of positive integers  $\ell$  such that  $\ell - 1$  divides 12 — we identify a distinguished Jacobi form  $Z^{(\ell)}$ , a finite group  $G^{(\ell)}$  and a family of vector-valued mock modular forms  $H_g^{(\ell)}$  for  $g \in G^{(\ell)}$ . The Jacobi forms  $Z^{(\ell)}$  satisfy an *extremal* condition formulated in terms

of unitary irreducible characters of the  $N = 4$  superconformal algebra (cf. Section 2.5), the mock modular form  $H^{(\ell)} = H_e^{(\ell)}$  is related to  $Z^{(\ell)}$  as  $H^{(2)}$  is to the elliptic genus of a  $K3$  surface (cf. Sections 2.4, 2.5), the Fourier coefficients of the *McKay–Thompson series*  $H_g^{(\ell)}$  support the existence of an infinite-dimensional graded module  $K^{(\ell)}$  for  $G^{(\ell)}$  playing a rôle analogous to that of  $K^{(2)}$  for  $G^{(2)} \simeq M_{24}$  (cf. Section 5.1), and the following *Umbral moonshine conjecture* is predicted to hold where  $R_\Gamma^{(\ell)}$  is an  $(\ell - 1)$ -vector-valued generalization (cf. Section 5.2) of the Rademacher sum construction  $R_\Gamma^{(2)}$  studied in [24].

For each  $g$  in  $G^{(\ell)}$  we have  $H_g^{(\ell)} = R_{\Gamma_g}^{(\ell)}$  where  $\Gamma_g$  is the invariance group of  $H_g^{(\ell)}$ .

In addition to the above properties with monstrous analogues we find the following new *discriminant property*. The exponents of the powers of  $q$  having non-vanishing coefficient in the Fourier development of  $H^{(\ell)}$  determine certain imaginary quadratic number fields and predict the existence of dual pairs of irreducible representations of  $G^{(\ell)}$  that are defined over these fields and irreducible over  $\mathbb{C}$ . Moreover, these dual pairs consistently appear as irreducible constituents in homogeneous  $G^{(\ell)}$ -submodules of  $K^{(\ell)}$  in such a way that the degree of the submodule determines the discriminant of the corresponding quadratic field (cf. Section 5.4).

A main result of this paper is Theorem 2.2, which states that the extremal condition formulated in Section 2.5 characterizes the Jacobi forms  $Z^{(\ell)}$  for  $\ell \in \{2, 3, 4, 5, 7, 13\}$ . To achieve this we establish a result of independent interest, which also serves to illustrate the depth of the characterization problem: We show in Theorem 2.3 that the existence of an extremal Jacobi form of index  $m - 1$  implies the vanishing of  $L(f, 1)$  for all new forms  $f$  of weight 2 and level  $m$ , where  $L(f, s)$  is the Dirichlet series naturally attached to  $f$ . According to the Birch–Swinnerton–Dyer conjecture the vanishing of  $L(f, 1)$  implies that the elliptic curve  $E_f$ , attached to  $f$  by Eichler–Shimura theory, has rational points of infinite order. Thus it is extremely unexpected that an extremal Jacobi form can exist for all but finitely many values of  $m$ , and using estimates [42] due to Ellenberg together with some explicit computations we are able to verify that the only possible values are those for which no non-zero weight two forms exist, i.e.,  $\Gamma_0(m)$  has genus zero. The last step in our proof of Theorem 2.2 is to check the finitely many corresponding finite-dimensional spaces of Jacobi forms, for which useful bases have been determined by Gritsenko in [43]. In this way we obtain that the



$Z^{(\ell)}$  for  $\ell \in \{2, 3, 4, 5, 7, 13\}$  are precisely the unique, up to scale, weak Jacobi forms of weight zero satisfying the extremal condition (2.37).

In contrast to the monstrous case the McKay–Thompson series  $H_g^{(\ell)}$  arising here are mock modular forms, and these are typically not in fact modular but become so after *completion* with respect to a *shadow* function  $S_g^{(\ell)}$ . It turns out that all the McKay–Thompson series  $H_g^{(\ell)}$  for fixed  $\ell \in \Lambda$  have shadows that are (essentially) proportional to a single vector-valued *unary theta function*  $S^{(\ell)}$  (cf. Section 2.2) and so it is in a sense the six *moonlight shadows*  $S^{(\ell)}$  for  $\ell \in \Lambda$  that provide the irreducible information required to uncover the structure that we reveal in this paper. We therefore refer to the phenomena investigated here as *Umbral moonshine*.

According to the Oxford English Dictionary, a flame or light that is *lambent* is playing “lightly upon or gliding over a surface without burning it, like a “tongue of fire”; shining with a soft clear light and without fierce heat.” And since the light of Umbral moonshine is apparently of this nature, we call the six values in  $\Lambda = \{2, 3, 4, 5, 7, 13\}$  *lambent*, and we refer to the index  $\ell \in \Lambda$  as the *lambency* of the connections relating the *Umbral group*  $G^{(\ell)}$  to the *Umbral forms*  $Z^{(\ell)}$  and  $H_g^{(\ell)}$ .

The rest of this paper is organized as follows. In Section 2 we discuss properties of Jacobi forms, Siegel forms, mock modular forms and mock theta functions. We explain two closely related ways in which Jacobi forms determine mock modular forms, one involving the decomposition into characters of the  $N = 4$  superconformal algebra and the other involving a decomposition of meromorphic Jacobi forms into mock modular forms following [44] and [45]. In Section 2.5 we introduce the Jacobi forms  $Z^{(\ell)}$  of weight 0 and index  $\ell - 1$  for lambent  $\ell$  and their associated vector-valued mock modular forms  $H^{(\ell)}$ . We prove (Theorem 2.2) that these functions are characterized by the extremal property (2.37) and we establish the connection (Theorem 2.6) to critical values of automorphic  $L$ -functions. We note that the coefficients in the  $q$ -expansions of the mock modular forms  $H^{(\ell)}$  appear to be connected to the dimensions of irreducible representations of groups  $G^{(\ell)}$  which we introduce and study in Section 3. In Sections 3.5 and 3.6 we discuss the remarkable fact that some of these groups manifest both the McKay correspondence relating ADE Dynkin diagrams to finite subgroups of  $SU(2)$  as well as a generalization of his monstrous  $E_8$  observation. In Section 4, we discuss analytic properties of the McKay–Thompson series  $H_g^{(\ell)}$  which are obtained by twisting the mock modular forms  $H^{(\ell)}$  by elements  $g \in G^{(\ell)}$ . We determine the proposed McKay–Thompson series  $H_g^{(\ell)} = (H_{g,r}^{(\ell)})$  precisely in terms of modular forms of weight 2 for all but a

few  $g$  occurring for  $\ell \in \{7, 13\}$ , and we find that we can identify many of the component functions  $H_{g,r}^{(\ell)}$  either with ratios of products of eta functions or with classical mock theta functions introduced by Ramanujan (and others). In Section 5, we collect our observations into a set of conjectures. These include the analogue of Thompson’s conjecture (cf. Section 5.1), the Umbral counterpart to the Conway–Norton moonshine conjecture (cf. Section 5.2), and a precise formulation of the discriminant property mentioned above (cf. Section 5.4). In Section 5.5 we mention some possible connections between our results, the geometry of complex surfaces, and string theory.

Our conventions for modular forms appear in Appendix A, the character tables of the Umbral groups  $G^{(\ell)}$  appear in Appendix B, tables of Fourier coefficients of low degree for all the proposed McKay–Thompson series  $H_g^{(\ell)}$  appear in Appendix C, and tables describing the  $G^{(\ell)}$ -module structures implied (for low degree) by the  $H_g^{(\ell)}$  are collected in Appendix D.

It is important to mention that much of the data presented in the tables of Appendix C was first derived using certain vector-valued generalizations of the Rademacher sum construction that was applied to the functions of monstrous moonshine in [25], and in [24] to the functions attached to  $M_{24}$  via the observation of Eguchi–Ooguri–Tachikawa. In particular, these vector-valued Rademacher sums played an indispensable rôle in helping us arrive at the groups  $G^{(\ell)}$  specified in Section 3, especially for  $\ell > 3$ , and also allowed us to formulate and test hypotheses regarding the modularity of the (vector-valued) functions  $H_g^{(\ell)}$ , including eta product expressions and the occurrences of classical mock theta functions; considerations which ultimately developed into the discussion of Section 4. A detailed discussion of the Rademacher construction is beyond the scope of this article but a full treatment is to be the focus of forthcoming work. The fact that the Rademacher sum approach proved so powerful may be taken as strong evidence in support of the *Umbral moonshine conjecture*, Conjecture 5.4.

## 2. Automorphic forms

In this section we discuss the modular objects that play a rôle in the connection between mock modular forms and finite groups that we will develop later in the paper. We also establish our notation and describe various relationships between Jacobi forms, theta functions, and vector-valued mock modular forms, including mock theta functions.

In what follows, we take  $\tau$  in the upper half-plane  $\mathbb{H}$  and  $z \in \mathbb{C}$ , and adopt the shorthand notation  $e(x) = e^{2\pi ix}$ . We also define  $q = e(\tau)$  and  $y =$

$e(z)$  and write

$$(2.1) \quad \gamma\tau = \frac{a\tau + b}{c\tau + d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

for the natural action of  $SL_2(\mathbb{Z})$  on  $\mathbb{H}$  and

$$(2.2) \quad \gamma(\tau, z) = \left( \frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right)$$

for the action of  $SL_2(\mathbb{Z})$  on  $\mathbb{H} \times \mathbb{C}$ .

### 2.1. Mock modular forms

Mock theta functions were first introduced in 1920 by Ramanujan in his last letter to Hardy. This letter contained 17 examples divided into four of *order 3*, ten of *order 5* and three of *order 7*. Ramanujan did not define what he meant by the term *order* and to this day there seems to be no universally agreed upon definition. In this paper, we use the term *order* only as a historical label. Ramanujan wrote his mock theta functions as what he termed “Eulerian series” that today would be recognized as specializations of  $q$ -hypergeometric series. A well studied example is the order 3 mock theta function

$$(2.3) \quad f(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \cdots (1+q^n)^2} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q; q)_n^2},$$

where we have introduced the  $q$ -Pochhammer symbol

$$(2.4) \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k).$$

Interest in and applications of mock theta functions has burgeoned during the last decade following the work of Zwegers [44] who found an intrinsic definition of mock theta functions and their near modular behaviour, and many applications of his work can be found in combinatorics [46, 47], characters of infinite-dimensional Lie superalgebras [48, 49], topological field theory [50–53], the computation of quantum invariants of 3-dimensional manifolds [54], and the counting of black hole states in string theory [45]. Descriptions of this breakthrough and some of the history of mock theta functions can be found in [55–57].

Mock theta functions are now understood as a special case of more general objects known as mock modular forms. A holomorphic function  $h(\tau)$  on  $\mathbb{H}$  is called a (*weakly holomorphic*) *mock modular form* of weight  $k$  for a discrete group  $\Gamma$  (e.g., a congruence subgroup of  $SL_2(\mathbb{Z})$ ) if it has at most exponential growth as  $\tau \rightarrow \alpha$  for any  $\alpha \in \mathbb{Q}$ , and if there exists a holomorphic modular form  $f(\tau)$  of weight  $2 - k$  on  $\Gamma$  such that the *completion* of  $h$  given by

$$(2.5) \quad \hat{h}(\tau) = h(\tau) + (4i)^{k-1} \int_{-\bar{\tau}}^{\infty} (z + \tau)^{-k} \overline{f(-\bar{z})} dz$$

is a (non-holomorphic) modular form of weight  $k$  for  $\Gamma$  for some multiplier system  $\nu$  say. In this case the function  $f$  is called the *shadow* of the mock modular form  $h$ . Even though  $h$  is not a modular form, it is common practice to call  $\nu$  the multiplier system of  $h$ . One can show that  $\nu$  is the conjugate of the multiplier system of  $f$ . In most of the examples in this paper we will deal with vector-valued mock modular forms so that the completion in fact transforms as  $\nu(\gamma)\hat{h}(\gamma\tau)(c\tau + d)^{-k} = \hat{h}(\tau)$  in case the weight is  $k$  for all  $\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma$  where  $\nu$  is a matrix-valued function on  $\Gamma$ .

The completion  $\hat{h}(\tau)$  satisfies interesting differential equations. For instance, completions of mock modular forms were identified as weak Maass forms (non-holomorphic modular forms which are eigenfunctions of the Laplace operator) in [46] as a part of their solution to the longstanding Andrews–Dragonette conjecture. Note that we have the identity

$$(2.6) \quad 2^{1-k} \pi \Im(\tau)^k \frac{\partial \hat{h}(\tau)}{\partial \bar{\tau}} = -2\pi i \overline{f(\tau)}$$

when  $f$  is the shadow of  $h$ .

Thanks to Zwegler’s work we may define a *mock theta function* to be a  $q$ -series  $h = \sum_n a_n q^n$  such that for some  $\lambda \in \mathbb{Q}$  the assignment  $\tau \mapsto q^\lambda h|_{q=\epsilon(\tau)}$  defines a mock modular form of weight  $1/2$  whose shadow is a unary (i.e., attached to a quadratic form in one variable) theta series of weight  $3/2$ .

In this paper we add one more rôle for mock theta functions to the list mentioned earlier; namely we conjecture that specific sets of mock theta functions appear as McKay–Thompson series associated to (also conjectural) infinite-dimensional modules for a sequence of groups  $G^{(\ell)}$  which we refer to as the *Umbral groups* and label by the *lambent* integers  $\ell \in \Lambda = \{2, 3, 4, 5, 7, 13\}$ , which are just those positive integers that are one greater than a divisor of 12.

Many of the mock theta functions that appear later in this paper appear either in Ramanujan’s last letter to Hardy or in his lost notebook [58]. These

include an order 2 mock theta function

$$(2.7) \quad \mu(q) = \sum_{n \geq 0} \frac{(-1)^n q^{n^2} (q; q^2)_n}{(-q^2; q^2)_n^2}$$

and an order 8 mock theta function

$$(2.8) \quad U_0(q) = \sum_{n \geq 0} \frac{q^{n^2} (-q; q^2)_n}{(-q^4; q^4)_n},$$

both of which appear at lambency 2 in connection with  $G^{(2)} \simeq M_{24}$ . The function  $f(q)$  of (2.3) together with

$$(2.9) \quad \begin{aligned} \phi(q) &= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q^2)(1+q^4)\cdots(1+q^{2n})}, \\ \chi(q) &= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q+q^2)(1-q^2+q^4)\cdots(1-q^n+q^{2n})}, \\ \omega(q) &= \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(1-q)^2(1-q^3)^2\cdots(1-q^{2n+1})^2}, \\ \rho(q) &= \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(1+q+q^2)(1+q^3+q^6)\cdots(1+q^{2n+1}+q^{4n+2})} \end{aligned}$$

constitute five order 3 mock theta functions appearing at lambency 3, and the four order 10 mock theta functions

$$(2.10) \quad \begin{aligned} \phi_{10}(q) &= \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q; q^2)_{n+1}}, \\ \psi_{10}(q) &= \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)/2}}{(q; q^2)_{n+1}}, \\ X(q) &= \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(-q; q)_{2n}}, \\ \chi_{10}(q) &= \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}}{(-q; q)_{2n+1}} \end{aligned}$$

appear at lambency 5.

More mock theta functions were found later by others. At lambency 4 we will encounter order 8 mock theta functions discussed in [59] with  $q$ -expansions

$$\begin{aligned}
 (2.11) \quad S_0(\tau) &= \sum_{n \geq 0} \frac{q^{n^2}(-q; q^2)_n}{(-q^2; q^2)_n}, \\
 S_1(\tau) &= \sum_{n \geq 0} \frac{q^{n(n+2)}(-q; q^2)_n}{(-q^2; q^2)_n}, \\
 T_0(\tau) &= \sum_{n \geq 0} \frac{q^{(n+1)(n+2)}(-q^2; q^2)_n}{(-q; q^2)_n}, \\
 T_1(\tau) &= \sum_{n \geq 0} \frac{q^{n(n+1)}(-q^2; q^2)_n}{(-q; q^2)_n}.
 \end{aligned}$$

It is curious to note that the order is divisible by the lambency in each example.

## 2.2. Jacobi forms

We now discuss Jacobi forms following [60]. We say a holomorphic function  $\phi : \mathbb{H} \times \mathbb{C} \rightarrow \mathbb{C}$  is an *unrestricted Jacobi form* of weight  $k$  and index  $m$  for  $SL_2(\mathbb{Z})$  if it transforms under the Jacobi group  $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$  as

$$(2.12) \quad \phi(\tau, z) = (c\tau + d)^{-k} e(-m \frac{cz^2}{c\tau + d}) \phi(\gamma(\tau, z)),$$

$$(2.13) \quad \phi(\tau, z) = e(m(\lambda^2\tau + 2\lambda z)) \phi(\tau, z + \lambda\tau + \mu),$$

where  $\gamma \in SL_2(\mathbb{Z})$  and  $\lambda, \mu \in \mathbb{Z}$ . In what follows, we refer to the transformations (2.12) and (2.13) as the *modular* and *elliptic* transformations, respectively. The invariance of  $\phi(\tau, z)$  under  $\tau \rightarrow \tau + 1$  and  $z \rightarrow z + 1$  implies a Fourier expansion

$$(2.14) \quad \phi(\tau, z) = \sum_{n, r \in \mathbb{Z}} c(n, r) q^n y^r$$

and the elliptic transformation can be used to show that  $c(n, r)$  depends only on the *discriminant*  $r^2 - 4mn$  and  $r \pmod{2m}$ , and so we have  $c(n, r) = C(r^2 - 4mn, \tilde{r})$  for some function  $D \mapsto C(D, \tilde{r})$  where  $\tilde{r} \in \{-m, \dots, m-1\}$ , for example. An unrestricted Jacobi form is called a *weak Jacobi form*, a (*strong*) *Jacobi form*, or a *Jacobi cusp form* according as the Fourier coefficients satisfy  $c(n, r) = 0$  whenever  $n < 0$ ,  $C(D, \tilde{r}) = 0$  whenever  $D > 0$ , or

$C(D, \tilde{r}) = 0$  whenever  $D \geq 0$ , respectively. In a slight departure from [60] we denote the space of weak Jacobi forms of weight  $k$  and index  $m$  by  $J_{k,m}$ .

In what follows, we will need two further generalizations of the above definitions. The first is straightforward and replaces  $SL_2(\mathbb{Z})$  by a finite index subgroup  $\Gamma \subset SL_2(\mathbb{Z})$  in the modular transformation law. The second is more subtle and leads to *meromorphic Jacobi forms* which obey the modular and elliptic transformation laws but are such that the functions  $z \mapsto \phi(\tau, z)$  are allowed to have poles lying at values of  $z \in \mathbb{C}$  that map to torsion points of the elliptic curve  $\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$ . Our treatment of meromorphic Jacobi forms follows [44, 45].

A property of (weak) Jacobi forms that will be important for us later is that they admit an expansion in terms of the *index  $m$  theta functions*

$$(2.15) \quad \theta_r^{(m)}(\tau, z) = \sum_{n \in \mathbb{Z}} q^{(2mn+r)^2/4m} y^{2mn+r},$$

given by

$$(2.16) \quad \phi(\tau, z) = \sum_{r \pmod{2m}} \tilde{h}_r(\tau) \theta_r^{(m)}(\tau, z)$$

in case the index of  $\phi$  is  $m$ , where the *theta-coefficients*  $\tilde{h}_r(\tau)$  constitute the components of a vector-valued modular form of weight  $k - 1/2$  when  $k$  is the weight of  $\phi$ . Recall that a vector-valued function  $\tilde{h} = (\tilde{h}_r)$  is called a *vector-valued modular form* of weight  $k$  for  $\Gamma \subset SL_2(\mathbb{Z})$  if

$$(2.17) \quad \tilde{h}_r(\tau) = \frac{1}{(c\tau + d)^k} \sum_s \nu_{rs}(\gamma) \tilde{h}_s(\tau)$$

for all  $\gamma \in \Gamma$  and  $\tau \in \mathbb{H}$  for some matrix-valued function  $\nu = (\nu_{rs})$  on  $\Gamma$  called the *multiplier system* for  $\tilde{h}$ .

The modular transformation law (2.12) with  $\gamma = -I_2$  implies that  $\phi(\tau, -z) = (-1)^k \phi(\tau, z)$ . Combining this with the identity  $\theta_{-r}^{(m)}(\tau, z) = \theta_r^{(m)}(\tau, -z)$  we see that  $\tilde{h}_r(\tau) = (-1)^k \tilde{h}_{-r}(\tau)$  and in particular we can recover a weak Jacobi form of weight  $k$  and index  $m$  from the  $m - 1$  theta-coefficients  $\{\tilde{h}_1, \dots, \tilde{h}_{m-1}\}$  in case  $k$  is odd.

In what follows, we will encounter only weight 0 and weight 1 Jacobi forms; a typical such form will be denoted by  $\phi(\tau, z)$  or  $\psi(\tau, z)$  according as the weight is 0 or 1 and we will write the theta-expansion of a weight 1

Jacobi form  $\psi$  as

$$(2.18) \quad \psi(\tau, z) = \sum_{r=1}^{m-1} \tilde{h}_r(\tau) \hat{\theta}_r^{(m)}(\tau, z),$$

where  $\hat{\theta}_r^{(m)}(\tau, z) = \theta_{-r}^{(m)}(\tau, z) - \theta_r^{(m)}(\tau, z)$  (cf. (2.15)) for  $r \in \{1, 2, \dots, m-1\}$ .

### 2.3. Meromorphic Jacobi forms

We now explain a connection between the vector-valued mock modular forms we shall consider in this paper and meromorphic Jacobi forms. We specialize our discussion to weight 1 meromorphic Jacobi forms of a particular form that arise in our pairing of Jacobi forms with groups  $G^{(\ell)}$ , and later in our computation of McKay–Thompson series; namely, we consider meromorphic weight 1 and index  $m$  Jacobi forms which can be written as

$$(2.19) \quad \psi(\tau, z) = \Psi_{1,1}(\tau, z) \phi(\tau, z)$$

for some weight 0 index  $m-1$  (holomorphic) weak Jacobi form  $\phi$  where  $\Psi_{1,1}$  is the specific meromorphic Jacobi form of weight 1 and index 1 given by

$$(2.20) \quad \Psi_{1,1}(\tau, z) = -i \frac{\theta_1(\tau, 2z) \eta(\tau)^3}{(\theta_1(\tau, z))^2} = \frac{y+1}{y-1} - (y^2 - y^{-2})q + \dots$$

We note that  $\psi(\tau, z)$  has a simple pole at  $z=0$  with residue  $\phi(\tau, 0)/\pi i$ . Since  $\phi(\tau, z)$  is a weak Jacobi form of weight 0 the function  $\tau \mapsto \phi(\tau, 0)$  is a modular form of weight 0 (with no poles at any cusps) and is hence equal to a constant; we denote this constant by

$$(2.21) \quad \chi = \phi(\tau, 0).$$

It was shown by Zwegers [44] that meromorphic Jacobi forms have a modified theta-expansion (cf. (2.16)) in terms of vector-valued mock modular forms; in [45] this expansion was recast as follows. Define the averaging operator

$$(2.22) \quad \text{Av}^{(m)} \left[ F(y) \right] = \sum_{k \in \mathbb{Z}} q^{mk^2} y^{2mk} F(q^k y),$$



which takes a function of  $y = e(z)$  with polynomial growth and returns a function of  $z$  which transforms like an index  $m$  Jacobi form under the elliptic transformations (2.13). Now define the *polar part* of  $\psi = \Psi_{1,1}\phi$  by

$$(2.23) \quad \psi^P(\tau, z) = \chi \text{Av}^{(m)} \left[ \frac{y+1}{y-1} \right],$$

where  $\chi = \phi(\tau, 0)$  and define the *finite part* of  $\psi$  by

$$(2.24) \quad \psi^F(\tau, z) = \psi(\tau, z) - \psi^P(\tau, z).$$

The term finite is appropriate because with the polar part subtracted the finite part no longer has a pole at  $z = 0$ .

It follows from the analysis in [45] that  $\psi^F(\tau, z)$  is a weight 1 index  $m$  *mock Jacobi form*, meaning that it has a theta-expansion

$$(2.25) \quad \psi^F(\tau, z) = \sum_{r=1}^{m-1} h_r(\tau) \hat{\theta}_r^{(m)}(\tau, z)$$

(cf. (2.18)) where the theta-coefficients  $h_r$  comprise the components of a vector-valued mock modular form of weight  $1/2$ .

In the above we have again used the fact that  $\psi$ ,  $\psi^P$  and hence also  $\psi^F$  pick up a minus sign under the transformation  $z \rightarrow -z$ . Moreover, the vector-valued mock modular forms obtained in this way always have shadow function given by the *unary theta series*

$$(2.26) \quad S_r^{(m)}(\tau) = \frac{1}{2\pi i} \frac{\partial}{\partial z} \theta_r^{(m)}(\tau, z) \Big|_{z=0} = \sum_{n \in \mathbb{Z}} (2mn + r) q^{(2mn+r)^2/4m}.$$

To see why this is so, and for later use, we introduce the functions

$$(2.27) \quad \mu_j^{(m)}(\tau, z) = (-1)^{1+2j} \sum_{k \in \mathbb{Z}} q^{mk^2} y^{2mk} \frac{(yq^k)^{-2j} + (yq^k)^{-2j+1} + \dots + (yq^k)^{1+2j}}{1 - yq^k}$$

for  $m$  a positive integer and  $2j \in \{0, 1, \dots, m-1\}$ . Note that  $\mu_0^{(m)}(\tau, z)$  is proportional to the polar part of  $\psi$ , as identified in (2.23),

$$(2.28) \quad \mu_0^{(m)}(\tau, z) = \text{Av}^{(m)} \left[ \frac{y+1}{y-1} \right].$$

**Remark 2.1.** The function  $\mu_0^{(2)}$  is closely related to the Appell–Lerch sum  $\mu(t, z)$  which features prominently in [44].

The  $\mu_0^{(m)}$  enjoy the following relation to the modular group  $SL_2(\mathbb{Z})$ . Define the *completion* of  $\mu_0^{(m)}(\tau, z)$  by setting

$$(2.29) \quad \hat{\mu}^{(m)}(\tau, \bar{\tau}, z) = \mu_0^{(m)}(\tau, z) + \frac{1}{\sqrt{2m}} \frac{1}{(4i)^{1/2}} \sum_{r=-m}^{m-1} \theta_r^{(m)}(\tau, z) \\ \times \int_{-\bar{\tau}}^{i\infty} (z + \tau)^{-1/2} \overline{S_r^{(m)}(-\bar{z})} dz.$$

Then  $\hat{\mu}^{(m)}$  transforms like a Jacobi form of weight 1 and index  $m$  for  $SL_2(\mathbb{Z})$  but is not holomorphic. Therefore, from the transformation of the polar part

$$(2.30) \quad \psi^P(\tau, z) = \chi \mu_0^{(m)}(\tau, z)$$

we see that the shadow of  $h = (h_r)$  is given by  $\chi S^{(m)} = (\chi S_r^{(m)})$ , as we claimed. This means that the vector-valued mock modular forms arising in this way are closely related to mock theta functions: By the definition given in Section 2.1 we have  $h_r(\tau) = q^\lambda M_r$  for some  $\lambda \in \mathbb{Q}$  with  $M_r$  a mock theta function.

## 2.4. Superconformal algebra

In Section 2.3 we saw how to associate a vector-valued mock modular form  $(h_r)$  to a weight 1 meromorphic Jacobi form  $\psi$  satisfying  $\psi = \Psi_{1,1}\phi$  for some weak Jacobi form  $\phi$  via the theta-expansion of the finite part of  $\psi$ . It will develop that the weight 0 forms  $\phi$  of relevance to us have a close relation to the representation theory of the 2-dimensional  $N = 4$  superconformal algebra. To see this recall (cf. [16–18]) that this algebra contains subalgebras isomorphic to the affine Lie algebra  $\hat{\mathfrak{sl}}_2$  and the Virasoro algebra, and in a unitary representation the former of these acts with level  $m - 1$ , for some integer  $m > 1$ , and the latter with central charge  $c = 6(m - 1)$ , and the unitary irreducible highest weight representations  $V_{h,j}^{(m)}$  are labelled by the two *quantum numbers*  $h$  and  $j$  which are the eigenvalues of  $L_0$  and  $\frac{1}{2}J_0^3$ , respectively, when acting on the highest weight state. (We adopt a normalization of the  $SU(2)$  current  $J^3$  such that the zero mode  $J_0^3$  has integer eigenvalues.) In the Ramond sector of the superconformal algebra there are

two types of highest weight representations: the *massless* (or Bogomol'nyi–Prasad–Sommerfield (BPS), *supersymmetric*) ones with  $h = \frac{m-1}{4}$  and  $j \in \{0, \frac{1}{2}, \dots, \frac{m-1}{2}\}$ , and the *massive* (or non-BPS, *non-supersymmetric*) ones with  $h > \frac{m-1}{4}$  and  $j \in \{\frac{1}{2}, 1, \dots, \frac{m-1}{2}\}$ . Their (*Ramond*) *characters*, defined as

$$(2.31) \quad \text{ch}_{h,j}^{(m)}(\tau, z) = \text{tr}_{V_{h,j}^{(m)}} \left( (-1)^{J_0^3} y^{J_0^3} q^{L_0 - c/24} \right),$$

are given by

$$(2.32) \quad \text{ch}_{\frac{m-1}{4},j}^{(m)}(\tau, z) = (\Psi_{1,1}(\tau, z))^{-1} \mu_j^{(m)}(\tau, z)$$

and

$$(2.33) \quad \text{ch}_{h,j}^{(m)}(\tau, z) = (-1)^{2j+1} (\Psi_{1,1}(\tau, z))^{-1} q^{h - \frac{m-1}{4} - \frac{j^2}{m}} \hat{\theta}_{2j}^{(m)}(\tau, z)$$

in the massless and massive cases, respectively, [17] where the function  $\mu_j^{(m)}(\tau, z)$  is defined as in (2.27) and  $\hat{\theta}_r^{(m)}(\tau, z)$  is as in the sentence following (2.18).

We can use the above results to derive a decomposition of an arbitrary weight 0 index  $m - 1$  Jacobi form  $\phi(\tau, z)$  into  $N = 4$  characters as follows. Set  $\psi(\tau, z) = \Psi_{1,1}(\tau, z)\phi(\tau, z)$  as in Section 2.3 and write

$$(2.34) \quad h_r(\tau) = \sum_n c_r(n - r^2/4m) q^{n - r^2/4m}$$

for the Fourier expansion of the theta-coefficient  $h_r$  of the finite part  $\psi^F$  of  $\psi$  (cf. (2.25)). Then use (2.24), (2.25) and (2.30) along with (2.32) and (2.33) to obtain

$$(2.35) \quad \begin{aligned} \phi = & \chi \text{ch}_{\frac{m-1}{4},0}^{(m)} + \sum_{r=1}^{m-1} (-1)^{r+1} c_r \left( -\frac{r^2}{4m} \right) \left( \text{ch}_{\frac{m-1}{4},\frac{r}{2}}^{(m)} + 2\text{ch}_{\frac{m-1}{4},\frac{r-1}{2}}^{(m)} + \text{ch}_{\frac{m-1}{4},\frac{r-2}{2}}^{(m)} \right) \\ & + \sum_{r=1}^{m-1} (-1)^{r+1} \sum_{n=1}^{\infty} c_r \left( n - \frac{r^2}{4m} \right) \text{ch}_{\frac{m-1}{4}+n,\frac{r}{2}}^{(m)}, \end{aligned}$$

where  $\chi = \phi(\tau, 0)$  is the constant such that  $\chi S^{(m)}$  is the shadow of  $h = (h_r)$  (cf. Section 2.3). In deriving (2.35) we have used the relation

$$(2.36) \quad \mu_{\frac{r-2}{2}}^{(m)} + 2\mu_{\frac{r-1}{2}}^{(m)} + \mu_{\frac{r}{2}}^{(m)} = (-1)^{r+1} q^{-\frac{r^2}{4m}} \hat{\theta}_r^{(m)}$$

subject to the understanding that  $\mu_{-\frac{1}{2}}^{(m)} = \text{ch}_{\frac{m-1}{4},-\frac{1}{2}}^{(m)} = 0$ .

One way a weak Jacobi form of weight 0 and index  $m - 1$  having integer coefficients can arise in nature is as the elliptic genus of a 2-dimensional  $N = 4$  superconformal field theory with central charge  $c = 6(m - 1)$ . There is a unique weak Jacobi form of weight 0 and index 1 up to scale and this (suitably scaled) turns out to be the elliptic genus of a superconformal field theory attached to a  $K3$  surface. Then the above analysis at  $m = 2$  recovers the mock modular form  $H^{(2)}$  (the vectors have  $m - 1 = 1$  components in this case) exhibiting the connection to the Mathieu group  $M_{24}$  observed by Eguchi–Ooguri–Tachikawa in [15].

In the next section we will construct a distinguished family of extremal weight 0 weak Jacobi forms  $\phi^{(\ell)}(\tau, z)$  with corresponding vector-valued weight 1/2 mock modular forms  $H^{(\ell)}(\tau)$  according to the procedure (2.19) to (2.25) of Section 2.3. In Section 3 we will specify finite groups  $G^{(\ell)}$  for which the forms  $H^{(\ell)}$  will serve as generating functions for the graded dimensions of conjectural bi-graded infinite-dimensional  $G^{(\ell)}$ -modules.

## 2.5. Extremal Jacobi forms

In this section we identify the significance of the divisors of 12 from the point of view of the  $N = 4$  superconformal algebra. We introduce the notion of an extremal (weak) Jacobi form of weight 0 and integral index and we prove that the space  $J_{0,m-1}^{\text{ext}}$  of extremal forms with index  $m - 1$  has dimension 1 if  $m - 1$  divides 12, and has dimension 0 otherwise. Several of the extremal Jacobi forms we identify have appeared earlier in the literature in the context of studying decompositions of elliptic genera of Calabi–Yau manifolds [43, 48, 61] and in a recent study of the connection between black hole counting in superstring theory and mock modular forms [45].

For  $m$  a positive integer and  $\phi$  a weak Jacobi form with weight 0 and index  $m - 1$  say  $\phi$  is *extremal* if it admits an expression

$$(2.37) \quad \phi = a_{\frac{m-1}{4},0} \text{ch}_{\frac{m-1}{4},0}^{(m)} + a_{\frac{m-1}{4},\frac{1}{2}} \text{ch}_{\frac{m-1}{4},\frac{1}{2}}^{(m)} + \sum_{0 < r < m} \sum_{\substack{n \in \mathbb{Z} \\ r^2 - 4mn < 0}} a_{\frac{m-1}{4}+n,\frac{r}{2}} \text{ch}_{\frac{m-1}{4}+n,\frac{r}{2}}^{(m)}$$

for some  $a_{h,j} \in \mathbb{C}$  (cf. (2.35)) where the  $N = 4$  characters are as defined in (2.32) and (2.33). Write  $J_{0,m-1}^{\text{ext}}$  for the subspace of  $J_{0,m-1}$  consisting of extremal weak Jacobi forms. Note that the extremal condition restricts both the massless and massive  $N = 4$  representations that can appear, for generally there are non-zero massive characters  $\text{ch}_{\frac{m-1}{4}+n,\frac{r}{2}}^{(m)}$  with  $r^2 - 4mn \geq 0$ .

We observe here that the extremal condition has a very natural interpretation in terms of the mock modular forms of weight  $1/2$  attached to weak Jacobi forms of weight  $0$  via the procedure detailed in Section 2.3. By comparing with (2.35) we find that the condition (2.37) on a weak Jacobi form  $\phi$  of index  $m - 1$  is equivalent to requiring that the corresponding vector-valued mock modular form  $(h_r)$  obtained from the theta-expansion (2.25) of the finite part of the weight  $1$  Jacobi form  $\psi = \Psi_{1,1}\phi$  has a single polar term  $q^{-\frac{1}{4m}}$  in the first component  $h_1$  and has all other components vanishing as  $\tau \rightarrow i\infty$ .

Our main result in this section is the following characterization of extremal Jacobi forms.

**Theorem 2.2.** *If  $m$  is a positive integer then  $\dim J_{0,m-1}^{\text{ext}} = 1$  in case  $m - 1$  divides  $12$  and  $\dim J_{0,m-1}^{\text{ext}} = 0$  otherwise.*

In preparation for the proof of Theorem 2.2 we now summarize (aspects of) a useful construction given in [43]. The graded ring

$$(2.38) \quad J_{0,*} = \bigoplus_{m \geq 1} J_{0,m-1}$$

of (weak) Jacobi forms of weight  $0$  and integral index has an ideal

$$(2.39) \quad J_{0,*}(q) = \bigoplus_{m > 1} J_{0,m-1}(q) = \left\{ \phi \in J_{0,*} \mid \phi(\tau, z) = \sum_{\substack{n,r \in \mathbb{Z} \\ n > 0}} c(n, r) q^n y^r \right\}$$

consisting of Jacobi forms that vanish in the limit as  $\tau \rightarrow i\infty$  (i.e., have vanishing coefficient of  $q^0 y^r$ , for all  $r$ , in their Fourier development). This ideal is principal and generated by a weak Jacobi form of weight  $0$  and index  $6$  given by

$$(2.40) \quad \zeta(\tau, z) = \frac{\theta_1(\tau, z)^{12}}{\eta(\tau)^{12}}$$

(cf. Appendix A for  $\theta_1$  and  $\eta$ ). Gritsenko shows [43] that for any positive integer  $m$  the quotient  $J_{0,m-1}/J_{0,m-1}(q)$  is a vector space of dimension  $m - 1$  admitting a basis consisting of weight  $0$  index  $m - 1$  weak Jacobi forms  $\varphi_n^{(m)}$  (denoted  $\psi_{0,m-1}^{(n)}$  in [43]) for  $1 \leq n \leq m - 1$  such that the coefficient of  $q^0 y^k$

in  $\varphi_n^{(m)}$  vanishes for  $|k| > n$  but does not vanish for  $|k| = n$ . In fact, Gritsenko works in the subring  $J_{0,*}^{\mathbb{Z}}$  of Jacobi forms having integer Fourier coefficients and his  $\varphi_n^{(m)}$  furnish a  $\mathbb{Z}$ -basis for the  $\mathbb{Z}$ -module  $J_{0,*}^{\mathbb{Z}}/J_{0,*}^{\mathbb{Z}}(q)$ .

We show now that there are no non-zero extremal Jacobi forms in the ideal  $J_{0,*}(q)$ .

**Lemma 2.3.** *If  $\phi$  is an extremal weak Jacobi form belonging to  $J_{0,*}(q)$  then  $\phi = 0$ .*

*Proof.* If  $\phi$  belongs to  $J_{0,*}(q)$  then the coefficients of  $q^0 y^k$  in the Fourier development of  $\phi$  vanish for all  $k$ . This implies the vanishing of  $a_{\frac{m-1}{4},0}$  and  $a_{\frac{m-1}{4},\frac{1}{2}}$  in (2.37) where  $m-1$  is the index of  $\phi$ , and this in turn implies that the meromorphic Jacobi form  $\psi = \Psi_{1,1}\phi$  of weight 1 and index  $m$  coincides with its finite part  $\psi = \psi^F = \sum_r h_r \hat{\theta}_r^{(m)}$  (cf. (2.21), (2.23) and (2.25)) and has theta-coefficients  $h_r$  that remain bounded as  $\tau \rightarrow i\infty$ . In particular,  $\psi$  is a (strong) Jacobi form of weight 1 and integral index but the space of such forms vanishes according to [62], so  $\phi$  must vanish also.  $\square$

As a consequence of Lemma 2.3 we obtain that there are no extremal Jacobi forms with vanishing massless contribution at spin  $j = 1/2$ .

**Lemma 2.4.** *If  $\phi$  is an extremal weak Jacobi form of index  $m-1$  such that  $a_{\frac{m-1}{4},\frac{1}{2}} = 0$  in (2.37) then  $\phi = 0$ .*

*Proof.* If  $\phi$  is of weight 0 and index  $m-1$  satisfying (2.37) then  $\phi = a + b(y + y^{-1}) + O(q)$  as  $\tau \rightarrow i\infty$  where  $a = a_{\frac{m-1}{4},0}$  and  $b = -a_{\frac{m-1}{4},\frac{1}{2}}$ . Following [43] we set

$$(2.41) \quad \tilde{\phi}(\tau, z) = \exp(-8\pi^2(m-1)G_2(\tau)z^2)\phi(\tau, z),$$

where  $G_2(\tau) = -\frac{1}{24} + \sum_{n>0} \sigma_1(n)q^n$  is the unique up to scale mock modular form of weight 2 for  $SL_2(\mathbb{Z})$  (with shadow a constant function) and consider the Taylor expansion  $\tilde{\phi}(\tau, z) = \sum_{n \geq 0} f_n(\tau)z^n$ . Then  $f_0(\tau) = \chi = a + 2b$  and more generally  $f_n(\tau)$  is a modular form of weight  $n$  for  $SL_2(\mathbb{Z})$ . In particular  $f_2(\tau) = 0$ . On the other hand, the constant term of  $f_2(\tau)$  is  $\frac{1}{3}\pi^2(m-1)\chi - 4\pi^2b$  so  $(m-1)\chi = 12b$ . By hypothesis  $b = 0$  so  $\chi = 0$  and  $\phi$  belongs to  $J_{0,m-1}(q)$  and thus vanishes according to Lemma 2.3.  $\square$

Applying Lemma 2.4 we obtain that the dimension of  $J_{0,m-1}^{\text{ext}}$  is at most 1 for any  $m$ .

