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Automorphy of Calabi–Yau threefolds of Borcea–Voisin type over $\mathbb{Q}$

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We consider certain Calabi–Yau threefolds of Borcea–Voisin type defined over $\mathbb{Q}$. We will discuss the automorphy of the Galois representations associated to these Calabi–Yau threefolds. We construct such Calabi–Yau threefolds as the quotients of products of $K3$ surfaces S and elliptic curves by a specific involution. We choose $K3$ surfaces S over $\mathbb{Q}$ with non-symplectic involution σ acting by -1 on $H^{2,0}(S)$. We fish out $K3$ surfaces with the involution σ from the famous 95 families of $K3$ surfaces in the list of Reid [32], and of Yonemura [43], where Yonemura described hypersurfaces defining these $K3$ surfaces in weighted projective 3-spaces.

Our first result is that for all but few (in fact, nine) of the 95 families of $K3$ surfaces S over $\mathbb{Q}$ in Reid–Yonemura's list, there are subsets of equations defining quasi-smooth hypersurfaces which are of Delsarte or Fermat type and endowed with non-symplectic involution σ. One implication of this result is that with this choice of defining equation, (S, σ) becomes of CM type.

Let E be an elliptic curve over $\mathbb{Q}$ with the standard involution ι, and let X be a standard (crepant) resolution, defined over $\mathbb{Q}$, of the quotient threefold $E \times S/\iota \times \sigma$, where (S, σ) is one of the above $K3$ surfaces over $\mathbb{Q}$ of CM type. One of our main results is the automorphy of the L-series of X.

The moduli spaces of these Calabi–Yau threefolds are Shimura varieties. Our result shows the existence of a CM point in the moduli space.

We also consider the L-series of mirror pairs of Calabi–Yau threefolds of Borcea–Voisin type, and study how L-series behave under mirror symmetry.

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1. Introduction

We will address the automorphy of the Galois representations associated to certain Calabi–Yau threefolds of Borcea–Voisin type over $\mathbb{Q}$. Here by the automorphy, we mean the Langlands reciprocity conjecture which claims that the L-series (of the ℓ-adic étale cohomology group) of the Calabi–Yau threefolds over $\mathbb{Q}$ come from automorphic representations. For our Calabi–Yau threefolds, we will show that these representations arise as induced automorphic cuspidal representations of $GL_2(K)$ of some abelian number fields K.

Our Calabi–Yau threefolds were previously considered by Voisin [41] and also by Borcea [6] from the point of view of geometry and also toward physics (mirror symmetry) applications.

We now describe briefly the Borcea–Voisin construction of Calabi–Yau threefolds over $\mathbb{C}$. Let E be any elliptic curve with involution ι, and let S be a $K3$ surface with involution σ acting by -1 on $H^{2,0}(S)$. The quotient threefold $E \times S/\iota \times \sigma$ is singular, but the singularities are all cyclic quotient singularities, and there is an explicit crepant resolution, which yields a smooth Calabi–Yau threefold X.

To find our $K3$ surfaces, we use the famous 95 families of $K3$ surfaces which can be given by weighted homogeneous equations in weighted projective 3-spaces. They are classified by M. Reid [32] (see also Iano–Fletcher [18]), and also in Yonemura [43]. Yonemura gave explicit equations for these surfaces as weighted hypersurfaces $h(x_0, x_1, x_2, x_3) = 0$ using toric methods, and we will use Yonemura's list throughout this paper.

We first fish out, from the list of Yonemura, $K3$ surfaces S having the required involutions σ acting on the holomorphic 2-forms of the surfaces as multiplication by -1. Earlier, Borcea [6] found 48 such pairs (S, σ). We will find additional $41 + (3)$ such pairs (S, σ) (our involutions may have a different formula from Borcea's examples), bringing the total to 92 pairs (S, σ).

Nikulin [29] classified all $K3$ surfaces (S, σ) over $\mathbb{C}$ with non-symplectic involution σ by triplets of integers (r, a, δ), and found that there are 75 triplets up to deformation. In this paper, we calculate only the invariants r and a for our 92 examples and realize at least 40 Nikulin triplets (r, a, δ). As the task of calculating δ is more involved, especially because we often need a $\mathbb{Z}$-basis for $\mathrm{Pic}\,(S)$, we leave the determination of the invariant δ to a future publication(s). Since $\delta \in \{0, 1\}$, the number of triplets realized may increase somewhat.

For 86 of our 92 pairs of (S, σ) above, we find a representative hypersurface defining equation for S of Delsarte type over $\mathbb{Q}$, that is, the equation consists exactly of four monomials with rational coefficients. Since S needs to be quasi-smooth, we put a condition on the defining equation (see Section 2.2). Then our new S has the same singularity configuration as the original hypersurface. (We should call attention why we only have 86 pairs: What happens to the remaining 6 pairs? This is because for the six weights, $K3$ surfaces have involution but cannot be realized as quasi-smooth hypersurfaces in four monomials.)

Thus, we obtain $K3$ surfaces S of Delsarte type. Recall that a cohomology group of a variety is of CM type if its Hodge group is commutative; and a variety is of CM type if all its cohomology groups are of CM type (see Zarhin [44]). In general, the computation of Hodge groups is notoriously difficult, and this is definitely not the direction we will pursue. Instead, we will follow the argument similar to the one in Livné–Schütt–Yui [25]: a Delsarte surface S can be realized as a quotient of a Fermat surface by some finite group. Since we know that Fermat (hyper) surfaces are of CM type, it follows that a Delsarte surface is also of CM type.

It is known [6, 33] that over $\mathbb{C}$ the moduli spaces of Nikulin's $K3$ families are Shimura varieties. Recently, the rationality of the moduli spaces of all but two out of the 75 Nikulin's $K3$ families has been established by Ma [26, 27], combined with the results of Kondo [21], and Dolgachev and Kondo [14].

Our results give explicit CM points in these moduli spaces defined over $\mathbb{Q}$; we do not know what their fields of definition (or moduli) are in the Shimura variety.

Next we take a product $E \times S$, where E is an elliptic curve over $\mathbb{Q}$ with the -1-involution ι, and S is a $K3$ surface of CM type over $\mathbb{Q}$ with involution σ as above. Take the quotient $E \times S/\iota \times \sigma$. Let X be a crepant resolution of the quotient threefold $E \times S/\iota \times \sigma$. Then X is a smooth Calabi–Yau threefold. We first show that X has a model defined over $\mathbb{Q}$. Then we will establish the automorphy of the Galois representations associated to X, in support of the Langlands reciprocity conjecture. We show that X is of CM type if and only if E also has complex multiplication.

This generalizes the work by Livné and Yui [24] on the modularity of the non-rigid Calabi–Yau threefold over $\mathbb{Q}$ obtained from the quotient $E \times S/\iota \times \sigma$, where S is a singular $K3$ surface with involution σ (and hence of CM type).

We also construct mirror partners $X^\vee$ (if they exist) of our Calabi–Yau threefolds using the Borcea–Voisin construction. In fact, 57 of the 95 families of $K3$ surfaces S of Reid and Yonemura have mirror partners $S^\vee$ within the list. We show that all these 57 families have subfamilies with involution σ and a CM point rational over $\mathbb{Q}$. Then the quotients of the products $E \times S^{\vee}/\iota \times \sigma^{\vee}$ give rise to mirror partners of $E \times S/\iota \times \sigma$.

From the point of view of mirror symmetry computations, our results supply particularly convenient base points both in the moduli space and in the mirror moduli space: they are defined over $\mathbb{Q}$, and their ℓ-adic étale cohomological Galois representations are attached to some automorphic forms whose L-series are known.

2. $K3$ surfaces

2.1. $K3$ surfaces with involution

Let S be a $K3$ surface over $\mathbb{C}$. Then $H^2(S,\mathbb{Z})$ is torsion-free and the intersection pairing gives it the structure of a lattice, even and unimodular, of rank 22 and signature $(3,19)$. By the classification theorem of such lattices, up to isometry,

$$H^2(S,\mathbb{Z}) \simeq U^3 \oplus (-E_8)^2,$$

where U is the usual hyperbolic lattice of rank 2 and E_8 is the unique even unimodular lattice of rank 8.

Let $\mathrm{Pic}\,(S)$ be the Picard lattice of S. It is torsion free and finitely generated, and together with the intersection pairing it can be identified as the sublattice $\mathrm{Pic}\,(S) = H^2(S,\mathbb{Z}) \cap H^{1,1}(S)$ of $H^2(S,\mathbb{Z})$. We define the transcendental lattice of S, denoted by $T(S)$, to be the orthogonal complement of $\mathrm{Pic}\,(S)$ in $H^2(S,\mathbb{Z})$, i.e., $T(S) := \mathrm{Pic}\,(S)^{\perp}$ in $H^2(S,\mathbb{Z})$, with respect to the intersection pairing.

Consider now a pair (S,σ), where S is a $K3$ surface and σ is an involution of S acting by -1 on $H^{2,0}(S)$. Let $\mathrm{Pic}\,(S)^{\sigma}$ denote the sublattice of $\mathrm{Pic}\,(S)$ fixed by σ. Let $(\mathrm{Pic}\,(S)^{\sigma})^* := \mathrm{Hom}(\mathrm{Pic}\,(\mathrm{S})^{\sigma},\mathbb{Z})$ be the dual lattice of $\mathrm{Pic}\,(S)^{\sigma}$. Let $T(S)_0 = (\mathrm{Pic}\,(S)^{\sigma})^{\perp}$ be the orthogonal complement of $\mathrm{Pic}\,(S)^{\sigma}$ in $H^2(S,\mathbb{Z})$, and let $T(S)_0^*$ be the dual lattice of $T(S)_0$. From the assumption that σ acts as -1 on the holomorphic 2-forms of S, one can show that it acts by -1 on $T(S)_0$ (and by 1 on $\mathrm{Pic}\,(S)^{\sigma}$).

Consider the quotient groups $(\mathrm{Pic}\,(S)^{\sigma})^*/\mathrm{Pic}\,(S)^{\sigma}$ and $T(S)_0^*/T(S)_0$. Since $H^2(S,\mathbb{Z})$ is unimodular, the two quotient abelian groups are canonically isomorphic:

$$(\mathrm{Pic}\,(S)^{\sigma})^*/\mathrm{Pic}\,(S)^{\sigma} \simeq T(S)_0^*/T(S)_0.$$

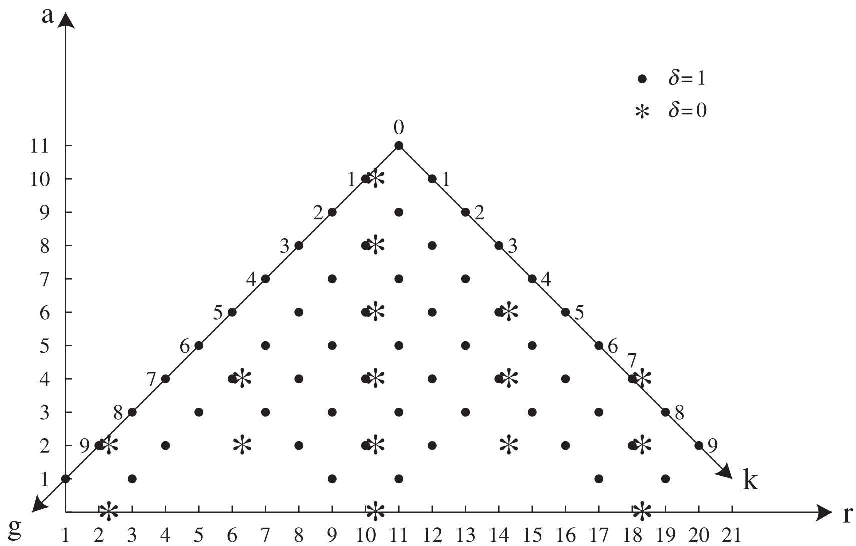

Figure 1: Nikulin's pyramid.

On $(\mathrm{Pic}\,(S)^\sigma)^*/\mathrm{Pic}\,(S)^\sigma$, σ acts by 1, whereas on $T(S)_0^*/T(S)_0$ it acts by -1. Then the only finite abelian groups where $+1$ is -1 are the $(\mathbb{Z}/2\mathbb{Z})^a$ for some a. This shows that

$$(\mathrm{Pic}\,(S)^\sigma)^*/\mathrm{Pic}\,(S)^\sigma \simeq (\mathbb{Z}/2\mathbb{Z})^a \quad \text{for some non-negative integer } a.$$

Nikulin [29, 30] has classified such pairs (S, σ).

Theorem 2.1 (Nikulin). *The pair (S, σ) of a $K3$ surface S with non-symplectic involution σ is determined, up to deformation, by a triplet (r, a, δ), where $r = \mathrm{rank\ Pic}\,(S)^\sigma$, $(\mathrm{Pic}\,(S)^\sigma)^*/\mathrm{Pic}\,(S)^\sigma \simeq (\mathbb{Z}/2\mathbb{Z})^a$, and $\delta = 0$ if $(x^*)^2 \in \mathbb{Z}$ for any $x^* \in (\mathrm{Pic}\,(S)^\sigma)^*$, and 1 otherwise.*

There are in total 75 triplets (r, a, δ), as shown in figure 1.

The moduli space of (S, σ) with given triplet (r, a, δ) is a bounded symmetric domain of type IV having dimension $20 - r$.

For a given pair (S, σ) of a $K3$ surface S with involution σ, we now consider the geometric structure of the fixed part S^σ of S under σ (i.e., the part where σ acts as identity). We follow Voisin [41] for this exposition.

Proposition 2.2. *There are three types for* S^σ*:*

(I) *For* $(r,a,\delta) \neq (10,10,0)$, $(10,8,0)$, S^σ *is a disjoint union of a smooth curve* C_g *of genus* g *and* k *rational curves* L_i*:*

$$S^\sigma = C_g \cup L_1 \cup \cdots \cup L_k.$$

(II) *For* $(r,a,\delta) = (10,10,0)$, $S^\sigma = \emptyset$.

(III) *For* $(r,a,\delta) = (10,8,0)$, S^σ *is a disjoint union of two elliptic curves* C_1 *and* $\bar{C}_1$*:*

$$S^\sigma = C_1 \cup \bar{C}_1.$$

Furthermore, in the case (I), the genus g *and the number* k *of rational curves can be determined in terms of the triplet* (r,a,δ) *as follows:*

$$g = \frac{1}{2}(22 - r - a),$$

and

$$k = \frac{1}{2}(r - a).$$

Equivalently, (r,a) *and* (g,k) *are related by the identities:*

$$r = 11 - g + k, \quad a = 11 - g - k.$$

Remark 2.1. Since σ is a non-symplectic involution, the quotient S/σ is either a rational surface or Enriques surface. It is an Enriques surface if and only if $S^\sigma = \emptyset$, i.e., $(r,a,\delta) = (10,10,0)$. Note that if σ is a symplectic involution, then σ has eight fixed points and the minimal resolution of S/σ is again a $K3$ surface.

2.2. Realization of $K3$ surfaces as hypersurfaces over $\mathbb{Q}$

We are interested in finding defining equations over $\mathbb{Q}$ for pairs (S,σ) of $K3$ surfaces S with non-symplectic involution σ. For this, we appeal to the famous 95 families of $K3$ surfaces of M. Reid [32] (see also Iano–Fletcher [18]) and of Yonemura [43]. All these 95 families of $K3$ surfaces are realized in weighted projective 3-spaces $\mathbb{P}^3(w_0,w_1,w_2,w_3)$. Reid determined 95 possible weights (w_0,w_1,w_2,w_3), and singularities as they are all determined by the weights. Then Yonemura described concrete families of hypersurfaces defining them, using toric constructions.

We first recall a result of Borcea [6]. Here we say that $Q = (w_0, w_1, w_2, w_3)$ is *normalized* if $\gcd(w_i, w_j, w_k) = 1$ for every distinct i, j, k. Also, we assume that w_i's are ordered in such a way that $w_0 \geq w_1 \geq w_2 \geq w_3$.

Proposition 2.3 (Borcea). *Assume that $Q = (w_0, w_1, w_2, w_3)$ is normalized and $w_0 = w_1 + w_2 + w_3$. Then there are in total 48 weights (w_0, w_1, w_2, w_3) giving rise to pairs (S, σ) of $K3$ surfaces S with involution σ acting by -1 on $H^{2,0}(S)$. More precisely, if w_0 is odd, there are 29 weights, and if w_0 is even, there are 19 weights.*

S may be realized as the minimal resolution of a hypersurface S_0 of degree $2w_0$ in $\mathbb{P}^3(w_0, w_1, w_2, w_3)$ of the form

$$x_0^2 = f(x_1, x_2, x_3),$$

where $deg(x_i) = w_i$ for $0 \leq i \leq 3$. A non-symplectic involution σ on S_0 is defined by $\sigma(x_0) = -x_0$, and f is a homogeneous polynomial in the variables x_1, x_2, x_3 of degree $2w_0$.

By abuse of notation, we often write σ for the involution on S_0 as well as that induced on S by desingularization.

Our first result is to extend the list of Borcea by adding more weights that yield $K3$ surfaces (S, σ) with involution σ.

Theorem 2.4. *There are in total $92 = 48 + 44$ normalized weights (w_0, w_1, w_2, w_3) giving rise to pairs (S, σ) of $K3$ surfaces S with non-symplectic involution σ defined by $\sigma(x_i) = -x_i$ for some single variable x_i. In other words, we have 44 new weights (i.e., not in the list of Borcea) yielding $K3$ surfaces with involution σ.*

We divide the 92 cases into two groups:

(i) *The 48 weights of Borcea and defining equations for quasi-smooth $K3$ surfaces S_0 are tabulated in tables* B.1 *to* B.3 *in* Appendix B.

(ii) *The additional 44 weights and defining equations $F(x_0, x_1, x_2, x_3) = 0$ for quasi-smooth $K3$ surfaces S_0 are tabulated in tables* B.4 *to* B.7 *in* Appendix B.

Proof. The proof of Theorem 2.4 is done by case by case analysis. Yonemura [43] determined hypersurface equations defining Reid's 95 families of $K3$ surfaces using toric geometry. We use his list of equations to find $K3$ surfaces with non-symplectic involutions.

If the defining equation contains the term x_0^2 or $x_0^2x_i$, then we can define involution σ by $\sigma(x_0) = -x_0$ just as in Borcea's 48 cases. If the defining equation contains the term $x_0^2x_i + x_0x_j^m$ (say), then we remove $x_0x_j^m$ to define the involution $\sigma(x_0) = -x_0$ (see tables B.4 and B.7).

For the equations in table B.5, we change x_0^3 to $x_0^2x_1$ to define an involution by $\sigma(x_0) = -x_0$.

For the equations in table B.6, we choose variables other than x_0 (and remove several terms if necessary) to define an involution.

Note that in each of the 92 cases, the quotient S/σ is a rational or Enriques surface. Hence σ is a non-symplectic involution. □

Remark 2.2. Among the 95 $K3$ weights of Reid, there are three cases $\#15, \#53, \#54$ where we find no obvious involution; that is, there is no involution σ on S acting as $\sigma(x_i) = -x_i$ for some variable x_i. These cases are tabulated in table B.8 in Appendix B.

For our arithmetic purposes, it is useful that we can compute the zeta-functions of S explicitly. One of such classes of varieties are those defined by equations of *Delsarte type* (i.e., equations consisting of the same number of monomials as the variables) named after Delsarte (see [25], Section 4). Hypersurfaces of Delsarte type are finite quotients of Fermat varieties. Our next task is to find the subset of the 92 cases of Theorem 2.4 which can be defined by equations of Delsarte type, namely

(1) for each S of [43], find an equation $h(x_0, x_1, x_2, x_3)$ consisting exactly of four monomials; and

(2) make sure that the hypersurface obtained in (1) is quasi-smooth.

Conditions (1) and (2) give a restriction on the form of S, but many of its geometric properties are unchanged. For instance, the types of singularities on S remain the same as the original hypersurfaces $h(x_0, x_1, x_2, x_3) = 0$ of [43].

Theorem 2.5. *There are* 86 *weights* (w_0, w_1, w_2, w_3) *for which there exists a quasi-smooth* $K3$ *surface* S_0 *in* $\mathbb{P}^3(w_0, w_1, w_2, w_3)$ *defined by a Delsarte equation with an involution. Moreover, this involution defines a non-symplectic involution* σ *on the minimal resolution* S *of* S_0.

(a) *If* (S, σ) *is one of the* 48 *pairs determined in* Proposition 2.3 *other than* $\#90, \#91, \#93$*, then* S_0 *can be defined by an equation over* $\mathbb{Q}$ *of four* *monomials*

$$x_0^2 = f(x_1, x_2, x_3) \subset \mathbb{P}^3(w_0, w_1, w_2, w_3).$$

The equation is obtained by removing several terms from the equation of Yonemura [43]*, where f is a homogeneous polynomial over $\mathbb{Q}$ of degree $w_0 + w_1 + w_2 + w_3$ (cf. tables* B.1 *to* B.3*).*

(b) *Let (S, σ) be one of the additional* 38 *pairs determined in* Theorem 2.4 (ii) *other than #85, #90, #91, #93, #94 and #95. Then S_0 can be defined by an equation $F(x_0, x_1, x_2, x_3) = 0$ over $\mathbb{Q}$ consisting of four monomials of degree $w_0 + w_1 + w_2 + w_3$. In most cases, $F(x_0, x_1, x_2, x_3)$ can be chosen as*

$$F(x_0, x_1, x_2, x_3) = x_0^2 x_i + f(x_1, x_2, x_3).$$

The weights and equations are listed in tables B.4 to B.6 in Appendix B.

Proof. This can be proved by case by case checking of the list of equations of Yonemura [43]. In both (a) and (b), we transform S into a Delsarte type by removing several terms of its original defining equation. In doing so, we make sure that the condition (2) above is satisfied so that the new surface is also quasi-smooth.

Condition (2) is met if for each variable x_i ($0 \le i \le 3$), the set of monomials containing x_i takes one of the following forms:

$$x_i^n,\ x_i^n x_j,\ x_i^n + x_i x_j^m,\ x_i^n x_k + x_i x_j^m$$

for some j and k ($j \neq k$) different from i.

This choice for the defining hypersurface preserves the configuration of singularities on S (i.e., types and the number of singularities) as the original hypersurfaces. The point is that for each of the 86 families, we can find a defining equation which consists of four monomials containing x_i^n or $x_i^n x_j$ ($i \neq j$) with nonzero coefficients. □

Remark 2.3. Our list of defining equations for S does not cover all possible equations of Delsarte type. For instance, in case #19 of weight $(3, 2, 2, 1)$ in table B.4, we may also choose an equation $x_0^2 x_1 + x_1^3 x_2 + x_2^4 + x_3^8 = 0$.

Remark 2.4. For the 95 families of quasi-smooth weighted $K3$ hypersurfaces, Yonemura [43] described the number of parameters (i.e., the number of monomials) for their defining equations. The minimum number was four, but often equations contain more than four monomials. Our result shows that except for the six cases #85, #94, #95 of table B.7 and #90, #91, #93 of Table B.1, the minimum number of parameters is attained.

Example 2.6. • Consider #42 in Yonemura = #3 in Borcea. The weight is $(5,3,1,1)$ and a hypersurface is given by

$$x_0^2 = f(x_1,x_2,x_3) = x_1^3x_2 + x_1^3x_2 + x_2^{10} + x_3^{10}$$

of degree 10. We can remove the second monomial $x_1^3x_3$. The singularity is of type A_2.

- Consider #78 in Yonemura = #10 in Borcea. The weight is $(11,6,4,1)$ and a hypersurface is given by

$$x_0^2 = f(x_1,x_2,x_3) = x_1^3x_2 + x_1^3x_3^4 + x_1x_2^4 + x_2^5x_3^2 + x_3^{22}$$

of degree 22. We can remove the second and the fourth monomials $x_1^3x_3^4$ and $x_2^5x_3^2$. The singularity is of type $A_1 + A_3 + A_5$.

- Consider #19 in Yonemura. The weight is $(3,2,2,1)$ and a hypersurface is given by

$$F(x_0,x_1,x_2,x_3) = x_0^2x_1 + x_0^2x_2 + x_0^2x_3^2 + x_1^4 + x_2^4 + x_3^8$$

of degree 8. We can remove the first or the second monomials $x_0^2x_1$ or $x_0^2x_2$, the third monomial $x_0^2x_3^2$. The involution is given by $x_0 \to -x_0$. The singularity is of type $4A_1 + A_2$.

Remark 2.5. The cases #85, #90, #91, #93, #94 and #95 of Yonemura cannot be realized as quasi-smooth hypersurfaces in four monomials with involution σ.

For instance, consider the case #85 of weight $(5,4,3,2)$ and degree 14. All the possible monomials of degree 14 are

$$x_0^2x_1,\quad x_0^2x_3^2,\quad x_0x_1x_2x_3,\quad x_1x_2^3,\quad x_0x_2x_3^3,\ x_1^3x_3,\quad x_1^2x_2^2,$$
$$x_1^2x_3^3,\quad x_1x_2^2x_3^2,\quad x_1x_3^5,\ x_2^4x_3,\quad x_2^2x_3^4,\quad x_3^7.$$

To make the polynomial quasi-smooth and defined by four monomials, we remove the monomials

$$x_0^2x_3^2,\quad x_0x_1x_2x_3,\quad x_0x_2x_3^3,\quad x_1^2x_2^2,\ x_1^2x_3^3,\quad x_1x_2^2x_3^2,\quad x_2^2x_3^4.$$

Then we obtain monomials

$$x_0^2x_1, \quad x_1x_2^3, \quad x_1^3x_3, \quad x_1x_3^5, \quad x_2^4x_3, \quad x_3^7.$$

There are no four monomials from this set such that their sum defines a quasi-smooth polynomial.

Note that if we allow more than four monomials, we can define a non-symplectic involution on this surface. For example, the surface defined by

$$x_0^2x_1 + x_0^2x_3^2 + x_1^3x_3 + x_1^2x_3^2 + x_2x_3^5 + x_2^4x_3 + x_3^7 = 0$$

is quasi-smooth and endowed with an involution $\sigma(x_0) = -x_0$.

2.3. $K3$ surfaces of CM type

Recall a definition of a CM type variety.

Definition 2.1. A cohomology group of a variety is said to be *of CM type* if its Hodge group is commutative, and a variety is said to be *of CM type* if all its cohomology groups are of CM type [44].

Theorem 2.7. *Let (S, σ) be one of the 86 pairs of $K3$ surfaces S with involution σ. Then (S, σ) is defined over $\mathbb{Q}$ and it is of CM type.*

Proof. We use Shioda's result (see, for instance, [36]). Let $X_n^m : x_0^m + x_1^m + \cdots + x_{n+1}^m = 0 \subset \mathbb{P}^{n+1}$ be the Fermat variety of degree m and dimension n. Let μ_m denote the group of mth roots of unity (in $\mathbb{C}$). Then the eigenspaces of the action of $(\mu_m)^{n+2}$ on the middle cohomology group of X_n^m are one-dimensional, and this action commutes with the Hodge group. Hence the Hodge group is commutative, and so the Fermat (hyper)surfaces are of CM type. Since a pair (S, σ) is a finite quotient of a Fermat surface, it is of CM type. □

2.4. Computations of Nikulin's invariants for $K3$ surfaces of Borcea type

Recall that a $K3$ surface S with involution σ is determined up to deformation by a triplet (r, a, δ), where r is the rank of $\mathrm{Pic}\,(S)^\sigma$. In this section, we compute r and a for $K3$ surfaces of Borcea type. By Proposition 2.2, r and

a can be computed through the fixed locus S^σ:

$$r = 11 - g + k, \quad a = 11 - g - k.$$

We note that the direct computation of r often requires a basis for $\mathrm{Pic}\,(S)$ or at least for $\mathrm{Pic}\,(S) \otimes \mathbb{Q}$. Since $\mathrm{Pic}\,(S)$ is usually difficult to determine, the fixed locus S^σ is often easier to handle than the Picard group.

In what follows, first we explain a general algorithm of computing g and k (and hence r and a). To describe the algorithm in detail, we choose $K3$ surfaces of Borcea type, namely those defined by $x_0^2 = f(x_1, x_2, x_3)$. After that, we explain how to compute r directly by looking at the σ-action on $\mathrm{Pic}\,(S)$ (see Theorem 2.11). It has a merit that we can obtain a closed formula for r.

2.4.1. Algorithm for the computation of g and k We explain how to compute g and k (and then r) for our $K3$ surfaces S. In this section and the next, we choose S_0 to be a surface in $\mathbb{P}^3(w_0, w_1, w_2, w_3)$ defined by the equation

$$x_0^2 = f(x_1, x_2, x_3) \quad \text{or} \quad x_0^2 x_i = f(x_1, x_2, x_3),$$

where σ acts on it by $\sigma(x_0) = -x_0$. We assume that $Q = (w_0, w_1, w_2, w_3)$ is normalized. The algorithm described in this section also works for more general $K3$ surfaces in $\mathbb{P}^3(Q)$ with non-symplectic involution.

