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On stringy invariants of GUT vacua

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We investigate aspects of certain stringy invariants of singular elliptic fibrations which arise in engineering Grand Unified Theories in F-theory. In particular, we exploit the small resolutions of the total space of these fibrations provided recently in the physics literature to compute "stringy characteristic classes", and find that numerical invariants obtained by integrating such characteristic classes are predetermined by the topology of the base of the elliptic fibration. Moreover, we derive a simple (dimension independent) formula for pushing forward powers of the exceptional divisor of a blowup, which one may use to reduce any integral (in the sense of Chow cohomology) on a small resolution of a singular elliptic fibration to an integral on the base. We conclude with a remark on the cohomology of small resolutions of GUT vacua, where we conjecture that certain simple formulas for their Hodge numbers may be given solely in terms of the first Chern class and Hodge numbers of the base.

1. Introduction

F-theory provides a geometric platform for engineering Grand Unified Theories within the framework of string theory [1–3]. The geometric apparatus of F-theory is an elliptic fibration $\varphi : Y \to B$, whose total space Y is a Calabi–Yau fourfold and whose base B is a compact smooth algebraic variety over $\mathbb{C}$. Crucial to the geometry and the associated physics of the fibration are its singular fibers, which lie over a hypersurface in the base referred to as the *discriminant locus* of the fibration. Physically significant aspects of the singular fibers include the fact that they encode the structure of gauge theories associated with D-branes wrapping components of the discriminant locus over which they appear. The standard procedure for the realization of a desired gauge group/singular fiber is to introduce a Weierstrass elliptic fibration in Tate form:

$$Y : (y^2z + a_1xyz + a_3yz^2 = x^3 + a_2x^2z + a_4xz^2 + a_6z^3) \subset \mathbb{P}(\mathscr{E}),$$

where $\pi : \mathbb{P}(\mathscr{E}) \to B$ is the projective bundle of *lines* in $\mathscr{E} = \mathscr{O}_B \oplus \mathscr{L}^2 \oplus \mathscr{L}^3$ for some suitably ample line bundle $\mathscr{L} \to B$[1]. Choosing x, y, and z to be respective sections of $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1) \otimes \pi^*\mathscr{L}^2$, $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1) \otimes \pi^*\mathscr{L}^3$, $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$, and each a_i to be a section of $\pi^*\mathscr{L}^i$ then realizes Y as the zero-scheme associated with the vanishing of a section of $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(3) \otimes \pi^*\mathscr{L}^6$, and naturally determines a proper surjective morphism $\varphi : Y \to B$ such that the generic fiber is an elliptic curve. When the a_is are suitably generic Y is a smooth hypersurface in $\mathbb{P}(\mathscr{E})$ with nodal and cuspidal cubics as its singular fibers, which correspond to abelian gauge groups in the physical theory. Such an elliptic fibration we will refer to as a *smooth Weierstrass fibration*. To realize non-abelian gauge groups, one resorts to Tate's algorithm [4], which renders conditions on the coefficient sections a_i which inflict singularities on the total space of the fibration Y in such a way that a resolution of the codimension

[1]For Y to be Calabi–Yau one must take $\mathscr{L}$ to be the anti-canonical bundle $\mathscr{O}(-K) \to B$.

one singularities of Y produce the desired singular fiber over a certain component of the discriminant locus Δ, which is given by

$$\Delta : (4F^3 + 27G^2 = 0) \subset B,$$

where

$$\begin{cases} F = -\frac{1}{48}(b_2^2 - 24b_4) \\ G = -\frac{1}{864}(36b_2b_4 - b_2^3 - 216b_6) \\ b_2 = a_1^2 + 4a_2 \\ b_4 = a_1a_3 + 2a_4 \\ b_6 = a_3^2 + 4a_6. \end{cases}$$

For example, to engineer an $\mathrm{SU}(n)$ gauge group one imposes that the a_is vanish to certain orders along a divisor $D_{\mathrm{GUT}} : (w = 0) \subset B$ in such a way that the total space Y is inflicted with a surface of A_{n-1} singularities over D_{GUT}. The discriminant locus then factors as:

$$\Delta : (w^n g = 0) \subset B,$$

where $\Delta' : (g = 0)$ is generically irreducible. Such an elliptic fibration we will refer to as an $\mathrm{SU}(n)$ *elliptic fibration*. Tate's algorithm then ensures that a split I_n fiber (a chain of n rational curves) appears generically over the divisor $D_{\mathrm{SU}(n)}$ upon a resolution of the codimension one singularities of Y. In this note, we consider the cases of SU(5), SO(10) and E_6 vacua. For the SO(10) and E_6 cases, the coefficients of the Tate form are tweaked in such a way that a split I_1^* fiber (in the SO(10) case) and split IV^* fiber (in the E_6 case) appears over D_{GUT} after a resolution of the codimension one singularities of Y, as is necessary for the associated physical theory. But due to the exotic nature of singularities, a crepant resolution of singularities in *all* codimensions must transpire for a well-defined physical theory to be associated with the fibration. Thus enters the theory of singular varieties, their resolutions, and associated invariants. When the base of the fibration is a toric variety, so too is the ambient projective bundle $\mathbb{P}(\mathscr{O}_B \oplus \mathscr{L}^2 \oplus \mathscr{L}^3)$ in which it resides, thus resolution procedures via toric methods are readily available [5,6]. Over non-toric bases the algorithmic methods of toric geometry no longer apply, and often one must seek a resolution "by hand". As it is not clear at present which bases give rise to phenomenologically realistic vacua, we prefer to work in the general setting where the only assumption we make on the base is that such an elliptic fibration exists. In the case of SU(5) models, Esole and Yau [7] performed an explicit resolution procedure yielding six distinct small (and thus crepant) resolutions of the total space of the fibration which they showed all differ by flop transitions. Most notably, they make no assumption on the base and do not assume any Calabi–Yau hypothesis. They also provided a detailed analysis of the singular fiber structure of the resolution varieties, and found that the structure of enhancement was different than what had been previously conjectured in the physics literature (and found fibers not on the list of Kodaira as well). For SO(10) and E_6 fibrations, small resolution procedures over arbitrary bases were recently presented in the physics literature [8, 9].

Invariants associated with a small resolution of a "GUT" singular elliptic fibration may be viewed as invariants associated with the original *singular* fourfold Y, following the convolution of mirror symmetry and algebraic geometry which has congealed into a theory of "stringy invariants" of singular varieties. From this perspective, a resolution variety is an auxiliary object which allows us to glean information from the original, singular variety. In particular, if $f : \widetilde{X} \to X$ is a crepant resolution[2] of a (not too) singular variety X then invariants of $\widetilde{X}$ are often deemed "stringy invariants" of X provided the invariant is independent of the chosen crepant resolution. For example, Kontsevich [10] showed using motivic integration that the Hodge numbers $h^{p,q}$ of a crepant resolution are independent of the chosen crepant resolution, leading Batyrev to define a notion of *stringy Hodge numbers* [11]. But since crepant resolutions do not always exist, intrinsic definitions are sought of stringy invariants which agree with the associated invariant of a crepant resolution when one exists. Somewhat recently the notion of *stringy Chern classes* were independently defined by Aluffi [12] and de Fernex *et al.* [13]. Aluffi's approach uses his theory of "celestial integration" to define stringy Chern classes while de Fernex *et al.* used Kontsevich's theory of motivic integration, yet the seemingly two different approaches reproduce the same class[3]. An indication that the moniker "stringy Chern class" is justified lies in the fact that integrating $c_{\mathrm{str}}(X)$ yields the *stringy Euler characteristic* $\chi_{\mathrm{str}}(X)$ as defined by Batyrev [14], i.e.,

$$\int_X c_{\mathrm{str}}(X) = \chi_{\mathrm{str}}(X),$$

[2]That is, $f^* K_X = K_{\widetilde{X}}$.

[3]Both approaches require that we work over an algebraically closed field of characteristic zero.

which we may view as a "stringy" version of the Poincaré–Hopf theorem. Moreover, if $f : \widetilde{X} \to X$ is a crepant resolution then $c_{\mathrm{str}}(X) = f_* c(\widetilde{X})$ (f_* is the proper push forward map associated with f) and $c_{\mathrm{str}}(X) = c(X)$ for smooth X, as should any appropriate generalization of Chern class. If $X \hookrightarrow M$ is a regular embedding with M smooth and $N_X M$ denotes the normal bundle to X in M, then the class

$$\frac{c(TM)}{c(N_X M)} \cap [X]$$

is denoted by $c_{\mathrm{F}}(X)$ and is referred to as the *Fulton class* of X, which coincides with the Chern class of a smooth representative of the rational equivalence class of X. The difference $c_{\mathrm{str}}(X) - c_{\mathrm{F}}(X)$ we will refer to as the *stringy Milnor class* of X, which we will denote by $\mathcal{M}_{\mathrm{str}}(X)$ (compare with the definition of Milnor class [15]). The stringy Milnor class is then a "stringy" invariant supported on the singular locus of X which measures the deviation of $c_{\mathrm{str}}(X)$ from the total homology Chern class of a smooth variety in the same rational equivalence class as X. For one interested in numerical invariants then $\int_X \mathcal{M}_{\mathrm{str}}(X)$ measures the deviation of $\chi_{\mathrm{str}}(X)$ from the topological Euler characteristic of a smooth variety in the same rational equivalence class as X.