**Proposition 2.5.** *We have  $\dim J_{0,m-1}^{\text{ext}} \leq 1$  for any positive integer  $m$ .*

*Proof.* If  $\phi'$  and  $\phi''$  are two elements of  $J_{0,m-1}^{\text{ext}}$  then there exists  $c \in \mathbb{C}$  such that  $\phi = \phi' - c\phi''$  is extremal and has vanishing  $a_{\frac{m-1}{4}, \frac{1}{2}}$  in (2.37). Then  $\phi = 0$  according to Lemma 2.4 and thus  $\phi'$  belongs to the linear span of  $\phi''$ .  $\square$

We now present a result which ultimately shows that there are only finitely many  $m$  for which  $J_{0,m-1}^{\text{ext}} \neq \{0\}$ , and also illustrates the depth of the characterization problem. In preparation for it let us write  $S_2(m)$  for the space of cusp forms of weight 2 for  $\Gamma_0(m)$  and recall that  $f \in S_2(m)$  is called a *newform* if it is a Hecke eigenform, satisfying  $T(n)f = \lambda_f(n)f$  whenever  $(n, m) = 1$  for some  $\lambda_f(n) \in \mathbb{C}$ , that is uniquely specified (up to scale) in  $S_2(m)$  by its Hecke eigenvalues  $\{\lambda_f(n) \mid (n, m) = 1\}$  (cf. [63, Section IX.7]). Write  $S_2^{\text{new}}(m)$  for the subspace of  $S_2(m)$  spanned by newforms. Given  $f \in S_2(m)$  define  $L(f, s) = \sum_{n>0} a_f(n)n^{-s}$  for  $\Re(s) > 1$  when  $f(\tau) = \sum_{n>0} a_f(n)q^n$ . Then  $L(f, s)$  is called the *Dirichlet L-function* attached to  $f$  and admits an analytic continuation to  $s \in \mathbb{C}$  (cf. [63, Section IX.4] or [64, Section 3.6].) To a newform  $f$  in  $S_2(m)$  (i.e., a Hecke eigenform in  $S_2^{\text{new}}(m)$ ) Eichler–Shimura theory attaches an elliptic curve  $E_f$  defined over  $\mathbb{Q}$  (cf. [63, Section XI.11]), and the Birch–Swinnerton-Dyer conjecture predicts that  $E_f$  has rational points of infinite order whenever  $L(f, s)$  vanishes at  $s = 1$  (cf. [65, 66]).

**Theorem 2.6.** *If  $m$  is a positive integer and  $J_{0,m-1}^{\text{ext}} \neq \{0\}$  then  $L(f, 1) = 0$  for every  $f \in S_2^{\text{new}}(m)$ .*

*Proof.* Let  $m$  be a positive integer. For  $L = \sqrt{2m}\mathbb{Z}$  the spaces  $H_{k,L}^+$  and  $S_{2-k,L^-}$  of [67, Section 3] are naturally isomorphic to  $\mathbb{J}_{k+1/2,m}$  and  $S_{5/2-k,m}^{\text{skew}}$ , respectively, where  $\mathbb{J}_{k,m}$  denotes the space of weak (pure) mock Jacobi forms of weight  $k$  and index  $m$  (cf. [45, Section 7.2]) and  $S_{k,m}^{\text{skew}}$  denotes the space of cuspidal skew-holomorphic Jacobi forms of weight  $k$  and index  $m$  (cf. [68]). Translating the construction (3.9) of [67] into this language, and taking  $k = 1/2$ , we obtain a pairing

$$(2.42) \quad \{ \cdot, \cdot \} : S_{2,m}^{\text{skew}} \times \mathbb{J}_{1,m} \rightarrow \mathbb{C}.$$

If  $J_{0,m-1}^{\text{ext}} \neq \{0\}$  then let  $\phi \in J_{0,m-1}^{\text{ext}}$  be a non-zero extremal Jacobi form and set  $\psi = \Psi_{1,1}\phi$  as in Section 2.3. Then the finite part  $\psi^F$  of  $\psi$  belongs

to  $\mathbb{J}_{1,m}$  and the shadow of  $\psi^F$  (cf. [45, Section 7.2]) is the cuspidal skew-holomorphic Jacobi form  $\chi\sigma^{(m)} \in S_{2,m}^{\text{skew}}$  where  $\chi = \phi(\tau, 0)$  and

$$(2.43) \quad \sigma^{(m)}(\tau, z) = \sum_{r \bmod 2m} \overline{S_r^{(m)}(\tau)} \theta_r^{(m)}(\tau, z).$$

Consider the linear functional  $\lambda_\phi$  on  $S_{2,m}^{\text{skew}}$  defined by setting  $\lambda_\phi(\varphi) = \{\varphi, \psi^F\}$ . According to the definition of (2.42) (i.e., (3.9) of [67]) we have

$$(2.44) \quad \lambda_\phi(\varphi) = C_m \langle \varphi, \chi\sigma^{(m)} \rangle$$

for some (non-zero) constant  $C_m$  (depending only on  $m$ ) where  $\langle \cdot, \cdot \rangle$  denotes the Petersson inner product on  $S_{2,m}^{\text{skew}}$  (cf. [68]). On the other hand, inspection reveals that  $h_{\pm 1}(\tau) = \mp a_{\frac{m-1}{4}, \frac{1}{2}} q^{-1/4m} (1 + O(q))$  when  $\psi^F = \sum_{r \bmod 2m} h_r \theta_r^{(m)}$  so

$$(2.45) \quad \lambda_\phi(\varphi) = -2a_{\frac{m-1}{4}, \frac{1}{2}} C_\varphi(1, 1)$$

according to Proposition 3.5 of [67] where  $\varphi(\tau, z) = \sum C_\varphi(\Delta, r) \bar{q}^{\Delta/4m} q^{r^2/4m} y^r$  is the Fourier expansion of  $\varphi$  (cf. [68]). Applying Lemma 2.4 we deduce from (2.45) that  $\lambda_\phi(\varphi) = 0$  for  $\varphi \in S_{2,m}^{\text{skew}}$  if and only if  $C_\varphi(1, 1) = 0$ .

Now let  $f$  be a newform in  $S_2(m)$ . If  $\Lambda(f, s) = (2\pi)^{-s} m^{s/2} \Gamma(s) L(f, s)$  then either  $\Lambda(f, s) = \Lambda(f, 2-s)$  or  $\Lambda(f, s) = -\Lambda(f, 2-s)$ . In the latter case  $L(f, 1)$  necessarily vanishes. In the former case Theorem 1 of [69] attaches a skew-holomorphic Jacobi form  $\varphi_f \in P_{2,m}^{\text{skew}}$  to  $f$  having the same eigenvalues as  $f$  under the Hecke operators  $T(n)$  for  $(n, m) = 1$ . (Cf. [69] for the action of Hecke operators on skew-holomorphic Jacobi forms.) Here  $P_{2,m}^{\text{skew}}$  denotes the subspace of  $S_{2,m}^{\text{skew}}$  spanned by cuspidal Hecke eigenforms which are not of the trivial type: say  $\varphi \in S_{2,m}^{\text{skew}}$  is of the *trivial type* if the Fourier coefficients  $C_\varphi(\Delta, r)$  are non-vanishing only when  $\Delta$  is a perfect square. Cusp forms of the trivial type are orthogonal to  $P_{2,m}^{\text{skew}}$  with respect to the Petersson inner product (cf. the proof of Theorem 3 in [69]) and  $\sigma^{(m)}$  is a cusp form of the trivial type, so we have  $\lambda_\phi(\varphi_f) = 0$  by virtue of (2.44) and hence  $C_{\varphi_f}(1, 1) = 0$  according to the previous paragraph. The proposition now follows by taking  $(\Delta, r) = (1, 1)$  in the formula at the bottom of p. 503 of [68].  $\square$

Applying Theorem 2.6 we see, for example, that there are no non-zero extremal Jacobi forms of index 10, for it is known that  $L(f, 1) \neq 0$  for  $f = \eta(\tau)^2 \eta(11\tau)^2 \in S_2(11)$ . (Cf. [68, p. 504] where the corresponding Jacobi form  $\varphi_f$  is computed explicitly.)



Duke showed [70] that there is a constant  $C$  such that at least  $Cm/\log^2 m$  newforms  $f \in S_2(m)$  satisfy  $L(f, 1) \neq 0$  when  $m$  is a sufficiently large prime, and Ellenberg proved an effective version of this result [42] that is valid also for composite  $m$ . We will apply the formulas of Ellenberg momentarily in order to deduce an upper bound on the  $m$  for which  $J_{0,m-1}^{\text{ext}} \neq \{0\}$ , but since these formulas give better results for values of  $m$  with fewer divisors we first show that  $J_{0,m-1}^{\text{ext}}$  must vanish whenever  $m$  is divisible by more than one prime, and whenever  $m = p^\nu$  is a prime power with  $\nu > 2$ .

**Proposition 2.7.** *If  $m$  is divisible by more than one prime then  $J_{0,m-1}^{\text{ext}} = \{0\}$ .*

*Proof.* Suppose that  $m = p^\nu d$  with  $p$  prime,  $\nu > 0$ ,  $d > 1$  and  $(p, d) = 1$ . With notation as in the proof of Theorem 2.6 we suppose  $\phi \in J_{0,m-1}^{\text{ext}}$  is non-zero and consider the linear functional  $\lambda_\phi : S_{2,m}^{\text{skew}} \rightarrow \mathbb{C}$ . The Atkin–Lehner involution  $W_{p^\infty}$  (cf. [45, Section 4.4]) acts on  $S_{2,m}^{\text{skew}}$  in such a way that

$$(2.46) \quad (W_{p^\infty} \varphi)(\tau, z) = \sum_{r \pmod{2m}} \overline{g_{r^*}(\tau)} \theta_r^{(m)}(\tau, z),$$

when  $\varphi = \sum_{r \pmod{2m}} \overline{g_r} \theta_r^{(m)}$  is the theta-decomposition of  $\varphi \in S_{2,m}^{\text{skew}}$  and where the map  $r \mapsto r^*$  is defined so that  $r^*$  is the unique solution (mod  $2m$ ) to  $r^* \equiv -r \pmod{2p^\nu}$  and  $r^* \equiv r \pmod{2d}$ .

By the definition of  $\lambda_\phi$  we have  $\lambda_\phi(\varphi) = \{\varphi, \psi^F\} = \overline{\chi} \sum_r \langle g_r, S_r^{(m)} \rangle$  with  $\varphi$  as above since the shadow of  $\psi^F$  is  $\chi \sigma^{(m)} = \chi \sum_r S_r^{(m)} \theta_r^{(m)}$ . Taking  $\varphi = W_{p^\infty} \sigma^{(m)}$  we see that  $\lambda_\phi(W_{p^\infty} \sigma^{(m)}) = \overline{\chi} \sum_r \langle S_{r^*}^{(m)}, S_r^{(m)} \rangle$  is not zero according to the computation

$$(2.47) \quad \langle S_{r^*}^{(m)}, S_r^{(m)} \rangle = \frac{1}{2^6 \pi m^{3/2}} (\delta_{r^*, r \pmod{2m}} - \delta_{r^*, -r \pmod{2m}})$$

which can be obtained using the Rankin–Selberg formula. (Cf., e.g., (3.15) of [45].) A very similar computation is carried out in Proposition 10.2 of [45].) On the other hand, for  $\varphi = W_{p^\infty} \sigma^{(m)}$  we have  $C_\varphi(1, 1) = 0$  since  $r^* \not\equiv \pm 1 \pmod{2m}$  for  $r \equiv \pm 1 \pmod{2m}$ , and  $S_r^{(m)} = O(q^{2/4m})$  for  $r \not\equiv \pm 1 \pmod{2m}$ . So  $\lambda_\phi(W_{p^\infty} \sigma^{(m)}) = 0$  in light of (2.45). This contradiction proves the claim.  $\square$

**Proposition 2.8.** *If  $m$  is a prime power  $m = p^\nu$  and  $\nu > 2$  then  $J_{0,m-1}^{\text{ext}} = \{0\}$ .*

*Proof.* The proof is very similar to that of Proposition 2.7, computing  $\lambda_\phi(\varphi)$  two ways using (2.44) and (2.45), but taking now  $\varphi(\tau, z) = \sigma^{(p)}(\tau, p^\mu z)$  or  $\varphi(\tau, z) = \sigma^{(p^2)}(\tau, p^\mu z)$  according as  $\nu = 2\mu + 1$  or  $\nu = 2\mu + 2$  with  $\mu > 0$ . The transformation  $\varphi(\tau, z) \mapsto \varphi(\tau, tz)$  maps Jacobi forms of index  $m$  to Jacobi forms of index  $t^2m$ , and is one of the Hecke-like operators of [45, Section 4.4]. We leave the remaining details — the expression of  $\varphi$  in terms of  $S_r^{(m)}$  and  $\theta_r^{(m)}$ , and an application of (2.47) — to the reader.  $\square$

We require one more result in advance of the proof of Theorem 2.2.

**Lemma 2.9.** *If  $m = p^2$  for some prime  $p$  and there exists a newform  $f \in S_2(p)$  with  $L(f, 1) \neq 0$  then  $J_{0, m-1}^{\text{ext}} = \{0\}$ .*

*Proof.* Let  $m, p$  and  $f \in S_2(p)$  be as in the statement of the lemma. Then, as in the proof of Theorem 2.6, there exists a uniquely determined  $\varphi_f \in P_{2,p}^{\text{skew}}$  with the same Hecke eigenvalues as  $f$  according to the results of [68, 69], and since  $L(f, 1) \neq 0$  we have  $C_{\varphi_f}(1, 1) \neq 0$ . Now set  $\varphi'_f = \varphi_f|V_p$  where  $V_p$  is the Hecke-like operator (cf. [45, (4.37)]) that maps Jacobi forms of index  $m$  to forms of index  $pm$ , and whose action is given explicitly by

$$(2.48) \quad C_{\varphi|V_p}(\Delta, r) = \sum_{d|\left(\frac{\Delta-r^2}{4p^2}, r, p\right)} dC_\varphi\left(\frac{\Delta}{d^2}, \frac{r}{d}\right)$$

in our special case that  $\varphi = \varphi_f \in J_{2,p}^{\text{skew}}$  when  $\varphi = \sum C_\varphi(\Delta, r)\bar{q}^{\Delta/4p}q^{r^2/4p}y^r$  is the Fourier expansion of  $\varphi$ .

To prove that  $J_{0, m-1}^{\text{ext}} = \{0\}$  suppose that  $\phi$  is a non-zero extremal form of index  $m - 1 = p^2 - 1$  and consider the functional  $\lambda_\phi : S_{2,p^2}^{\text{skew}} \rightarrow \mathbb{C}$  constructed in the proof of Theorem 2.6. Since  $C_{\varphi_f}(1, 1) \neq 0$  we have  $C_{\varphi'_f}(1, 1) \neq 0$  according to (2.48). So  $\lambda_\phi(\varphi'_f) \neq 0$  thanks to (2.45) and Lemma 2.4. On the other hand, the Hecke-like operator  $V_p$  commutes with all Hecke operators  $T(n)$  (with, in our case,  $(n, p) = 1$ ) and so restricts to a map  $P_{2,p}^{\text{skew}} \rightarrow P_{2,p^2}^{\text{skew}}$ . Thus  $\varphi'_f$  is a Hecke-eigenform in  $P_{2,p^2}^{\text{skew}}$  and we have  $\lambda_\phi(\varphi'_f) = 0$  by virtue of (2.44) and the fact that  $P_{2,p^2}^{\text{skew}}$  is orthogonal to the cusp form  $\sigma^{(p^2)}$  of the trivial type. This contradiction completes the proof.  $\square$

*Proof of Theorem 2.2.* We have that  $\dim J_{0, m-1}^{\text{ext}} \leq 1$  for all  $m$  according to Proposition 2.5, and  $\dim J_{0, m-1}^{\text{ext}} = 0$  unless  $m = p$  or  $m = p^2$  for some prime  $p$  by virtue of Propositions 2.7 and 2.8. Applying Theorem 1 of [42] with

$\sigma = 13/25\pi$ , for example, shows that some  $f \in S_2^{\text{new}}(p)$  satisfies  $L(f, 1) \neq 0$  whenever  $p$  is a prime greater than or equal to 3001. The tables of arithmetic data on newforms posted at [71] give the non-vanishing or otherwise for every newform of level up to 5134, and inspecting the entries for the 430 prime levels less than 3000 we see that  $S_2(p)$  has a newform  $f$  with  $L(f, 1) \neq 0$  for every prime  $p$  so long as  $S_2(p)$  has a newform. We have  $S_2^{\text{new}}(p) = S_2(p) = \{0\}$  just when  $p \in \{2, 3, 5, 7, 13\}$ , so we conclude from Theorem 2.6 that  $J_{0,m-1}^{\text{ext}} = \{0\}$  for every prime  $m = p$  such that  $p - 1$  does not divide 12. Also, we conclude from Lemma 2.9 that  $J_{0,m-1}^{\text{ext}} = \{0\}$  whenever  $m = p^2$  is the square of a prime and  $p - 1$  does not divide 12. We inspect the tables of [71] again to find newforms with non-vanishing critical central value at levels  $7^2 = 49$  and  $13^2 = 169$ , and apply Theorem 2.6 to conclude  $J_{0,48}^{\text{ext}} = J_{0,168}^{\text{ext}} = \{0\}$ .

So we now require to determine the dimension of  $J_{0,m-1}^{\text{ext}}$  for  $m \in \{2, 3, 4, 5, 7, 9, 13, 25\}$ , which are the primes and squares of primes  $m$  for which  $S_2(m) = \{0\}$ . For this we utilize the basis  $\{\varphi_n^{(m)}\}$  for  $J_{0,*}$  determined by Gritsenko [43], and discussed, briefly, after the statement of Theorem 2.2 above. In more detail, the ring  $J_{0,*}$  is finitely generated, by  $\varphi_1^{(2)}$ ,  $\varphi_1^{(3)}$  and  $\varphi_1^{(4)}$ , where

$$(2.49) \quad \begin{aligned} \varphi_1^{(2)} &= 4(f_2^2 + f_3^2 + f_4^2), \\ \varphi_1^{(3)} &= 2(f_2^2 f_3^2 + f_3^2 f_4^2 + f_4^2 f_2^2), \\ \varphi_1^{(4)} &= 4f_2^2 f_3^2 f_4^2, \end{aligned}$$

and  $f_i(\tau, z) = \theta_i(\tau, z)/\theta_i(\tau, 0)$  for  $i \in \{2, 3, 4\}$  (cf. A.2). If we work over  $\mathbb{Z}$  then we must include  $\varphi_1^{(5)}$

$$(2.50) \quad \varphi_1^{(5)} = \frac{1}{4}(\varphi_1^{(4)} \varphi_1^{(2)} - (\varphi_1^{(3)})^2)$$

as a generator also, so that  $J_{0,*}^{\mathbb{Z}} = \mathbb{Z}[\varphi_1^{(2)}, \varphi_1^{(3)}, \varphi_1^{(4)}, \varphi_1^{(5)}]$ . Following [43] we define

$$(2.51) \quad \begin{aligned} \varphi_1^{(7)} &= \varphi_1^{(3)} \varphi_1^{(5)} - (\varphi_1^{(4)})^2, \\ \varphi_1^{(9)} &= \varphi_1^{(3)} \varphi_1^{(7)} - (\varphi_1^{(5)})^2, \\ \varphi_1^{(13)} &= \varphi_1^{(5)} \varphi_1^{(9)} - 2(\varphi_1^{(7)})^2, \end{aligned}$$

and define  $\varphi_1^{(m)}$  for the remaining positive integers  $m$  according to the following recursive procedure. For  $(12, m - 1) = 1$  and  $m > 5$  we set

$$(2.52) \quad \varphi_1^{(m)} = (12, m - 5)\varphi_1^{(m-4)}\varphi_1^{(5)} + (12, m - 3)\varphi_1^{(m-2)}\varphi_1^{(3)} \\ - 2(12, m - 4)\varphi_1^{(m-3)}\varphi_1^{(4)}.$$

For  $(12, m - 1) = 2$  and  $m > 10$  we set

$$(2.53) \quad \varphi_1^{(m)} = \frac{1}{2}((12, m - 5)\varphi_1^{(m-4)}\varphi_1^{(5)} + (12, m - 3)\varphi_1^{(m-2)}\varphi_1^{(3)} \\ - 2(12, m - 4)\varphi_1^{(m-3)}\varphi_1^{(4)}).$$

For  $(12, m - 1) = 3$  and  $m > 9$  we set

$$(2.54) \quad \varphi_1^{(m)} = \frac{2}{3}(12, m - 4)\varphi_1^{(m-3)}\varphi_1^{(4)} + \frac{1}{3}(12, m - 7)\varphi_1^{(m-6)}\varphi_1^{(7)} \\ - (12, m - 5)\varphi_1^{(m-4)}\varphi_1^{(5)}.$$

For  $(12, m - 1) = 4$  and  $m > 16$  we set

$$(2.55) \quad \varphi_1^{(m)} = \frac{1}{4}((12, m - 13)\varphi_1^{(m-12)}\varphi_1^{(13)} + (12, m - 5)\varphi_1^{(m-4)}\varphi_1^{(5)} \\ - (12, m - 9)\varphi_1^{(m-8)}\varphi_1^{(9)}).$$

For  $(12, m - 1) = 6$  and  $m > 18$  we set

$$(2.56) \quad \varphi_1^{(m)} = \frac{1}{3}(12, m - 4)\varphi_1^{(m-3)}\varphi_1^{(4)} + \frac{1}{6}(12, m - 7)\varphi_1^{(m-6)}\varphi_1^{(7)} \\ - \frac{1}{2}(12, m - 5)\varphi_1^{(m-4)}\varphi_1^{(5)}.$$

Finally, for  $(12, m - 1) = 12$  and  $m > 24$  we set<sup>1</sup>

$$(2.57) \quad \varphi_1^{(m)} = \frac{1}{6}(12, m - 4)\varphi_1^{(m-3)}\varphi_1^{(4)} - \frac{1}{4}(12, m - 5)\varphi_1^{(m-4)}\varphi_1^{(5)} \\ + \frac{1}{12}(12, m - 7)\varphi_1^{(m-6)}\varphi_1^{(7)}.$$

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<sup>1</sup>Our expression gives one possible correction for a small error in the last line on p. 10 of [43].

The  $\varphi_2^{(m)}$  are defined by setting

$$(2.58) \quad \begin{aligned} \varphi_2^{(3)} &= (\varphi_1^{(2)})^2 - 24\varphi_1^{(3)}, \\ \varphi_2^{(4)} &= \varphi_1^{(2)}\varphi_1^{(3)} - 18\varphi_1^{(4)}, \\ \varphi_2^{(5)} &= \varphi_1^{(2)}\varphi_1^{(4)} - 16\varphi_1^{(5)} \end{aligned}$$

and

$$(2.59) \quad \varphi_2^{(m)} = (12, m - 4)\varphi_1^{(m-3)}\varphi_1^{(4)} - (12, m - 5)\varphi_1^{(m-4)}\varphi_1^{(5)} - (12, m - 1)\varphi_1^{(m)}$$

for  $m > 5$ , and the remaining  $\varphi_n^{(m)}$  for  $2 \leq m \leq 25$  are given by

$$(2.60) \quad \begin{aligned} \varphi_n^{(m)} &= \varphi_{n-1}^{(m-3)}\varphi_1^{(4)}, \\ \varphi_{m-2}^{(m)} &= (\varphi_1^{(2)})^{m-3}\varphi_1^{(3)}, \\ \varphi_{m-1}^{(m)} &= (\varphi_1^{(2)})^{m-1}, \end{aligned}$$

where the first equation of (2.60) holds for  $3 \leq n \leq m - 3$ .

Next we calculate that  $\varphi_1^{(m)}$  defines a non-zero element of  $J_{0,m-1}^{\text{ext}}$  when  $m - 1$  divides 12, so  $\dim J_{0,m-1}^{\text{ext}} = 1$  for  $m \in \{2, 3, 4, 5, 7, 13\}$ , and it remains to show that  $J_{0,m-1}^{\text{ext}} = \{0\}$  for  $m \in \{9, 25\}$ . To do this we first observe that a weight zero form with vanishing Fourier coefficients of  $q^0 y^k$  for  $|k| > 1$  necessarily has the form

$$(2.61) \quad c_{0,1}\varphi_1^{(m)} + \sum_{i=1}^{\lfloor \frac{m-1}{6} \rfloor} \sum_{j=1}^{m-6i-1} c_{i,j} \zeta^i \varphi_j^{(m-6i)}$$

for some  $c_{i,j} \in \mathbb{C}$ . One has to show that the only such linear combination satisfying the extremal condition (2.37) necessarily has  $c_{i,j} = 0$  for all  $i, j$  (including  $(i, j) = (0, 1)$ ). An explicit inspection of the coefficients of terms  $q^n y^r$  with  $r^2 - 4mn \geq 0$  for  $n = 1, 2, 3, 4$  is sufficient to establish that  $\dim J_{0,m-1}^{\text{ext}} = 0$  when  $m = 9$ . For  $m = 25$  we notice that there are at least as many possible polar terms in the above sum as the number of parameters  $c_{i,j}$ . Indeed, by explicitly solving the system of linear equations we find that there is no accidental cancellation and the only solution to (2.61) that also satisfies the extremal condition (2.37) is zero. In this way we conclude that there is no solution to (2.37) in  $J_{0,m-1}$  for  $m \in \{9, 25\}$ . This completes the proof of the Theorem.  $\square$

The quantity  $Z^{(2)}(\tau, z) = 2\varphi_1^{(2)}(\tau, z)$  is equal to the elliptic genus of a(ny)  $K3$  surface and the factor of two relating  $Z^{(2)}$  to  $\varphi_1^{(2)}$  is required in the  $K3/M_{24}$  connection in order for the mock modular form  $H^{(2)} = (H_1^{(2)})$  derived from  $Z^{(2)}$  to have coefficients compatible with an interpretation as dimensions of representations of  $M_{24}$  for which the corresponding McKay–Thompson series  $H_g^{(2)}$  have integer Fourier coefficients. Inspired by this we define the *Umbral Jacobi forms*

$$(2.62) \quad Z^{(\ell)}(\tau, z) = 2\varphi_1^{(\ell)}(\tau, z)$$

for  $\ell \in \Lambda = \{2, 3, 4, 5, 7, 13\}$ . We also set  $\chi^{(\ell)}$  to be the constant  $Z^{(\ell)}(\tau, 0)$  and find that  $\chi^{(\ell)} = 24/(\ell - 1)$  for all  $\ell \in \Lambda$ .

**Remark 2.10.** The identity  $(m - 1)\chi = 12b$  for  $\phi = a + b(y + y^{-1}) + O(q)$  established in Lemma 2.4 shows that there is no such Jacobi form with  $b = 1$  and  $a$  an integer unless  $m - 1$  divides 12. In particular, there is no extremal Jacobi form  $\phi \in J_{0, m-1}^{\text{ext}}$  such that  $a_{\frac{m-1}{4}, \frac{1}{2}} = 1$  and  $a_{\frac{m-1}{4}, 0}$  is an integer unless  $m - 1$  divides 12. Inspecting the  $Z^{(\ell)}$  we conclude that the divisors of 12 are exactly the values of  $m - 1$  for which such a Jacobi form exists.

Following the discussion in Sections 2.3 and 2.4, each of the Umbral Jacobi forms  $Z^{(\ell)}$  leads to an  $(\ell - 1)$ -vector-valued mock modular form  $H^{(\ell)} = (H_r^{(\ell)})$  through the decomposition of  $Z^{(\ell)}$  into  $N = 4$  characters, or equivalently through the decomposition of  $\psi^{(\ell)} = \Psi_{1,1} Z^{(\ell)}$  into its polar and finite parts and the theta-expansion of the finite part,

$$(2.63) \quad \psi^{(\ell), F}(\tau, z) = \sum_{r=1}^{\ell-1} H_r^{(\ell)}(\tau) \hat{\theta}_r^{(\ell)}(\tau, z).$$

As we have explained above, the extremal condition translates into a natural condition on the vector-valued mock modular forms  $H^{(\ell)}$ : The requirement that  $Z^{(\ell)}$  takes the form (2.37) is equivalent to requiring that the only polar term in  $H^{(\ell)} = (H_r^{(\ell)})$  is  $-2q^{-\frac{1}{4\ell}}$  in the component  $H_1^{(\ell)}$  and all the other components  $H_r^{(\ell)}$  for  $r \neq 1$  vanish as  $\tau \rightarrow i\infty$ , so that

$$(2.64) \quad H_r^{(\ell)}(\tau) = -2\delta_{r,1} q^{-\frac{1}{4\ell}} + O(q^{\frac{1}{4\ell}})$$

for  $0 < r < \ell$ .

Some low-order terms in the Fourier expansions of the component functions  $H_r^{(\ell)}$  obtained from the extremal forms  $Z^{(\ell)}$  for  $\ell \in \{2, 3, 4\}$  are given

as follows:

(2.65)

$$H_1^{(2)}(\tau) = 2q^{-1/8} (-1 + 45q + 231q^2 + 770q^3 + 2277q^4 + 5796q^5 + \dots),$$

(2.66)

$$H_1^{(3)}(\tau) = 2q^{-1/12} (-1 + 16q + 55q^2 + 144q^3 + 330q^4 + 704q^5 + \dots),$$

$$H_2^{(3)}(\tau) = 2q^{2/3} (10 + 44q + 110q^2 + 280q^3 + 572q^4 + 1200q^5 + \dots),$$

$$H_1^{(4)}(\tau) = 2q^{-1/16} (-1 + 7q + 21q^2 + 43q^3 + 94q^4 + 168q^5 + \dots),$$

(2.67)  $H_2^{(4)}(\tau) = 2q^{3/4} (8 + 24q + 56q^2 + 112q^3 + 216q^4 + 392q^5 + \dots),$

$$H_3^{(4)}(\tau) = 2q^{7/16} (3 + 14q + 28q^2 + 69q^3 + 119q^4 + 239q^5 + \dots).$$

As a prelude to the next section we note that the coefficients 16, 55 and 144 appearing in  $H_1^{(3)}$  are dimensions of irreducible representations of the Mathieu group  $M_{12}$  and that  $\chi^{(3)} = 12$  is the dimension of the defining permutation representation of  $M_{12}$  just as  $\chi^{(2)} = 24$  is the dimension of the defining permutation representation of  $M_{24}$ . We further note that the low-lying coefficients appearing in  $H_2^{(3)}$  are dimensions of faithful irreducible representations of a group  $2.M_{12}$ . Here the notation  $2.G$  denotes a group with a  $\mathbb{Z}/2\mathbb{Z}$  normal subgroup such that  $2.G/(\mathbb{Z}/2\mathbb{Z}) = G$ . Note also that  $2.M_{12}$  has a faithful (and irreducible) 12-dimensional signed permutation representation (cf. Section 3.2). The pattern that the coefficients of  $H_r^{(\ell)}$  for  $r$  odd are dimensions of representations of a group  $\bar{G}^{(\ell)}$  with a permutation representation of dimension  $\chi^{(\ell)}$  and the coefficients of  $H_r^{(\ell)}$  for  $r$  even are dimensions of faithful representations of a group  $G^{(\ell)} = 2.\bar{G}^{(\ell)}$  with a signed permutation representation of degree  $\chi^{(\ell)}$  persists for all  $\ell \in \Lambda$ ; detailed descriptions of the (unsigned) permutation and signed permutation representations of  $\bar{G}^{(\ell)}$  and  $G^{(\ell)}$  are given in Sections 3.3 and 3.4.

The leading terms in the  $q$ -expansions of the mock modular forms  $H_1^{(\ell)}$  that correspond via the procedure described earlier to the Jacobi forms  $Z^{(\ell)}$  for  $\ell \in \{5, 7, 13\}$  are

(2.68)

$$H_1^{(5)}(\tau) = 2q^{-1/20} (-1 + 4q + 9q^2 + 20q^3 + 35q^4 + 60q^5 + \dots),$$

$$H_1^{(7)}(\tau) = 2q^{-1/28} (-1 + 2q + 3q^2 + 5q^3 + 10q^4 + 15q^5 + 21q^6 + \dots),$$

$$H_1^{(13)}(\tau) = 2q^{-1/52} (-1 + q + q^2 + q^4 + q^5 + 2q^6 + 3q^7 + \dots).$$

To avoid clutter we refrain from describing low order terms in the  $q$ -expansions of  $H_r^{(\ell)}$  for  $\ell \in \{5, 7, 13\}$  and  $r > 1$  here, but these can be read off from the 1A entries in Tables C.8–C.10, C.12–C.16, and C.18–C.28.

We conclude this section with a comparison of our condition (2.37) with other notions of extremal in the literature. A notion of *extremal holomorphic conformal field theory* was given in [27] following earlier related work on vertex operator superalgebras in [72]. Extremal CFT's have central charge  $c = 24k$  with  $k$  a positive integer and are defined in such a way that their partition function satisfies

$$(2.69) \quad Z = q^{-k} \prod_{n>1}^{\infty} \frac{1}{1-q^n} + O(q) \prod_{n>0}^{\infty} \frac{1}{1-q^n}$$

as  $\tau \rightarrow i\infty$  where the term  $O(q)$  is presumed to be a series in  $q$  with integer coefficients, so that the second term  $O(q) \prod_{n>0} (1-q^n)^{-1}$  in (2.69) represents an integer combination of irreducible characters of the Virasoro algebra. The first term in (2.69) is the vacuum character of the Virasoro algebra, generated by the vacuum state, and thus any Virasoro primaries above the vacuum are only allowed to contribute positive powers of  $q$  to the partition function. At this time the only known example of an extremal CFT is that determined by the monster VOA  $V^\natural$  whose partition function has  $k = 1$  in (2.69) so this notion is, at the very least, good at singling out extraordinary structure.

If  $\phi$  is a weak Jacobi form of weight 0 and index  $m - 1$  that is extremal in our sense then we have

$$(2.70) \quad \phi = a_{\frac{m-1}{4}, 0} \text{ch}_{\frac{m-1}{4}, 0}^{(m)} + a_{\frac{m-1}{4}, \frac{1}{2}} \text{ch}_{\frac{m-1}{4}, \frac{1}{2}}^{(m)} + \sum_{0 < r < m} O(q) \frac{\hat{\theta}_r^{(m)}}{\Psi_{1,1}}$$

as  $\tau \rightarrow i\infty$  for some  $a_{\frac{m-1}{4}, 0}$  and  $a_{\frac{m-1}{4}, \frac{1}{2}}$ , and the third term in (2.70) is a natural counterpart to the second term in (2.69) since the massive  $N = 4$  characters (in the Ramond sector with  $(-1)^F$  insertion) are all of the form  $q^\alpha \hat{\theta}_r^{(m)} / \Psi_{1,1}$  for some  $\alpha$  and some  $r$ . It is harder to argue that the massless  $N = 4$  contributions in (2.70) are in direct analogy with the first term in (2.69) since the vacuum state in a superconformal field theory with  $N = 4$  supersymmetry will generally (i.e., when the index is greater than 1) give rise to non-zero massless  $N = 4$  character contributions with spin greater



than  $1/2$ . We remark that the stronger condition

$$(2.71) \quad \phi = a_{\frac{m-1}{4},0} \text{ch}_{\frac{m-1}{4},0}^{(m)} + \sum_{0 < r < m} O(q) \frac{\hat{\theta}_r^{(m)}}{\Psi_{1,1}}$$

has no solutions according to Lemma 2.4. According to Theorem 2.2 the six Jacobi forms  $Z^{(\ell)}$  of Umbral moonshine are the unique solutions to (2.70) having  $a_{\frac{m-1}{4},\frac{1}{2}} = -2$ .

A notion of *extremal conformal field theory with  $N = (2, 2)$  superconformal symmetry* was introduced in [73] and the Ramond sector partition function of such an object defines a weak Jacobi form of weight 0 and some index but will generally not coincide with a Jacobi form that is extremal in our sense since, as has been mentioned, our functions are free from contributions arising from states in the Ramond sector that are related by spectral flow to the vacuum state in the Neveu–Schwarz sector, except in the case of index 1.

Despite the absence of a notion of extremal conformal field theory underlying our extremal Jacobi forms it is interesting to reflect on the fact that the one known example of an extremal CFT is that (at  $k = 1$ ) determined by the moonshine VOA of [12] with partition function  $Z(\tau) = J(\tau)$ . This function encodes dimensions of irreducible representations of the monster and, as we have seen to some extent above and will see in more detail below, the quantities which play the same role with respect to the groups  $G^{(\ell)}$  of Umbral moonshine are precisely the mock modular forms  $H^{(\ell)}(\tau) = (H_r^{(\ell)}(\tau))$  which arise naturally from the extremal Jacobi forms  $Z^{(\ell)}(\tau, z)$  according to the procedure of Section 2.3. One of the main motivations for the notion of extremal CFT introduced in [27] was a possible connection to pure (chiral) gravity theory in 3-dimensional Anti-de Sitter space via the AdS/CFT correspondence and it is interesting to compare this with the important rôle that Rademacher sum constructions play in monstrous moonshine [25], Umbral moonshine at  $\ell = 2$  [24], and in Umbral moonshine more generally (cf. Sections 1 and 5.2). We refer to [28, 29, 74–85] for further discussions on the AdS/CFT correspondence and extremal CFT's.

## 2.6. Siegel modular forms

Siegel modular forms are automorphic forms which generalize modular forms by replacing the modular group  $SL_2(\mathbb{Z})$  by the *genus  $n$  Siegel modular group*  $Sp_{2n}(\mathbb{Z})$  and the upper half-plane  $\mathbb{H}$  by the *genus  $n$  Siegel upper half-plane*

$\mathbb{H}_n$ . Here we restrict ourselves to the case  $n = 2$ . Define

$$(2.72) \quad J = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix}$$

with  $I_2$  the unit  $2 \times 2$  matrix and let  $Sp_4(\mathbb{Z})$  be the group of  $4 \times 4$  matrices  $\gamma \in M_4(\mathbb{Z})$  satisfying  $\gamma J \gamma^t = J$ . If we write  $\gamma$  in terms of  $2 \times 2$  matrices with integer entries  $A, B, C, D$ ,

$$(2.73) \quad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

then the condition  $\gamma J \gamma^t = J$  becomes

$$(2.74) \quad AB^t = BA^t, \quad CD^t = DC^t, \quad AD^t - BC^t = 1.$$

Just as  $SL_2(\mathbb{Z})$  has a natural action on  $\mathbb{H}$ , the (genus 2) Siegel modular group  $Sp_4(\mathbb{Z})$  has a natural action on the (genus 2) Siegel upper half-plane,  $\mathbb{H}_2$ , defined as the set of  $2 \times 2$  complex, symmetric matrices

$$(2.75) \quad \Omega = \begin{pmatrix} \tau & z \\ z & \sigma \end{pmatrix}$$

obeying

$$(2.76) \quad \text{Im}(\tau) > 0, \quad \text{Im}(\sigma) > 0, \quad \det(\text{Im}(\Omega)) > 0.$$

The action of  $\gamma \in Sp_4(\mathbb{Z})$  is given by

$$(2.77) \quad \gamma \Omega = (A\Omega + B)(C\Omega + D)^{-1}$$

when  $\gamma$  is given by (2.73). A *Siegel modular form* of weight  $k$  for  $\Gamma \subset Sp_4(\mathbb{Z})$  is a holomorphic function  $F : \mathbb{H}_2 \rightarrow \mathbb{C}$  obeying the transformation law

$$(2.78) \quad F((A\Omega + B)(C\Omega + D)^{-1}) = \det(C\Omega + D)^k F(\Omega)$$

for  $\gamma \in \Gamma$ . We can write a *Fourier–Jacobi decomposition* of  $F(\Omega)$  in terms of  $p = e(\sigma)$  as

$$(2.79) \quad F(\Omega) = \sum_{m=0}^{\infty} \varphi_m(\tau, z) p^m$$

and the transformation law for  $F(\Omega)$  can be used to show that the Fourier–Jacobi coefficient  $\varphi_m(\tau, z)$  is a Jacobi form of weight  $k$  and index  $m$ . (This

is one of the main motivations for the notion of Jacobi form; see [86] for an early analysis with applications to affine Lie algebras.) We can thus write

$$(2.80) \quad \varphi_m(\tau, z) = \sum_{\substack{n, r \in \mathbb{Z} \\ 4mn - r^2 \geq 0}} c(m, n, r) q^n y^r$$

and then

$$(2.81) \quad F(\Omega) = \sum_{m, n, r} c(m, n, r) p^m q^n y^r.$$

A special class of Siegel modular forms, called *Spezialschar* by Maass, arise by taking the Jacobi–Fourier coefficients  $\varphi_m$  to be given by  $\varphi_m = \varphi_1 |V_m$  where  $\varphi_1 \in J_{k,1}$  and  $V_m$  is the *Hecke-like operator* defined so that if  $\varphi_1 = \sum_{n,r} c(n, r) q^n y^r$  then

$$(2.82) \quad \varphi_1 |V_m = \sum_{n,r} \left( \sum_{j|(n,r,m)} j^{k-1} c\left(\frac{nm}{j^2}, \frac{r}{j}\right) \right) q^n y^r.$$

It develops that the function

$$(2.83) \quad F(\Omega) = \sum_{m=0}^{\infty} (\varphi_1 |V_m)(\tau, z) p^m$$

is a weight  $k$  Siegel modular form known as the *Saito–Kurokawa* or *additive lift* of the weight  $k$  and index 1 Jacobi form  $\varphi_1$ . An important example arises by taking the additive lift of the Jacobi form

$$(2.84) \quad \varphi_{10,1}(\tau, z) = -\eta(\tau)^{18} \theta_1(\tau, z)^2$$

(cf. Appendix A) which produces the *Igusa cusp form*

$$(2.85) \quad \Phi_{10}(\Omega) = \sum_{m=1}^{\infty} (\varphi_{10,1} |V_m)(\tau, z) p^m.$$

The Igusa cusp form also admits a product representation obtained from the *Borchers* or *exponential lift* of the Umbral Jacobi form  $Z^{(2)} = 2\varphi_{2,1}$

((2.49)) which is

$$(2.86) \quad \Phi_{10}(\Omega) = pqy \prod_{\substack{m,n,r \in \mathbb{Z} \\ (m,n,r) > 0}} (1 - p^m q^n y^r)^{c^{(2)}(4mn-r^2)},$$

where the coefficients  $c^{(2)}$  are defined by the Fourier expansion of  $Z^{(2)}$

$$(2.87) \quad Z^{(2)}(\tau, z) = \sum_{n,r} c^{(2)}(4n - r^2) q^n y^r$$

and where the condition  $(m, n, r) > 0$  is that either  $m > 0$  or  $m = 0$  and  $n > 0$  or  $m = n = 0$  and  $r < 0$ .