Since S_0 is singular, S is chosen to be the minimal resolution of S_0 as in

$$\begin{array}{ccc} \mathbb{P}^3(w_0, w_1, w_2, w_3) & \longleftarrow & \tilde{\mathbb{P}^3}(w_0, w_1, w_2, w_3) \\ \cup & & \cup \\ S_0 & \longleftarrow & S \end{array}$$

where $\tilde{\mathbb{P}^3}(w_0, w_1, w_2, w_3)$ is a partial resolution of $\mathbb{P}^3(w_0, w_1, w_2, w_3)$ so that S is non-singular. The curve C_g in Proposition 2.2 is the strict transform of the curve defined by $x_0 = 0$ on S_0 which is isomorphic to the curve $f(x_1, x_2, x_3) = 0$ in $\mathbb{P}^2(w_1, w_2, w_3)$. The rational curves L_i in Proposition 2.2 are among those defined by letting either x_1, x_2 or x_3 be zero, or from the exceptional divisors arising in the desingularization. The procedure is as follows.

(i) The curve $\{x_0 = 0\}$ on S_0 is fixed by σ. Since $f(x_1, x_2, x_3) = 0$ is quasi-smooth (and hence smooth) in $\mathbb{P}^2(w_1, w_2, w_3)$, its strict transform C_g is also fixed by σ. The genus g can be calculated from $d := \deg f$ and weight (w_1, w_2, w_3) once it is normalized (see examples below). For instance, one can use the formula, which can be found in, e.g., Iano-Fletcher [18]:

$$g = \frac{1}{2}\left(\frac{d^2}{w_1 w_2 w_3} - d\sum_{i>j}\frac{\gcd(w_i, w_j)}{w_i w_j} + \sum_{i=1}^{3}\frac{\gcd(d, w_i)}{w_i} - 1\right).$$

(ii) The locus $\{x_1 = 0\}$, $\{x_2 = 0\}$ or $\{x_3 = 0\}$ may be fixed by σ depending on (w_0, w_1, w_2, w_3). For instance, consider the case where w_0 is odd. If w_2 and w_3 are even (and w_3 is necessarily odd), then the locus $\{x_3 = 0\}$ is fixed by σ. In this case, the strict transform of the locus $\{x_3 = 0\}$ is also point-wise fixed by σ. It can be shown that this locus is a rational curve on S and contributes to a curve L_i of Proposition 2.2.

(iii) The rest of the curves L_i (for S^σ) are exceptional divisors in the minimal resolution $S_0 \leftarrow S$. To find the divisors where σ acts as identity, we look carefully at the σ-action around the singularities of S_0 fixed by σ. Every singularity $P = (x_0 : x_1 : x_2 : x_3)$ on S_0 has at least two coordinates zero. Hence P is fixed by σ if and only if
- it has three zero coordinates, or
- it has exactly two zero coordinates with $x_0 = 0$, or
- it has exactly two zero coordinates and two non-zero coordinates x_0, x_i, and if $d = \gcd(w_0, w_i) \geq 2$, then w_0/d is odd and w_i/d is even.

Combining information obtained in (ii) and (iii), we can calculate the number k and hence the invariants r and a by the formula given in Proposition 2.2.

2.4.2. Computation of g and k for surfaces $x_0^2 = f(x_1, x_2, x_3)$ We consider surfaces $x_0^2 = f(x_1, x_2, x_3)$ in detail. Since the minimal resolution S is a $K3$ surface, P is a cyclic quotient singularity of type $A_{n+1,n}$ (or simply A_n) for some positive integer n. We obtain n exceptional divisors (i.e., irreducible components in the exceptional locus) by resolving P. Each exceptional divisor is isomorphic to $\mathbb{P}^1$ and σ acts on it either as identity or as a non-trivial involution, which depends on the singularity at P. In the first case, we say that the divisor is *ramified*. The detail is explained in the following lemmas.

Lemma 2.8. *Let $S_0 :\ x_0^2 = f(x_1, x_2, x_3)$ be one of the* 48 *$K3$ surfaces defined in* Proposition 2.3. *If $w_1 \geq 2$ and $P = (0, 1, 0, 0)$ is on S_0, then P is a cyclic quotient singularity of type A_{w_1, w_1-1} $(= A_{w_1-1})$. Among the exceptional divisors arising from P, ramified divisors appear alternately and there* *are* $\left[\dfrac{w_1-1}{2}\right]$ *of them, where* $[x]$ *denotes the integer part of* x*. The same assertion holds for singularities* $(0,0,1,0)$ *and* $(0,0,0,1)$*.*

Proof. When $P=(0,1,0,0)$ is a singularity, (x_0,x_2) or (x_0,x_3) gives a pair of local coordinates above P, depending on the polynomial $f(x_1,x_2,x_3)$. Assume that (x_0,x_2) is a coordinate system above P. Since P is a cyclic quotient singularity of type A_{w_1,w_1-1}, the μ_{w_1}-action above P can be written as

$$(x_0,x_2)\longmapsto(\zeta^{w_1-1}x_0,\zeta x_2)$$

with $\zeta\in\mu_{w_1}$. (In other words, P is the singularity of the quotient $\mathbb{A}^2/\mu_{w_1}$ at the origin.) By the quotient map $\pi_0: S_0\longrightarrow S_0/\sigma$, where S_0/σ is in $\mathbb{P}^3(2w_0,w_1,w_2,w_3)$, P is mapped to $\pi_0(P)\in S_0/\sigma$. It is a singularity locally described by the group action

$$(y_0,x_2)\longmapsto(\zeta^{2(w_1-1)}y_0,\zeta x_2)$$

with $y_0=x_0^2$. Hence $\pi_0(P)$ is a singularity of type $A_{w_1,2(w_1-1)}$ (or precisely, A_{w_1,w_1-2}).

In order to see how σ acts on the exceptional divisors, let $E_1+E_2+E_3+\cdots+E_{w_1-1}$ be the exceptional divisors on S arising from P. Here we set E_1 to be the divisor intersecting with (the strict transforms) of the curves passing through P. Consider the continued fractional expansions

$$\frac{w_1}{w_1-1}=2-\cfrac{1}{2-\cfrac{1}{2-\cfrac{1}{2-\cfrac{1}{\cdots}}}}\quad\text{and}\quad\frac{w_1}{2(w_1-1)}=1-\cfrac{1}{4-\cfrac{1}{1-\cfrac{1}{4-\cfrac{1}{\cdots}}}}.$$

If $\pi: S\longrightarrow S/\sigma$ denotes the quotient map, the continued fractions show that

$$\pi(E_i)^2=\begin{cases}-1 & \text{if } i \text{ is odd,}\\ -4 & \text{if } i \text{ is even.}\end{cases}$$

By the projection formula, we see that σ acts as identity (resp. -1) on E_i when $\pi(E_i)^2=-4$ (resp. when $\pi(E_i)^2=-1$). Therefore the ramified exceptional divisors are those E_i's with even i and there are in total $[\frac{w_1-1}{2}]$ of them. □

Next, we discuss singularities with exactly two zero coordinates, one of which is $x_0 = 0$.

Lemma 2.9. *Let* $S_0 :\ x_0^2 = f(x_1, x_2, x_3)$ *be one of the* 48 $K3$ *surfaces defined in* Proposition 2.3. *If* $\gcd(w_1, w_2) \geq 2$, *then* $P = (0, x_1, x_2, 0)$ *with* $x_1 x_2 \neq 0$ *is a singularity of type* $A_{2,1}$ *and fixed by* σ. *There arises one exceptional divisor from* P *and it is not ramified under the quotient* $S \longrightarrow S/\sigma$. *The same assertion holds for singularities of the form* $(0, x_1, 0, x_3)$ *and* $(0, 0, x_2, x_3)$.

Proof. Write $d := \gcd(w_1, w_2)$. When $P = (0, x_1, x_2, 0)$ is a singularity, (x_0, x_3) gives a local coordinate system above P; that is, P is a singularity of the quotient of an affine plane by the group action $\zeta \in \mu_d$ defined by

$$(x_0, x_3) \longmapsto (\zeta^{w_0} x_0, \zeta^{w_3} x_3).$$

Since f is quasi-smooth, f contains a term without x_3. Comparing the degree of monomials in $x_0^2 = f(x_1, x_2, x_3)$, one finds that $2w_0$ is divisible by d. Since the weight is normalized, we see that $\gcd(d, w_0) = \gcd(d, w_3) = 1$ and hence $d \mid 2$. If $d \geq 2$, then d must be 2.

Since $d = 2$, the relation $w_0 = w_1 + w_2 + w_3$ implies $w_0 \equiv w_3 \pmod 2$ and the μ_2 action above can be written as

$$(x_0, x_3) \longmapsto (-x_0, -x_3).$$

This means that P is a singularity of type $A_{2,1}$. If $\pi_0 : S_0 \longrightarrow S_0/\sigma$ denotes the quotient map, then $\pi_0(P)$ is the singularity associated with the group action

$$(y_0, x_3) \longmapsto (y_0, -x_3),$$

where $y_0 = x_0^2$. This shows in fact that $\pi_0(P) \in S_0/\sigma$ is not a singularity. Hence if E is the exceptional divisor arising from P and $\pi : S \longrightarrow S/\sigma$ is the quotient map, $\pi(E)$ should be a (-1)-curve. Therefore E is not ramified in π. □

Lastly, we look at the singularity with two zero coordinates and $x_0 \neq 0$.

Lemma 2.10. *Let* $S_0 :\ x_0^2 = f(x_1, x_2, x_3)$ *be one of the* 48 $K3$ *surfaces defined in* Proposition 2.3. *If* $d := \gcd(w_0, w_1) \geq 2$, *then* $P = (x_0, x_1, 0, 0)$ *with* $x_0 x_1 \neq 0$ *is a singularity of* S_0. *It is fixed by* σ *if and only if* w_0/d *is odd and* w_1/d *is even. Let* $E_1 + \cdots + E_{d-1}$ *denote the exceptional divisors arising from* P. *Then* σ *acts on* E_i *as identity (resp. by* -1*) if* i *is odd (resp.*

even). There are in total $\left[\dfrac{d}{2}\right]$ *divisors with odd* i*. The same assertion holds for singularities* $(x_0, 0, x_2, 0)$ *and* $(x_0, 0, 0, x_3)$*.*

Proof. Since S_0 is quasi-smooth, one knows that $P = (x_0, x_1, 0, 0)$ with $x_0 x_1 \neq 0$ is a singularity if and only if $d = \gcd(w_0, w_1) \geq 2$. Let $w_0 = du_0$ and $w_1 = du_1$ with $\gcd(u_0, u_1) = 1$. To see if P is fixed by σ, there are three cases to consider: $(u_0, u_1) = (\text{even}, \text{odd})$, (odd, odd), $(\text{odd}, \text{even})$. Suppose that there is a t such that $t^{w_0} = -1$ and $t^{w_1} = 1$. If u_0 is even and u_1 is odd, then $t^{u_0 u_1 d} = (t^{u_0 d})^{u_1} = (-1)^{u_1} = -1$ and $t^{u_0 u_1 d} = (t^{u_1 d})^{u_0} = 1$, which is absurd. By the same reason as above, $(u_0, u_1) = (\text{odd, odd})$ cannot happen either. If u_0 is odd and u_1 is even, then let ζ be a primitive $2d$th root of unity. We have $\zeta^{w_0} = (\zeta^d)^{u_0} = (-1)^{u_0} = -1$ and $\zeta^{w_1} = (\zeta^d)^{u_1} = (-1)^{u_1} = 1$. Hence there does exist a t satisfying $t^{w_0} = -1$ and $t^{w_1} = 1$, and P is fixed by σ in this case.

Choose (x_2, x_3) as a local coordinate system above P. P is isomorphic to the singularity of the quotient by the group action μ_d defined by

$$(x_2, x_3) \longmapsto (\zeta^{w_2} x_2, \zeta^{w_3} x_3),$$

where ζ runs through μ_d. Since the weight is normalized, $\gcd(d, w_2) = \gcd(d, w_3) = 1$ and the μ_d action above can be written as

$$(x_2, x_3) \longmapsto (\zeta^{d-1} x_2, \zeta x_3).$$

P is mapped to $(x_0, x_1, 0, 0) \in S_0/\sigma$ and the group action around this point is

$$(x_2, x_3) \longmapsto (\zeta^{d-1} x_2, \zeta x_3)$$

as above. But, since $\gcd(2w_0, w_1) = \gcd(2du_0, du_1) = 2d \gcd(u_0, u_1/2) = 2d$, ζ now runs through μ_{2d}. Hence this singularity on S_0/σ is of type $A_{2d,d-1}$.

Consider two continued fractions

$$\frac{d}{d-1} = 2 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{\cdots}}}} \quad \text{and} \quad \frac{2d}{d-1} = 4 - \cfrac{1}{1 - \cfrac{1}{4 - \cfrac{1}{1 - \cfrac{1}{\cdots}}}}.$$

This shows that if $E_1 + E_2 + E_3 + \cdots$ are exceptional divisor arising from P, then only the exceptional divisors E_i with odd i are fixed by σ. Therefore ramified exceptional divisors appear alternately and there are $[\frac{d}{2}]$ such divisors. □

Recall that r is the rank of $\mathrm{Pic}\,(S)^\sigma$. If we have good knowledge of $\mathrm{Pic}\,(S)$ and the σ action on it, then we can compute r without knowing S^σ. The following theorem shows that we can in fact find a closed formula for r by taking this approach.

Theorem 2.11. *Let (S, σ) be one of the 48 $K3$ surfaces defined in* Proposition 2.3 *as the minimal resolution of a hypersurface*

$$S_0 : \; x_0^2 = f(x_1, x_2, x_3) \subset \mathbb{P}^3(Q),$$

where $Q = (w_0, w_1, w_2, w_3)$ and f is a homogeneous polynomial of degree $w_0 + w_1 + w_2 + w_3$. Recall $r = \mathrm{rank}\,\mathrm{Pic}\,(S)^\sigma$. Let $r(Q)$ denote the number of exceptional divisors in the resolution $S \longrightarrow S_0$. Assume that $\mathrm{rank}\,\mathrm{Pic}\,(S_0)^\sigma = 1$.

(a) *If w_0 is odd, then there exists at most one odd weight w_i such that $\gcd(w_0, w_i) \geq 2$ and in such a case, $\gcd(w_0, w_i) = w_i$. We have*

$$r = \begin{cases} r(Q) - w_i + 2 & \text{if } \gcd(w_0, w_i) \geq 2 \text{ for some odd weight } w_i \ (1 \leq i \leq 3), \\ r(Q) + 1 & \text{otherwise.} \end{cases}$$

(b) *If w_0 is even, then let $d_i = \gcd(w_0, w_i)$. We have*

$$r = r(Q) + 1 - \sum_{i=1}^{3} (d_i - 1)\left(\frac{2d_i}{w_i} - 1\right).$$

Proof. Since $\mathrm{rank}\,\mathrm{Pic}\,(S_0)^\sigma = 1$, $\mathrm{Pic}\,(S_0) \otimes \mathbb{Q}$ is generated by the hyperplane section $\{x_0 = 0\}$, which is fixed by σ. Its strict transform on S is also fixed by σ and gives a one-dimensional subspace of $\mathrm{Pic}\,(S)^\sigma \otimes \mathbb{Q}$. The rest of it is generated by exceptional divisors.

Possible singularities for S_0 are either of the form $(0 : x_1 : x_2 : x_3)$ with one or two zero coordinates or of the form $(x_0 : x_1 : x_2 : x_3)$ with $x_0 \neq 0$ and exactly two zero coordinates. Since the points with $x_0 = 0$ are fixed by σ, every exceptional divisor E arising from a singularity $(0 : x_1 : x_2 : x_3)$ satisfies $\sigma(E) = E$. (σ acts on E as ± 1.) Such exceptional divisors form part of a basis for $\mathrm{Pic}\,(S)^\sigma$.

Consider a singularity $(x_0 : x_1 : x_2 : x_3)$ with $x_0 x_i \neq 0$, $\gcd(w_0, w_i) \geq 2$ and other coordinates zero. It is fixed by σ if and only if $t^{w_0} = -1$ and

$t^{w_i} = 1$ for some t. As $x_0x_i \neq 0$ and other coordinates are zero, $f(x_1, x_2, x_3)$ contains a monomial solely in x_i. Hence $w_i \mid 2w_0$, where $2w_0 = \deg f$. Now we divide the proof according as the parity of w_0.

(a) Assume that w_0 is odd. Then the normality of weight Q and the equality $w_0 = w_1 + w_2 + w_3$ imply that there is exactly one odd w_i for $1 \leq i \leq 3$. For simplicity, assume that w_1 is odd and w_2 and w_3 are even.

(i) If $\gcd(w_0, w_1) \geq 2$, then $(x_0 : x_1 : 0 : 0)$ are singular points. Since $1 = (t^{w_1})^{w_0} = (t^{w_0})^{w_1} = -1$, none of the singularities is fixed by σ. But, if we consider a σ-conjugate pair, it is invariant under σ. There are two singularities of the form $(x_0 : x_1 : 0 : 0)$, each of which is of type A_{w_1-1}. (As $w_1 \mid 2w_0$ and w_1 is odd, we have $\gcd(w_0, w_1) = w_1$.) Totally, there are $2(w_1 - 1)$ exceptional divisors arising from these singularities and $w_1 - 1$ conjugate pairs contribute to $r = \operatorname{rank} \operatorname{Pic}(S)^\sigma$.

If $\gcd(w_0, w_1) = 1$, then $(x_0 : x_1 : 0 : 0)$ is not a singularity.

(ii) Consider the case where $\gcd(w_0, w_2) \geq 2$ and w_2 is even. $(x_0 : 0 : x_2 : 0)$ is a singularity. By letting $t = -1$, we see $t^{w_0} = -1$ and $t^{w_2} = 1$. Hence $(x_0 : 0 : x_2 : 0)$ is fixed by σ and so are the exceptional divisors arising from this point. This means that all exceptional divisors contribute to r. The same argument is valid for the singularities $(x_0 : 0 : 0 : x_3)$ with even w_3.

Therefore it follows from (i) and (ii) that $r = r(Q) + 1$ if there is no odd w_i with $\gcd(w_0, w_i) \geq 2$, and

$$r = r(Q) + 1 - (w_i - 1) = r(Q) - w_i + 2$$

if $\gcd(w_0, w_i) \geq 2$ for some odd weight w_i.

(b) Assume now that w_0 is even. Then the normality of weight Q and the equality $w_0 = w_1 + w_2 + w_3$ imply that there is exactly one even weight w_i for $1 \leq i \leq 3$. For simplicity, let w_1 be even, and w_2 and w_3 be odd.

(i) Consider the point $(x_0 : x_1 : 0 : 0)$ with w_1 even. Since $\gcd(w_0, w_1) \geq 2$, it is a singularity. We see $w_1 \mid 2w_0$.

If $w_1 \mid w_0$, then $d_1 = \gcd(w_0, w_1) = w_1$ and $\{(x_0 : x_1 : 0 : 0)\}$ consists of two points, each of which is a singularity of type A_{w_1-1}. There arise $2(w_1 - 1)$ exceptional divisors from them and $w_1 - 1$ conjugate pairs are fixed by σ. This means that the rank r is the number of exceptional divisors minus $w_1 - 1$.

If $w_1 \not| w_0$, then $\mathrm{lcm}\,(w_0, w_1) = 2w_0$ and $(x_0 : x_1 : 0 : 0)$ is a singularity of type A_{d_1-1}. This point is fixed by σ and so are the exceptional divisors arising from it.
In summary, the rank r is less than the number of exceptional divisors by

$$-(d_1 - 1)\left(\frac{2w_0}{\mathrm{lcm}\,(w_0, w_1)} - 1\right) = -(d_1 - 1)\left(\frac{2d_1}{w_1} - 1\right).$$

(ii) Consider the points $(x_0 : 0 : x_2 : 0)$ with w_2 odd. They are singularities if and only if $\gcd(w_0, w_2) \geq 2$. Here $w_2 \mid 2w_0$ implies $w_2 \mid w_0$ and $d_2 = \gcd(w_0, w_2) = w_2$. Such singularities are of type A_{d_2-1}. The multiplicity of $(x_0 : 0 : x_2 : 0)$ is $2w_0/\mathrm{lcm}\,(w_0, w_2) = 2$ and they are σ-conjugate. Among the $2(w_2 - 1)$ exceptional divisors, $w_2 - 1$ conjugate pairs are fixed by σ. Hence the rank r is less than the number of exceptional divisors by

$$-(d_2 - 1)\left(\frac{2d_2}{w_2} - 1\right).$$

The same argument holds for the points $(x_0 : 0 : 0 : x_3)$ with odd w_3.
Therefore the asserted formula of (b) follows from (i) and (ii).

□

Example 2.12. For a generic choice of S_0, one has $\mathrm{rank}\,\mathrm{Pic}\,(S_0)^\sigma = 1$ and Theorem 2.11 gives a convenient way to calculate invariant r.

(1) For $Q = (7, 3, 2, 2)$, we find that w_0 is odd, $r(Q) = 9$ and no odd weight w_i with $\gcd(w_0, w_i) \geq 2$. Hence $r = 9 + 1 = 10$.

(2) For $Q = (15, 10, 3, 2)$, we find that w_0 is odd, $r(Q) = 11$ and $\gcd(15, 3) = 3$. Hence $r = 11 - 3 + 2 = 10$.

(3) For $Q = (8, 4, 3, 1)$, we find that w_0 is even, $r(Q) = 8$ and $d_1 = 4$. Hence $r = 8 + 1 - (4 - 1)(2 \cdot 4/4 - 1) = 6$.

(4) For $Q = (10, 5, 3, 2)$, we find that w_0 is even, $r(Q) = 12$, $d_1 = 5$ and $d_3 = 2$. Hence $r = 12 + 1 - (5 - 1)(2 \cdot 5/5 - 1) - (2 - 1)(2 \cdot 2/2 - 1) = 8$.

(5) For $Q = (24, 16, 5, 3)$, we find that w_0 is even, $r(Q) = 15$, $d_1 = 8$ and $d_3 = 3$. Hence $r = 15 + 1 - (8 - 1)(2 \cdot 8/16 - 1) - (3 - 1)(2 \cdot 3/3 - 1) = 14$.

Example 2.13. Proposition 2.2 tells that we can calculate r and a by knowing S^σ, but the computation of a by finding a $\mathbb{Z}$-basis for $\mathrm{Pic}\,(S)^\sigma$ is rather involved.

Take a look at the $K3$ surface #8 in Yonemura defined by the equation

$$S_0 : x_0^2 = x_1^4 + x_2^6 + x_3^{12} \subset \mathbb{P}^3(6,3,2,1).$$

The involution is defined by $\sigma(x_0) = -x_0$.

We see that S_0 is quasi-smooth and the minimal resolution S is a $K3$ surface. S_0 has four singularities, P_1, P_1', P_2 and P_2' as follows:

Singularity	Type	Exceptional divisors
$P_1 := (1:1:0:0)$	$A_{3,2}$	$E_1 + E_2$
$P_1' := (-1:1:0:0)$	$A_{3,2}$	$E_1' + E_2'$
$P_2 := (1:0:1:0)$	$A_{2,1}$	E_3
$P_2' := (-1:0:1:0)$	$A_{2,1}$	E_3'

No singularity is fixed by σ. There is a curve, C', defined by $x_0 = 0$. Its strict transform, C_7, on S is of genus 7 and ramified under σ. We have

$$S^\sigma = C_7.$$

Hence $g = 7$ and $k = 0$. This implies that $r = 11 - 7 + 0 = 4$ and $a = 11 - 7 - 0 = 4$. As $a = 4$, the intersection matrix of a $\mathbb{Z}$-basis for $\mathrm{Pic}\,(S)^\sigma$ should have determinant $\pm 2^4$.

We look for a basis for $\mathrm{Pic}\,(S)^\sigma$. An immediate choice for a set of four divisors on S fixed by σ is $E_1 + E_1'$, $E_2 + E_2'$, $E_3 + E_3'$ and C_7. But the determinant of their intersection matrix is calculated as

$$\begin{vmatrix} -4 & 2 & 0 & 0 \\ 2 & -4 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 12 \end{vmatrix} = -2^6 3^2.$$

Hence they do not form a $\mathbb{Z}$-basis for $\mathrm{Pic}\,(S)^\sigma$.

We now consider another set of divisors by replacing C_7 with the rational curve D defined by $x_3 = 0$. As a divisor, D is also fixed by σ. As D is a rational curve on a $K3$ surface, $D^2 = -2$. Since the curve $\{x_3 = 0\}$ on S_0 passes through every singularity,

$$D.(E_1 + E_1') = D.(E_3 + E_3') = 2, \quad D.(E_2 + E_2') = 0.$$

Hence the determinant of the intersection matrix of these divisors is

$$\begin{vmatrix} -4 & 2 & 0 & 2 \\ 2 & -4 & 0 & 0 \\ 0 & 0 & -4 & 2 \\ 2 & 0 & 2 & -2 \end{vmatrix} = -2^4.$$

This agrees with the above calculation of $a = 4$; thus we see that

$$E_1 + E_1', \quad E_2 + E_2', \quad E_3 + E_3' \quad \text{and} \quad D$$

form a $\mathbb{Z}$-basis for $\mathrm{Pic}\,(S)^\sigma$.

It is not too difficult to find some subgroup of $\mathrm{Pic}\,(S_0)^\sigma$, but often difficult to describe $\mathrm{Pic}\,(S_0)^\sigma$ completely. Above calculations give us a clue to determine $\mathrm{rank}\,\mathrm{Pic}\,(S_0)^\sigma$. Let S be the minimal resolution of S_0 and let $\mathbb{E}$ denote the subgroup of $\mathrm{Pic}\,(S)$ generated by the exceptional divisors of the resolution. We have

$$\mathrm{Pic}\,(S)^\sigma \otimes \mathbb{Q} \cong \mathrm{Pic}\,(S_0)^\sigma \otimes \mathbb{Q} \oplus \mathbb{E}^\sigma \otimes \mathbb{Q}.$$

Groups $\mathbb{E}$ and $\mathbb{E}^\sigma$ are easily describable and the rank of $\mathrm{Pic}\,(S)^\sigma$ may be computed from the fixed part S_0^σ by Proposition 2.2 as

$$\mathrm{rank}\,\mathrm{Pic}\,(S_0)^\sigma = \mathrm{rank}\,\mathrm{Pic}\,(S)^\sigma - \mathrm{rank}\,\mathbb{E}^\sigma = 11 + k - g - \mathrm{rank}\,\mathbb{E}^\sigma.$$

Hence if one finds this number of divisors in $\mathrm{Pic}\,(S_0)^\sigma$, then they form a basis for $\mathrm{Pic}\,(S_0)^\sigma$ over $\mathbb{Q}$.

Corollary 2.14. *Let (S, σ) be one of the 48 $K3$ surfaces (considered in* Proposition 2.3*) with involution σ defined by a hypersurface of the form $x_0^2 = f(x_1, x_2, x_3)$ where σ acts by $\sigma(x_0) = -x_0$. Let $S^\sigma = C_g \cup L_1 \cup \cdots \cup L_k$ be the decomposition in connected components of S^σ, where C_g is a smooth genus g curve and $L_1, \ldots, L_k$ are rational curves.*

Suppose that f is defined by three monomials, so that S is of Delsarte type. Then the Jacobian variety $J(C_g)$ of C_g is also of CM type.

Proof. In this case C_g is defined by putting $x_0 = 0$. So C_g is defined by three monomials and is realized as a Fermat quotient. □

In the next section, we discuss another type (non-Borcea type) of $K3$ surfaces. Noting the differences from the case $x_0^2 = f(x_1, x_2, x_3)$, we sketch the outline of our algorithm.

3. Computations of Nikulin's invariants for $K3$ surfaces of non-Borcea type

3.1. Computations of r and a

In this section, we compute r and a for $K3$ surfaces of non-Borcea type, namely for a quasi-smooth $K3$ surface S_0 in $\mathbb{P}^3(w_0, w_1, w_2, w_3)$ defined by the equation

$$x_0^2 x_i = f(x_1, x_2, x_3) \tag{3.1}$$

for some i $(= 1, 2, 3)$. Let S be the minimal resolution of S_0. Write C_g for the strict transform of the curve defined by $x_0 = 0$, which is isomorphic to the curve $f(x_1, x_2, x_3) = 0$ in $\mathbb{P}^2(w_1, w_2, w_3)$. We assume that $f(x_1, x_2, x_3) = 0$ is quasi-smooth (hence smooth) in $\mathbb{P}^2(w_1, w_2, w_3)$ after normalization of the weight.

