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M_{24}-twisted product expansions are Siegel modular forms

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Cheng constructed product expansions from twists of elliptic genera of symmetric powers of $K3$ surfaces that are related to M_{24} moonshine. We study which of them are Siegel modular forms. If the predicted level is non-composite, they are modular, and their powers can be represented as products of rescaled Borcherds products.

1. Introduction

In [13], Conway and Norton described a phenomenon that later became famous as "Monstrous Moonshine" and was completely resolved by Borcherds [6] after parts had been proven in [18]. He showed that (twisted) McKay–Thompson series of the "Monster Algebra" are modular forms. Monstrous moonshine connects the realm of operator algebras, more precisely vertex operator algebras [25], with the world of modular forms [8]. This connection was stimulating to both. In [16], Eguchi et al. discovered another moonshine phenomenon originating in elliptic genera of $K3$, which they linked to the Mathieu group M_{24}. They conjectured that it gives rise to mock modular forms. In this paper, we address a question on modularity that arises in this context.

Mock modular forms generalize holomorphic elliptic modular forms [3, 29, 33, 34], and have been successfully applied, e.g., in physics and combinatorics — see [1, 2, 4, 5, 14, 24, 26–28] and references therein, to name some examples. The findings of Eguchi et al. relate the first few Fourier coefficients of a mock modular form arising naturally from the elliptic genus of $K3$ to sums of dimensions of representations of M_{24}. Motivated by this observation, they conjectured that all Fourier coefficients are decomposable in such a way. This conjecture attracted much interest in physics and mathematics; see, for example, [22]. In the mean time, the initial observation made in [16] has been confirmed by Gannon [20].

It is commonly believed that the above modularity results come from a vertex operator algebra like object, which carries an M_{24} action. In [21], it was shown that there is no $K3$ sigma model with such an action. In light of these results, a relation to VOAs themselves has become rather unlikely from a physical perspective, which emphasizes the role of $K3$. But as the behavior of the series in questions is so close to what is expected from a connection to VOAs, many believe that there is some notion of "weak super VOA" behind them. While the questions on decompositions of Fourier coefficients of mock modular forms posed in [16] were answered in the affirmative, the potential connection to vertex operators algebras, and thus string theory, remains to be examined.

Cheng suggested to construct M_{24} twisted elliptic genera for symmetric powers of $K3$ surfaces [12], which must be modular if a vertex operator algebra as above exists. She gave explicit product expansions Φ_g attached to conjugacy classes g of M_{24}, and conjectured they are Siegel modular forms of degree 2. They are, she argued, related to the 1/4-BPS spectrum of the $\mathrm{K3} \times T^2$-compactified type II string theory [12]. In contrast to the expansions given in [15], they are, however, not modular by construction.

We briefly set up notation to state Conjecture 1.1 and Theorem 1.2. Conjugacy classes (or pairs of conjugacy classes) of M_{24} that appear are labeled according to ATLAS [9] by

$$\begin{aligned}&1\mathrm{A}, 2\mathrm{A}, 2\mathrm{B}, 3\mathrm{A}, 3\mathrm{B}, 4\mathrm{A}, 4\mathrm{B}, 4\mathrm{C}, 5\mathrm{A}, 6\mathrm{A}, 6\mathrm{B},\\ &7\mathrm{AB}, 8\mathrm{A}, 10\mathrm{A}, 11\mathrm{A}, 12\mathrm{A}, 12\mathrm{B}, 14\mathrm{AB}, 15\mathrm{AB}, 21\mathrm{AB}, 23\mathrm{AB}.\end{aligned} \tag{1.1}$$

Siegel modular forms generalize the notion of elliptic modular forms, and appear, for example, in applications to moduli problems [23] and state counting in the theory of quantum black holes [14]. We define Siegel modular forms of degree 2: Let

$$\mathbb{H}_2 = \{Z \in \mathrm{Mat}_2(\mathbb{C}) \,:\, Z^{\mathrm{T}} = Z,\, \mathfrak{Im}(Z) \text{ positive definite}\}$$

be the Siegel upper half space of degree 2. A meromorphic Siegel modular form (of degree 2) of weight $k \in \mathbb{Z}$ for a finite index subgroup $\Gamma \subseteq \mathrm{Sp}_2(\mathbb{Z})$ and a character χ of Γ is a meromorphic function $\Phi : \mathbb{H}_2 \to \mathbb{C}$ that satisfies

$$\Phi\big((AZ+B)(CZ+D)^{-1}\big) = \det(CZ+D)^k \chi\big(\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)\big)\, \Phi(Z)$$

for all $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$. The subgroups $\Gamma_0^{(2)}(N)$ $(0 < N \in \mathbb{Z})$ of all matrices in $\mathrm{Sp}_2(\mathbb{Z})$ with $C \in N \cdot \mathrm{Mat}_2(\mathbb{Z})$ play a major role in the present work. We say a Siegel modular form has level N if it is a Siegel modular form for $\Gamma_0^{(2)}(N)$ and some character.

Table 1: The level N_g and the weight k_g of Φ_g. Precise expressions for Φ_g can be found in table 2 on page 475. A proof is given in Theorem 5.2 and Corollary 5.3.

g	N_g	k_g
1A	1	10
2A	2	6
2B	4	4
3A	3	4
3B	9	2
4A	8	2
4B	4	3
4C	16	1
5A	5	2
7AB	7	1
8A	8	$\frac{1}{2}$
11A	11	0
23AB	23	-1

In [10], Cheng and Duncan discussed more thoroughly the twisted elliptic genera that appeared in [12]. They made the following precise conjecture.