According to Theorem 2.1 in [87] the Umbral Jacobi form  $Z^{(\ell)}$  defines a Siegel modular form  $\Phi^{(\ell)}$  on the *paramodular group*  $\Gamma_{\ell-1}^+ < Sp_4(\mathbb{Q})$  via the Borchers lift for each  $\ell \in \Lambda$ . (We refer the reader to [87] for details.) For small values of  $\ell$  these Siegel forms  $\Phi^{(\ell)}$  have appeared in the literature (in particular in the work [87, 88] of Gritsenko–Nikulin). Taking  $\ell \in \Lambda$  and defining  $c^{(\ell)}(n, r)$  so that

$$(2.88) \quad Z^{(\ell)}(\tau, z) = \sum_{n,r} c^{(\ell)}(n, r) q^n y^r,$$

we have the exponential lift  $\Phi^{(\ell)}$  of weight  $k = c^{(\ell)}(0, 0)/2$  for  $\Gamma_{\ell-1}^+$  given by

$$(2.89) \quad \Phi^{(\ell)}(\Omega) = p^{A(\ell)} q^{B(\ell)} y^{C(\ell)} \prod_{(m,n,r) > 0} (1 - p^m q^n y^r)^{c^{(\ell)}(mn,r)},$$

where

$$(2.90) \quad A(\ell) = \frac{1}{24} \sum_r c^{(\ell)}(0, r), \quad B(\ell) = \frac{1}{2} \sum_{r>0} r c^{(\ell)}(0, r),$$

$$C(\ell) = \frac{1}{4} \sum_r r^2 c^{(\ell)}(0, r).$$

Note that for  $\ell \in \{2, 3, 4, 5\}$  we have  $\Phi^{(\ell)} = (\Delta_k)^2$  in the notation of [87] where  $k$  is given by  $k = (7 - \ell)/(\ell - 1)$ . These four functions  $\Delta_k$  appear as denominator functions for Lorentzian Kac–Moody Lie (super)algebras in [87, Section 5.1] (see also [89]) and in connection with mirror symmetry for  $K3$  surfaces in [87, Section 5.2]. We refer the reader to Section 5.5 for more discussion on this.

### 3. Finite groups

In this section we introduce the *Umbral groups*  $G^{(\ell)}$  for  $\ell \in \Lambda = \{2, 3, 4, 5, 7, 13\}$ . It will develop that the representation theory of  $G^{(\ell)}$  is intimately related to the vector-valued mock modular form  $H^{(\ell)}$  of Section 2.5.

We specify the abstract isomorphism types of the groups explicitly in Section 3.1. Each group admits a quotient  $\bar{G}^{(\ell)}$  which is naturally realized as a permutation group on  $24/(\ell - 1)$  points. We construct these permutations explicitly in Section 3.3. In order to construct the  $G^{(\ell)}$  we use signed permutations — a notion we discuss in Section 3.2 — and explicit generators for the  $G^{(\ell)}$  are finally obtained in Section 3.4. The signed permutation constructions are then used (again in Section 3.4) to define characters for  $G^{(\ell)}$  — the *twisted Euler characters* — which turn out to encode the automorphy of the mock modular forms  $H_g^{(\ell)}$  that are described in detail in Section 4. In Sections 3.5 to 3.6 we describe curious connections between the  $G^{(\ell)}$  and certain Dynkin diagrams.

#### 3.1. Specification

For  $\ell \in \Lambda$  let abstract groups  $G^{(\ell)}$  and  $\bar{G}^{(\ell)}$  be as specified in Table 1. We will now explain the notation used therein (moving from right to left).

We write  $n$  as a shorthand for  $\mathbb{Z}/n\mathbb{Z}$  (in the second and third rows of Table 1) and  $\text{Sym}_n$  denotes the symmetric group on  $n$  points. We write  $\text{Alt}_n$  for the alternating group on  $n$  points, which is the subgroup of  $\text{Sym}_n$  consisting of all even permutations.

We say that  $G$  is a *double cover* of a group  $H$  and write  $G \simeq 2.H$  (cf. [90, Section 5.2]) in case  $G$  has a subgroup  $Z$  of order 2 that is normal (and therefore central, being of order two) with the property that  $G/Z$  is isomorphic to  $H$ . We say that  $G$  is a *non-trivial* double cover of  $H$  if it is a double cover that is not isomorphic to the direct product  $2 \times H$ . (We do not usually write  $G \simeq 2.H$  unless  $G$  is a non-trivial cover of  $H$ .) The cyclic group of order 4 is a non-trivial double cover of the group of order 2.

Table 1: The Umbral Groups.

$\ell$	2	3	4	5	7	13
$G^{(\ell)}$	$M_{24}$	$2.M_{12}$	$2.AGL_3(2)$	$GL_2(5)/2$	$SL_2(3)$	4
$\bar{G}^{(\ell)}$	$M_{24}$	$M_{12}$	$AGL_3(2)$	$PGL_2(5)$	$L_2(3)$	2

For  $n$  a positive integer and  $q$  a prime power we write  $\mathrm{GL}_n(q)$  for the general linear group of invertible  $n \times n$  matrices with coefficients in the finite field  $\mathbb{F}_q$  with  $q$  elements, and we write  $\mathrm{SL}_n(q)$  for the subgroup consisting of matrices with determinant 1. We write  $\mathrm{PGL}_n(q)$  for the quotient of  $\mathrm{GL}_n(q)$  by its centre but we adopt the ATLAS convention (cf. [90, Section 2.1]) of writing  $L_n(q)$  as a shorthand for  $\mathrm{PSL}_n(q)$  — being the quotient of  $\mathrm{SL}_n(q)$  by its centre — since this group is typically simple. In the case that  $q = 3$  and  $n$  is even the centre of  $\mathrm{SL}_n(q)$  has order 2 and  $\mathrm{SL}_n(q) \simeq 2.L_n(q)$  is a non-trivial double cover of  $L_n(q)$ .

In case  $q = 5$  the centre of  $\mathrm{GL}_n(q)$  is a cyclic group of order 4 and so  $\mathrm{GL}_n(5)$  has a unique central subgroup of order 2. We write  $\mathrm{GL}_n(5)/2$  for the quotient of  $\mathrm{GL}_n(5)$  by this subgroup. In case  $n = 2$  the exceptional isomorphism  $\mathrm{PGL}_2(5) \simeq \mathrm{Sym}_5$  tells us then that  $\mathrm{GL}_2(5)/2$  is a double cover of the symmetric group  $\mathrm{Sym}_5$ . For  $n \geq 4$  there are two isomorphism classes of non-trivial double covers of  $\mathrm{Sym}_n$  with the property that the central subgroup of order 2 is contained in the commutator subgroup of the double cover — these are the so-called *Schur double covers* (cf. [90, Section 4.1, Section 6.7]). The group  $\mathrm{GL}_2(5)/2$  is curious in that its only subgroup of index 2 is a direct product  $2 \times \mathrm{Alt}_5$  and its commutator subgroup is a copy of  $\mathrm{Alt}_5$ , so it is a non-trivial double cover of  $\mathrm{Sym}_5$  that is not isomorphic to either of the Schur double covers.

The symbols  $\mathrm{AGL}_n(q)$  denote the *affine linear group* generated by the natural action of  $\mathrm{GL}_n(q)$  on  $\mathbb{F}_q^n$  and the translations  $x \mapsto x + v$  for  $v \in \mathbb{F}_q^n$ . The group  $\mathrm{AGL}_n(q)$  may be realized as a subgroup of  $\mathrm{GL}_{n+1}(q)$  so in particular  $\mathrm{AGL}_3(2)$  embeds in  $\mathrm{GL}_4(2)$ . The group  $\mathrm{GL}_4(2)$  is unusual amongst the  $\mathrm{GL}_n(2)$  in that it admits a non-trivial double cover  $2.\mathrm{GL}_4(2)$ , which is a manifestation of the exceptional isomorphism  $\mathrm{GL}_4(2) \simeq \mathrm{Alt}_8$ . There are two conjugacy classes of subgroups of  $2.\mathrm{Alt}_8$  of order 2688, which is twice the order of  $\mathrm{AGL}_3(2)$ . The groups in both classes are isomorphic to a particular non-trivial double cover of  $\mathrm{AGL}_3(2)$  and this is the group we denote  $2.\mathrm{AGL}_3(2)$  in Table 1.

The symbols  $M_{24}$  and  $M_{12}$  denote the sporadic simple Mathieu groups attached to the binary and ternary Golay codes, respectively. The group  $M_{24}$  is the automorphism group of the binary Golay code, which is the unique self-dual linear binary code of length 24 with minimum weight 8 (cf. [91]). The ternary Golay code is the unique self-dual linear ternary code of length 12 with minimum weight 6 (cf. [91]) and its automorphism group is a non-trivial double cover of  $M_{12}$ . There is a unique such group up to isomorphism (cf. [90]) which we denote by  $2.M_{12}$  in Table 1.

Observe that for  $\ell > 2$  the group  $G^{(\ell)}$  has a unique central subgroup of order 2. Then  $\bar{G}^{(\ell)}$  may be described for  $\ell > 2$  by  $\bar{G}^{(\ell)} = G^{(\ell)}/2$  and  $G^{(\ell)}$  is a non-trivial double cover of  $\bar{G}^{(\ell)}$  for each  $\ell > 2$ . It will develop in Section 3.2 that  $\bar{G}^{(\ell)}$  is a permutation group of degree  $n = 24/(\ell - 1)$  for each  $\ell \in \Lambda$ . Since  $G^{(2)} \simeq M_{24}$  has trivial centre and is already a permutation group of degree  $24 = 24/(2 - 1)$  we set  $\bar{G}^{(2)} = G^{(2)}$ .

Generalizing the notation used above for double covers we write  $G \simeq N.H$  to indicate that  $G$  fits into a short exact sequence

$$(3.1) \quad 1 \rightarrow N \rightarrow G \rightarrow H \rightarrow 1$$

for some groups  $N$  and  $H$ . We write  $G \simeq N:H$  for a group of the form  $N.H$  for which the sequence (3.1) splits (i.e., in case  $G$  is a semi-direct product  $N \rtimes H$ ). Then we have  $AGL_3(2) \simeq 2^3:L_2(7)$  according to the exceptional isomorphism  $L_3(2) \simeq L_2(7)$  where  $2^3$  denotes an elementary abelian group of order 8.

### 3.2. Signed permutations

The group of *signed permutations of degree  $n$*  or *hyperoctahedral group of degree  $n$* , denoted here by  $\text{Oct}_n$ , is a semi-direct product  $2^n:\text{Sym}_n$  where  $2^n$  denotes an elementary abelian group of order  $2^n$ . We may realize it explicitly as the subgroup of invertible linear transformations of an  $n$ -dimensional vector space generated by sign changes and permutations in a chosen basis. (This recovers  $2^n:\text{Sym}_n$  so long as the vector space is defined over a field of characteristic other than 2. In case the characteristic is 2 the sign changes are trivial and we recover  $\text{Sym}_n$ .) When we write  $\text{Oct}_n$  we usually have in mind the data of fixed subgroups  $N \simeq 2^n$  and  $H \simeq \text{Sym}_n$  such that  $N$  is normal and  $\text{Oct}_n/N \simeq H$ , such as are given by sign changes and coordinate permutations, respectively, when we realize  $\text{Oct}_n$  explicitly as described above. In what follows, we assume such data to be chosen for each  $n$  and write  $\text{Oct}_n \rightarrow \text{Sym}_n$  for the composition  $\text{Oct}_n \rightarrow \text{Oct}_n/N \simeq \text{Sym}_n$ .

We will show in Section 3.4 that each  $G^{(\ell)}$  for  $\ell \in \Lambda$  admits a realization as a subgroup of  $\text{Oct}_n$  for  $n = 24/(\ell - 1)$  such that the image of  $G^{(\ell)}$  under the map  $\text{Oct}_n \rightarrow \text{Sym}_n$  is  $\bar{G}^{(\ell)}$ .

In Section 3.4 we use the following modification of the usual cycle notation for permutations to denote elements of  $\text{Oct}_n$ . Suppose  $\Omega$  is a set with  $n$  elements and  $V$  is the vector space generated over a field  $\mathbf{k}$  by the symbols  $\{e_x\}_{x \in \Omega}$ . Whereas the juxtaposition  $xy$  occurring in a cycle  $(\dots xy \dots)$

indicates a coordinate permutation mapping  $e_x$  to  $e_y$ , we write  $x\bar{y}$  to indicate that a sign change is applied so that  $e_x$  is mapped to  $-e_y$ . Then, for example, if  $\Omega = \{\infty, 0, 1, 2, 3, 4\}$  we write  $\sigma = (\infty 2\bar{3}4)(\bar{0})$  for the element of  $\text{GL}(V)$  determined by

$$(3.2) \quad \sigma : e_\infty \mapsto e_2, \quad e_2 \mapsto -e_3, \quad e_3 \mapsto e_4, \quad e_4 \mapsto -e_\infty, \quad e_0 \mapsto -e_0.$$

(We think of the bar over the 3 in  $(\dots 2\bar{3}4 \dots)$  as ‘‘occurring between’’ the 2 and 3.) We call the symbol 1 a *fixed point* of  $\sigma$  and we call 0 an *anti-fixed point*.

We define the *signed permutation character* of  $\text{Oct}_n$  by setting

$$(3.3) \quad \begin{aligned} \chi : \text{Oct}_n &\rightarrow \mathbb{Z} \\ g &\mapsto \chi_g = h_{g,+} - h_{g,-}, \end{aligned}$$

where  $h_{g,+}$  is the number of fixed points of  $g$  and  $h_{g,-}$  is the number of anti-fixed points. Then  $\chi$  is just the character of the ordinary representation of  $\text{Oct}_n$  furnished by  $V$  that obtains when the field  $\mathbf{k}$  is taken to be  $\mathbb{C}$ . Writing  $\bar{\chi}$  for the composition of  $\text{Oct}_n \rightarrow \text{Sym}_n$  with the usual permutation character of  $\text{Sym}_n$  we have

$$(3.4) \quad \begin{aligned} \bar{\chi} : \text{Oct}_n &\rightarrow \mathbb{Z} \\ g &\mapsto \bar{\chi}_g = h_{g,+} + h_{g,-} \end{aligned}$$

and we call  $\bar{\chi}$  the *unsigned permutation character* of  $\text{Oct}_n$ . Then the signed and unsigned permutation characters  $\chi$  and  $\bar{\chi}$  together encode the number of fixed and anti-fixed points for each  $g \in \text{Oct}_n$ .

Let us take  $\mathbf{k} = \mathbb{C}$  in the realization of  $\text{Oct}_n$  as a subgroup of  $\text{GL}(V)$ . Then to each element  $g \in \text{Oct}_n$  we assign a *signed permutation Frame shape*  $\Pi_g$  which encodes the eigenvalues of the corresponding (necessarily diagonalizable) linear transformation of  $V$  in the following way. An expression

$$(3.5) \quad \Pi_g = \prod_{k \geq 1} k^{m_g(k)}$$

with  $m_g(k) \in \mathbb{Z}$  and  $m_g(k) = 0$  for all but finitely many  $k$  indicates that  $g$  defines a linear transformation with  $m_g(k)$  eigenvalues equal to  $e(j/k)$  for each  $0 \leq j < k$  in the case that all the  $m_g(k)$  are non-negative. If some  $m_g(k)$  are negative then we find the number of eigenvalues equal to  $\xi$  say by looking at how many copies of  $\xi$  are present in  $\Pi_g^+ = \prod_{k \geq 1, m_g(k) > 0} k^{m_g(k)}$  and subtracting the number of copies indicated by  $\Pi_g^- = \prod_{k \geq 1, m_g(k) < 0} k^{-m_g(k)}$ .



Observe that the signed permutation character of  $\text{Oct}_n$  is recovered from the signed permutation Frame shapes via  $\chi_g = m_g(1)$ . Also, the Frame shape is invariant under conjugacy.

The map  $\text{Oct}_n \rightarrow \text{Sym}_n$  (which may be realized by “ignoring” sign changes) furnishes a (non-faithful) permutation representation of  $\text{Oct}_n$  on  $n$  points; we call it the *unsigned permutation representation* of  $\text{Oct}_n$ . We write  $g \mapsto \bar{\Pi}_g$  for the corresponding cycle shapes and call them the *unsigned permutation Frame shapes* attached to  $\text{Oct}_n$ . Observe that the unsigned permutation character of  $\text{Oct}_n$  is recovered from the unsigned permutation Frame shapes via  $\bar{\chi}_g = \bar{m}_g(1)$  when  $\bar{\Pi}_g = \prod_{k \geq 1} k^{\bar{m}_g(k)}$ .

Observe that we obtain a faithful permutation representation  $\text{Oct}_n \rightarrow \text{Sym}_{2n}$  by regarding  $g \in \text{Oct}_n$  as a permutation of the  $2n$  points  $\{\pm e_x\}_{x \in \Omega}$ . We denote the corresponding cycle shapes  $\tilde{\Pi}_g$  and call them the *total permutation Frame shapes* attached to  $\text{Oct}_n$ . Then the total permutation Frame shapes can be recovered from the signed and unsigned permutation Frame shapes for if we define a formal product on Frame shapes by setting

$$(3.6) \quad \Pi \Pi' = \prod_{k \geq 1} k^{m(k)+m'(k)}$$

in case  $\Pi = \prod_{k \geq 1} k^{m(k)}$  and  $\Pi' = \prod_{k \geq 1} k^{m'(k)}$  then we have  $\tilde{\Pi}_g = \Pi_g \bar{\Pi}_g$  for all  $g \in \text{Oct}_n$ .

Suppose that  $G$  is a subgroup of  $\text{Oct}_n$  with the property that the intersection  $G \cap N$  has order 2. (This will be the case for  $G = G^{(\ell)}$  when  $\ell \in \Lambda$  and  $\ell \neq 2$ .) Let  $z$  be the unique and necessarily central involution in  $G \cap N$  and write  $\bar{G}$  for the image of  $G$  under the map  $\text{Oct}_n \rightarrow \text{Sym}_n$ . Then the conjugacy class of  $zg$  depends only on the conjugacy class of  $g$  and we obtain an involutory map  $[g] \mapsto [zg]$  on the conjugacy classes of  $G$ . We call  $[g]$  and  $[zg]$  *paired* conjugacy classes and we say that  $[g]$  is *self-paired* if  $[g] = [zg]$ . Observe that  $z$  must act as  $-1$  times the identity in any faithful irreducible representation of  $G$  so if  $g \mapsto \chi(g)$  is the character of such a representation then  $\chi(zg) = -\chi(g)$ . In particular,  $\chi(g) = 0$  if  $g$  is self-paired.

### 3.3. Realization part I

In this section, we construct the groups  $\bar{G}^{(\ell)}$  as subgroups of  $\text{Sym}_n$  where  $n = 24/(\ell - 1)$ . This construction will be nested in the sense that each group  $\bar{G}^{(\ell)}$  will be realized as a subgroup of some  $\bar{G}^{(\ell')}$  for  $\ell' - 1$  a divisor of  $\ell - 1$ .

For  $\ell = 2$  let  $\Omega^{(2)}$  be a set with 24 elements and let  $\mathcal{G} \subset \mathcal{P}(\Omega^{(2)})$  be a copy of the Golay code on  $\Omega^{(2)}$ . Then the subgroup of the symmetric group

$\text{Sym}_{\Omega^{(2)}}$  of permutations of the set  $\Omega^{(2)}$  that preserves  $\mathcal{G}$  is isomorphic to  $M_{24}$ . So we may set

$$\bar{G}^{(2)} = \{\sigma \in \text{Sym}_{\Omega^{(2)}} \mid \sigma(C) \in \mathcal{G}, \forall C \in \mathcal{G}\}.$$

For  $\ell = 3$  the largest proper divisor of  $\ell - 1 = 2$  is  $1 = 2 - 1$ . Choose a partition of  $\Omega^{(2)}$  into  $2/1 = 2$  subsets of  $24/2 = 12$  elements such that each 12-element set belongs to  $\mathcal{G}$  and denote it  $P^{(3)} = \{\Omega^{(3)}, \Omega^{(3)'}\}$ . Equivalently, take  $\Omega^{(3)}$  to be the symmetric difference of the sets of fixed-points of elements  $\sigma, \sigma' \in \bar{G}^{(2)}$  of cycle shape  $1^8 2^8$  such that  $\sigma\sigma'$  has order 6. (That is,  $\Omega^{(3)}$  consists of the points that are fixed by  $\sigma$  or  $\sigma'$  but not both. It turns out that if the product  $\sigma\sigma'$  has order 6 then its cycle shape is  $1^2 2^2 3^2 6^2$  and so the fixed-point sets of  $\sigma$  and  $\sigma'$  have intersection of size 2 and  $\Omega^{(3)}$  and  $\Omega^{(3)'} = \Omega^{(2)} \setminus \Omega^{(3)}$  both have 12 elements.) Then the group of permutations of  $\Omega^{(3)}$  induced by the subgroup of  $\bar{G}^{(2)}$  that fixes this partition is a copy of  $\bar{G}^{(3)} \simeq M_{12}$ . Explicitly, and to explain what we mean here by fix, if  $\bar{G}_{P^{(3)}}^{(2)}$  denotes the subgroup of  $\bar{G}^{(2)}$  that fixes the partition  $P^{(3)}$  in the sense that

$$(3.7) \quad \bar{G}_{P^{(3)}}^{(2)} = \left\{ \sigma \in \bar{G}^{(2)} \mid \sigma(C) = C, \forall C \in P^{(3)} \right\},$$

and if  $\varrho : \bar{G}_{P^{(3)}}^{(2)} \rightarrow \text{Sym}_{\Omega^{(3)}}$  denotes the natural map obtained by restricting elements of  $\bar{G}_{P^{(3)}}^{(2)}$  to  $\Omega^{(3)}$ ,

$$(3.8) \quad \begin{aligned} \varrho : \bar{G}_{P^{(3)}}^{(2)} &\rightarrow \text{Sym}_{\Omega^{(3)}} \\ \sigma &\mapsto \sigma|_{\Omega^{(3)}}, \end{aligned}$$

then  $\bar{G}^{(3)}$  is the image of  $\varrho$ . The kernel of this map is the subgroup

$$(3.9) \quad \bar{G}_{P^{(3)}, \Omega^{(3)}}^{(2)} = \left\{ \sigma \in \bar{G}_{P^{(3)}}^{(2)} \mid \sigma(x) = x, \forall x \in \Omega^{(3)} \right\}$$

comprised of permutations in  $\bar{G}_{P^{(3)}}^{(2)}$  that fix every element of  $\Omega^{(3)}$  so we have

$$\bar{G}^{(3)} = \varrho \left( \bar{G}_{P^{(3)}}^{(2)} \right) \simeq \bar{G}_{P^{(3)}}^{(2)} / \bar{G}_{P^{(3)}, \Omega^{(3)}}^{(2)} \simeq M_{12}.$$

For  $\ell = 4$  the largest proper divisor of  $\ell - 1 = 3$  is  $1 = 2 - 1$ . Choose a partition of  $\Omega^{(2)}$  into  $3/1 = 3$  subsets of  $24/3 = 8$  elements that belong to  $\mathcal{G}$  and denote it  $P^{(4)} = \{\Omega^{(4)}, \Omega^{(4)'}, \Omega^{(4)''}\}$ . Equivalently, choose  $\Omega^{(4)}$  and  $\Omega^{(4)'}$  to be the fixed-point sets of respective elements  $\sigma$  and  $\sigma'$  of order 2 in  $\bar{G}^{(2)}$  having cycle shapes  $1^8 2^8$  and disjoint fixed-point sets and the

property that  $\sigma\sigma'$  has order 3. Then the group of permutations of  $\Omega^{(4)}$  induced by the subgroup of  $\bar{G}^{(2)}$  that fixes this partition is a copy of the group  $\bar{G}^{(4)} \simeq AGL_3(2)$  and we have

$$\bar{G}^{(4)} = \varrho \left( \bar{G}_{P^{(4)}}^{(2)} \right) \simeq \bar{G}_{P^{(4)}}^{(2)} / \bar{G}_{P^{(4)}, \Omega^{(4)}}^{(2)} \simeq AGL_3(2),$$

where  $\bar{G}_{P^{(4)}}^{(2)}$ , the map  $\varrho$  and the subgroup  $\bar{G}_{P^{(4)}, \Omega^{(4)}}^{(2)}$  are defined according to the natural analogues of (3.7)–(3.9), respectively.

For  $\ell = 5$  the largest proper divisor of  $\ell - 1 = 4$  is  $2 = 3 - 1$ . Choose a partition  $P^{(5)} = \{\Omega^{(5)}, \Omega^{(5)'}\}$  of  $\Omega^{(3)}$  into  $4/2 = 2$  subsets of  $12/2 = 6$  elements such that neither of the sets in  $P^{(5)}$  is contained in a set of size 8 in  $\mathcal{G}$ . Equivalently, take  $\Omega^{(5)}$  to be the symmetric difference of the sets of fixed-points of elements  $\sigma, \sigma' \in \bar{G}^{(3)}$  of cycle shape  $1^4 2^4$  such that  $\sigma\sigma'$  has order 6. (This condition forces the fixed-point sets of  $\sigma$  and  $\sigma'$  to have a single point of intersection so that  $\Omega^{(5)}$  and  $\Omega^{(5)'} = \Omega^{(3)} \setminus \Omega^{(5)}$  both have 6 elements.) Then the group of permutations of  $\Omega^{(5)}$  induced by the subgroup of  $\bar{G}^{(3)}$  that fixes this partition is a copy of  $\bar{G}^{(5)} \simeq PGL_2(5)$ .

$$\bar{G}^{(5)} = \varrho \left( \bar{G}_{P^{(5)}}^{(3)} \right) \simeq \bar{G}_{P^{(5)}}^{(3)} / \bar{G}_{P^{(5)}, \Omega^{(5)}}^{(3)} \simeq PGL_2(5)$$

For  $\ell = 7$  the largest proper divisor of  $\ell - 1 = 6$  is  $3 = 4 - 1$ . Choose a partition  $P^{(7)} = \{\Omega^{(7)}, \Omega^{(7)'}\}$  of  $\Omega^{(4)}$  into  $6/3 = 2$  subsets of  $8/2 = 4$  elements such that  $\Omega^{(7)}$  (and therefore also  $\Omega^{(7)'} = \Omega^{(4)} \setminus \Omega^{(7)}$ ) is the set of fixed points of an element of order 2 in  $\bar{G}^{(4)}$  having cycle shape  $1^4 2^2$ . (There are three conjugacy classes of elements of order 2 in  $\bar{G}^{(4)}$  but for just one of these classes do the elements have the cycle shape  $1^4 2^2$ .) Then the group of even permutations of  $\Omega^{(7)}$  induced by the subgroup of  $\bar{G}^{(4)}$  that fixes this partition is a copy of the group  $\bar{G}^{(7)} \simeq L_2(3)$ , which is of course isomorphic to the alternating group on four points.

$$\bar{G}^{(7)} = \varrho \left( \bar{G}_{P^{(7)}}^{(4)} \right) \cap Alt_{\Omega^{(7)}} \simeq L_2(3).$$

(The quotient  $\bar{G}_{P^{(7)}}^{(4)} / \bar{G}_{P^{(7)}, \Omega^{(7)}}^{(4)}$  is isomorphic to the full symmetric group on 4 points so the restriction to even permutations is not redundant.)

The next largest divisor of  $7 - 1 = 6$  is  $2 = 3 - 1$ . An alternative construction of  $\bar{G}^{(7)}$  is obtained by choosing a partition  $P^{(7)} = \{\Omega^{(7)}, \Omega^{(7)'}, \Omega^{(7)''}\}$  of  $\Omega^{(3)}$  into  $6/2 = 3$  subsets of  $12/3 = 4$  elements such that  $\Omega^{(7)}$  and  $\Omega^{(7)'}$  are the fixed-point sets of respective elements  $\sigma$  and  $\sigma'$  of order 2 in  $\bar{G}^{(3)}$  having cycle shapes  $1^4 2^4$  and disjoint fixed-point sets and the property that  $\sigma\sigma'$  has

order 3. Then the group of permutations of  $\Omega^{(7)}$  induced by the subgroup of  $\bar{G}^{(3)}$  that fixes this partition is again a copy of the group  $\bar{G}^{(7)} \simeq L_2(3)$ :

$$\bar{G}^{(7)} = \varrho \left( \bar{G}_{P^{(7)}}^{(3)} \right) \simeq \bar{G}_{P^{(7)}}^{(3)} / \bar{G}_{P^{(7)}, \Omega^{(7)}}^{(3)} \simeq L_2(3).$$

For  $\ell = 13$  the largest proper divisor of  $\ell - 1 = 12$  is  $6 = 7 - 1$ . Choose a partition  $P^{(13)} = \{\Omega^{(13)}, \Omega^{(13)'}\}$  of  $\Omega^{(7)}$  into  $12/6 = 2$  subsets of  $4/2 = 2$  elements such that  $\Omega^{(13)}$  is the orbit of an element of order 2 in  $\bar{G}^{(7)}$ . (That is, choose any partition of  $\Omega^{(7)}$  into subsets of size 2. The elements of order 2 in  $\bar{G}^{(7)}$  all have cycle shape  $2^2$ .) Then the group of permutations of  $\Omega^{(13)}$  induced by the subgroup of  $\bar{G}^{(7)}$  that fixes this partition is a copy of the group  $\bar{G}^{(13)} \simeq \text{Sym}_2$ :

$$\bar{G}^{(13)} = \varrho \left( \bar{G}_{P^{(13)}}^{(7)} \right) \simeq \bar{G}_{P^{(13)}}^{(7)} / \bar{G}_{P^{(13)}, \Omega^{(13)}}^{(7)} \simeq \text{Sym}_2.$$

The next largest divisor of  $13 - 1 = 12$  is  $4 = 5 - 1$ . An alternative construction of  $\bar{G}^{(13)}$  is obtained by choosing a partition  $P^{(13)} = \{\Omega^{(13)}, \Omega^{(13)'}, \Omega^{(13)''}\}$  of  $\Omega^{(5)}$  into  $12/4 = 3$  subsets of  $6/3 = 2$  elements such that  $\Omega^{(13)}$  and  $\Omega^{(13)'}$  are the fixed-point sets of respective elements  $\sigma$  and  $\sigma'$  of order 2 in  $\bar{G}^{(3)}$  having cycle shapes  $1^2 2^2$  and disjoint fixed-point sets and the property that  $\sigma\sigma'$  has order 3. Then the group of permutations of  $\Omega^{(13)}$  induced by the subgroup of  $\bar{G}^{(5)}$  that fixes this partition is a copy of the group  $\bar{G}^{(13)} \simeq \text{Sym}_2$ .

$$\bar{G}^{(13)} = \varrho \left( \bar{G}_{P^{(13)}}^{(5)} \right) \simeq \bar{G}_{P^{(13)}}^{(5)} / \bar{G}_{P^{(13)}, \Omega^{(13)}}^{(5)} \simeq \text{Sym}_2.$$

**Remark 3.1.** The group  $\bar{G}_{P^{(\ell)}, \Omega^{(\ell)}}^{(\ell')}$  is trivial in every instance except for when  $(\ell', \ell)$  is  $(2, 4)$  or  $(4, 7)$  so apart from these cases we have  $\bar{G}^{(\ell)} \simeq \bar{G}_{P^{(\ell)}}^{(\ell')}$ . The group  $\bar{G}_{P^{(4)}, \Omega^{(4)}}^{(2)}$  has order 8 and  $\bar{G}_{P^{(7)}, \Omega^{(7)}}^{(4)}$  has order 4.

**Remark 3.2.** A *Steiner system* with parameters  $(t, k, n)$  is the data of an  $n$ -element set  $\Omega$  together with a collection of subsets of  $\Omega$  of size  $k$ , called the *blocks* of the system, with the property that any  $t$ -element subset of  $\Omega$  is contained in a unique block. It is well known that  $M_{24} \simeq \bar{G}^{(2)}$  may be realized as the automorphism group of a Steiner system with parameters  $(5, 8, 24)$  and  $M_{12} \simeq \bar{G}^{(3)}$  may be described as the automorphism group of a Steiner system with parameters  $(5, 6, 12)$ . Indeed, for  $\bar{G}^{(2)} \simeq M_{24}$  we may take the blocks to be the  $\binom{24}{5} / \binom{8}{5} = 759$  sets of size 8 in a copy  $\mathcal{G}$  of the Golay code on the 24 element set  $\Omega = \Omega^{(2)}$  and for  $\bar{G}^{(3)} \simeq M_{12}$  regarded as the subgroup

of  $\bar{G}^{(2)}$  preserving a set  $\Omega^{(3)}$  of size 12 in  $\mathcal{G}$  we may take the blocks to be the  $\binom{12}{5}/\binom{6}{5} = 132$  subsets of size 6 that arise as an intersection  $C \cap \Omega^{(3)}$  for  $C \in \mathcal{G}$  having size 8. The group  $\bar{G}^{(4)} \simeq AGL_3(2)$  also admits such a description: it is the automorphism group of the unique Steiner system with parameters  $(3, 4, 8)$ . The blocks of such a system constitute the  $\binom{8}{3}/\binom{4}{3} = 14$  codewords of weight 4 in the unique doubly even linear binary code of length 8, the length 8 *Hamming code*.

**Remark 3.3.** Following [92, Ch. 11] let  $n$  be a divisor of 12, take  $n$  cards labelled by 0 through  $n - 1$  and consider the group  $M_n < \text{Sym}_n$  generated by the *reverse shuffle*  $r_n : t \mapsto n - 1 - t$  and the *Mongean shuffle*  $s_n : t \mapsto \min\{2t, 2n - 1 - 2t\}$ . Then the notation is consistent with that used above, for the group  $M_{12}$  just defined is indeed isomorphic to the Mathieu group of permutations on 12 points. So we have  $M_{12} \simeq \bar{G}^{(3)}$  according to Table 1 and it turns out that in fact  $M_n \simeq \bar{G}^{(\ell)}$  whenever  $n = 24/(\ell - 1)$ , and so this shuffle construction recovers all of the  $\bar{G}^{(\ell)}$  except for  $G^{(2)} \simeq M_{24}$  and  $G^{(4)} \simeq AGL_3(2)$  (i.e., all  $\bar{G}^{(\ell)}$  for  $\ell$  odd).

### 3.4. Realization part II

Having constructed the  $\bar{G}^{(\ell)}$  as permutation groups we now describe the Umbral groups  $G^{(\ell)}$  as groups of signed permutations. We retain the notation of Section 3.3.

To construct  $G^{(3)}$  choose an element  $\tau$  of order 11 in  $\bar{G}^{(3)}$  and (re)label the elements of  $\Omega^{(3)}$  so that

$$\Omega^{(3)} = \{\infty, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, X\}$$

and the action of  $\tau$  on  $\Omega^{(3)}$  is given by  $x \mapsto x + 1$  modulo 11 (where  $X = 10$ ). Then for any set  $C \subset \Omega^{(3)}$  of size 4 there is a unique element  $\sigma \in \bar{G}^{(3)}$  with cycle shape  $1^4 2^4$  such that the fixed-point set of  $\sigma$  is precisely  $C$ . In case  $C = \{0, 1, 4, 9\}$  for example  $\sigma = (\infty 6)(2X)(35)(78)$  and  $\bar{G}^{(3)}$  is generated by  $\sigma$  and  $\tau$ . Let  $\hat{\Omega}^{(3)} = \{e_x \mid x \in \Omega^{(3)}\}$  be a basis for a 12-dimensional vector space over  $\mathbb{C}$  say, let  $\hat{\tau}$  be the element of the hyperoctahedral group  $\text{Oct}_{\hat{\Omega}^{(3)}} \simeq \text{Oct}_{12}$  given by  $\hat{\tau} : e_x \mapsto e_{x+1}$ , so that  $\hat{\tau} = (0123456789X)$  in (signed) cycle notation (cf. Section 3.2), and set  $\hat{\sigma} = (\infty 6)(\bar{2}\bar{X})(35)(\bar{7}\bar{8})$ . Then for  $G^{(3)}$  the group generated by  $\hat{\sigma}$  and  $\hat{\tau}$  we have that  $G^{(3)} \simeq 2.M_{12}$  and the image of  $G^{(3)}$  under the map  $\text{Oct}_{\hat{\Omega}^{(3)}} \rightarrow S_{\Omega^{(3)}}$  is  $\bar{G}^{(3)}$ .

To construct  $G^{(4)}$  choose an element  $\tau$  of order 7 in  $\bar{G}^{(4)}$  and (re)label the elements of  $\Omega^{(4)}$  so that

$$\Omega^{(4)} = \{\infty, 0, 1, 2, 3, 4, 5, 6\}$$

and the action of  $\tau$  on  $\Omega^{(4)}$  is given by  $x \mapsto x + 1$  modulo 7. There is a unique conjugacy class of elements of  $\bar{G}^{(4)}$  having cycle shape  $1^4 2^2$  and the fixed-point sets of these elements are the 14 subsets of  $\Omega^{(4)}$  of size 4 comprising the blocks of a  $(3, 4, 8)$  Steiner system preserved by  $\bar{G}^{(4)}$  (cf. Remark 3.2). The particular block  $C = \{0, 1, 3, 6\}$  is the unique one with the property that  $\sigma$  and  $\tau$  generate  $\bar{G}^{(4)}$  in case  $\sigma$  is any (of the 3) involution(s) with  $C$  as its fixed-point set. Take  $\sigma = (\infty 5)(24)$  and let  $\hat{\Omega}^{(4)} = \{e_x \mid x \in \Omega^{(4)}\}$  be a basis for an 8-dimensional vector space over  $\mathbb{C}$  say, let  $\hat{\tau}$  be the element of the hyperoctahedral group  $\text{Oct}_{\hat{\Omega}^{(4)}} \simeq \text{Oct}_8$  given by  $\hat{\tau} : e_x \mapsto e_{x+1}$ , so that  $\hat{\tau} = (0123456)$ , and set  $\hat{\sigma} = (\infty 5)(\bar{0})(\bar{1})(24)$ . Then for  $G^{(4)}$  the group generated by  $\hat{\sigma}$  and  $\hat{\tau}$  we have that  $G^{(4)} \simeq 2.AGL_3(2)$  and the image of  $G^{(4)}$  under the map  $\text{Oct}_{\hat{\Omega}^{(4)}} \rightarrow S_{\Omega^{(4)}}$  is  $\bar{G}^{(4)}$ .

For the remaining  $\ell \geq 5$  we can construct  $G^{(\ell)}$  from  $G^{(3)}$  by proceeding in analogy with the constructions of  $\bar{G}^{(\ell)}$  from  $\bar{G}^{(3)}$  (and its subgroups) given earlier in this section. Concretely, for  $\ell = 5$  we may set

$$G^{(5)} = \hat{\varrho} \left( G_{\hat{P}^{(5)}}^{(3)} \right) \simeq G_{\hat{P}^{(5)}}^{(3)} / G_{\hat{P}^{(5)}, \hat{\Omega}^{(5)}}^{(3)} \simeq \text{GL}_2(5)/2,$$

where  $\hat{\Omega}^{(5)} = \{e_x \mid x \in \Omega^{(5)}\}$  is a basis for  $24/(5 - 1) = 6$ -dimensional vector space over  $\mathbb{C}$ , we define  $\hat{C} = \{e_x \mid x \in C\}$  for  $C \subset \Omega^{(5)}$  and set  $\hat{P}^{(5)} = \{\hat{\Omega}^{(5)}, \hat{\Omega}^{(5)'}\}$  for  $P^{(5)} = \{\Omega^{(5)}, \Omega^{(5)'}\}$  as in the construction of  $\bar{G}^{(5)}$  given above, the groups  $G_{\hat{P}^{(5)}}^{(3)}$  and  $G_{\hat{P}^{(5)}, \hat{\Omega}^{(5)}}^{(3)}$  are defined by

$$(3.10) \quad G_{\hat{P}^{(5)}}^{(3)} = \left\{ \sigma \in G^{(3)} \mid \sigma(\text{Span } \hat{C}) \subset \text{Span } \hat{C}, \forall \hat{C} \in \hat{P}^{(5)} \right\},$$

$$(3.11) \quad G_{\hat{P}^{(5)}, \hat{\Omega}^{(5)}}^{(3)} = \left\{ \sigma \in G_{\hat{P}^{(5)}}^{(3)} \mid \sigma(e_x) = e_x, \forall x \in \Omega^{(5)} \right\}$$

and  $\hat{\varrho}$  is the map  $G_{\hat{P}^{(5)}}^{(3)} \rightarrow \text{Oct}_{\hat{\Omega}^{(5)}}$  obtained by restriction.

$$(3.12) \quad \begin{aligned} \hat{\varrho} : G_{\hat{P}^{(5)}}^{(3)} &\rightarrow \text{Oct}_{\hat{\Omega}^{(5)}} \\ \sigma &\mapsto \sigma|_{\text{Span } \hat{\Omega}^{(5)}} \end{aligned}$$

In direct analogy with this we have that

$$G^{(7)} = \hat{\rho} \left( G_{\hat{P}^{(7)}}^{(3)} \right) \simeq G_{\hat{P}^{(7)}}^{(3)} / G_{\hat{P}^{(7)}, \hat{\Omega}^{(7)}}^{(3)} \simeq SL_2(3),$$

$$G^{(13)} = \hat{\rho} \left( G_{\hat{P}^{(13)}}^{(5)} \right) \simeq G_{\hat{P}^{(13)}}^{(5)} / G_{\hat{P}^{(13)}, \hat{\Omega}^{(13)}}^{(5)} \simeq 4,$$

when  $G_{\hat{P}^{(\ell)}}^{(\ell')}$  and  $G_{\hat{P}^{(\ell)}, \hat{\Omega}^{(\ell')}}^{(\ell')}$  are defined as in (3.10) and (3.11), respectively, for  $P^{(\ell)} = \{\Omega^{(\ell)}, \dots\}$  as specified in the construction of  $\bar{G}^{(\ell')}$  given above, and  $\hat{\rho}$  as in (3.12).