Since $Q = (w_0, w_1, w_2, w_3)$ is normalized, every fixed point in S_0^σ must have at least one zero coordinate. There are four cases to consider.

(i) The curve $\{x_0 = 0\}$ on S_0 is fixed by σ. Since $f(x_1, x_2, x_3) = 0$ is quasi-smooth in $\mathbb{P}^2(w_1, w_2, w_3)$, its strict transform C_g is also fixed by σ. The genus g can be calculated from $\deg f$ and weight (w_1, w_2, w_3) as in the previous section.

The rational curves L_i of $S^\sigma = C_g \cup L_1 \cup \cdots \cup L_k$ are obtained by letting x_1, x_2 or x_3 be zero, or from the exceptional divisors arising in the desingularization.

(ii) As in the case of $K3$ surfaces of Borcea type, the one-dimensional locus $\{x_1 = 0\}$, $\{x_2 = 0\}$ or $\{x_3 = 0\}$ may be fixed by σ. In addition to them, there may be another one-dimensional locus fixed by σ; it has two zero coordinates, one of which is x_i of (3.1).

For instance, consider a surface $x_0^2 x_1 = f(x_1, x_2, x_3)$ with $f(0, 0, x_3) = 0$. If w_0 is odd and w_3 is even, then the locus $(x_0 : 0 : 0 : x_3)$ is a line and fixed by σ. Its strict transform on S is also fixed by σ, which gives one L_i of Proposition 2.2.

(iii) The rest of the curves in L_i's are obtained from exceptional divisors in the resolution $S_0 \leftarrow S$. The singularities discussed in Lemmas 2.8, 2.9 and 2.10 also exist on surfaces (3.1) and by the same procedure as described there, we can find those divisors fixed identically by σ. (The proof of Lemma 2.9 needs a little modification; see Lemma 3.2).

(iv) In addition to the singularities of (iii), we now have a singularity $(1 : 0 : 0 : 0)$ on the surface $x_0^2 x_i = f(x_1, x_2, x_3)$. The exceptional divisors arising from it and fixed by σ are determined as follows.

Lemma 3.1. *Let* $S_0 :\ x_0^2 x_i = f(x_1, x_2, x_3)$ *be one of the* $K3$ *surfaces obtained in* Theorem 2.4. *Then* $P = (1, 0, 0, 0) \in S_0$ *is a (cyclic quotient) singularity of type* $A_{w_0, w_0 - 1}$. *Let* $E_1 + \cdots + E_{w_0 - 1}$ *be the exceptional divisors on* S *arising from* $P \in S_0$. *On the quotient* S_0/σ, *we see that* $\sigma(P)$ *is a singularity of type* A_{2w_0, w_0-1} *or* $A_{w_0, 2(w_0-1)}$.

(1) *If* $\sigma(P)$ *is of type* A_{2w_0, w_0-1}, *then* σ *acts on* E_i *as identity if and only if* i *is odd. There exist* $\left[\frac{w_0}{2}\right]$ *such divisors in total, where* $[x]$ *denotes the integer part of* x *as before.*

(2) *If* $\sigma(P)$ *is of type* $A_{w_0, 2(w_0-1)}$, *then* σ *acts on* E_i *as identity if and only if* i *is even. There exist* $\left[\frac{w_0 - 1}{2}\right]$ *such divisors in total.*

The type of singularity at $\sigma(P)$ *is determined as follows according to the parity of weights:*

	w_0	w_i	w_j	w_k	*Type of* $\sigma(P)$
(*a*)	*Even*	*Even*	*Odd*	*Odd*	A_{2w_0, w_0-1}
(*b*)	*Even*	*Odd*	*Odd*	*Odd*	A_{2w_0, w_0-1}
(*c*)	*Odd*	*Odd*	*Even*	*Odd*	A_{2w_0, w_0-1}
(*c*)′	*Odd*	*Odd*	*Odd*	*Even*	A_{2w_0, w_0-1}
(*d*)	*Odd*	*Even*	*Even*	*Odd*	$A_{w_0, 2(w_0-1)}$
(*d*)′	*Odd*	*Even*	*Odd*	*Even*	$A_{w_0, 2(w_0-1)}$

Proof. Since $w_0 \geq 2$ and S is $K3$, $P = (1, 0, 0, 0)$ is a cyclic quotient singularity of type A_{w_0, w_0-1}. From the equation $x_0^2 x_i = f(x_1, x_2, x_3)$, we can choose variables x_j and x_k, different from x_0 and x_i, as local parameters around P. The μ_{w_0}-action at P is then written as

$$(x_j, x_k) \longmapsto (\zeta^{w_j} x_j, \zeta^{w_k} x_k).$$

As $\deg f = 2w_0 + w_i$ and Q is a $K3$ weight, we have $2w_0 + w_i = w_0 + w_i + w_j + w_k$. This implies $w_0 = w_j + w_k$ and, because Q is normalized, $\gcd(w_0, w_j) = \gcd(w_0, w_k) = 1$. In particular, the congruence $w_j \alpha \equiv w_k \pmod{w_0}$ has solution $\alpha \equiv -1 \pmod{w_0}$. Now P is mapped to $(1, 0, 0, 0) \in S_0/\sigma$ and the group action around this point is

$$(x_j, x_k) \longmapsto (\xi^{w_j} x_j, \xi^{w_k} x_k),$$

where ξ ranges over μ_{2w_0}. To find out the type of singularity at $\sigma(P)$, we divide the case according to the parity of weights. The cases (c) and $(c)'$, (d) and $(d)'$ are essentially the same, so we discuss cases (a) to (d).

In (a) and (b), w_0 is even. Since $w_0 = w_j + w_k$, both w_j and w_k are odd. Hence weight $(2w_0, w_i, w_j, w_k)$ is normalized and the singularity $(1,0,0,0) \in S_0/\sigma$ is of type A_{2w_0,w_0-1}. As in Lemma 2.10, we see that σ acts on E_i as identity if and only if i is odd, and there are $[w_0/2]$ such divisors.

In (c), w_0 is odd. Since $w_0 = w_j + w_k$, w_j and w_k have different parity. Because w_i is odd, the weight $(2w_0, w_i, w_j, w_k)$ is normalized. Hence the singularity $(1,0,0,0) \in S_0/\sigma$ is of type A_{2w_0,w_0-1}. As in Lemma 2.10, σ acts on E_i as identity if and only if i is odd, and there are $[w_0/2]$ such divisors.

In (d), w_0 is odd. Since $w_0 = w_j + w_k$, either w_j or w_k is even, say w_j is even. Because w_i is even in this case, $(2w_0, w_i, w_j, w_k)$ is not yet normalized. By normalization

$$\mathbb{P}^3(2w_0, w_i, w_j, w_k) \cong \mathbb{P}^3\Big(w_0, \frac{w_i}{2}, \frac{w_j}{2}, w_k\Big)$$

the group action around $(1,0,0,0) \in S_0/\sigma$ is now written as

$$(x_j, x_k) \longmapsto (\xi^{w_j/2} x_j, \xi^{w_k} x_k)$$

where ξ ranges over μ_{w_0}. As $\gcd(w_0, w_j/2) = \gcd(w_0, w_k) = 1$ and $w_0 = w_j + w_k$, we have the congruence

$$\frac{w_j}{2} 2(w_0 - 1) \equiv w_k \pmod{w_0}.$$

This shows that $(1,0,0,0) \in S_0/\sigma$ is of type $A_{w_0,2(w_0-1)}$ (to be more precise, type A_{w_0,w_0-2}). As in Lemma 2.9, σ acts on E_i as identity if and only if i is even, and there are $[(w_0-1)/2]$ such divisors. □

Lemma 3.2. *Let* $S_0 :\ x_0^2 x_i = f(x_1, x_2, x_3)$ *be one of the* $K3$ *surfaces obtained in* Theorem 2.4. *If* w_j *and* w_k *are the weight other than* w_0 *and* w_i, *then* $\gcd(w_j, w_k) = 1$. *If* $d = \gcd(w_i, w_j) \geq 2$, *then* d *must be* 2 *and* $P = (0, x_i, x_j, 0)$ *with* $x_i x_j \neq 0$ *is a singularity of type* $A_{2,1}$ *fixed by* σ. *There arises one exceptional divisor from* P *and it is not ramified under the quotient* $S \longrightarrow S/\sigma$.

Proof. Let $d = \gcd(w_j, w_k)$. The relation $w_0 = w_j + w_k$ implies $d \mid w_0$. But this is not possible as the weight Q is normalized unless $d = 1$.

Let $d = \gcd(w_i, w_j)$. As in the proof of Lemma 2.9, f contains a term without w_k. By the relation $\deg f = 2w_0 + w_i$, we see that $d \mid 2w_0$. Since Q is normalized, $d \mid 2$ and hence $d = 2$. The rest of the proof is similar to Lemma 2.9. □

We combine (ii), (iii) and (iv) to calculate the number of rational curves L_i. Then Proposition 2.2 gives the value for r and a.

Example 3.3. Consider the $K3$ surface #60 in Yonemura. Dropping several monomials, we choose the equation

$$S_0 : \; x_0^2 x_2 + x_1^3 + x_1 x_2^3 + x_3^{18} = 0 \subset \mathbb{P}^3(7, 6, 4, 1).$$

The involution is defined by $\sigma(x_0) = -x_0$. We see that S_0 is quasi-smooth and the minimal resolution S is a $K3$ surface. S_0 has three singularities, P_1, P_2 and P_3 as follows:

Singularity	Type	Exceptional divisors
$P_1 := (1 : 0 : 0 : 0)$	$A_{7,6}$	$E_1 + E_2 + E_3 + E_4 + E_5 + E_6$
$P_2 := (0 : 0 : 1 : 0)$	$A_{4,3}$	$E_7 + E_8 + E_9$
$P_3 := (0 : -1 : 1 : 0)$	$A_{2,1}$	E_{10}

Every singularity is fixed by σ, and E_2, E_4, E_6 and E_8 are ramified under σ (acting on the minimal resolution S).

There are two curves on S_0 fixed by σ, namely C' defined by $x_0 = 0$ and L' defined by $x_3 = 0$. C' has genus 3 and L' is a projective line. Their strict transforms C $(= C_3)$ and L on S are ramified under σ. We have

$$S^\sigma = C_3 \cup E_2 \cup E_4 \cup E_6 \cup E_8 \cup L.$$

Hence $g = 3$ and $k = 5$. This implies $r = 13$ and $a = 3$.

Example 3.4. Consider the $K3$ surface #89 in Yonemura. Dropping several monomials, we choose the equation

$$S_0: \ x_0^2x_3 + x_1^3x_2 + x_1x_2^4 + x_3^{11} = 0 \subset \mathbb{P}^3(5,3,2,1).$$

The involution is defined by $\sigma(x_0) = -x_0$. We see that S_0 is quasi-smooth and the minimal resolution S is a $K3$ surface. S_0 has three singularities, P_1, P_2 and P_3 as follows:

Singularity	Type	Exceptional divisors
$P_1 := (1:0:0:0)$	$A_{5,4}$	$E_1 + E_2 + E_3 + E_4$
$P_2 := (0:1:0:0)$	$A_{3,2}$	$E_5 + E_6$
$P_3 := (0:0:1:0)$	$A_{2,1}$	E_7

Every singularity is fixed by σ, and E_2, E_4 and E_6 are ramified under σ (acting on the minimal resolution S).

There are two curves on S_0 fixed by σ, namely C' defined by $x_0 = 0$ and L' defined by $x_1 = x_3 = 0$. C' has genus 5 and L' is a projective line. Their strict transforms C $(= C_5)$ and L on S are ramified under σ. We have

$$S^\sigma = C_5 \cup E_2 \cup E_4 \cup E_6 \cup L.$$

Hence $g = 5$ and $k = 4$. This implies $r = 10$ and $a = 2$.

Corollary 3.5. *Let (S, σ) be one of the* 86 *$K3$ surfaces of Delsarte type with involution σ in* Theorem 2.5. *Let C_g be the genus g curve in the fixed locus $S^\sigma = C_g \cup L_1 \cdots \cup L_k$ (where L_i are rational curves). Then C_g is of CM type in the sense that its Jacobian variety $J(C_g)$ is a CM abelian variety of dimension g.*

Proof. If σ acts as $\sigma(x_i) = -x_i$ on S, then C_g is defined by letting $x_i = 0$. It is a curve of Delstarte type and hence of CM type. ☐

3.2. Realization of Nikulin's invariants

We briefly discuss how many Nikulin's triplets (r, a, δ) can be realized by our $K3$ surfaces. To realize as many triplets as possible, we introduce more involutions than those considered in previous sections.

First, we summarize the results of previous sections where σ acts on the variable x_0 (with the highest weight among x_i's).

Theorem 3.6. *Let* (S, σ) *be one of the* 92 $K3$ *surfaces in* Theorem 2.4 *with involution* $\sigma(x_0) = -x_0$. *Among the* 75 *triplets* (r, a, δ) *of Nikulin, at least* 29 *triplets are realized with such* $K3$ *surfaces. See* tables B.1 *to* B.5 *and* B.7 *of* Appendix B *for the list of defining equations of* S_0 *and the values for* (r, a)*; in most cases,* S_0 *is defined by*

$$x_0^2 = f(x_1, x_2, x_3) \quad \textit{or} \quad x_0^2 x_i = f(x_1, x_2, x_3).$$

Remark 3.1. We say "at least 29" because we computed only the invariants r and a. If we are to find δ, we should have a $\mathbb{Z}$-basis for $\mathrm{Pic}\,(S)$ or calculate intersection numbers of divisors on S. We leave this as a future problem. Once δ is calculated, the number of realizable triplets may increase.

When the weight $Q = (w_0, w_1, w_2, w_3)$ is fixed and σ is defined by $\sigma(x_0) = -x_0$ on the surface $x_0^2 = f(x_1, x_2, x_3)$ or $x_0^2 x_i = f(x_1, x_2, x_3)$, no matter what quasi-smooth equation we choose for $f(x_1, x_2, x_3)$, the fixed locus S^σ is the same. Hence changes in equation $f(x_1, x_2, x_3)$ do not lead to any new pairs of r and a.

On the other hand, even with the same S_0, changing the involution σ may change the fixed locus S^σ and thus also r and a may change. We use this approach by letting σ act on some variable x_i other than x_0.

Theorem 3.7. *Let* (S, σ) *be one of the* 92 $K3$ *surfaces in* Theorem 2.4 *having an involution* σ *on a variable* x_i *other than* x_0. *They are listed on* table B.9 *in* Appendix B *with the values of* r *and* a. *Compared with the triplets obtained in* Theorem 3.6*, at least* 14 *more triplets are realized with such* σ *actions. Among the* 75 *triplets* (r, a, δ) *of Nikulin, the total number of triplets we realize is at least* 40. *The values for* r *and* a *of such* 40 *triplets are as follows:*

r	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
a	1	0	1			2	3	6	1	0	1	6	3	2	5	2	1	0	1	2
		2				4	7	8	9	2	9	8	5	4	7	6	3	2	3	
										4			9	6			5	4		
										6										
										8										

Proof. The σ-fixed locus S_0^σ can be determined by the same method as in previous subsections. In particular, the singularities of S_0 are independent of the choice of σ. We use Lemmas 2.8 to 2.10 and 3.1 to find which exceptional divisors are fixed by involution σ. □

Remark 3.2. In some cases (say, Case #57 of table B.4), the σ-action on x_i $(i \neq 0)$ gives the same S^σ as the action by $\sigma(x_0) = -x_0$. Such cases are omitted from table B.9.

Remark 3.3. Hisanori Ohashi has communicated us an idea of even more different types of involutions on S_0. We thank him for the idea of different involutions. We plan to work on it in a subsequent paper.

4. Calabi–Yau threefolds of Borcea–Voisin type

4.1. Construction of Calabi–Yau threefolds of Borcea–Voisin type

In this section, we recall the Borcea–Voisin construction of Calabi–Yau threefolds.

Let E be an elliptic curve with the standard involution ι, and let (S, σ) be a pair of a $K3$ surface S with involution σ acting by -1 on $H^{2,0}(S)$. By the classification theorem of Nikulin, the deformation class of such pairs (S, σ) is determined by a triplet (r, a, δ) as we discussed in Section 2.

Now we consider the product $E \times S$, and the quotient threefolds

$$E \times S/\iota \times \sigma.$$

Obviously, this quotient is singular, having cyclic quotient singularities. We resolve singularities to obtain a smooth crepant resolution, denoted by $X = X(r, a, \delta)$, which is a Calabi–Yau threefold; we call it a *Calabi–Yau threefold of Borcea–Voisin type*. It is plain that a Calabi–Yau threefold of Borcea–Voisin type is equipped with the following two fibrations: the elliptic fibration with constant fiber E induced from the projection $E \times S/\iota \times \sigma \to S/\sigma$, and the $K3$ fibration with the constant fiber S induced from the projection $E \times S/\iota \times \sigma \to E/\iota$.

Proposition 4.1 (Borcea [6]). *The Hodge numbers of the Calabi–Yau threefold $X = X(r, a, \delta)$ of Borcea–Voisin type are determined by the given triplet (r, a, δ) and by the data from the fixed locus S^σ: Indeed,*

$$h^{1,1}(X) = 5 + 3r - 2a = 1 + r + 4(k + 1),$$

$$h^{2,1}(X) = 65 - 3r - 2a = 1 + (20 - r) + 4g$$

where k, g are described in Proposition 2.2. *The Euler characteristic of X is*

$$e(X) = 2(h^{1,1}(X) - h^{2,1}(X)) = 2(r - 10).$$

[Voisin [41]**].** *Put $N := 1 + k$ and $N' := g$. That is, N is the number of components, and N' is the sum of genera of components, of S^σ. Then*

$$h^{1,1}(X) = 11 + 5N - N',$$
$$h^{2,1}(X) = 11 + 5N' - N$$

and the Euler characteristic of X is

$$e(X) = 2(h^{1,1}(X) - h^{2,1}(X)) = 12(N - N').$$

Now we discuss briefly resolution of singularities; detailed discussions are in Section 6. As above, let $\iota : E \to E$ be the standard involution. The fixed part E^ι consists of four points $\{P_1, P_2, P_3, P_4\}$.

We consider the generic case (I) when the fixed part S^σ of S is given by

$$S^\sigma = C_g \cup L_1 \cup L_2 \cup \cdots \cup L_k$$

where C_g is a smooth curve of genus $g \geq 1$ and L_i is a rational curve for each $i = 1, \ldots, k$.

Proposition 4.2. *The quotient threefold $E \times S/\iota \times \sigma$ has singularities along $\{P_i\} \times S^\sigma$ $(i = 1, 2, 3, 4)$. Each singular locus is a cyclic quotient singularity by a group action of order* 2.

By resolving singularities, we obtain a smooth Calabi–Yau threefold X:

$$E \times S/\iota \times \sigma \leftarrow X.$$

The exceptional divisors are four copies of a union of ruled surfaces

$$S^\sigma \times \mathbb{P}^1 := (C_g \times \mathbb{P}^1) \cup (L_1 \times \mathbb{P}^1) \cup \cdots \cup (L_k \times \mathbb{P}^1).$$

4.2. Realization of Calabi–Yau threefolds of Borcea–Voisin type as hypersurfaces over $\mathbb{Q}$

The construction of Calabi–Yau threefolds of Borcea–Voisin type we have discussed so far are geometric in nature. In order to study arithmetic of these Calabi–Yau threefolds, we wish to have defining equations by, e.g., hypersurfaces or complete intersections defined over $\mathbb{Q}$ in weighted projective spaces. We require that the zero loci of these equations define singular Calabi–Yau threefolds and whose resolution would be birationally equivalent to our Calabi–Yau threefolds of Borcea–Voisin type. In fact, the constructions of such singular models have been already carried out in Goto–Kloosterman–Yui [16], using the so-called *twist maps*. Now we will briefly recall such constructions.

We start with examples. Let $\mathbb{P}^2(k+1,k,1)$ be a weighted projective 2-space with weight$(k+1,k,1)$ of degree $2(k+1)$. Let $\mathbb{P}^3(w_0,w_1,w_2,w_3)$ be a weighted projective 3-space with weight (w_0,w_1,w_2,w_3) of degree $d := \sum_{i=0}^3 w_i$. A *twist map* may be defined as follows. Assume that $\gcd(k+1,w_0)=1$. Let

$$\begin{aligned}\Phi : \mathbb{P}^2(k+1,k,1) \times \mathbb{P}^3(w_0,w_1,w_2,w_3)\\ \mapsto \mathbb{P}^4(kw_0,w_0,(k+1)w_1,(k+1)w_2,(k+1)w_3)\end{aligned}$$

be a map into a weighted projective 4-space with degree $(k+1)d$. The twist map was defined in Goto–Kloosterman–Yui [16] explicitly, and is given by

$$\begin{aligned}\Phi : ((y_0:y_1:y_2),(x_0:x_1:x_2:x_3)) \mapsto \Bigg(y_1\left(\frac{x_0}{y_0}\right)^{k/k+1} : y_2\left(\frac{x_0}{y_0}\right)^{k/k+1}\\ : x_1 : x_2 : x_3 \Bigg).\end{aligned}$$

This will produce many singular Calabi–Yau threefolds defined by hypersurfaces in weighted projective 4-spaces.

Take $k=1$. Then we have an elliptic curve

$$E_2 : y_0^2 = y_1^4 + y_2^4 \subset \mathbb{P}^2(2,1,1).$$

If we take $k=2$, we obtain an elliptic curve

$$E_3 : y_0^2 = y_1^3 + y_2^6 \subset \mathbb{P}^2(3,2,1).$$

Both E_2 and E_3 have complex multiplication, by $\mathbb{Z}[\sqrt{-1}]$ and $\mathbb{Z}[\sqrt{-3}]$, respectively.

Proposition 4.3 (Borcea [6]**).** *Let E_2 and E_3 be elliptic curves defined above. Let $S_0 : x_0^2 = f(x_1,x_2,x_3) \subset \mathbb{P}^3(w_0,w_1,w_2,w_3)$ be one of the* 40 *K*3 *surfaces from* tables B.1 *and* B.3. *Let S be the minimal resolution of S_0.*

(a) *Suppose that w_0 is odd, and that $w_0 = w_1+w_2+w_3$. Then there is a twist map*

$$\mathbb{P}^2(2,1,1) \times \mathbb{P}^3(w_0,w_1,w_2,w_3) \cdots \longrightarrow \mathbb{P}^4(w_0,w_0,2w_1,2w_2,2w_3)$$

given by

$$(y_0 : y_1 : y_2) \times (x_0 : x_1 : x_2 : x_3) \mapsto \left(y_1 \left(\frac{x_0}{y_0}\right)^{1/2} : y_2 \left(\frac{x_0}{y_0}\right)^{1/2} : x_1 : x_2 : x_3 \right).$$

The product $E_2 \times S_0$ *maps generically* $2:1$ *to the hypersurface of degree* $4w_0$ *of the form*

$$z_0^4 + z_1^4 = f(z_2, z_3, z_4)$$

where f *is a homogeneous polynomial over* $\mathbb{Q}$ *of degree* $4w_0$*. This is a singular model for a Calabi–Yau threefold* $E_2 \times S/\iota \times \sigma$ *in* $\mathbb{P}^4(w_0, w_0, 2w_1, 2w_2, 2w_3)$*.*

(b) *Suppose that* w_0 *is even but not divisible by* 3*, and that* $w_0 = w_1 + w_2 + w_3$*. Then there is a twist map*

$$\mathbb{P}^2(3,2,1) \times \mathbb{P}^3(w_0, w_1, w_2, w_3) \cdots \longrightarrow \mathbb{P}^4(2w_0, w_0, 3w_1, 3w_2, 3w_3)$$

given by

$$(y_0 : y_1 : y_2) \times (x_0 : x_1 : x_2 : x_3) \mapsto \left(y_1 \left(\frac{x_0}{y_0}\right)^{2/3} : y_2 \left(\frac{x_0}{y_0}\right)^{1/3} : x_1 : x_2 : x_3 \right).$$

The product $E_3 \times S_0$ *maps generically* $2:1$ *to the hypersurface of degree* $6w_0$ *of the form*

$$z_0^3 + z_1^6 = f(z_2, z_3, z_4),$$

where f *is a homogeneous polynomial over* $\mathbb{Q}$ *of degree* $6w_0$*. This is a singular model for a Calabi–Yau threefold* $E_3 \times S/\iota \times \sigma$ *in* $\mathbb{P}^4(2w_0, w_0, 3w_1, 3w_2, 3w_3)$*.*

Problem 1. When w_0 is divisible by 6, describe the twist map explicitly and construct hypersurface equations defining the Calabi–Yau threefolds modifying twist maps.

Proposition 4.4. *There are in total* 40 *Calabi–Yau threefolds corresponding to* $K3$ *surfaces* S_0 *in* tables B.1 *and* B.2 *which are realized by quasi-smooth hypersurfaces over* $\mathbb{Q}$ *in weighted projective* 4*-spaces by the above construction.*

Theorem 4.5. *Suppose that S is the minimal resolution of one of the* 45 *$K3$ surfaces of* tables B.1 *to* B.3 *excluding* $\#90, \#91, \#93$. *Then the associated* 45 *Calabi–Yau threefolds of Borcea–Voisin type constructed above are all of CM type.*

Proof. We follow Borcea [6] Proposition 1.2. Let h be the rational Hodge structure of $H^3(E\times S,\mathbb{Q})$, and let h_E and h_S denote, respectively, the Hodge structures of $H^1(E,\mathbb{Q})$ and $H^2(S,\mathbb{Q})$. Then

$$(H^3(E\times S,\mathbb{Q}),h)=(H^1(E,\mathbb{Q}),h_E)\otimes(H^2(S,\mathbb{Q}),h_S).$$

(See Voisin [41], Thèoréme 11.38.) Since the fixed locus of S^σ are given by a curve C_g $(g\geq 1)$ and rational curves L_i $(i=1,\dots,k)$, the rational polarized Hodge structure h_X of a Calabi–Yau threefold $X=\widetilde{E\times S/\iota\times\sigma}$ is given by the rational sub-Hodge structure of $(H^3(E\times S,\mathbb{Q}),h)$ together with those arising from exceptional divisors associated to the curve C_g of genus $g\geq 1$ in the fixed locus S^σ. We get

$$\begin{aligned}(H^3(X,\mathbb{Q}),h_X)\simeq{}&(V_-\cap H^2(S,\mathbb{Q}),h_-)\otimes(H^1(E,\mathbb{Q}),h_E)\\&\oplus(H^1(C_g,\mathbb{Q}),h_{C_g}),\end{aligned}$$

where (V_-,h_-) denotes the restricted Hodge structure on $H^{1,1}(S)$ of the -1 eigenspace, and h_{C_g} is the Hodge structure of $H^1(C_g,\mathbb{Q})$. Note that there is an associated Abel–Jacobi map $H^{1,0}(C_g)\to H^{2,1}(X)$ and we identify its image with $H^{1,0}(C_g)$ in $H^{2,1}(X)$ (see Clemens and Griffiths [7]). Then h_X is of CM type if and only if h_E and h_S (more precisely, h_- and h_{C_g}) are all of CM type. Now with our choices of E and S, $E=E_2$ or E_3 is of CM type, and S is of CM type (in particular, C_g is of CM type (cf. Lemma 5.13 below)). Therefore, h_E and h_S (h_- and h_{C_g}) are all of CM type, and hence X is of CM type (Cf. Borcea [5], Proposition 1.2). □

Remark 4.1. More examples of CM type Calabi–Yau threefolds of Borcea–Voisin type may be obtained by taking any CM type elliptic curves E. In fact, one notices that any elliptic curve E can be embedded in $\mathbb{P}^2(2,1,1)$, with equation

$$E: x_0^2=x_2(x_1^3+ax_0x_2+bx_2^3)\quad\text{with } a,b\in\mathbb{Q}.$$

In particular, elliptic curves with CM by a quadratic field but with j-invariant in $\mathbb{Q}$ can be realized in this way.

Rohde [33] (Example A.1.9) gave four examples of elliptic curves over $\mathbb{Q}$ with complex multiplication in the usual projective 2-space.

For the additional 41 pairs (S, σ) of $K3$ surfaces with involution σ of Theorem 2.5 (b) (see tables B.4 to B.6), the situation is slightly different from the above cases. Calabi–Yau threefolds X are birational to hypersurfaces over $\mathbb{Q}$, but they are not quasi-smooth. More precisely, we have the following result.