Conjecture 1.1 Cheng, Duncan [10]. *For all conjugacy classes g that are given in* (1.1)*, the product Φ_g (defined in* (4.2)*) is a Siegel modular form of level N_g, where N_g is given in table 1.*

The affirmative answer to this conjecture would mean significant support for the idea that a vertex operator algebra like object with M_{24} action exists. It also has an interpretation in terms of root multiplicities of generalized Kac–Moody algebras [12]. This is connected to one of the key steps in Borcherds proof of Monstrous moonshine [6], where Φ_g is replaced by $j(\tau) - j(\sigma)$.

The cases $g \in \{1\text{A}, 2\text{A}, 3\text{A}, 4\text{B}\}$ of Conjecture 1.1 were solved using Borcherds products (see, e.g., [11]). We are able to resolve all cases for which N_g is a prime power. More precisely, we prove the following result:

Theorem 1.2. *For*

$$g \in \big\{1\text{A}, 2\text{A}, 2\text{B}, 3\text{A}, 3\text{B}, 4\text{A}, 4\text{B}, 4\text{C}, 5\text{A}, 7\text{AB}, 8\text{A}, 11\text{A}, 23\text{AB}\big\},$$

Conjecture 1.1 is true.

We remark that the weights k_g of Φ_g, given in table 1, are not to be confused with the weights k_g that are given in table 1 of [10]. The latter are simply half the number of cycles in the conjugacy class g. Already in the case $g = 1\mathrm{A}$ the two quantities, which then equal 10 and 12, are not the same. Concerning the conjugacy classes with composite N_g attached to them, we can currently not say much. We still expect that the Φ_g are modular in these cases. But they cannot be represented as products of Borcherds products in the same way as the Φ_g for prime powers N_g can.

The functions Φ_g given by Cheng and Duncan are product expansions. In the case $g \in \{1\mathrm{A}, 2\mathrm{A}, 3\mathrm{A}, 4\mathrm{B}\}$ they were known to be Borcherds products. Borcherds products, studied in [7], are the only basic construction of modular product expansions that are currently known. Clearly, products of Borcherds products are modular, too, and so are rescaled Borcherds functions $\Phi(NZ)$ $(0 < N \in \mathbb{Z})$, where $\Phi(Z)$ is a Borcherds product. Thus, it is natural to believe that all Φ_g, if they are modular, are products of rescaled Borcherds products. If they are not, but Conjecture 1.1 is still true, it would provide a novel basic construction of modular product expansions, which is unlikely. For this reason, we have studied the question whether or not the Φ_g are products of rescaled Borcherds products.

Our proof of Theorem 1.2 relies on linearization of product expansions. We associate to each product expansions Φ a vector $\mathcal{E}(\Phi)$ in a $\mathbb{Z}$-module E that is defined in Section 2. For any $0 < N \in \mathbb{Z}$, we can identify explicitly a submodule $E_{\mathrm{Bor}}(N) \subset E$ associated with products of rescaled Borcherds products of level N. Proving Theorem 1.2 then amounts to checking whether the rank one module $\mathbb{Z} \cdot \mathcal{E}(\Phi_g)$ has non-trivial intersection with $E_{\mathrm{Bor}}(N_g)$. However, it is a difficult task to compute $\mathcal{E}_{\mathrm{Bor}}(N)$, which, if N is not squarefree, involves computation of Fourier expansions of elliptic modular forms of non-squarefree level at all cusps. We employ Sage [32] and [31] to do these computations.

The paper is organized as follows. Section 2 contains preliminaries on modular forms and product expansions. In Section 3, we recall the theory of Borcherds products that we will make use of. We also introduce rescaled Borcherds products. Section 4 contains a revision of the material in [10] that is relevant to this paper. The proof of Theorem 5.2 is discussed in Section 5. Final remarks are given in Section 6.

We have included three appendices. Appendix A contains a description of how we found, by computer methods, the solutions to Cheng's and Duncan's modularity problem. In Appendix B, we briefly describe numerical tests performed to verify data in table 6. Most tables included in this paper are given in Appendix C.

2. Preliminaries

Denote the space of elliptic modular forms of weight k for a finite index subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ by $\mathrm{M}_k(\Gamma)$. We write $\mathrm{M}^!_k(\Gamma)$ for the space of weakly holomorphic modular forms of weight k. Let $\mathbb{H} \subset \mathbb{C}$ be the Poincaré upper half plane. The weight k slash action is denoted by $|_k$. Precise definitions and a useful introduction into the subject can be found in [8].

We write $\mathrm{J}_{k,m}(\Gamma) \subseteq \mathrm{J}^{(!)}_{k,m}(\Gamma) \subseteq \mathrm{J}^!_{k,m}(\Gamma)$ for the space of Jacobi forms, weak Jacobi forms, and weakly holomorphic Jacobi forms of weight k and index m for $\Gamma \ltimes \mathbb{Z}^2$. Jacobi forms are functions on the Jacobi upper half plane $\mathbb{H}^{\mathrm{J}} := \mathbb{H} \times \mathbb{C}$. The reader is referred to [17] for definitions.