For $\varphi : Y \to B$ a GUT elliptic fibration the integer $\int_Y \mathcal{M}_{\mathrm{str}}(Y)$ is physically relevant for the computation of 3-brane tadpoles, as it modifies the Euler characteristic of the total space of a smooth Weierstrass fibration $\psi : Z \to B$ to obtain the Euler characteristic of a crepant resolution of Y. In the case of SU(5), SO(10) and E_6 fibrations, we relate this invariant of Y to invariants of the base B via the proper pushforward map φ_*, as $\int_Y \mathcal{M}_{\mathrm{str}}(Y) = \int_B \varphi_* \mathcal{M}_{\mathrm{str}}(Y)$. We recall that for $\psi : Z \to B$ a smooth Weierstrass fibration it is known that

$$\psi_* c(Z) = \frac{12L}{1+6L} c(B),$$

where L is the first Chern class of the line bundle $\mathscr{L} \to B$ used to define a smooth Weierstrass fibration [16]. As such, we may arrive at an expression for $\varphi_* \mathcal{M}_{\mathrm{str}}(Y)$ by subtracting $\frac{12L}{1+6L} c(B)$ from $\varphi_* c_{\mathrm{str}}(Y)$, the outcome of which we record in the following:

Proposition 1.1. *Let* $\varphi : Y \to B$ *be an* SU(5), SO(10) *or* E_6 *elliptic fibration with* $\dim(B) \leq 3$, $D = D_{\mathrm{GUT}} \in A^* B$ *and let* $L = c_1(\mathscr{L})$. *Then* $\varphi_* \mathcal{M}_{\mathrm{str}}$

(Y) is a multiple (in A^*B) of $c(D)$, in particular,

$$(*) \qquad \varphi_*\mathcal{M}_{\mathrm{str}}(Y) = W \cdot c(D),$$

where

$$W = \begin{cases} \dfrac{5(36L^3+42L^2+16L-31LD-30L^2D-6D)}{(1+6L)(1+6L-5D)(1+L)} \\ \qquad \text{(SU(5) case)} \\ \dfrac{4(108L^3+84L^2+21L+10D^2+45D^2L-77DL-144DL^2-8D)}{(1+6L)(1+6L-5D)(1+2L-D)} \\ \qquad \text{(SO(10) case)} \\ \dfrac{3(252L^3+162L^2-378L^2D-165LD+30L-12D+140LD^2+30D^2)}{(1+6L)(1+6L-5D)(1+3L-2D)} \\ \qquad (\mathrm{E}_6 \text{ case}). \end{cases}$$

In the SU(5) and E_6 cases, $\int_B \varphi_*\mathcal{M}_{\mathrm{str}}(Y)$ over a base of dimension three coincides with $\chi_{\mathrm{str}}(Y) - \chi(Z)$ for Z the total space of a smooth Weierstrass fibration as computed in the physics literature [9, 17]. We also note that the assumption on the dimension of the base is required as the small resolutions we exploit to compute stringy Chern classes resolve the singularities of Y only up to codimension three in the base.

From the F-theory perspective, many physically significant quantities may be expressed in terms of integrals (or rather, intersection numbers) on a crepant resolution $\widetilde{Y}$ of the singular fourfold Y. In each of the small resolutions we consider, a resolution variety $\widetilde{Y}$ results from the taking the proper transform of the original singular elliptic fibration Y under a sequence of at least four blowups, thus keeping track of intersection data on $\widetilde{Y}$ can be rather complicated [9, 17]. As such, the methods used in deriving Proposition 1.1 (namely Lemmas 2.1, 2.2 and the push forward formula of [18]) may be used to bypass intersection theory on $\widetilde{Y}$ and reduce *any* integral (in the sense of Chow cohomology) on $\widetilde{Y}$ to an integral on the base B. In what follows we do not assume the total space of the fibration is Calabi–Yau, as the Calabi–Yau case is easily recovered by letting $\mathscr{L} = \mathscr{O}(-K_B)$.

2. A little blowup calculus

In this section, we recall a lemma of Aluffi regarding Chern classes of blowups along with a derivation of a formula for pushing forward powers of the exceptional divisor of a blowup under (somewhat) mild assumptions. We combine both results in the following:

Lemma 2.1 (Aluffi,·). *Let $f : \widetilde{X} \to X$ be the blowup of a smooth variety X along a smooth complete intersection $V : (F_1 = F_2 = \cdots = F_k = 0) \subset X$, let $E \in A^*\widetilde{X}$ be the class of the exceptional divisor and let $U_i \in A^*X$ be the class of $F_i = 0$. Then*

$$c(T_{\widetilde{X}}) = \frac{(1+E)(1+f^*U_1 - E)\cdots(1+f^*U_k - E)}{(1+f^*U_1)\cdots(1+f^*U_k)} \cdot f^*c(T_X) \tag{$\dagger$}$$

and

$$f_*(E^n) = \sum_{i=1}^{k} \left(\prod_{j\neq i} \frac{U_j}{U_j - U_i} \right) U_i^n \tag{$\dagger\dagger$}$$

for all $n \geq 0$.

As $U_j - U_i$ is not necessarily invertible in A^*X, the quantities $\frac{1}{U_j - U_i}$ appearing in formula (††) are formal objects which end up (formally) canceling to give a well-defined class in A^*X. We prove only formula (††) as a proof of (†) can be found in [19]. The proof is merely a simple observation about the *Segre class* of V in X, which is denoted $s(V, X)$ (for more on Segre classes, see also [20]).

Proof. By the birational invariance of Segre classes $f_*s(E, \widetilde{X}) = s(V, X)$, where f_* is the proper pushforward associated with the map $f : \widetilde{X} \to X$. Then from the fact that E and V are both regularly embedded in $\widetilde{X}$ and X, respectively, we have

$$s(E, \widetilde{X}) = \frac{E}{1+E} = E - E^2 + E^3 - \cdots$$

and

$$s(V, X) = \prod_{i=1}^{k} \frac{U_i}{1+U_i},$$

thus

$$f_*(E - E^2 + E^3 - \cdots) = \prod_{i=1}^{k} \frac{U_i}{1+U_i}.$$

Matching terms of like dimension we see that $f_*(E^n)$ equals the coefficient of t^n in the series $(-1)^{n+1} \prod_{i=1}^{k} \frac{tU_i}{1+tU_i}$, which (one can prove by induction) is precisely $\sum_{i=1}^{k} \left(\prod_{j\neq i} \frac{U_j}{U_j - U_i} \right) U_i^n$. □

A nice feature of formula (††) is that it gives a dimension independent way of pushing forward classes in the Chow ring of the blowup $A^*\widetilde{X}$, i.e., a class that is given in terms of a rational expression such as formula (†) may be pushed forward in terms of another rational expression. For example, assume we are under the hypotheses of Lemma 2.1, let $k = 2$, $U_1 = U$, $U_2 = V$, and let $T \in A^*(X)$ [4]. Then, for $n \geq 0$ we have

$$
\begin{aligned}
f_*\left(\frac{E^n}{1+T-E}\right) &= f_*\left(\frac{E^n}{1+T}\left(\frac{1}{1-\frac{E}{1+T}}\right)\right) \\
&= f_*\left((1+T)^{n-1}\sum_{m=n}^{\infty}\left(\frac{E}{1+T}\right)^m\right) \\
&\overset{\text{by }(\dagger\dagger)}{=} \frac{(1+T)^{n-1}}{V-U}\left(V\sum_{m=n}^{\infty}\left(\frac{U}{1+T}\right)^m - U\sum_{m=n}^{\infty}\left(\frac{V}{1+T}\right)^m\right) \\
&= \frac{(1+T)^{n-1}}{V-U}\left(\left(\frac{U}{1+T}\right)^n\frac{V}{1-\frac{U}{1+T}} - \left(\frac{V}{1+T}\right)^n\frac{U}{1-\frac{V}{1+T}}\right) \\
&= \frac{1}{V-U}\left(\frac{U^nV}{1+T-U} - \frac{V^nU}{1+T-V}\right).
\end{aligned}
$$

A general formula for arbitrary k is derived in a similar fashion, which we record in the following:

Lemma 2.2. *Under the assumptions of Lemma 2.1, let $T \in A^*(X)$. Then*

$$
f_*\left(\frac{E^n}{1+T\pm E}\right) = \sum_{i=1}^{k}\left(\prod_{j\neq i}\frac{U_j}{U_j-U_i}\right)\frac{U_i^n}{1+T\pm U_i}
$$

for $n \geq 0$.