Theorem 4.6. *Let (S, σ) be (the minimal resolution of) one of the 41 pairs of $K3$ surfaces with involution σ of Theorem 2.5 (b), which is not in the list of Borcea. Let E be an elliptic curve over $\mathbb{Q}$ with involution ι with or without CM. Take the product $E \times S$ and consider the quotient threefold $E \times S/\iota \times \sigma$. Resolving singularities, we obtain a smooth Calabi–Yau threefold X over $\mathbb{Q}$. Further, X is of CM type if and only if E is of CM type.*

About the realization of X as a hypersurface, the following holds.

(a) *If (S, σ) is one of the $K3$ surfaces listed in* tables B.4 *and* B.5 *other than #22 and #58, then S is birational to $x_0^2 x_i + f(x_1, x_2, x_3) = 0$ for some $i \neq 0$ and X is birational to a (non-quasi-smooth) hypersurface over $\mathbb{Q}$ defined by*

$$\begin{cases} z_{i+1}(z_0^4 + z_1^4) + f(z_2, z_3, z_4) = 0, & \text{if } w_0 \text{ is odd and } E = E_2, \\ z_{i+1}(z_0^3 + z_1^6) + f(z_2, z_3, z_4) = 0, & \text{if } w_0 \text{ is even but not divisible} \\ & \text{by 3 and } E = E_3. \end{cases}$$

(b) *Let (S, σ) be one of the $K3$ surfaces listed in* table B.6 *other than #16. If we choose $E = E_2$, then X is birational to the following (non-quasi-smooth) hypersurface over $\mathbb{Q}$:*

$$\begin{cases} (z_0^4 + z_1^4)^2 + z_2^3 + z_3^4 + z_4^6 = 0 & \text{in } \#2, \\ (z_0^4 + z_1^4)^2 + z_2^3 + z_2 z_3^3 + z_3 z_4^4 = 0 & \text{in } \#52, \\ z_3(z_0^4 + z_1^4)^2 + z_2^3 + z_2 z_4^3 + z_3^3 z_4 = 0 & \text{in } \#84. \end{cases}$$

Proof. Since the singular locus of $E \times S/\iota \times \sigma$ is defined over $\mathbb{Q}$, resolving singularities, we obtain a smooth Calabi–Yau threefold X over $\mathbb{Q}$. Here X is not necessarily of CM type. By the same argument as for proof of Theorem 4.6, X is of CM type if and only if each component, E and S, is of CM type. Since we already know that S is of CM type, X is of CM type if and only if E is a CM elliptic curve over $\mathbb{Q}$.

(a) If we choose an appropriate elliptic curve E, then the twist map of Proposition 4.3 works for $x_0^2 = -f(x_1, x_2, x_3)/x_i$. Depending on the parity of w_0, we obtain the equations as claimed.

(b) Since the variable associated with the involution σ carries an odd weight, we may choose $E = E_2$. Then the twist map of Proposition 4.3 (a) works for $x_0^2 = \sqrt{-f(x_1, x_2, x_3)}$ or $x_0^2 = \sqrt{-f(x_1, x_2, x_3)/x_i}$ and we obtain the equations as asserted. □

Remark 4.2. Note that $K3$ surfaces S_0 realized by Yonemura in weighted projective 3-spaces are often singular. To have smooth $K3$ surfaces, we ought to consider minimal resolutions S. The involution σ is lifted to S and we use S to carry out the above construction of Calabi–Yau threefolds X. The procedure is shown as follows:

$$\begin{array}{ccccc} E \times S_0 & \longleftarrow & E \times S & & \\ & & \downarrow & & \\ & & E \times S/\iota \times \sigma & \longleftarrow & X. \end{array}$$

Remark 4.3. The above constructions work with any elliptic curves, not only with E_2 and E_3.

4.3. Singularities and resolutions on Calabi–Yau threefolds of Borcea–Voisin type

Let S_0 be a $K3$ surface defined by a weighted hypersurface

$$S_0 : x_0^2 = f(x_1, x_2, x_3) \subset \mathbb{P}^3(w_0, w_1, w_2, w_3)$$

of degree $\deg(f) = w_0 + w_1 + w_2 + w_3$. The involution σ is given by $\sigma(x_0) = -x_0$. The singularities on S_0 are determined by the weight. Let S be the minimal resolution of S_0. The involution σ is extended to S. Let S^σ be the fixed part of S by σ.

Let E be an elliptic curve

$$E_2 : y_0^2 = y_1^4 + y_2^4 \subset \mathbb{P}^2(2, 1, 1)$$

or

$$E_3 : y_0^2 = y_1^3 + y_2^6 \subset \mathbb{P}^2(3, 2, 1),$$

defined in Section 4.2. The involution ι is given by $\iota(y_0) = -y_0$. The fixed part E^ι consists of four points

$$E^\iota = \{P_1, P_2, P_3, P_4\}.$$

We use for E either E_2 if w_0 is even, or E_3 if w_0 is odd. Take the quotient threefold $E \times S/\iota \times \sigma$.

Remark 4.4. In fact, the above statement is true for any elliptic curve

$$E : y^2 = x^3 + ax + b \in \mathbb{P}^2(1,1,1) \quad \text{with } a, b \in \mathbb{Q}.$$

E has the involution $y \to -y$ and the fixed points consists of 4 points. Thus, there is no need to confine our discussions to E_2 or E_3. We will get extra singularities working in weighted projective spaces, but this is not intrinsic to the Borcea–Voisin construction.

Let X be a smooth resolution of $E \times S/\iota \times \sigma$. Then the singular loci $\{P_i\} \times S^\sigma$ are determined from the weight of S_0 and the singularity data of the ambient space.

Here are examples.

Example 4.7. Let

$$S_0 : x_0^2 = x_1^5 + x_2^5 + x_3^{10} \subset \mathbb{P}^3(5,2,2,1).$$

(This is #6 in Yonemura = #2 in Borcea.) Let

$$E_2 : y_0^2 = y_1^4 + y_2^4 \subset \mathbb{P}^2(2,1,1).$$

- Then S_0 is a singular $K3$ surface and the singular locus is:

$$\mathrm{Sing}(S_0) = \{(0 : x_1 : x_2 : 0) \,|\, x_1^5 + x_2^5 = 0\} = \{Q_1, Q_2, Q_3, Q_4, Q_5\}$$

where every Q_i is a cyclic quotient singularity of type A_1.

- Let C' be a curve on S_0 defined by $x_0 = 0$, that is,

$$C' = \{x_0 = 0\} : x_1^5 + x_2^5 + x_3^{10} = 0 \subset \mathbb{P}^2(2,2,1).$$

Since $\mathbb{P}^2(2,2,1) \simeq \mathbb{P}^2(1,1,1)$, C' is identified with

$$C' : x_1^5 + x_2^5 + x_3^5 = 0 \subset \mathbb{P}^2$$

which is a smooth curve of genus 6.

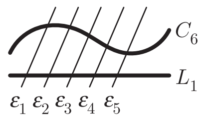

Figure 2: Exceptional divisors and the fixed locus.

• Let L' be a curve on S defined by $x_3 = 0$, that is,

$$L' = \{ x_3 = 0 \} : x_0^2 = x_1^5 + x_2^5 \subset \mathbb{P}^2(5, 2, 2).$$

Since $\mathbb{P}^2(5, 2, 2) \simeq \mathbb{P}^2(5, 1, 1)$, L' is identified with

$$L' : x_0 = x_1^5 + x_2^5 \subset \mathbb{P}^2(5, 1, 1)$$

and hence L' is a rational curve.

• We see that

$$C' \cap L' = \{Q_1, Q_2, Q_2, Q_4, Q_5\}.$$

• Let S be the minimal resolution of S_0. The involution σ is lifted to S. Let C_6 and L_1 be the respective strict transforms of C' and L' to the minimal resolution S. Let $\mathcal{E}_1, \ldots, \mathcal{E}_5$ be the exceptional divisors on S arising from singularities Q_i, $(i = 1, \ldots, 5)$, respectively.

Then C_6 and L_1 are fixed by σ, but not the exceptional divisors $\mathcal{E}_i$ for any $i \in \{1, 2, \ldots, 5\}$. Hence

$$S^\sigma = C_6 \cup L_1$$

and we see $g = 6$ and $k = 1$ (so $r = 6$, $a = 4$ in Nikulin's notation). The resolution picture is given in figure 2, where the curves in boldface are fixed by σ.

Proposition 4.8. *Let X be a crepant resolution of the quotient threefold $E_2 \times S/\iota \times \sigma$ of* Example 4.7. *Then X is a Calabi–Yau threefold corresponding to the triplet* $(6, 4, 0)$, *and its exceptional divisors are four copies of the ruled surfaces*

$$(C_6 \times \mathbb{P}^1) \cup (L \times \mathbb{P}^1).$$

Furthermore, X is of CM type, and has a (quasi-smooth) model

$$z_0^4 + z_1^4 = z_2^5 + z_3^5 + z_4^{10} \subset \mathbb{P}^4(5, 5, 4, 4, 2).$$

The Hodge numbers are given by

$$h^{1,1}(X) = 15,\ h^{2,1}(X) = 39.$$

Example 4.9. Consider the surface

$$S_0 : x_0^2 = x_1^3 x_3 + x_2^7 + x_3^{28} \subset \mathbb{P}^3(14, 9, 4, 1).$$

This is #45 in Yonemua = #36 in Borcea. Let

$$E_3 : y_0^2 = y_1^3 + y_2^6 \subset \mathbb{P}^2(3, 2, 1).$$

• The surface S_0 is a singular $K3$ surface. There are two singular points:

$$Q := (0:1:0:0) \quad \text{of type } A_{9,8},$$

and

$$R := (1:0:1:0) \quad \text{of type } A_{2,1}.$$

• Let C' be the curve on S_0 defined by $x_0 = 0$

$$C' = \{x_0 = 0\} : x_1^3 x_3 + x_2^7 + x_3^{28} = 0 \subset \mathbb{P}^2(9, 4, 1).$$

Then C' is a quasi-smooth curve with singularity Q.

• No other curves defined by $x_i = 0$ $(i \neq 0)$ are fixed by the involution σ.

• Let S be the minimal resolution of S_0. Then S is a smooth $K3$ surface and the involution σ is lifted to S. Let C_6 be the strict transform of C' to S; it has genus 6. Let $\mathcal{E}_1, \ldots, \mathcal{E}_8$ be exceptional divisors arising from singularity Q. Let $\mathcal{E}_9$ be the exceptional divisor arising from R. Then $\mathcal{E}_{2i}$ $(i = 1, 2, 3, 4)$ and $\mathcal{E}_9$ are fixed by σ, but others are not.

Put $L_i := \mathcal{E}_{2i}$ $(i = 1, 2, 3, 4)$ and $L_5 := \mathcal{E}_9$. Then

$$S^\sigma = C_6 \cup L_1 \cup \cdots \cup L_5.$$

So $g = 6$ and $k = 5$ (so $r = 10, a = 0$ in Nikulin's notation). The resolution picture is given in figure 3, where the curves in boldface are fixed by σ.

• The quotient threefold $E_3 \times S/\iota \times \sigma$ has singularities along $\{P_i\} \times S^\sigma$ where $E_3^\iota = \{P_1, P_2, P_3, P_4\}$.

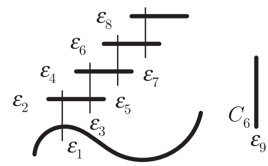

Figure 3: Exceptional divisors and the fixed locus.

Summarizing the above, we have

Proposition 4.10. *A crepant resolution X of the quotient threefold $E_3 \times S/\iota \times \sigma$ of* Example 4.9 *is a Calabi–Yau threefold corresponding to the triplet $(10, 0, 0)$, and its exceptional divisors are four copies of*

$$(C_6 \times \mathbb{P}^1) \cup (L_1 \times \mathbb{P}^1) \cup \cdots \cup (L_5 \times \mathbb{P}^1).$$

Furthermore, X is of CM type, and has a (quasi-smooth) model

$$z_0^3 + z_1^6 = z_2^3 z_4 + z_3^7 + z_4^{28} \subset \mathbb{P}^4(28, 14, 27, 12, 3).$$

The Hodge numbers are given by

$$h^{1,1}(X) = 35, \quad h^{2,1}(X) = 35.$$

Example 4.11. Consider the surface

$$S_0 : x_0^2 = x_1^3 + x_2^{10} + x_2^{15} \subset \mathbb{P}^3(15, 10, 3, 2).$$

This is #11 in Yonemura = #18 in Borcea. Let

$$E_2 : y_0^2 = y_1^4 + y_2^4 \subset \mathbb{P}^2(2, 1, 1).$$

- S_0 is a singular $K3$ surface and singularities are

$$Q_1, Q_2, Q_3 = (0 : x_1 : 0 : x_3) \quad \text{of type } A_{2,1},$$
$$R := (x_0 : x_1 : 0 : 0) \quad \text{of type } A_{5,4},$$

and

$$T_1, T_2 := (x_0 : 0 : x_2 : 0) \quad \text{of type } A_{3,2}.$$

- Let C' be the curve on S_0 defined by $x_0 = 0$:

$$C' = \{x_0 = 0\} : x_1^3 + x_2^{10} + x_3^{15} = 0 \subset \mathbb{P}^2(10, 3, 2).$$

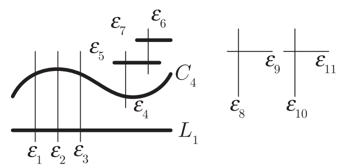

Figure 4: Exceptional divisors and the fixed locus.

Via the isomorphism $\mathbb{P}^2(10, 3, 2) \simeq \mathbb{P}^2(5, 3, 1)$, C' is identified with

$$C' : x_1^3 + x_2^5 + x_3^{15} = 0 \subset \mathbb{P}^2(5, 3, 1).$$

• Let L' be the curve on S_0 defined by $x_2 = 0$:

$$L' = \{x_2 = 0\} : x_0^2 = x_1^3 + x_3^{15} \subset \mathbb{P}^2(15, 10, 2).$$

Via the isomorphism $\mathbb{P}^2(15, 10, 2) \simeq \mathbb{P}^2(3, 1, 1)$, L' is identified with

$$L' : x_0 = x_1^3 + x_3^3 \subset \mathbb{P}^2(3, 1, 1).$$

• C' has genus 4 and L' is rational with

$$C' \cap L' = \{Q_1, Q_2, Q_3\} \quad \text{and} \quad R \in L'.$$

• Let S be the minimal resolution of S_0. The involution σ is lifted to S. Let C_4 and L_1 be the strict transforms of C' and L' on S, respectively. Let $\mathcal{E}_i\,(i = 1, 2, 3)$ be the exceptional divisors arising from singularities $Q_i\,(i = 1, 2, 3)$, $\mathcal{E}_{4+j}\,(j = 0, 1, 2, 3)$ be the exceptional divisors arising from R, and $\mathcal{E}_{8+t}\,(t = 0, 1, 2, 3)$ be the exceptional divisors arising from singularities T_1, T_2.

• $C, L_1, \mathcal{E}_5 =: L_2$ and $\mathcal{E}_7 =: L_3$ are fixed by σ, but all others are not. Hence

$$S^\sigma = C_4 \cup L_1 \cup L_2 \cup L_3.$$

So $g = 4$ and $k = 3$ (so $r = 10, a = 4$ in Nikulin's notation). The resolution picture is given in figure 4, where the curves in boldface are fixed by σ.

• The quotient threefold $E_2 \times S/\iota \times \sigma$ has singularities $\{P_i\} \times S^\sigma$ $(i = 1, 2, 3, 4)$ where $E_2^\iota = \{P_1, P_2, P_3, P_4\}$.

Proposition 4.12. *A crepant resolution X of the quotient threefold $E_2 \times S/\iota \times \sigma$ of* Example 4.11 *is a Calabi–Yau threefold corresponding to the* *triplet* $(10,4,0)$*, and its exceptional divisors are four copies of ruled surfaces:*

$$(C_4 \times \mathbb{P}^1) \cup (L_1 \times \mathbb{P}^1) \cup (L_2 \times \mathbb{P}^1) \cup (L_3 \times \mathbb{P}^1).$$

Furthermore, X is of CM type, and has a (quasi-smooth) model

$$z_0^4 + z_1^4 = z_2^3 + z_3^{10} + z_4^{15} \subset \mathbb{P}^4(15,15,20,6,4).$$

The Hodge numbers are given by

$$h^{1,1}(X) = 27,\ h^{2,1}(X) = 27.$$

5. Automorphy of Calabi–Yau threefolds of Borcea–Voisin type over $\mathbb{Q}$

5.1. The L-series

Let X be a Calabi–Yau variety defined over $\mathbb{Q}$ of dimension d where $d \leq 3$. Hence X is an elliptic curve for $d = 1$, a $K3$ surface for $d = 2$ and a Calabi–Yau threefold for $d = 3$.

We may assume that X has defining equations with integer coefficients. A prime p is said to be *good* if the reduction $X_p = X \otimes \mathbb{F}_p$ is smooth and defines a Calabi–Yau variety over $\mathbb{F}_p$. A prime p is said to be *bad* if it is not a good prime. There are only finitely many bad primes and we denote by S the product of bad primes. Then a Calabi–Yau variety X has an integral model over $\mathbb{Z}[1/S]$.

Put $\bar{X} := X \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}$. We will consider the Galois representation associated to the ℓ-adic étale cohomology groups $H^i_{\text{et}}(\bar{X}, \mathbb{Q}_\ell)$ $(0 \leq i \leq 2d)$ of X, where ℓ is a prime.

The absolute Galois group $G_{\mathbb{Q}} := \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $\bar{X}$. For each i, $0 \leq i \leq 2d$, one has a Galois representation on the cohomology group $H^i_{\text{et}}(\bar{X}, \mathbb{Q}_\ell)$ where ℓ is a prime different from p. This defines a continuous ℓ-adic representation $\rho : G_{\mathbb{Q}} \to GL_r(\mathbb{Q}_\ell)$ of some finite rank r' where $r' = \dim_{\mathbb{Q}_\ell} H^i_{\text{et}}(\bar{X}, \mathbb{Q}_\ell) = B_i(X)$ (the ith Betti number of X).

For a good prime p the structure of this Galois representation can be studied by passing to the reduction $X_p = X \otimes \mathbb{F}_p$. The Frobenius morphism Frob_p induces a $\mathbb{Q}_\ell$-linear map $\rho(\text{Frob}_p)$ on $H^i_{\text{et}}(\bar{X}, \mathbb{Q}_\ell)$ $(i,\ 0 \leq i \leq 2d)$. Let

$$P^i_p(X, \rho, t) := \det(1 - \rho(\text{Frob}_p)\, t \,|\, H^i_{\text{et}}(\bar{X}, \mathbb{Q}_\ell))$$

be the characteristic polynomial of $\rho(\mathrm{Frob}_p)$, where t is an indeterminate. By the validity of the Weil conjectures, one knows that

- $P_p^i(X,\rho,t) \in 1+\mathbb{Z}[t]$ has degree $B_i(X)$.

- The reciprocal roots of $P_p^i(X,\rho,t)$ are algebraic integers with complex absolute value $p^{i/2}$ (the Riemann Hypothesis for X_p).

- The zeta-function of X_p is a rational function of t over $\mathbb{Q}$ and is given by

$$\zeta(X_p,t) = \frac{P_p^1(X,\rho,t)P_p^3(X,\rho,t)\cdots P_p^{2d-1}(X,\rho,t)}{P_p^0(X,\rho,t)P_p^2(X,\rho,t)\cdots P_p^{2d}(X,\rho,t)}.$$

Now putting all local data together, we can define the (incomplete) global L-series and the (incomplete) zeta-function of X.

Definition 5.1. For each $i, 0 \le i \le 2d$, we define the ith (incomplete) L-series by

$$L_i(X,s) := L(H^i_{\mathrm{et}}(\bar{X},\mathbb{Q}_\ell),s) = \prod_{p\notin S}\frac{1}{P_p^i(X,\rho,p^{-s})}.$$

The (Hasse–Weil) zeta-function of X is then defined by

$$\zeta(X,s) = \frac{\prod_{i=0}^d L_{2i-1}(X,s)}{\prod_{i=1}^d L_{2i}(X,s)}.$$

The use of the terminology of "incomplete" L-series is based on the fact that it does not include a few Euler factors corresponding to bad primes. We can also define Euler factors for primes $p \in S$ to complete the L-series bringing in the *Gamma factor* corresponding to the prime at infinity, and also factors corresponding to bad primes.

We denote by $\zeta(\mathbb{Q},s) = \sum_{n=1}^\infty \frac{1}{n^s} = \prod_{p:\mathrm{prime}} \frac{1}{1-p^{-s}}$ the Riemann zeta-function.

Example 5.1. We consider an elliptic curve E defined over $\mathbb{Q}$. Let S be a set of bad primes. Then one knows that the L-series of E is given by

$$L_0(E,s) = \zeta(\mathbb{Q},s),\ L_2(E,s) = \zeta(\mathbb{Q},s-1),$$

$$L_1(E,s)=L(H^1(E),s)=\prod_{p\notin S}P_p^1(E,\rho,p^{-s})^{-1}=\prod_{p\notin S}\frac{1}{1-a_pp^{-s}+pp^{-2s}},$$

where $a_p=p+1-\#E(\mathbb{F}_p)=\mathrm{trace}(\rho(\mathrm{Frob}_p))$.

The zeta-function of E is then given by

$$\zeta(E,s)=\frac{L_1(E,s)}{\zeta(\mathbb{Q},s)\zeta(\mathbb{Q},s-1)}.$$

These assertions are true a priori for good primes, but they can be extended to include bad primes and also prime at infinity.

This is a classical result and can be found, for instance, in Silverman [38].

Example 5.2. We consider a $K3$ surface S defined over $\mathbb{Q}$. The zeta-function of S is given by

$$\zeta(S,s)=\frac{L_1(S,s)L_3(S,s)}{L_0(S,s)L_2(S,s)L_4(S,s)}=\frac{1}{L_0(S,s)L_2(S,s)L_4(S,s)},$$

where $L_4(S,s)=L_0(S,s-2)$ by Poincaré duality. The $L_2(S,s)$ factors as a product

$$L_2(S,s)=L(H^2(S,\mathbb{Q}_\ell),s)=L(NS(S)\otimes\mathbb{Q}_\ell,s)L(T(S)\otimes\mathbb{Q}_\ell,s)$$

in accordance with the decomposition $H^2_{\mathrm{et}}(\bar{S},\mathbb{Q}_\ell)=(NS(S)\oplus T(S))\otimes\mathbb{Q}_\ell$ where $NS(S)$ is the Néron–Severi group spanned by algebraic cycles and $T(S)$ is its orthogonal complement, and this decomposition is Galois invariant. Also this factorization is independent of the choice of ℓ. The Tate conjecture [39] (Theorem 5.6) further asserts that

$$NS(S)\otimes_{\mathbb{Q}}\mathbb{Q}_\ell=H^2_{\mathrm{et}}(\bar{S},\mathbb{Q}_\ell)^G$$

where G denotes the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. (We follow the proof in [39], due to D. Ramakrishnan. It rests on the facts: (1) the existence of an abelian variety A and the absolute Hodge cycle on $S\times A$ inducing an injection $H^2_{\mathrm{et}}(S,\mathbb{Q}_\ell)\hookrightarrow H^2_{\mathrm{et}}(A,\mathbb{Q}_\ell)$ (Deligne [11]), (2) the theorem of Faltings that the Tate conjecture is true for A, and (3) the theorem of Lefschetz that rational classes of type $(1,1)$ are algebraic.) Therefore, the Picard number $\rho(S)$ of S is equal to the dimension of the G-invariant subspace $H^2_{\mathrm{et}}(\bar{S},\mathbb{Q}_\ell)^G$. With the validity of the Tate conjecture the zeta-function of S takes the form

$$\zeta(S,s) = [\zeta(\mathbb{Q},s)\zeta(\mathbb{Q},s-2)\zeta(\mathbb{Q},s-1)^{\rho(S)}L(T(S)\otimes\mathbb{Q}_\ell,s)]^{-1}.$$

Now $NS(\bar{S}) \neq NS(S)$ in general. In that case, not all algebraic cycles in $NS(\bar{S})$ are defined over $\mathbb{Q}$, let $\mathbb{L}$ be the smallest algebraic number field over which all $\rho(S)$ algebraic cycles are defined. Let $\zeta(\mathbb{L},s)$ denote the Dedekind zeta-function of $\mathbb{L}$, that is,

$$\zeta(\mathbb{L},s) = \sum_{I\subseteq\mathcal{O}_\mathbb{L}} \frac{1}{N_{\mathbb{L}/\mathbb{Q}}(I)^s} = \prod_{P\subseteq\mathcal{O}_\mathbb{L}} \frac{1}{1-N_{\mathbb{L}/\mathbb{Q}}(P)^{-s}}$$

where $\mathcal{O}_\mathbb{L}$ is the ring of integers of $\mathbb{L}$, I (resp. P) is an ideal (resp. prime ideal) of $\mathcal{O}_\mathbb{L}$, and $N(I)$ (resp. $N(P)$) denotes the norm. Then $\zeta(\mathbb{L},s)$ is a product of the Artin L-functions over the irreducible complex representations of the Galois group, whereas only some need to occur in $NS(\bar{S})$ and the multiplicity of an irreducible representation in $NS(\bar{S})_\mathbb{C}$ depends on the geometry. Then the zeta-function of S is of the form

$$\zeta(S,s) = [\zeta(\mathbb{Q},s)\zeta(\mathbb{Q},s-2)\zeta(\mathbb{L},s-1)^t L(\rho,s)L(T(S)\otimes\mathbb{Q}_\ell,s)]^{-1},$$

where the exponent t is some integer $1 \leq t \leq \rho(\bar{S})$, which is rather difficult to determine explicitly, and $L(\rho,s)$ is the Artin L-series of the irreducible complex representation. (In general, the automorphy of the Artin L-function is still an open problem.)

For example, let S be a $K3$ surface with $NS(\bar{S})_\mathbb{Q} \cong \mathbb{Q}^2$ so $\rho(\bar{S}) = 2$. Let $\mathbb{L}$ be a quadratic extension of $\mathbb{Q}$ so that $NS(S_\mathbb{L})_\mathbb{Q} \cong \mathbb{Q}$, and the Galois group acts trivially on it, but acts by a non-trivial character on a complementary one-dimensional subspace. Then

$$L(NS(\bar{S}),s) = \zeta(\mathbb{L},s-1) = \zeta(\mathbb{Q},s-1)L(\mathbb{L},s-1),$$

where $L(\mathbb{L},s)$ is the Dirichlet L-function of $\mathbb{L}$.

Example 5.3. We now consider a Calabi–Yau threefold X over $\mathbb{Q}$. The zeta-function of X is given by

$$\begin{aligned}\zeta(X,s) &= \frac{L_1(X,s)L_3(X,s)L_5(X,s)}{L_0(X,s)L_2(X,s)L_4(X,s)L_6(X,s)} \\ &= \frac{L_3(X,s)}{L_0(X,s)L_2(X,s)L_4(X,s)L_6(X,s)},\end{aligned}$$

where $L_6(X,s) = L_0(X,s-3)$, $L_4(X,s) = L_2(X,s-1)$ by Poincaré duality. The zeta-function of X is of the form

$$\zeta(X,s) = \frac{L_3(X,s)}{\zeta(\mathbb{Q},s)\zeta(\mathbb{Q},s-3)L_2(X,s)L_2(X,s-1)}.$$

Conjecture 5.4 (Langlands reciprocity conjecture [23]**).** *Let X be a Calabi–Yau variety defined over $\mathbb{Q}$. Then the zeta-function $\zeta(X,s)$ is automorphic.*

Remark 5.1. For our Calabi–Yau varieties over $\mathbb{Q}$, we know the form of their zeta-functions, and the Riemann zeta-function $\zeta(s) = \zeta(\mathbb{Q},s)$ (and its translates) are trivially automorphic as they correspond to the identity representation. The automorphy question for our $K3$ surfaces and our Calabi–Yau varieties over $\mathbb{Q}$ is then for the automorphy of the L-series $L_i(X,s)$ for each i, $0 \le i \le \dim(X)$.

Are there any automorphic forms (representations) such that $L_i(X,s)$ for each i ($0 \le i \le \dim(X)$) are determined by the L-series of such automorphic objects?

First we will discuss some examples in support of the Langlands reciprocity conjecture.

5.2. Elliptic curves over $\mathbb{Q}$

For dimension 1 Calabi–Yau varieties over $\mathbb{Q}$, we have the well-known celebrated results of Wiles [42], Taylor and Wiles [40].