Fourier expansions of elliptic modular forms and weak Jacobi forms are central in this paper. Write $q := e(\tau)$, $\zeta := e(z)$, $e(x) := \exp(2\pi i\, x)$, where τ and (τ, z) are coordinates of the Poincaré and the Jacobi upper half-plane, respectively. Write π_{FE} for the projection of the Fourier expansion to those terms with integral exponents:

$$\pi_{\mathrm{FE}}\Bigg(\sum_{n \in \mathbb{Q}} c(n)\, q^n\Bigg) := \sum_{n \in \mathbb{Z}} c(n)\, q^n \quad \text{and}$$
$$\pi_{\mathrm{FE}}\Bigg(\sum_{n \in \mathbb{Q},\, r \in \mathbb{Z}} c(n,r)\, q^n \zeta^r\Bigg) := \sum_{n \in \mathbb{Z}\, r \in \mathbb{Z}} c(n,r)\, q^n \zeta^r.$$

Fix a cusp $\mathfrak{c}$ of $\mathbb{H}$ and a matrix $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, such that $\gamma\infty = \mathfrak{c}$. For a modular form f of weight k or a weak Jacobi form ϕ of weight k and index m we write

$$\pi_{\mathrm{FE}}(f_{\mathfrak{c}}) := \pi_{\mathrm{FE}}(f|_k\gamma) \quad \text{and} \quad \pi_{\mathrm{FE}}(\phi_{\mathfrak{c}}) := \pi_{\mathrm{FE}}(\phi|_{k,m}\gamma).$$

This notation is well defined, since the left-hand sides only depend on $\mathfrak{c}$, but not on γ.

For $0 < N \in \mathbb{Z}$, denote the set of cusps of $\Gamma_0(N) \backslash \mathbb{H}$ by $\mathcal{C}(N)$. Given such a cusp $\mathfrak{c} \in \mathcal{C}(N)$, we write $h_{\mathfrak{c}}(N)$ and $e_{\mathfrak{c}}(N)$ for the width and the denominator of $\mathfrak{c}$, respectively. Fix some $0 \le f_{\mathfrak{c}}(N) \le e_{\mathfrak{e}}(N)$ such that $\mathfrak{c}$ is $\Gamma_0(N)$-equivalent to $f_{\mathfrak{c}} \,/\, e_{\mathfrak{c}}$. Set $N_{\mathfrak{c}}(N) := N e_{\mathfrak{c}}(N)^{-1}$. We will write $\pi_{\mathrm{FE}}(f_{\mathfrak{c}})$, $\mathfrak{c} \in \mathcal{C}(N)$, if f is a modular form for $\Gamma_0(N)$.

2.1. Elliptic modular forms

As a shorthand notation, we will write $\mathrm{M}_k(N)$ for $\mathrm{M}_k(\Gamma_0(N))$. Fix k and N, and let $d = \dim \mathrm{M}_k(N)$. Recall that the echelon basis of elliptic modular forms of weight k for $\Gamma_0(N)$ consists of elements $f_{k,N;1}, \dots, f_{k,N;d} \in \mathrm{M}_k(N)$ with Fourier expansions

$$f_{k,N;d}(\tau) = \sum_n c(f_{k,N;d}; n)\, q^n$$

such the matrix $\big(c(f^{(d)}_{k,N}; n)\big)_{1 \le i \le d;\, 0 \le n}$ has row reduced echelon form. Define

$$\mathrm{M}_k(N)[c_1, \dots, c_d] = \sum_{i=1}^{d} c_i\, f_{k,N;i}.$$

We use this notation frequently in Section 5, and in particular, in table 2. We will also identify elliptic modular forms with coordinate (column) vectors with respect to this basis.

Example 2.1. The echelon basis of $\mathrm{M}_2(11)$ has form

$$f_{2,11;1}(\tau) = 1 + 12q^2 + 12q^3 + 12q^4 + 12q^5 + O(q^6),$$
$$f_{2,11;2}(\tau) = q - 2q^2 - q^3 + 2q^4 + q^5 + O(q^6).$$

Correspondingly, we have

$$\mathrm{M}_2(11)[c_1, c_2] = c_1 + c_2 q + (12c_1 - 2c_2)q^2 + (12c_1 - c_2)q^3 + (12c_1 + 2c_2)q^4 + (12c_1 + c_2)q^5 + O(q^6).$$

Proposition 2.2. *Suppose that $f \in \mathrm{M}_k(N)$ for some $0 < N \in \mathbb{Z}$. If $\mathfrak{c} \in \mathcal{C}(N)$ has a representative of the form $1 / e \in \mathbb{Q}$, then $\pi_{\mathrm{FE}}(f_{\mathfrak{c}}) \in \mathrm{M}_k(\Gamma)$, where*

$$\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) : a - d \equiv 0 \pmod{\gcd(e, e^{-1}N)} \right\}.$$

In particular, if the odd part of N is squarefree and the even part divides 8, then $\pi_{\mathrm{FE}}(f_{\mathfrak{c}}) \in \mathrm{M}_k(\Gamma_0(N))$ for all $\mathfrak{c} \in \mathcal{C}(N)$.

Table 2: We have $\Phi_g = \prod_i B_{N_{g,i}}[\phi_{g,i}, n_{g,i}]$. A proof is given in Theorem 5.2.