As one can see by glancing at formula (†), such pushforward formulas are all that is needed to pushforward Chern classes of blowups, whose derivation involves only simple manipulations of rational expressions and geometric series. Moreover, such manipulations are manifestly independent of the dimension of X.

The small resolutions of GUT vacua we exploit to compute stringy characteristic classes are all obtained by a sequence of blowups along smooth

[4]Here and throughout, we often fail to distinguish between classes and their pullbacks.

complete intersections, and then taking the proper transform of the total space of the singular elliptic fibration Y under the blowups. As such, each blowup in the resolution satisfies the hypotheses of Lemma 2.1. Thus if $\widetilde{Y}$ is a small resolution of Y then pushing forward a class $\gamma \in A^*\widetilde{Y}$ to A^*B amounts to applying of formula (††) until one arrives at a class in $A^*\mathbb{P}(\mathscr{E})$, and then applying the pushforward formula for classes in a projective bundle which was first derived in [18].

We conclude this section with an illustration of our "calculus" applied to the small resolution of the Whitney umbrella.

Example 2.3. Consider the surface

$$W : (y^2 w = x^3 + zx^2) \subset \mathbb{P}^3,$$

i.e., the so-called "Whitney umbrella". The singular locus of W is

$$\mathrm{Sing}(W) : (x = y = 0) \subset \mathbb{P}^3.$$

Now let $f : \widetilde{\mathbb{P}}^3 \to \mathbb{P}^3$ be the blowup of $\mathbb{P}^3$ along $\mathrm{Sing}(W)$. It is readily checked that the proper transform $\widetilde{W}$ of W is a small resolution of W. The stringy Chern class $c_{\mathrm{str}}(W)$ is then equal to $f_*c(\widetilde{W})$. By Lemma 2.1 we have

$$c(\widetilde{W}) = \frac{(1+E)(1+H-E)^2(1+H)^4 \cdot [\widetilde{W}]}{(1+H)^2(1+\widetilde{W})}, \tag{2.1}$$

where E denotes the exceptional divisor of the blowup and H denotes (the pullback of) the hyperplane class in $\mathbb{P}^3$. Since $[\widetilde{W}] = 3H - E \in A^*\widetilde{\mathbb{P}}^3$, (2.1) then simplifies to

$$\begin{aligned} c(\widetilde{W}) &= \frac{(1+E)(1+H-E)^2(1+H)^2 \cdot (3H-E)}{1+3H-E} \\ &= \frac{\alpha_0 + \alpha_1 E + \alpha_2 E^2 + \alpha_3 E^3}{1+3H-E}, \end{aligned}$$

with

$$\begin{aligned} &\alpha_0 = 3H + 12H^2 + 18H^3, \quad \alpha_1 = -1 - 7H - 12H^2 - 4H^3, \\ &\alpha_2 = 1 - H - 12H^2 - 17H^3, \quad \alpha_3 = 1 + 7H + 11H^2. \end{aligned}$$

Then by Lemma 2.2 and the projection formula we have

$$f_*\left(\frac{\alpha_0}{1+3H-E}\right)=\frac{\alpha_0}{1+2H},\quad f_*\left(\frac{\alpha_1 E}{1+3H-E}\right)=0$$

$$f_*\left(\frac{\alpha_2 E^2}{1+3H-E}\right)=\frac{-\alpha_2 H^2}{1+2H},\quad f_*\left(\frac{\alpha_3 E^3}{1+3H-E}\right)=\frac{-\alpha_3 2H^3}{1+2H}.$$

Putting things all together yields

$$c_{\mathrm{str}}(W)=f_*c(\widetilde{W})=\frac{3H+11H^2+17H^3}{1+2H}=3[\mathbb{P}^2]+5[\mathbb{P}^1]+7[\mathbb{P}^0].$$

In particular, we see that the stringy Euler characteristic of the Whitney umbrella is 7.

3. The fibrations under consideration

Let B be a smooth compact complex algebraic variety of dimension at most three endowed with a line bundle $\mathscr{L}\to B$. As mentioned in Section 1, Grand Unified Theories are engineered in F-theory via an elliptic fibration in Tate form:

$$Y:(y^2z+a_1xyz+a_3yz^2=x^3+a_2x^2z+a_4xz^2+a_6z^3)\subset\mathbb{P}(\mathscr{E}),$$

where $\pi:\mathbb{P}(\mathscr{E})\to B$ is the projective bundle of *lines* in $\mathscr{E}=\mathscr{O}_B\oplus\mathscr{L}^2\oplus\mathscr{L}^3$, the a_is are global sections of $\mathscr{L}^i$ and x, y and z are chosen such that the equation for Y corresponds to the zero-locus of a section of $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(3)\otimes\pi^*\mathscr{L}^6$. The physics of F-theory requires that Y be an anti-canonical divisor of $\mathbb{P}(\mathscr{E})$ (which is achieved by taking $\mathscr{L}=\mathscr{O}(-K_B)$), but is not necessary for our considerations thus we make no assumption on $\mathscr{L}$ other than the fact that suitably generic a_is exist. Such a hypersurface naturally determines a proper surjective morphism $\varphi:Y\to B$ such that the generic fiber is an elliptic curve. The elliptic fiber degenerates to a nodal cubic over a generic point of the the discriminant locus $\Delta\subset B$, and enhances to a cusp over a curve in Δ[5]. As nodal and cuspidal singular fibers correspond to abelian gauge groups in F-theory, one engineers a particular non-abelian gauge group by introducing singularities into the total space Y of the elliptic fibration in such a way that a resolution of the codimension one singularities of Y will result in the corresponding singular fiber appearing over a divisor $D_{\mathrm{SU}(5)}:(w=0)$

[5]We recall that the precise definition of Δ was given in Section 1.

of the base. In particular, Tate's algorithm prescribes the precise orders of vanishing along D_{GUT} that each coefficient section a_i must satisfy in order to realize a particular gauge group. For example, in the SU(5) case, Tate's algorithm renders the following (re)definitions of the a_is (for $i \neq 1$):

$$a_2 = \beta_4 w, \quad a_3 = \beta_3 w^2, \quad a_4 = \beta_2 w^3, \quad a_6 = \beta_0 w^5.$$

Each β_j is then necessarily a section of $\mathscr{L}^{6-j} \otimes \mathscr{L}_{\mathrm{GUT}}^{j-5}$ ($\mathscr{L}_{\mathrm{GUT}}$ is the line bundle corresponding to the divisor D_{GUT}) and the new equation for Y then becomes

$$Y : (y^2 z + a_1 xyz + \beta_3 w^2 yz^2 = x^3 + \beta_4 w x^2 z + \beta_2 w^3 x z^2 + \beta_0 w^5 z^3) \subset \mathbb{P}(\mathscr{E}).$$

Upon such redefinitions of the a_is we will refer to such a fibration as an *SU*(5) *elliptic fibration*.

For the SO(10) case, Tate's algorithm prescribes the same definitions of the a_is except a_1, which is now required to vanish to order one along D_{GUT}, i.e., $a_1 = \vartheta w$, where ϑ is a generic section of $\mathscr{L} \otimes \mathscr{L}_{\mathrm{GUT}}^{-1}$ independent from β_4. The new equation for Y then becomes

$$Y : (y^2 z + \vartheta w xyz + \beta_3 w^2 yz^2 = x^3 + \beta_4 w x^2 z + \beta_2 w^3 x z^2 + \beta_0 w^5 z^3) \subset \mathbb{P}(\mathscr{E}).$$

We will refer to such a fibration as an *SO(10) elliptic fibration*.

For the E_6 case, Tate's algorithm prescribes the same definitions of the a_is as in the SO(10) case except a_2, which is now required to vanish to order two along D_{GUT}, i.e., $a_2 = \eta w^2$, where η is a generic section of $\mathscr{L}^2 \otimes \mathscr{L}_{\mathrm{GUT}}^{-2}$ independent from β_3. The new equation for Y then becomes

$$Y : (y^2 z + \vartheta w xyz + \beta_3 w^2 yz^2 = x^3 + \eta w^2 x^2 z + \beta_2 w^3 x z^2 + \beta_0 w^5 z^3) \subset \mathbb{P}(\mathscr{E}).$$

We will refer to such a fibration as an E_6 *elliptic fibration*[6].

The singular loci of SU(5), SO(10) and E_6 elliptic fibrations coincide with the smooth complete intersection[7]

$$Y_{\mathrm{sing}} : (x = y = w = 0) \subset \mathbb{P}(\mathscr{E}).$$

[6]We apologize for the possible confusion, as these "E_6 elliptic fibrations" are different from the smooth family of elliptic fibrations previously referred to in the physics literature also as "E_6 elliptic fibrations" [21, 26].