Theorem 5.5. *Let E be an elliptic curve defined over $\mathbb{Q}$. Then there exists a normalized weight 2 new form f_E of level N_E which is an eigenvector of the Hecke operators, such that*

$$L_1(E,s) = L(f_E,s).$$

Here N_E is the conductor of E and f_E has the q-expansion ($q = \mathrm{e}^{2\pi \mathrm{i}z}$, $z = x+iy$ with $y>0$)

$$f_E = q + a_2q^2 + \cdots + a_pq^p + \cdots,$$

where a_p is the same as defined in Example 5.1*.*

In terms of Galois representations, let $\rho_{E,\ell}$ be an ℓ-adic representation of Gal($\overline{\mathbb{Q}}/\mathbb{Q}$) on the ℓ-adic Tate module $T_\ell(E)$ of $E/\mathbb{Q}$. Then $\rho_{E,\ell}$ is modular *for some ℓ. That is, there exists a cusp form f_E and a representation ρ_{f_E} of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $\rho_{E,\ell} = \rho_{f_E}$. We will denote this representation simply by ρ_E.*

Remark 5.2. If E is an elliptic curve over $\mathbb{Q}$ with CM by an imaginary quadratic field $K/\mathbb{Q}$, then $L_1(E,s)$ is equal to a Hecke L-series of K with a suitable Grossencharacter of K (Deuring [12]).

5.3. $K3$ surfaces over $\mathbb{Q}$ of CM type

For dimension 2 Calabi–Yau varieties, namely, $K3$ surfaces, our results on automorphy is formulated as follows. We can establish the automorphy of $K3$ surfaces over $\mathbb{Q}$ of CM type. This generalizes the result of Shioda and Inose [35] for singular $K3$ surfaces, and also the results of Livné–Schütt–Yui [25] for certain $K3$ surfaces with non-symplectic group actions.

Theorem 5.6. *Let (S,σ) be one of the 86 pairs (S,σ) of $K3$ surfaces in* Theorem 2.5. *Then (S,σ) is defined over $\mathbb{Q}$, and there exists a quadruple $(\rho, \mathbb{K}, \iota, \chi)$ with the following properties:*

(a) *ρ is an (Artin) Galois representation of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, and the degree of ρ is $\rho(\bar{S})$ (the geometric Picard number of S),*

(b) *$\mathbb{K}$ is a CM abelian extension of $\mathbb{Q}$,*

(c) *$\iota : \mathbb{K} \to \mathbb{C}$ is an embedding,*

(d) *χ is a Hecke character of $\mathbb{K}$ of ∞ type $z \to \iota(z)^2$,*

(e) $\dim \rho + [\mathbb{K} : \mathbb{Q}] = 22$ *where* $[\mathbb{K} : \mathbb{Q}] = \dim T(\bar{S})_{\mathbb{Q}}$.

such that the zeta-function $\zeta(S,s)$ and the $L_2(S,s)$ of $S/\mathbb{Q}$ are given by

$$\zeta(S/\mathbb{Q}, s) = [\zeta(\mathbb{Q}, s)\zeta(\mathbb{Q}, s-2)L_2(S,s)]^{-1}$$

where

$$L_2(S,s) = L(\rho, s-1)L(\chi, s).$$

Proof. We follow an argument similar to the one in [25], to which we refer for further details. Since S is a surface defined over $\mathbb{Q}$, its $\mathbb{Q}_\ell$-cohomology is 1-dimensional in dimensions 1 and 4, which contribute the factors $\zeta(s)$ and $\zeta(s-2)$, respectively. Since S is a $K3$ surface the first and the third cohomology groups vanish, giving no contribution to the L-function. The second cohomology group is a direct sum $N_S \oplus T_S$ of the algebraic part, spanned by the subspace N_S of algebraic cycles and its orthogonal complement T_S, called the space of transcendental cycles. This direct sum decomposition is Galois invariant. Since N_S is the $\mathbb{Q}_\ell$-span of the image by the cycle map of the Néron–Severi group $NS(\bar{S})$ (with scalars extended to $\mathbb{Q}_\ell$), the Galois group acts on $NS(\bar{S})$ through a finite quotient. Hence it acts on N_S by the Tate twist $\rho(1)$ of the corresponding Artin representation ρ.

To obtain the last factor we first consider the cohomology with complex coefficients. Since the defining equation for S uses only four monomials, it is a Delsarte surface. Hence it is a quotient of a surface in $\mathbb{P}^3$ with a (homogeneous) diagonal equation $\sum_{i=1,\dots,4} a_i w_i^r = 0$ by some diagonal action of roots of unity. Moreover in our case, the monomials in the (diagonal) equation for S have coefficients 1, which implies that the a_i's can also be taken to be all 1's. Weil's calculation (see [25], Section 6) gives that over an appropriate cyclotomic field the Galois representation on the transcendental cycles is a sum of one-dimensional representations coming from Jacobi sums, of infinity type as in the statement of Theorem 5.3, which the absolute Galois group permutes transitively. The Theorem follows. (For detailed discussion on Jacobi sums, the reader is referred to the Appendix section below, or Gouvêa and Yui [17].) □

Corollary 5.7. *Let (S, σ) be as in* Theorem 5.6*. Then S has CM by a cyclotomic field $\mathbb{K} = \mathbb{Q}(\zeta_t)$ for some t. (Here ζ_t denotes a primitive t-th root of unity.)*

Proof. This follows from Theorem 5.6. S is realized by a finite quotient of some Fermat surface of some degree, however this finite group may have a rather large order and it requires more work to determine its precise form. The field $\mathbb{K}$ that corresponds to the transcendental cycles is isomorphic to $T(S) \otimes \mathbb{Q} \simeq \mathbb{Q}(\zeta_t)$ for some t. Moreover, it is generated by Jacobi sum Grossencharacters of $\mathbb{Q}(\zeta_t)$. This is because the Galois representation defined by $T(S)$ is a sum of one-dimensional representations induced from the Jacobi sum Grossencharacters corresponding to the unique character $\mathbf{a}$ with $\|\mathbf{a}\| = 0$ (See [25]). □

In general, the automorphy of the Artin L-function is still a conjecture. However, in our cases, we have the following result.

Corollary 5.8. *Let (S, σ) be as in* Theorem 5.6*. Then the Artin L-function $L(\rho, s)$ is automorphic.*

Proof. We know that S is dominated by some Fermat surface

$$\mathcal{F}_m : x_0^m + x_1^m + x_2^m + x_3^m = 0 \subset \mathbb{P}^3$$

of degree m. Here we review Shioda's treatment (cf. Shioda [37] or Gouvêa–Yui [17]). The cohomology group $H^2(\mathcal{F}_m, \mathbb{Q}_\ell)$ is the direct sum of one-dimensional spaces. More precisely, let

$$H^2(\mathcal{F}_m, \mathbb{Q}_\ell) = \oplus_{\alpha \in \{0\} \cup \mathfrak{A}_m} V(\alpha), \quad \dim V(\alpha) = 1,$$

where

$$\mathfrak{A}_m := \left\{ \mathbf{a} = (a_0, a_1, a_2, a_3) \in (\mathbb{Z}/m\mathbb{Z})^4 \mid a_i \neq 0, \quad \sum_{i=0}^{3} a_i = 0 \in \mathbb{Z}/m\mathbb{Z} \right\}.$$

This implies that the Néron–Severi group $NS(\mathcal{F}_m)$ and the group of transcendental cycles $T(\mathcal{F}_m)$ of $\mathcal{F}_m$ are also described as direct sums of one-dimensional spaces:

$$NS(\mathcal{F}_m) \otimes \mathbb{Q}_\ell = \oplus_{\alpha \in \{0\} \cup \mathfrak{B}_m} V(\alpha)$$

and

$$T(\mathcal{F}_m) \otimes \mathbb{Q}_\ell = \oplus_{\alpha \in \mathfrak{C}_m} V(\alpha),$$

where

$$\mathfrak{B}_m := \left\{ \mathbf{a} = (a_0, a_1, a_2, a_3) \in \mathfrak{A}_m \mid \sum_{i=0}^{3} \left\langle \frac{t a_i}{m} \right\rangle = 2 \right. \\ \left. \text{for all } t \text{ such that } (t, m) = 1 \right\},$$

and

$$\mathfrak{C}_m := \mathfrak{A}_m \setminus \mathfrak{B}_m.$$

Since S is realized as a Fermat quotient of some Fermat surface $\mathcal{F}_m$ by a finite group, say, H, the irreducible Galois representation ρ is also induced from one-dimensional subspaces belonging to $NS(\mathcal{F}_m)$, which are invariant under the action of H, and hence $L(\rho, s)$ is automorphic. (Indeed, working through all the examples, we are able to show that Artin L-functions are indeed automorphic.) □

5.4. Calabi–Yau threefolds over $\mathbb{Q}$ of Borcea–Voisin type

For dimension 3 Calabi–Yau varieties over $\mathbb{Q}$ of Borcea–Voisin type, our automorphy results are formulated in the following theorems.

Theorem 5.9. *Let (S, σ) be one of the 86 pairs of $K3$ surfaces with involution given in* Theorem 2.5. *Let E be an elliptic curve over $\mathbb{Q}$ with involution ι. Let X be a crepant resolution of the quotient threefold $E \times S/\iota \times \sigma$ with a model defined over $\mathbb{Q}$. Then X is automorphic.*

We reformulate the above assertion in more concrete fashion as follows.

Theorem 5.10. *Let (S, σ) be (the minimal resolution of) one of the 86 $K3$ surfaces with involution σ listed in* Theorem 2.5. *Then S is defined over $\mathbb{Q}$, and is of Delsarte type, and hence S is of CM type. Let E be an elliptic curve E_2 or E_3 with involution ι (or any elliptic curve with complex multiplication). Consider the quotient threefold $E \times S/\iota \times \sigma$, and let X be its crepant resolution. Then X is a Calabi–Yau threefold and has a model defined over $\mathbb{Q}$.*

Furthermore, the following assertions hold:

- *X is of CM type,*
- *The L-series $L_2(X, s)$ and $L_3(X, s)$ are automorphic.*
- *The zeta-function $\zeta(X, s)$ is automorphic, and hence X is automorphic.*

Since all elliptic curves E defined over $\mathbb{Q}$ are modular, without the assumption that E is of CM type, we have the following more general results.

Theorem 5.11. *Let (S, σ) be one of the 86 pairs of $K3$ surfaces with involution σ defined over $\mathbb{Q}$ in* Theorem 2.5. *Let (E, ι) be an elliptic curve defined over $\mathbb{Q}$. Let X be a crepant resolution of the quotient threefold $E \times S/\iota \times \sigma$, which has a model defined over $\mathbb{Q}$.*

Then the following assertions hold:

(a) *Let E be an elliptic curve $\mathbb{Q}$. Then there is the automorphic representation ρ_E. Equivalently, there is a cusp form f_E of weight 2 associated to ρ_E.*

(b) *Take S to be a $K3$ surface of CM type (cf.* Theorem 2.7*). There is an Artin representation ρ of an algebraic extension $\mathbb{K}$ over $\mathbb{Q}$ where* *we put* $m := [\mathbb{K} : \mathbb{Q}]$*. Put* $G_{\mathbb{Q}} = Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ *and* $G_{\mathbb{K}} = Gal(\overline{\mathbb{Q}}/K)$*. Let* $\rho_{G_{\mathbb{K}}}$ *be the compatible system of* 1*-dimensional* ℓ*-adic representation of* $G_{\mathbb{K}}$*. Then the* m*-dimensional Galois representation associated to the group of transcendental cycles* $T(S)^{\sigma}$ *is given by* $Ind_{G_{\mathbb{K}}}^{G_{\mathbb{Q}}}\rho_{G_{\mathbb{K}}}$*. We denote by* $f_{T(S)}$ *the "fictitious" automorphic form associated to this representation (though we cannot write it down explicitly).*

(c) *Let* X *be a Calabi–Yau threefold over* $\mathbb{Q}$ *of Borcea–Voisin type. Then there is the* $2m$*-dimensional Galois representation*

$$\pi := \rho_E \otimes Ind_{G_{\mathbb{K}}}^{G_{\mathbb{Q}}}\rho_{G_{\mathbb{K}}} \simeq Ind_{G_{\mathbb{K}}}^{G_{\mathbb{Q}}}(\rho_{G_{\mathbb{K}}} \otimes Res_{G_{\mathbb{K}}}^{G_{\mathbb{Q}}}\rho_E).$$

π *is an automorphic cuspidal irreducible representation of* $GL(2, \mathbb{K})$*, and*

$$\begin{aligned} L(\pi, s) &= L(Ind_{G_{\mathbb{K}}}^{G_{\mathbb{Q}}}\pi, s) = L(\pi_E \otimes Ind_{Gal(\overline{\mathbb{Q}}/K)}^{Gal(\overline{\mathbb{Q}}/\mathbb{Q})}\rho, s) \\ &= L(f_E \otimes f_{T(S)}, s). \end{aligned}$$

(d) *The Galois representation associated to the twisted sectors (the exceptional divisors) is automorphic.*

Therefore, X *is automorphic.*

5.5. Preparation for proof of automorphy for Calabi–Yau threefolds of Borcea–Voisin type

We need to compute the cohomology groups of our Calabi–Yau threefolds of Borcea–Voisin type. For this, first we compute the cohomology groups of the product $E \times S$.

Lemma 5.12. *The Künneth formula for the product* $E \times S$ *gives:*

$$H^i(E \times S, \mathbb{Q}_\ell) = \oplus_{p+q=i} H^p(E, \mathbb{Q}_\ell) \otimes H^q(S, \mathbb{Q}_\ell)$$

for $0 \leq i \leq 6$*. Then for each* i*,* $0 \leq i \leq 3$*, we obtain*

- $H^0(E \times S, \mathbb{Q}_\ell) = \mathbb{Q}_\ell$.
- $H^1(E \times S, \mathbb{Q}_\ell) = H^1(E, \mathbb{Q}_\ell) \otimes \mathbb{Q}_\ell$.
- $H^2(E \times S, \mathbb{Q}_\ell) = \mathbb{Q}_\ell \otimes H^2(S, \mathbb{Q}_\ell) \oplus \mathbb{Q}_\ell \otimes \mathbb{Q}_\ell$.
- $H^3(E \times S, \mathbb{Q}_\ell) = H^1(E, \mathbb{Q}_\ell) \otimes H^2(S, \mathbb{Q}_\ell)$.

The higher cohomologies for $i = 4, 5, 6$ can be determined by Poincaré duality.

Proof. These follow from the definition of E and S and the Künneth formula. In fact, we have

$$\begin{aligned}
H^0(E \times S, \mathbb{Q}_\ell) &= H^0(E, \mathbb{Q}_\ell) \otimes H^0(S, \mathbb{Q}_\ell) = \mathbb{Q}_\ell. \\
H^1(E \times S, \mathbb{Q}_\ell) &= H^1(E, \mathbb{Q}_\ell) \otimes H^0(S, \mathbb{Q}_\ell) \oplus H^0(E, \mathbb{Q}_\ell) \otimes H^1(S, \mathbb{Q}_\ell) \\
&= H^1(E, \mathbb{Q}_\ell) \otimes \mathbb{Q}_\ell. \\
H^2(E \times S, \mathbb{Q}_\ell) &= H^0(E, \mathbb{Q}_\ell) \otimes H^2(S, \mathbb{Q}_\ell) \oplus H^1(E, \mathbb{Q}_\ell) \otimes H^1(S, \mathbb{Q}_\ell) \\
&\quad \oplus H^2(E, \mathbb{Q}_\ell) \otimes H^0(S, \mathbb{Q}_\ell) = \mathbb{Q}_\ell \otimes H^2(S, \mathbb{Q}_\ell) \oplus \mathbb{Q}_\ell \otimes \mathbb{Q}_\ell. \\
H^3(E \times S, \mathbb{Q}_\ell) &= H^0(E, \mathbb{Q}_\ell) \otimes H^3(S, \mathbb{Q}_\ell) \oplus H^1(E, \mathbb{Q}_\ell) \otimes H^2(S, \mathbb{Q}_\ell) \\
&\quad \oplus H^2(E, \mathbb{Q}_\ell) \otimes H^1(S, \mathbb{Q}_\ell) \oplus H^3(E, \mathbb{Q}_\ell) \otimes H^0(S, \mathbb{Q}_\ell) \\
&= H^1(E, \mathbb{Q}_\ell) \otimes H^2(S, \mathbb{Q}_\ell).
\end{aligned}$$

□

In order to determine the cohomology groups for the Calabi–Yau threefolds $E \times S/\iota \times \sigma$ of Borcea–Voisin type, we need to compute the cohomology groups of "non-twisted" sector and the cohomology groups arising from singularities of "twisted" sector. For this we will use the orbifold Dolbeault cohomology theory developed by Chen and Ruan [8]. We will give a brief description of orbifold cohomology formulas relevant to our calculations.

Definition 5.2. Let $X_0 = E \times S/\iota \times \sigma$ be a singular Calabi–Yau threefold of Borcea–Voisin type. Then the cohomology group of X_0 is given by

$$H^{p,q}_{\text{orb}}(\bar{X}_0) = H^{p,q}(E \times S) \oplus H^{p-1,q-1}((E \times S)^{\iota \times \sigma})$$

for $0 \le p, q \le 3$ with $p + q = 3$, where $\bar{X}_0 = X_0 \otimes \mathbb{C}$.

The twisted sectors are the cohomology groups that correspond to $h \neq 1$ (the second term), and the non-twisted sector corresponds to $h = 1$ (the first term).

First we calculate the non-twisted sector of the cohomology.

Lemma 5.13. *Let $X_0 := E \times S/\iota \times \sigma$ be a singular Calabi–Yau threefold over $\mathbb{C}$ of Borcea–Viosin type. Then we have*

- $H^{1,0}(X_0) = H^{1,0}(E)^\iota \otimes \mathbb{C} = 0.$
- $H^{2,0}(X_0) = \mathbb{C} \otimes H^{2,0}(S)^\sigma = 0.$

- $H^{3,0}(X_0) = \mathbb{C}$.
- $H^{1,1}(X_0) = \mathbb{C} \otimes H^{1,1}(S)^{\sigma=1} \oplus \mathbb{C}$.
- $H^{2,1}(X_0) = \mathbb{C} \oplus H^{1,1}(S)^{\sigma=-1} \otimes \mathbb{C}$.

Therefore, the Hodge numbers of the singular Calabi–Yau threefold X_0 are given by

$$h^1(X_0) = 0,\ h^{2,0}(X_0) = 0,\ h^{3,0}(X_0) = 1$$

and

$$h^{1,1}(X_0) = 1 + r,\ h^{2,1}(X_0) = 1 + (20 - r),$$

where $r = \mathrm{rank} NS(S)^\sigma$.

Proof. Let ω_E and ω_S be the non-trivial holomorphic 1-form and 2-form on E and S, respectively. Then $\omega_E \wedge \omega_S$ descends to a holomorphic 3-form on X. Now the involution $\iota : E \to E$ acts on ω_E non-symplectically, and the involution $\sigma : S \to S$ acts on ω_S non-symplectically. This gives

$$H^0(X_0, \Omega^3_{X_0}) = \mathbb{C}$$

so that $h^{3,0}(X_0) = 1$, indeed. $H^{3,0}(X_0)$ is spanned by $\omega_E \times \omega_S$. Then the Künneth formula gives that

$$H^0(X_0, \Omega^1_{X_0}) = H^0(E, \Omega^1_E)^\iota \otimes \mathbb{C} = 0,$$

so that $h^1(X_0) = 0$. Also we have

$$H^0(X_0, \Omega^2_{X_0}) = \mathbb{C} \otimes H^0(S, \Omega^2_S)^\sigma = 0,$$

so that $h^{2,0}(X_0) = 0$. Hence X_0 is indeed a (singular) Calabi–Yau threefold. Now we compute $H^{1,1}(X_0) = H^2(X_0)$.

$$\begin{aligned} H^{1,1}(X_0) &= H^{0,0}(E)^\iota \otimes H^{1,1}(S)^{\sigma=1} \oplus H^{1,1}(E)^\iota \otimes H^{0,0}(S)^\sigma \\ &= \mathbb{C} \otimes H^{1,1}(S)^{\sigma=1} \oplus \mathbb{C} \end{aligned}$$

so that $h^{1,1}(X_0) = 1 + r$. Now note that

$$H^{3,0}(X_0) = (H^{1,0}(E) \otimes H^{2,0}(S)^{\iota\times\sigma}),$$

so that $h^{3,0}(X_0) = 1$. By the Künneth formula, we get

$$H^{2,1}(X_0) = (H^{1,0}(E) \otimes H^{1,1}(S) \oplus H^{0,1}(E) \otimes H^{2,0}(S))^{\iota\times\sigma}.$$

Let $H^{1,1}(S,\mathbb{C})^{\sigma=-1}$ denote the -1 eigenspace for the action of σ on $H^{1,1}(S)$. Since ι induces -1 on $H^1(E)$ and σ acts by -1 on $H^{2,0}(S)$, this gives that

$$H^{2,1}(X) = \mathbb{C}\otimes H^{1,1}(S)^{\sigma=-1}\oplus\mathbb{C}$$

and hence we have $h^{2,1}(X_0) = 1 + (20 - r)$. □

Now we pass onto a smooth resolution X of X_0. We need to calculate the cohomology groups of X, in particular, the twisted sectors.

Lemma 5.14. *Let $X = \widetilde{E\times S/\iota\times\sigma}$ be a smooth Calabi–Yau threefold over $\mathbb{C}$ of Borcea–Voisin type. Let S^σ be the fixed locus of S of σ. Then the twisted sectors consist of 4 copies of S^σ, and*

$$h^{0,0}(S^\sigma) = k + 1,\ h^{1,0}(S^\sigma) = g.$$

Therefore,

$$h^{1,0}(X) = h^{2,0}(X) = 0,\ h^{3,0}(X) = 1,$$
$$h^{1,1}(X) = 1 + r + 4(k + 1),\ h^{2,1}(X) = 1 + (20 - r) + 4g.$$

Proof. The Hodge numbers $h^{1,0}, h^{2,0}$ and $h^{3,0}$ of X are the same as those for the singular X_0. For $h^{1,1}(X)$ and $h^{2,1}(X)$ we need to bring in resolutions of singularities. The twisted sectors consist of 4 copies of S^σ. We know that $S^\sigma = C_g\cup L_1\cup\cdots\cup L_k$ for $(r,a,\delta)\neq(10,10,0),(10,8,0)$ and $S^\sigma = C_1\cup\tilde{C}_1$ for $(r,a,\delta) = (10,8,0)$. Therefore, $h^{1,1}(X) = 1 + r + 4(k+1)$. For $h^{2,1}(X)$, the contribution from the twisted sectors is the four copies of $\mathbb{P}^1\times C_g$, and hence $h^{2,1}(X) = 1 + (20 - r) + 4g$. □

Remark 5.3. Voisin [41] gave more geometrical computations for the Hodge numbers $h^{1,1}(X)$ and $h^{2,1}(X)$. We will recall briefly her calculations. Recall that the fixed locus of ι on E consists of four points $\{P_i,\ i = 1,\ldots,4\}$, and that the fixed locus of S under the action of σ is $S^\sigma = C_g\cup L_1\cup\cdots\cup L_k$ where C_g is a genus g curve and $L_i\,(i = 1,\ldots,k)$ are rational curves. Let N be the number of components in S^σ, that is, $N = k+1$, and let N' be the sum of genera of the components, that is, $N' = g$. The fixed point locus of the action $\iota\times\sigma$ on $E\times S$ consists of $4N$ curves $\{P_i\}\times C_g$, $\{P_i\}\times L_j$. We blow up $E\times S$ along these $4N$ curves to obtain a smooth Calabi–Yau threefold X with exceptional divisors arising from the $4N$ curves.

Now compute $h^{1,1}(X)$. First the exceptional divisors give $4N$ classes in $H^{1,1}(X)$. From the quotient surface S/σ we get $h^0 = 1 = h^4$, $h^1 = h^3 =$

0, $h^{2,0} = h^{0,2} = 0$ and $h^{1,1} = 10 + N - N'$. Then by the Künneth formula,

$$H^{1,1}(X) = \mathbb{C} - \text{span of } 4N \text{ exceptional divisors} \oplus H^{1,1}(S/\sigma) \oplus H^{1,1}(E).$$

Hence

$$h^{1,1}(X) = 4N + 10 + N - N' + 1 = 11 - 5N - N'.$$

For $h^{2,1}(X)$, we first note that the $4N$ curves give rise to the classes $H^{1,0}(C_g) \oplus_{j=1}^{k} H^{1,0}(L_j)$ in $H^{2,1}(X)$. Next, let $H^2(S)^-$ denote the -1 eigenspace for the action of σ on $H^2(S)$. Then again by the Künneth formula, we obtain

$$H^{2,1}(X) \simeq H^{1,0}(C_g) \oplus_{j=1}^{k} H^{1,0}(L_j) \oplus H^{1,1}(S)^{-1} \oplus H^{2,0}(S)$$

and this shows that

$$h^{2,1}(X) = 4N' + h^{1,1}(S)^{-} + 1 = 4N' + 10 + N' - N + 1 = 11 + 5N' - N.$$

We now compute the L-series of our Calabi–Yau threefolds of Borcea–Voisin type.

Theorem 5.15. *Let X be a Calabi–Yau threefold of Borcea–Voisin type, $X = E \times S/\iota \times \sigma$. The Betti numbers of X are given by*

$$B_0(X) = 1, \quad B_1(X) = 1, \quad B_2(X) = h^{1,1}(X) = 1 + r + 4(k+1),$$
$$B_3(X) = 2(1 + h^{2,1}(X)) = 2(1 + (20 - r) + 4g).$$

The ℓ-adic étale cohomological L-series $L_i(X, s)$ $(0 \le i \le 6)$ can be computed as follows:

- $L_0(X, s) = \zeta(\mathbb{Q}, s)$.
- $L_1(X, s) = 1$.
- $L_2(X, s) = \zeta(\mathbb{Q}, s-1)^{h^{1,1}(X)}$ *provided that all algebraic cycles in $NS(S)^\sigma$ are defined over $\mathbb{Q}$. Otherwise, let $t < r$ be the number of algebraic cycles in $NS(S)^\sigma$ that are defined over $\mathbb{Q}$ and let F be the smallest field of definition for all $\rho(\bar{S}) - t$ algebraic cycles in $NS(\bar{S})^\sigma \setminus NS(S)^\sigma$. Also suppose that all 4 points in E^ι are defined over $\mathbb{Q}$. Then $L_2(X, s) = \zeta(\mathbb{Q}, s-1)^{1+t+4(k+1)} L(\rho', s)$, where ρ' is an irreducible representation of dimension $r - t$ and $L(\rho', s)$ is its Artin L-function.*

  (Without knowing the field of definitions of algebraic cycles and four fixed points in E^ι explicitly, it is very difficult to write down an explicit formula for the L-function $L_2(X, s)$.)

- $L_3(X,s) = L(E\otimes\chi, s)L(E\otimes\rho, s)L(J(C_g), s-1)^4$.

The higher cohomologies are determined by Poincaré duality.

Proof. For the calculation of L-series, we ought to pass onto étale cohomology groups. Obivously,

$$L_0(X,s) = \zeta(\mathbb{Q}, s), \quad \text{and} \quad L_1(X,s) = 1.$$

For $L_2(X,s)$, note that

$$h^{1,1}(X) = 1 + r + 4(k+1).$$

So if all the r algebraic cycles in $NS(\bar{S})^\sigma$ are defined over $\mathbb{Q}$, then we have

$$L_2(X,s) = L_2(H^2(X,\mathbb{Q}_\ell), s) = \zeta(\mathbb{Q}, s-1)^{h^{1,1}(X)}.$$

Otherwise, t algebraic cycles are defined over $\mathbb{Q}$ so the Galois group acts trivially on $1+t+4(k+1)$ algebraic cycles so that the exponent is $1+t+4(k+1)$. But the Galois group acts non-trivially on the $r-t$-dimensional subspace of algebraic cycles and this gives rise to an irreducible representation of dimension $r-t$ and the Artin L-function. Therefore,

$$L_2(X,s) = \zeta(\mathbb{Q}, s-1)^{1+t+4(k+1)}L(\rho', s).$$

Finally for $L_3(X,s)$, note that

$$H^3(X,\mathbb{Q}_\ell) = H^1(E,\mathbb{Q}_\ell)\otimes H^2(S,\mathbb{Q}_\ell).$$

Under the action of ι, $H^1(E,\mathbb{Q}_\ell)$ is the direct sum of two eigenspaces:

$$H^1(E,\mathbb{Q}_\ell) = H^1(E,\mathbb{Q}_\ell)^{\iota=1}\oplus H^1(E,\mathbb{Q}_\ell)^{\iota=-1} = H^1(E,\mathbb{Q}_\ell)^{\iota=-1}.$$

Similarly, under the action of σ, $H^2(S,\mathbb{Q}_\ell)$ is the direct sum of two eigenspaces:

$$\begin{aligned} H^2(S,\mathbb{Q}_\ell) &= H^2(S,\mathbb{Q}_\ell)^{\sigma=1}\oplus H^2(S,\mathbb{Q}_\ell)^{\sigma=-1} \\ &= (H^{1,1}(\bar{S})^{\sigma=1}\otimes\mathbb{Q}_\ell)\oplus(H^{1,1}(\bar{S})^{\sigma=-1}\otimes\mathbb{Q}_\ell) \\ &= (NS(S)^{\sigma=1}\otimes\mathbb{Q}_\ell)\oplus(T(S)^{\sigma=-1}\otimes\mathbb{Q}_\ell). \end{aligned}$$

For the twisted sector, singularities occur along S^σ and for each singularity, its smooth resolution is the sum of four copies of the ruled surface $\mathbb{P}^1\times S^\sigma$.