g	$N_{g,i}$	$n_{g,i}$	$\mathrm{tc}_0(\phi_{g,i})$	$\mathrm{tc}_2(\phi_{g,i})$
1A	1	1	24	0
2A	2	1	8	$\mathrm{M}(2)_2[\frac{4}{3}]$
2B	1	1	-12	0
	2	1	12	0
	4	1	0	$\mathrm{M}_2(4)[2, -16]$
3A	3	1	6	$\mathrm{M}(3)_2[\frac{3}{2}]$
3B	1	1	-8	0
	3	1	8	$\mathrm{M}_2(3)[2]$
4A	1	1	-6	0
	2	1	2	$\mathrm{M}(2)_2[-\frac{2}{3}]$
	4	1	4	$\mathrm{M}(4)_2[\frac{2}{3}, 16]$
	8	1	0	$\mathrm{M}(8)_2[2, 0, -16]$
4B	4	1	4	$\mathrm{M}(4)_2[\frac{5}{3}, 8]$
4C	1	1	-3	0
	2	1	-3	$\mathrm{M}(2)_2[\frac{1}{4}]$
	4	1	6	$\mathrm{M}(4)_2[\frac{7}{4}, -14]$
	4	2	0	$\mathrm{M}(4)_2[1, -8]$
5A	5	1	4	$\mathrm{M}(5)_2[\frac{5}{3}]$
7AB	7	1	3	$\mathrm{M}(7)_2[\frac{7}{4}]$
8A	1	1	$\frac{3}{2}$	0
	2	1	$-\frac{5}{2}$	$\mathrm{M}(2)_2[\frac{1}{3}]$
	4	1	1	$\mathrm{M}(4)_2[\frac{1}{6}, -12]$
	8	1	2	$\mathrm{M}(8)_2[\frac{4}{3}, 8, 0]$
	8	2	0	$\mathrm{M}(8)_2[1, 0, -8]$
11A	11	1	2	$\mathrm{M}(11)_2[\frac{11}{6}, 0]$
23AB	23	1	1	$\mathrm{M}(23)_2[\frac{23}{12}, \frac{46}{11}, -\frac{23}{11}]$

Proof. Let $\mathfrak{c} \in \mathcal{C}(N)$ with representative $1\,/\,e$, as in the statement. By definition,

$$\pi_{\mathrm{FE}}(f_{\mathfrak{c}}) = \pi_{\mathrm{FE}}\left(f\big|_k \begin{pmatrix} 1 & 0 \\ e & 1 \end{pmatrix} \right) = \sum_{n \pmod{h_{\mathfrak{c}}(N)}} f\big|_k \begin{pmatrix} 1 & 0 \\ e & 1 \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

For $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and any choices of $n'(n)$, we have

$$\pi_{\mathrm{FE}}(f_{\mathfrak{c}})\big|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \sum_{n \,(\mathrm{mod}\, h_{\mathfrak{c}}(N))} \left(f\big|_k \begin{pmatrix} 1 & 0 \\ e & 1 \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right.$$
$$\left. \times \begin{pmatrix} 1 & -n'(n) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -e & 1 \end{pmatrix} \right) \Big|_k \begin{pmatrix} 1 & 0 \\ e & 1 \end{pmatrix} \begin{pmatrix} 1 & n'(n) \\ 0 & 1 \end{pmatrix}.$$

We shall want to choose $n'(n)$ such that it runs through a system of representatives mod $h_{\mathfrak{c}}(N)$ as n does, and the bottom left entry of the inner matrix product is divisible by N. Then we can sum over n' instead of n, and we will have shown that

$$\pi_{\mathrm{FE}}(f_{\mathfrak{c}})\big|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pi_{\mathrm{FE}}(f_{\mathfrak{c}}),$$

which proves the desired statement.

The divisibility condition on the bottom left entry of the inner matrix product is equivalent to

$$ea\, n'(n) \equiv a - d - eb - edn \pmod{e^{-1}N}.$$

By the condition on $a - d$, this is the same as

$$\frac{e}{\gcd(e, e^{-1}N)} a n'(n) \equiv \frac{a - d - eb - edn}{\gcd(e, e^{-1}N)} \pmod{\gcd(e^2, N)^{-1} N}.$$

We have $h_{\mathfrak{c}}(N) = \gcd(e^2, N)^{-1} N$, so the congruence condition is a condition mod $h_{\mathfrak{c}}(N)$. Using the Chinese Remainder Theorem, we can and will assume that N is a prime power. It is straightforward to check that either $e = \gcd(e, e^{-1}N)$ or $\gcd(e, e^{-1}N)^{-1} f$ is a unit mod $h_{\mathfrak{c}}(N)$. Since, further, a is coprime to N, we can choose

$$n'(n) \equiv a^{-1} \left(\left(\frac{e}{\gcd(e, e^{-1}N)} \right)^{-1} \frac{a - d}{\gcd(e, e^{-1}N)} - b - dn \right) \pmod{h_{\mathfrak{c}}(N)}.$$

Because d is coprime to N, too, we find that $n'(n)$ runs through a system of representatives mod $h_{\mathfrak{c}}(N)$ as n does. This completes the proof. □

From the previous proposition, we find that

$$\pi_{\mathrm{FE}}(\,\cdot\,_{\mathfrak{c}}) : \mathrm{M}_k(N) \to \mathrm{M}_k(N)$$

for $N \in \{2, 3, 4, 5, 7, 8, 11, 23\}$. In table 6, we have given these maps as matrices $\Pi_{\mathrm{FE}}(k, N, \mathfrak{c})$ acting on coordinate (column) vectors with respect to the echelon basis. They have been computed by the method described in [31], which uses algebraic methods. In Appendix B, we describe additional, numerical tests that have been performed to verify their correctness.