[7]Though the singular loci of SU(5), SO(10) and E_6 elliptic fibrations coincide, it is the *scheme* structure on the singular locus of each fibration determined by the ideal generated by the partial derivatives of a local defining equation which distinguishes the singularities from one another.

4. Stringy Chern classes

As mentioned in Section 1, stringy Chern classes were defined independently by Aluffi and de Fernex *et al.* using two different technologies which produce the same class. Aluffi's approach was in terms of his theory of "celestial integration", while de Fernex *et al* make use of Kontsevich's theory of motivic integration[8]. When a crepant resolution $f : \widetilde{X} \to X$ of a singular variety X exists, we have $c_{\mathrm{str}}(X) = f_* c(\widetilde{X})$, and is independent of the resolution. As such, we may now exploit the resolution procedures recently provided in the physics literature (which we outline in the Appendix) along with Lemmas 2.1 and 2.2 to compute stringy Chern classes of SU(5), SO(10) and E_6 elliptic fibrations, then we pushforward these classes to the base using the pushforward formula of [18]. More precisely, for $\varphi : Y \to B$ an SU(5), SO(10) or E_6 elliptic fibration and $f : \widetilde{Y} \to Y$ a small resolution of Y we compute $\widetilde{\varphi}_* c(\widetilde{Y})$, where $\widetilde{\varphi} = \varphi \circ f$. We give details of the SU(5) case only as the SO(10) and E_6 cases follow *mutatis mutandis.*

Let $\widetilde{Y}$ be one of the small resolutions of an SU(5) elliptic fibration as given in Section A.1, $H = c_1(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1))$ and let $L = c_1(\mathscr{L})$. By adjunction and Lemma 2.1 (†) we have

$$\begin{aligned} \iota_* c(\widetilde{Y}) &= \frac{(1+E_4)(1+\mathcal{Y}_2-E_4)(1+V-E_4)\cdot[\widetilde{Y}]}{(1+\mathcal{Y}_2)(1+V)(1+[\widetilde{Y}])} f_4^* c(\widetilde{\mathbb{P}}^{(3)}(\mathscr{E})) \\ &= \frac{(1+E_4)(1+\mathcal{Y}_2-E_4)(1+V-E_4)(T-E_4)}{(1+\mathcal{Y}_2)(1+V)(1+T-E_4)} f_4^* c(\widetilde{\mathbb{P}}^{(3)}(\mathscr{E})), \end{aligned}$$

where $T = 3H + 6L - 2E_1 - 2E_2 - E_3$ and $\iota : \widetilde{Y} \hookrightarrow \widetilde{\mathbb{P}}(\mathscr{E})$ is the inclusion. By the projection formula,

$$\begin{aligned} f_{4*} c(\widetilde{Y}) = f_{4*} &\left(\frac{(1+E_4)(1+\mathcal{Y}_2-E_4)(1+V-E_4)(T-E_4)}{(1+T-E_4)} \right) \\ &\times \frac{c(\widetilde{\mathbb{P}}^{(3)}(\mathscr{E}))}{(1+\mathcal{Y}_2)(1+V)}. \end{aligned}$$

[8]As Aluffi often does, we put quotes around celestial integration as it is not defined with respect to some sort of measure, and thus is not an honest "integral". However, it has many points of contact with motivic integration and has integral-like properties such as a change of variable formula with respect to birational maps.

Now let $C = \frac{(1+E_4)(1+\mathcal{Y}_2-E_4)(1+V-E_4)(T-E_4)}{(1+T-E_4)}$ and let $\alpha_0, \ldots, \alpha_3$ be the classes obtained by expanding the numerator of C as a polynomial in E_4:

$$(1+E_4)(1+\mathcal{Y}_2-E_4)(1+V-E_4)(T-E_4) = \alpha_0 + \alpha_1 E_4 + \alpha_2 E_4^2 + \alpha_3 E_4^3 + \alpha_4 E_4^4.$$

Lemma 2.2 (along with the projection formula and the fact that $f_* f^* \alpha = \alpha$ for any blowup f) then gives

$$f_{4*}(C) = \sum_{i=0}^{4} \frac{\alpha_i}{V-\mathcal{Y}_2}\left(\frac{\mathcal{Y}_2^i V}{1+T-\mathcal{Y}_2} - \frac{V^i \mathcal{Y}_2}{1+T-V}\right).$$

Computing three more pushforwards[9] in the same manner then yields the stringy Chern class of Y:

$$c_{\mathrm{str}}(Y) = c(Z) + X \cdot [Y_{\mathrm{sing}}],$$

where $\psi : Z \to B$ is a smooth Weierstrass fibration, $[Y_{\mathrm{sing}}] = (H+2L)(H+3L)D$ is the class of the singular locus of Y and X is a (lengthy) rational expression in H, L and $D := D_{\mathrm{GUT}}$ multiplied by $\pi^* c(B)$ which we do not write explicitly. The stringy Milnor class of Y is then $X \cdot [Y_{\mathrm{sing}}]$ and we recall that

$$c(Z) = \frac{(1+H)(1+H+2L)(1+H+3L)(3H+6L)}{1+3H+6L}\pi^* c(B).$$

As the pushforward to the base of $c(Z)$ has been computed in, e.g., [16], computing $\varphi_* c_{\mathrm{str}}(Y)$ amounts to computing $\varphi_* \mathcal{M}_{\mathrm{str}}(Y) = \varphi_*(X \cdot [Y_{\mathrm{sing}}])$. For this, we view $X \cdot [Y_{\mathrm{sing}}]$ as a class in $A^* \mathbb{P}(\mathscr{E})$ and push it forward to the base via the pushforward formula for classes in a projective bundle first derived in [18]. To apply the pushforward formula of [18] to the case of a projective bundle of the form $\mathbb{P}(\mathscr{O} \oplus \mathscr{L}^2 \oplus \mathscr{L}^3)$, we first expand $X \cdot [Y_{\mathrm{sing}}]$ as a series in H:

$$X \cdot [Y_{\mathrm{sing}}] = \nu_0 + \nu_1 H + \cdots .$$

Then we consider the following expression (viewed as a function of H):

$$F(H) = \frac{(X \cdot [Y_{\mathrm{sing}}] - \gamma)}{H^2},$$

[9]For the full calculation we wrote a Maple program which implements Lemma 2.2, which may be found at www.math.fsu.edu/∼hoeij/files/BlowUp

where $\gamma = \nu_0 + \nu_1 H$. Then the pushforward to the base of $X \cdot [Y_{\mathrm{sing}}]$ is precisely

$$3 \cdot F|_{-3L} - 2 \cdot F|_{-2L} = W \cdot c(D),$$

where W is as given in the conclusion of Proposition 1.1. Thus

$$\varphi_* c_{\mathrm{str}}(Y) = \psi_* c(Z) + W \cdot c(D),$$

from which Proposition 1.1 immediately follows. Upon integration of $\varphi_* c_{\mathrm{str}}(Y)$ we arrive at the following expressions for stringy Euler characteristics for SU(5), SO(10) and E_6 elliptic fibrations in terms of Chern classes of B, D and $\mathscr{L}$ (see tables 1–6).

Table 1: Stringy Euler characteristics of singular SU(5) elliptic fibrations.

dim(B)	$\chi_{\mathrm{str}}(Y)$
1	$12L$
2	$12Lc_1 - 72L^2 + 80LD - 30D^2$
3	$12Lc_2 - 72L^2c_1 + 432L^3 + (80Lc_1 - 830L^2)D$ $+(555L - 30c_1)D^2 - 120D^3$

Table 2: Stringy Euler characteristics of singular Calabi–Yau SU(5) elliptic fibrations.

dim(B)	$\chi_{\mathrm{str}}(Y)$
1	$12c_1$
2	$80c_1D - 60c_1^2 - 30D^2$
3	$288 + 360c_1^3 - 750c_1^2D + 525c_1D^2 - 120D^3$

Table 3: Stringy Euler characteristics of singular SO(10) elliptic fibrations.

dim(B)	$\chi_{\mathrm{str}}(Y)$
1	$12L$
2	$12Lc_1 - 72L^2 + 84LD - 32D^2$
3	$12Lc_2 - 72L^2c_1 + 432L^3 + (84Lc_1 - 840L^2)D$ $+(560L - 32c_1)D^2 - 120D^3$

Table 4: Stringy Euler characteristics of singular Calabi–Yau SO(10) elliptic fibrations.

dim(B)	$\chi_{\text{str}}(Y)$
1	$12c_1$
2	$84c_1D - 60c_1^2 - 32D^2$
3	$288 + 360c_1^3 - 756c_1^2D + 528c_1D^2 - 120D^3$

Table 5: Stringy Euler characteristics of singular E_6 elliptic fibrations.

dim(B)	$\chi_{\text{str}}(Y)$
1	$12L$
2	$12Lc_1 - 72L^2 + 90LD - 36D^2$
3	$12Lc_2 - 72L^2c_1 + 432L^3 + (-864L^2 + 90Lc_1)D$ $+(585L - 36c_1)D^2 - 126D^3$