So we have

$$\begin{aligned}
L_3(X,s) &= L(H^3(X,\mathbb{Q}_\ell),s)\\
&= L((H^1(E,\mathbb{Q}_\ell)\otimes H^2(S,\mathbb{Q}_\ell))^{\iota\times\sigma},s)\times L(H^3(\mathbb{P}^1\otimes J(C_g),\mathbb{Q}_\ell),s)^4\\
&= L((H^1(E,\mathbb{Q}_\ell)^{\iota=-1}\otimes H^{1,1}(S)^{\sigma=-1})\otimes\mathbb{Q}_\ell,s)\\
&\quad\times L((H^1(E,\mathbb{Q}_\ell)^{\iota=1}\otimes H^{1,1}(S)^{\sigma=1})\otimes\mathbb{Q}_\ell,s)\\
&\quad\times L(H^3(\mathbb{P}^1\otimes J(C_g),\mathbb{Q}_\ell),s)^4\\
&= L(\rho_E\otimes\chi,s)L(\rho_E\otimes\rho,s)L(J(C_g),s-1)^4.
\end{aligned}$$

□

5.6. Proof of automorphy of Calabi–Yau threefolds of Borcea–Voisin type

Finally, we can give proofs for Theorems 5.7 to 5.9 on the automorphy of Calabi–Yau threefolds over $\mathbb{Q}$ of Borcea–Voisin type.

Definition 5.3. (a) We will denote the orthogonal complement of $NS(S)^\sigma$ in $H^{1,1}(S)^{\sigma=-1}$ by $T(S)^{\sigma=-1}$. Its ℓ-adic realization $T(S)^{\sigma=-1}\otimes\mathbb{Q}_\ell\subset H^2(S,\mathbb{Q}_\ell)$ is called the $K3$ *motive* and denoted by $\mathcal{M}_S$. This is the unique motive with $h^{0,2}(\mathcal{M}_S)=1$.

(Note that $H^{1,1}(S)^{\sigma=1}$ gives rise to motives $\mathcal{M}_A$, which are all algebraic in the sense that $h^{0,2}(\mathcal{M}_A)=0$.)

(b) We will call the submotive $H^1(E,\mathbb{Q}_\ell)^{\iota=-1}\otimes(T(S)^{\sigma=-1}\otimes\mathbb{Q}_\ell)$ of $H^3(X,\mathbb{Q}_\ell)$ the *Calabi–Yau motive* of X, and denote by $\mathcal{M}_X$.

Proof. Here we will give proof for Theorem 5.8.

- X is of CM type by Theorem 4.5.
- $L_2(X,s)$ is automorphic by Proposition 5.14.
  For $L_3(X,s)$, we need to show that the L-series associated to the exceptional divisor arising from the singular loci $\{P_i\}\times C_g$ $(i=1,2,3,4)$ is automorphic. The exceptional divisor is given by the 4 copies of the ruled surface $\mathbb{P}^1\times C_g$. Now C_g is a component of S^σ where S is a finite quotient of a Fermat or diagonal surface, hence C_g is again expressed in terms of a diagonal or quasi-diagonal curve in weighted projective 2-space (see Corollaries 2.14 and 3.5). Hence, the Jacobian variety $J(C_g)$ of C_g is also of CM type, and hence $L(\mathbb{P}^1\otimes J(C_g),s)=L(J(C_g),s-1)$ is automorphic.

- $\zeta(X,s)$ is automorphic, as the factors $L_i(X,s)$ $(0 \le i \le 6)$ are all automorphic.

□

Proof. Now we will prove Theorem 5.9. Here E can be any elliptic curve, but S is of CM type. The resulting Calabi–Yau threefolds are not necessarily of CM type.

- An elliptic curve factor E is modular by the results of Wiles *et al.* So there is an automorphic representation ρ_E of dimension 2 associated to $H^1(E,\mathbb{Q}_\ell)$.
- The assertion of (b) is proved in Theorem 5.6.
- We know that the $G_{\mathbb{K}}$-Galois representation $\rho_{G_{\mathbb{K}}}$ on $T(S)$ is a direct sum of one-dimensional representations (coming from Jacobi sums), which the Galois group $\mathrm{Gal}(\mathbb{K}/\mathbb{Q})$ permutes transitively. This induces an m-dimensional irreducible Galois representation $\mathrm{Ind}_{G_{\mathbb{K}}}^{G_{\mathbb{Q}}}\rho_{G_{\mathbb{K}}}$. Hence we obtain the $2m$-dimensional Galois representation $\pi := \rho_E \otimes \mathrm{Ind}_{G_{\mathbb{K}}}^{G_{\mathbb{Q}}}$ $\rho_{G_{\mathbb{K}}}$, which is isomorphic to $\mathrm{Ind}_{G_{\mathbb{K}}}^{G_{\mathbb{Q}}}(\rho_{G_{\mathbb{K}}} \otimes \mathrm{res}_{G_{\mathbb{K}}}^{G_{\mathbb{Q}}}\rho_E)$. Hence

$$L(\pi,s) = L(\rho_E \otimes \mathrm{Ind}_{G_{\mathbb{K}}}^{G_{\mathbb{Q}}}\rho_{G_{\mathbb{K}}}, s) = L(f_E \otimes f_{T(S)}, s).$$

  In terms of the local p-factors, the above L-series is given as follows.

  The Euler p-factor of the L-series can be written as follows, for good prime p. Let $L_{E,p}(s)$ be the p-factor of $L(E,s)$. Then

$$L_{E,p}(s) = (1-\alpha_1 p^{-s})(1-\alpha_2 p^{-s}),$$

  where α_1, α_2 are conjugate algebraic integers with complex absolute value $p^{1/2}$.

  Let $L_{T(S),p}(s)$ be the p-factor of $L(T(S),s)$. Then

$$L_{T(S),p}(s) = \prod_{i=1}^{t}(1-\beta_i p^{-s}),$$

  where β_i are algebraic integers with complex absolute value p such that β_i/p is not a root of unity.

  Now for $X = E \times \widetilde{S/\iota} \times \sigma$, let $L_{\pi,p}(s)$ be the p-factor of the $L(\pi,s)$. Then

$$L_{\pi,p}(s) = \prod_{i=1}^{t}(1-\alpha_1\beta_i p^{-s})(1-\alpha_2\beta_i p^{-s}).$$

- The L-series $L(E \otimes \chi, s) = L(\rho_E \otimes \chi, s)$ is automorphic, as E (or ρ_E) is automorphic, and χ is automorphic since it is induced by a GL_1-representation of some cyclotomic field over $\mathbb{Q}$.

  Similarly, the L-series $L((E \otimes H^{1,1}(S)^{\sigma=1}) \otimes \mathbb{Q}_\ell, s)$ is automorphic as E is automorphic, and the representation on $H^{1,1}(S)^{\sigma=1} \otimes \mathbb{Q}_\ell$ is also induced by a GL_1-representation of some cyclotomic field over $\mathbb{Q}$.
- The L-series of $\mathbb{P}^1 \times J(C_g)$ is automorphic as that of $J(C_g)$ is automorphic.
- The L-series $L_3(X, s)$ is automorphic as each component is automorphic. □

We will give a representation theoretic proof (involving base change and automorphic induction) for the automorphy results of our Calabi–Yau threefolds of Borcea–Voisin type in the appendix.

Remark 5.4. In motivic formulation, our automorphy results for $K3$ surfaces and our Calabi–Yau threefolds may be reformulated as follows: *The L-series of the $K3$-motive is automorphic, and the L-series of the Calabi–Yau motive is automorphic.*

For our $K3$ surfaces S over $\mathbb{Q}$, the $K3$ motive $T(S)^\sigma \otimes \mathbb{Q}_\ell$ is a submotive of $H^2(S, \mathbb{Q}_\ell)$, and the L-series $L_2(S, s)$ factors as

$$L_2(S, s) = L(NS(S)^\sigma \otimes \mathbb{Q}_\ell, s) L(T(S)^\sigma \otimes \mathbb{Q}_\ell, s).$$

The automorphy of $L_2(S, s)$ then boils down to the automorphy of each L-factor. But we know that both factors are automorphic by Theorem 5.6 and its corollaries.

For Calabi–Yau threefolds X of Borcea–Voisin type, the Calabi–Yau motive $H^1(E, \mathbb{Q}_\ell) \otimes (T(S)^\sigma \otimes \mathbb{Q}_\ell)$ is a submotive of $H^3(X, \mathbb{Q}_\ell)$, which appears as a factor of $L_3(X, s)$. The automorphy of $L_3(X, s)$ again boils down to the automorphy of the L-series of the Calabi–Yau motive. Indeed, the other factors of $L_3(X, s)$ are expressed in terms of the L-series of the tensor product of ρ_E and the motives $\mathcal{M}_A$ of $K3$ surfaces with $h^{2,0}(\mathcal{M}_A) = 0$. These motives are all automorphic as they are induced from GL_1-representations of some cyclotomic fields over $\mathbb{Q}$.

6. Mirror symmetry for Calabi–Yau threefolds of Borcea–Voisin type

6.1. Mirror symmetry for $K3$ surfaces

There are several versions of mirror symmetry for $K3$ surfaces:

- Arnold's strange duality. This version is discussed by Dolgachev, Arnold and by others in relation to singularity theory. It is formulated for lattice polarized $K3$ surfaces as follows: a pair of lattice polarized $K3$ surfaces $(S, S^\vee)$ is said to be a *mirror* pair if

  $$\mathrm{Pic}\,(S)^{\perp}_{H^2(S,\mathbb{Z})} = U \oplus \mathrm{Pic}\,(S^\vee)$$

  as lattices. In terms of the Picard numbers,

  $$22 - \rho(S) = 2 + \rho(S^\vee) \quad \Leftrightarrow \quad \rho(S^\vee) = 20 - \rho(S).$$

  (See, for instance, Dolgachev [13].)

- Berglund–Hübsch–Krawitz mirror symmetry (Berglund and Hübsch [4] and Krawitz [22]). This version of mirror symmetry is for finite quotients of hypersurfaces in weighted projective 3-spaces. Mirror symmetry for these $K3$ surfaces are addressed in the articles by Artebani–Boissière and Sarti [2] and Comparin *et al.* [10]. It is formulated as follows: let W be a quasihomogeneous invertible polynomial together with a group G of diagonal automorphisms. (Here an "invertible" polynomial means that it has the same number of monomials as variables. Thus their zero loci define Delsarte surfaces.) Let Y_W be the hypersurface $\{W = 0\}$ in a weighted projective 3-space, then the orbifold Y_W/G defines a $K3$ surface. Now define the polynomial W^T by transposing the exponent matrix of W. Then W^T is again invertible and let G^T be the dual group of G. Then the orbifold Y_{W^T}/G^T is again a $K3$ surface. The Berglund–Hübsch–Krawitz mirror symmetry is that Y_W/G and Y_{W^T}/G^T form a mirror pair of $K3$ surfaces.

These two versions of mirror symmetry for $K3$ surfaces are shown to coincide for certain $K3$ surfaces in [10].

Now we consider mirror symmetry for pairs (S, σ) of $K3$ surfaces with involution σ classified by Nikulin in terms of triplets (r, a, δ). Let (S, σ) be a pair of a $K3$ surface with involution σ corresponding to a triplet (r, a, δ). Then the mirror pair $(S^\vee, \sigma^\vee)$ corresponds to the triplet $(20 - r, a, \delta)$.

For the Nikulin pyramid given in Section 2, the mirror is placed at the vertical line $r = 10$, corresponding to the symmetry $(r, a, \delta) \leftrightarrow (20 - r, a, \delta)$. It should be remarked that mirrors do not exist for the points located at the utmost right outerlayer of the pyramid, (the so-called the "pale region"), that is, (r, a, δ) is one of the following triplets $(20, 2, 1)$, $(19, 3, 1)$, $(18, 4, 1)$, $(18, 4, 0)$, $(17, 5, 1)$, $(16, 6, 1)$, $(15, 7, 1)$, $(14, 8, 1)$, $(13, 9, 1)$ and $(12, 10, 1)$. However, one particular triplet $(14, 6, 0)$ is not in this region, but does not have a mirror partner.

6.2. Mirror symmetry of Calabi–Yau threefolds of Borcea–Voisin type

Now we consider our Calabi–Yau threefolds of Borcea–Voisin type obtained as crepant resolutions of quotient threefolds $E \times S/\iota \times \sigma$. Mirror symmetry for these Calabi–Yau threefolds has been discussed by Voisin [41] and also by Borcea [6].

Theorem 6.1. *Given a Calabi–Yau threefold of Borcea–Voisin type* $X = X(r, a, \delta) = \widetilde{E \times S/\iota \times \sigma}$, *there is a mirror family of Calabi–Yau threefolds* $X^\vee = X(20 - r, a, \delta) = \widetilde{E \times S^\vee/\iota \times \sigma^\vee}$ *such that*

$$e(X^\vee) = -e(X).$$

Mirror symmetry for Calabi–Yau threefolds X is purely determined by mirror symmetry for the $K3$ components S.

Borcea's formulation of mirror symmetry is:

$$h^{1,1}(X^\vee) = 5 + 3(20 - r) - 2a = 65 - 3r - 2a = h^{2,1}(X),$$
$$h^{2,1}(X^\vee) = 65 - 3(20 - r) - 2a = 5 + 3r - 2a = h^{1,1}(X)$$

and

$$e(X^\vee) = -12(r - 10) = -e(X).$$

That is, mirror symmetry interchanges r by $20 - r$.

Voisin's formulation of mirror symmetry is given as follows: recall that the fixed part S^σ of S under σ ia a disjoint union of a genus-g curve and k

rational curves on S. Put

$$N := 1 + k = \textit{the number of components of } S^\sigma,$$

and

$$N' := \textit{the sum of genera of components of } S^\sigma.$$

Then

$$h^{1,1}(X) = 11 + 5N - N',$$
$$h^{2,1}(X) = 11 + 5N' - N$$

and

$$e(X) = 12(N - N').$$

Mirror symmetry interchanges N and N'.

$$h^{1,1}(X^\vee) = 11 + 5N' - N,$$
$$h^{2,1}(X^\vee) = 11 + 5N - N'$$

and

$$e(X^\vee) = 12(N' - N) = -e(X).$$

Remark 6.1. The mirror symmetry in the above theorem is merely a numerical check for the topological mirror symmetry that the Hodge numbers of $X(r, a, \delta)$ and $X(20 - r, a, \delta)$ are indeed "mirrored". Mirror symmetry for $X = X(r, a, \delta)$ indeed comes from the mirror symmetry of the $K3$ surface component. Also the mirror of $X(r, a, \delta)$ occurs in a family, so mirror symmetry does not relate one Calabi–Yau threefold to another Calabi–Yau threefold, rather mirror symmetry deals with families.

Remark 6.2. For the Calabi–Yau threefolds corresponding to the 11 $K3$ surfaces corresponding to the triplets (r, a, δ) located at the utmost right outerlayer of the Nikulin's pyramid (called the "pale region" by Borcea) plus the triplet $(14, 6, 0)$, mirror partners do not exist.

Rohde [33] and Garbagnati–van Geemen [15] considered those Calabi–Yau threefolds of Borcea–Voisin type whose $K3$ surface components have only rational curves in their fixed loci by non-symplectic involution (i.e., no curves with higher genera). A reason for not having mirror partners is the non-existence of boundary points in the complex structure moduli space of the Calabi–Yau threefold where the variation of Hodge structures on H^3 has maximal unipotent monodromy, and hence there is no way of defining mirror maps.

6.3. Mirror pairs of $K3$ surfaces

Now we consider the 95 $K3$ surfaces in the list of Reid and Yonemura. Belcastro [3] determined the Picard lattices for these 95 $K3$ surfaces, and showed that the set of these 95 $K3$ surfaces are not closed under mirror symmetry.

We can fish out those $K3$ surfaces with involution σ which are closed under mirror symmetry.

Lemma 6.2 (Belcastro [3]). *The set of the 95 $K3$ surfaces of Reid and Yonemura is not closed under mirror symmetry. Among them, the 57 $K3$ surfaces have mirror partners within the list.*

Lemma 6.3. *All 57 $K3$ surfaces S have non-symplectic involutions σ acting as -1 on $H^{2,0}(S)$, and their mirror partners $S^\vee$ also have non-symplectic involutions $\sigma^\vee$ acting as -1 on $H^{2,0}(S^\vee)$.*

Proof. We tabulate the 57 $K3$ surfaces with involutions, and their mirror partners in tables B.10 and B.10. □

6.4. Examples of mirror pairs of Calabi–Yau threefolds of Borcea–Voisin type

Example 6.4. Let $E = E_2$ be the elliptic curve with involution ι as in Section 4.3, and let S_0 be the $K3$ surface, #14 in Yonemura and #26 in Borcea, given by

$$S_0 : x_0^2 = x_1^3 + x_2^7 + x_3^{42} \subset \mathbb{P}^3(21, 14, 6, 1)$$

of degree 42 and involution $\sigma(x_0) = -x_0$. Let S be the minimal resolution of S_0. S has Nikulin's triplet $(10, 0, 0)$. Thus, S is its own mirror. Recall from Example 6.4 that the fixed locus S^σ is $S^\sigma = C_6 \cup L_1 \cup \cdots \cup L_5$. Also S is of CM type. This is because S is dominated by the Fermat surface of degree 42. Hence the field $\mathbb{K}$ corresponding to the transcendental cycles $T(S)$ of S is the cyclotomic field $\mathbb{Q}(\zeta_{42})$ with $[\mathbb{K} : \mathbb{Q}] = \varphi(42) = 12$. (Here ζ_{42} is a primitive 42th root of unity and φ is the Euler phi-function.) Note that $10 = 22 - 12 =$

r, so that $NS(S) \cong NS(S)^\sigma$. (Or equivalently, $12 = 22 - r = 22 - 10$ so that $T(S) \cong T(S)^\sigma$.) The $K3$-motive is automorphic and hence S is automorphic by Theorem 4.5.

The Calabi–Yau threefold $X = E_2 \widetilde{\times S/\iota \times} \sigma$ has a birational model defined over $\mathbb{Q}$

$$X : z_0^4 + z_1^4 = z_2^3 + z_3^7 + z_4^{42} \subset \mathbb{P}^4(21, 21, 28, 12, 2)$$

of degree 84. Since E and S are both of CM type, so is X. The Hodge numbers and the Euler characteristic are

$$h^{1,1}(X) = 35 = 1 + 10 + 4(1 + 5),\ h^{2,1}(X) = 35 = 1 + 10 + 4 \cdot 6,\ e(X) = 0,$$

so that X is its own topological mirror.

Obviously $L_i(X, s)$ and $L_{6-i}(X, s)$ for $i = 0, 1, 2$ are all automorphic. To show the automorphy of $L_3(X, s)$, we have only to show the automorphy of the L-series corresponding to the Calabi–Yau motive $H^1(E, \mathbb{Q}_\ell) \otimes T(S)^\sigma \otimes \mathbb{Q}_\ell$. The Galois representation associated to the Calabi–Yau motive has dimension 24, and is given by the tensor product of the two-dimensional Galois representation associated to $H^1(E, \mathbb{Q}_\ell)$ and the 12-dimensional irreducible Galois representation associated to $T(S)^\sigma \otimes \mathbb{Q}_\ell$ induced from a Jacobi sum Grossencharacter of $\mathbb{K} = \mathbb{Q}(\zeta_{42})$. Hence it is automorphic. We repeat the argument in motivic formulation. The Calabi–Yau motive $\mathcal{M}_X$ has dimension $\varphi(84) = 24$, and the Jacobi sum Grossencharacter of $K = \mathbb{Q}(\zeta_{84})$ gives GL_1-representations and its automorphic induction gives rise to the GL_{24} irreducible representation for $\mathcal{M}_X$ over $\mathbb{Q}$, and hence it is modular (automorphic). (Compare the Calabi–Yau motive $\mathcal{M}_X$ with the Ω-motive constructed by Schimmrigk in [34].)

Example 6.5. Let $E = E_2$ be the elliptic curve with involution ι as in Section 4.3 and let S_0 be the $K3$ surface, #40 in Yonemura and = #5 in Borcea, given by

$$S_0 : x_0^2 = x_1^3 x_2 + x_1^3 x_3^2 + x_2^7 - x_3^{14} \subset \mathbb{P}^2(7, 4, 2, 1)$$

of degree 14 with involution $\sigma(x_0) = -x_0$. Its minimal resolution S has Nikulin's triplet $(7, 3, 0)$. By Theorem 2.5, we may remove the monomial $x_1^3 x_2^2$ from the defining equation, we get

$$S_0 : x_0^2 = x_1^3 x_2 + x_2^7 - x_3^{14}$$

making S_0 of CM type. This is a weighted hypersurface of degree 14, and $\mathrm{lcm}(3,2,14) = 42$, and hence S_0 is dominated by the Fermat surface of degree 42 (cf. [16], Corollary 8.1). The field $\mathbb{K}$ corresponding to $T(S)$ is the cyclotomic field $\mathbb{Q}(\zeta_{42})$ of degree $\varphi(42) = 12$, and we obtain the induced Galois representation of dimension 12. Thus, the $K3$-motive is automorphic, and hence S is automorphic.

In this case, $r = 7 \neq 10 = 22 - 12$ so $NS(S)^\sigma \ncong NS(S)$. (Or equivalently, $22 - r = 22 - 7 = 15 \neq 12 = \varphi(42)$ so $T(S)^\sigma \ncong T(S)$.)

The Calabi–Yau threefold X has a birational model defined over $\mathbb{Q}$:

$$X : z_0^4 + z_1^4 = z_2^3 z_3 + z_3^7 - z_4^{14} \subset \mathbb{P}^4(7,7,8,4,2)$$

of degree 28, and $\mathrm{lcm}(4,3,14) = 84$. (See [16], Theorem 9.2.) Since E and S are of CM type, so is X. The Hodge numbers and the Euler characteristic are

$$\begin{aligned} h^{1,1}(X) &= 20 = 1 + 7 + 4(2+1), \\ h^{2,1}(X) &= 38 = 1 + (20 - 7) + 4 \cdot 6, \\ e(X) &= -36. \end{aligned}$$

Now we apply Theorem 5.9. We pass from $\mathbb{Q}(\zeta_{42})$ to $\mathbb{Q}(\zeta_{84})$ to take $H^1(E_2)$ into account. The Calabi–Yau motive $\mathcal{M}_X$ has dimension $24 = \varphi(84)$. Indeed, the Jacobi sum Grossencharacter of $\mathbb{K} = \mathbb{Q}(\zeta_{84})$ gives rise to the GL_{24} irreducible automorphic cuspidal representation for the Calabi–Yau motive $\mathcal{M}_X$ over $\mathbb{Q}$. Hence the Calabi–Yau motive $\mathcal{M}_X$ is automorphic. Hence $L_3(X,s)$ is automorphic, and consequently X is automorphic.

To find a mirror family of Calabi–Yau threefolds, we first look for a mirror $S^\vee$ of $K3$ surface S. We may take for $S^\vee$ the K3 surface #47 in Yonemura =#24 in Borcea. $S^\vee$ is a $K3$ surface defined by

$$S^\vee : x_0^2 = x_1^3 + x_1 x_2^7 + x_2^9 x_3^2 + x_3^{14} \subset \mathbb{P}^3(21,14,4,3)$$

of degree 42. It has a non-symplectic involution $\sigma^\vee$ that sends x_0 to $-x_0$. The pair $(S^\vee, \sigma^\vee)$ corresponds to the triplet $(13,3,0)$. By Theorem 4.3, we may remove the monomial $x_2^9 x_3^2$ from the defining equation, which makes $S^\vee$ to be of CM type. So $S^\vee : x_0^2 = x_1^3 + x_1 x_2^7 + x_3^{14}$ is a weighted hypersurface of degree 28. Since $\mathrm{lcm}(2,7,4) = 28$, the field $\mathbb{K}$ corresponding to $T(S^\vee)$ is the cyclotomic field $\mathbb{Q}(\zeta_{28})$ of degree $\varphi(28) = 12$, and we obtain the induced Galois representation of dimension 12. Thus, the $K3$-motive is automorphic, and hence $S^\vee$ is automorphic. In this case, $22 - r = 22 - 13 = 9 \neq 12 = \varphi(28)$ so $T(S^\vee)^\sigma \ncong T(S^\vee)$. (Or equivalently, $r = 13 \neq 10 = 22 - 12$ so that $NS(S^\vee)^\sigma \ncong NS(S^\vee)$.)

A candidate for mirror family for X might be a deformation of $E_2 \times \widetilde{S^\vee/\iota \times \sigma^\vee}$. One member of this mirror family denoted by, $X^\vee$, may be chosen to have a birational model defined over $\mathbb{Q}$ by the following equation:

$$X^\vee : z_0^4 + z_1^4 = z_2^3 + z_2 z_3^7 + z_4^{14} \subset \mathbb{P}^4(21, 21, 28, 8, 6)$$

of degree 84. The Hodge numbers and the Euler characteristic of $X^\vee$ are

$$\begin{aligned} h^{1,1}(X^\vee) &= 38 = 1 + 13 + 4(5+1), \\ h^{2,1}(X^\vee) &= 20 = 1 + (20 - 13) + 4 \cdot 3, \\ e(X^\vee) &= 36. \end{aligned}$$

We pass from $\mathbb{Q}(\zeta_{28})$ to $\mathbb{Q}(\zeta_{56})$ to take $H^1(E_2)$ into account. Then the Calabi–Yau motive $\mathcal{M}_{X^\vee}$ has dimension $24 = \varphi(56)$. Again, by Theorem 5.9, the Jacobi sum Grossencharacter of $\mathbb{K} = \mathbb{Q}(\zeta_{28})$ gives rise to a GL_1 representations for $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{K})$ and its automorphic induction yields the GL_{24} irreducible cuspidal automorphic representation for the Calabi–Yau motive $\mathcal{M}_{X^\vee}$ over $\mathbb{Q}$. Hence the Calabi–Yau motive $\mathcal{M}_{X^\vee}$ is automorphic. Consequently, we conclude that $L_3(X^\vee, s)$ is automorphic, and hence the automorphy of $X^\vee$.

For these two examples, it happens that $L(\mathcal{M}_X, \rho, s) = L(\mathcal{M}_{X^\vee}, \rho^\vee, s)$. That is the L-series of the Calabi–Yau motives of X and $X^\vee$ coinside.

6.5. Automorphy and mirror symmetry for Calabi–Yau threefolds of Borcea–Voisin type

Mirror symmetry for Calabi–Yau threefolds is not the correspondence for one Calabi–Yau threefold to another, rather it is a correspondence between families. At the moment, we do not know how to compute the zeta-functions and L-series of a deformation family of mirror Calabi–Yau threefolds. So we will consider one particular member of this mirror family and compare the L-series of the Calabi–Yau motives.

Theorem 6.6. *Let (S, σ) be one of the* 57 *$K3$ surfaces in* Lemma 6.2 *with involution σ, which are closed under mirror symmetry. Then S is of CM type. Let $X = X(r, a, \delta)$ be a Calabi–Yau threefold corresponding to a triplet (r, a, δ) as in* Section 4.1, *so $X = E \times \widetilde{S/\iota \times \sigma}$. Then a mirror family of Calabi–Yau threefolds exists and corresponds to a triplet $(20 - r, a, \delta)$, and may be obtained as a deformation of a crepant resolution of the quotient*

$E \times S^\vee/\iota \times \sigma^\vee$, where $\sigma^\vee$ is a non-symplectic involution on $S^\vee$. Then a special member $X^\vee$ of this mirror family has the following properties:

(a) *$X^\vee$ has a model defined over $\mathbb{Q}$ provided that E is defined over $\mathbb{Q}$.*

(b) *$X^\vee$ is of CM type if and only if E is of CM type.*

(c) *If $X^\vee$ is of CM type, then $X^\vee$ is automorphic.*

Observation 1. Under the situation of the above theorem, we have

(d) If the $K3$ motives of S and $S^\vee$ are isomorphic (in the sense that they correspond to the same Jacobi sum Grossencharacter), then they have the same L-series. Furthermore, the Calabi–Yau motives of X and $X^\vee$ are invariant under mirror symmetry.