Example 2.3. The first column of $\Pi_{\mathrm{FE}}(2, 8; \frac{1}{2})$ equals $(-\frac{1}{8}\; 3\; -3)^{\mathrm{T}}$. Therefore, we have

$$\pi_{\mathrm{FE}}\left(\mathrm{M}_2(8)[1,0,0]_{\frac{1}{2}}\right) = \mathrm{M}_2(8)\left[-\frac{1}{8}, 3, -3\right].$$

More concretely, this means

$$\pi_{\mathrm{FE}}\left(\mathrm{M}_2(8)[1,0,0]_{\frac{1}{2}}\right) = -\frac{1}{8} + 3q - 3q^2 + 1332q^3 + 2172q^4 + 46026q^5 + O(q^6)$$

2.2. Jacobi forms of index 1

In this paper, we only use weak Jacobi forms of weight 0 and index 1. Such a Jacobi form ϕ has a Fourier expansion

$$\sum_{0 \geq n,\, r \in \mathbb{Z}} c(\phi;\, 4n - r^2)\, q^n \zeta^r.$$

Recall the weak Jacobi forms $\phi_{0,1}$ and $\phi_{-2,1}$ defined in [17], Theorem 9.3. In the proof of Theorem 5.2, we will need the initial Fourier expansions of $\phi_{-2,1}$ and $\phi_{0,1}$. They equal

$$\phi_{-2,1}(\tau, z) = \zeta - 2 + \zeta^{-1} + O(q), \quad \phi_{0,1}(\tau, z) = \zeta + 10 + \zeta^{-1} + O(q). \tag{2.1}$$

Fix $0 < N \in \mathbb{Z}$. Any $\phi \in \mathrm{J}^!_{0,1}(\Gamma_0(N))$ can be written as

$$\phi(\tau, z) = tc_0(\phi)(\tau)\,\frac{\phi_{0,1}(\tau, z)}{12} + tc_2(\phi)(\tau)\,\phi_{-2,1}(\tau, z),$$

where $tc_0(\phi) \in \mathrm{M}^!_0(\Gamma_0(N))$ and $tc_2(\phi) \in \mathrm{M}^!_2(\Gamma_0(N))$ are the (rescaled) 0th and 2nd Taylor coefficient of ϕ. One can directly verify that ϕ is a weak

Jacobi form in the sense of [17] if $tc_0(\phi) \in \mathbb{C} = \mathrm{M}_0(\Gamma_0(N))$ and $tc_2(\phi) \in \mathrm{M}_2(\Gamma_0(N))$. This yields an isomorphism of vector spaces

$$\mathrm{J}^{(!)}_{0,1}(\Gamma_0(N)) \to \mathrm{M}_0(\Gamma_0(N)) \times \mathrm{M}_2(\Gamma_0(N)), \quad \phi \mapsto \big(tc_0(\phi), tc_2(\phi)\big). \tag{2.2}$$

We will use this in order to reduce our considerations to elliptic modular forms.

In order to apply Theorem 3.1, which describes the product expansions that arise via Borcherds's construction, we have to compute Fourier expansions, $\pi_{\mathrm{FE}}(\phi_{\mathfrak{c}})$, at all cusps $\mathfrak{c}$ for weak Jacobi forms ϕ. Since $\phi_{0,1}$ and $\phi_{-2,1}$ are weak Jacobi forms of level 1, we get

$$\begin{aligned}\pi_{\mathrm{FE}}(\phi_{\mathfrak{c}}) &= \pi_{\mathrm{FE}}(tc_0(\phi)_{\mathfrak{c}})\,\frac{\phi_{0,1}}{12} + \pi_{\mathrm{FE}}(tc_2(\phi)_{\mathfrak{c}})\,\phi_{-2,1} \\ &= tc_0(\phi)\,\frac{\phi_{0,1}}{12} + \phi_{\mathrm{FE}}(tc_2(\phi)_{\mathfrak{c}})\,\phi_{-2,1}.\end{aligned} \tag{2.3}$$

2.3. Siegel product expansions

Siegel modular forms (of genus 2) are certain functions on the Siegel upper half space

$$\mathbb{H}_2 = \{Z \in \mathrm{Mat}_2(\mathbb{C}) : Z^{\mathrm{T}} = Z,\, \mathfrak{Im}(Z) \text{ positive definite}\}$$

We will write $Z = \left(\begin{smallmatrix}\tau_1 & z\\ z & \tau_2\end{smallmatrix}\right)$ for the entries of Z, and $q_1 := e(\tau_1)$, $\zeta := e(z)$, and $q_2 := e(\tau_2)$ for the corresponding Fourier expansion variables.

Write $\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right)$, with $A, B, C, D \in \mathrm{Mat}_2(\mathbb{R})$ for an element $\gamma \in \mathrm{Sp}_2(\mathbb{R})$.

Definition 2.4. Let $f : \mathbb{H}_2 \to \mathbb{C}$ be a holomorphic function. We call f a Siegel modular form of weight k with character χ for $\Gamma \subset \mathrm{Sp}_2(\mathbb{Z})$ if and only if

$$\Phi((AZ+B)(CZ+D)^{-1}) = \det(CZ+D)^k \chi\left(\begin{pmatrix}A & B\\ C & D\end{pmatrix}\right)\Phi(Z)$$

for all $\gamma \in \Gamma$.

We refer the reader to [19] for details on Siegel modular forms. For $0 < N \in \mathbb{Z}$, we define

$$\Gamma^{(2)}_0(N) := \left\{\begin{pmatrix}A & B\\ C & D\end{pmatrix} : C \equiv 0 \in \mathrm{Mat}_2(\mathbb{Z}) \pmod{N}\right\}.$$

Fix $0 < N_\Phi \in \mathbb{Z}$ and $e_{q_1}(\Phi)$, $e_\zeta(\Phi)$, $e_{q_2}(\Phi) \in \mathbb{Z}$. Given weak Jacobi forms

$$\psi[\Phi, d](\tau, z) := \sum_{n \geq 0, r \in \mathbb{Z}} c[\Phi, d](4n - r^2)\, q^n \zeta^r \in \mathrm{J}^{(!)}_{0,1}(\Gamma_0(N_\Phi)) \tag{2.4}$$

for $0 < d \in \mathbb{Z}$ that have integral Fourier coefficients and vanish for all but finitely many d, we can define an absolutely convergent product expansion

$$\Phi(Z) = q_1^{e_{q_1}(\Phi)} \zeta^{e_\zeta(\Phi)} q_2^{e_{q_2}(\Phi)} \prod_{d \,|\, n_\Phi} \prod_{(n,r,m)>0} (1 - (q_1^n \zeta^r q_2^m)^d)^{c[\Phi,d](4nm - r^2)}.$$