We conclude this section with explicit signed permutation presentations of the  $G^{(\ell)}$  as subgroups of the  $\text{Oct}_{\hat{\Omega}^{(\ell)}}$  for  $\ell \geq 3$ . The presentations for  $\ell = 3$  and  $\ell = 4$  were obtained above, and the remaining ones can be found in a similar manner. In each case we label the index set  $\Omega^{(\ell)}$  so that  $\Omega^{(\ell)} = \{\infty, 0, \dots, n-1\}$  where  $n = 24/(\ell-1) - 1 = (25-\ell)/(\ell-1)$  and seek a presentation for which the coordinate permutation  $e_x \mapsto e_{x+1}$  (with indices read modulo  $n$ ) is an element of order  $n$  in  $G^{(\ell)}$  (although we must take this element to be trivial in case  $\ell = 13$ ).

$$(3.13) \quad G^{(3)} = \langle (\infty 6)(\bar{2}\bar{X})(35)(\bar{7}\bar{8}), (0123456789X) \rangle,$$

$$(3.14) \quad G^{(4)} = \langle (\infty 5)(\bar{0})(\bar{1})(24), (0123456) \rangle,$$

$$(3.15) \quad G^{(5)} = \langle (\infty \bar{0})(3\bar{1})(2\bar{4}), (01234) \rangle,$$

$$(3.16) \quad G^{(7)} = \langle (\infty \bar{0})(\bar{1}2), (012) \rangle,$$

$$(3.17) \quad G^{(13)} = \langle (\infty \bar{0}) \rangle.$$

Equipped now with explicit realizations  $G^{(\ell)} < \text{Oct}_n$  of the Umbral groups  $G^{(\ell)}$  as signed permutation groups we define symbols  $\Pi_g^{(\ell)}$ ,  $\chi_g^{(\ell)}$ ,  $\bar{\Pi}_g^{(\ell)}$  and  $\bar{\chi}_g^{(\ell)}$  as follows for  $\ell \in \Lambda$  and  $g \in G^{(\ell)}$ . We write  $\Pi_g^{(\ell)}$  for the signed permutation Frame shape attached to  $g \in G^{(\ell)}$  (when regarded as an element of  $\text{Oct}_n$ ) as defined in Section 3.2 and we write  $\bar{\Pi}_g^{(\ell)}$  for the cycle shape attached to the image of  $g \in G^{(\ell)}$  under the composition  $G^{(\ell)} \rightarrow \text{Oct}_n \rightarrow \text{Sym}_n$ . We define  $g \mapsto \chi_g^{(\ell)}$  by restricting the signed permutation character (cf. (3.3)) to  $G^{(\ell)}$  and we define  $g \mapsto \bar{\chi}_g^{(\ell)}$  by restricting the unsigned permutation character (cf. (3.4)) to  $G^{(\ell)}$ . We call  $\chi_g^{(\ell)}$  the *signed twisted Euler character* attached to  $g \in G^{(\ell)}$  and we call  $\bar{\chi}_g^{(\ell)}$  the *unsigned twisted Euler character* attached to  $g \in G^{(\ell)}$ .

As is explained in Section 3.2 the data  $g \mapsto \Pi_g^{(\ell)}$  is sufficient to determine the cycle shapes  $\bar{\Pi}_g^{(\ell)}$  and the twisted Euler characters  $\chi_g^{(\ell)}$  and  $\bar{\chi}_g^{(\ell)}$ . We will attach vector-valued mock modular forms  $H_g^{(\ell)}$  to each  $g \in G^{(\ell)}$  in Section 4

and it will develop that the shadows and multiplier systems of these functions are encoded in the Frame shapes  $\Pi_g^{(\ell)}$ . We list the Frame shapes  $\Pi_g^{(\ell)}$  and  $\bar{\Pi}_g^{(\ell)}$  and the twisted Euler characters  $\chi_g^{(\ell)}$  and  $\bar{\chi}_g^{(\ell)}$  for all  $g \in G^{(\ell)}$  and  $\ell \in \Lambda$  in the tables of B.2.

We conclude this section with an extraordinary property relating the Frame shapes  $\Pi_g^{(\ell)}$  and  $\bar{\Pi}_g^{(\ell)}$  attached to the Umbral groups  $G^{(\ell)}$  at  $\ell = 2$  and  $\ell = 4$  which can be verified explicitly using the tables of B.2.

**Proposition 3.4.** *Let  $g \in G^{(4)}$  and suppose that the Frame shape  $\Pi_g^{(4)} = \prod_k k^{m_g(k)}$  is a cycle shape, so that  $m_g(k) \geq 0$  for all  $k$ . Then there exists  $g' \in G^{(2)}$  with  $o(g') = 2o(g)$  such that*

$$(3.18) \quad \bar{\Pi}_{g'}^{(2)} = \prod_k k^{\bar{m}_g(k)} (2k)^{m_g(k)},$$

when  $\bar{\Pi}_g^{(4)} = \prod_k k^{\bar{m}_g(k)}$  except in case  $g$  is of class  $4B$ .

As will be explained in Section 4 this result implies direct relationships between the functions  $H_{g'}^{(2)}$  and  $H_g^{(4)}$  when  $g$  and  $g'$  are as in the statement of the proposition.

### 3.5. Dynkin diagrams part I

The *McKay correspondence* [93] relates finite subgroups of  $SU(2)$  to the affine Dynkin diagrams of ADE type by associating irreducible representations of the finite groups to nodes of the corresponding diagrams and by now is well understood in terms of resolutions of *simple singularities*  $\mathbb{C}^2/G$  for  $G < SU(2)$  [94, 95]. A more mysterious Dynkin diagram correspondence also due to McKay is his *monstrous  $E_8$  observation* [93] (see also [96, Section 14]) which associates nine of the conjugacy classes of the monster group to the nodes of the affine  $E_8$  Dynkin diagram and extends to similar correspondences relating certain subgroups of the Monster to other affine Dynkin diagrams [97]. We find analogues of both McKay's Dynkin diagram observations manifesting in the groups  $G^{(\ell)}$ , as we shall now explain.

For  $\ell \in \Lambda = \{2, 3, 4, 5, 7, 13\}$  the number  $(25 - \ell)/(\ell - 1)$  is an odd integer  $p$  such that  $p + 1$  divides 24 and is a prime in case  $\ell$  is not 13. Recall the construction of  $\bar{G}^{(\ell)}$  as permutations of  $\Omega^{(\ell)}$  from Section 3.3. By inspection, with the assistance of [98], we obtain the following lemma.

**Lemma 3.5.** *Let  $\ell \in \{2, 3, 4, 5, 7\}$  and set  $p = (25 - \ell)/(\ell - 1)$ . Then the group  $\bar{G}^{(\ell)}$  has a unique conjugacy class of subgroups isomorphic to  $L_2(p)$*



that act transitively on the degree  $p + 1 = 24/(\ell - 1)$  permutation representation of  $\bar{G}^{(\ell)}$  on  $\Omega^{(\ell)}$ .

Armed with Lemma 3.5 we pick a transitive subgroup of  $\bar{G}^{(\ell)}$  isomorphic to  $L_2(p)$  for each  $\ell$  in  $\{2, 3, 4, 5, 7\}$  — these being the cases that  $(25 - \ell)/(\ell - 1)$  is prime — and denote it  $\bar{L}^{(\ell)}$ . For future reference we note that there are two conjugacy classes of subgroups of  $\bar{G}^{(\ell)}$  isomorphic to  $L_2(p)$  in case  $\ell \in \{3, 4\}$  but in each case only one of these classes contains subgroups acting transitively.

**Lemma 3.6.** *For  $\ell \in \{3, 4\}$  and  $p = (25 - \ell)/(\ell - 1)$  there is a unique conjugacy class of subgroups of  $\bar{G}^{(\ell)}$  isomorphic to  $L_2(p)$  that does not act transitively in the degree  $p + 1 = 24/(\ell - 1)$  permutation representation of  $\bar{G}^{(\ell)}$  on the set  $\Omega^{(\ell)}$ .*

It is a result due to Galois (proven in a letter to Chevalier written on the eve of his deadly duel [92, Ch. 10]) that the group  $L_2(p)$  has no transitive permutation representation of degree less than  $p + 1$  if  $p > 11$ . However for the remaining primes  $p$  not exceeding 11 there are transitive permutation representations on exactly  $p$  points, and in fact we have the following statement.

**Lemma 3.7.** *Let  $p \in \{2, 3, 5, 7, 11\}$  and set  $\bar{L} = L_2(p)$ . Then there is a subgroup  $\bar{D} < \bar{L}$  of index  $p$  with the property that  $\bar{L} = \langle \sigma \rangle \bar{D}$  for  $\sigma$  an element of order  $p$  in  $\bar{L}$ , so that every element  $g \in \bar{L}$  admits a unique expression  $g = sd$  where  $s \in \langle \sigma \rangle$  and  $d \in \bar{D}$ .*

**Remark 3.8.** See [92, Ch. 10] for applications of the result of Lemma 3.7 to exceptional isomorphisms among finite simple groups of small order, and see [99, 100] for an application of the case that  $p = 11$  to the truncated icosahedron (which describes the structure of buckminsterfullerenes, a.k.a. buckyballs).

According to Lemma 3.7 we obtain a permutation representation of degree  $p$  for  $\bar{L}$  by taking the natural action of  $\bar{L}$  on cosets of  $\bar{D}$  and this action is transitive since  $\sigma$  induces a  $p$ -cycle. Using this result we choose a copy of  $\bar{D}$  in  $\bar{L} = \bar{L}^{(\ell)}$  for each  $\ell \in \{3, 4, 5, 7\}$  and denote it  $\bar{D}^{(\ell)}$ . (Such subgroups are uniquely determined up to isomorphism, but there are as many conjugacy classes of subgroups of  $\bar{L}$  isomorphic to  $\bar{D}$  as there are conjugacy classes of elements of order  $p$  in  $\bar{L}$ , which is to say there are 2 classes in case  $p \in \{11, 7\}$  and a unique class in case  $p \in \{5, 3\}$ .) We describe the groups

Table 2: The Umbral Groups and Dynkin Data.

$\ell$	2	3	4	5	7	13
$p$	23	11	7	5	3	1
$G^{(\ell)}$	$M_{24}$	$2.M_{12}$	$2.AGL_3(2)$	$GL_2(5)/2$	$SL_2(3)$	4
$L^{(\ell)}$		$SL_2(11)$	$SL_2(7)$		$SL_2(3)$	
$D^{(\ell)}$		$2.Alt_5$	$2.Sym_4$		$Q_8$	
$\bar{G}^{(\ell)}$	$M_{24}$	$M_{12}$	$AGL_3(2)$	$PGL_2(5)$	$L_2(3)$	2
$\bar{L}^{(\ell)}$	$L_2(23)$	$L_2(11)$	$L_2(7)$	$L_2(5)$	$L_2(3)$	
$\bar{D}^{(\ell)}$		$Alt_5$	$Sym_4$	$Alt_4$	$2^2$	
$\Delta^{(\ell)}$		$\hat{E}_8$	$\hat{E}_7$	$\hat{E}_6$	$\hat{D}_4$	

$\bar{D}^{(\ell)}$  in Table 2 and observe that each one is isomorphic to a finite group  $\bar{D}_0 < SO_3(\mathbb{R})$ . As such there is a corresponding finite group  $D_0 < SU(2)$  that maps onto  $\bar{D}_0$  via the 2-fold covering  $SU(2) \rightarrow SO_3(\mathbb{R})$ , and a corresponding Dynkin diagram  $\Delta^{(\ell)}$  according to the McKay correspondence. We list the Dynkin diagrams  $\Delta^{(\ell)}$  also in Table 2.

Traditionally finite subgroups of  $SO_3(\mathbb{R})$  are called *ternary*, owing to the fact that their elements are described using  $3 \times 3$  matrices, and finite subgroups of  $SU(2)$  are called *binary*. The map  $SU(2) \rightarrow SO_3(\mathbb{R})$  determines a correspondence between binary and ternary groups whereby the ternary polyhedral groups (of orientation preserving symmetries of regular polyhedra) correspond to binary groups of the form  $2.Alt_5$ ,  $2.Sym_4$  and  $2.Alt_4$  (depending upon the polyhedron), a dihedral subgroup  $Dih_n < SO_3(\mathbb{R})$  corresponds to a generalized quaternion group of order  $4n$  in  $SU(2)$ , and a binary cyclic group corresponds to a ternary cyclic group of the same order. Thus, given a finite subgroup  $\bar{D}_0 < SO_3(\mathbb{R})$ , we may speak of the *associated binary group*  $D_0 < SU(2)$ .

Next we observe that each group  $G^{(\ell)}$  contains a subgroup  $D^{(\ell)}$  isomorphic to the binary group associated to  $\bar{D}^{(\ell)}$  (when  $\bar{D}^{(\ell)}$  is regarded as a subgroup of  $SO_3(\mathbb{R})$ ) for all of the above cases except when  $\ell = p = 5$ . More particularly, for  $\ell \in \{3, 7\}$  there is a unique conjugacy class of subgroups isomorphic to the binary group attached to  $\bar{D}^{(\ell)}$  while for  $\ell = 4$  there are two such conjugacy classes and for  $\ell = 5$  there are none. For each  $\ell \in \{3, 4, 7\}$  we pick a subgroup isomorphic to the binary group attached to  $\bar{D}^{(\ell)}$  and denote it  $D^{(\ell)}$  and we display the (isomorphism types of the) groups  $D^{(\ell)}$  in Table 2. We write  $Q_8$  there for the quaternion group of order 8.

To see how the  $D^{(\ell)}$  arise in  $G^{(\ell)}$  (and fail to do so in the case that  $\ell = 5$ ) recall from Lemma 3.5 that  $\bar{G}^{(\ell)}$  contains a unique transitive subgroup isomorphic to  $L_2(p)$  up to conjugacy for  $\ell \in \{3, 4, 5, 7\}$ . The preimage of such a subgroup under the natural map  $G^{(\ell)} \rightarrow \bar{G}^{(\ell)}$  (cf. Section 3.4) is a double cover of  $L_2(p)$  that is in fact isomorphic to  $SL_2(p)$  (cf. Section 3.1) except when  $\ell = 5$ . In case  $\ell = 5$  we have  $G^{(5)} \simeq \mathrm{GL}_2(5)/2$  which has the same order as  $SL_2(5)$  but is not isomorphic to it. We see then that for  $\ell \in \{3, 4, 7\}$  we may find a copy of  $SL_2(p)$  in the preimage of  $\bar{L}^{(\ell)}$  under the map  $G^{(\ell)} \rightarrow \bar{G}^{(\ell)}$ ; we do so and denote it  $L^{(\ell)}$ . Then we may take  $D^{(\ell)}$  to be the preimage of  $\bar{D}^{(\ell)}$  under the map  $L^{(\ell)} \rightarrow \bar{L}^{(\ell)}$ . The fact that there is no  $SL_2(5)$  in  $G^{(5)}$  explains why there is no group  $D^{(5)}$ .

Observe that the rank of  $\Delta^{(\ell)}$  is given by  $11 - \ell$  for each  $\ell \in \{3, 4, 5, 7\}$ . This may be taken as a hint to the following uniform construction of the  $\Delta^{(\ell)}$ . Starting with the (finite type)  $E_8$  Dynkin diagram, being star shaped with three *branches*, construct a sequence of diagrams iteratively by removing the end node from a branch of maximal length at each iteration. In this way we obtain the sequence  $E_8, E_7, E_6, D_5, D_4, A_3, A_2, A_1$ , and it is striking to observe that our list  $\Delta^{(\ell)}$ , obtained by applying the McKay correspondence to distinguished subgroups of the  $G^{(\ell)}$ , is a subsequence of the one obtained from this by affinization.

We summarize the main observations of this section as follows:

*For  $\ell \in \{3, 4, 5, 7\}$  the group  $\bar{G}^{(\ell)}$  admits a distinguished isomorphism class of subgroups  $\bar{D}^{(\ell)}$ . This connects  $\bar{G}^{(\ell)}$  to a Dynkin diagram of rank  $11 - \ell$ , for the group  $D^{(\ell)}$  is the image in  $SO_3(\mathbb{R})$  of a finite subgroup  $D^{(\ell)}$  in  $SU(2)$  which corresponds to the affinization  $\Delta^{(\ell)}$  of a Dynkin diagram of rank  $11 - \ell$  according to McKay's correspondence. The Dynkin diagrams arising belong to a naturally defined sequence.*

### 3.6. Dynkin diagrams part II

Recall from Lemma 3.6 that the cases  $\ell \in \{3, 4\}$  are distinguished in that for such  $\ell$  the group  $\bar{G}^{(\ell)}$  has a unique conjugacy class of subgroups isomorphic to  $L_2(p)$  and not acting transitively in the degree  $p + 1$  permutation representation (cf. Section 3.3). Since such an  $L_2(p)$  subgroup acts non-trivially on  $\Omega^{(\ell)}$  it follows from Lemma 3.7 that it must have one fixed point and act transitively on the remaining  $p$  points of  $\Omega^{(\ell)}$ , and thus we have witnesses within  $G^{(\ell)}$  to the exceptional degree  $p$  permutation representations of  $L_2(p)$  in case  $\ell \in \{3, 4\}$  and  $p \in \{11, 7\}$ . For these two special cases of lambency

3 and 4 we find direct analogues of McKay’s monstrous  $E_8$  observation [93] attaching certain conjugacy classes of  $G^{(\ell)}$  to the nodes of  $\Delta^{(\ell)}$ . At lambency 5 and 7 (where  $p$  is 5 and 3, respectively) we find similar analogues where the diagram  $\Delta^{(\ell)}$  is replaced by one obtained via folding with respect to a diagram automorphism of order 3.

In case  $\ell = 3$  let  $T$  denote the conjugacy class of elements  $g$  of order 2 in  $G^{(3)} \simeq 2.M_{12}$  such that  $g$  has four fixed points in both the signed and unsigned permutation representations of  $G^{(3)}$ . This is the conjugacy class labelled  $2B$  in Table B.2 and we have  $\chi_g^{(3)} = \bar{\chi}_g^{(3)} = 4$  in the notation of Table B.8. Then  $T^2 = \{gh \mid g, h \in T\}$  is a union of conjugacy classes of  $G^{(3)}$  and in fact there are exactly 9 of the 26 conjugacy classes of  $G^{(3)}$  that appear. In (3.19) we use these classes (and the notation of Table B.2) to label the affine  $E_8$  Dynkin diagram.

$$(3.19) \quad \begin{array}{ccccccccccc} & & & & & & 3B & & & & & & \\ & & & & & & | & & & & & & \\ 1A & \text{---} & 2B & \text{---} & 3A & \text{---} & 4C & \text{---} & 5A & \text{---} & 6C & \text{---} & 4B & \text{---} & 2C \end{array}$$

Observe that the labelling of (3.19) recovers the highest root labelling when the classes are replaced with their orders. In (3.20) we replace the labels of (3.19) with the cycle shapes attached to these classes via the total permutation action (cf. Section 3.2) of  $G^{(3)}$  on the  $24 = 2 \cdot 24 / (\ell - 1)$  elements  $\{\pm e_i \mid i \in \Omega^{(3)}\}$  appearing in the signed permutation representation of  $G^{(3)}$  (cf. Section 3.4).

$$(3.20) \quad \begin{array}{ccccccccccc} & & & & & & 3^8 & & & & & & \\ & & & & & & | & & & & & & \\ 1^{24} & \text{---} & 1^8 2^8 & \text{---} & 1^6 3^6 & \text{---} & 1^4 2^2 4^4 & \text{---} & 1^4 5^4 & \text{---} & 1^2 2^2 3^2 6^2 & \text{---} & 2^4 4^4 & \text{---} & 2^{12} \end{array}$$

Explicitly, the cycle shape attached to  $g \in G^{(3)}$  is the total permutation Frame shape  $\tilde{\Pi}_g^{(3)} = \Pi_g^{(3)} \bar{\Pi}_g^{(3)}$  realized as the product (cf. Section 3.2) of the signed and unsigned permutation Frame shapes attached to  $G^{(3)}$  (cf. Table B.8).

**Remark 3.9.** The conjugacy class labelled  $2C$  in Table B.2 consists of elements of the form  $gz$  where  $g$  belongs to the class  $2B$  (denoted  $T$  above) and  $z$  is the central involution of  $G^{(3)}$ , so we obtain exactly the nine conjugacy classes of (3.19) by considering products  $gh$  where  $g$  and  $h$  are involutions in the class  $2C$ .

**Remark 3.10.** In [101] an analogue of McKay’s monstrous  $E_8$  observation is found for  $M_{24}$ . Namely, it is observed that there are exactly nine conjugacy classes of elements of  $M_{24}$  having representatives of the form  $gh$  where  $g$  and  $h$  belong to the class labelled 2A in Table B.1 — this is the class with cycle shape  $1^8 2^8$  in the defining degree 24 permutation representation — and the nodes of the affine  $E_8$  Dynkin diagram can be labelled by these nine classes in such a way that their orders recover the highest root labelling. This condition leaves some ambiguity as to the correct placement of the two classes of order 2, and similarly for the pairs of orders 3 and 4, but an analogue of the procedure described in [102] is shown to determine a particular choice that recovers the original correspondence for the monster group via the modular functions of monstrous moonshine. Observe that if we express the  $G^{(2)} \simeq M_{24}$  labelling of the affine  $E_8$  diagram given in [101] using cycle shapes to name the conjugacy classes then we recover exactly the labelling (3.20) obtained here from  $G^{(3)} \simeq 2.M_{12}$ .

In case  $\ell = 4$  let  $T$  denote the conjugacy class of elements  $g$  of order 2 in  $G^{(4)}$  such that  $g$  has four fixed points in the unsigned permutation representation of  $G^{(4)}$  but has two fixed points and two anti-fixed points in the signed permutation representation. This is the conjugacy class labelled 2C in Table B.3 and we have  $\chi_g^{(4)} = 0$  and  $\bar{\chi}_g^{(4)} = 4$  in the notation of Table B.9. Then  $T^2$  is the union of eight of the sixteen conjugacy classes of  $G^{(4)}$  and we use these classes (and the notation of Table B.3) to label the nodes of the affine  $E_7$  Dynkin diagram in (3.21):

$$(3.21) \quad \begin{array}{ccccccccc} & & & & 2B & & & & \\ & & & & | & & & & \\ & & & & 4C & & & & \\ 1A & \text{---} & 2C & \text{---} & 3A & \text{---} & 4C & \text{---} & 6A & \text{---} & 4A & \text{---} & 2A \end{array}$$

In (3.22) we replace the labels of (3.21) with the cycle shapes  $\bar{\Pi}_g^{(4)}$  (Table B.9) attached to these classes via the degree 8 permutation action of  $G^{(4)}$  on  $\Omega^{(4)}$ . The orders of these permutations are the orders of the images of the corresponding elements of  $G^{(4)}$  under the map  $G^{(4)} \rightarrow \bar{G}^{(4)}$  so the labelling (3.22) demonstrates that we recover the highest root labelling of the affine  $E_7$  Dynkin diagram when we replace the conjugacy classes of (3.21) with the orders of their images in  $\bar{G}^{(4)}$ :

$$(3.22) \quad \begin{array}{ccccccccc} & & & & 2^4 & & & & \\ & & & & | & & & & \\ & & & & 1^2 2^1 4^1 & & & & \\ 1^8 & \text{---} & 1^4 2^2 & \text{---} & 1^2 3^2 & \text{---} & 1^2 2^1 4^1 & \text{---} & 1^2 3^2 & \text{---} & 2^4 & \text{---} & 1^8 \end{array}$$

In (3.23) we replace the labels of (3.21) with the cycle shapes of degree 24 given by the products  $\prod_g^{(4)} \bar{\Pi}_g^{(4)} \bar{\Pi}_g^{(4)}$  (cf. Table B.9) for  $g$  an element of  $G^{(4)}$  arising in (3.21). Observe that all the cycle shapes in (3.23) are *balanced*, meaning that they are invariant (and well-defined) under the operation  $\prod_{k \geq 1} k^{m(k)} \mapsto \prod_{k \geq 1} (N/k)^{m(k)}$  for some integer  $N > 1$ , and constitute a subset of the cycle shapes attached to  $G^{(3)}$  in (3.20). Observe also that the order two symmetry of the affine  $E_7$  diagram that identifies the two long branches is realized by squaring: the cycle shapes obtained by squaring permutations represented by the cycle shapes on the right-hand branch of (3.23) are just those that appear on the left-hand branch:

$$(3.23) \quad 1^{24} \text{ --- } 1^8 2^8 \text{ --- } 1^6 3^6 \text{ --- } 1^4 2^2 4^4 \text{ --- } 2^{12} \text{ --- } 1^2 2^2 3^2 6^2 \text{ --- } 2^4 4^4 \text{ --- } 1^8 2^8$$

**Remark 3.11.** It is interesting to note that while the conjugacy class  $2C$  is stable under multiplication by the central involution there are just eight conjugacy classes of  $G^{(4)}$  that are contained in  $T^2$  when  $T$  is the class labelled  $4A$  in Table B.3. In fact, the eight conjugacy classes appearing are  $8A$  together with all those of (3.21) except for  $4C$ . Thus we obtain analogues of the labelings (3.21), (3.22) and (3.23) where  $4C$ ,  $1^2 2^1 4^1$  and  $1^4 2^2 4^4$ , respectively, are replaced by  $8A$ ,  $4^2$  and  $4^2 8^2$ , respectively. Under this correspondence the highest root labelling is again obtained by taking orders in  $\bar{G}^{(4)}$ , but while the eta product attached to  $4^2 8^2$  is multiplicative it only has weight 2 (i.e., less than 4) and so does not appear in [101].

**Remark 3.12.** The correspondence of [101] may be viewed as attaching the nine multiplicative eta products of weight at least 4 to the nodes of the affine  $E_8$  Dynkin diagram when we regard a cycle shape  $\prod_{k \geq 1} k^{m(k)}$  as a shorthand for the *eta product* function  $\prod_{k \geq 1} \eta(k\tau)^{m(k)}$  (cf. A.1). Observe that the cycle shapes appearing in (3.23) are just those whose corresponding eta products are multiplicative, have weight at least 4, and have level dividing 24. (The eta products defined by  $1^4 5^4$  and  $3^8$  have level 5 and 9, respectively.)

For  $\ell = 5$  the Dynkin diagram  $\Delta^{(5)} = \hat{E}_6$  admits a  $\text{Sym}_3$  group of automorphisms. Folding by either of the 3-fold symmetries we obtain the affine Dynkin diagram of type  $G_2$ . Let  $T$  be either of the two conjugacy class of elements  $g$  of order 4 in  $G^{(5)}$  such that  $g^2$  is central. Then we have  $\chi_g^{(5)} = \bar{\chi}_g^{(5)} = 0$  in the notation of Table B.10 and  $T$  is either the class labelled  $4A$  in Table B.4 or the class labelled  $4B$ , and there are exactly three of the

fourteen conjugacy classes of  $G^{(5)}$  that are subsets of  $T^2$ . In (3.24) we use these classes (and the notation of Table B.4) to label the affine  $G_2$  Dynkin diagram:

$$(3.24) \quad 2A \text{ --- } 2C \text{ --- } 6A$$

In (3.25) we replace the labels of (3.24) with the cycle shapes  $\bar{\Pi}_g^{(5)}$  (cf. Table B.10) attached to these classes via the degree 6 permutation action of  $G^{(5)}$  on  $\Omega^{(5)}$ . The orders of these permutations are the orders of the images of the corresponding elements of  $G^{(5)}$  under the map  $G^{(5)} \rightarrow \bar{G}^{(5)}$  so the labelling (3.25) demonstrates that we recover the highest root labelling of the affine  $G_2$  Dynkin diagram — which is the labelling induced from the highest root labelling of  $\hat{E}_6$  — when we replace the conjugacy classes of (3.24) with the orders of their images in  $\bar{G}^{(5)}$ :

$$(3.25) \quad 1^6 \text{ --- } 1^2 2^2 \text{ --- } 3^2$$

For  $\ell = 7$  the Dynkin diagram  $\Delta^{(7)} = \hat{D}_4$  admits a  $\text{Sym}_4$  group of automorphisms. Folding by any 3-fold symmetry we again obtain the affine Dynkin diagram of type  $G_2$ . Let  $T$  be the unique conjugacy class of elements of order 4 in  $G^{(7)}$ . Then we have  $\chi_g^{(7)} = \bar{\chi}_g^{(7)} = 0$  in the notation of Table B.11 and  $T$  is the class labelled  $4A$  in Table B.5, and there are exactly three of the seven conjugacy classes of  $G^{(7)}$  that have a representative of the form  $gh$  for some  $g, h \in T$  (and  $T^2$  is the union of these three conjugacy classes). In (3.26) we use these classes (and the notation of Table B.5) to label the affine  $G_2$  Dynkin diagram:

$$(3.26) \quad 1A \text{ --- } 4A \text{ --- } 2A$$

In (3.27) we replace the labels of (3.26) with the cycle shapes  $\bar{\Pi}_g^{(7)}$  (cf. Table B.11) attached to these classes via the degree 4 permutation action of  $G^{(7)}$  on  $\Omega^{(7)}$ . The orders of these permutations are the orders of the images of the corresponding elements of  $G^{(7)}$  under the map  $G^{(7)} \rightarrow \bar{G}^{(7)}$  so the labelling (3.27) demonstrates that we recover the labelling of the affine  $G_2$  Dynkin diagram that is induced by the highest root labelling of  $\hat{D}_4$  when

we replace the conjugacy classes of (3.26) with the orders of their images in  $\bar{G}^{(7)}$ :

$$(3.27) \quad 1^4 \text{ --- } 2^2 \begin{array}{c} \text{---} \text{---} \\ \text{---} \text{---} \\ \text{---} \text{---} \end{array} 1^4$$

We summarize the results of this section with the following statement.

*For  $\ell \in \{3, 4, 5, 7\}$  there is an analogue of McKay's monstrous  $E_8$  observation that relates  $G^{(\ell)}$  to (a folding of) the affinization of a Dynkin diagram of rank  $11 - \ell$ , and the diagrams arising are precisely those that appear in Section 3.5.*

We conclude by mentioning that McKay's monstrous observation has partially been explained, using VOA theory, by the work of Sakuma [103] and Lam–Yamada–Yamauchi [104, 105]. Important related work appears in [106, 107].

#### 4. McKay–Thompson series

In Section 2.5 we made the observation that the first few positive degree Fourier coefficients of the vector-valued mock modular forms  $H^{(\ell)} = (H_r^{(\ell)})$  coincide (up to a factor of 2) with dimensions of irreducible representations of the group  $G^{(\ell)}$  described in the previous section. This observation suggests the possibility that

$$(4.1) \quad \begin{aligned} H_r^{(\ell)}(\tau) &= \sum_{k \in \mathbb{Z}} c_r^{(\ell)}(k - r^2/4\ell) q^{k - r^2/4\ell} \\ &= -2\delta_{r,1} q^{-1/4\ell} + \sum_{\substack{k \in \mathbb{Z} \\ r^2 - 4k\ell < 0}} \dim K_{r, k - r^2/4\ell}^{(\ell)} q^{k - r^2/4\ell} \end{aligned}$$

for some  $\mathbb{Z} \times \mathbb{Q}$ -graded infinite-dimensional  $G^{(\ell)}$ -module  $K^{(\ell)} = \bigoplus_{r,d} K_{r,d}^{(\ell)}$ . To further test this possibility we would like to see if there are similar vector-valued mock modular forms  $H_g^{(\ell)} = (H_{g,r}^{(\ell)})$  whose positive degree Fourier coefficients recover characters of representations of  $G^{(\ell)}$ . In other words, for an element  $g$  of the group  $G^{(\ell)}$  we would like to see if we can find a mock



modular form  $H_g^{(\ell)}$  compatible with the hypothesis that

$$(4.2) \quad \begin{aligned} H_{g,r}^{(\ell)}(\tau) &= \sum_k c_{g,r}^{(\ell)}(k - r^2/4\ell) q^{k-r^2/4\ell} \\ &= -2\delta_{r,1} q^{-1/4\ell} + \sum_{\substack{k \in \mathbb{Z} \\ r^2 - 4k\ell < 0}} \text{tr}_{K_{r,k-r^2/4\ell}^{(\ell)}}(g) q^{k-r^2/4\ell} \end{aligned}$$

for a hypothetical bi-graded  $G^{(\ell)}$ -module  $K^{(\ell)}$ . We refer to such a generating function as a *McKay–Thompson series*. Notice that we recover the generating functions of the  $\dim K_{r,d}^{(\ell)}$  when  $g$  is the identity element. Moreover, the McKay–Thompson series attached to  $g \in G^{(\ell)}$  is invariant under conjugation, since the trace is such, so  $H_g^{(\ell)}$  depends only on the conjugacy class  $[g]$  of  $g$ .

Having such McKay–Thompson series for each conjugacy class of  $G^{(\ell)}$  not only provides strong evidence for the existence of the  $G^{(\ell)}$ -module  $K^{(\ell)}$  but in fact it uniquely specifies it up to  $G^{(\ell)}$ -module isomorphism. This is because, since there are as many irreducible representations as conjugacy classes of a finite group, given the characters  $\text{tr}_{K_{r,d}^{(\ell)}}(g)$  for all  $[g] \subset G^{(\ell)}$  we can simply invert the character table to have a unique decomposition of the conjectural module  $K_{r,d}^{(\ell)}$  into irreducible representations. What is not clear, a priori, is that we will end up with a decomposition into non-negative integer multiples of irreducible representations of  $G^{(\ell)}$ , but remarkably it appears that this property holds for all  $\ell \in \Lambda$  and all bi-degrees  $(r, d)$ . For evidence in support of this see the explicit decompositions for small degrees tabulated in Appendix D.

In this section we construct a set of vector-valued mock modular forms  $H_g^{(\ell)} = (H_{g,r}^{(\ell)})$  for each lambency  $\ell$  and we formulate a precise conjecture implying (4.2) in Section 5.1.

#### 4.1. Forms of higher level

In Section 2 we have discussed the relation between certain vector-valued mock modular forms, meromorphic Jacobi forms of weight 1 and Jacobi forms of weight 0 under the group  $SL_2(\mathbb{Z})$ . In order to investigate the McKay–Thompson series of the groups  $G^{(\ell)}$  we need to generalize the discussion in Section 2.5 to modular forms of higher level and consider forms transforming under  $\Gamma_0(N)$  (cf. (A.2)) with  $N > 1$ .

As in Section 2 we would like to consider  $(\ell - 1)$ -vector-valued mock modular forms  $(H_{g,r}^{(\ell)})$  for a group  $\Gamma_0(N_g) < SL_2(\mathbb{Z})$  with shadow given by the

unary theta series  $S^{(\ell)}$  (cf. (2.26)). The levels  $N_g$  will be specified explicitly for all  $g$  in Section 4.8. A first difference between the case of  $N_g = 1$  and  $N_g > 1$  is the following. Since the components of  $S^{(\ell)} = (S_r^{(\ell)})$  have more than one orbit under  $\Gamma_0(N_g)$  when  $N_g$  is even it is natural to consider mock modular forms with shadows given by  $(\chi_{g,r}^{(\ell)} S_r^{(\ell)})$  where the  $\chi_{g,r}^{(\ell)}$  are not necessarily equal for different values of  $r$ . It will develop that in the cases of interest to us  $\chi_{g,r}^{(\ell)}$  depends only on  $r$  modulo 2. In fact, we will find that

$$(4.3) \quad \chi_{g,r}^{(\ell)} = \begin{cases} \chi_g^{(\ell)}, & \text{for } r \equiv 0 \pmod{2}, \\ \bar{\chi}_g^{(\ell)}, & \text{for } r \equiv 1 \pmod{2}, \end{cases}$$

where  $\chi_g^{(\ell)}$  and  $\bar{\chi}_g^{(\ell)}$  are as defined in Section 3.4 and as tabulated in Appendix B.2.

Group theoretically the multiplicities  $\bar{\chi}_g^{(\ell)}$  and  $\chi_g^{(\ell)}$  appearing in the shadow of  $H_g^{(\ell)}$  are determined by the number of fixed points and anti-fixed points in the signed permutation representation of the group  $G^{(\ell)}$ , as explained in Section 3.2. For instance, we have  $\bar{\chi}_g^{(\ell)} = \chi_g^{(\ell)} = \chi^{(\ell)}$  for the identity element  $g = e$ . From this interpretation we can deduce that the vanishing of  $\bar{\chi}_g^{(\ell)}$  implies the vanishing of  $\chi_g^{(\ell)}$ , while it is possible to have  $\chi_g^{(\ell)} = 0$  and  $\bar{\chi}_g^{(\ell)} \neq 0$ . It will also turn out that

$$(4.4) \quad \bar{\chi}_g^{(\ell)} = \chi_g^{(\ell)} \quad \text{unless } 2|N_g.$$

For later use we define the combinations

$$(4.5) \quad \chi_{g,+}^{(\ell)} = \frac{1}{2}(\bar{\chi}_g^{(\ell)} + \chi_g^{(\ell)}), \quad \chi_{g,-}^{(\ell)} = \frac{1}{2}(\bar{\chi}_g^{(\ell)} - \chi_g^{(\ell)}),$$

which enumerate the number of fixed and anti-fixed points, respectively, in the signed permutation representation of  $G^{(\ell)}$  (cf. (3.3) and (3.4)). Of course in the cases where  $\bar{\chi}_g^{(\ell)} = \chi_g^{(\ell)} = 0$  the function  $H_g^{(\ell)} = (H_{g,r}^{(\ell)})$  has vanishing shadow and is a vector-valued modular form in the usual sense.

Interestingly, just as in the  $SL_2(\mathbb{Z})$  case that was considered in Section 2.3, the higher level vector-valued mock modular forms with shadows as described above are again closely related to the finite parts of meromorphic Jacobi forms of weight 1. More explicitly, from the transformation (2.29) and the simple fact  $\theta_r^{(\ell)}(\tau, z + 1/2) = (-1)^r \theta_r^{(\ell)}(\tau, z)$  it is not difficult to check that the function

$$(4.6) \quad \psi_g^{(\ell)}(\tau, z) = \chi_{g,+}^{(\ell)} \mu_0^{(\ell)}(\tau, z) - \chi_{g,-}^{(\ell)} \mu_0^{(\ell)}(\tau, z + 1/2) + \sum_{r=1}^{\ell-1} H_{g,r}^{(\ell)}(\tau) \hat{\theta}_r^{(\ell)}(\tau, z)$$

transforms as a Jacobi form of weight 1 and index  $\ell$  under the congruence subgroup  $\Gamma_0(N_g)$  when  $H_g^{(\ell)}$  is a mock modular form of weight  $1/2$  for  $\Gamma_0(N_g)$  with shadow  $S_g^{(\ell)} = (\chi_{g,r}^{(\ell)} S_r^{(\ell)})$ . Observe that, for those  $g$  with both  $\chi_{g,+}^{(\ell)}$  and  $\chi_{g,-}^{(\ell)}$  being non-zero, the weak Jacobi form  $\psi_g^{(\ell)}(\tau, z)$  has a pole not only at  $z = 0$  but also at  $z = 1/2$ . From

$$(4.7) \quad \mu_0^{(\ell)}(\tau, z) = \text{Av}^{(\ell)} \left[ \frac{y+1}{y-1} \right], \quad -\mu_0^{(\ell)}(\tau, z + 1/2) = \text{Av}^{(\ell)} \left[ \frac{1-y}{1+y} \right],$$

we see that the last two terms of the decomposition given in (4.6) have the interpretation as the polar parts at the poles  $z = 0$  and  $1/2$ , respectively. In other words, we again have a decomposition  $\psi_g^{(\ell)} = \psi_g^{(\ell),P} + \psi_g^{(\ell),F}$  into polar and finite parts given by

$$(4.8) \quad \begin{aligned} \psi_g^{(\ell),P}(\tau, z) &= \chi_{g,+}^{(\ell)} \mu_0^{(\ell)}(\tau, z) - \chi_{g,-}^{(\ell)} \mu_0^{(\ell)}(\tau, z + 1/2), \\ \psi_g^{(\ell),F}(\tau, z) &= \sum_{r=1}^{\ell-1} H_{g,r}^{(\ell)}(\tau) \hat{\theta}_r^{(\ell)}(\tau, z), \end{aligned}$$

and the components  $H_{g,r}^{(\ell)}$  of the mock modular form  $H_g^{(\ell)}$  may again be interpreted as the theta-coefficients of a meromorphic Jacobi form; namely,  $\psi_g^{(\ell)}(\tau, z)$ .

Moreover, analogous to the  $SL_2(\mathbb{Z})$  case, the mock modular form  $H_g^{(\ell)}$  also enjoys a close relationship with a weight 0 index  $\ell - 1$  weak Jacobi form  $Z_g^{(\ell)}$  which admits a decomposition into characters of the  $N = 4$  superconformal algebra at level  $\ell - 1$ . To see this, observe that

$$(4.9) \quad \psi_g^{(\ell)}(\tau, z) = \frac{\chi_{g,+}^{(\ell)}}{\bar{\chi}_g^{(\ell)}} \Psi_{1,1}(\tau, z) Z_g^{(\ell)}(\tau, z) - \frac{\chi_{g,-}^{(\ell)}}{\bar{\chi}_g^{(\ell)}} \Psi_{1,1}(\tau, z + 1/2) Z_g^{(\ell)}(\tau, z + 1/2)$$

when  $\bar{\chi}_g^{(\ell)} \neq 0$  and

$$(4.10) \quad \psi_g^{(\ell)}(\tau, z) = \Psi_{1,1}(\tau, z) Z_g^{(\ell)}(\tau, z)$$

when  $\bar{\chi}_g^{(\ell)} = 0$ , where  $\Psi_{1,1}(\tau, z)$  is as in (2.20) and  $Z_g^{(\ell)}$  is the weak Jacobi form of weight 0 and index  $\ell - 1$  given by

$$(4.11) \quad Z_g^{(\ell)}(\tau, z) = \frac{1}{\Psi_{1,1}(\tau, z)} \left( \bar{\chi}_g^{(\ell)} \mu_0^{(\ell)}(\tau, z) + \sum_{0 < r < \ell} \left( 1 + \frac{\bar{\chi}_g^{(\ell)} - \chi_{g,r}^{(\ell)}}{\chi_g^{(\ell)}} \right) H_{g,r}^{(\ell)}(\tau) \hat{\theta}_r^{(\ell)}(\tau, z) \right)$$

for  $\chi_g^{(\ell)} \neq 0$  and

$$(4.12) \quad Z_g^{(\ell)}(\tau, z) = \frac{1}{\Psi_{1,1}(\tau, z)} \left( \bar{\chi}_g^{(\ell)} \mu_0^{(\ell)}(\tau, z) + \sum_{0 < r < \ell} H_{g,r}^{(\ell)}(\tau) \hat{\theta}_r^{(\ell)}(\tau, z) \right)$$

otherwise.