The two examples 6.4 and 6.5 are in support of this observation. It appears that when the original Calabi–Yau threefold and a member of its mirror Calabi–Yau threefolds are both of CM type and are realized as finite quotients of the same Fermat or quasi-diagonal hypersurface, then the Calabi–Yau motives are the same and hence are invariant under mirror symmetry.

6.6. Berglund–Hübsch–Krawitz mirror symmetry for Calabi–Yau threefolds

Here are other examples of Calabi–Yau threefolds of CM type due to Kelly [19]. For the computations of zeta-functions and L-series, we use the method developed in Goto *et al.* [16].

Consider the polynomials

$$F_A : x_0^8 + x_1^8 + x_2^4 + x_3^3 + x_4^6 = 0,$$
$$F_{A'} : x_0^8 + x_1^8 + x_2^4 + x_3^3 + x_3 x_4^4 = 0.$$

Both are hypersurfaces of degree 24 in the weighted projective 4-space $\mathbb{P}^4(3, 3, 6, 8, 4)$. Let $\zeta = \zeta_{24}$ be a primitive 24th root of unity. Both F_A and $F_{A'}$ are covered by the Fermat hypersurface of degree 24 (see Theorem 9.2 in [16]), and hence F_A and $F_{A'}$ are both of CM type.

Let $J_{F_A} = \mathrm{Aut}(F_A) \cap \mathbb{C}^*$. Then J_{F_A} is generated by $(\zeta^3, \zeta^3, \zeta^6, \zeta^8, \zeta^4) \in (\mathbb{C}^*)^5$. Define the group $SL(F_A) := \{ (\lambda_0, \lambda_1, \ldots, \lambda_4) \in \mathrm{Aut}(F_A) \mid \prod_{j=0}^4 \lambda_j = 1 \}$. Fix a group G so that $J_{F_A} \subseteq G \subseteq SL(F_A)$. Put $\tilde{G} := G/J_{F_A}$. Define $Z_{A,G} := X_{F_A}/\tilde{G}$. Then $Z_{A,G}$ is a Calabi–Yau threefold (orbifold).

For our F_A and F'_A, choose G and G' to be the same group given by

$$G = G' =< (\zeta^3, \zeta^3, \zeta^6, \zeta^8, \zeta^4), (\zeta^{18}, 1, \zeta^6, 1, 1), (1, 1, \zeta^{12}, 1, \zeta^{12}) > .$$

Then $Z_{A,G}$ and $Z_{A',G'}$ are Calabi–Yau threefolds which are in the same family of hypersurfaces in $\mathbb{P}^4(3, 3, 6, 8, 4)/\tilde{G}$. Since both are realized as finite quotients of the Fermat hypersurface of degree 24, both $Z_{A,G}$ and $Z_{A',G'}$ are of CM type.

The Hodge numbers are given by:

$$h^{1,1}(Z_{A,G}) = 7,\ h^{2,1}(Z_{A,G}) = 55$$

and

$$h^{1,1}(Z_{A',G'}) = 55,\ h^{2,1}(Z_{A',G'}) = 7.$$

Now recall the construction of the Berglund–Hübsch–Krawitz mirrors of these Calabi–Yau threefolds. Let

$$F_{A^T} = F_A : x_0^8 + x_1^8 + x_2^4 + x_3^3 + Y_4^6 = 0 \subset \mathbb{P}^4(3, 3, 6, 8, 4),$$
$$F_{(A')^T} : x_0^8 + x_1^8 + x_2^4 + x_2^3 x_3 + x_4^4 = 0 \subset \mathbb{P}^4(1, 1, 2, 2, 2).$$

Then F_{A^T} is a hypersurface of degree 24 in the weighted projective 4-space $\mathbb{P}^4(3, 3, 8, 6, 4)$ but $F_{(A')^T}$ is a hypersurface of degree 8 in the weighted projective 4-space $\mathbb{P}^4(1, 1, 2, 2, 2)$. The groups J_{F_A}, $J_{F_{(A')^T}}$, G^T, $(G')^T$ are computed:

$$J_{F_A} =< (\zeta^3, \zeta^3, \zeta^6, \zeta^8, \zeta^4) >;\ J_{F_{(A')^T}} =< (\zeta^3, \zeta^3, \zeta^6, \zeta^6, \zeta^6) >;$$
$$G^T = J_{F_A};\ (G')^T =< (\zeta^3, \zeta^3, \zeta^6, \zeta^6, \zeta^6), (1, 1, 1, \zeta^{12}, \zeta^{12}) > .$$

Then taking the quotients, we obtain Calabi–Yau orbifolds Z_{A^T,G^T} and $Z_{(A')^T,(G')^T}$ which are the topological mirrors of $Z_{A,T}$ and $Z_{A',G'}$, respectively.

$$h^{1,1}(Z_{A^T,G^T}) = 55,\ h^{2,1}(Z_{A^T,G^T}) = 7$$

and

$$h^{1,1}(Z_{(A')^T,(G')^T}) = 7,\ h^{2.1}(Z_{(A')^T,(G')^T}) = 55.$$

The Berglund–Hübsch–Krawitz mirror symmetry is that $Z_{A,G}$ and Z_{A^T,G^T} are mirror partners in the sense of interchanging Hodge numbers. Similarly, $Z_{A',G'}$ and $Z_{(A')^T,(G')^T}$ are mirror pairs. However, the latter two do not live in the same weighted projective 4-spaces.

Theorem 6.7 (Kelly [19]). *Let $Z_{A,G}$ and $Z_{A',G'}$ be the Calabi–Yau orbifolds constructed above. Let Z_{A^T,G^T} and $Z_{(A')^T,(G')^T}$ be Berglund–Hübsch–Krawitz mirrors, respectively. If $G = G'$, then Z_{A^T,G^T} and $Z_{(A')^T,(G')^T}$ are birational.*

Proposition 6.8. *Both $Z_{A,G}$ and $Z_{A',G'}$ are of CM type and hence automorphic. The mirrors Z_{A^T,G^T} and $Z_{(A')^T,(G')^T}$ are again of CM type and hence automorphic. The L-series of the Calabi–Yau motives of $Z_{A,G}$ and Z_{A^T,G^T} are invariant under the mirror symmetry. Similar assertions hold for $Z_{A',G'}$ and $Z_{(A')^T,(G')^T}$.*

Proof. We have only to show the last claim. Since the Calabi–Yau motives of Calabi–Yau threefolds $Z_{A,G}$ and Z_{A^T,G^T} come from the unique Fermat motive associated to the weight of the same Fermat hypersurface, the Calabi–Yau motive is invariant under the mirror symmetry. For $Z_{A',G'}$ and $Z_{(A')^T,(G')^T}$, they do not sit in the same family of hypersurfaces, but they are birational. The Calabi–Yau motives are left invariant under birational map. The L-series of the Calabi–Yau motives are invariant under mirror symmetry. For details about Fermat motives, see the Appendix section below or Goto *et al.* [16], and Kadir and Yui [20]. □

Remark 6.3. Rohde [33] (see Appendix A, page 209) constructed many examples of Calabi–Yau threefolds of CM type (CMCY threefolds), by Borcea–Voisin construction. The automorphy of his CMCY 3-folds should follow by studying Galois representations associated to them. This is left to the reader for exercise.

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Appendix A. Base change and automorphic induction, and Rankin–Selberg *L*-series of convolution

Appendix A.1. Base change and automorphic induction maps

For the proof of automorphy of our Calabi–Yau threefolds via representation theory, we need the three ingredients, (the existence of) base change and automorphic induction maps for solvable extensions over $\mathbb{Q}$, and the Rankin–Selberg L-series of convolution.

In this subsection, we will explain the result of Arthur and Clozel [1] on base change and automorphic induction proved for cyclic extensions of prime degree over $\mathbb{Q}$, and their generalization by Rajan [31] (see also Murty [M93]) to solvable extensions over $\mathbb{Q}$.

Definition A.1. Let k be a number field with the ring $\mathcal{O}_k$ of integers. Let K be a Galois extension of k with the ring of integers $\mathcal{O}_K$ and Galois group $G = \mathrm{Gal}(K/k)$. If ρ is an irreducible (finite-dimensional) representation of G, we can associate to it a Dirichlet series with Euler product, called the Artin L-series $L(s, \rho, K/k)$ as follows. Let v be a (finite) place of $\mathcal{O}_k$, p_v the associated prime ideal in $\mathcal{O}_k$, q_v the cardinality of the residue field $\mathcal{O}_k/p_v$, and Φ_v the conjugacy class of Frobenius elements attached to p_v, for v unramified in the extension K/k. Let $\mathfrak{S}$ be the finite set of (finite) places ramified in K/k. The Artin L-series is defined by

$$L(s, \rho, K/k) = \prod_{v \notin \mathfrak{S}} \frac{1}{\det(1_\rho - q_v^{-s} \rho(\Phi_v))}.$$

The definition of the Artin L-series can be extended to arbitrary representations of G by additivity:

$$L(s, \rho_1 \oplus \rho_2, K/k) = L(s, \rho_1, K/k) L(s, \rho_2, K/k).$$

Let $\mathbb{A}_k$ be the adele ring of k and $\mathfrak{A}(GL_n(\mathbb{A}_k))$ be the set of automorphic representations of $GL_n(\mathbb{A}_k)$ for some n.

The Langlands philosophy predicts that an Artin L-series should be equal to an L-series associated to some automorphic form (e.g., cusp form) on GL_n. More concretely, for each ρ, the Langlands reciprocity conjecture states that there exists an automorphic representation $\pi(\rho) \in \mathfrak{A}(GL_n(\mathbb{A}_K))$ ($n = \deg(\rho)$) such that

$$L(s, \rho, K/k) = L(s, \pi(\rho)).$$

We assert that the Artin L-functions of the Calabi–Yau threefolds of Borcea–Voisin type which are of CM type are indeed automorphic.

Now we need to introduce the notion of "base change" and "automorphic induction".

Lemma A.1. *Let H be a subgroup of G, and let K^H be the fixed subfield of K by H. Let ψ be an Artin representation of $Gal(K/K^H) = H$. Let $L(s, \psi, K/K^H)$ be the Artin L-series of the extension K/K^H. Then the Artin L-series is invariant under induction, that is, if Ind_G^H is the induced representation, then*

$$L(s, Ind_G^H \psi, K/K^H) = L(s, \psi, K/K^H).$$

When $L(s, \rho, K/k) = L(s, \pi(\rho))$, then $L(s, \rho|_H, K/K^H) = L(s, \rho \otimes \mathrm{Ind}_H^G \mathbf{1}, K/k)$. But $\mathrm{Ind}_H^G \mathbf{1} = \mathrm{reg}_H$ is nothing but the permutation representation on the cosets of H in G. Let $\pi \in \mathfrak{A}(GL_n(\mathbb{A}_k))$. For each unramified π_v, let $A_v \in GL_n(\mathbb{C})$ be a semi-simple conjugacy class defined by the representation π. If v is unramified in K, define

$$L_v(s, B(\pi)) = \det(1 - A_v \otimes \mathrm{reg}_H(\sigma_v) N v^{-s})^{-1}$$

where σ_v is the Artin symbol of v.

Conjecture A.2. *(a) (Base change) There exists a base change map*

$$B : \mathfrak{A}(GL_n(\mathbb{A}_K)) \to \mathfrak{A}(GL_n(\mathbb{A}(K^H)))$$

and the Artin L-series $L(s, B(\pi), K/K^H)$ such that its v-factor coincides with $L_v(s, B(\pi))$ defined above.

(b) *(Automorphic induction) Now let ψ be a representation of H. Then there exists an automorphic induction map*

$$I : \mathfrak{A}(GL_n(\mathbb{A}_{K^H})) \to \mathfrak{A}(GL_{nr}(\mathbb{A}_k))$$

such that for $I(\pi) \in \mathfrak{A}(GL_{nr}(\mathbb{A}_k))$,

$$L(s, I(\pi)) = L(s, Ind_H^G, K/k).$$

Here $n = deg(\psi)$, and $r = [G : H]$.

We now recall a theorem of Arthur and Clozel [1] on the existence of base change and automorphic induction maps for GL_n, when K/k is a cyclic extension of prime degree, and representations are automorphic cuspidal representations.

Theorem A.3 (Arthur–Clozel). *Suppose that K/k is a cyclic extension of prime degree ℓ. Let π and Π denote cuspidal unitary automorphic representations of $GL_n(\mathbb{A}_k)$ and $GL_n(\mathbb{A}_K)$, respectively. Then*

- *the base change lift of π, denoted by $B(\pi)$, exists, and it is an automorphic representation in $\mathfrak{A}(GL_n(\mathbb{A}_K))$,*

- *the automorphic induction $I(\Pi)$ of Π exists, and it is an automorphic representation in $\mathfrak{A}(GL_{n\ell}(\mathbb{A}_k))$.*

A.2. Rankin–Selberg L-series of convoluton

We can reformulate the Arthur–Clozel theorem in terms of the L-series. In this subsection, we will consider Rankin–Selberg L-series of convolution. These L-series are needed from the fact that the eigenvalues of the Frobenius morphism of our Calabi–Yau threefolds of Borcea–Voisin type are given by tensor products of eigenvalues of those of the components. We need to consider Rankin–Selberg L-series of convolution.

Let π and π' be two cuspidal, unitary automorphic representations of $GL_n(\mathbb{A}_k)$ and $GL_m(\mathbb{A}_k)$, respectively. Let $\mathfrak{S}$ be a finite set of primes of k such that π and π' are unramified outside $\mathfrak{S}$. Let $L(s, \pi \otimes \pi')$ be the Rankin–Selberg L-series of convolution. Then the result of Arthur and Clozel mentioned above is formulated in terms of the Rankin–Selberg L-series as follows:

Lemma A.4. *Let K/k be cyclic extension of prime degree, and suppose that $\pi \in \mathfrak{A}(GL_n(\mathbb{A}_k))$ and $\Pi \in \mathfrak{A}(GL_m(\mathbb{A}_K))$ are cuspidal unitary automorphic representations, respectively. Then the Rankin–Selberg L-series satisfies the formal identity:*

$$L(s, B(\pi) \otimes \Pi) = L(s, \pi \otimes I(\Pi)).$$

A.3. Generalizations of base change and automorphic induction to solvable extensions over $\mathbb{Q}$

Arthur and Clozel's results are proved for cyclic extensions of prime degree over $\mathbb{Q}$. For our application, we need base change and automorphic induction results for abelian extensions (e.g., cyclotomic fields) over $\mathbb{Q}$. In fact, the existence of base change and automorphic induction is established for solvable extensions over $\mathbb{Q}$ by Rajan [31], see also Murty [28].

A.4. Weighted Jacobi sums and Fermat motives

We recall now the definition of weighted Jacobi sums and weighted Fermat motives from Gouvêa and Yui [17].

We consider a weighted Fermat hypersurface of dimension $n+1$, degree m and a weight $\mathbf{w} = (w_0, w_1, \ldots, w_{n+1})$ defined by

$$x_0^{m_0} + x_1^{m_1} + \cdots + x_{n+1}^{m_{n+1}} = 0 \subset \mathbb{P}^n(\mathbf{w})$$

where $m_i w_i = m$ for every i, $0 \le i \le n+1$.

If $\mathbf{w} = (1, 1, \ldots, 1)$, this is nothing but the Fermat hypersurface of dimension $n+1$ and degree m.

Definition A.2. (a) Let $\mathbb{K} = \mathbb{Q}(\zeta_m)$ be the mth cyclotomic field over $\mathbb{Q}$, $\mathcal{O}_{\mathbb{K}}$ the ring of integers of $\mathbb{K}$. Let $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_{\mathbb{K}})$. For every $x \in \mathcal{O}_{\mathbb{K}}$ relatively prime to $\mathfrak{p}$, let $\chi_{\mathfrak{p}}(x \bmod \mathfrak{p}) = (\frac{x}{\mathfrak{p}})$ be the mth power residue symbol on $\mathbb{K}$. If $x \equiv 0 \pmod{\mathfrak{p}}$, we put $\chi_{\mathfrak{p}}(x \bmod \mathfrak{p}) = 0$. Let $(w_0, w_1, w_2, \ldots, w_{n+1})$ be a weight. Define the set

$$\begin{aligned} &\mathfrak{A}_d(w_0, w_1, \ldots, w_{n+1}) \\ &:= \left\{ \mathbf{a} = (a_0, a_1, \ldots, a_{n+1}) \,|\, a_i \in (w_i\mathbb{Z}/m\mathbb{Z}), a_i \neq 0, \sum_{i=0}^{n+1} a_i = 0 \right\}. \end{aligned}$$

For each $\mathbf{a} = (a_0, a_1, \ldots, a_{n+1}) \in \mathfrak{A}_d(w_0, w_1, \ldots, w_{n+1})$, the *weighted Jacobi sum* is defined by

$$j_{\mathfrak{p}}(\mathbf{a}) = j_{\mathfrak{p}}(a_0, a_1, \ldots, a_{n+1}) = (-1)^n \sum \chi_{\mathfrak{p}}(v_1)^{a_1} \chi_{\mathfrak{p}}(v_2)^{a_2} \cdots \chi_{\mathfrak{p}}(v_{n+1})^{a_{n+1}}$$

where the sum is taken over $(v_1, v_2, \ldots, v_{n+1}) \in (\mathcal{O}_{\mathbb{K}}/\mathfrak{p})^\times \times \cdots \times (\mathcal{O}_{\mathbb{K}}/\mathfrak{p})^\times$ subject to the linear relation $1 + v_1 + v_2 + \cdots + v_{n+1} \equiv 0 \pmod{\mathfrak{p}}$.

Weighted Jacobi sums are elements of $\mathcal{O}_{\mathbb{K}}$ with complex absolute value equal to $q^{n/2}$ where $q = |\operatorname{Norm} \mathfrak{p}| \equiv 1 \pmod{m}$.

(b) The Galois group $\operatorname{Gal}(\mathbb{K}/\mathbb{Q}) \simeq (\mathbb{Z}/m\mathbb{Z})^\times$ acts on weighted Jacobi sums, by multiplication by $t \in (\mathbb{Z}/m\mathbb{Z})^\times$ on each component of $\mathbf{a}$. Let A denote the $(\mathbb{Z}/m\mathbb{Z})^\times$-orbit of $\mathbf{a}$. For a weighted Jacobi sum $j_{\mathfrak{p}}(\mathbf{a})$, the $(\mathbb{Z}/m\mathbb{Z})^\times$-orbit of $j_{\mathfrak{p}}(\mathbf{a})$ is called the weighted *Fermat motive*, and denoted by $\mathcal{M}_A$.

To each $\mathbf{a} = (a_0, a_1, \ldots, a_{n+1}) \in \mathfrak{A}_d(w_0, w_1, \ldots, w_{n+1})$, define the *length* of $\mathbf{a}$ to be

$$\|\mathbf{a}\| := \left(\frac{1}{m} \sum_{i=0}^{n+1} a_i \right) - 1.$$

Via cohomological realizations of these motives, we can compute the numerical characters of $\mathcal{M}_A$.

• The ith Betti number is

$$B_i(\mathcal{M}_A) := \dim_{\mathbb{Q}_\ell} H^i(\overline{\mathcal{M}_A}, \mathbb{Q}_\ell) = \begin{cases} \#A & \text{if } i = n, \\ 1 & \text{if } i \text{ is even and } A = [0], \\ 0 & \text{otherwise.} \end{cases}$$

• The (i,j)th Hodge number is

$$h^{i,j}(\mathcal{M}_A) := \dim_{\mathbb{C}} H^j(\overline{\mathcal{M}_A}, \Omega^i) = \begin{cases} \#\{\mathbf{a} \in A \,|\, \|\mathbf{a}\| = i\} & \text{if } i + j = n, \\ 1 & \text{if } A = [0], \\ 0 & \text{otherwise} \end{cases}$$

where we put $\overline{\mathcal{M}_A} := \mathcal{M}_A \otimes \mathbb{C}$.

For the Fermat hypersurface of dimension $n+1$ and degree m, we simply write $\mathfrak{A}_n$ for $\mathfrak{A}(1, 1, \ldots)$.

Lemma A.5. (a) *Let S be a $K3$ surface of degree d in a weighted projective 3-space $\mathbb{P}^3(w_0, w_1, w_2, w_3)$ and suppose that S is dominated by a Fermat* *surface (so S is of CM type). Then there is the unique motive $\mathcal{M}_{\mathbf{w}}$ associated to the weight $\mathbf{w} = (w_0, w_1, w_2, w_3)$ such that $h^{0,2}(\mathcal{M}_{\mathbf{w}}) = 1$ and $B_2(\mathcal{M}_{\mathbf{w}}) = \varphi(d)$. For all other motives $h^{0,2}(\mathcal{M}_A) = 0$.*

(b) *Let X be a Calabi–Yau threefold of degree d in a weighted projective 4-space $\mathbb{P}^4(w_0, w_1, w_2, w_3, w_4)$, and suppose that X is dominated by a Fermat threefold (so X is of CM type). Then there is the unique motive $\mathcal{M}_{\mathbf{w}}$ associated to the weight $\mathbf{w} = (w_0, w_1, \ldots, w_4)$ such that $h^{0,3}(\mathcal{M}_{\mathbf{w}}) = 1$ and $B_3(\mathcal{M}_{\mathbf{w}}) = \varphi(d)$. For all other motives, $h^{0,3}(\mathcal{M}_A) = 0$.*

Here φ denotes the Euler φ-function.

Proposition A.6. *Under the situation of* Lemma 7.5*, the following assertions hold.*

(a) *The Fermat motive $\mathcal{M}_{\mathbf{w}}$ associated to the weight contains the $K3$ motive $\mathcal{M}_S$ as a submotive.*

(b) *The Fermat motive $\mathcal{M}_{\mathbf{w}}$ associated to the weight contains the Calabi–Yau motive $\mathcal{M}_X$ as a submotive.*

Proof. S is realized as the quotient of a Fermat surface $\mathcal{F}_m$ by some finite subgroup G of the automorphism group of $\mathcal{F}_m$. That is, S is birationally equivalent to $\mathcal{F}_m/G$. Furthermore, the transcendental part of $H^2(S)$ can be identified with the transcendental part of $H^2(\mathcal{F}_m)$ that is invariant under G. We know that $H^2(\mathcal{F}_m)$ is a direct sum of one-dimensional subspaces. The Fermat motive associated to the weight is the unique motive of Hodge type $(0, 2)$, and hence its G-invariant transcendental part must contain the $K3$ motive, the unique motive $\mathcal{M}_S$ of S of Hodge type $(0, 2)$.

Similarly for the Calabi–Yau motive $\mathcal{M}_X$, it corresponds to the tensor product $E \otimes T(S)^\sigma$ and it must be contained in the G-invariant part of the Fermat motive associated to the weight, which is the unique motive of X of Hodge type $(0, 3)$. □

(Compare the Fermat motive associated to the weight to the Ω motive defined by Schimmrigk in [34].)

Proposition A.7. *Let (S, σ) be one of the* 86 *$K3$ surfaces with involution σ defined in* Theorem 2.5 *by a hypersurface over $\mathbb{Q}$ in a weighted projective* 3*-space $\mathbb{P}^3(w_0, w_1, w_2, w_3)$. Then S is of CM type. The L-series of S is determined by the Jacobi sum Grossencharacter of some cyclotomic field $\mathbb{K} := \mathbb{Q}(\zeta_d)$ over $\mathbb{Q}$.*

(a) *Let* $\mathbf{w} = (w_0, w_1, w_2, w_3)$ *be the weight defining* S*, and let* $\mathcal{M}_{\mathbf{w}}$ *be the unique motive associated to* $\mathbf{w}$*. Let* $j_{\mathfrak{p}}(\mathbf{w})$ *be the Jacobi sum associated to it. Then* $j_{\mathfrak{p}}(\mathbf{w})$ *is an algebraic integer in* $\mathcal{O}_{\mathbb{K}}$ *with absolute value* $|\,Norm\,\mathfrak{p}\,|$*. The motive* $\mathcal{M}_{\mathbf{w}}$ *associated to* $\mathbf{w}$ *is transcendental and corresponds to the single* $(\mathbb{Z}/d\mathbb{Z})^{\times}$*-orbit of* $j_{\mathfrak{p}}(\mathbf{w})$*. Therefore the Galois representation associated to* $\mathcal{M}_{\mathbf{w}}$ *is induced by a* GL_1 *automorphic representation of* $\mathbb{K} = \mathbb{Q}(\zeta_d)$*, and it is irreducible over* $\mathbb{Q}$ *of dimension* $\varphi(d)$*. Consequently, the Galois representation of* $\mathcal{M}_S$ *is the automorphic induction of the* GL_1 *Grossencharacter representation of* $\mathbb{K}$*.*

In other words, $\mathcal{M}_{\mathbf{w}}$ *is automorphic, that is,* $L(\mathcal{M}_{\mathbf{w}}, s)$ *is determined by an automorphic representation over* $\mathbb{Q}$*.*

(b) *Let* $\mathcal{M}_A$ *be a motive associated to* S *other than* $\mathcal{M}_{\mathbf{w}}$*. Then* $\mathcal{M}_A$ *is automorphic, that is,* $L(\mathcal{M}_A, s)$ *is the Artin* L*-function determined by an automorphic representation over* $\mathbb{Q}$*.*

Proof. Our $K3$ surface S is defined by a hypersurface of degree d over $\mathbb{Q}$ in a weighted projective 3-space. S is of CM type. The characteristic polynomial of the Frobenius of the motive $\mathcal{M}_{\mathbf{w}}$ has reciprocal roots $j_{\mathfrak{p}}(\mathbf{w})$ and its Galois conjugates, that is, the $(\mathbb{Z}/d\mathbb{Z})^{\times}$-orbit of $j_{\mathfrak{p}}(\mathbf{w})$. We know that $j_{\mathfrak{p}}(\mathbf{w})$ and its Galois conjugates are elements of the cyclotomic field $\mathbb{K} = \mathbb{Q}(\zeta_d)$. The restriction $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{K})$ is a sum of GL_1-dimensional representations corresponding to Jacobi sum Grossencharacters.

For (a), the automorphic induction process yields the automorphic representation $I(\mathcal{M}_{\mathbf{w}})$ in $\mathfrak{A}(GL_{\varphi(d)}(\mathbb{Q}))$, which is irreducible over $\mathbb{Q}$ of dimension $\varphi(d)$. Therefore, $\mathcal{M}_{\mathbf{w}}$ is automorphic.

For (b), a similar argument establishes the automorphy of $\mathcal{M}_A$. □

Now we consider Calabi–Yau threefolds of Borcea–Voisin type, and establish their automorphy.