All product expansions that show up in this paper are of this form. Write E for the space of functions $\mathbb{Z}_{>0} \to \mathrm{M}_0(N_\Phi) \times \mathrm{M}_2(N_\Phi)$ with finite support. We obtain $\mathcal{E}(\Phi) \in E$:

$$\mathcal{E}(\Phi) : \mathbb{Z}_{>0} \longrightarrow \mathrm{M}_0(N_\Phi) \times \mathrm{M}_2(N_\Phi),$$
$$\mathcal{E}(\Phi)(d) = \begin{cases} (tc_0(\psi[\Phi, d]), tc_2(\psi[\Phi, d])), & \text{if } d \,|\, n_\Phi; \\ 0, & \text{otherwise.} \end{cases}$$

Product expansions can be represented by $\mathcal{E}(\Phi)$ and the triple $(e_{q_1}(\Phi), e_\zeta(\Phi), e_{q_2}(\Phi))$.

Observe that

$$\mathcal{E}(\Phi\Phi') = \mathcal{E}(\Phi) + \mathcal{E}(\Phi') \tag{2.5}$$

for product expansions Φ and Φ'. That is, the map $\mathcal{E}$ linearizes our problem of identifying Cheng's and Duncan's product expansion as products of rescaled Borcherds products.

3. Rescaled Borcherds products

We recall a special case of a theorem by Cléry and Gritsenko [11], which is a refinement and reformulation in terms of Jacobi forms, of Borcherds's result [7] on product expansions for Siegel modular forms.

For a triple (n, r, m) of integers, $(n, r, m) > 0$ means that $n > 0$, or $n = 0$ and $m > 0$, or $n = m = 0$ and $r < 0$.

Theorem 3.1 Cléry–Gritsenko. *Let* $\phi(\tau, z) = \sum_{n,r} c(4n - r^2)\, q^n \zeta^r \in J^!_{0,1}(\Gamma_0(N))$. *Write* $c(\phi_\mathfrak{c}; \Delta)$ *for the Fourier coefficients of* ϕ *at* $\mathfrak{c}$, *and* *assume that for all* $\mathfrak{c} \in \mathcal{C}(N)$ *we have* $h_{\mathfrak{c}}(N) N_{\mathfrak{c}}(N)^{-1}\, c(\phi_{\mathfrak{c}}; \Delta) \in \mathbb{Z}$, *if* $\Delta \le 0$. *Let*

$$B_N[\phi](Z) = q_1^{e_{q_1}(\phi)} \zeta^{e_\zeta(\phi)} q_2^{e_{q_2}(\phi)} \times \prod_{\mathfrak{c} \in \mathcal{C}(N)} \prod_{(n,r,m)>0} \left(1 - (q_1^n \zeta^r q_2^m)^{N_{\mathfrak{c}}(N)}\right)^{h_{\mathfrak{c}}(N) N_{\mathfrak{c}}(N)^{-1}\, c(\phi_{\mathfrak{c}};\, 4nm - r^2)}$$

where

$$e_{q_1}(\phi) = e_{q_1}(B_N[\phi]) = \frac{1}{24} \sum_{\substack{\mathfrak{c} \in \mathcal{C}(N) \\ l \in \mathbb{Z}}} h_{\mathfrak{c}}(N) c(\phi_{\mathfrak{c}};\, -l^2),$$

$$e_\zeta(\phi) = e_\zeta(B_N[\phi]) = \frac{1}{2} \sum_{\substack{\mathfrak{c} \in \mathcal{C}(N) \\ 0 < l \in \mathbb{Z}}} l\, h_{\mathfrak{c}}(N) c(\phi_{\mathfrak{c}};\, -l^2),$$

$$e_{q_2}(\phi) = e_{q_2}(B_N[\phi]) = \frac{1}{4} \sum_{\substack{\mathfrak{c} \in \mathcal{C}(N) \\ l \in \mathbb{Z}}} l^2 h_{\mathfrak{c}}(N) c(\phi_{\mathfrak{c}};\, -l^2).$$

Set

$$k = \frac{1}{2} \sum_{\mathfrak{c} \in \mathcal{C}(N)} h_{\mathfrak{c}}(N) N_{\mathfrak{c}}(N)^{-1}\, c_{\mathfrak{c}}(0).$$

Then $B_N[\phi]$ *is a meromorphic Siegel modular form of weight* k *with character for* $\Gamma_0^{(2)}(N)$.

Given $0 < n \in \mathbb{Z}$, we write $B_N[\phi, n](Z) = B_N[\phi](nZ)$ for a *rescaled Borcherds product*, which has level nN.