Anticipating their relation to the conjugacy classes  $[g]$  of  $G^{(\ell)}$ , another comment on the weight 0 forms  $Z_g^{(\ell)}$  is in order here. As discussed in Section 3.2, if  $\ell \in \Lambda$  and  $\ell > 2$  then  $G^{(\ell)}$  has a unique central element  $z$  of order 2 and for any  $g \in G^{(\ell)}$  the conjugacy classes  $[g]$  and  $[zg]$  are said to be paired, and a class  $[g]$  is said to be self-paired if  $[g] = [zg]$ . We have

$$(4.13) \quad \bar{\chi}_g^{(\ell)} = \bar{\chi}_{zg}^{(\ell)}, \quad \chi_g^{(\ell)} = -\chi_{zg}^{(\ell)},$$

and as a consequence  $Z_g^{(\ell)}(\tau, z) = Z_{zg}^{(\ell)}(\tau, z)$ . Therefore, while the vector-valued mock modular forms  $H_g^{(\ell)}$  and the weight 1 meromorphic Jacobi forms  $\psi_g^{(\ell)}(\tau, z)$  are generally distinct for different conjugacy classes  $[g]$  of the group  $G^{(\ell)}$ , the weight 0 index  $\ell - 1$  weak Jacobi forms  $Z_g^{(\ell)}$  cannot distinguish between two classes that are paired in the above sense and are more naturally associated to the group  $\bar{G}^{(\ell)}$  which is the quotient of  $G^{(\ell)}$  by the subgroup  $\langle z \rangle$ .

Equipped with the  $H^{(\ell)}$  defined in terms of decompositions of Jacobi forms as discussed in Section 2.5 it will develop that a convenient way to specify the vector-valued mock modular forms  $H_g^{(\ell)} = (H_{g,r}^{(\ell)})$  corresponding to non-identity conjugacy classes of  $G^{(\ell)}$  will be to specify certain sets of weight 2 modular forms. To see how a weight 2 modular form is naturally associated with a vector-valued modular form with the properties described above, observe that we can eliminate the presence of the polar part in the weight 1 Jacobi form  $\psi_g^{(\ell)}(\tau, z)$  (cf. (4.6)) by taking a linear combination with  $\psi^{(\ell)}(\tau, z)$ . To be more precise, note that

$$(4.14) \quad \begin{aligned} \hat{\psi}_g^{(\ell)}(\tau, z) &= \sum_{r=1}^{\ell-1} \hat{H}_{g,r}^{(\ell)}(\tau) \hat{\theta}_r^{(\ell)}(\tau, z) \\ &= \psi_g^{(\ell)}(\tau, z) - \frac{\chi_{g,+}^{(\ell)}}{\chi^{(\ell)}} \psi^{(\ell)}(\tau, z) + \frac{\chi_{g,-}^{(\ell)}}{\chi^{(\ell)}} \psi^{(\ell)}(\tau, z + 1/2) \end{aligned}$$

is a weight 1 index  $\ell$  Jacobi form for  $\Gamma_0(N_g)$  with no poles in  $z$ , and the functions

$$(4.15) \quad \hat{H}_{g,r}^{(\ell)}(\tau) = H_{g,r}^{(\ell)}(\tau) - \frac{\chi_{g,r}^{(\ell)}}{\chi^{(\ell)}} H_r^{(\ell)}(\tau)$$

are the components of a vector-valued modular form  $\hat{H}_g^{(\ell)}$  for  $\Gamma_0(N_g)$  in the usual sense (i.e., a mock modular form with vanishing shadow). From this we readily conclude that the function

$$(4.16) \quad F_g^{(\ell)}(\tau) = \sum_{r=1}^{\ell-1} \hat{H}_{g,r}^{(\ell)}(\tau) S_r^{(\ell)}(\tau) = -\frac{1}{4\pi i} \frac{\partial}{\partial z} \hat{\psi}_g^{(\ell)}(\tau, z) \Big|_{z=0}$$

is a weight 2 modular form for the group  $\Gamma_0(N_g)$ .

For general values of  $\ell$ , specifying the weight 2 form  $F_g^{(\ell)}(\tau)$  is not sufficient to specify the whole vector-valued modular form  $\hat{H}_g^{(\ell)}$  since we have evidently collapsed information in taking the particular combination (4.16). However, for  $\ell \in \{2, 3\}$  we are in the privileged situation that specifying the weight 2 form  $F_g^{(\ell)}(\tau)$  completely specifies all the components of  $H_g^{(\ell)}$ , as will be explained in more detail in the following section. For  $\ell > 3$  we need more weight 2 forms, and we will also consider

$$(4.17) \quad F_g^{(\ell),2}(\tau) = \sum_{r=1}^{\ell-1} (-1)^{r+1} \hat{H}_{g,r}^{(\ell)}(\tau) S_{\ell-r}^{(\ell)}(\tau).$$

In the next section we give our concrete proposals for the McKay–Thompson series  $H_g^{(\ell)}$  for all but a few of the conjugacy classes  $[g]$  arising. We give closed expressions for all the  $H_g^{(\ell)}$  in case  $\ell \in \{2, 3, 4, 5\}$  and we partially determine the  $H_g^{(\ell)}$  for  $\ell = \{7, 13\}$ . Although we do not offer analytic expressions for all the  $H_g^{(\ell)}$  with  $\ell = 7$  or 13 we have predictions for the low degree terms in the Fourier developments of all the McKay–Thompson series at all lambencies  $\ell \in \Lambda$  and these are detailed in the tables of Appendix C.

### 4.2. Lambency two

When  $\ell = 2$  the vector-valued mock modular form  $H_g^{(2)}$  has only one component  $H_{g,1}^{(2)}$  and our conjecture relating the  $H_g^{(2)}$  and the group  $G^{(2)}$  is nothing but the conjecture relating (scalar-valued) mock modular forms and

the largest Mathieu group  $M_{24}$  that has been investigated recently [15, 19–22, 24, 108, 109]. See also [41, 110] for reviews and [111, 112] for related discussions. Explicit expressions for the McKay–Thompson series arising from the  $M_{24}$ -module that is conjectured to underlie this connection have been proposed in [19–22]. As mentioned before, one convenient way to express them is via a set of weight 2 modular forms  $F_g^{(2)}(\tau)$ . In this case (4.15) and

Table 3: The list of weight 2 modular forms  $F_g^{(2)}(\tau)$  for  $\Gamma_0(N_g)$ .

$[g]$	$N_g$	$F_g^{(2)}(\tau)$
1A	1	0
2A	2	$-16\Lambda_2$
2B	4	$24\Lambda_2 - 8\Lambda_4 = -2\eta(\tau)^8/\eta(2\tau)^4$
3A	3	$-6\Lambda_3$
3B	9	$-2\eta(\tau)^6/\eta(3\tau)^2$
4A	8	$-4\Lambda_2 + 6\Lambda_4 - 2\Lambda_8 = -2\eta(2\tau)^8/\eta(4\tau)^4$
4B	4	$4(\Lambda_2 - \Lambda_4)$
4C	16	$-2\eta(\tau)^4\eta(2\tau)^2/\eta(4\tau)^2$
5A	5	$-2\Lambda_5$
6A	6	$2(\Lambda_2 + \Lambda_3 - \Lambda_6)$
6B	36	$-2\eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^2/\eta(6\tau)^2$
7AB	7	$-\Lambda_7$
8A	8	$\Lambda_4 - \Lambda_8$
10A	20	$-2\eta(\tau)^3\eta(2\tau)\eta(5\tau)/\eta(10\tau)$
11A	11	$\frac{2}{5}(-\Lambda_{11} + 11f_{11})$
12A	24	$-2\eta(\tau)^3\eta(4\tau)^2\eta(6\tau)^3/\eta(2\tau)\eta(3\tau)\eta(12\tau)^2$
12B	144	$-2\eta(\tau)^4\eta(4\tau)\eta(6\tau)/\eta(2\tau)\eta(12\tau)$
14AB	14	$\frac{1}{3}(\Lambda_2 + \Lambda_7 - \Lambda_{14} + 14f_{14})$
15AB	15	$\frac{1}{4}(\Lambda_3 + \Lambda_5 - \Lambda_{15} + 15f_{15})$
21AB	63	$\frac{1}{3}(-7\eta(\tau)^3\eta(7\tau)^3/\eta(3\tau)\eta(21\tau) + \eta(\tau)^6/\eta(3\tau)^2)$
23AB	23	$\frac{1}{11}(-\Lambda_{23} + 23f_{23,a} + 69f_{23,b})$

(4.16) simply reduces to

$$(4.18) \quad H_{g,1}^{(2)}(\tau) = \frac{\chi_g^{(2)}}{24} H_1^{(2)}(\tau) + \frac{F_g^{(2)}(\tau)}{\eta(\tau)^3},$$

where we have used  $\chi^{(2)} = 24$  and  $S_1^{(2)}(\tau) = \eta(\tau)^3$ . For later use and for the sake of completeness we collect the explicit expressions for  $F_g^{(2)}(\tau)$  for all conjugacy classes  $[g] \subset M_{24}$  in Table 3. They are given in terms of eta quotients and standard generators of the weight 2 modular forms of level  $N$ . Among the latter are those denoted here by  $\Lambda_N(\tau)$  and  $f_N(\tau)$  and given explicitly in Appendix A.

### 4.3. Lambency three

We would like to specify the two components  $H_{g,1}^{(3)}(\tau)$  and  $H_{g,2}^{(3)}(\tau)$  of the vector-valued mock modular form  $(H_{g,r}^{(3)}(\tau))$  which we propose to be the McKay–Thompson series arising from the  $G^{(3)}$ -module  $K^{(3)}$ . As mentioned earlier, to specify  $H_{g,1}^{(3)}(\tau)$  and  $H_{g,2}^{(3)}(\tau)$  it is sufficient to specify the weight 2 forms defined in (4.16). To see this, recall that to any given conjugacy class  $[g]$  we can associate a conjugacy class  $[zg]$  such that (4.13) holds. From this we obtain

$$(4.19) \quad \begin{aligned} H_{g,1}^{(3)}(\tau) &= \frac{\bar{\chi}_g^{(3)}}{\chi^{(3)}} H_1^{(3)}(\tau) + \frac{1}{2} \frac{\eta(4\tau)^2}{\eta(2\tau)^5} (F_g^{(3)}(\tau) + F_{zg}^{(3)}(\tau)), \\ H_{g,2}^{(3)}(\tau) &= \frac{\chi_g^{(3)}}{\chi^{(3)}} H_2^{(3)}(\tau) + \frac{1}{4} \frac{\eta(2\tau)}{\eta(\tau)^2 \eta(4\tau)^2} (F_g^{(3)}(\tau) - F_{zg}^{(3)}(\tau)), \end{aligned}$$

where we have used the eta quotient expressions

$$(4.20) \quad S_1^{(3)}(\tau) = \frac{\eta(2\tau)^5}{\eta(4\tau)^2}, \quad S_2^{(3)}(\tau) = 2 \frac{\eta(\tau)^2 \eta(4\tau)^2}{\eta(2\tau)},$$

for the components of the unary theta series at  $\ell = 3$ .

The explicit expressions for  $F_g^{(3)}(\tau)$  are listed in Table 4.

We also note that for the classes  $3B$  and  $6B$  we also have the following alternative expressions for the McKay–Thompson series in terms of eta

Table 4: The list of weight 2 modular forms  $F_g^{(3)}(\tau)$  for  $\Gamma_0(N_g)$ .

$[g]$	$N_g$	$F_g^{(3)}(\tau)$
1A	1	0
2A	4	0
4A	16	$-2\eta(\tau)^4\eta(2\tau)^2/\eta(4\tau)^2$
2B	2	$-16\Lambda_2$
2C	4	$16(\Lambda_2 - \Lambda_4/3)$
3A	3	$-6\Lambda_3$
6A	12	$-9\Lambda_2 - 2\Lambda_3 + 3\Lambda_4 + 3\Lambda_6 - \Lambda_{12}$
3B	9	$8\Lambda_3 - 2\Lambda_9 + 2\eta^6(\tau)/\eta^2(3\tau)$
6B	36	$-2\eta(\tau)^5\eta(3\tau)/\eta(2\tau)\eta(6\tau)$
4B	8	$-2\eta(2\tau)^8/\eta(4\tau)^4$
4C	4	$-8\Lambda_4/3$
5A	5	$-2\Lambda_5$
10A	20	$\sum_{d 20} c_{10A}(d)\Lambda_d + \frac{20}{3}f_{20}$
12A	144	$-2\eta(\tau)\eta(2\tau)^5\eta(3\tau)/\eta(4\tau)^2\eta(6\tau)$
6C	6	$2(\Lambda_2 + \Lambda_3 - \Lambda_6)$
6D	12	$-5\Lambda_2 - 2\Lambda_3 + \frac{5}{3}\Lambda_4 + 3\Lambda_6 - \Lambda_{12}$
8AB	32	$-2\eta(2\tau)^4\eta(4\tau)^2/\eta(8\tau)^2$
8CD	8	$-2\Lambda_2 + \frac{5}{3}\Lambda_4 - \Lambda_8$
20AB	80	$-2\eta(2\tau)^7\eta(5\tau)/\eta(\tau)\eta(4\tau)^2\eta(10\tau)$
11AB	11	$-\frac{2}{5}\Lambda_{11} - \frac{33}{5}f_{11}$
22AB	44	$\sum_{d 44} c_{22AB}(d)\Lambda_d(\tau) - \frac{11}{5}\sum_{d 4} c'_{22AB}(d)f_{11}(d\tau) + \frac{22}{3}f_{44}(\tau)$
		$c_{10A}(d) = -5, \frac{5}{3}, -\frac{2}{3}, 1, -\frac{1}{3}$ for $d = 2, 4, 5, 10, 20$
		$c_{22AB}(d) = -\frac{11}{5}, \frac{11}{15}, -\frac{2}{15}, \frac{1}{5}, -\frac{1}{15}$ for $d = 2, 4, 11, 22, 44$
		$c'_{22AB}(d) = 1, 4, 8$ for $d = 1, 2, 4$



quotients:

$$(4.21) \quad \begin{aligned} H_{3B,1}^{(3)}(\tau) &= H_{6B,1}^{(3)}(\tau) = -2 \frac{\eta(\tau)\eta(6\tau)^5}{\eta(3\tau)^3\eta(12\tau)^2}, \\ H_{3B,2}^{(3)}(\tau) &= -H_{6B,2}^{(3)}(\tau) = -4 \frac{\eta(\tau)\eta(12\tau)^2}{\eta(3\tau)\eta(6\tau)}. \end{aligned}$$

Coincidences with Ramanujan’s mock theta functions will be discussed in Section 4.7.

#### 4.4. Lambency four

The McKay–Thompson series for lambency 4 display an interesting relation with those for lambency 2. To see this, notice the following relation among the theta functions:

$$(4.22) \quad S_1^{(4)}(2\tau) - S_3^{(4)}(2\tau) = S_1^{(2)}(\tau).$$

Given this relation it is natural to consider the function

$$(4.23) \quad H_{g,*}^{(4)}(\tau) := H_{g,1}^{(4)}(\tau) - H_{g,3}^{(4)}(\tau)$$

for each conjugacy class  $[g]$  of  $G^{(4)}$ . Note that  $H_{g,1}^{(4)}(\tau)$  and  $H_{g,3}^{(4)}(\tau)$  can be reconstructed from  $H_{g,*}^{(4)}(\tau)$  since they are  $q$ -series of the form  $q^{-1/16}$  times a series of even or odd powers of  $q^{1/2}$ , respectively. Explicitly, we have

$$(4.24) \quad \begin{aligned} H_{g,1}^{(4)}(\tau) &= \frac{1}{2} \left( H_{g,*}^{(4)}(\tau) + e \left( \frac{1}{16} \right) H_{g,*}^{(4)}(\tau + 1) \right), \\ H_{g,3}^{(4)}(\tau) &= \frac{1}{2} \left( -H_{g,*}^{(4)}(\tau) + e \left( \frac{1}{16} \right) H_{g,*}^{(4)}(\tau + 1) \right). \end{aligned}$$

In order to obtain an expression for  $H_{g,*}^{(4)}(\tau)$  we rely on the following two observations. First, recall in Section 3.4 we have observed a relation between the Frame shapes of  $g \in G^{(4)}$  and  $g' \in G^{(2)}$ . It turns out that for a pair of group elements  $g \in G^{(4)}$  and  $g' \in G^{(2)}$  related in the way described in Proposition 3.4 their McKay–Thompson series  $H_g^{(4)}(2\tau)$  and  $H_{g'}^{(2)}(\tau)$  are

also related in a simple way. As examples of this we have

$$\begin{aligned}
 (4.25) \quad & H_{1A,*}^{(4)}(\tau) = H_{2A,*}^{(4)}(\tau) = H_{2A}^{(2)}(\tau/2), \\
 & H_{2B,*}^{(4)}(\tau) = H_{4A}^{(2)}(\tau/2), \\
 & H_{2C,*}^{(4)}(\tau) = H_{4B}^{(2)}(\tau/2), \\
 & H_{3A,*}^{(4)}(\tau) = H_{6A,*}^{(4)}(\tau) = H_{6A}^{(2)}(\tau/2), \\
 & H_{4C,*}^{(4)}(\tau) = H_{8A}^{(2)}(\tau/2), \\
 & H_{6BC,*}^{(4)}(\tau) = H_{12A}^{(2)}(\tau/2), \\
 & H_{7AB,*}^{(4)}(\tau) = H_{14AB,*}^{(4)}(\tau) = H_{14AB}^{(2)}(\tau/2).
 \end{aligned}$$

This leaves us just the classes  $4A$ ,  $4B$ , and  $8A$  that are not related to any element of  $G^{(2)} \simeq M_{24}$  in the way described in Proposition 3.4. All of these classes have  $\chi_g^{(4)} = \bar{\chi}_g^{(4)} = 0$ . A second observation is that all the  $H_{g,r}^{(3)}(\tau)$  for those classes  $[g] \subset G^{(3)}$  with  $\chi_g^{(3)} = \bar{\chi}_g^{(3)} = 0$  have an expression in terms of eta quotients according to Table 4, and so do the  $H_g^{(2)}(\tau)$  for those classes  $[g] \subset G^{(2)}$  with  $\chi_g^{(2)} = 0$ . This is consistent with our expectation that the shadow of the vector-valued mock modular form  $(H_{g,r}^{(\ell)}(\tau))$  is given by  $(\chi_{g,r}^{(\ell)} S_r^{(\ell)}(\tau))$ , and hence  $(H_{g,r}^{(\ell)}(\tau))$  is nothing but a vector-valued modular form in the usual sense when  $\chi_g^{(\ell)} = \bar{\chi}_g^{(\ell)} = 0$ . Given this it is natural to ask whether we can find similar eta quotient expressions for  $H_{g,*}^{(4)}(\tau)$  with  $\chi_g^{(\ell)} = \bar{\chi}_g^{(\ell)} = 0$  when  $\ell > 3$ . We find

$$\begin{aligned}
 (4.26) \quad & H_{4A,*}^{(4)}(\tau) = -2 \frac{\eta(\tau/2)\eta(\tau)^2}{\eta(2\tau)^2}, \\
 & H_{4B,*}^{(4)}(\tau) = -2 \frac{\eta(\tau/2)\eta(\tau)^4}{\eta(\tau)^2\eta(4\tau)^2}, \\
 & H_{8A,*}^{(4)}(\tau) = -2 \frac{\eta(\tau)^3}{\eta(\tau/2)\eta(4\tau)}.
 \end{aligned}$$

Together with (4.24) and Table 3, the above equations completely specify  $H_{g,1}^{(4)}(\tau)$  and  $H_{g,3}^{(4)}(\tau)$  for all conjugacy classes  $[g]$  of  $G^{(4)}$ . We are left to determine the second components  $H_{g,2}^{(4)}(\tau)$ . To start with, we have  $H_2^{(4)}(\tau) = H_{1A,2}^{(4)}(\tau) = -H_{2A,2}^{(4)}(\tau)$  given in terms of the decomposition of the weight 0 Jacobi form as explained in Section 2.5. To determine the rest, notice that

(4.13) and (4.16) implies that

$$(4.27) \quad S_2^{(4)}(\tau) \left( H_{g,2}^{(4)}(\tau) - \frac{\chi_{g,2}^{(4)}}{\chi^{(4)}} H_2^{(4)}(\tau) \right)$$

should be a weight 2 modular form. Employing this consideration we arrive at

$$(4.28) \quad \begin{aligned} H_{3A,2}^{(4)}(\tau) &= -H_{6A,2}^{(4)}(\tau) \\ &= \frac{1}{4} H_2^{(4)}(\tau) + \frac{1}{2\eta(2\tau)^3} (-3\Lambda_2(\tau) - 4\Lambda_3(\tau) + \Lambda_6(\tau)), \\ H_{7AB,2}^{(4)}(\tau) &= -H_{14AB,2}^{(4)}(\tau) \\ &= \frac{1}{8} H_2^{(4)}(\tau) + \frac{1}{12\eta(2\tau)^3} (-7\Lambda_2(\tau) - 4\Lambda_7(\tau) + \Lambda_{14}(\tau) + 28f_{14}(\tau)), \end{aligned}$$

where we have used  $S_2^{(4)}(\tau) = 2\eta(2\tau)^3$  and the hypothesis that

$$(4.29) \quad H_{g,2}^{(4)}(\tau) = 0$$

for all classes not in the set  $\{1A, 2A, 3A, 6A, 7AB, 14AB\}$ . This finishes our proposal for the McKay–Thompson series  $(H_{g,r}^{(4)})$  at  $\ell = 4$ . Coincidences between some of the  $H_{g,r}^{(4)}$  and Ramanujan’s mock theta functions will be discussed in Section 4.7.

#### 4.5. Lambency five

As mentioned before, for  $\ell = 5$  it is no longer sufficient to specify the weight 2 forms  $F_g^{(\ell)}(\tau)$  as defined in (4.16). In this case we also need to specify the weight 2 forms  $F_g^{(\ell),2}(\tau)$  defined in (4.17).

With the data of  $F_g^{(5)}(\tau)$  and  $F_g^{(5),2}(\tau)$  we can determine the  $H_{g,r}^{(5)}$  using the relations

$$(4.30) \quad H_{g,r}^{(5)}(\tau) = \hat{H}_{g,r}^{(5)}(\tau) + \frac{\chi_{g,r}^{(5)}}{6} H_r^{(5)}(\tau)$$

Table 5: The list of weight 2 modular forms  $F_g^{(5)}(\tau)$  on  $\Gamma_0(N_g)$  and the other set of weight 2 modular forms  $F_g^{(5),2}(\tau)$ .

$[g]$	$N_g$	$F_g^{(5)}(\tau)$	$F_g^{(5),2}(\tau)$
1A	1	0	0
2A	4	0	0
2B	4	$16(\Lambda_2 - \Lambda_4/3)$	$-\frac{8}{3}\eta(\tau)^8/\eta(\tau/2)^4$
2C	2	$-16\Lambda_2$	$e(\frac{1}{4})F_{2B}^{(5),2}(\tau+1)$
3A	9	$-2\eta^6(\tau)/\eta^2(3\tau)$	$-2\frac{\eta(\tau)^8\eta(3\tau/2)^2\eta(6\tau)^2}{\eta(\tau/2)^2\eta(2\tau)^2\eta(3\tau)^4}$
6A	36	$-2\eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^2/\eta(6\tau)^2$	$2\eta(\tau/2)^2\eta(\tau)^2\eta(3\tau)^2/\eta(3\tau/2)^2$
4AB	16	$-2\eta(2\tau)^{14}/\eta(\tau)^4\eta(4\tau)^6$	$-16\eta(\tau)^2\eta(4\tau)^6/\eta(2\tau)^4$
4CD	8	$-8/3\Lambda_4(\tau)$	$-\frac{32}{3}\eta(2\tau)^2\eta(4\tau)^2/\eta(\tau)^2$
5A	5	$-2\Lambda_5$	$e(\frac{1}{4})F_{10A}^{(5),2}(\tau+1)$
10A	20	$\sum_{d 20} c_{10A}(d)\Lambda_d - \frac{40}{3}f_{20}$	$\sum_{d 20} c_{10A,2}(d)\Lambda_d(\tau/4) + \frac{10}{3}f_{20}(\tau/4)$
12AB	144	$-2\eta(\tau)^2\eta(2\tau)^2\eta(6\tau)^4/\eta(3\tau)^2\eta(12\tau)^2$	$-4\eta(\tau)^2\eta(2\tau)^2\eta(12\tau)^2/\eta(6\tau)^2$
		$c_{10A}(d) = -5, \frac{5}{3}, -\frac{2}{3}, 1, -\frac{1}{3}$ for $d = 2, 4, 5, 10, 20$	
		$c_{10A,2}(d) = -\frac{5}{4}, \frac{5}{24}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{24}$ for $d = 2, 4, 5, 10, 20$	

and

$$\begin{aligned}
 \hat{H}_{g,1}^{(5)}(\tau)S_1^{(5)}(\tau) + \hat{H}_{g,3}^{(5)}(\tau)S_3^{(5)}(\tau) &= \frac{1}{2} \left( F_g^{(5)}(\tau) + F_{zg}^{(5)}(\tau) \right), \\
 \hat{H}_{g,1}^{(5)}(\tau)S_4^{(5)}(\tau) + \hat{H}_{g,3}^{(5)}(\tau)S_2^{(5)}(\tau) &= \frac{1}{2} \left( F_g^{(5),2}(\tau) + F_{zg}^{(5),2}(\tau) \right), \\
 \hat{H}_{g,2}^{(5)}(\tau)S_2^{(5)}(\tau) + \hat{H}_{g,4}^{(5)}(\tau)S_4^{(5)}(\tau) &= \frac{1}{2} \left( F_g^{(5)}(\tau) - F_{zg}^{(5)}(\tau) \right), \\
 \hat{H}_{g,2}^{(5)}(\tau)S_3^{(5)}(\tau) + \hat{H}_{g,4}^{(5)}(\tau)S_1^{(5)}(\tau) &= \frac{1}{2} \left( -F_g^{(5),2}(\tau) + F_{zg}^{(5),2}(\tau) \right),
 \end{aligned}
 \tag{4.31}$$

where  $g$  and  $zg$  again form a pair satisfying (4.13).

Moreover, notice that in  $F_g^{(5),2}(\tau)$  the part of the sum involving even  $r$  has an expansion in  $q^{1/4+k}$  for non-negative integers  $k$ , while the sum over odd  $r$  has an expansion in  $q^{3/4+k}$ . As a result, it is sufficient to specify  $F_g^{(\ell),2}(\tau)$  for one of the paired classes  $g$  and  $zg$  as they are related to each other by

$$F_{zg}^{(5),2}(\tau) = e\left(\frac{1}{4}\right) F_g^{(\ell),2}(\tau+1).
 \tag{4.32}$$

Using (4.30) to (4.32), the data recorded in Table 5 are sufficient to explicitly specify all the mock modular forms  $H_g^{(5)}(\tau)$  for all  $[g] \subset G^{(5)}$  at lambency 5.

### 4.6. Lambencies seven and thirteen

For  $\ell = 7$ , apart from the  $1A$  and  $2A$  classes whose McKay–Thomson series have been given in terms of the weight 0 Jacobi form  $Z^{(7)}$  in Section 2.5, there are five more classes  $3AB$ ,  $4A$ , and  $6AB$  whose McKay–Thomson series  $H_g^{(7)}$  we would like to identify. We specify the weight 2 forms associated to these classes. Notice that these expressions are not sufficient to completely determine all the components  $H_{g,r}^{(7)}$ . Nevertheless, they provide strong evidence for the mock modular properties of the proposed McKay–Thompson series.

$$\begin{aligned}
 (4.33) \quad & F_{4A}^{(7)}(\tau) = -2 \frac{\eta(\tau)^4 \eta(2\tau)^2}{\eta(4\tau)^2}, \\
 & F_{4A}^{(7),2}(\tau) = 4 \frac{\eta(\tau)^6 \eta(4\tau)^2}{\eta(2\tau)^4}, \\
 & F_{3AB}^{(7)}(\tau) = -6\Lambda_3(\tau), \\
 & F_{6AB}^{(7)}(\tau) = -9\Lambda_2(\tau) - 2\Lambda_3(\tau) + 3\Lambda_4(\tau) + 3\Lambda_6(\tau) - \Lambda_{12}(\tau), \\
 & F_{6AB}^{(7),2}(4\tau) = -\frac{9}{4}\Lambda_2(\tau) - \Lambda_3(\tau) + \frac{3}{8}\Lambda_4(\tau) + \frac{3}{4}\Lambda_6(\tau) - \frac{1}{8}\Lambda_{12}(\tau).
 \end{aligned}$$

Similarly, for lambency 13 we have

$$\begin{aligned}
 (4.34) \quad & F_{4AB}^{(13)}(\tau) = -2 \frac{\eta(2\tau)^{14}}{\eta(\tau)^4 \eta(4\tau)^6}, \\
 & F_{4AB}^{(13),2}(\tau) = -16 \frac{\eta(\tau)^2 \eta(4\tau)^6}{\eta(2\tau)^4}.
 \end{aligned}$$

### 4.7. Mock theta functions

As mentioned in Section 2.1 we observe that in many cases across different lambencies the components  $H_{g,r}^{(\ell)}$  of the McKay–Thompson series coincide with known mock theta functions. In particular, we often encounter Ramanujan’s mock theta functions, and always in such a way that the order is divisible by the lambency. In this section we will list these conjectural identities between  $H_{g,r}^{(\ell)}$  and mock theta functions identified previously in the literature.

For  $\ell = 2$  two of the functions  $H_g^{(2)}(\tau)$  are related to Ramanujan's mock theta functions of orders 2 and 8 (cf. (2.7)) through

$$(4.35) \quad \begin{aligned} H_{4B}^{(2)}(\tau) &= -2q^{-1/8}\mu(q), \\ H_{8A}^{(2)}(\tau) &= -2q^{-1/8}U_0(q). \end{aligned}$$

For  $\ell = 3$  we encounter the following order 3 mock theta functions of Ramanujan:

$$(4.36) \quad \begin{aligned} H_{2B,1}^{(3)}(\tau) &= H_{2C,1}^{(3)}(\tau) = H_{4C,1}^{(3)}(\tau) = -2q^{-1/12}f(q^2), \\ H_{6C,1}^{(3)}(\tau) &= H_{6D,1}^{(3)}(\tau) = -2q^{-1/12}\chi(q^2), \\ H_{8C,1}^{(3)}(\tau) &= H_{8D,1}^{(3)}(\tau) = -2q^{-1/12}\phi(-q^2), \\ H_{2B,2}^{(3)}(\tau) &= -H_{2C,2}^{(3)}(\tau) = -4q^{2/3}\omega(-q), \\ H_{6C,2}^{(3)}(\tau) &= -H_{6D,2}^{(3)}(\tau) = 2q^{2/3}\rho(-q). \end{aligned}$$

(Cf. (2.9).) The description of the shadow of  $H_g^{(3)}$  is consistent with the fact that, among the seven order 3 mock theta functions of Ramanujan,  $f(\tau)$ ,  $\phi(\tau)$ ,  $\psi(\tau)$  and  $\chi(\tau)$  form a group with the same shadow ( $S_1^{(3)}(\tau)$ ) while the other three  $\omega(\tau)$ ,  $\nu(\tau)$  and  $\rho(\tau)$  form another group with another shadow ( $S_2^{(3)}(\tau)$ ). Moreover, various relations among these order 3 mock theta functions can be obtained as consequences of the above identification.

For  $\ell = 4$  we encounter the following order 8 mock theta functions:

$$(4.37) \quad \begin{aligned} H_{2C,1}^{(4)}(\tau) &= q^{-\frac{1}{16}}(-2S_0(q) + 4T_0(q)), \\ H_{2C,3}^{(4)}(\tau) &= q^{\frac{7}{16}}(2S_1(q) - 4T_1(q)), \\ H_{4C,1}^{(4)}(\tau) &= -2q^{-\frac{1}{16}}S_0(q), \\ H_{4C,3}^{(4)}(\tau) &= 2q^{\frac{7}{16}}S_1(q). \end{aligned}$$

(Cf. (2.11).) Comparing with (2.7) we see that our proposal implies the identities

$$(4.38) \quad \begin{aligned} \mu(q) &= S_0(q^2) - 2T_0(q^2) + q(S_1(q^2) - 2T_1(q^2)) \\ &= U_0(q) - 2U_1(q), \\ U_0(q) &= S_0(q^2) + qS_1(q^2), \\ U_1(\tau) &= T_0(q^2) + qT_1(q^2) \end{aligned}$$

between different mock theta functions of Ramanujan. See, for instance, [113] for a collection of such identities.

For  $\ell = 5$  we encounter four of Ramanujan's order 10 mock theta functions:

$$(4.39) \quad \begin{aligned} H_{2BC,1}^{(5)}(\tau) &= H_{4CD,1}^{(5)}(\tau) = -2q^{-\frac{1}{20}} X(q^2), \\ H_{2BC,3}^{(5)}(\tau) &= H_{4CD,3}^{(5)}(\tau) = -2q^{-\frac{9}{20}} \chi_{10}(q^2), \\ H_{2C,2}^{(5)}(\tau) &= -H_{2B,2}^{(5)}(\tau) = 2q^{-\frac{1}{5}} \psi_{10}(-q), \\ H_{2C,4}^{(5)}(\tau) &= -H_{2B,4}^{(5)}(\tau) = -2q^{\frac{1}{5}} \phi_{10}(-q). \end{aligned}$$

(Cf. (2.10).)

#### 4.8. Automorphy

In this section we discuss the automorphy of the proposed McKay–Thompson series  $H_g^{(\ell)} = (H_{g,r}^{(\ell)})$ . As mentioned in Section 4.1, the function  $H_g^{(\ell)}$  is a vector-valued mock modular form with shadow  $(\chi_{g,r} S_r^{(\ell)})$  for some  $\Gamma_g \subset SL_2(\mathbb{Z})$  with a certain (matrix-valued) multiplier  $\nu_g$ . In this subsection we will specify the group  $\Gamma_g$  and the multiplier  $\nu_g$ . The multipliers we specify here can be verified explicitly using the data given in Sections 4.2 to 4.6 (except for the few conjugacy classes at  $\ell = 7$  and 13 for which the McKay–Thompson series  $H_g^{(\ell)}$  have not been completely determined).

We find that the automorphy of the vector-valued function  $H_g^{(\ell)}$  is governed, in way that we shall describe presently, by the signed permutation representation of  $G^{(\ell)}$  arising from the construction (as a subgroup of  $\text{Oct}_m$  for  $m = 24/(\ell - 1)$ ) given in Section 3.4. We use this representation to define the symbols  $n_g | h_g$  which appear in the second row of each Tables C.1 to C.28, and also in the twisted Euler character Tables B.7–B.12, and we will explain below how to use these symbols to determine the multiplier system for each  $H_g^{(\ell)}$ . It will develop also that the twisted Euler character  $g \mapsto \chi_g^{(\ell)}$  (cf. Tables B.7–B.12) attached to the signed permutation representation of  $G^{(\ell)}$  determines the shadow of  $H_g^{(\ell)}$ .

Recall from Section 3.4 that the signed permutation representation of  $G^{(\ell)}$  naturally induces a permutation representation of the same degree, and this permutation representation factors through the quotient  $\bar{G}^{(\ell)} = G^{(\ell)}/2$  of  $G^{(\ell)}$  by its unique central subgroup of order 2 (except in case  $\ell = 2$  when the signed and unsigned permutation representations coincide, and we have  $G^{(2)} = \bar{G}^{(2)}$ ). Let  $g \mapsto \bar{g}$  denote the natural map  $G^{(\ell)} \rightarrow \bar{G}^{(\ell)}$  and observe that (for  $\ell > 2$ ) if the unique central involution of  $G^{(\ell)}$  belongs

to the cyclic subgroup  $\langle g \rangle$  generated by  $g$  then  $o(g) = 2o(\bar{g})$  and otherwise  $o(g) = o(\bar{g})$ . We say that  $g \in G^{(\ell)}$  is *split* over  $\bar{G}^{(\ell)}$  in case  $o(g) = o(\bar{g})$  and we call  $g$  *non-split* otherwise. (By this definition every element of  $G^{(2)} = \bar{G}^{(2)}$  is split.)

Recall from Section 3.4 that  $g \mapsto \Pi_g^{(\ell)}$  denotes the map attaching signed permutation Frame shapes to elements of  $G^{(\ell)}$  and  $g \mapsto \bar{\Pi}_g^{(\ell)}$  denotes the Frame shapes (actually cycle shapes) arising from the (unsigned) permutation representation (on  $m = 24/(\ell - 1)$  points). Taking a formal product of Frame shapes  $\tilde{\Pi}_g^{(\ell)} = \Pi_g^{(\ell)} \bar{\Pi}_g^{(\ell)}$  (defined so that  $j^{m_1} j^{m_2} = j^{m_1+m_2}$ , &c.) we obtain the Frame shape of  $g \in G^{(\ell)}$  regarded as a permutation of the  $2m$  points  $\{\pm e_i\}$  for  $i \in \Omega^{(\ell)}$  (cf. Section 3.4). In particular,  $\tilde{\Pi}_g$  is a cycle shape and none of the exponents appearing in  $\tilde{\Pi}_g$  are negative. Given  $g \in G^{(\ell)}$  and  $\tilde{\Pi}_g = j_1^{m_1} \cdots j_k^{m_k}$  with  $j_1 < \cdots < j_k$  and  $m_i > 0$  define  $N_g = j_1 j_k$ . That is, set  $N_g$  to be the product of the shortest and longest cycle lengths appearing in a cycle decomposition for  $g$  regarded as a permutation on the  $2m$  points  $\{\pm e_i\}$ . Now define the symbols  $n_g|h_g$  by setting  $n_g = o(\bar{g})$  for all  $g \in G^{(\ell)}$  and all  $\ell \in \Lambda$ , and by setting  $h_g = N_g/n_g$  for all  $g \in G^{(\ell)}$  and  $\ell \in \Lambda$  except when  $\ell = 4$  and  $g$  is non-split in which case set  $h_g = N_g/2n_g$ . The symbols  $n_g|h_g$  are specified in Tables B.7 to B.12, and also in Tables C.1 to C.28, in the rows labelled  $\Gamma_g$ . We omit the  $|h_g$  when  $h_g = 1$ , so  $n_g$  is a shorthand for  $n_g|1$  in these tables.

The significance of the value  $n_g$  is that it is the minimal positive integer  $n$  for which  $H_g^{(\ell)}$  is a mock modular form (of weight  $1/2$ ) on  $\Gamma_0(n)$ , and the significance of  $h_g$  is that, as we shall see momentarily, it is the minimal positive integer  $h$  for which the multiplier for  $H_g^{(\ell)}$  coincides with the conjugate multiplier for the (vector-valued) cusp form  $S^{(\ell)}$  when restricted to  $\Gamma_0(nh)$  (for  $n = n_g$ ). Since  $n_g h_g \leq N_g$  for all  $g$  the multiplier for  $H_g^{(\ell)}$  coincides with the conjugate multiplier for  $S^{(\ell)}$  when regarded as a mock modular form on  $\Gamma_0(N_g)$  for all  $g$ . It is very curious that this coincidence of multipliers extends to the larger group  $\Gamma_0(N_g/2)$  in the case that  $\ell = 4$  and  $g$  is non-split (i.e.,  $o(g) = 2o(\bar{g})$ ).

Given a pair of positive integers  $(n, h)$  we define a matrix-valued function  $\rho_{n|h}^{(\ell)}$  on  $\Gamma_0(n)$  as follows. For each  $\ell \in \Lambda$  let the integer  $v^{(\ell)}$  be as specified

Table 6: Admissible  $v^{(\ell)}$ .

$\ell$	2	3	4	5	7	13
$v^{(\ell)}$	1	5	3	7	1	7



in Table 6. When  $h$  divides  $n$  we set

$$(4.40) \quad \rho_{n|h}^{(\ell)}(\gamma) = e\left(-v^{(\ell)} \frac{cd}{nh}\right) I_{\ell-1},$$

where  $I_{\ell-1}$  denotes the  $(\ell - 1) \times (\ell - 1)$  identity matrix. When  $h$  does not divide  $n$  and  $n$  is even we set

$$(4.41) \quad \rho_{n|h}^{(\ell)}(\gamma) = e\left(-v^{(\ell)} \frac{cd}{nh} \frac{(n, h)}{n}\right) J_{\ell-1}^{c(d+1)/n} K_{\ell-1}^{c/n},$$

where  $(n, h)$  denotes the greatest common divisor of  $n$  and  $h$ , and  $J_{\ell-1}$  and  $K_{\ell-1}$  are the  $(\ell - 1) \times (\ell - 1)$  matrices given by

$$(4.42) \quad J_{\ell-1} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (-1)^\ell \end{pmatrix}, \quad K_{\ell-1} = \begin{pmatrix} 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & 1 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 & 0 \end{pmatrix},$$

and when  $h$  does not divide  $n$  and  $n$  is odd we set

$$(4.43) \quad \rho_{n|h}^{(\ell)}(\gamma) = e\left(-v^{(\ell)} \frac{cd}{nh} \frac{n}{(n, h)}\right) J_{\ell-1}^{c(d+1)/n} K_{\ell-1}^{c/n}.$$

Now the multiplier system  $\nu_g^{(\ell)}$  for the Umbral mock modular form  $H_g^{(\ell)}$  is the matrix-valued function on  $\Gamma_0(n) = \Gamma_0(n_g)$  given simply by

$$(4.44) \quad \nu_g^{(\ell)} = \nu_{n|h}^{(\ell)} = \rho_{n|h}^{(\ell)} \overline{\sigma^{(\ell)}},$$

where  $n = n_g$  and  $h = h_g$  and  $\sigma^{(\ell)} = (\sigma_{ij}^{(\ell)})$  denotes the (matrix-valued) multiplier system for the (vector-valued) theta series  $S^{(\ell)}$  (cf. (2.26)) and satisfies

$$(4.45) \quad \sigma^{(\ell)}(\gamma) S^{(\ell)}(\gamma\tau) \text{jac}(\gamma, \tau)^{3/4} = S^{(\ell)}(\tau)$$

for  $\gamma \in \Gamma_0(1) = SL_2(\mathbb{Z})$  where  $\text{jac}(\gamma, \tau) = (c\tau + d)^{-2}$  in case  $\gamma$  has lower row  $(c, d)$ .