Table 6: Stringy Euler characteristics of singular Calabi–Yau E_6 elliptic fibrations.

dim(B)	$\chi_{\text{str}}(Y)$
1	$12c_1$
2	$90c_1D - 60c_1^2 - 36D^2$
3	$288 + 360c_1^3 - 774c_1^2D + 549c_1D^2 - 126D^3$

5. Hirzebruch series of a small resolution $\widetilde{Y}$

Let X be a complex smooth variety, then its *Hirzebruch series* is defined as

$$\mathscr{H}_y(X) = \mathscr{H}_0(X) + \mathscr{H}_1(X)y + \mathscr{H}_2(X)y^2 + \cdots = \prod_{i=1}^{\dim(X)} (1 + ye^{-\lambda_i})\frac{\lambda_i}{1 - e^{-\lambda_i}},$$

where λ_i are the Chern roots of the tangent bundle of X and $\mathscr{H}_q(X) = \mathrm{ch}(\Omega_X^q(X))\mathrm{td}(X)$, i.e., the Chern character of the qth exterior power of the cotangent bundle of X multiplied by the Todd class of X. Integrating each term in the Hirzebruch series then yields Hirzebruch's $\chi(y)$ characteristic [22]:

$$\chi(y) = \chi_0 + \chi_1 y + \chi_2 y^2 + \cdots ,$$

where $\chi_q = \int_X \mathscr{H}_q(X)$. By the celebrated Hirzebruch–Riemann–Roch theorem (later generalized by Grothendieck),

$$\chi_q = \sum_{i=0}^{\dim(X)} (-1)^i h^{i,q},$$

where $h^{p,q}$ are the Hodge numbers of X. As the Hodge numbers of string vacua are particularly important in the context of string theory, we were naturally motivated to compute Hirzebruch series of GUT vacua. Now it turns out that the Hirzebruch series of a variety satisfies many of the same properties as its Chern polynomial, thus many formulas for Chern classes may be immediately converted into a formula for a Hirzebruch series, which we now explain.

Let $\mathscr{E} \to X$ be a vector bundle over a smooth variety. Define the Chern-ext character of $\mathscr{E}$ to be

$$\mathrm{ch}_{\mathrm{ext}}(\mathscr{E}) = 1 + \mathrm{ch}(\mathscr{E})y + \mathrm{ch}(\Lambda^2\mathscr{E})y^2 + \cdots ,$$

which can be interpreted as just the Chern character of the *total λ-class* of $\mathscr{E}$ [23]. In [24] it was shown that if

$$0 \longrightarrow \mathscr{A} \longrightarrow \mathscr{B} \longrightarrow \mathscr{C} \longrightarrow 0$$

is an exact sequence of vector bundles, then $\mathrm{ch}_{\mathrm{ext}}$ satisfies the *Whitney property*, i.e.,

$$\mathrm{ch}_{\mathrm{ext}}(\mathscr{B}) = \mathrm{ch}_{\mathrm{ext}}(\mathscr{A})\mathrm{ch}_{\mathrm{ext}}(\mathscr{C}).$$

The fact that the usual Chern character is additive with respect to exact sequences is encoded in the degree one piece of the formula above for the Chern-ext character. As is well known, Todd classes also satisfy the Whitney property. Thus we can associate with any vector bundle $\mathscr{E} \to X$ a *Hirzebruch series*

$$\mathscr{H}_y(\mathscr{E}) = \mathrm{ch}_{\mathrm{ext}}(\mathscr{E}^\vee)\mathrm{td}(\mathscr{E}),$$

and this Hirzebruch series satisfies the Whitney property as well, i.e., given an exact sequence of vector bundles $0 \longrightarrow \mathscr{A} \longrightarrow \mathscr{B} \longrightarrow \mathscr{C} \longrightarrow 0$ we have

$$\mathscr{H}_y(\mathscr{B}) = \mathscr{H}_y(\mathscr{A})\mathscr{H}_y(\mathscr{C}).$$

The Hirzebruch series of a smooth variety X is then $\mathscr{H}_y(TX) \cap [X]$, and if $X \hookrightarrow M$ is a regular embedding into a smooth variety M then adjunction holds, i.e.,

$$\mathscr{H}_y(X) = \frac{\mathscr{H}_y(TM)}{\mathscr{H}_y(N)} \cap [X],$$

where N is the bundle which restricts to the normal bundle of X in M. Moreover, since the Chern character and Todd class are both defined in terms of symmetric functions in Chern roots, the Hirzebruch series satisfies all the properties which characterize Chern classes (such as the projection formula and functoriality $\mathscr{H}_y(f^*\mathscr{E}) = f^*\mathscr{H}_y(\mathscr{E})$) but with a different normalization condition. The normalization condition can be recovered from the definition: If $\mathscr{L} \to X$ is a line bundle and $L = c_1(\mathscr{L})$ is its first Chern class then

$$\mathscr{H}_y(\mathscr{L}) = \frac{(1+ye^{-L})L}{1-e^{-L}}.$$

Thus given a Chern class formula that was derived using only its characterizing properties then you immediately have a formula for its Hirzebruch series as well[10]. For example, the Hirzebruch series version of Aluffi's formula for blowing up Chern classes is the following:

Lemma 5.1. *Let $f : \widetilde{X} \to X$ be the blowup of a smooth variety X along a smooth complete intersection $V : (F_1 = F_2 = \cdots = F_k = 0) \subset X$, let $E \in A^*\widetilde{X}$ be the class of the exceptional divisor and let $U_i \in A^*X$ be the class of $F_i = 0$. Then*

$$\mathscr{H}_y(T_{\widetilde{X}}) = \frac{\mathscr{H}_y(\mathscr{O}(E))\mathscr{H}_y(\mathscr{O}(f^*U_1 - E))\cdots\mathscr{H}_y(\mathscr{O}(f^*U_k - E))}{(1+y)\mathscr{H}_y(\mathscr{O}(f^*U_1))\cdots\mathscr{H}_y(\mathscr{O}(f^*U_k))} \cdot f^*\mathscr{H}_y(T_X).$$

As such, we may compute Hirzebruch series of small resolutions of GUT vacua just as we computed their Chern classes. Furthermore, if $g : X \to V$ is a proper morphism we define $g_*\mathscr{H}_y(X)$ in the obvious way:

$$g_*\mathscr{H}_y(X) = g_*\mathscr{H}_0(X) + g_*\mathscr{H}_1(X)y + \cdots$$

Moreover, if $f : \widetilde{Y} \to Y$ is a small resolution of a GUT elliptic fibration we will then (tentatively) refer to $f_*\mathscr{H}_y(\widetilde{Y})$ as the *stringy Hirzebruch series of* Y, which we will denote by $\mathscr{H}_y^{\mathrm{str}}(Y)$. The analog of Proposition 1.1 is then recorded in the following.

[10]However, one must be careful when converting Chern class formulas into Hirzebruch series formulas, as $c(\mathscr{O}) = 1$, while $\mathscr{H}_y(\mathscr{O}) = 1 + y$.

Proposition 5.2. *Let $\varphi : Y \to B$ be an* SU(5)*,* SO(10) *or* E_6 *elliptic fibration with* $\dim(B) \leq 3$*, $D = D_{\mathrm{GUT}} \in A^*B$ and let $L = c_1(\mathscr{L})$. Then*

$$\varphi_* \mathscr{H}_y^{\mathrm{str}}(Y) = \left(1 - y + \frac{(1+y)(ye^{-5L} - e^{-L})}{(1+ye^{-6L})} + P\right) \cdot \mathscr{H}_y(B),$$

where $P = P_1 y + P_2 y^2 + P_3 y^3 + P_4 y^4$ with P_i a polynomial in D and L for $i = 1, \ldots, 4$, each of which is listed in table 7.

Remark 5.3. The degree zero part of $\mathscr{H}_y(X)$ for smooth X encodes the following classical invariants

$$\int_X \mathscr{H}_y(X) = \begin{cases} \chi_{\mathrm{top}}(X) & \text{for } y = -1, \\ \chi_a(X) & \text{for } y = 0, \\ \sigma(X) & \text{for } y = -1, \end{cases}$$

where $\chi_{\mathrm{top}}(X)$, $\chi_a(X)$ and $\sigma(X)$ denote the topological Euler characteristic, the arithmetic genus, and the signature of X, respectively. If X is singular and admits a crepant resolution $f : \widetilde{X} \to X$, then the degree zero part of $f_* \mathscr{H}_y(\widetilde{X})$ coincides with the degree zero part of $T_{y*}^{\mathrm{str}}(X)$, the stringy motivic

Table 7: Polynomial coefficients of P in the SU(5) case as in Proposition 5.2.