The map $\mathcal{E}(B_N[\phi, n]) \in E$ associated to $B_N[\phi, n]$ is defined by

(3.1)

$$\mathcal{E}(B_N[\phi, n])(d) = \sum_{\mathfrak{c} \in \mathcal{C}(N), N_{\mathfrak{c}}(N) = d/n} \frac{h_{\mathfrak{c}}(N)}{N_{\mathfrak{c}}(N)} \cdot \left(\pi_{\mathrm{FE}}(tc_0(\phi)_{\mathfrak{c}}), \pi_{\mathrm{FE}}(tc_2(\phi)_{\mathfrak{c}})\right).$$

The level $N(B[\phi, n])$ of modular forms in the image equals N. Furthermore, we have

(3.2) $$e_{q_1}(B_N[\phi, n]) = n\, e_{q_1}(B_N[\phi]), \quad e_\zeta(B_N[\phi, n]) = n\, e_\zeta(B_N[\phi]), \text{ and } e_{q_2}(B_N[\phi, n]) = n\, e_{q_2}(B_N[\phi]).$$

4. Siegel product expansions by Cheng–Duncan

Consider the following family of weak Jacobi forms $\mathcal{Z}_g$, the M_{24}-twisted elliptic genus.

$$\mathcal{Z}_g(\tau, z) := \chi(g)\,\frac{\phi_{0,1}(\tau, z)}{12} + \widetilde{T}_g(\tau)\,\phi_{-2,1}(\tau, z),$$

where $\widetilde{T}_g$ is given in table 2 of [10] and, partially, in table 3. We have $tc_0(\mathcal{Z}_g) = \chi(g)$ and $tc_2(\mathcal{Z}_g) = \widetilde{T}_g$. By Proposition 3.1 in [10], the $\mathcal{Z}_g$ are weak Jacobi forms of weight 0 and index 1 for the Jacobi group $\Gamma_0(N_g) \ltimes \mathbb{Z}^2$. That is, $\widetilde{T}_g$ is a modular form of level dividing N_g. We write

$$\mathcal{Z}_g(\tau, z) = \sum_{n\geq 0, r\in\mathbb{Z}} c_g(4n - r^2)\, q^n \zeta^r \tag{4.1}$$

for the Fourier expansion of $\mathcal{Z}_g$.

In Section 4 of [10], Cheng and Duncan give the following product expansion

$$\Phi_g(Z) := q_1 \zeta q_2 \prod_{(n,r,m)>0} \exp\left(-\sum_{k=1}^{\infty} \frac{c_{g^k}(4nm - r^2)}{k} \,(q_1^n \zeta^r q_2^m)^k \right).$$

The product runs over triples of integers (n, r, m). Recall that $(n, r, m) > 0$ means that $n > 0$, or $n = 0$ and $m > 0$, or $n = m = 0$ and $r < 0$. In analogy with Formula (3.9) of [12], we can rewrite the above as

$$\Phi_g(Z) = q_1 \zeta q_2 \prod_{d\,|\,n_g} \prod_{(n,r,m)>0} \left(1 - (q_1^n \zeta^r q_2^m)^d\right)^{c_{g,d}(4nm - r^2)}, \tag{4.2}$$

where

$$c_{g,d}(D) := -d^{-1} \sum_{d'\,|\,d} \mu(d/d') c_{g^{d'}}(D)$$

and the Möbius function is denoted by μ. The $c_{g,d}$ are Fourier coefficients of a weak Jacobi form of weight 0 and index 1:

$$\mathcal{Z}_{g,d}(\tau, z) = \sum_{n\geq 0,\, r\in\mathbb{Z}} c_{g,d}(4n - r^2)\, q^n \zeta^r := -d^{-1} \sum_{d'\,|\,d} \mu(d/d') \mathcal{Z}_{g^{d'}}(\tau, z). \tag{4.3}$$

Hence Φ_g is a product expansion to which we can associate $\mathcal{E}(\Phi_g)$ in the sense of Section 2.3. It equals

$$\mathcal{E}(\Phi_g)(d) = \begin{cases} -d^{-1} \sum_{d' \mid d} \mu(d/d') \cdot \left(\chi(g^{d'}), \widetilde{T}_{g^{d'}}\right), & \text{if } d \mid n_g; \\ 0, & \text{otherwise.} \end{cases} \tag{4.4}$$

Precise expressions for all g that we treat in this paper are given in table 5.

5. Proof of the main theorem

We will prove Theorem 5.2, which implies our main result, by applying the following proposition to solutions given in table 2.

Proposition 5.1. *Suppose that Φ is a product expansion in the sense of Section 2.3 satisfying, for some $0 < N \in \mathbb{Z}$ and some $0 < r \in \mathbb{Z}$,*

$$\mathcal{E}(\Phi) = \sum_{i=1}^{r} \mathcal{E}(B_{N_i}[\phi_i, n_i]),$$

where $\phi_i \in \mathrm{J}^{(!)}_{0,1}(N_i)$ and $n_i N_i \mid N$ for all $1 \le i \le r$. If

$$e_{q_1}(\Phi) = \sum_i e_{q_1}(B_{N_i}[\phi_i, n_i]), \quad e_\zeta(\Phi) = \sum_i e_\zeta(B_{N_i}[\phi_i, n_i]),$$
$$e_{q_2}(\Phi) = \sum_i e_{q_2}(B_{N_i}[\phi_i, n_i]),$$

then

$$\Phi = \prod_{i=1}^{r} B_{N_i}[\phi_i, n_i].$$

In particular, Φ is a Siegel modular forms of level N.