Recall from Section 3.4 that the character of  $G^{(\ell)}$  attached to its signed permutation representation is denoted  $g \mapsto \chi_g^{(\ell)}$  in Tables B.7–B.12 and that of the (unsigned) permutation representation is denoted  $g \mapsto \bar{\chi}_g^{(\ell)}$ . Define  $\chi_{g,r}^{(\ell)}$  for  $0 < r < \ell$  by setting  $\chi_{g,r}^{(\ell)} = \chi_g^{(\ell)}$  in case  $r$  is even and  $\chi_{g,r}^{(\ell)} = \bar{\chi}_g^{(\ell)}$  otherwise. Then the shadow of  $H_g^{(\ell)}$  is the function  $S_g^{(\ell)} = (S_{g,r}^{(\ell)})$  with

components related to those of  $S^{(\ell)}$  by  $S_{g,r}^{(\ell)} = \chi_{g,r}^{(\ell)} S_r^{(\ell)}$ . In particular,  $H_g^{(\ell)}$  is a vector-valued modular form of weight  $1/2$  for  $\Gamma_0(n_g)$  when  $\chi_g^{(\ell)} = \bar{\chi}_g^{(\ell)} = 0$ .

To summarize, we claim that our proposed McKay–Thompson series  $H_g^{(\ell)}$  are such that if we define  $\hat{H}_g^{(\ell)} = (\hat{H}_{g,r}^{(\ell)})$  by setting

$$(4.46) \quad \hat{H}_{g,r}^{(\ell)}(\tau) = H_{g,r}(\tau) - \frac{\chi_{g,r}^{(\ell)}}{\sqrt{2\ell}} \frac{1}{(4i)^{1/2}} \int_{-\bar{\tau}}^{i\infty} (z + \tau)^{-1/2} S_r^{(\ell)}(z) dz$$

for  $0 < r < \ell$  then  $\hat{H}_g^{(\ell)}$  is invariant for the weight  $1/2$  action of  $\Gamma_0(n)$  on  $(\ell - 1)$ -vector-valued functions on  $\mathbb{H}$  given by

$$(4.47) \quad \left( \hat{H}_g^{(\ell)} \Big|_{1/2, n|h} \gamma \right) (\tau) = \nu_{n|h}(\gamma) \hat{H}_g^{(\ell)}(\gamma\tau) \text{jac}(\gamma, \tau)^{1/4},$$

where  $n = n_g$  and  $h = h_g$  and  $\nu_{n|h}$  is defined by (4.44). This statement completely describes the (conjectured) automorphy of the mock modular forms  $H_g^{(\ell)}$  (cf. Section 5.3).

## 5. Conjectures

We have described the Umbral forms  $Z^{(\ell)}$ ,  $H^{(\ell)}$  and  $\Phi^{(\ell)}$  in Section 2 and the Umbral groups  $G^{(\ell)}$  in Section 3 and we have introduced families  $\{H_g^{(\ell)} \mid g \in G^{(\ell)}\}$  of vector-valued mock modular forms in Section 4. The discussions of those sections clearly demonstrate the distinguished nature of these objects, and we have mentioned some coincidences relating the groups  $G^{(\ell)}$  and the forms  $H_g^{(\ell)}$  directly. In this section we present, in a more systematic fashion, evidence that the relationship between the  $G^{(\ell)}$  and the  $H^{(\ell)}$  is more than coincidental. Our observations lead naturally to conjectures that we hope will serve as first steps in revealing the structural nature of the mechanism underlying Umbral moonshine.

### 5.1. Modules

As was mentioned in Section 4, after comparison of the character tables (cf. Appendix B.1) of the Umbral groups  $G^{(\ell)}$  with the Fourier coefficient tables (cf. Appendix C) for the forms  $H_g^{(\ell)}$  it becomes apparent that the low-degree Fourier coefficients of  $H^{(\ell)} = H_e^{(\ell)}$  may be interpreted as degrees of representations of  $G^{(\ell)}$  in such a way that the corresponding coefficients of  $H_g^{(\ell)}$  are recovered by substituting character values at  $g$ ; we have tabulated evidence for this in the form of explicit combinations of irreducible representations in Appendix D. This observation suggests the existence

of bi-graded  $G^{(\ell)}$ -modules  $K^{(\ell)} = \bigoplus K_{r,d}^{(\ell)}$  whose bi-graded dimensions are recovered via Fourier coefficients from the vector-valued mock modular forms  $H^{(\ell)} = (H_r^{(\ell)})$ .

**Conjecture 5.1.** *We conjecture that for  $\ell \in \Lambda = \{2, 3, 4, 5, 7, 13\}$  there exist naturally defined  $\mathbb{Z} \times \mathbb{Q}$ -graded  $G^{(\ell)}$ -modules*

$$(5.1) \quad K^{(\ell)} = \bigoplus_{\substack{r \in \mathbb{Z} \\ 0 < r < \ell}} K_r^{(\ell)} = \bigoplus_{\substack{r, k \in \mathbb{Z} \\ 0 < r < \ell}} K_{r, k-r^2/4\ell}^{(\ell)}$$

such that the graded dimension of  $K^{(\ell)}$  is related to the vector-valued mock modular form  $H^{(\ell)} = (H_r^{(\ell)})$  by

$$(5.2) \quad H_r^{(\ell)}(\tau) = -2\delta_{r,1}q^{-1/4\ell} + \sum_{\substack{k \in \mathbb{Z} \\ r^2 - 4k\ell < 0}} \dim \left( K_{r, k-r^2/4\ell}^{(\ell)} \right) q^{k-r^2/4\ell}$$

for  $q = e(\tau)$  and such that the vector-valued mock modular forms  $H_g^{(\ell)} = (H_{g,r}^{(\ell)})$  described in Section 4 (and partially in the tables of Appendix C) are recovered from  $K^{(\ell)}$  via graded trace functions according to

$$(5.3) \quad H_{g,r}^{(\ell)}(\tau) = -2\delta_{r,1}q^{-1/4\ell} + \sum_{\substack{k \in \mathbb{Z} \\ r^2 - 4k\ell < 0}} \text{tr}_{K_{r, k-r^2/4\ell}^{(\ell)}}(g)q^{k-r^2/4\ell}.$$

**Remark 5.2.** Recall that a *superspace* is a  $\mathbb{Z}/2\mathbb{Z}$ -graded vector space and if  $V = V_0 \oplus V_1$  is such an object and  $T : V \rightarrow V$  is a linear operator preserving the grading then the *supertrace* of  $T$  is given by  $\text{str}_V T = \text{tr}_{V_0} T - \text{tr}_{V_1} T$  where  $\text{tr}_W T$  denotes the usual trace of  $T$  on  $W$ . Since the coefficient of  $q^{-1/4\ell}$  in  $H_{g,1}^{(\ell)}$  is  $c_{g,1}^{(\ell)}(-1/4\ell) = -2$  for all  $g \in G^{(\ell)}$  for all  $\ell$  it is natural to expect that this term may be interpreted as the supertrace of  $g \in G^{(\ell)}$  on a trivial  $G^{(\ell)}$ -supermodule  $K_{1,-1/4\ell}^{(\ell)}$  with vanishing even part and 2-dimensional odd part. Thus Conjecture 5.1 implies the existence of a  $G^{(\ell)}$ -supermodule  $K^{(\ell)} = \bigoplus K_{r,d}^{(\ell)}$  such that  $K_{r,d}^{(\ell)}$  is purely even or purely odd according as  $d$  is positive or negative, and such that  $H_{g,r}^{(\ell)} = \sum_k \text{str}_{K_{r, k-r^2/4\ell}^{(\ell)}}(g)q^{k-r^2/4\ell}$  for each  $0 < r < \ell$  and  $g \in G^{(\ell)}$ .

**Remark 5.3.** For  $\ell \in \Lambda$  and  $0 < r, s < \ell$  we have that  $r^2 \equiv s^2 \pmod{4\ell}$  implies  $r = s$  so the first index  $r$  in  $K_{r,d}^{(\ell)}$  can be deduced from  $d$  since it is the unique  $0 < r < \ell$  such that  $d + r^2/4\ell$  is an integer. Thus we may dispense with the first of the two gradings on the conjectural  $G^{(\ell)}$ -modules  $K^{(\ell)}$  and

regard them as  $\mathbb{Q}$ -graded  $K^{(\ell)} = \bigoplus_d K_d^{(\ell)}$  by the rationals of the form  $d = n - r^2/4\ell$  for  $n \in \mathbb{Z}$  and  $0 < r < \ell$  without introducing any ambiguity.

## 5.2. Moonshine

In the case of monstrous moonshine the McKay–Thompson series  $T_g$  for  $g$  in the monster group have the astonishing property that they all serve as generators for the function fields of their invariance groups (cf. [1]). In other words, if  $\Gamma_g$  is the subgroup of  $PSL_2(\mathbb{R})$  consisting of the isometries  $\gamma : \mathbb{H} \rightarrow \mathbb{H}$  such that  $T_g(\gamma\tau) = T_g(\tau)$  for all  $\tau \in \mathbb{H}$  then  $T_g$  induces an isomorphism from the compactification of  $\Gamma_g \backslash \mathbb{H}$  to the Riemann sphere (being the one point compactification of  $\mathbb{C}$ ). In particular,  $\Gamma_g$  is a genus zero subgroup of  $PSL_2(\mathbb{R})$ , and so this property is commonly referred to as the *genus zero property* of monstrous moonshine.

By now there are many methods extant for constructing graded vector spaces with algebraic structure whose graded dimensions are modular functions — suitable classes of vertex algebras, for example, serve this purpose (cf. [114, 115]) — and so one can expect to obtain analogues of the McKay–Thompson series  $T_g$  by equipping such an algebraic structure with the action of a group. But there is no guarantee that such a procedure will result in functions that have the genus zero property of monstrous moonshine, and so it is this genus zero property that distinguishes the observations of McKay, Thompson, Conway and Norton regarding the Monster group from any number of more generic connections between finite groups and modular functions. In what follows, we will propose a conjecture that may be regarded as the natural analogue of the Conway–Norton conjecture for Umbral moonshine.

Suppose that  $T : \mathbb{H} \rightarrow \mathbb{C}$  is a holomorphic function with invariance group  $\Gamma < PSL_2(\mathbb{R})$  and a simple pole in  $q = e(\tau)$  as  $\tau \rightarrow i\infty$ . Suppose also that  $\Gamma$  is commensurable with  $PSL_2(\mathbb{Z})$  and that the *translation subgroup*  $\Gamma_\infty$ , consisting of the elements of  $\Gamma$  with upper-triangular preimages in  $SL_2(\mathbb{R})$ , is generated by  $\tau \mapsto \tau + 1$ . It is shown in [25] that such a function  $T$  is a generator for the field of  $\Gamma$ -invariant functions on  $\mathbb{H}$  if and only if  $T(\tau)$  coincides with the weight 0 *Rademacher sum*

$$(5.4) \quad R_\Gamma(\tau) = \text{Reg} \left( \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} q^{-1} \Big|_0 \gamma \right)$$

attached to  $\Gamma$ . The sum here is over representatives for the cosets of  $\Gamma_\infty$  in  $\Gamma$  and  $f \mapsto f|_0\gamma$  denotes the weight 0 action of  $\gamma$  on holomorphic functions

(viz.,  $(f|_0\gamma)(\tau) = f(\gamma\tau)$ ). We write  $\text{Reg}(\cdot)$  to indicate a regularization procedure first realized by Rademacher (for the case that  $\Gamma = \text{PSL}_2(\mathbb{Z})$ ) in [26]. According then to the result of [25] the genus zero property of monstrous moonshine may be reformulated in the following way.

For each element  $g$  in the monster group the McKay–Thompson series  $T_g$  satisfies  $T_g = R_{\Gamma_g}$  when  $\Gamma_g$  is the invariance group of  $T_g$ .

This may be compared with the article [24] which considers the mock modular forms  $H_g^{(2)}$  attached to the largest Mathieu group  $G^{(2)} \simeq M_{24}$  via the observation of Eguchi–Ooguri–Tachikawa and applies some of the philosophy of [25] to the problem of finding a uniform construction for these functions. The solution developed in [24] is that the function  $H_g^{(2)}$  coincides with the weight  $1/2$  Rademacher sum

$$(5.5) \quad R_{n|h}^{(2)}(\tau) = \text{Reg} \left( \sum_{\gamma \in \Gamma_\infty \setminus \Gamma_0(n)} -2q^{-1/8} \Big|_{1/2, n|h} \gamma \right),$$

where  $n = n_g$  and  $h = h_g$  are integers determined by the defining permutation representation of  $M_{24}$  (cf. Table B.7) and  $f \mapsto f|_{1/2, n|h} \gamma$  denotes a certain weight  $1/2$  action of  $\Gamma_0(n)$  determined by  $n$  and  $h$ . (We refer to [24] for full details.) By comparison with the previous paragraph we thus arrive at a direct analogue of the genus zero property of monstrous moonshine that holds for all the Umbral forms with  $\ell = 2$ .

For each element  $g$  in the largest Mathieu group the McKay–Thompson series  $H_g^{(2)}$  satisfies  $H_g^{(2)} = R_{n|h}^{(2)}$  when  $n = n_g$  and  $h = h_g$ .

We conjecture that this genus zero property extends to all the functions of Umbral moonshine.

**Conjecture 5.4 (Umbral moonshine).** *We conjecture that for each  $\ell \in \Lambda$  and each  $g \in G^{(\ell)}$  we have  $H_g^{(\ell)} = R_{n|h}^{(\ell)}$  where  $n = n_g$  and  $h = h_g$  are as specified in Appendix B.2 and*

$$(5.6) \quad R_{n|h}^{(\ell)} = \text{Reg} \left( \sum_{\gamma \in \Gamma_\infty \setminus \Gamma_0(n)} \left( \begin{array}{c} -2q^{-1/4\ell} \\ 0 \\ \vdots \\ 0 \end{array} \right) \Big|_{1/2, n|h} \gamma \right)$$

is a vector-valued generalization of the Rademacher sum  $R_{n|h}^{(2)}$  adapted to the weight  $1/2$  action of  $\Gamma_0(n)$  on  $(\ell - 1)$ -vector-valued functions on  $\mathbb{H}$  that is defined in (4.47).

### 5.3. Modularity

A beautiful feature of the Umbral moonshine conjecture is that it implies the precise nature of the modularity of the Thompson series  $H_g^{(\ell)}$ . We record these implications explicitly as conjectures in this short section.

**Conjecture 5.5.** *We conjecture that the graded supertrace functions (5.3) for fixed  $\ell \in \Lambda$  and  $g \in G^{(\ell)}$  and varying  $0 < r < \ell$  define the components of a vector-valued mock modular form  $H_g^{(\ell)}$  of weight  $1/2$  on  $\Gamma_0(n_g)$  with shadow function  $S_g^{(\ell)} = (S_{g,r}^{(\ell)}) = (\chi_{g,r}^{(\ell)} S_r^{(\ell)})$  where  $S^{(\ell)} = (S_r^{(\ell)})$  is the vector-valued theta series described in Section 2.3, the  $\chi_{g,r}^{(\ell)}$  are determined from the twisted Euler characters of  $G^{(\ell)}$  (cf. Appendix B.2) by  $\chi_{g,r}^{(\ell)} = \bar{\chi}_g^{(\ell)}$  for  $r$  odd and  $\chi_{g,r}^{(\ell)} = \chi_g^{(\ell)}$  for  $r$  even, and  $n_g$  denotes the order of the image of  $g \in G^{(\ell)}$  in the factor group  $\bar{G}^{(\ell)}$  (cf. Sections 3.1 and 3.4).*

Recall from Section 4.8 that for  $\ell \in \Lambda$  and a pair  $(n, h)$  of positive integers we have the matrix-valued function  $\nu_{n|h}^{(\ell)}$  on  $\Gamma_0(n)$  defined in (4.44). Recall also that we have attached a pair  $(n_g, h_g)$  to each  $g \in G^{(\ell)}$  for each  $\ell \in \Lambda$  in Section 4.8.

**Conjecture 5.6.** *We conjecture that the multiplier system of  $H_g^{(\ell)}$  is given by  $\nu_{n|h}^{(\ell)}$  when  $n = n_g$  and  $h = h_g$  for all  $g \in G^{(\ell)}$  for all  $\ell \in \Lambda$ .*

### 5.4. Discriminants

One of the most striking features of Umbral moonshine is the apparently intimate relation between the number fields on which the irreducible representations of  $G^{(\ell)}$  are defined and the discriminants of the vector-valued mock modular form  $H^{(\ell)}$ . We discuss the evidence for this relation and formulate conjectures about it in this section.

First we observe that the discriminants of the components  $H_r^{(\ell)}$  of the mock modular form  $H^{(\ell)} = H_e^{(\ell)}$  determine some important properties of the representations of  $G^{(\ell)}$ , where we say that an integer  $D$  is a *discriminant of  $H^{(\ell)}$*  if there exists a term  $q^d = q^{-\frac{D}{4\ell}}$  with non-vanishing Fourier coefficient in

at least one of the components. The following result can be verified explicitly using the tables in Appendices B and C.

**Proposition 5.7.** *Let  $\ell \in \Lambda$ . If  $n > 1$  is an integer satisfying*

- (1) *there exists an element of  $G^{(\ell)}$  of order  $n$ , and*
- (2) *there exists an integer  $\lambda$  that is co-prime to  $n$  such that  $D = -n\lambda^2$  is a discriminant of  $H^{(\ell)}$ ,*

*then there exists at least one pair of irreducible representations  $\varrho$  and  $\varrho^*$  of  $G^{(\ell)}$  and at least one element  $g \in G^{(\ell)}$  such that  $\text{tr}_{\varrho}(g)$  is not rational but*

$$(5.7) \quad \text{tr}_{\varrho}(g), \text{tr}_{\varrho^*}(g) \in \mathbb{Q}(\sqrt{-n})$$

*and  $n$  divides  $o(g)$ .*

The finite list of integers  $n$  satisfying the two conditions of Proposition 5.7 is given in Table 7.

From now on we say that an irreducible representation  $\varrho$  of the Umbral group  $G^{(\ell)}$  is of *type  $n$*  if  $n$  is an integer satisfying the two conditions of Proposition 5.7 and the character values of  $\varrho$  generate the field  $\mathbb{Q}(\sqrt{-n})$ . Evidently, irreducible representations of type  $n$  come in pairs  $(\varrho, \varrho^*)$  with  $\text{tr}_{\varrho^*}(g)$  the complex conjugate of  $\text{tr}_{\varrho}(g)$  for all  $g \in G^{(\ell)}$ . The list of all irreducible representations of type  $n$  is given in Table 7. (Cf. Appendix B.1 for the character tables of the  $G^{(\ell)}$  and our notation for irreducible representations.)

Recall that the *Frobenius–Schur indicator* of an irreducible ordinary representation of a finite group is 1,  $-1$  or 0 according as the representation admits an invariant symmetric bilinear form, an invariant skew-symmetric bilinear form, or no invariant bilinear form, respectively. The representations

Table 7: The irreducible representations of type  $n$ .

$\ell$	$n$	$(\varrho, \varrho^*)$
2	7,15,23	$(\chi_3, \chi_4), (\chi_5, \chi_6), (\chi_{10}, \chi_{11}), (\chi_{12}, \chi_{13}), (\chi_{15}, \chi_{16})$
3	5,8,11,20	$(\chi_4, \chi_5), (\chi_{16}, \chi_{17}), (\chi_{20}, \chi_{21}), (\chi_{22}, \chi_{23}), (\chi_{25}, \chi_{26})$
4	3,7	$(\chi_2, \chi_3), (\chi_{13}, \chi_{14}), (\chi_{15}, \chi_{16})$
5	4	$(\chi_8, \chi_9), (\chi_{10}, \chi_{11}), (\chi_{12}, \chi_{13})$
7	3	$(\chi_2, \chi_3), (\chi_6, \chi_7)$
13	4	$(\chi_3, \chi_4)$

admitting no invariant bilinear form are precisely those whose character values are not all real. We can now state the next observation.

**Proposition 5.8.** *For each  $\ell \in \Lambda$  an irreducible representation  $\varrho$  of  $G^{(\ell)}$  has Frobenius–Schur indicator 0 if and only if it is of type  $n$  for some  $n$ .*

The Schur index of an irreducible representation  $\varrho$  of a finite group  $G$  is the smallest positive integer  $s$  such that there exists a degree  $s$  extension  $k$  of the field generated by the character values  $\text{tr}_\varrho(g)$  for  $g \in G$  such that  $\varrho$  can be realized over  $k$ . Inspired by Proposition 5.8 we make the following conjecture.

**Conjecture 5.9.** *If  $\varrho$  is an irreducible representation of  $G^{(\ell)}$  of type  $n$  then the Schur index of  $\varrho$  is equal to 1.*

In other words, we conjecture that the irreducible  $G^{(\ell)}$ -representations of type  $n$  can be realized over  $\mathbb{Q}(\sqrt{-n})$ . For  $\ell = 2$  this speculation is in fact a theorem, since it is known [116] that the Schur indices for  $M_{24}$  are always 1. For  $\ell = 3$  it is also known [116] that the Schur indices for  $\bar{G}^{(3)} = M_{12}$  are also always 1. Moreover, the representations of  $G^{(3)} \simeq 2.\bar{G}^{(3)}$  with characters  $\chi_{16}$  and  $\chi_{17}$  in the notation of Table B.2 have been constructed explicitly over  $\mathbb{Q}(\sqrt{-2})$  in [117]. Finally, Proposition 5.8 constitutes a non-trivial consistency check for Conjecture 5.9 since the Schur index is at least 2 for a representation with Frobenius–Schur indicator equal to  $-1$ .

Armed with the preceding discussion we are now ready to state our main observation for the discriminant property of Umbral moonshine. For the purpose of stating this we temporarily write  $K_{r,d}^{(\ell)}$  for the ordinary representation of  $G^{(\ell)}$  with character  $g \mapsto c_{g,r}^{(\ell)}(d)$  where the coefficients  $c_{g,r}^{(\ell)}(d)$  are assumed to be those given in C.

**Proposition 5.10.** *Let  $n$  be one of the integers in Table 7 and let  $\lambda_n$  be the smallest positive integer such that  $D = -n\lambda_n^2$  is a discriminant of  $H^{(\ell)}$ . Then  $K_{r,-D/4\ell}^{(\ell)} = \varrho_n \oplus \varrho_n^*$  where  $\varrho_n$  and  $\varrho_n^*$  are dual irreducible representations of type  $n$ . Conversely, if  $\varrho$  is an irreducible representation of type  $n$  and  $-D$  is the smallest positive integer such that  $K_{r,-D/4\ell}^{(\ell)}$  has  $\varrho$  as an irreducible constituent then there exists an integer  $\lambda$  such that  $D = -n\lambda^2$ .*

Extending this we make the following conjecture.



**Conjecture 5.11.** *If  $D$  is a discriminant of  $H^{(\ell)}$  which satisfies  $D = -n\lambda^2$  for some integer  $\lambda$  then the representation  $K_{r,-D/4\ell}^{(\ell)}$  has at least one dual pair of irreducible representations of type  $n$  arising as irreducible constituents.*

We conclude this section with conjectures arising from the observation (cf. Appendix D) that the conjectural  $G^{(\ell)}$ -module  $K_{r,d}^{(\ell)}$  is typically isomorphic to several copies of a single representation. Say a  $G$ -module  $V$  is a *doublet* if it is isomorphic to the direct sum of two copies of a single representation of  $G$ .

**Conjecture 5.12.** *For  $\ell \in \Lambda = \{2, 3, 4, 5, 7, 13\}$  the representation  $K_{r,-D/4\ell}^{(\ell)}$  is a doublet if and only if  $D \neq -n\lambda^2$  for any integer  $\lambda$  for any  $n$  satisfying the conditions of Proposition 5.7.*

To see some evidence for Conjecture 5.12 one can inspect the proposed decompositions of the representations  $K_{r,d}^{(\ell)}$  in the tables in Appendix D for the following discriminants:

- $-D = 7, 15, 23, 63, 135, 175, 207$  for  $\ell = 2$ ,
- $-D = 8, 11, 20, 32, 44, 80$  for  $\ell = 3$ ,
- $-D = 7, 12, 28, 63, 108$  for  $\ell = 4$ ,
- $-D = 4, 16, 64, 144, 196$  for  $\ell = 5$ ,
- $-D = 3\lambda^2$ ,  $\lambda = 1, \dots, 9$ ,  $\lambda \neq 7$  for  $\ell = 7$ ,
- $-D = 4\lambda^2$ ,  $\lambda = 1, \dots, 11$  for  $\ell = 13$ .

## 5.5. Geometry and physics

Beyond the conjectures already mentioned above several interesting and important questions remain regarding the structural nature of Umbral moonshine. In the case that  $\ell = 2$  there are strong indications that a deep relationship to  $K3$  surfaces — extending in some way the relation [118, 119] between finite groups of  $K3$  surface symplectomorphisms and subgroups of  $M_{23}$  — is responsible for the relationship between  $G^{(2)} \simeq M_{24}$  and  $H^{(2)}$ . One such indication is the fact that the Jacobi form  $Z^{(2)}$ , from which  $H^{(2)}$  may be obtained by decomposing with respect to superconformal characters, coincides with the elliptic genus of a(ny) complex  $K3$  surface. It is

natural then to ask if there are analogous geometric interpretations for the remaining extremal Jacobi forms  $Z^{(\ell)}$  for  $\ell \in \Lambda$ , and a positive answer to this question will be a first step in determining the geometric significance of the Umbral groups  $G^{(\ell)}$  and the attached mock modular forms  $H_g^{(\ell)}$ .

In a series of papers [87–89, 120, 121] Gritsenko–Nikulin develop applications of Siegel modular forms to mirror symmetry for  $K3$  surfaces (cf. [122]). Many of the Siegel forms arising are realized as additive or exponential lifts of (weak) Jacobi forms, and amongst many examples the exponential lifts  $\Phi^{(\ell)}$  of the particular forms  $Z^{(\ell)}$  for  $\ell \in \{2, 3, 4, 5\} \subset \Lambda$  appear in connection with explicitly defined families of lattice polarized  $K3$  surfaces in [87]. It is natural to ask if there is an analogous relationship between the Umbral  $Z^{(\ell)}$  and suitably defined mirror families of  $K3$  surfaces for the remaining  $\ell$  in  $\Lambda$ .

As mentioned in the introduction, monstrous moonshine involves aspects of conformal field theory and string theory and there are several hints that Umbral moonshine will also play a role in string theory. The most obvious hint at  $\ell = 2$  is through the fact mentioned above that the weak Jacobi form  $Z^{(2)}$  coincides with the elliptic genus of a  $K3$  surface and the fact that  $K3$  surfaces play a prominent role in the study of superstring compactification. The Siegel form  $\Phi^{(2)} = (\Delta_5)^2$  which is the multiplicative lift of the weak Jacobi form  $Z^{(2)}$  also plays a distinguished role in type II string theory on  $K3 \times E$  where  $E$  is an elliptic curve and  $1/\Phi^{(2)}$  occurs as the generating function which counts the number of  $1/4$  BPS dyon states [123]. BPS states in a supersymmetric theory are states that are annihilated by some of the supercharges of the theory; the  $1/4$  BPS states of interest here are annihilated by 4 of the 16 supercharges of a theory with  $N = 4$  supersymmetry in four spacetime dimensions and are termed *dyons* because they necessarily carry both electric and magnetic charges. The relation between the zeros of  $\Phi^{(2)}$  and the wall-crossing phenomenon in this theory has been studied in [124–127] and this relation leads to a connection between the counting of BPS black hole states in string theory and mock modular forms [45]. For a pedagogical review of this material see [38, 128].

There are several hints that Umbral moonshine for  $\ell > 2$  will also play a role in string theory. The first of these occurs in the study of  $N = 2$  dual pairs of string theories [129, 130]. An  $N = 2$  dual pair consists of a compactification of the heterotic string on a product of a  $K3$  surface with an elliptic curve  $E$  with a specific choice of  $E_8 \times E_8$  gauge bundle on  $K3 \times E$  and a conjectured dual description of this model in terms of IIA string theory on a Calabi–Yau 3-fold  $X$  which admits a  $K3$  fibration. The low-energy description of such theories involves a vector-multiplet moduli space which

is a special Kähler manifold of the form

$$(5.8) \quad \mathcal{M}_{vm}^{s+2,2} = \frac{SU(1,1)}{U(1)} \times \mathcal{N}^{s+2,2},$$

where  $\mathcal{N}^{s+2,2}$  is the quotient  $\Gamma \backslash \mathcal{H}^{s+1,1}$  of the generalized upper half-plane

$$(5.9) \quad \mathcal{H}^{s+1,1} = O(s+2, 2; \mathbb{R}) / (O(s+2) \times O(2))$$

by an arithmetic subgroup  $\Gamma$  in  $SO(s+2, 2; \mathbb{R})$ , with  $\Gamma$  depending on the specific model in question. One-loop string computations in the heterotic string lead to automorphic forms on  $\mathcal{N}^{s+2,2}$  via a generalized theta lift constructed in [131, 132], cf. [133] for a review. In models with  $s = 1$  this leads to automorphic forms on  $\Gamma \backslash O(3, 2; \mathbb{R}) / (O(3) \times O(2))$ , that is to Siegel modular forms on the genus 2 Siegel upper half-space  $\mathbb{H}_2 \cong \mathcal{H}^{2,1}$ .

A particular example of such an  $N = 2$  dual pair was studied in [134, 135] with  $s = 1$  and involving a dual description in terms of Type II string theory on a Calabi–Yau 3-fold with Hodge numbers  $(h^{1,1}, h^{2,1}) = (4, 148)$ . In the heterotic description the arithmetic subgroup which appears is the paramodular group  $\Gamma_2$ , the Siegel modular form which occurs is the exponential lift of  $Z^{(3)}$  described in Section 2.6 and the Calabi–Yau 3-fold is also elliptically fibred and the elliptic fibre is of  $E_7$  type. We thus see many of the ingredients of  $\ell = 3$  Umbral moonshine appearing in the context of a specific dual pair of string theories with  $N = 2$  spacetime supersymmetry. It will be very interesting to investigate whether other aspects of  $\ell = 3$  Umbral moonshine can be realized in these models and whether  $N = 2$  dual pairs exist which exhibit elements of Umbral moonshine for  $\ell > 3$ .

A second way that elements of Umbral moonshine for  $\ell > 2$  are likely to appear in string theory involves a generalization of the generating function discussed above that counts 1/4 BPS states. Type II string theory on  $K3 \times E$  preserves  $N = 4$  spacetime supersymmetry. There exist orbifold versions of this model that also preserve  $N = 4$  spacetime supersymmetry that are known in the physics literature as CHL models. To construct such a model one chooses a  $K3$  surface with a  $\mathbb{Z}/m\mathbb{Z}$  hyper-Kähler automorphism and constructs the orbifold theory  $(K3 \times E) / (\mathbb{Z}/m\mathbb{Z})$  where  $\mathbb{Z}/m\mathbb{Z}$  is a freely acting symmetry realized as the product of a hyper-Kähler automorphism of the  $K3$  surface and an order  $m$  translation along  $E$ . For  $m = 2, 3, 4$  it is possible to find  $K3$  surfaces with  $(\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/m\mathbb{Z})$  symmetry, and in this case one can construct a CHL model that utilizes the first  $\mathbb{Z}/m\mathbb{Z}$  factor in the orbifold construction and has the second  $\mathbb{Z}/m\mathbb{Z}$  factor acting as a symmetry which preserves the holomorphic 3-form of the Calabi–Yau space ( $K3 \times$

$E)/(\mathbb{Z}/m\mathbb{Z})$ . It was proposed in [136] that the generating function which counts 1/4 BPS states weighted by an element of  $\mathbb{Z}/m\mathbb{Z}$  is the reciprocal of the Siegel form  $\Phi^{(m+1)}$  for  $m \in \{2, 3, 4\}$ . In both this construction and in the study of  $N = 2$  dual pairs the appearance of some of the Umbral Siegel forms  $\Phi^{(\ell)}$  can be anticipated, although many details remain to be worked out. The action of the Umbral groups  $G^{(\ell)}$  remains more elusive, and deeper insight into possible connections between string theory and Umbral moonshine will undoubtedly require progress in understanding the actions of these groups in terms of their action on BPS states.

We plan to elaborate further on the topics mentioned above in future work.

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## Appendix A. Modular forms

### A.1. Dedekind eta function

The *Dedekind eta function*, denoted  $\eta(\tau)$ , is a holomorphic function on the upper half-plane defined by the infinite product

$$\eta(\tau) = q^{1/24} \prod_{n \geq 1} (1 - q^n),$$

where  $q = e(\tau) = e^{2\pi i \tau}$ . It is a modular form of weight  $1/2$  for the modular group  $SL_2(\mathbb{Z})$  with multiplier  $\epsilon : SL_2(\mathbb{Z}) \rightarrow \mathbb{C}^*$ , which means that

$$\epsilon(\gamma)\eta(\gamma\tau)\text{jac}(\gamma, \tau)^{1/4} = \eta(\tau)$$

for all  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ , where  $\text{jac}(\gamma, \tau) = (c\tau + d)^{-2}$ . The *multiplier system*  $\epsilon$  may be described explicitly as

$$(A.1) \quad \epsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} e(-b/24), & c = 0, d = 1, \\ e(-(a+d)/24c + s(d, c)/2 + 1/8), & c > 0, \end{cases}$$

where  $s(d, c) = \sum_{m=1}^{c-1} (d/c)((md/c))$  and  $((x))$  is 0 for  $x \in \mathbb{Z}$  and  $x - [x] - 1/2$  otherwise. We can deduce the values  $\epsilon(a, b, c, d)$  for  $c < 0$ , or for  $c = 0$  and  $d = -1$ , by observing that  $\epsilon(-\gamma) = \epsilon(\gamma)e(1/4)$  for  $\gamma \in SL_2(\mathbb{Z})$ .

Let  $T$  denote the element of  $SL_2(\mathbb{Z})$  such that  $\text{tr}(T) = 2$  and  $T\tau = \tau + 1$  for  $\tau \in \mathbb{H}$ . Observe that

$$\epsilon(T^m \gamma) = \epsilon(\gamma T^m) = e(-m/24)\epsilon(\gamma)$$

for  $m \in \mathbb{Z}$ .

### A.2. Jacobi theta functions

We define the *Jacobi theta functions*  $\theta_i(\tau, z)$  as follows for  $q = e(\tau)$  and  $y = e(z)$ :

$$\begin{aligned}\theta_1(\tau, z) &= -iq^{1/8}y^{1/2} \prod_{n=1}^{\infty} (1 - q^n)(1 - yq^n)(1 - y^{-1}q^{n-1}), \\ \theta_2(\tau, z) &= q^{1/8}y^{1/2} \prod_{n=1}^{\infty} (1 - q^n)(1 + yq^n)(1 + y^{-1}q^{n-1}), \\ \theta_3(\tau, z) &= \prod_{n=1}^{\infty} (1 - q^n)(1 + yq^{n-1/2})(1 + y^{-1}q^{n-1/2}), \\ \theta_4(\tau, z) &= \prod_{n=1}^{\infty} (1 - q^n)(1 - yq^{n-1/2})(1 - y^{-1}q^{n-1/2}).\end{aligned}$$

Note that there are competing conventions for  $\theta_1(\tau, z)$  in the literature and our normalization may differ from another by a factor of  $-1$  (or possibly  $\pm i$ ).

### A.3. Higher level modular forms

The congruence subgroups of the modular group  $SL_2(\mathbb{Z})$  that are most relevant for this paper are

$$(A.2) \quad \Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}), c = 0 \pmod{N} \right\}.$$

For  $N > 1$  a (non-zero) modular form of weight 2 for  $\Gamma_0(N)$  is given by

$$(A.3) \quad \begin{aligned}\Lambda_N(\tau) &= Nq\partial_q \log \left( \frac{\eta(N\tau)}{\eta(\tau)} \right) \\ &= \frac{N(N-1)}{24} \left( 1 + \frac{24}{N-1} \sum_{k>0} \sigma(k)(q^k - Nq^{Nk}) \right),\end{aligned}$$

where  $\sigma(k)$  is the divisor function  $\sigma(k) = \sum_{d|k} d$ .

Observe that a modular form on  $\Gamma_0(N)$  is a modular form on  $\Gamma_0(M)$  whenever  $N|M$ , and for some small  $N$  the space of forms of weight 2 is spanned by the  $\Lambda_d(\tau)$  for  $d$  a divisor of  $N$ . By contrast, in the case that

$N = 11$  we have the *newform*

$$(A.4) \quad f_{11}(\tau) = \eta^2(\tau)\eta^2(11\tau),$$

which is a cusp form of weight 2 for  $\Gamma_0(11)$  that is not a multiple of  $\Lambda_{11}(\tau)$ . We meet the newforms

$$(A.5) \quad f_{14}(\tau) = \eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau),$$

$$(A.6) \quad f_{15}(\tau) = \eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau),$$

$$(A.7) \quad f_{20}(\tau) = \eta(2\tau)^2\eta(10\tau)^2,$$

at  $N = 14, 15$  and  $20$ , respectively, and together with  $f_N$  the functions  $\Lambda_d(\tau)$  for  $d|N$  span the space of weight 2 forms on  $\Gamma_0(N)$  for  $N = 11, 14, 15$ .

For  $N = 23$  there is a 2-dimensional space of newforms. We may use the basis

$$(A.8) \quad f_{23,a}(\tau) = \frac{\eta(\tau)^3\eta(23\tau)^3}{\eta(2\tau)\eta(46\tau)} + 3\eta(\tau)^2\eta(23\tau)^2 + 4\eta(\tau)\eta(2\tau)\eta(23\tau)\eta(46\tau) \\ + 4\eta(2\tau)^2\eta(46\tau)^2 \\ f_{23,b}(\tau) = \eta(\tau)^2\eta(23\tau)^2$$

satisfying  $f_{23,a} = q + O(q^3)$  and  $f_{23,b} = q^2 + O(q^3)$ . (The normalized Hecke-eigenforms of weight 2 for  $\Gamma_0(23)$  are  $f_{23,a} - \frac{1}{2}(1 \pm \sqrt{5})f_{23,b}$ .)

For  $N = 44$  there is a unique newform up to a multiplicative constant. It satisfies

$$f_{44}(\tau) = q + q^3 - 3q^5 + 2q^7 - 2q^9 - q^{11} - 4q^{13} - 3q^{15} + 6q^{17} + 8q^{19} \\ + 2q^{21} - 3q^{23} + 4q^{25} - 5q^{27} + O(q^{28}).$$

See [137] and Chapter 4.D of [138] for more details. A discussion of the ring of weak Jacobi forms of higher level can be found in [139].

## Appendix B. Characters

In Appendix B.1 we give character tables (with power maps and Frobenius–Schur indicators) for each group  $G^{(\ell)}$ . These were computed with the aid of the computer algebra package GAP4 [98] using the explicit presentations for the  $G^{(\ell)}$  that appear in Section 3.4. We use the abbreviations  $a_n = \sqrt{-n}$  and  $b_n = (-1 + \sqrt{-n})/2$  in these tables.

The tables in Appendix B.2 furnish the Frame shapes  $\Pi_g^{(\ell)}$  and character values  $\chi_g^{(\ell)}$  attached to the signed permutation representations (cf. Section 3.2) of the groups  $G^{(\ell)}$  given in Section 3.4. These Frame shapes and character values can easily be computed by hand; we detail them here explicitly since they can be used to define symbols  $n_g|h_g$  which encode the automorphy of the vector-valued mock modular forms  $H_g^{(\ell)}$  according to the prescription of Section 4.8. The symbols  $n_g|h_g$  are given in the rows labelled  $\Gamma_g$  in Appendix B.2.







Table B.3: Character table of  $G^{(4)} \simeq 2.AGL_3(2)$ .

$[g]$	FS	1A	2A	2B	4A	4B	2C	3A	6A	6B	6C	8A	4C	7A	14A	7B	14B
$[g^2]$		1A	1A	1A	2A	2B	1A	3A	3A	3A	3A	4A	2C	7A	7A	7B	7B
$[g^3]$		1A	2A	2B	4A	4B	2C	1A	2A	2B	2B	8A	4C	7B	14B	7A	14A
$[g^7]$		1A	2A	2B	4A	4B	2C	3A	6A	6B	6C	8A	4C	1A	2A	1A	2A
$\chi_1$	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\chi_2$	o	3	3	-1	-1	-1	-1	0	0	0	0	1	1	$b_7$	$b_7$	$\overline{b_7}$	$\overline{b_7}$
$\chi_3$	o	3	3	-1	-1	-1	-1	0	0	0	0	1	1	$\overline{b_7}$	$\overline{b_7}$	$b_7$	$b_7$
$\chi_4$	+	6	6	2	2	2	2	0	0	0	0	0	0	-1	-1	-1	-1
$\chi_5$	+	7	7	-1	-1	-1	-1	1	1	1	1	-1	-1	0	0	0	0
$\chi_6$	+	8	8	0	0	0	0	-1	-1	-1	-1	0	0	1	1	1	1
$\chi_7$	+	7	-1	3	-1	-1	-1	1	1	-1	-1	1	-1	0	0	0	0
$\chi_8$	+	7	-1	-1	-1	-1	3	1	1	-1	-1	-1	1	0	0	0	0
$\chi_9$	+	14	14	-2	2	-2	2	-1	-1	1	1	0	0	0	0	0	0
$\chi_{10}$	+	21	21	-3	1	1	-3	0	0	0	0	-1	1	0	0	0	0
$\chi_{11}$	+	21	21	-3	-3	1	1	0	0	0	0	1	-1	0	0	0	0
$\chi_{12}$	+	8	-8	0	0	0	0	2	-2	0	0	0	0	1	-1	1	-1
$\chi_{13}$	o	8	-8	0	0	0	0	-1	1	$a_3$	$\overline{a_3}$	0	0	1	-1	1	-1
$\chi_{14}$	o	8	-8	0	0	0	0	-1	1	$\overline{a_3}$	$a_3$	0	0	1	-1	1	-1
$\chi_{15}$	o	24	-24	0	0	0	0	0	0	0	0	0	0	$\overline{b_7}$	$\overline{b_7}$	$b_7$	$b_7$
$\chi_{16}$	o	24	-24	0	0	0	0	0	0	0	0	0	0	$b_7$	$b_7$	$\overline{b_7}$	$\overline{b_7}$

Table B.4: Character table of  $G^{(5)} \simeq \text{GL}_2(5)/2$ .