	SU(5) case
P_1	$20D^3 + 15D^2 - 95LD^2 - 40LD + 145L^2D$
P_2	$-140D^3 - 45D^2 + 650LD^2 + 120DL - 975DL^2$
P_3	$380D^3 + 75D^2 - 1760LD^2 + 2635DL^2 - 200LD$
P_4	$-740D^3 - 105D^2 + 280LD - 5125DL^2 + 3425LD^2$

Table 8: Polynomial coefficients of P in the SO(10) case as in Proposition 5.2.

	SO(10) case
P_1	$20D^3 + 16D^2 - 96LD^2 + 147L^2D - 42LD$
P_2	$-140D^3 - 48D^2 + 656LD^2 - 987L^2D + 126LD$
P_3	$380D^3 + 80D^2 - 1776LD^2 + 2667L^2D - 210LD$
P_4	$-740D^3 - 112D^2 + 294LD - 5187L^2D + 3456LD^2$

Table 9: Polynomial coefficients of P in the E_6 case as in Proposition 5.2.

	E_6 case
P_1	$21D^3+18D^2+\frac{303}{2}L^2D-45LD-\frac{201}{2}LD^2$
P_2	$-147D^3-54D^2-\frac{2031}{2}L^2D+135LD+\frac{1371}{2}LD^2$
P_3	$399D^3+90D^2-\frac{3711}{2}LD^2+\frac{5487}{2}L^2D-225LD$
P_4	$-777D^3-126D^2+\frac{7221}{2}LD^2-\frac{10671}{2}L^2D+315LD$

Todd class transformation defined in [27]. As such, in the case of a singular GUT elliptic fibration Y we have

$$\int_Y \mathscr{H}_y^{\mathrm{str}}(Y)\Big|_{y=-1} = \chi_{\mathrm{str}}(Y).$$

Moreover, we may view the degree zero part of $\mathscr{H}_y^{\mathrm{str}}(Y)$ evaluated at $y=0,1$ as "stringy" versions of the arithmetic genus and signature of Y, respectively.

6. A remark on stringy Hodge numbers

To our knowledge, the Hodge numbers of small resolutions of GUT vacua (which we recall coincide with the stringy Hodge numbers of singular GUT elliptic fibrations) have yet to be computed. One difficulty is that elliptic fibrations rarely appear as ample divisors of some smooth ambient variety, thus the Lefschetz hyperplane theorem may not be envoked to identify their upper cohomology with that of some smooth variety. However, in the case of smooth Weierstrass fibrations over toric Fano threefolds, it is indeed the case that the upper cohomology of these elliptic fourfolds coincides with that of the ambient $\mathbb{P}^2$-bundle in which they reside (even though they are not ample divisors) [26]. We speculate that this is due to the fact that even though a smooth Weierstrass fibration Y over a toric Fano threefold is not an ample divisor, perhaps

$$(\ddagger) \qquad H^q(\mathbb{P}(\mathscr{E}),\Omega^p_{\mathbb{P}(\mathscr{E})}(-Y)) = H^q(Y,\Omega^{p-1}_Y(-Y)) = 0$$

for $p+q<\dim(\mathbb{P}(\mathscr{E}))$, which would then imply the result of Lefschetz[11]. Thus if one could show that property (‡) holds for small resolutions of GUT vacua , then the Hodge numbers for any small resolution of a singular Weierstrass fibration would be easily computable via Hirzebruch–Riemann–Roch, although they are not ample divisors. Moreover, using Hodge–Deligne polynomials one would be able to obtain formulas for their Hodge numbers strictly in terms of the Hodge numbers and first Chern class of the base. For example, let $\varphi : Y \to B$ be a smooth Calabi–Yau Weierstrass fibration with B a toric Fano threefold (as defined in Section 1). Then by Hirzebruch–Riemann–Roch and the pushforward formula of [18] we have the following relations between the Hodge numbers of Y [25]:

(‡‡)

$$h^{1,2}-h^{1,1}-h^{1,3}=-40-60c_1(B)^3, \quad h^{2,2}-2h^{2,1}=204+240c_1(B)^3.$$

Now as mentioned earlier, the Hodge numbers $h^{1,1}$ and $h^{1,2}$ coincide with the corresponding Hodge numbers of the total space of the projective bundle $\pi:\mathbb{P}(\mathscr{E})\to B$ in which it resides, which are easily computed in terms of the Hodge numbers of B using Hodge–Deligne polynomials. The Hodge numbers $h^{2,2}$ and $h^{1,3}$ then could immediately be written in terms of the Hodge numbers of B and $c_1(B)^3$ by equations (‡‡).

To be more precise, we recall that if X a smooth projective variety then the Hodge–Deligne polynomial of X is simply

$$E(X)=E(X;u,v):=\sum_{p,q}(-1)^{p+q}h^{p,q}(X)u^pv^q.$$

For example, we have

$$E(\mathbb{P}^n)=1+uv+\cdots+(uv)^n.$$

It is a fact that for a Zariski locally trivial fibration $f:X\to Z$ with fiber F we have $E(X)=E(Z)\cdot E(F)$, thus the Hodge–Deligne polynomial of the ambient projective bundle $\mathbb{P}(\mathscr{E})$ is

$$E(\mathbb{P}(\mathscr{E}))=(1+uv+(uv)^2)\cdot E(B).$$

[11]In the algebraic proof of the Lefschetz hyperplane theorem, it is property (‡) which ultimately yields the conclusion of the theorem. Although the ampleness (or “amplitude”) of Y is sufficient (by Kodaira’s vanishing theorem) to conclude property (‡), it is not necessary.

In particular if B is Fano, then

$$E(B) = 1 + h^{1,1}(B)uv - h^{1,2}(B)(u^2v + uv^2) + h^{1,1}(B)u^2v^2 + u^3v^3,$$

thus

$$h^{1,1}(\mathbb{P}(\mathscr{E})) = h^{1,1}(B) + 1, \quad h^{1,2}(\mathbb{P}(\mathscr{E})) = h^{1,2}(B).$$

Equations (‡‡) then yield the following formulas for the non-trivial Hodge numbers of a smooth Weierstrass fibration Y solely in terms of the Hodge numbers of B and $c_1(B)^3$:

$$\begin{cases} h^{1,1}(Y) = h^{1,1}(B) + 1, \\ h^{1,2}(Y) = h^{1,2}(B), \\ h^{1,3}(Y) = 39 + 60c_1(B)^3 - h^{1,1}(B) + h^{1,2}(B), \\ h^{2,2}(Y) = 204 + 240c_1(B)^3 + 2h^{1,2}(B). \end{cases}$$

We reiterate that these formulas are correct for B any of the 18 toric Fano threefolds. We speculate that not only are these formulas correct for any smooth Calabi–Yau Weierstrass fibration over *any* Fano threefold base, but similar formulas may be derived analogously for small resolutions of singular GUT elliptic fibrations over Fano threefolds, as one can easily compute Hodge–Deligne polynomials of blowups provided the Hodge–Deligne polynomial of the variety which is being blown up and the Hodge–Deligne polynomial of the center of the blowup are known. But indeed, it still remains uncertain whether or not the upper cohomology of small resolutions of GUT vacua coincide with that of the blown up projective bundles in which they reside, or better yet, if property (‡) holds.

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Appendix A. Small resolutions of GUT vacua

We now recall the resolution procedures for SU(5), SO(10) and E_6 elliptic fibrations as first presented in the physics literature [7–9] (we follow very closely the global description of a resolution presented in [17]).

Appendix A.1. SU(5) resolution(s)

First blowup: Let $f_1 : \widetilde{\mathbb{P}}^{(1)}(\mathscr{E}) \to \mathbb{P}(\mathscr{E})$ be the blowup of $\mathbb{P}(\mathscr{E})$ along the singular locus $Y_{\mathrm{sing}} : (x = y = w = 0)$ of Y and denote the exceptional divisor by E_1. The sections x, y and w then pullback as

$$x \mapsto \delta_1 x_1, \quad y \mapsto \delta_1 y_1, \quad w \mapsto \delta_1 w_1,$$

where δ_1 is a regular section $\mathscr{O}(E_1)$ (this also determines the classes $[x_1 = 0]$, $[y_1 = 0]$ and $[w_1 = 0]$). The class of the proper transform $Y^{(1)}$ of Y is then $[Y^{(1)}] = 3H + 6L - 2E_1$, where $H = c_1(\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1))$ and $L = c_1(\mathscr{L})$. When $L = c_1(B)$, the proper transform $Y^{(1)}$ of Y is an anti-canonical divisor of $\widetilde{\mathbb{P}}^{(1)}(\mathscr{E})$, as $c_1(\widetilde{\mathbb{P}}^{(1)}(\mathscr{E})) = f_1^* c_1(\mathbb{P}(\mathscr{E})) + (1-3)E_1 = 3H + c_1(B) + 5L - 2E_1$. By Lemma 2.1 (†), we have

$$c(\widetilde{\mathbb{P}}^{(1)}(\mathscr{E})) = \frac{(1+E_1)(1+H+2L-E_1)(1+H+3L-E_1)(1+D-E_1)}{(1+H+2L)(1+H+3L)(1+D)} \times f_1^* c(\mathbb{P}(\mathscr{E})),$$

where $D := c_1(\mathscr{L}_{\mathrm{GUT}})$.