Lemma A.8. *Let* $\mathbb{K} = \mathbb{Q}(\zeta_d)$ *be the* d*-th cyclotomic field over* $\mathbb{Q}$*. Let* ψ *be a Jacobi sum Grossencharacter of* $\mathbb{K}$*. Let* $\phi \in \mathfrak{A}(GL_2(\mathbb{A}_{\mathbb{Q}}))$ *be an automorphic representation. Then there is the base change representation* $B_{\mathbb{K}/\mathbb{Q}}(\phi) \in \mathfrak{A}(GL_2(\mathbb{A}_{\mathbb{K}}))$*.*

Furthermore, there exists the automorphic induction

$$I(B_{\mathbb{K}/\mathbb{Q}}(\phi)) \otimes \psi \in \mathfrak{A}(GL_{2\varphi(d)}(\mathbb{A}_{\mathbb{Q}})).$$

Finally, the Rankin–Selberg L*-series is given by*

$$L(s, \psi \otimes B_{\mathbb{K}/\mathbb{Q}}(\phi)) = L(s, I(B_{\mathbb{K}/\mathbb{Q}}(\phi)) \otimes \psi).$$

Proposition A.9. *Let $d_1, d_2 \in \mathbb{N}$. Let $\mathbb{K}_1 = \mathbb{Q}(\zeta_{d_1})$ and $\mathbb{K}_2 = \mathbb{Q}(\zeta_{d_2})$ be d_1th and d_2th cyclotomic fields over $\mathbb{Q}$. Let ψ_1 and ψ_2 be Jacobi sum Grossen-characters of $\mathbb{K}_1$ and $\mathbb{K}_2$, respectively. Then they are automorphic forms in $\mathfrak{A}(GL_1(\mathbb{A}_{\mathbb{K}_1}))$ and $\mathfrak{A}(GL_1(\mathbb{A}_{\mathbb{K}_2}))$, respectively. Consider the induced automorphic representation $\psi_1 \otimes \psi_2$.*

(a) *If $\mathbb{K}_1 \not\supseteq \mathbb{K}_2$, then $\psi_1 \otimes \psi_2$ corresponds to an automorphic representation in $\mathfrak{A}(GL_1(\mathbb{A}_{\mathbb{K}_1\mathbb{K}_2}))$, and*

$$L(s, \psi_1 \otimes \psi_2) = L(s, \psi_1)L(s, \psi_2).$$

(b) *If $\mathbb{K}_1 = \mathbb{K}_2$, then $\psi_1 \otimes \psi_2$ corresponds to the induced automorphic representation $I(\psi_1 \otimes \psi_2)$ in $\mathfrak{A}(GL_2(\mathbb{A}_{\mathbb{K}_1}))$, and*

$$L(s, \psi_1 \otimes \psi_2) = L(s, I(\psi_1 \otimes \psi_2)).$$

(c) *If $\mathbb{K}_1 \supset \mathbb{K}_2$ but $\mathbb{K}_1 \neq \mathbb{K}_2$, then ψ_2 corresponds to a representation of a subgroup, H, of Gal$(\mathbb{K}_1/\mathbb{Q}) := G$, and $\psi_1 \otimes \psi_2$ corresponds to the induced representation in $\mathfrak{A}(GL_1(\mathbb{A}_{\mathbb{K}_1}))$, and*

$$L(s, \psi_1 \otimes \psi_2) = L(s, \psi_1 \otimes Ind_H^G \psi_2).$$

To prove these results, we apply the base change and automorphic induction method of Arthur and Clozel (and Rajan) to our situation. Also, confer Murty [28].

Appendix B. Tables

Some clarifications might be in order how to read the tables.

- In the tables B.1 to B.3, we use two numbering systems, one from Borcea $B\#$ and the other from Yonemura $Y\#$. We matched up the numbers in two lists.
- We use two sets of notations for variables, one is $x_0, x_1, \ldots, x_n$, and the other is $x, y, z, w, \ldots$. In relevant tables, we indicated identification of these two sets of variables.
- The equations in tables B.1 to B.3 are taken from Borcea's paper [6]. "Terms removed" indicates the terms we may remove from the equations (or deformation) in [6] to create those of Fermat or Delsarte type.

Table B.1: $K3$ weights in Borcea's list with odd w_0.

$Y\#$	$B\#$	(w_0, w_1, w_2, w_3)	$f(x_1, x_2, x_3) = f(y, z, w)$	r	a	Terms removed from equations of [B]
5	1	$(3, 1, 1, 1)$	$y^6 + z^6 + w^6$	1	1	
6	2	$(5, 2, 2, 1)$	$y^5 + z^5 + w^{10}$	6	4	
42	3	$(5, 3, 1, 1)$	$y^3z + z^{10} + w^{10}$	3	1	y^3w
32	4	$(7, 3, 2, 2)$	$y^4z + z^7 + w^7$	10	6	y^4w
40	5	$(7, 4, 2, 1)$	$y^3z + z^7 + w^{14}$	7	3	y^3w^2
33	6	$(9, 4, 3, 2)$	$y^4w + z^6 + w^9$	10	6	y^3z^2
39	7	$(9, 5, 3, 1)$	$y^3z + z^6 + w^{18}$	7	3	y^3w^3
12	8	$(9, 6, 2, 1)$	$y^3 + z^9 + w^{18}$	6	2	
75	9	$(11, 5, 4, 2)$	$y^4w + z^5w + w^{11}$	13	5	y^2z^3
78	10	$(11, 6, 4, 1)$	$y^3z + yz^4 + w^{22}$	10	2	y^3w^4, z^5w^2
82	11	$(11, 7, 3, 1)$	$y^3w + yz^5 + w^{22}$	9	1	z^7w
76	12	$(13, 6, 5, 2)$	$y^4w + yz^4 + w^{13}$	14	4	z^4w^3
77	13	$(13, 7, 5, 1)$	$y^3z + z^5w + w^{26}$	11	1	y^3w^5
81	14	$(13, 8, 3, 2)$	$y^3w + yz^6 + w^{13}$	13	3	z^8w
29	15	$(15, 6, 5, 4)$	$y^5 + z^6 + yw^6$	12	6	z^2w^5
34	16	$(15, 7, 6, 2)$	$y^4w + z^5 + w^{15}$	14	4	
38	17	$(15, 8, 6, 1)$	$y^3z + z^5 + w^{30}$	11	1	y^3w^6
11	18	$(15, 10, 3, 2)$	$y^3 + z^{10} + w^{15}$	10	4	
50	19	$(15, 10, 4, 1)$	$y^3 + yz^5 + w^{30}$	9	1	z^7w^2
90	20	$(17, 7, 6, 4)$	$y^4z + y^2w^5 + z^5w + zw^7$	17	3	no Delsarte form
93	21	$(17, 10, 4, 3)$	$y^3z + yz^6 + yw^8 + z^7w^2 + zw^{10}$	16	2	no Delsarte form
91	22	$(19, 8, 6, 5)$	$y^4z + yz^5 + yw^6 + z^3w^4$	18	2	no Delsarte form
92	23	$(19, 11, 5, 3)$	$y^3z + yw^9 + z^7w$	17	1	zw^{11}
47	24	$(21, 14, 4, 3)$	$y^3 + yz^7 + w^{14}$	13	3	z^9w^2
49	25	$(21, 14, 5, 2)$	$y^3 + z^8w + w^{21}$	14	2	
14	26	$(21, 14, 6, 1)$	$y^3 + z^7 + w^{42}$	10	0	
73	27	$(25, 10, 8, 7)$	$y^5 + yz^5 + zw^6$	19	1	
83	28	$(27, 18, 5, 4)$	$y^3 + yw^9 + z^{10}w$	17	1	z^2w^{11}
46	29	$(33, 22, 6, 5)$	$y^3 + z^{11} + zw^{12}$	18	0	

- The equations in table B.4 are taken from Yonemura's paper [43]. "Terms removed" indicates the terms we may remove to specialize the equations into Delsarte type. In Case #1, there is no choice of

Table B.2: $K3$ weights in Borcea's list with even w_0.

$Y\#$	$B\#$	(w_0, w_1, w_2, w_3)	$f(x_1, x_2, x_3)$ $= f(y, z, w)$	r	a	Terms removed from equations of [B]
7	30	$(4, 2, 1, 1)$	$y^4 + z^8 + w^8$	2	2	
37	31	$(8, 4, 3, 1)$	$y^4 + yz^4 + w^{16}$	6	4	z^5w
44	32	$(8, 5, 2, 1)$	$y^3w + z^8 + w^{16}$	6	2	y^2z^3
36	33	$(10, 5, 3, 2)$	$y^4 + yz^5 + w^{10}$	8	6	z^6w
9	34	$(10, 5, 4, 1)$	$y^4 + z^5 + w^{20}$	6	4	
35	35	$(14, 7, 4, 3)$	$y^4 + z^7 + yw^7$	10	6	zw^8
45	36	$(14, 9, 4, 1)$	$y^3w + z^7 + w^{28}$	10	0	
74	37	$(16, 7, 5, 4)$	$y^4w + yz^5 + w^8$	14	4	z^4w^3
79	38	$(16, 9, 5, 2)$	$y^3z + z^6w + w^{16}$	14	2	y^2w^7
30	39	$(20, 8, 7, 5)$	$y^5 + z^5w + w^8$	14	4	
80	40	$(22, 13, 5, 4)$	$y^3z + z^8w + w^{11}$	18	0	

Table B.3: $K3$ weights in Borcea's list with w_0 divisible by 6.

$Y\#$	$B\#$	(w_0, w_1, w_2, w_3)	$f(x_1, x_2, x_3)$ $= f(y, z, w)$	r	a	Terms removed from equations of [B]
8	41	$(6, 3, 2, 1)$	$y^4 + z^6 + w^{12}$	4	4	
10	42	$(6, 4, 1, 1)$	$y^3 + z^{12} + w^{12}$	2	0	
31	43	$(12, 5, 4, 3)$	$y^4z + z^6 + w^8$	10	6	y^3w^3
41	44	$(12, 7, 3, 2)$	$y^3z + z^8 + w^{12}$	10	4	y^2w^5
13	45	$(12, 8, 3, 1)$	$y^3 + z^8 + w^{24}$	6	2	
43	46	$(18, 11, 4, 3)$	$y^3w + z^9 + w^{12}$	14	2	
51	47	$(18, 12, 5, 1)$	$y^3 + z^7w + w^{36}$	10	0	
48	48	$(24, 16, 5, 3)$	$y^3 + z^9w + w^{16}$	14	2	

equations of the form $x_0^2 = f(x_1, x_2, x_3)$ or $x_0^2x_i = f(x_1, x_2, x_3)$.

- In table B.5, the equations are taken from Yonemura's paper [43]. In order to make the equations into Delsarte type, we slightly generalize the original equations and then remove some terms, if necessary. In other words, we first add a few terms to the original equation and

Table B.4: Delsarte-type $K3$ surfaces with involutions $\sigma(x) = -x$, NOT in Borcea's list, after removal of several terms.

$Y\#$	(w_0, w_1, w_2, w_3)	$f(x_1, x_2, x_3) = f(y, z, w)$	r	a	Terms removed from equations of [Y]
1	$(1,1,1,1)$	$x^4+y^4+z^4+w^4$	8	8	
19	$(3,2,2,1)$	$x^2y+y^4+z^4+w^8$	10	6	x^2w^2 and x^2z
20	$(9,8,6,1)$	$x^2z+y^3+z^4+w^{24}$	10	6	x^2w^6
21	$(2,1,1,1)$	$x^2y+y^5+z^5+w^5$	6	4	x^2z, x^2w
22	$(6,5,3,1)$	$x^2z+y^3+z^5+w^{15}$	10	4	x^2w^3
23	$(5,3,2,2)$	$x^2z+y^4+z^6+w^6$	12	6	x^2w
24	$(5,4,2,1)$	$x^2z+y^3+z^6+w^{12}$	10	4	x^2w^2
25	$(4,3,1,1)$	$x^2z+y^3+z^9+w^9$	6	2	x^2w
26	$(9,5,4,2)$	$x^2w+y^4+z^5+w^{10}$	14	4	
27	$(11,8,3,2)$	$x^2w+y^3+z^8+w^{12}$	14	2	
28	$(10,7,3,1)$	$x^2w+y^3+z^7+w^{21}$	11	1	
55	$(7,6,5,2)$	$x^2y+y^3w+z^4+w^{10}$	14	4	x^2w^3
56	$(11,8,6,5)$	$x^2y+y^3z+z^5+w^6$	19	1	
57	$(9,6,5,4)$	$x^2y+y^4+z^4w+w^6$	18	2	xz^3
58	$(6,5,4,1)$	$x^2z+y^3w+z^4+w^{16}$	14	2	x^2w^4, xy^2
59	$(8,7,5,1)$	$x^2z+y^3+z^4w+w^{21}$	14	2	x^2w^5
60	$(7,6,4,1)$	$x^2z+y^3+yz^3+w^{18}$	13	3	x^2w^4, z^4w^2
61	$(11,7,6,4)$	$x^2z+y^4+z^4w+w^7$	18	2	
62	$(8,5,4,3)$	$x^2z+y^4+yw^5+z^5$	14	4	xw^4, z^2w^4
63	$(4,3,2,1)$	$x^2z+y^3w+z^5+w^{10}$	10	4	x^2w^2, xy^2, y^2z^2
64	$(10,7,4,3)$	$x^2z+y^3w+z^6+w^8$	18	0	xy^2
65	$(14,11,5,3)$	$x^2z+y^3+z^6w+w^{11}$	18	0	
66	$(3,2,1,1)$	$x^2z+y^3w+z^7+w^7$	7	3	x^2w, xy^2, y^3z
67	$(9,7,3,2)$	$x^2z+y^3+yw^7+z^7$	13	3	xw^6, zw^9
68	$(13,10,4,3)$	$x^2z+y^3+yz^5+w^{10}$	17	1	z^6w^2
69	$(7,4,3,2)$	$x^2w+y^4+yz^4+w^8$	14	4	xz^3, z^4w^2
70	$(8,5,3,2)$	$x^2w+y^3z+z^6+w^9$	14	2	xy^2, y^2w^4
71	$(7,4,3,1)$	$x^2w+y^3z+z^5+w^{15}$	11	1	xy^2, y^3w^3
72	$(7,5,2,1)$	$x^2w+y^3+yz^5+w^{15}$	9	1	xz^4, z^7w
86	$(9,7,5,4)$	$x^2y+y^3w+z^5+zw^5$	19	1	xw^4
87	$(5,4,3,1)$	$x^2z+y^3w+yz^3+w^{13}$	13	3	x^2w^3, xy^2, z^4w
88	$(11,9,5,2)$	$x^2z+y^3+yw^9+z^5w$	17	1	xw^8, zw^{11}
89	$(5,3,2,1)$	$x^2w+y^3z+yz^4+w^{11}$	10	2	xy^2, xz^3, y^3w^2, z^5w

Table B.5: $K3$ surfaces with involution $\sigma(x) = -x$, NOT in Borcea's list, after change and/or removal of several terms.

$Y\#$	(w_0, w_1, w_2, w_3)	$f(x_1, x_2, x_3) = f(y, z, w)$	r	a	Terms removed from equations of [Y]
3	$(2, 2, 1, 1)$	$x^2y + y^3 + z^6 + w^6$	7	7	$x^3 \to x^2y$
4	$(4, 4, 3, 1)$	$x^2y + y^3 + z^4 + w^{12}$	7	7	$x^3 \to x^2y$
17	$(5, 5, 3, 2)$	$x^2y + y^3 + z^5 + zw^6$	12	6	$xw^5, yw^5, x^3 \to x^2y$
18	$(3, 3, 2, 1)$	$x^2y + y^3 + z^4w + w^9$	10	6	$xz^3, yz^3, x^3 \to x^2y$

Table B.6: $K3$ surfaces with a different kind of involution.

$Y\#$	(w_0, w_1, w_2, w_3)	$F(x_0, x_1, x_2, x_3)$ $= F(x, y, z, w)$	r	a	Terms removed	Involution
2	$(4, 3, 3, 2)$	$x^3 + y^4 + z^4 + w^6$	10	8	None	$y \to -y$
16	$(8, 7, 6, 3)$	$x^3 + y^3w + z^4 + w^8$	14	6	None	$z \to -z$
52	$(12, 9, 8, 7)$	$x^3 + y^4 + xz^3 + zw^4$	19	3	None	$y \to -y$
84	$(9, 7, 6, 5)$	$x^3 + xz^3 + y^3z + yw^4$	20	2	z^2w^3	$w \to -w$

Table B.7: $K3$ surfaces with involution $\sigma(x) = -x$, but not realized as quasi-smooth hypersurfaces in four monomials.

$Y\#$	(w_0, w_1, w_2, w_3)	$F(x_0, x_1, x_2, x_3) = F(x, y, z, w)$	r	a
85	$(5, 4, 3, 2)$	$x^2y + x^2w^2 + y^3w + y^2z^2 + yw^5 + z^4w + w^7$	15	5
90	$(17, 7, 6, 4)$	$x^2 + y^4z + y^2w^5 + z^5w + zw^7$	17	3
91	$(19, 8, 6, 5)$	$x^2 + y^4z + yz^5 + yw^6 + z^3w^4$	18	2
93	$(17, 10, 4, 3)$	$x^2 + y^3z + yz^6 + yw^8 + z^7w^2 + zw^{10}$	16	2
94	$(7, 5, 4, 3)$	$x^2y + y^3z + y^2w^3 + z^4w + zw^5$	18	2
95	$(7, 5, 3, 2)$	$x^2z + y^3w + yz^4 + yw^6 + z^5w + zw^7$	16	2

Table B.8: $K3$ weights with no obvious involution.

$Y\#$	(w_0, w_1, w_2, w_3)	$F(x_0, x_1, x_2, x_3) = F(x, y, z, w)$
15	$(5, 4, 3, 3)$	$x^3 + y^3z + y^3w + z^5 + w^5$
53	$(6, 5, 4, 3)$	$x^3 + y^3w + y^2z^2 + xz^3 + z^3w^2 + w^6$
54	$(7, 6, 5, 3)$	$x^3 + y^3w + yz^3 + z^3w^2 + w^7$

Table B.9: Nikulin's invariant associated with other types of involutions.

$Y\#$	(w_0,w_1,w_2,w_3)	$F(x_0,x_1,x_2,x_3)$ $= F(x,y,z,w)$	$\sigma(x_i)=-x_i$	r	a
2	$(4,3,3,2)$	$x^3+y^4+z^4+w^6$	y	10	8
	$(4,3,3,2)$	$x^3+y^4+z^4+w^6$	w	18	4
3	$(2,2,1,1)$	$x^2y+y^3+z^6+w^6$	z	10	8
4	$(4,4,3,1)$	$x^2y+y^3+z^4+w^{12}$	z	14	6
5	$(3,1,1,1)$	$x^2+y^6+z^6+w^6$	y	9	9
6	$(5,2,2,1)$	$x^2+y^5+z^5+w^{10}$	w	6	4
	$(5,2,2,1)$	$x^2+y^5+yz^4+w^{10}$	z	10	8
7	$(4,2,1,1)$	$x^2+y^4+z^8+w^8$	y	10	6
	$(4,2,1,1)$	$x^2+y^4+z^8+w^8$	w	10	8
8	$(6,3,2,1)$	$x^2+y^4+z^6+w^{12}$	y	12	6
	$(6,3,2,1)$	$x^2+y^4+z^6+w^{12}$	z	12	8
9	$(10,5,4,1)$	$x^2+y^4+z^5+w^{20}$	y	14	4
	$(10,5,4,1)$	$x^2+y^4+z^5+w^{20}$	w	14	4
10	$(6,4,1,1)$	$x^2+y^3+z^{12}+w^{12}$	z	10	8
12	$(9,6,2,1)$	$x^2+y^3+z^9+w^{18}$	w	6	2
13	$(12,8,3,1)$	$x^2+y^3+z^8+w^{24}$	z	14	6
16	$(8,7,6,3)$	$x^3+y^3w+z^4+w^8$	z	14	6
17	$(5,5,3,2)$	$x^2y+y^3+z^5+zw^6$	w	17	5
18	$(3,3,2,1)$	$x^2y+y^3+z^4w+w^9$	z	14	6
19	$(3,2,2,1)$	$x^2y+y^4+z^4+w^8$	z	10	8
23	$(5,3,2,2)$	$x^2z+y^4+z^6+w^6$	w	12	6
29	$(15,6,5,4)$	$x^2+y^5+z^6+yw^6$	w	18	4
31	$(12,5,4,3)$	$x^2+y^4z+z^6+w^8$	y	18	4
33	$(9,4,3,2)$	$x^2+y^4w+z^6+w^9$	y	14	6
	$(9,4,3,2)$	$x^2+y^4w+z^6+w^9$	z	10	6
36	$(10,5,3,2)$	$x^2+y^4+yz^5+w^{10}$	w	16	6
37	$(8,4,3,1)$	$x^2+y^4+yz^4+w^{16}$	z	10	8
39	$(9,5,3,1)$	$x^2+y^3z+z^6+w^{18}$	w	15	7
40	$(7,4,2,1)$	$x^2+y^3z+z^7+w^{14}$	w	7	3
41	$(12,7,3,2)$	$x^2+y^3z+z^8+w^{12}$	w	15	7
42	$(5,3,1,1)$	$x^2+y^3z+z^{10}+w^{10}$	w	11	9

Table B.9: Nikulin's invariant associated with other types of involutions.

$Y\#$	(w_0, w_1, w_2, w_3)	$F(x_0, x_1, x_2, x_3)$ $= F(x, y, z, w)$	$\sigma(x_i) = -x_i$	r	a
44	$(8, 5, 2, 1)$	$x^2 + y^3w + z^8 + w^{16}$	z	14	6
52	$(12, 9, 8, 7)$	$x^3 + y^4 + xz^3 + zw^4$	y	20	2
	$(12, 9, 8, 7)$	$x^3 + y^4 + xz^3 + zw^4$	w	19	3
75	$(11, 5, 4, 2)$	$x^2 + y^4w + z^5w + w^{11}$	y	13	5
84	$(9, 7, 6, 5)$	$x^3 + xz^3 + y^3z + yw^4$	w	20	2

then remove several terms to make the equation into a form of Delsarte type. This procedure is indicated as "terms changed/removed."

For instance, in the case #17, we first add a term x^2y and then remove x^3, xw^5 and yw^5. In effect, this procedure interchanges x^3 with x^2y, and remove xw^5 and yw^5. It results in a new equation $x^2y + y^3 + z^5 + zw^6$.

- In table B.6, the equations are taken from Yonemura's paper [43]. For the $K3$ surfaces on this table, there is no way to define the involution $\sigma(x) = -x$ by using equations of Delsarte type. We therefore define an involution on some other variable. This alternative involution is indicated in the column "involution." Nikulin's invariants r and a for such (S, σ) are calculated in table B.9.

- The $K3$ surfaces in Table B.7 are taken from Yonemura's paper [43]. They have involution $\sigma(x) = -x$, but no matter what terms we add or remove from the equations, we cannot transform them into quasi-smooth equations of Delsarte type (cf. Remark 2.5).

- The $K3$ surfaces in table B.8 are also taken from Yonemura's paper [43]. For them, we do not know how to define a non-symplectic involution on the $K3$ surfaces by preserving their quasi-smoothness (even if we allow more than four monomials in the equations).

- Table B.9 lists $K3$ surfaces with involution at x_1, x_2 or x_3 (i.e., not at the variable x_0 of highest weight). The variable we choose is indicated under the column $\sigma(x_i) = -x_i$. For some $K3$ surfaces, we consider two involutions.

Table B.10: $K3$ surfaces and their mirror partners.

$S:Y\#$	$S:B\#$	Weight	r	$S^\vee Y\#$	$S^\vee B\#$	$20-r$	Weight for mirror
1		$(1,1,1,1)$	8	56		12	$(11,8,6,5)$
				73	27	12	$(25,10,8,7)$
4		$(4,4,3,1)$	10	4		10	$(4,4,3,1)$
5	1	$(3,1,1,1)$	1	52		19	$(12,9,8,7)$
6	2	$(5,2,2,1)$	6	26		14	$(9,5,4,2)$
				34	16	14	$(15,7,6,2)$
				76	12	14	$(13,6,5,2)$
8	41	$(6,3,2,1)$	3	64		17	$(10,7,4,5)$
9		$(10,5,4,1)$	10	9		10	$(10,5,4,1)$
				71		10	$(7,4,3,1)$
10	42	$(6,4,1,1)$	2	65		18	$(14,11,5,3)$
				46	29	18	$(53,22,6,5)$
				80	40	18	$(22,13,5,4)$
11	17	$(15,10,3,2)$	12	24		8	$(5,4,2,1)$
12	8	$(9,6,2,1)$	6	27		14	$(11,8,3,2)$
				49	25	14	$(21,14,5,2)$
13	45	$(12,7,3,1)$	8	20		12	$(9,8,6,1)$
				59		12	$(8,7,5,1)$
14	26	$(21,14,6,1)$	10	14	26	10	$(21,14,6,1)$
				28		10	$(10,7,3,1)$
				45	36	10	$(14,9,4,1)$
				51	47	10	$(18,12,5,1)$
20		$(9,8,6,1)$	12	17		8	$(12,8,3,1)$
				72		8	$(7,5,2,1)$
21		$(2,1,1,1)$	2	30	39	18	$(20,8,7,5)$
				86		18	$(9,7,5,4)$
22		$(6,5,3,1)$	10	22		10	$(6,5,3,1)$
24		$(5,4,2,1)$	8	11	18	12	$(15,10,3,2)$
25		$(4,3,1,1)$	8	43	46	12	$(18,11,4,3)$
				48	48	12	$(24,16,5,3)$
				88		12	$(11,9.5,2)$
26		$(9,5,4,2)$	14	6	2	6	$(5,2,2,1)$
27		$(11,8,3,2)$	14	12	8	6	$(9,6,2,1)$
28		$(10,7,3,1)$	10	14	26	10	$(21,14,6,1)$
				28		10	$(10,7,3,1)$
				45	36	10	$(14,9,4,1)$
				51	47	10	$(18,12,5,1)$
30	39	$(20,8,7,5)$	18	21		2	$(2,1,1,1)$
32	4	$(7,3,2,2)$	10	10	42	10	$(7,3,2,2)$
34	16	$(15,7,6,2)$	14	6	2	6	$(5,2,2,1)$
35	35	$(14,7,4,3)$	16	66		4	$(3,2,1,1)$
37	31	$(8,4,3,1)$	9	58		11	$(6,5,4,1)$

Table B.10: $K3$ surfaces and their mirror partners (continued).

$S:$ $Y\#$	$S:$ $B\#$	Weight	r	$S^\vee:$ $Y\#$	$S^\vee:$ $B\#$	$20-r$	Weight for mirror
38	17	$(15,8,6,1)$	11	50	19	9	$(15,10,4,1)$
				82	11	9	$(11,7,3,1)$
39	7	$(9,5,3,1)$	9	60		11	$(7,6,4,1)$
40	5	$(7,4,2,1)$	7	81	14	13	$(13,8,3,2)$
42	3	$(5,3,1,1)$	3	68		17	$(13,10,4,3)$
				83	28	17	$(27,18,5,4)$
				92	23	17	$(19,11,5,3)$
43	46	$(18,11,4,3)$	16	25		4	$(4,3,1,1)$
45	36	$(14,9,4,1)$	10	14	26	10	$(21,14,6,1)$
				28		10	$(10,7,3,1)$
				45	36	10	$(14,9,4,1)$
				51	47	10	$(18,12,5,1)$
46	29	$(33,22,6,5)$	18	10	42	2	$(6,4,1,1)$
48	48	$(24,16,5,3)$	16	25		4	$(4,3,1,1)$
49	25	$(21,14,5,2)$	14	12	8	6	$(9,6,2,1)$
50	19	$(15,10,4,1)$	9	38	17	11	$(15,8,6,1)$
				77	13	11	$(13,7,5,1)$
51	47	$(18,12,5,1)$	10	14	26	10	$(21,12,6,1)$
				28		10	$(10,7,3,1)$
				45	36	10	$(14,9,4,1)$
				51	47	10	$(18,12,5,1)$
52		$(12,9,8,7)$	19	5		1	$(3,1,1,1)$
56		$(11,8,6,5)$	19	1		1	$(1,1,1,1)$
58		$(6,5,4,1)$	11	37	31	9	$(8,4,3,1)$
59		$(8,7,5,1)$	12	13	45	8	$(12,8,3,1)$
				72		8	$(7,5,2,1)$
60		$(7,6,4,1)$	11	39	7	9	$(9,5,3.1)$
64		$(10,7,4,3)$	17	7	30	3	$(4,2,1,1)$
65		$(14,11,5,3)$	17	10	42	3	$(6,4,1,1)$
66		$(3,2,1,1)$	4	35	35	16	$(14,7,4,3)$
68		$(13,10,4,3)$	17	42	3	3	$(5,3,1,1)$
71		$(7,4,3,1)$	10	9	34	10	$(10,5,4,1)$
				71		10	$(7,4,3,1)$
72		$(7,5,2,1)$	8	20		12	$(9,8,6,1)$
				59		12	$(8,7,5,1)$

Table B.10: $K3$ surfaces and their mirror partners.

$S:Y\#$	$S:B\#$	Weight	r	$S^\vee Y\#$	$S^\vee B\#$	$20-r$	Weight for mirror
73	27	$(25,10,8,7)$	19	1		1	$(1,1,1,1)$
76	12	$(13,6,5,2)$	14	6	2	6	$(5,2,2,1)$
77	13	$(13,7,5,1)$	11	50	19	9	$(15,10,4,1)$
				82	11	9	$(11,7,3,1)$
78	10	$(11,6,4,1)$	10	10	42	10	$(11,6,4,1)$
80	40	$(22,13,5,4)$	18	10	42	2	$(6,4,1,1)$
81	14	$(13,8,3,2)$	13	40	5	7	$(7,4,2,1)$
82	11	$(11,7,3,1)$	9	38	17	11	$(15,8,6,1)$
				77	13	11	$(13,7,5,1)$
83	28	$(27,18,5,4)$	17	42	3	3	$(5,3,1,1)$
86		$(9,7,5,4)$	18	21		2	$(2,1,1,1)$
87		$(5,4,3,1)$	10	87		10	$(1,3,4,5)$
92	23	$(19,11,5,3)$	17	42	3	3	$(5,3,1,1)$

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