$[g]$	FS	1A	2A	2B	2C	3A	6A	5A	10A	4A	4B	4C	4D	12A	12B
$[g^2]$		1A	1A	1A	1A	3A	3A	5A	5A	2A	2A	2C	2C	6A	6A
$[g^3]$		1A	2A	2B	2C	1A	2A	5A	10A	4B	4A	4D	4C	4B	4A
$[g^5]$		1A	2A	2B	2C	3A	6A	1A	2A	4A	4B	4C	4D	12A	12B
$\chi_1$	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\chi_2$	+	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1
$\chi_3$	+	4	4	0	0	1	1	-1	-1	2	2	0	0	-1	-1
$\chi_4$	+	4	4	0	0	1	1	-1	-1	-2	-2	0	0	1	1
$\chi_5$	+	5	5	1	1	-1	-1	0	0	1	1	-1	-1	1	1
$\chi_6$	+	5	5	1	1	-1	-1	0	0	-1	-1	1	1	-1	-1
$\chi_7$	+	6	6	-2	-2	0	0	1	1	0	0	0	0	0	0
$\chi_8$	o	1	-1	1	-1	1	-1	1	-1	$a_1$	$-a_1$	$a_1$	$-a_1$	$a_1$	$-a_1$
$\chi_9$	o	1	-1	1	-1	1	-1	1	-1	$-a_1$	$a_1$	$-a_1$	$a_1$	$-a_1$	$a_1$
$\chi_{10}$	o	4	-4	0	0	1	-1	-1	1	$2a_1$	$-2a_1$	0	0	$-a_1$	$a_1$
$\chi_{11}$	o	4	-4	0	0	1	-1	-1	1	$-2a_1$	$2a_1$	0	0	$a_1$	$-a_1$
$\chi_{12}$	o	5	-5	1	-1	-1	1	0	0	$a_1$	$-a_1$	$-a_1$	$a_1$	$a_1$	$-a_1$
$\chi_{13}$	o	5	-5	1	-1	-1	1	0	0	$-a_1$	$a_1$	$a_1$	$-a_1$	$-a_1$	$a_1$
$\chi_{14}$	+	6	-6	-2	2	0	0	1	-1	0	0	0	0	0	0

Table B.5: Character table of  $G^{(7)} \simeq SL_2(3)$ .

$[g]$	FS	1A	2A	4A	3A	6A	3B	6B
$[g^2]$		1A	1A	2A	3B	3A	3A	3B
$[g^3]$		1A	2A	4A	1A	2A	1A	2A
$\chi_1$	+	1	1	1	1	1	1	1
$\chi_2$	○	1	1	1	$b_3$	$\overline{b_3}$	$\overline{b_3}$	$b_3$
$\chi_3$	○	1	1	1	$\overline{b_3}$	$b_3$	$b_3$	$\overline{b_3}$
$\chi_4$	+	3	3	-1	0	0	0	0
$\chi_5$	-	2	-2	0	-1	1	-1	1
$\chi_6$	○	2	-2	0	$-\overline{b_3}$	$b_3$	$-\overline{b_3}$	$\overline{b_3}$
$\chi_7$	○	2	-2	0	$-\overline{b_3}$	$\overline{b_3}$	$-\overline{b_3}$	$b_3$

Table B.6: Character table of  $G^{(13)} \simeq 4$ .

$[g]$	FS	1A	2A	4A	4B
$[g^2]$		1A	1A	2A	2A
$\chi_1$	+	1	1	1	1
$\chi_2$	+	1	1	-1	-1
$\chi_3$	○	1	-1	$a_1$	$\overline{a_1}$
$\chi_4$	○	1	-1	$\overline{a_1}$	$a_1$

### B.2. Euler characters

The tables in this section describe the Frame shapes  $\Pi_g^{(\ell)}$  and twisted Euler characters  $\chi_g^{(\ell)}$  attached to each group  $G^{(\ell)}$  via the signed permutation representations given in Section 3.4. The rows labelled  $\overline{\Pi}_g^{(\ell)}$  and  $\overline{\chi}_g^{(\ell)}$  describe the corresponding data for the (unsigned) permutation representations. According to the discussion of Section 4.8 the Frame shapes  $\Pi_g^{(\ell)}$  and  $\overline{\Pi}_g^{(\ell)}$  (or even just the  $\Pi_g^{(\ell)}$ ) can be used to define symbols  $n_g|h_g$  which encode the automorphy of the vector-valued mock modular form  $H_g^{(\ell)}$ ; these symbols are given in the rows labelled  $\Gamma_g$ . We write  $n_g$  here as a shorthand for  $n_g|1$ .

Table B.7: Twisted Euler characters and Frame shapes at  $\ell = 2$ .

$[g]$	1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B
$\Gamma_g$	1	2	2 2	3	3 3	4 2	4	4 4	5	6	6 6
$\chi_g^{(2)}$	24	8	0	6	0	0	4	0	4	2	0
$\Pi_g^{(2)}$	1 <sup>24</sup>	1 <sup>8</sup> 2 <sup>8</sup>	2 <sup>12</sup>	1 <sup>6</sup> 3 <sup>6</sup>	3 <sup>8</sup>	2 <sup>4</sup> 4 <sup>4</sup>	1 <sup>4</sup> 2 <sup>2</sup> 4 <sup>4</sup>	4 <sup>6</sup>	1 <sup>4</sup> 5 <sup>4</sup>	1 <sup>2</sup> 2 <sup>2</sup> 3 <sup>2</sup> 6 <sup>2</sup>	6 <sup>4</sup>
$[g]$	7AB	8A	10A	11A	12A	12B	14AB	15AB	21AB	23AB	
$\Gamma_g$	7	8	10 2	11	12 2	12 12	14	15	21 3	23	
$\chi_g^{(2)}$	3	2	0	2	0	0	1	1	0	1	
$\Pi_g^{(2)}$	1 <sup>3</sup> 7 <sup>3</sup>	1 <sup>2</sup> 2 <sup>1</sup> 4 <sup>1</sup> 8 <sup>2</sup>	2 <sup>2</sup> 10 <sup>2</sup>	1 <sup>2</sup> 11 <sup>2</sup>	2 <sup>1</sup> 4 <sup>1</sup> 6 <sup>1</sup> 12 <sup>1</sup>	12 <sup>2</sup>	1 <sup>1</sup> 2 <sup>1</sup> 7 <sup>1</sup> 14 <sup>1</sup>	1 <sup>1</sup> 3 <sup>1</sup> 5 <sup>1</sup> 15 <sup>1</sup>	3 <sup>1</sup> 21 <sup>1</sup>	1 <sup>1</sup> 23 <sup>1</sup>	

We have  $\chi_g^{(2)} = \chi_1(g) + \chi_2(g)$  in the notation of Table B.1.

Table B.8: Twisted Euler characters and Frame shapes at  $\ell = 3$ .

$[g]$	1A	2A	4A	2B	2C	3A	6A	3B	6B	4B	4C	5A	10A	12A	6C	6D	8AB	8CD	20AB	11AB	22AB
$\Gamma_g$	1	1 4	2 8	2	2 2	3	3 4	3 3 3	12	4 2	4	5	5 4	6 24	6	6 2	8 4	8	10 8	11	11 4
$\bar{\chi}_g^{(3)}$	12	12	0	4	4	3	3	0	0	4	2	2	2	0	1	1	0	2	0	1	1
$\chi_g^{(3)}$	12	-12	0	4	-4	3	-3	0	0	0	2	-2	0	0	1	-1	0	0	0	1	-1
$\bar{\Pi}_g^{(3)}$	$1^{12}$	$1^{12}$	$2^6$	$1^{424}$	$1^{424}$	$1^{333}$	$1^{333}$	$3^4$	$2^24^2$	$1^{442}$	$1^{252}$	$1^{252}$	$6^2$	$1^{213}6^1$	$1^{213}6^1$	$4^18^1$	$1^{22}18^1$	$2^110^1$	$1^111^1$	$1^111^1$	$1^111^1$
$\Pi_g^{(3)}$	$1^{12}$	$2^{12}$	$4^6$	$1^{424}$	$2^8$	$1^{333}$	$2^36^3$	$3^4$	$6^4$	$2^24^2$	$1^{252}$	$2^210^2$	$1^{22}$	$1^{213}6^1$	$1^{213}6^1$	$4^18^1$	$4^18^1$	$4^18^1$	$4^120^1$	$1^111^1$	$2^122^1$
	$1^{12}$	$2^6$	$2^6$	$1^{14}$	$1^{14}$	$1^{333}$	$1^{333}$	$3^4$	$3^4$	$2^24^2$	$1^{252}$	$1^{252}$	$6^2$	$1^{213}6^1$	$1^{213}6^1$	$4^18^1$	$4^18^1$	$4^18^1$	$2^110^1$	$1^111^1$	$1^111^1$

We have  $\chi_g^{(3)} = \chi_{18}(g)$  and  $\bar{\chi}_g^{(3)} = \chi_1(g) + \chi_2(g)$  in the notation of Table B.2.

Table B.9: Twisted Euler characters and Frame shapes at  $\ell = 4$ .

$[g]$	1A	2A	2B	4A	4B	2C	3A	6A	6BC	8A	4C	7AB	14AB
$\Gamma_g$	1	1 2	2 2	2 4	4 4	2	3	3 2	6 2	4 8	4	7	7 2
$\bar{\chi}_g^{(4)}$	8	8	0	0	0	4	2	2	0	0	2	1	1
$\chi_g^{(4)}$	8	-8	0	0	0	0	2	-2	0	0	0	1	-1
$\bar{\Pi}_g^{(4)}$	1 <sup>8</sup>	1 <sup>8</sup>	2 <sup>4</sup>	2 <sup>4</sup>	4 <sup>2</sup>	1 <sup>4</sup> 2 <sup>2</sup>	1 <sup>2</sup> 3 <sup>2</sup>	1 <sup>2</sup> 3 <sup>2</sup>	2 <sup>1</sup> 6 <sup>1</sup>	4 <sup>2</sup>	1 <sup>2</sup> 2 <sup>1</sup> 4 <sup>1</sup>	1 <sup>1</sup> 7 <sup>1</sup>	1 <sup>1</sup> 7 <sup>1</sup>
$\Pi_g^{(4)}$	1 <sup>8</sup>	$\frac{2^8}{1^8}$	2 <sup>4</sup>	$\frac{4^4}{2^4}$	4 <sup>2</sup>	2 <sup>4</sup>	1 <sup>2</sup> 3 <sup>2</sup>	$\frac{2^2 6^2}{1^2 3^2}$	2 <sup>1</sup> 6 <sup>1</sup>	$\frac{8^2}{4^2}$	4 <sup>2</sup>	1 <sup>1</sup> 7 <sup>1</sup>	$\frac{2^1 14^1}{1^1 7^1}$

We have  $\chi_g^{(4)} = \chi_{12}$  and  $\bar{\chi}_g^{(4)} = \chi_1(g) + \chi_8(g)$  in the notation of Table B.3.

Table B.10: Twisted Euler characters and Frame shapes at  $\ell = 5$ .

$[g]$	1A	2A	2B	2C	3A	6A	5A	10A	4AB	4CD	12AB
$\Gamma_g$	1	1 4	2 2	2	3 3	3 12	5	5 4	2 8	4	6 24
$\bar{\chi}_g^{(5)}$	6	6	2	2	0	0	1	1	0	2	0
$\chi_g^{(5)}$	6	-6	-2	2	0	0	1	-1	0	0	0
$\bar{\Pi}_g^{(5)}$	1 <sup>6</sup>	1 <sup>6</sup>	1 <sup>2</sup> 2 <sup>2</sup>	1 <sup>2</sup> 2 <sup>2</sup>	3 <sup>2</sup>	3 <sup>2</sup>	1 <sup>1</sup> 5 <sup>1</sup>	1 <sup>1</sup> 5 <sup>1</sup>	2 <sup>3</sup>	1 <sup>2</sup> 4 <sup>1</sup>	6 <sup>1</sup>
$\Pi_g^{(5)}$	1 <sup>6</sup>	$\frac{2^6}{1^6}$	$\frac{2^4}{1^2}$	1 <sup>2</sup> 2 <sup>2</sup>	3 <sup>2</sup>	$\frac{6^2}{3^2}$	1 <sup>1</sup> 5 <sup>1</sup>	$\frac{2^1 10^1}{1^1 5^1}$	$\frac{4^3}{2^3}$	2 <sup>1</sup> 4 <sup>1</sup>	$\frac{12^1}{6^1}$

We have  $\chi_g^{(5)} = \chi_{14}(g)$  and  $\bar{\chi}_g^{(5)} = \chi_1(g) + \chi_6(g)$  in the notation of Table B.4.

Table B.11: Twisted Euler characters and Frame shapes at  $\ell = 7$ .

$[g]$	1A	2A	4A	3AB	6AB
$\Gamma_g$	1	1 4	2 8	3	3 4
$\bar{\chi}_g^{(7)}$	4	4	0	1	1
$\chi_g^{(7)}$	4	-4	0	1	-1
$\bar{\Pi}_g^{(7)}$	1 <sup>4</sup>	1 <sup>4</sup>	2 <sup>2</sup>	1 <sup>1</sup> 3 <sup>1</sup>	1 <sup>1</sup> 3 <sup>1</sup>
$\Pi_g^{(7)}$	1 <sup>4</sup>	$\frac{2^4}{1^4}$	$\frac{4^2}{2^2}$	1 <sup>1</sup> 3 <sup>1</sup>	$\frac{2^1 6^1}{1^1 3^1}$

We have  $\chi_g^{(7)} = \chi_6(g) + \chi_7(g)$  in the notation of Table B.5.



Table B.12: Twisted Euler characters and Frame shapes at  $\ell = 13$ .

$[g]$	1A	2A	4AB
$\Gamma_g$	1	1 4	2 8
$\bar{\chi}_g^{(13)}$	2	2	0
$\chi_g^{(13)}$	2	-2	0
$\bar{\Pi}_g^{(13)}$	$1^2$	$1^2$	$2^1$
$\Pi_g^{(13)}$	$1^2$	$\frac{2^2}{1^2}$	$\frac{4^1}{2^1}$

We have  $\chi_g^{(13)} = \chi_3(g) + \chi_4(g)$  in the notation of Table B.6.

### Appendix C. Coefficients

In this section, we furnish tables of Fourier coefficients of small degree for the vector-valued mock modular forms  $H_g^{(\ell)}$  that we attach to the conjugacy classes of the groups  $G^{(\ell)}$  for  $\ell \in \Lambda$ . For each  $\ell$  and  $0 < r < \ell$  we give a table that displays the coefficients of  $H_{g,r}^{(\ell)}$  for ( $g$  ranging over a set of representatives for) each conjugacy class  $[g]$  in  $G^{(\ell)}$ . The first row of each table labels the conjugacy classes, and the first column labels exponents of  $q$  (or rather  $q^{1/4\ell}$ ), so that for the table captioned  $H_{g,r}^{(\ell)}$  (for some  $\ell \in \Lambda$  and  $0 < r < \ell$ ) the entry in the row labelled  $d$  and the column labelled  $nZ$  is the coefficient of  $q^{d/4\ell}$  in the Fourier expansion of  $H_{g,r}^{(\ell)}$  for  $[g] = nZ$ . Occasionally the functions  $H_g^{(\ell)}$  and  $H_{g'}^{(\ell)}$  coincide for non-conjugate  $g$  and  $g'$  and when this happens we condense information into a single column, writing  $7AB$  in Table C.1, for example, to indicate that the entries in that column are Fourier coefficients for both  $H_{7A}^{(2)}$  and  $H_{7B}^{(2)}$ .

C.1. Lambency two

Table C.1: McKay–Thompson series  $H_{g,1}^{(2)}$ .

$[g]$	1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B	7AB	8A	10A	11A	12A	12B	14AB	15AB	21AB	23AB	
$\Gamma_g$	1	2	2 2	3	3 3	4 2	4	4 4	5	6	6 6	7	8	10 2	11	12 2	12 12	14	15	21 3	23	
-1	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
7	90	-6	10	0	6	-6	2	2	0	0	-2	-1	-2	0	2	0	2	1	0	-1	0	-2
15	462	14	-18	-6	0	-2	-2	6	2	2	0	0	-2	2	0	-2	0	0	-1	0	0	2
23	1540	-28	20	10	-14	4	-4	-4	0	2	2	0	0	0	0	-2	2	0	0	0	0	-1
31	4554	42	-38	0	12	-6	2	-6	-6	0	4	4	-2	2	0	0	0	0	0	0	0	0
39	11592	-56	72	-18	0	-8	8	0	2	-2	0	0	0	0	2	-2	0	0	0	2	0	0
47	27830	86	-90	20	-16	6	-2	6	0	-4	0	-2	2	0	0	0	0	2	0	-2	0	0
55	61686	-138	118	0	30	6	-10	-2	6	0	-2	2	-2	-2	-2	0	-2	2	0	2	0	0
63	131100	188	-180	-30	0	-4	4	-12	0	2	0	-3	0	0	2	2	0	-1	0	0	0	0
71	265650	-238	258	42	-42	-14	10	10	-10	2	6	0	-2	-2	0	-2	-2	0	2	0	0	0
79	521136	336	-352	0	42	0	-8	16	6	0	2	0	-4	-2	0	0	0	0	0	0	0	2
87	988770	-478	450	-60	0	18	-14	-6	0	-4	0	6	2	0	2	0	0	-2	0	0	0	0
95	1830248	616	-600	62	-70	-8	8	-16	8	-2	-6	0	0	0	2	2	2	0	2	0	0	0
103	3303630	-786	830	0	84	-18	22	6	0	0	-4	-6	2	0	0	0	0	-2	0	0	0	2
111	5844762	1050	-1062	-90	0	10	-6	18	-18	6	0	0	2	-2	0	0	0	0	0	0	0	2
119	10139734	-1386	1334	118	-110	22	-26	-10	4	6	2	-4	-2	4	0	-2	2	0	-2	2	0	0
127	17301060	1764	-1740	0	126	-12	12	-28	0	0	6	0	0	0	-4	0	2	0	0	0	0	0
135	29051484	-2212	2268	-156	0	-36	28	12	14	-4	0	0	-4	-2	0	0	0	0	0	0	0	0
143	48106430	2814	-2850	170	-166	14	-18	38	0	-6	8	-2	0	-2	2	2	2	0	0	-1	0	0
151	78599556	-3612	3540	0	210	36	-36	-20	-24	0	-6	0	0	0	2	0	-2	0	0	0	0	0
159	126894174	4510	-4482	-228	0	-18	14	-42	14	4	0	-6	-2	4	0	-2	2	0	2	0	0	0
167	202537080	-5544	5640	270	-282	-40	48	16	0	6	6	4	4	0	-2	2	-2	0	0	-2	0	0
175	319927608	6936	-6968	0	300	24	-16	48	18	0	4	-7	4	2	0	0	0	-1	0	-1	0	0
183	500376870	-8666	8550	-360	0	54	-58	-18	0	-8	0	0	-2	0	4	0	0	0	0	0	0	0
191	775492564	10612	-10556	400	-392	-28	28	-60	-36	-8	-8	0	0	4	0	-4	0	0	0	0	0	0
199	1191453912	-12936	13064	0	462	-72	64	32	12	0	-10	12	-4	4	0	0	2	0	0	0	0	0
207	1815754710	15862	-15930	-510	0	22	-34	78	0	10	0	0	-6	0	0	-2	0	0	0	0	0	-1
215	2745870180	-19420	19268	600	-600	84	-76	-36	30	8	8	-10	4	-2	-2	0	0	-2	0	2	0	0
223	4122417420	23532	-23460	0	660	-36	36	-84	0	0	12	2	0	0	0	0	0	-2	0	2	0	0
231	6146311620	-28348	28548	-762	0	-92	100	36	-50	-10	0	-6	4	-2	-2	-2	0	2	-2	0	0	0
239	9104078592	34272	-34352	828	-840	48	-40	96	22	-12	-8	0	4	-2	4	0	0	0	0	-2	0	0
247	13401053820	-41412	41180	0	966	108	-116	-44	0	0	-10	0	-4	0	0	0	-2	0	0	0	0	-2

C.2. Lambency three

Table C.2: McKay–Thompson series  $H_{g,1}^{(3)}$ .

$[g]$	1A	2A	4A	2B	2C	3A	6A	3B	6B	4B	4C	5A	10A	12A	6C	6D	8AB	8CD	20AB	11AB	22AB	
$\Gamma_g$	1	1 4	2 8	2	2 2	3	3 4	3 3	3 12	4 2	4	5	5 4	6 24	6	6 2	8 4	8	10 8	11	11 4	
-1	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
11	32	32	8	0	0	-4	-4	2	2	0	0	2	2	2	0	0	0	0	0	-2	-1	-1
23	110	110	-10	-2	-2	2	2	2	6	-2	0	0	0	0	2	-2	-2	2	0	0	0	0
35	288	288	8	0	0	0	0	0	-6	0	0	-2	-2	2	0	0	0	0	-2	2	2	2
47	660	660	-20	4	4	-6	-6	6	6	-4	4	0	0	-2	-2	4	0	0	0	0	0	0
59	1408	1408	32	0	0	4	4	4	4	0	0	-2	-2	-4	0	0	0	2	0	0	0	0
71	2794	2794	-30	-6	-6	4	4	-8	-8	2	-6	4	0	0	0	2	-2	0	0	0	0	0
83	5280	5280	40	0	0	-12	-12	6	6	0	0	0	0	-2	0	0	0	0	0	0	0	0
95	9638	9638	-58	6	6	8	8	2	2	-10	6	-2	-2	2	0	0	-2	2	2	2	2	2
107	16960	16960	80	0	0	4	4	-14	-14	0	0	0	0	2	0	0	0	0	0	-2	-2	-2
119	29018	29018	-102	-6	-6	-16	-16	8	8	10	-6	-2	-2	0	0	0	2	2	0	0	0	0
131	48576	48576	112	0	0	12	12	6	6	0	0	6	6	-2	0	0	0	0	2	0	0	0
143	79530	79530	-150	10	10	6	6	-24	-24	-6	10	0	0	0	-2	-2	2	2	0	0	0	0
155	127776	127776	200	0	0	-24	-24	18	18	0	0	-4	-4	2	0	0	0	0	0	0	0	0
167	202050	202050	-230	-14	-14	18	18	12	12	10	-14	0	0	4	-2	-2	2	2	0	2	2	2
179	314688	314688	272	0	0	12	12	-30	-30	0	0	-2	-2	2	0	0	0	0	2	0	0	0
191	483516	483516	-348	12	12	-36	-36	24	24	-12	12	6	6	0	0	0	-4	0	2	0	0	0
203	733920	733920	440	0	0	24	24	12	12	0	0	0	0	-4	0	0	0	0	0	0	0	0
215	1101364	1101364	-508	-12	-12	16	16	-44	-44	20	-12	-6	-6	-4	0	0	-4	4	2	0	0	0
227	1635680	1635680	600	0	0	-52	-52	32	32	0	0	0	0	0	0	0	0	0	0	2	2	2
239	2406116	2406116	-740	20	20	38	38	20	20	-20	20	-4	-4	4	2	4	0	0	0	-2	-2	-2
251	3507680	3507680	888	0	0	20	20	-64	-64	0	0	10	10	0	0	0	0	0	-2	0	0	0
263	5071000	5071000	-1040	-24	-24	-68	-68	46	46	16	-24	0	0	-2	0	0	-4	0	0	0	0	0
275	7274464	7274464	1208	0	0	52	52	28	28	0	0	-6	-6	-4	0	0	0	0	-2	-1	-1	-1
287	10359030	10359030	-1450	22	22	30	30	-84	-84	-26	22	0	0	-4	-2	-2	-2	0	0	0	0	0
299	14650176	14650176	1744	0	0	-96	-96	60	60	0	0	-4	-4	4	0	0	0	4	2	0	0	0
311	20585334	20585334	-2018	-26	-26	66	66	36	36	30	-26	14	14	-2	-2	-2	2	2	0	0	0	0
323	28747840	28747840	2320	0	0	40	40	-116	-116	0	0	0	0	4	0	0	0	0	0	0	0	0
335	39914402	39914402	-2750	34	34	-130	-130	86	86	-30	34	-8	-8	-2	-2	-2	2	2	0	0	0	0
347	55114400	55114400	3240	0	0	92	92	50	50	0	0	0	0	-6	0	0	0	0	0	0	0	0
359	75704904	75704904	-3712	-40	-40	54	54	-156	-156	32	-40	-6	-6	-4	2	2	-4	-4	-2	0	0	0

Table C.3: McKay–Thompson series  $H_{g,2}^{(3)}$ .

$[g]$	1A	2A	4A	2B	2C	3A	6A	3B	6B	4B	4C	5A	10A	12A	6C	6D	8AB	8CD	20AB	11AB	22AB	
$\Gamma_g$	1	1 4	2 8	2	2 2	3	3 4	3 3	3 12	4 2	4	5	5 4	6 24	6	6 2	8 4	8	10 8	11	11 4	
8	20	-20	0	-4	4	2	-2	-4	4	0	0	0	0	0	2	-2	0	0	0	0	-2	2
20	88	-88	0	8	-8	-2	2	4	-4	0	0	-2	2	0	2	-2	0	0	0	0	0	0
32	220	-220	0	-12	12	4	-4	4	-4	0	0	0	0	0	0	0	0	0	0	0	0	0
44	560	-560	0	16	-16	2	-2	-4	4	0	0	0	0	0	-2	2	0	0	0	0	-1	1
56	1144	-1144	0	-24	24	-8	8	4	-4	0	0	4	-4	0	0	0	0	0	0	0	0	0
68	2400	-2400	0	32	-32	6	-6	0	0	0	0	0	0	0	2	-2	0	0	0	0	2	-2
80	4488	-4488	0	-40	40	6	-6	-12	12	0	0	-2	2	0	2	-2	0	0	0	0	0	0
92	8360	-8360	0	56	-56	10	10	8	-8	0	0	0	0	0	2	-2	0	0	0	0	0	0
104	14696	-14696	0	-72	72	8	-8	8	-8	0	0	-4	4	0	0	0	0	0	0	0	0	0
116	25544	-25544	0	88	-88	2	-2	-16	16	0	0	4	-4	0	-2	2	0	0	0	0	2	-2
128	42660	-42660	0	-116	116	-18	18	12	-12	0	0	0	0	0	-2	2	0	0	0	0	2	-2
140	70576	-70576	0	144	-144	16	-16	4	-4	0	0	-4	4	0	0	0	0	0	0	0	0	0
152	113520	-113520	0	-176	176	12	-12	-24	24	0	0	0	0	0	4	-4	0	0	0	0	0	0
164	180640	-180640	0	224	-224	-26	26	16	-16	0	0	0	0	0	2	-2	0	0	0	0	-2	2
176	281808	-281808	0	-272	272	18	-18	12	-12	0	0	8	-8	0	-2	2	0	0	0	0	-1	1
188	435160	-435160	0	328	-328	10	-10	-32	32	0	0	0	0	0	2	-2	0	0	0	0	0	0
200	661476	-661476	0	-404	404	-42	42	24	-24	0	0	-4	4	0	-2	2	0	0	0	0	2	-2
212	996600	-996600	0	488	-488	30	-30	12	-12	0	0	0	0	0	2	-2	0	0	0	0	0	0
224	1482536	-1482536	0	-584	584	20	-20	-52	52	0	0	-4	4	0	4	-4	0	0	0	0	0	0
236	2187328	-2187328	0	704	-704	-50	50	40	-40	0	0	8	-8	0	2	-2	0	0	0	0	0	0
248	3193960	-3193960	0	-840	840	40	-40	28	-28	0	0	0	0	0	0	0	0	0	0	0	0	0
260	4629152	-4629152	0	992	-992	20	-20	-64	64	0	0	-8	8	0	-4	4	0	0	0	0	0	0
272	6650400	-6650400	0	-1184	1184	-78	78	48	-48	0	0	0	0	0	-2	2	0	0	0	0	-2	2
284	9490536	-9490536	0	1400	-1400	54	-54	24	-24	0	0	4	-4	0	2	-2	0	0	0	0	0	0
296	13441032	-13441032	0	-1640	1640	36	-36	-96	96	0	0	12	-12	0	4	-4	0	0	0	0	0	0
308	18920240	-18920240	0	1936	-1936	-100	100	68	-68	0	0	0	0	0	4	-4	0	0	0	0	-2	2
320	26457464	-26457464	0	-2264	2264	74	-74	44	-44	0	0	-6	6	0	-2	2	0	0	0	0	0	0
332	36792560	-36792560	0	2640	-2640	38	-38	-124	124	0	0	0	0	0	-6	6	0	0	0	0	2	-2
344	50865232	-50865232	0	-3088	3088	-140	140	88	-88	0	0	-8	8	0	-4	4	0	0	0	0	0	0
356	69966336	-69966336	0	3584	-3584	102	-102	48	-48	0	0	16	-16	0	2	-2	0	0	0	0	0	0

**C.3. Lambency four**

Table C.4: McKay–Thompson series  $H_{g,1}^{(4)}$ .

$[g]$	1A	2A	2B	4A	4B	2C	3A	6A	6BC	8A	4C	7AB	14AB
$\Gamma_g$	1	1 2	2 2	2 4	4 4	2	3	3 2	6 2	4 8	4	7	7 2
-1	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
15	14	14	-2	6	-2	-2	2	2	-2	2	-2	0	0
31	42	42	-6	-6	2	2	0	0	0	2	-2	0	0
47	86	86	6	6	-2	-2	-4	-4	0	-2	2	2	2
63	188	188	-4	-12	-4	4	2	2	2	0	0	-1	-1
79	336	336	0	16	0	-8	0	0	0	0	-4	0	0
95	616	616	-8	-16	0	8	-2	-2	-2	4	0	0	0
111	1050	1050	10	18	2	-6	6	6	-2	-2	2	0	0
127	1764	1764	-12	-28	-4	12	0	0	0	-4	0	0	0
143	2814	2814	14	38	-2	-18	-6	-6	2	2	-2	0	0
159	4510	4510	-18	-42	6	14	4	4	0	2	-2	2	2
175	6936	6936	24	48	0	-16	0	0	0	-4	4	-1	-1
191	10612	10612	-28	-60	-4	28	-8	-8	-4	-4	0	0	0
207	15862	15862	22	78	-2	-34	10	10	-2	2	-6	0	0
223	23532	23532	-36	-84	4	36	0	0	0	4	0	-2	-2
239	34272	34272	48	96	0	-40	-12	-12	0	0	4	0	0
255	49618	49618	-46	-126	-6	50	10	10	2	-6	2	2	2
271	70758	70758	54	150	-2	-66	0	0	0	6	-6	2	2
287	100310	100310	-74	-170	6	70	-10	-10	-2	6	-2	0	0
303	140616	140616	88	192	0	-72	18	18	-2	-4	8	0	0
319	195888	195888	-96	-232	-8	96	0	0	0	-4	0	0	0
335	270296	270296	104	272	0	-120	-22	-22	2	4	-8	-2	-2
351	371070	371070	-130	-306	6	126	18	18	2	6	-2	0	0
367	505260	505260	156	348	4	-140	0	0	0	-4	8	0	0
383	684518	684518	-170	-410	-10	174	-22	-22	-2	-10	2	2	2
399	921142	921142	182	486	-2	-202	28	28	-4	6	-10	-2	-2
415	1233708	1233708	-228	-540	12	220	0	0	0	8	-4	0	0
431	1642592	1642592	272	608	0	-248	-34	-34	2	-8	12	0	0
447	2177684	2177684	-284	-708	-12	292	32	32	4	-8	4	-2	-2
463	2871918	2871918	318	814	-2	-346	0	0	0	6	-14	0	0
479	3772468	3772468	-380	-908	12	380	-38	-38	-2	12	0	0	0
495	4932580	4932580	436	1020	4	-412	46	46	-2	-8	12	2	2
511	6425466	6425466	-486	-1174	-14	490	0	0	0	-14	2	-2	-2
527	8335418	8335418	538	1338	-6	-566	-52	-52	4	10	-14	0	0
543	10776290	10776290	-622	-1494	18	610	50	50	2	14	-6	0	0
559	13879290	13879290	714	1666	2	-678	0	0	0	-10	18	-2	-2
575	17818766	17818766	-786	-1898	-18	790	-58	-58	-6	-14	2	0	0
591	22798188	22798188	860	2148	-4	-900	72	72	-4	8	-20	0	0



Table C.6: McKay–Thompson series  $H_{g,3}^{(4)}$ .

$[g]$	1A	2A	2B	4A	4B	2C	3A	6A	6BC	8A	4C	7AB	14AB
$\Gamma_g$	1	1 2	2 2	2 4	4 4	2	3	3 2	6 2	4 8	4	7	7 2
7	6	6	6	-2	-2	-2	0	0	0	2	2	-1	-1
23	28	28	-4	4	-4	4	-2	-2	2	0	0	0	0
39	56	56	8	0	0	-8	2	2	2	-4	0	0	0
55	138	138	-6	2	2	10	0	0	0	-2	2	-2	-2
71	238	238	14	-10	-2	-10	-2	-2	2	2	2	0	0
87	478	478	-18	6	-2	14	4	4	0	2	-2	2	2
103	786	786	18	-6	2	-22	0	0	0	-2	-2	2	2
119	1386	1386	-22	10	2	26	-6	-6	2	2	2	0	0
135	2212	2212	36	-12	-4	-28	4	4	0	4	4	0	0
151	3612	3612	-36	20	-4	36	0	0	0	0	0	0	0
167	5544	5544	40	-16	0	-48	-6	-6	-2	-4	-4	0	0
183	8666	8666	-54	18	2	58	8	8	0	-2	2	0	0
199	12936	12936	72	-32	0	-64	0	0	0	4	4	0	0
215	19420	19420	-84	36	-4	76	-8	-8	0	0	-4	2	2
231	28348	28348	92	-36	4	-100	10	10	2	-4	-4	-2	-2
247	41412	41412	-108	44	4	116	0	0	0	0	4	0	0
263	59178	59178	138	-62	-6	-126	-12	-12	0	6	6	0	0
279	84530	84530	-158	66	-6	154	14	14	-2	2	-2	-2	-2
295	118692	118692	180	-68	4	-188	0	0	0	-8	-4	0	0
311	166320	166320	-208	88	8	216	-18	-18	2	-4	4	0	0
327	230092	230092	252	-108	-4	-244	16	16	0	8	4	2	2
343	317274	317274	-294	122	-6	282	0	0	0	2	-6	-1	-1
359	432964	432964	324	-132	4	-340	-20	-20	0	-8	-8	0	0
375	588966	588966	-378	150	6	390	24	24	0	-2	6	0	0
391	794178	794178	450	-190	-6	-430	0	0	0	10	10	0	0
407	1067220	1067220	-508	220	-12	500	-30	-30	2	0	-4	0	0
423	1423884	1423884	572	-228	4	-588	30	30	2	-12	-8	0	0
439	1893138	1893138	-654	266	10	666	0	0	0	-2	6	2	2
455	2501434	2501434	762	-326	-6	-742	-32	-32	0	10	10	-2	-2
471	3294256	3294256	-864	360	-8	848	40	40	0	4	-8	0	0
487	4314912	4314912	960	-392	8	-984	0	0	0	-12	-12	0	0
503	5633596	5633596	-1092	452	12	1108	-44	-44	0	0	8	-4	-4
519	7320670	7320670	1262	-522	-10	-1234	46	46	2	18	14	0	0
535	9483336	9483336	-1416	592	-16	1400	0	0	0	4	-8	2	2
551	12233330	12233330	1570	-646	10	-1598	-58	-58	-2	-18	-14	4	4
567	15734606	15734606	-1778	726	14	1798	62	62	-2	-6	10	-1	-1
583	20161302	20161302	2022	-850	-10	-1994	0	0	0	18	14	0	0
599	25761288	25761288	-2264	944	-16	2240	-72	-72	3	4	-12	0	0

## C.4. Lambency five

Table C.7: McKay–Thompson series  $H_{g,1}^{(5)}$ .

$[g]$	1A	2A	2B	2C	3A	6A	5A	10A	4AB	4CD	12AB
$\Gamma_g$	1	1 4	2 2	2	3 3	3 12	5	5 4	2 8	4	6 24
-1	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
19	8	8	0	0	2	2	-2	-2	4	0	-2
39	18	18	2	2	0	0	-2	-2	-6	2	0
59	40	40	0	0	-2	-2	0	0	4	0	-2
79	70	70	-2	-2	4	4	0	0	-6	-2	0
99	120	120	0	0	0	0	0	0	12	0	0
119	208	208	0	0	-2	-2	-2	-2	-8	0	-2
139	328	328	0	0	4	4	-2	-2	12	0	0
159	510	510	-2	-2	0	0	0	0	-18	-2	0
179	792	792	0	0	-6	-6	2	2	20	0	2
199	1180	1180	4	4	4	4	0	0	-24	4	0
219	1728	1728	0	0	0	0	-2	-2	24	0	0
239	2518	2518	-2	-2	-8	-8	-2	-2	-30	-2	0
259	3600	3600	0	0	6	6	0	0	40	0	-2
279	5082	5082	2	2	0	0	2	2	-42	2	0
299	7120	7120	0	0	-8	-8	0	0	48	0	0
319	9838	9838	-2	-2	10	10	-2	-2	-58	-2	2
339	13488	13488	0	0	0	0	-2	-2	72	0	0
359	18380	18380	4	4	-10	-10	0	0	-80	4	-2
379	24792	24792	0	0	12	12	2	2	84	0	0
399	33210	33210	-6	-6	0	0	0	0	-102	-6	0
419	44248	44248	0	0	-14	-14	-2	-2	116	0	2
439	58538	58538	2	2	14	14	-2	-2	-130	2	2
459	76992	76992	0	0	0	0	2	2	144	0	0
479	100772	100772	-4	-4	-16	-16	2	2	-168	-4	0
499	131160	131160	0	0	18	18	0	0	196	0	-2
519	169896	169896	8	8	0	0	-4	-4	-216	8	0
539	219128	219128	0	0	-22	-22	-2	-2	236	0	2
559	281322	281322	-6	-6	24	24	2	2	-270	-6	0
579	359712	359712	0	0	0	0	2	2	312	0	0
599	458220	458220	4	4	-24	-24	0	0	-336	4	0
619	581416	581416	0	0	28	28	-4	-4	372	0	0
639	735138	735138	-6	-6	0	0	-2	-2	-426	-6	0
659	926472	926472	0	0	-30	-30	2	2	476	0	2
679	1163674	1163674	10	10	34	34	4	4	-526	10	2
699	1457040	1457040	0	0	0	0	0	0	576	0	0
719	1819056	1819056	-8	-8	-42	-42	-4	-4	-644	-8	-2
739	2264376	2264376	0	0	42	42	-4	-4	724	0	-2



Table C.8: McKay–Thompson series  $H_{g,2}^{(5)}$ .

$[g]$	1A	2A	2B	2C	3A	6A	5A	10A	4AB	4CD	12AB
$\Gamma_g$	1	1 4	2 2	2	3 3	3 12	5	5 4	2 8	4	6 24
16	10	-10	2	-2	-2	2	0	0	0	0	0
36	30	-30	-2	2	0	0	0	0	0	0	0
56	52	-52	4	-4	4	-4	2	-2	0	0	0
76	108	-108	-4	4	0	0	-2	2	0	0	0
96	180	-180	4	-4	0	0	0	0	0	0	0
116	312	-312	-8	8	0	0	2	-2	0	0	0
136	488	-488	8	-8	-4	4	-2	2	0	0	0
156	792	-792	-8	8	0	0	2	-2	0	0	0
176	1180	-1180	12	-12	4	-4	0	0	0	0	0
196	1810	-1810	-14	14	-2	2	0	0	0	0	0
216	2640	-2640	16	-16	0	0	0	0	0	0	0
236	3868	-3868	-20	20	4	-4	-2	2	0	0	0
256	5502	-5502	22	-22	-6	6	2	-2	0	0	0
276	7848	-7848	-24	24	0	0	-2	2	0	0	0
296	10912	-10912	32	-32	4	-4	2	-2	0	0	0
316	15212	-15212	-36	36	-4	4	2	-2	0	0	0
336	20808	-20808	40	-40	0	0	-2	2	0	0	0
356	28432	-28432	-48	48	4	-4	2	-2	0	0	0
376	38308	-38308	52	-52	-8	8	-2	2	0	0	0
396	51540	-51540	-60	60	0	0	0	0	0	0	0
416	68520	-68520	72	-72	12	-12	0	0	0	0	0
436	90928	-90928	-80	80	-8	8	-2	2	0	0	0
456	119544	-119544	88	-88	0	0	4	-4	0	0	0
476	156728	-156728	-104	104	8	-8	-2	2	0	0	0
496	203940	-203940	116	-116	-12	12	0	0	0	0	0
516	264672	-264672	-128	128	0	0	2	-2	0	0	0
536	341188	-341188	148	-148	16	-16	-2	2	0	0	0
556	438732	-438732	-164	164	-12	12	2	-2	0	0	0
576	560958	-560958	182	-182	0	0	-2	2	0	0	0
596	715312	-715312	-208	208	16	-16	2	-2	0	0	0
616	907720	-907720	232	-232	-20	20	0	0	0	0	0
636	1148928	-1148928	-256	256	0	0	-2	2	0	0	0
656	1447904	-1447904	288	-288	20	-20	4	-4	0	0	0
676	1820226	-1820226	-318	318	-18	18	-4	4	0	0	0
696	2279520	-2279520	352	-352	0	0	0	0	0	0	0
716	2847812	-2847812	-396	396	20	-20	2	-2	0	0	0
736	3545636	-3545636	436	-436	-28	28	-4	4	0	0	0
756	4404384	-4404384	-480	480	0	0	4	-4	0	0	0

Table C.9: McKay–Thompson series  $H_{g,3}^{(5)}$ .