Second blowup: Let $f_2 : \widetilde{\mathbb{P}}^{(2)}(\mathscr{E}) \to \widetilde{\mathbb{P}}^{(1)}(\mathscr{E})$ be the blowup of $\widetilde{\mathbb{P}}^{(1)}(\mathscr{E})$ along

$$W^{(2)} : (x_1 = y_1 = \delta_1 = 0) \subset \widetilde{\mathbb{P}}^{(1)}(\mathscr{E}).$$

Denote the classes $[x_1 = 0] = H + 2L - E_1$ and $[y_1 = 0] = H + 3L - E_1$ by $\mathcal{X}_1$ and $\mathcal{Y}_1$, respectively, and denote the class of the exceptional divisor by E_2. The sections x_1, y_1 and δ_1 then pullback as

$$x_1 \mapsto \delta_2 x_2, \quad y_1 \mapsto \delta_2 y_2, \quad \delta_1 \mapsto \delta_2 \zeta_2,$$

where δ_2 is a regular section of $\mathscr{O}(E_2)$. The inverse image of Y through the first two blowups takes the form

$$(f_1 \circ f_2)^{-1}(Y) : \left(\zeta_2^2 \delta_2^4 \cdot (y_2 \tau z - \zeta_2 \delta_2 \chi) = 0\right) \subset \widetilde{\mathbb{P}}^{(2)}(\mathscr{E}),$$

where $\tau = y_2 + \zeta_2\beta_3 w_1^2 z + a_1 x_2$ and $\chi = x_2^3\delta_2 + \beta_4 w_1 x_2^2 z + \zeta_2\beta_2 w_1^3 x_2 z^2 + \zeta_2^2\beta_0 w_1^5 z^3$. The proper transform $Y^{(2)}$ is then given by $\{y_2\tau z - \zeta_2\delta_2\chi = 0\}$ and $[Y^{(2)}] = [Y^{(1)}] - 2E_2 = 3H + 6L - 2E_1 - 2E_2$, which again is an anti-canonical divisor of $\widetilde{\mathbb{P}}^{(2)}(\mathscr{E})$ when $L = c_1(B)$. By Lemma 2.1 (†), we have

$$c(\widetilde{\mathbb{P}}^{(2)}(\mathscr{E})) = \frac{(1+E_2)(1+\mathcal{X}_2)(1+\mathcal{Y}_2)(1+E_1-E_2)}{(1+\mathcal{X}_1)(1+\mathcal{Y}_1)(1+E_1)} f_2^* c(\widetilde{\mathbb{P}}^{(1)}(\mathscr{E})),$$

where $\mathcal{X}_2 = \mathcal{X}_1 - E_2$ and $\mathcal{Y}_2 = \mathcal{Y}_1 - E_2$.

Third and fourth blowups: The third and fourth blowups are obtained by blowing up codimension two loci which results in a small resolution of Y. Let $f_3 : \widetilde{\mathbb{P}}^{(3)}(\mathscr{E}) \to \widetilde{\mathbb{P}}^{(2)}(\mathscr{E})$ be the blowup of $\widetilde{\mathbb{P}}^{(2)}(\mathscr{E})$ along

$$W^{(3)} : (y_2 = u = 0) \subset \widetilde{\mathbb{P}}^{(2)}(\mathscr{E}),$$

and let $f_4 : \widetilde{\mathbb{P}}(\mathscr{E}) \to \widetilde{\mathbb{P}}^{(3)}(\mathscr{E})$ be the blowup of $\widetilde{\mathbb{P}}^{(3)}(\mathscr{E})$ along

$$W^{(4)} : (\tau = v = 0) \subset \widetilde{\mathbb{P}}^{(3)}(\mathscr{E}).$$

We denote the exceptional divisors by E_3 and E_4, respectively. The six resolutions are then obtained by taking the proper transform of $Y^{(2)}$ under these blowups corresponding to the following choices for the values of u and v and are shown in table A.1:

The sections y_2, u, τ and v then pullback as

$$y_2 \mapsto \delta_3 y_3, \quad u \mapsto \delta_3 u_3, \quad \tau \mapsto \delta_4 \tilde{y}, \quad v \mapsto \delta_4 v_4,$$

Table A.1: Choices for u and v (and their classes) corresponding to each of the six small resolutions of SU(5) vacua.

	u	v	$[u=0]$	$[v=0]$
1st resolution	δ_2	ζ_2	E_2	$E_1 - E_2$
2nd resolution	δ_2	χ	E_2	$3\mathcal{X}_2 + E_2$
3rd resolution	ζ_2	δ_2	$E_1 - E_2$	E_2
4th resolution	ζ_2	χ	$E_1 - E_2$	$3\mathcal{X}_2 + E_2$
5th resolution	χ	δ_2	$3\mathcal{X}_2 + E_2$	E_2
6th resolution	χ	ζ_2	$3\mathcal{X}_2 + E_2$	$E_1 - E_2$

where δ_3 and δ_4 are regular sections of $\mathscr{O}(E_3)$ and $\mathscr{O}(E_4)$, respectively. By Lemma 2.1 (†), we have

$$c(\widetilde{\mathbb{P}}^{(3)}(\mathscr{E})) = \frac{(1+E_3)(1+\mathcal{Y}_2-E_3)(1+U-E_3)}{(1+\mathcal{Y}_2)(1+U)} f_3^* c(\widetilde{\mathbb{P}}^{(2)}(\mathscr{E})),$$

and

$$c(\widetilde{\mathbb{P}}(\mathscr{E})) = \frac{(1+E_4)(1+\mathcal{Y}_2-E_4)(1+V-E_4)}{(1+\mathcal{Y}_2)(1+V)} f_4^* c(\widetilde{\mathbb{P}}^{(3)}(\mathscr{E})),$$

where U and V are the classes of $\{u=0\}$ and $\{v=0\}$, respectively. The pullback of the defining equation for Y under the four blowups (corresponding to the first resolution in table 1) is

$$\delta_4^3 v_4^2 \delta_3^5 u_3^4 (y_4 y_3 z - v_4 u_3 \chi) = 0,$$

thus the proper transform of Y is given by

$$\widetilde{Y}_{\mathrm{SU}(5)} : (y_4 y_3 z - v_4 u_3 \chi = 0) \subset \widetilde{\mathbb{P}}(\mathscr{E}),$$

and is of class

$$\begin{aligned}[\widetilde{Y}_{\mathrm{SU}(5)}] = [Y^{(3)}] - E_4 = [Y^{(2)}] - E_3 - E_4 = 3H + 6L - 2E_1 \\ - 2E_2 - E_3 - E_4.\end{aligned}$$

Moreover, $\widetilde{Y}_{\mathrm{SU}(5)}$ is easily checked to be a smooth hypersurface of $\widetilde{\mathbb{P}}(\mathscr{E})$ which is K-equivalent to the original singular fourfold Y, and is an anti-canonical divisor when $L = c_1(B)$.

Appendix A.2. SO(10) resolution

For the SO(10) case, we follow the procedure first presented in [8] which requires five blowups for the small resolution of Y, and only give the pullback maps associated with each blowup as everything else follows exactly as in the SU(5) case.

First and second blowups: The first and second blowups are precisely the same as in the SU(5) case and we use the same notations.

Third blowup: Let $g_3 : \widetilde{\mathbb{P}}^{(III)}(\mathscr{E}) \to \widetilde{\mathbb{P}}^{(2)}(\mathscr{E})$ be the blowup of $\widetilde{\mathbb{P}}^{(2)}(\mathscr{E})$ along

$$W'^{(3)} : (y_2 = \zeta_2 = \delta_2 = 0) \subset \widetilde{\mathbb{P}}^{(II)}(\mathscr{E}),$$

and denote the exceptional divisor by E_3. The sections y_2, ζ_2 and δ_2 then pullback as

$$y_2 \mapsto \delta_3 y_3, \quad \zeta_2 \mapsto \delta_3 \zeta_3, \quad \delta_2 \mapsto \delta_3 \xi_3,$$

where δ_3 is a regular section of $\mathscr{O}(E_3)$. The inverse image of the SO(10) fibration through the first three blowups takes the form

$$(g_1 \circ g_2 \circ g_3)^{-1}(Y) : \left(\delta_3^8 \xi_3^4 \zeta_3^2 \cdot (y_3 \tau_3 z - \zeta_3 \xi_3 \chi_3) = 0\right) \subset \widetilde{\mathbb{P}}^{(III)}(\mathscr{E}),$$

where $\tau_3 = \zeta_3 \vartheta w_1 x_2 \xi_3 \delta_3 + y_3 + \zeta_3 \beta_3 w_1^2 z$ and $\chi_3 = \zeta_3^2 \beta_0 w_1^5 z^3 \delta_3^2 + \delta_3 x_2^3 \xi_3 + \delta_3 \zeta_3 \beta_2 w_1^3 x_2 z^2 + \beta_4 w_1 x_2^2 z$. The proper transform $\widetilde{Y}^{(3)}$ is then given by $\{y_3 \tau_3 z - \zeta_3 \xi_3 \chi_3 = 0\}$.