$[g]$	1A	2A	2B	2C	3A	6A	5A	10A	4AB	4CD	12AB
$\Gamma_g$	1	1 4	2 2	2	3 3	3 12	5	5 4	2 8	4	6 24
11	8	8	0	0	2	2	-2	-2	-4	0	2
31	22	22	-2	-2	-2	-2	2	2	2	-2	2
51	48	48	0	0	0	0	-2	-2	0	0	0
71	90	90	2	2	0	0	0	0	6	2	0
91	160	160	0	0	-2	-2	0	0	-8	0	-2
111	270	270	-2	-2	0	0	0	0	6	-2	0
131	440	440	0	0	2	2	0	0	-4	0	2
151	700	700	4	4	-2	-2	0	0	8	4	2
171	1080	1080	0	0	0	0	0	0	-12	0	0
191	1620	1620	-4	-4	6	6	0	0	16	-4	-2
211	2408	2408	0	0	-4	-4	-2	-2	-12	0	0
231	3522	3522	2	2	0	0	2	2	18	2	0
251	5048	5048	0	0	2	2	-2	-2	-28	0	2
271	7172	7172	-4	-4	-4	-4	2	2	24	-4	0
291	10080	10080	0	0	0	0	0	0	-24	0	0
311	13998	13998	6	6	6	6	-2	-2	34	6	-2
331	19272	19272	0	0	-6	-6	2	2	-44	0	-2
351	26298	26298	-6	-6	0	0	-2	-2	42	-6	0
371	35600	35600	0	0	8	8	0	0	-48	0	0
391	47862	47862	6	6	-6	-6	2	2	62	6	2
411	63888	63888	0	0	0	0	-2	-2	-72	0	0
431	84722	84722	-6	-6	8	8	2	2	78	-6	0
451	111728	111728	0	0	-10	-10	-2	-2	-80	0	-2
471	146520	146520	8	8	0	0	0	0	96	8	0
491	191080	191080	0	0	10	10	0	0	-124	0	2
511	248008	248008	-8	-8	-14	-14	-2	-2	128	-8	2
531	320424	320424	0	0	0	0	4	4	-132	0	0
551	412088	412088	8	8	14	14	-2	-2	160	8	-2
571	527800	527800	0	0	-14	-14	0	0	-188	0	-2
591	673302	673302	-10	-10	0	0	2	2	198	-10	0
611	855616	855616	0	0	16	16	-4	-4	-216	0	0
631	1083444	1083444	12	12	-18	-18	4	4	248	12	2
651	1367136	1367136	0	0	0	0	-4	-4	-288	0	0
671	1719362	1719362	-14	-14	20	20	2	2	314	-14	-4
691	2155592	2155592	0	0	-22	-22	2	2	-332	0	-2
711	2694276	2694276	12	12	0	0	-4	-4	384	12	0
731	3357664	3357664	0	0	28	28	4	4	-440	0	4
751	4172746	4172746	-14	-14	-26	-26	-4	-4	470	-14	2

Table C.10: McKay–Thompson series  $H_{g,4}^{(5)}$ .

$[g]$	1A	2A	2B	2C	3A	6A	5A	10A	4AB	4CD	12AB
$\Gamma_g$	1	1 4	2 2	2	3 3	3 12	5	5 4	2 8	4	6 24
4	2	-2	2	-2	2	-2	2	-2	0	0	0
24	12	-12	-4	4	0	0	2	-2	0	0	0
44	20	-20	4	-4	-4	4	0	0	0	0	0
64	50	-50	-6	6	2	-2	0	0	0	0	0
84	72	-72	8	-8	0	0	2	-2	0	0	0
104	152	-152	-8	8	-4	4	2	-2	0	0	0
124	220	-220	12	-12	4	-4	0	0	0	0	0
144	378	-378	-14	14	0	0	-2	2	0	0	0
164	560	-560	16	-16	-4	4	0	0	0	0	0
184	892	-892	-20	20	4	-4	2	-2	0	0	0
204	1272	-1272	24	-24	0	0	2	-2	0	0	0
224	1940	-1940	-28	28	-4	4	0	0	0	0	0
244	2720	-2720	32	-32	8	-8	0	0	0	0	0
264	3960	-3960	-40	40	0	0	0	0	0	0	0
284	5500	-5500	44	-44	-8	8	0	0	0	0	0
304	7772	-7772	-52	52	8	-8	2	-2	0	0	0
324	10590	-10590	62	-62	0	0	0	0	0	0	0
344	14668	-14668	-68	68	-8	8	-2	2	0	0	0
364	19728	-19728	80	-80	12	-12	-2	2	0	0	0
384	26772	-26772	-92	92	0	0	2	-2	0	0	0
404	35624	-35624	104	-104	-16	16	4	-4	0	0	0
424	47592	-47592	-120	120	12	-12	2	-2	0	0	0
444	62568	-62568	136	-136	0	0	-2	2	0	0	0
464	82568	-82568	-152	152	-16	16	-2	2	0	0	0
484	107502	-107502	174	-174	18	-18	2	-2	0	0	0
504	140172	-140172	-196	196	0	0	2	-2	0	0	0
524	180940	-180940	220	-220	-20	20	0	0	0	0	0
544	233576	-233576	-248	248	20	-20	-4	4	0	0	0
564	298968	-298968	280	-280	0	0	-2	2	0	0	0
584	382632	-382632	-312	312	-24	24	2	-2	0	0	0
604	486124	-486124	348	-348	28	-28	4	-4	0	0	0
624	617112	-617112	-392	392	0	0	2	-2	0	0	0
644	778768	-778768	432	-432	-32	32	-2	2	0	0	0
664	981548	-981548	-484	484	32	-32	-2	2	0	0	0
684	1230732	-1230732	540	-540	0	0	2	-2	0	0	0
704	1541244	-1541244	-596	596	-36	36	4	-4	0	0	0
724	1921240	-1921240	664	-664	40	-40	0	0	0	0	0
744	2391456	-2391456	-736	736	0	0	-4	4	0	0	0

### C.5. Lambency seven

Table C.11:  $H_{g,1}^{(7)}$ .

$[g]$	1A	2A	4A	3AB	6AB
$\Gamma_g$	1	1 4	2 8	3	3 4
-1	-2	-2	-2	-2	-2
27	4	4	4	1	1
55	6	6	-2	0	0
83	10	10	2	-2	-2
111	20	20	-4	2	2
139	30	30	6	0	0
167	42	42	-6	0	0
195	68	68	4	2	2
223	96	96	-8	0	0
251	130	130	10	-2	-2
279	188	188	-12	2	2
307	258	258	10	0	0
335	350	350	-10	-4	-4
363	474	474	18	3	3
391	624	624	-16	0	0
419	826	826	18	-2	-2
447	1090	1090	-22	4	4
475	1410	1410	26	0	0
503	1814	1814	-26	-4	-4
531	2338	2338	26	4	4
559	2982	2982	-34	0	0
587	3774	3774	38	-6	-6
615	4774	4774	-42	4	4
643	5994	5994	42	0	0
671	7494	7494	-50	-6	-6
699	9348	9348	60	6	6
727	11586	11586	-62	0	0
755	14320	14320	64	-8	-8
783	17654	17654	-74	8	8
811	21654	21654	86	0	0
839	26488	26488	-88	-8	-8
867	32334	32334	94	9	9
895	39324	39324	-108	0	0
923	47680	47680	120	-8	-8
951	57688	57688	-128	10	10
979	69600	69600	136	0	0
1007	83760	83760	-152	-12	-12
1035	100596	100596	172	12	12

Table C.12:  $H_{g,2}^{(7)}$ .

$[g]$	1A	2A	4A	3AB	6AB
$\Gamma_g$	1	1 4	2 8	3	3 4
24	4	-4	0	-2	2
52	12	-12	0	0	0
80	20	-20	0	2	-2
108	32	-32	0	-1	1
136	48	-48	0	0	0
164	80	-80	0	2	-2
192	108	-108	0	-3	3
220	168	-168	0	0	0
248	232	-232	0	4	-4
276	328	-328	0	-2	2
304	444	-444	0	0	0
332	620	-620	0	2	-2
360	812	-812	0	-4	4
388	1104	-1104	0	0	0
416	1444	-1444	0	4	-4
444	1904	-1904	0	-4	4
472	2460	-2460	0	0	0
500	3208	-3208	0	4	-4
528	4080	-4080	0	-6	6
556	5244	-5244	0	0	0
584	6632	-6632	0	8	-8
612	8400	-8400	0	-6	6
640	10524	-10524	0	0	0
668	13224	-13224	0	6	-6
696	16408	-16408	0	-8	8
724	20436	-20436	0	0	0
752	25216	-25216	0	10	-10
780	31120	-31120	0	-8	8
808	38148	-38148	0	0	0
836	46784	-46784	0	8	-8
864	56976	-56976	0	-12	12
892	69432	-69432	0	0	0
920	84144	-84144	0	12	-12
948	101904	-101904	0	-12	12
976	122868	-122868	0	0	0
1004	148076	-148076	0	14	-14
1032	177656	-177656	0	-16	16
1060	213072	-213072	0	0	0

Table C.13:  $H_{g,3}^{(7)}$ .

$[g]$	1A	2A	4A	3AB	6AB
$\Gamma_g$	1	1 4	2 8	3	3 4
19	6	6	-2	0	0
47	12	12	4	0	0
75	22	22	-2	1	1
103	36	36	4	0	0
131	58	58	-6	-2	-2
159	90	90	2	0	0
187	132	132	-4	0	0
215	190	190	6	-2	-2
243	274	274	-6	1	1
271	384	384	8	0	0
299	528	528	-8	0	0
327	722	722	10	2	2
355	972	972	-12	0	0
383	1300	1300	12	-2	-2
411	1724	1724	-12	2	2
439	2256	2256	16	0	0
467	2938	2938	-22	-2	-2
495	3806	3806	22	2	2
523	4890	4890	-22	0	0
551	6244	6244	28	-2	-2
579	7940	7940	-28	2	2
607	10038	10038	30	0	0
635	12620	12620	-36	-4	-4
663	15814	15814	38	4	4
691	19722	19722	-46	0	0
719	24490	24490	50	-2	-2
747	30310	30310	-50	4	4
775	37362	37362	58	0	0
803	45908	45908	-68	-4	-4
831	56236	56236	68	4	4
859	68646	68646	-74	0	0
887	83556	83556	84	-6	-6
915	101436	101436	-92	6	6
943	122790	122790	102	0	0
971	148254	148254	-106	-6	-6
999	178566	178566	118	6	6
1027	214548	214548	-132	0	0
1055	257190	257190	142	-6	-6

Table C.14:  $H_{g,4}^{(7)}$ .

$[g]$	1A	2A	4A	3AB	6AB
$\Gamma_g$	1	1 4	2 8	3	3 4
12	4	-4	0	1	-1
40	12	-12	0	0	0
68	16	-16	0	-2	2
96	36	-36	0	0	0
124	48	-48	0	0	0
152	84	-84	0	0	0
180	116	-116	0	2	-2
208	180	-180	0	0	0
236	244	-244	0	-2	2
264	360	-360	0	0	0
292	480	-480	0	0	0
320	676	-676	0	-2	2
348	896	-896	0	2	-2
376	1224	-1224	0	0	0
404	1588	-1588	0	-2	2
432	2128	-2128	0	1	-1
460	2736	-2736	0	0	0
488	3588	-3588	0	0	0
516	4576	-4576	0	4	-4
544	5904	-5904	0	0	0
572	7448	-7448	0	-4	4
600	9500	-9500	0	2	-2
628	11892	-11892	0	0	0
656	14992	-14992	0	-2	2
684	18628	-18628	0	4	-4
712	23256	-23256	0	0	0
740	28688	-28688	0	-4	4
768	35532	-35532	0	3	-3
796	43560	-43560	0	0	0
824	53528	-53528	0	-4	4
852	65256	-65256	0	6	-6
880	79656	-79656	0	0	0
908	96564	-96564	0	-6	6
936	117196	-117196	0	4	-4
964	141360	-141360	0	0	0
992	170600	-170600	0	-4	4
1020	204848	-204848	0	8	-8
1048	245988	-245988	0	0	0

Table C.15:  $H_{g,5}^{(7)}$ .

$[g]$	1A	2A	4A	3AB	6AB
$\Gamma_g$	1	1 4	2 8	3	3 4
3	2	2	2	-1	-1
31	6	6	-2	0	0
59	14	14	-2	2	2
87	22	22	-2	-2	-2
115	36	36	4	0	0
143	56	56	0	2	2
171	82	82	2	-2	-2
199	126	126	-2	0	0
227	182	182	6	2	2
255	250	250	-6	-2	-2
283	354	354	2	0	0
311	490	490	-6	4	4
339	656	656	8	-4	-4
367	882	882	-6	0	0
395	1180	1180	4	4	4
423	1550	1550	-10	-4	-4
451	2028	2028	12	0	0
479	2638	2638	-10	4	4
507	3394	3394	10	-5	-5
535	4362	4362	-14	0	0
563	5562	5562	18	6	6
591	7032	7032	-16	-6	-6
619	8886	8886	14	0	0
647	11166	11166	-18	6	6
675	13940	13940	28	-7	-7
703	17358	17358	-26	0	0
731	21536	21536	24	8	8
759	26594	26594	-30	-10	-10
787	32742	32742	38	0	0
815	40180	40180	-36	10	10
843	49124	49124	36	-10	-10
871	59916	59916	-44	0	0
899	72852	72852	52	12	12
927	88296	88296	-56	-12	-12
955	106788	106788	52	0	0
983	128816	128816	-64	14	14
1011	154948	154948	76	-14	-14



Table C.16:  $H_{g,6}^{(7)}$ .

$[g]$	1A	2A	4A	3AB	6AB
$\Gamma_g$	1	1 4	2 8	3	3 4
20	4	-4	0	-2	2
48	4	-4	0	1	-1
76	12	-12	0	0	0
104	12	-12	0	0	0
132	32	-32	0	2	-2
160	36	-36	0	0	0
188	64	-64	0	-2	2
216	80	-80	0	2	-2
244	132	-132	0	0	0
272	160	-160	0	-2	2
300	252	-252	0	3	-3
328	312	-312	0	0	0
356	448	-448	0	-2	2
384	572	-572	0	2	-2
412	792	-792	0	0	0
440	992	-992	0	-4	4
468	1348	-1348	0	4	-4
496	1680	-1680	0	0	0
524	2220	-2220	0	-6	6
552	2776	-2776	0	4	-4
580	3600	-3600	0	0	0
608	4460	-4460	0	-4	4
636	5712	-5712	0	6	-6
664	7044	-7044	0	0	0
692	8892	-8892	0	-6	6
720	10932	-10932	0	6	-6
748	13656	-13656	0	0	0
776	16672	-16672	0	-8	8
804	20672	-20672	0	8	-8
832	25116	-25116	0	0	0
860	30856	-30856	0	-8	8
888	37352	-37352	0	8	-8
916	45564	-45564	0	0	0
944	54884	-54884	0	-10	10
972	66572	-66572	0	11	-11
1000	79848	-79848	0	0	0
1028	96256	-96256	0	-14	14

## C.6. Lambency thirteen

Table C.17:  $H_{g,1}^{(13)}$ .

$[g]$	1A	2A	4AB
$\Gamma_g$	1	1 4	2 8
-1	-2	-2	-2
51	2	2	2
103	2	2	-2
155	0	0	0
207	2	2	-2
259	2	2	2
311	4	4	0
363	6	6	2
415	6	6	-2
467	8	8	4
519	12	12	-4
571	14	14	2
623	14	14	-2
675	20	20	4
727	24	24	-4
779	28	28	4
831	36	36	-4
883	42	42	6
935	50	50	-6
987	62	62	6
1039	70	70	-6
1091	84	84	8
1143	102	102	-6
1195	118	118	6
1247	136	136	-8
1299	162	162	10
1351	190	190	-10
1403	216	216	8
1455	254	254	-10
1507	292	292	12
1559	336	336	-12
1611	392	392	12
1663	446	446	-14
1715	510	510	14
1767	592	592	-16
1819	672	672	16
1871	764	764	-16
1923	876	876	20

Table C.18:  $H_{g,2}^{(13)}$ .

$[g]$	1A	2A	4AB
$\Gamma_g$	1	1 4	2 8
48	0	0	0
100	2	-2	0
152	4	-4	0
204	4	-4	0
256	6	-6	0
308	8	-8	0
360	8	-8	0
412	12	-12	0
464	16	-16	0
516	20	-20	0
568	24	-24	0
620	32	-32	0
672	36	-36	0
724	48	-48	0
776	56	-56	0
828	68	-68	0
880	80	-80	0
932	100	-100	0
984	112	-112	0
1036	140	-140	0
1088	164	-164	0
1140	192	-192	0
1192	224	-224	0
1244	268	-268	0
1296	306	-306	0
1348	364	-364	0
1400	420	-420	0
1452	488	-488	0
1504	560	-560	0
1556	656	-656	0
1608	744	-744	0
1660	864	-864	0
1712	988	-988	0
1764	1134	-1134	0
1816	1292	-1292	0
1868	1484	-1484	0
1920	1676	-1676	0
1972	1920	-1920	0

Table C.19:  $H_{g,3}^{(13)}$ .

$[g]$	1A	2A	4AB
$\Gamma_g$	1	1 4	2 8
43	2	2	-2
95	2	2	2
147	4	4	0
199	6	6	2
251	8	8	-4
303	10	10	2
355	14	14	-2
407	18	18	2
459	22	22	-2
511	26	26	2
563	34	34	-2
615	44	44	4
667	52	52	-4
719	64	64	4
771	78	78	-2
823	96	96	4
875	114	114	-6
927	136	136	4
979	164	164	-4
1031	194	194	6
1083	230	230	-6
1135	270	270	6
1187	318	318	-6
1239	374	374	6
1291	434	434	-10
1343	506	506	10
1395	592	592	-8
1447	686	686	10
1499	792	792	-12
1551	914	914	10
1603	1054	1054	-10
1655	1214	1214	14
1707	1394	1394	-14
1759	1594	1594	14
1811	1822	1822	-14
1863	2084	2084	16
1915	2374	2374	-18
1967	2698	2698	18

Table C.20:  $H_{g,4}^{(13)}$ .

$[g]$	1A	2A	4AB
$\Gamma_g$	1	1 4	2 8
36	2	-2	0
88	4	-4	0
140	4	-4	0
192	8	-8	0
244	8	-8	0
296	12	-12	0
348	16	-16	0
400	22	-22	0
452	24	-24	0
504	36	-36	0
556	40	-40	0
608	52	-52	0
660	64	-64	0
712	80	-80	0
764	92	-92	0
816	116	-116	0
868	136	-136	0
920	168	-168	0
972	196	-196	0
1024	238	-238	0
1076	272	-272	0
1128	332	-332	0
1180	384	-384	0
1232	456	-456	0
1284	528	-528	0
1336	620	-620	0
1388	712	-712	0
1440	840	-840	0
1492	960	-960	0
1544	1120	-1120	0
1596	1280	-1280	0
1648	1484	-1484	0
1700	1688	-1688	0
1752	1952	-1952	0
1804	2216	-2216	0
1856	2544	-2544	0
1908	2888	-2888	0
1960	3304	-3304	0

Table C.21:  $H_{g,5}^{(13)}$ .

$[g]$	1A	2A	4AB
$\Gamma_g$	1	1 4	2 8
27	2	2	2
79	4	4	0
131	6	6	2
183	6	6	-2
235	10	10	2
287	14	14	-2
339	16	16	0
391	22	22	-2
443	30	30	2
495	36	36	-4
547	46	46	2
599	58	58	-2
651	68	68	4
703	86	86	-2
755	106	106	2
807	124	124	-4
859	152	152	4
911	184	184	-4
963	216	216	4
1015	258	258	-6
1067	308	308	4
1119	362	362	-6
1171	426	426	6
1223	502	502	-6
1275	584	584	8
1327	684	684	-8
1379	798	798	6
1431	920	920	-8
1483	1070	1070	10
1535	1238	1238	-10
1587	1422	1422	10
1639	1638	1638	-10
1691	1884	1884	12
1743	2156	2156	-12
1795	2468	2468	12
1847	2822	2822	-14
1899	3212	3212	16
1951	3660	3660	-16

Table C.22:  $H_{g,6}^{(13)}$ .

$[g]$	1A	2A	4AB
$\Gamma_g$	1	1 4	2 8
16	2	-2	0
68	4	-4	0
120	4	-4	0
172	8	-8	0
224	8	-8	0
276	16	-16	0
328	16	-16	0
380	24	-24	0
432	28	-28	0
484	38	-38	0
536	44	-44	0
588	60	-60	0
640	68	-68	0
692	88	-88	0
744	104	-104	0
796	132	-132	0
848	152	-152	0
900	190	-190	0
952	220	-220	0
1004	268	-268	0
1056	312	-312	0
1108	376	-376	0
1160	432	-432	0
1212	520	-520	0
1264	596	-596	0
1316	708	-708	0
1368	812	-812	0
1420	956	-956	0
1472	1092	-1092	0
1524	1280	-1280	0
1576	1460	-1460	0
1628	1696	-1696	0
1680	1932	-1932	0
1732	2236	-2236	0
1784	2536	-2536	0
1836	2924	-2924	0
1888	3308	-3308	0
1940	3792	-3792	0

Table C.23:  $H_{g,7}^{(13)}$ .

$[g]$	1A	2A	4AB
$\Gamma_g$	1	1 4	2 8
3	2	2	-2
55	2	2	2
107	4	4	0
159	8	8	0
211	10	10	-2
263	12	12	0
315	16	16	0
367	22	22	2
419	26	26	-2
471	34	34	2
523	44	44	0
575	54	54	2
627	68	68	-4
679	82	82	2
731	102	102	-2
783	124	124	4
835	148	148	-4
887	176	176	4
939	214	214	-2
991	256	256	4
1043	300	300	-4
1095	356	356	4
1147	420	420	-4
1199	494	494	6
1251	580	580	-8
1303	674	674	6
1355	786	786	-6
1407	918	918	6
1459	1060	1060	-8
1511	1226	1226	6
1563	1418	1418	-6
1615	1632	1632	8
1667	1874	1874	-10
1719	2150	2150	10
1771	2464	2464	-8
1823	2816	2816	12
1875	3214	3214	-14

Table C.24:  $H_{g,8}^{(13)}$ .

$[g]$	1A	2A	4AB
$\Gamma_g$	1	1 4	2 8
40	4	-4	0
92	4	-4	0
144	6	-6	0
196	6	-6	0
248	12	-12	0
300	12	-12	0
352	20	-20	0
404	24	-24	0
456	32	-32	0
508	36	-36	0
560	52	-52	0
612	56	-56	0
664	76	-76	0
716	88	-88	0
768	112	-112	0
820	128	-128	0
872	164	-164	0
924	184	-184	0
976	232	-232	0
1028	268	-268	0
1080	324	-324	0
1132	372	-372	0
1184	452	-452	0
1236	512	-512	0
1288	616	-616	0
1340	704	-704	0
1392	832	-832	0
1444	950	-950	0
1496	1120	-1120	0
1548	1268	-1268	0
1600	1486	-1486	0
1652	1688	-1688	0
1704	1956	-1956	0
1756	2220	-2220	0
1808	2568	-2568	0
1860	2896	-2896	0
1912	3336	-3336	0

Table C.25:  $H_{g,9}^{(13)}$ .

$[g]$	1A	2A	4AB
$\Gamma_g$	1	1 4	2 8
23	2	2	-2
75	4	4	0
127	4	4	0
179	6	6	2
231	8	8	0
283	12	12	0
335	14	14	-2
387	20	20	0
439	26	26	-2
491	30	30	2
543	40	40	0
595	50	50	2
647	60	60	0
699	74	74	2
751	90	90	-2
803	108	108	4
855	134	134	-2
907	158	158	2
959	188	188	-4
1011	226	226	2
1063	266	266	-2
1115	314	314	2
1167	372	372	-4
1219	436	436	4
1271	508	508	-4
1323	596	596	4
1375	692	692	-4
1427	802	802	6
1479	932	932	-4
1531	1074	1074	6
1583	1238	1238	-6
1635	1430	1430	6
1687	1640	1640	-8
1739	1878	1878	6
1791	2150	2150	-6
1843	2456	2456	8
1895	2800	2800	-8

Table C.26:  $H_{g,10}^{(13)}$ .

$[g]$	1A	2A	4AB
$\Gamma_g$	1	1 4	2 8
4	2	-2	0
56	0	0	0
108	4	-4	0
160	4	-4	0
212	8	-8	0
264	8	-8	0
316	12	-12	0
368	12	-12	0
420	20	-20	0
472	20	-20	0
524	32	-32	0
576	34	-34	0
628	48	-48	0
680	52	-52	0
732	72	-72	0
784	78	-78	0
836	104	-104	0
888	116	-116	0
940	148	-148	0
992	164	-164	0
1044	208	-208	0
1096	232	-232	0
1148	288	-288	0
1200	324	-324	0
1252	396	-396	0
1304	444	-444	0
1356	536	-536	0
1408	604	-604	0
1460	720	-720	0
1512	812	-812	0
1564	960	-960	0
1616	1080	-1080	0
1668	1268	-1268	0
1720	1428	-1428	0
1772	1664	-1664	0
1824	1872	-1872	0

Table C.27:  $H_{g,11}^{(13)}$ .

$[g]$	1A	2A	4AB
$\Gamma_g$	1	1 4	2 8
35	2	2	2
87	2	2	2
139	2	2	-2
191	4	4	0
243	4	4	0
295	8	8	0
347	10	10	-2
399	10	10	2
451	16	16	0
503	20	20	0
555	22	22	-2
607	28	28	0
659	36	36	0
711	44	44	0
763	54	54	-2
815	64	64	0
867	76	76	0
919	94	94	2
971	114	114	-2
1023	130	130	2
1075	156	156	0
1127	188	188	0
1179	216	216	-4
1231	254	254	2
1283	300	300	0
1335	346	346	2
1387	404	404	-4
1439	470	470	2
1491	542	542	-2
1543	630	630	2
1595	724	724	-4
1647	828	828	4
1699	954	954	-2
1751	1100	1100	4
1803	1250	1250	-6
1855	1428	1428	4

Table C.28:  $H_{g,12}^{(13)}$ .

$[g]$	1A	2A	4AB
$\Gamma_g$	1	1 4	2 8
12	0	0	0
64	2	-2	0
116	0	0	0
168	4	-4	0
220	0	0	0
272	4	-4	0
324	2	-2	0
376	8	-8	0
428	4	-4	0
480	12	-12	0
532	8	-8	0
584	16	-16	0
636	16	-16	0
688	24	-24	0
740	20	-20	0
792	36	-36	0
844	32	-32	0
896	48	-48	0
948	48	-48	0
1000	68	-68	0
1052	68	-68	0
1104	96	-96	0
1156	98	-98	0
1208	128	-128	0
1260	136	-136	0
1312	176	-176	0
1364	184	-184	0
1416	240	-240	0
1468	252	-252	0
1520	312	-312	0
1572	340	-340	0
1624	416	-416	0
1676	448	-448	0
1728	548	-548	0
1780	592	-592	0
1832	708	-708	0

## Appendix D. Decompositions

As explained in Section 4 (see also Section 5.1) our conjectural proposals for the Umbral McKay–Thompson series  $H_{g,r}^{(\ell)}(\tau) = \sum_d c_{g,r}^{(\ell)}(d)q^d$  (cf. Sections 4 and Appendix C) determine the  $G^{(\ell)}$ -modules  $K_{r,d}^{(\ell)}$  up to isomorphism for  $d > 0$ , at least for those values of  $d$  for which we can identify all the Fourier coefficients  $c_{g,r}^{(\ell)}(d)$ . In this section, we furnish tables of explicit decompositions into irreducible representations of  $G^{(\ell)}$  for  $K_{r,d}^{(\ell)}$ , for the first few values of  $d$ . The coefficient  $c_r^{(\ell)}(d)$  of  $H_r^{(\ell)} = H_{e,r}^{(\ell)}$  is non-zero only when  $d = n - r^2/4\ell$  for some integer  $n \geq 0$ . For each of the tables in this section the rows are labelled by the values  $4\ell d$ , so that the entry in row  $m$  and column  $\chi_i$  indicates the multiplicity of the irreducible representation of  $G^{(\ell)}$  with character  $\chi_i$  (in the notation of the character tables of Appendix B.1) appearing in the  $G^{(\ell)}$ -module  $K_{r,m/4\ell}^{(\ell)}$ . One can observe that these tables support Conjectures 5.1, 5.11 and 5.12, and also give evidence in support of the hypothesis that  $K_{r,d}^{(\ell)}$  has a decomposition into irreducible representations that factor through  $\bar{G}^{(\ell)}$  when  $r$  is odd, and has a decomposition into faithful irreducible representations of  $G^{(\ell)}$  when  $r$  is even.



D.1. Lambency two

Table D.1: Decomposition of  $K_1^{(2)}$ .

	$\chi_1$	$\chi_2$	$\chi_3$	$\chi_4$	$\chi_5$	$\chi_6$	$\chi_7$	$\chi_8$	$\chi_9$	$\chi_{10}$	$\chi_{11}$	$\chi_{12}$	$\chi_{13}$	$\chi_{14}$	$\chi_{15}$	$\chi_{16}$	$\chi_{17}$	$\chi_{18}$	$\chi_{19}$	$\chi_{20}$	$\chi_{21}$	$\chi_{22}$	$\chi_{23}$	$\chi_{24}$	$\chi_{25}$	$\chi_{26}$	
-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
31	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
47	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
55	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	2	0	0	0	2	2	2	2
63	0	0	0	0	0	0	0	0	0	0	1	1	0	1	1	1	2	0	2	2	2	2	2	4	2	2	6
71	0	0	0	0	0	0	0	0	2	2	2	0	0	2	2	2	4	4	4	4	6	6	8	12	10	10	24
79	0	0	0	0	2	2	0	2	2	0	4	4	4	6	4	4	2	8	10	8	14	12	22	24	26	40	
87	0	0	0	0	0	0	0	0	4	4	6	8	8	8	8	8	12	12	12	18	26	30	40	38	40	80	
95	0	2	0	2	2	2	4	4	6	6	8	10	14	18	14	14	16	26	30	28	44	44	70	80	84	136	
103	0	0	2	2	2	2	4	2	6	10	10	14	14	18	14	14	16	26	30	38	46	86	128	126	132	254	
111	0	0	0	0	8	8	4	6	14	16	16	24	24	22	24	24	34	38	46	58	80	86	128	126	132	254	
119	0	0	2	2	8	8	12	8	18	38	38	40	40	46	44	44	46	78	86	88	138	144	218	238	246	424	
127	0	2	2	2	18	18	18	22	36	50	50	72	72	68	72	72	100	122	140	170	232	252	378	382	400	742	
135	0	2	8	8	25	25	30	26	54	94	94	116	116	130	124	124	140	212	246	262	392	410	630	670	704	1222	
143	0	6	6	6	50	50	50	58	100	148	148	194	194	192	202	202	256	342	388	454	654	704	1044	1074	1120	2058	
151	0	4	18	18	68	68	80	72	150	252	252	318	318	346	332	332	394	582	664	722	1062	1116	1702	1800	1880	3320	
159	0	14	20	20	182	182	214	200	396	652	652	814	814	872	860	860	1020	1476	1684	1862	2742	2902	4384	4622	4828	8572	
175	2	32	55	55	314	314	328	346	640	988	988	1298	1336	1348	1348	1686	2302	3000	3000	4324	4616	6950	7204	7532	13620		
183	2	40	98	98	460	460	512	496	972	1590	1590	2020	2020	2144	2118	2118	2546	3638	4162	4624	6768	7166	10856	11376	11898	21204	

**D.2. Lambency three**Table D.2: Decomposition of  $K_1^{(3)}$ .

	$\chi_1$	$\chi_2$	$\chi_3$	$\chi_4$	$\chi_5$	$\chi_6$	$\chi_7$	$\chi_8$	$\chi_9$	$\chi_{10}$	$\chi_{11}$	$\chi_{12}$	$\chi_{13}$	$\chi_{14}$	$\chi_{15}$
-1	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0
35	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0
47	0	0	0	0	0	0	0	0	0	2	0	2	0	0	2
59	0	0	0	0	0	2	2	0	0	0	2	2	2	2	2
71	0	0	2	0	0	0	0	2	2	2	2	2	4	4	6
83	0	0	0	2	2	4	4	2	2	2	4	6	6	8	10
95	0	0	2	0	0	4	4	8	8	6	6	10	12	14	18
107	0	2	2	4	4	8	12	8	8	8	14	16	22	28	30

Table D.3: Decomposition of  $K_2^{(3)}$ .

	$\chi_{16}$	$\chi_{17}$	$\chi_{18}$	$\chi_{19}$	$\chi_{20}$	$\chi_{21}$	$\chi_{22}$	$\chi_{23}$	$\chi_{24}$	$\chi_{25}$	$\chi_{26}$
8	1	1	0	0	0	0	0	0	0	0	0
20	0	0	0	0	1	1	0	0	0	0	0
32	0	0	0	0	0	0	1	1	0	0	0
44	0	0	0	0	0	0	0	0	2	1	1
56	0	0	0	2	0	0	2	2	0	2	2
68	0	0	2	0	2	2	2	2	4	4	4
80	2	2	0	0	1	1	6	6	4	8	8
92	0	0	2	4	6	6	8	8	12	14	14
104	2	2	0	4	6	6	20	20	16	24	24
116	2	2	6	8	12	12	26	26	36	44	44

**D.3. Lambency four**Table D.4: Decomposition of  $K_1^{(4)}$ .

	$\chi_1$	$\chi_2$	$\chi_3$	$\chi_4$	$\chi_5$	$\chi_6$	$\chi_7$	$\chi_8$	$\chi_9$	$\chi_{10}$	$\chi_{11}$
-1	-2	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	2	0	0	0	0
31	0	0	0	0	0	0	0	0	0	0	2
47	0	0	0	0	0	2	0	0	2	2	0
63	0	1	1	0	2	0	0	2	2	2	4
79	0	0	0	2	2	2	4	0	4	6	4
95	0	2	2	2	2	4	2	4	6	8	12
111	2	2	2	6	6	6	8	6	10	18	14
127	0	4	4	6	10	10	6	12	18	26	30
143	2	6	6	14	14	18	18	10	32	46	40
159	4	10	10	18	24	26	20	26	44	68	76

Table D.5: Decomposition of  $K_3^{(4)}$ .

	$\chi_1$	$\chi_2$	$\chi_3$	$\chi_4$	$\chi_5$	$\chi_6$	$\chi_7$	$\chi_8$	$\chi_9$	$\chi_{10}$	$\chi_{11}$
7	0	1	1	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	2	0	0
39	0	0	0	0	2	0	0	0	0	2	0
55	0	0	0	2	0	0	0	2	2	2	2
71	0	2	2	0	2	2	0	0	2	4	4
87	2	0	0	2	2	2	4	4	6	6	8
103	0	2	2	2	6	6	4	2	6	14	12
119	2	2	2	8	4	8	6	8	18	20	22
135	2	8	8	8	14	14	12	10	20	36	34
151	4	6	6	18	16	20	20	22	42	54	56

Table D.6: Decomposition of  $K_2^{(4)}$ .

	$\chi_{12}$	$\chi_{13}$	$\chi_{14}$	$\chi_{15}$	$\chi_{16}$
12	0	1	1	0	0
28	0	0	0	1	1
44	2	0	0	2	2
60	0	2	2	4	4
76	2	2	2	8	8
92	6	4	4	14	14
108	6	9	9	24	24
124	14	14	14	40	40
140	24	20	20	66	66
156	32	36	36	104	104

#### D.4. Lambency five

Table D.7: Decomposition of  $K_1^{(5)}$ .

	$\chi_1$	$\chi_2$	$\chi_3$	$\chi_4$	$\chi_5$	$\chi_6$	$\chi_7$
-1	-2	0	0	0	0	0	0
19	0	0	2	0	0	0	0
39	0	0	0	2	0	2	0
59	0	0	2	0	2	2	2
79	0	2	2	4	2	2	4
99	2	0	6	2	6	4	6
119	0	2	6	8	8	10	10
139	4	2	14	10	14	12	16
159	2	6	14	20	20	22	26
179	8	4	28	22	36	32	40

Table D.8: Decomposition of  $K_3^{(5)}$ .

	$\chi_1$	$\chi_2$	$\chi_3$	$\chi_4$	$\chi_5$	$\chi_6$	$\chi_7$
11	0	0	0	2	0	0	0
31	0	0	0	0	2	0	2
51	0	0	2	2	2	2	2
71	2	0	4	2	4	4	4
91	0	2	4	6	6	8	8
111	2	2	10	8	12	10	14
131	4	4	14	16	18	18	22
151	8	4	24	22	30	30	34
171	8	10	34	38	44	46	54
191	14	14	58	52	68	64	82

Table D.9: Decomposition of  $K_2^{(5)}$ .

	$\chi_8$	$\chi_9$	$\chi_{10}$	$\chi_{11}$	$\chi_{12}$	$\chi_{13}$	$\chi_{14}$
16	0	0	0	0	1	1	0
36	0	0	1	1	1	1	2
56	2	2	2	2	2	2	2
76	0	0	4	4	4	4	6
96	2	2	6	6	8	8	8
116	2	2	10	10	12	12	18
136	4	4	16	16	22	22	22
156	6	6	26	26	32	32	42
176	12	12	40	40	50	50	56
196	13	13	60	60	74	74	94

Table D.10: Decomposition of  $K_4^{(5)}$ .

	$\chi_8$	$\chi_9$	$\chi_{10}$	$\chi_{11}$	$\chi_{12}$	$\chi_{13}$	$\chi_{14}$
4	1	1	0	0	0	0	0
24	0	0	0	0	0	0	2
44	0	0	0	0	2	2	0
64	0	0	2	2	1	1	4
84	2	2	2	2	4	4	2
104	0	0	4	4	6	6	10
124	4	4	8	8	10	10	8
144	1	1	13	13	14	14	22
164	6	6	18	18	26	26	24
184	6	6	30	30	34	34	50

### D.5. Lambency seven

Table D.11: Decomposition of  $K_1^{(7)}$ .

	$\chi_1$	$\chi_2$	$\chi_3$	$\chi_4$
-1	-2	0	0	0
27	2	1	1	0
55	0	0	0	2
83	0	2	2	2
111	2	0	0	6
139	4	4	4	6
167	2	2	2	12
195	8	6	6	16
223	6	6	6	26
251	12	14	14	30

Table D.12: Decomposition of  $K_3^{(7)}$ .

	$\chi_1$	$\chi_2$	$\chi_3$	$\chi_4$
19	0	0	0	2
47	2	2	2	2
75	2	1	1	6
103	4	4	4	8
131	2	4	4	16
159	8	8	8	22
187	10	10	10	34
215	16	18	18	46
243	22	21	21	70
271	34	34	34	94

Table D.13: Decomposition of  $K_5^{(7)}$ .

	$\chi_1$	$\chi_2$	$\chi_3$	$\chi_4$
3	0	1	1	0
31	0	0	0	2
59	2	0	0	4
87	0	2	2	6
115	4	4	4	8
143	6	4	4	14
171	6	8	8	20
199	10	10	10	32
227	18	16	16	44
255	18	20	20	64

Table D.14: Decomposition of  $K_2^{(7)}$ .

	$\chi_5$	$\chi_6$	$\chi_7$
24	2	0	0
52	2	2	2
80	2	4	4
108	6	5	5
136	8	8	8
164	12	14	14
192	20	17	17
220	28	28	28
248	36	40	40
276	56	54	54

Table D.15: Decomposition of  $K_4^{(7)}$ .

	$\chi_5$	$\chi_6$	$\chi_7$
12	0	1	1
40	2	2	2
68	4	2	2
96	6	6	6
124	8	8	8
152	14	14	14
180	18	20	20
208	30	30	30
236	42	40	40
264	60	60	60

Table D.16: Decomposition of  $K_6^{(7)}$ .

	$\chi_5$	$\chi_6$	$\chi_7$
20	2	0	0
48	0	1	1
76	2	2	2
104	2	2	2
132	4	6	6
160	6	6	6
188	12	10	10
216	12	14	14
244	22	22	22
272	28	26	26

**D.6. Lambency thirteen**Table D.17: Decomposition of  $K_1^{(13)}$ .

	$\chi_1$	$\chi_2$
-1	-2	0
51	2	0
103	0	2
155	0	0
207	0	2
259	2	0
311	2	2
363	4	2
415	2	4
467	6	2

Table D.18: Decomposition of  $K_3^{(13)}$ .

	$\chi_1$	$\chi_2$
43	0	2
95	2	0
147	2	2
199	4	2
251	2	6
303	6	4
355	6	8
407	10	8
459	10	12
511	14	12

Table D.19: Decomposition of  $K_5^{(13)}$ .

	$\chi_1$	$\chi_2$
27	2	0
79	2	2
131	4	2
183	2	4
235	6	4
287	6	8
339	8	8
391	10	12
443	16	14
495	16	20

Table D.20: Decomposition of  $K_2^{(13)}$ .

	$\chi_3$	$\chi_4$
48	0	0
100	1	1
152	2	2
204	2	2
256	3	3
308	4	4
360	4	4
412	6	6
464	8	8
516	10	10



Table D.21: Decomposition of  $K_4^{(13)}$ .

	$\chi_3$	$\chi_4$
36	1	1
88	2	2
140	2	2
192	4	4
244	4	4
296	6	6
348	8	8
400	11	11
452	12	12
504	18	18

Table D.22: Decomposition of  $K_6^{(13)}$ .

	$\chi_3$	$\chi_4$
16	1	1
68	2	2
120	2	2
172	4	4
224	4	4
276	8	8
328	8	8
380	12	12
432	14	14
484	19	19

Table D.23: Decomposition of  $K_7^{(13)}$ .

	$\chi_1$	$\chi_2$
3	0	2
55	2	0
107	2	2
159	4	4
211	4	6
263	6	6
315	8	8
367	12	10
419	12	14
471	18	16

Table D.24: Decomposition of  $K_9^{(13)}$ .

	$\chi_1$	$\chi_2$
23	0	2
75	2	2
127	2	2
179	4	2
231	4	4
283	6	6
335	6	8
387	10	10
439	12	14
491	16	14

Table D.25: Decomposition of  $K_{11}^{(13)}$ .

	$\chi_1$	$\chi_2$
35	2	0
87	2	0
139	0	2
191	2	2
243	2	2
295	4	4
347	4	6
399	6	4
451	8	8
503	10	10

Table D.26: Decomposition of  $K_8^{(13)}$ .

	$\chi_3$	$\chi_4$
40	2	2
92	2	2
144	3	3
196	3	3
248	6	6
300	6	6
352	10	10
404	12	12
456	16	16
508	18	18

Table D.27: Decomposition of  $K_{10}^{(13)}$ .

	$\chi_3$	$\chi_4$
4	1	1
56	0	0
108	2	2
160	2	2
212	4	4
264	4	4
316	6	6
368	6	6
420	10	10
472	10	10

Table D.28: Decomposition of  $K_{12}^{(13)}$ .

	$\chi_3$	$\chi_4$
12	0	0
64	1	1
116	0	0
168	2	2
220	0	0
272	2	2
324	1	1
376	4	4
428	2	2
480	6	6

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