Fourth and fifth blowups: The fourth and fifth blowups are obtained by blowing up codimension two loci which results in a small resolution of the SO(10) fibration. Let $g_4 : \widetilde{\mathbb{P}}^{(IV)}(\mathscr{E}) \to \widetilde{\mathbb{P}}^{(III)}(\mathscr{E})$ be the blowup of $\widetilde{\mathbb{P}}^{(2)}(\mathscr{E})$ along

$$W'^{(4)} : (y_3 = \zeta_3 = 0) \subset \widetilde{\mathbb{P}}^{(III)}(\mathscr{E}),$$

and let $g_5 : \widetilde{\mathbb{P}^{*}}(\mathscr{E}) \to \widetilde{\mathbb{P}}^{(IV)}(\mathscr{E})$ be the blowup of $\widetilde{\mathbb{P}}^{(IV)}(\mathscr{E})$ along

$$W'^{(5)} : (y_4 = \xi_3 = 0) \subset \widetilde{\mathbb{P}}^{(IV)}(\mathscr{E}),$$

where y_4 is the pullback of y_3 minus the exceptional divisor. We denote the exceptional divisors by E_4 and E_5, respectively. The sections y_3, ζ_3, y_4 and ξ_3 then pullback as

$$y_3 \mapsto \delta_4 y_4, \quad \zeta_3 \mapsto \delta_4 \zeta_4, \quad y_4 \mapsto \delta_5 y_5, \quad \xi_3 \mapsto \delta_5 \xi_5,$$

where δ_4 and δ_5 are regular sections of $\mathscr{O}(E_4)$ and $\mathscr{O}(E_5)$, respectively. The pullback of the defining equation for the SO(10) fibration under the five blowups is

$$\delta_4^3 \zeta_4^2 \delta_3^8 \delta_5^5 \xi_5^4 \cdot (y_5 y_4 z - \zeta_4 \xi_5 \chi_3) = 0,$$

thus the proper transform of the SO(10) fibration is then given by

$$\widetilde{Y}_{\mathrm{SO}(10)} : (y_5 y_4 z - \zeta_4 \xi_5 \chi_3 = 0) \subset \widetilde{\mathbb{P}}'(\mathscr{E}),$$

and is of class

$$[\widetilde{Y}_{\mathrm{SO}(10)}] = 3H + 6L - 2E_1 - 2E_2 - 2E_3 - E_4 - E_5.$$

Moreover, $\widetilde{Y}_{\mathrm{SO}(10)}$ is easily checked to be a smooth hypersurface of $\widetilde{\mathbb{P}}'(\mathscr{E})$ which is K-equivalent to the original singular SO(10) fibration, and is an anti-canonical divisor when $L = c_1(B)$.

Appendix A.3. $\mathbf{E_6}$ resolution

For the E_6 case we follow the procedure first presented in [9] which requires seven blowups for the small resolution of Y, and only give the pullback maps associated with each blowup as everything else follows exactly as in the SU(5) and SO(10) cases.

First three blowups: The first three blowups are precisely the same as in the SO(10) case and we use the same notations.

Fourth blowup: Let $h_4 : \widetilde{\mathbb{P}}^{(\diamondsuit)}(\mathscr{E}) \to \widetilde{\mathbb{P}}^{(III)}(\mathscr{E})$ be the blowup of $\widetilde{\mathbb{P}}^{(III)}(\mathscr{E})$ along

$$\mathscr{W}^{(4)} : (y_3 = \zeta_3 = \delta_3 = 0) \subset \widetilde{\mathbb{P}}^{(III)}(\mathscr{E}),$$

and denote the exceptional divisor by E_4. The sections y_3, ζ_3 and δ_3 then pullback as

$$y_3 \mapsto \delta_4 y_4, \quad \zeta_3 \mapsto \delta_4 \zeta_4, \quad \delta_3 \mapsto \delta_4 \xi_4,$$

where δ_4 is a regular section of $\mathscr{O}(E_4)$.

Final three blowups: The final three blowups are obtained by blowing up codimension two loci which results in a small resolution of the E_6 fibration. Let $h_5 : \widetilde{\mathbb{P}}^{(\clubsuit)}(\mathscr{E}) \to \widetilde{\mathbb{P}}^{(\diamondsuit)}(\mathscr{E})$ be the blowup of $\widetilde{\mathbb{P}}^{(\diamondsuit)}(\mathscr{E})$ along

$$\mathscr{W}^{(5)} : (y_4 = \xi_3 = 0) \subset \widetilde{\mathbb{P}}^{(\diamondsuit)}(\mathscr{E}),$$

let $h_6 : \widetilde{\mathbb{P}}^{(\heartsuit)}(\mathscr{E}) \to \widetilde{\mathbb{P}}^{(\clubsuit)}(\mathscr{E})$ be the blowup of $\widetilde{\mathbb{P}}^{(\clubsuit)}(\mathscr{E})$ along

$$\mathscr{W}^{(6)} : (y_5 = \xi_4 = 0) \subset \widetilde{\mathbb{P}}^{(\clubsuit)}(\mathscr{E}),$$

and let $h_7 : \widetilde{\mathbb{P}}^{(\spadesuit)}(\mathscr{E}) \to \widetilde{\mathbb{P}}^{(\heartsuit)}(\mathscr{E})$ be the blowup of $\widetilde{\mathbb{P}}^{(\heartsuit)}(\mathscr{E})$ along

$$\mathscr{W}^{(7)} : (y_6 = \zeta_4 = 0) \subset \widetilde{\mathbb{P}}^{(\heartsuit)}(\mathscr{E}),$$

where y_i denotes the pullback of y_{i-1} minus the exceptional divisor of the corresponding blowup. We denote the exceptional divisors by E_5, E_6 and

E_7, respectively. The sections y_4, ξ_3, y_5, ξ_5, y_6 and ζ_4 then pullback as

$$y_4 \mapsto \delta_5 y_5, \quad \xi_3 \mapsto \delta_5 \xi_5, \quad y_5 \mapsto \delta_6 y_6, \quad \xi_4 \mapsto \delta_6 \xi_6, \quad y_6 \mapsto \delta_7 y_7, \quad \zeta_4 \mapsto \delta_7 \zeta_7,$$

where δ_5, δ_6 and δ_7 are regular sections of $\mathscr{O}(E_5)$, $\mathscr{O}(E_6)$ and $\mathscr{O}(E_7)$, respectively. The pullback of the defining equation for the E_6 fibration under the seven blowups is

$$\delta_4^{12}\delta_6^8\xi_6^8\delta_5^5\xi_5^4\delta_7^3\zeta_7^2 \cdot (\zeta_7\xi_5\xi_6\delta_6\tau_7 - y_7\delta_5\delta_7 z\chi_7) = 0,$$

where $\tau_7 = \beta_0 w_1^5 z^3 \delta_4^3 \delta_7^2 \delta_6 \xi_6 \zeta_7^2 + \zeta_7 \delta_7 \delta_5 \eta w_1^2 x_2^2 z \delta_4^2 \delta_6 \xi_6 \xi_5 + \zeta_7 \delta_4 \delta_7 \beta_2 w_1^3 x_2 z^2 + \delta_5 x_2^3 \xi_5$ and $\chi_7 = y_7 \delta_5^2 + \delta_4 \delta_6 \xi_6 \xi_5 \zeta_7 \vartheta w_1 x_2 \delta_5 + \zeta_7 \beta_3 w_1^2 z$. The proper transform of the E_6 fibration is then given by

$$\widetilde{Y}_{\mathrm{E}_6} : (\zeta_7\xi_5\xi_6\delta_6\tau_7 - y_7\delta_5\delta_7 z\chi_7 = 0) \subset \widetilde{\mathbb{P}}^{(\spadesuit)}(\mathscr{E}),$$

and is of class

$$[\widetilde{Y}_{\mathrm{E}_6}] = 3H + 6L - 2E_1 - 2E_2 - 2E_3 - 2E_4 - E_5 - E_6 - E_7.$$

Moreover, $\widetilde{Y}_{\mathrm{E}_6}$ is easily checked to be a smooth hypersurface of $\widetilde{\mathbb{P}}^{(\spadesuit)}(\mathscr{E})$ which is K-equivalent to the original singular E_6 fibration, and is an anti-canonical divisor when $L = c_1(B)$.

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