Proof. By Property (2.5) of $\mathcal{E}$, we have

$$\mathcal{E}(\Phi) = \mathcal{E}\left(\prod_{i=1}^{r} B_{N_i}[\phi_i, n_i]\right).$$

By definition of $\mathcal{E}$, we then find

$$q_1^{-e_{q_1}(\Phi)}\zeta^{-e_\zeta(\Phi)}q_2^{-e_{q_2}(\Phi)}\Phi(Z) = q_1^{-\sum_i e_{q_1}(B[\phi_i,n_i])}\zeta^{-\sum_i e_\zeta(B[\phi_i,n_i])} \times q_2^{-\sum_i e_{q_2}(B[\phi_i,n_i])}\prod_{i=1}^{r} B[\phi_i,n_i](Z).$$

The conditions on e_{q_1}, e_ζ and e_{q_2} ensure that we have

$$\Phi = \prod_{i=1}^{r} B_{N_i}[\phi_i, n_i],$$

as desired. □

Theorem 5.2. *For*

$$g \in \{1\mathrm{A}, 2\mathrm{A}, 2\mathrm{B}, 3\mathrm{A}, 3\mathrm{B}, 4\mathrm{A}, 4\mathrm{B}, 4\mathrm{C}, 5\mathrm{A}, 7\mathrm{AB}, 8\mathrm{A}, 11\mathrm{A}, 23\mathrm{AB}\},$$

let $p_{3\mathrm{B}} = 3$, $p_{4\mathrm{A}} = 2$, $p_{4\mathrm{C}} = 8$, $p_{8\mathrm{A}} = 8$ *and* $p_g = 1$ *in all other cases. We have*

$$\Phi_g^{p_g} = \prod_i B_{N_{g,i}}[p_g\,\phi_{g,i}, n_{g,i}],$$

where $N_{g,i}$, $\phi_{g,i}$, *and* $n_{g,i}$ *are given in table 2 on page* 475, *and we let*

$$\phi_{g,i} = tc_0(\phi_{g,i})\,\frac{\phi_{0,1}}{12} + tc_2(\phi_{g,i})\,\phi_{-2,1}$$

in accordance with Isomorphism (2.2).

In particular, $\Phi_g^{p_g}$ *is a Siegel modular form of level* N_g *and weight* $p_g k_g$.

Corollary 5.3. *For*

$$g \in \{1\mathrm{A}, 2\mathrm{A}, 2\mathrm{B}, 3\mathrm{A}, 3\mathrm{B}, 4\mathrm{A}, 4\mathrm{B}, 4\mathrm{C}, 5\mathrm{A}, 7\mathrm{AB}, 8\mathrm{A}, 11\mathrm{A}, 23\mathrm{AB}\},$$

the product expansions Φ_g *define meromorphic Siegel modular forms of level* N_g *and weight* k_g.

In particular, Theorem 1.2 holds.

Proof. Theorem 5.2 tells us that $\Phi_g^{p_g}$ is modular by representing it as a product of rescaled Borcherds products. This implies modularity of Φ_g by the following argument.

Note that the product expansion $\Phi_g(Z)$ is convergent on some domain in $\mathbb{H}_2$, since its formal power, $\Phi_g^{p_g}$, converges on some domain. We shall have proved that

$$\Phi_g^{p_g}((AZ+B)(CZ+D)^{-1}) = \det(CZ+D)^{p_g k_g} \chi_g \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} \right)^{p_g} \Phi_g^{p_g}(Z)$$

for some character χ_g of $\Gamma_0^{(2)}(N_g)$ and all $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right) \in \Gamma_0^{(2)}(N_g)$. From this, we find that

$$\Phi_g((AZ+B)(CZ+D)^{-1}) = \det(CZ+D)^{k_g} \epsilon_g(\gamma) \chi_g(\gamma)\, \Phi_g(Z)$$

for some $\epsilon_g(\gamma)$, which is a character if $g \neq 8\mathrm{A}$, and a multiplier system otherwise. □

Proof of Theorem 5.2. Recall the description of projected cusp expansions given in Section 2.1 and the matrices $\Pi_{\mathrm{FE}}(k, N;\, \mathfrak{c})$ ($\mathfrak{c} \in \mathcal{C}(N)$) given in table 6 (cf. table 4 for cusp configurations). We identify elliptic modular forms $\mathrm{tc}_2(\phi_{g,i})$ with the coordinate (column) vector with respect to the echelon basis of $\mathrm{M}_2(\Gamma_0(N_{g,i}))$. Since $\mathrm{tc}_0(\phi_{g,i})$ is a constant, its cusp expansions are the same at all $\mathfrak{c} \in \mathcal{C}(N_{g,i})$.

To apply Proposition 5.1, we first verify that all $p_g \phi_{g,i}$ satisfy the assumptions of Theorem 3.1. Using (2.1) and table 6, it is straightforward to check that the Fourier coefficients $c(p_g \pi_{\mathrm{FE}}(\phi_{g,i})_{\mathfrak{c}}; \Delta)$ are integral for all $\Delta < 0$.

Next, we compute

$$p_g \mathcal{E}(\Phi_g) = \sum_i \mathcal{E}\big(B_{N_{g,i}}[p_g \phi_{g,i}, n_{g,i}]\big), \tag{5.1}$$

using the echelon basis of elliptic modular forms. By (3.1), we have

$$\mathcal{E}\big(B_{N_{g,i}}[p_g \phi_{g,i}, n_{g,i}]\big)(d) = \sum_{\substack{\mathfrak{c} \in \mathcal{C}(N_{g,i}) \\ N_{\mathfrak{c}}(N_{g,i}) = d \,/\, n_{g,i}}} \frac{h_{\mathfrak{c}}(N_{g,i})}{N_{\mathfrak{c}}(N_{g,i})} p_g \times \left(\mathrm{tc}_0(\phi_{g,i}), \Pi_{\mathrm{FE}}(2, N_{g,i}; \mathfrak{c})\, \mathrm{tc}_2(\phi_{g,i}) \right).$$

Formulas for e_{q_1}, e_ζ and e_{q_2} can be derived from Theorem 3.1 and (3.2).

$$e_{q_1}\big(B_{N_{g,i}}[p_g \phi_{g,i}, n_{g,i}]\big) = \frac{n_{g,i}}{24} \sum_{\substack{\mathfrak{c} \in \mathcal{C}(N_{g,i}) \\ l \in \mathbb{Z}}} h_{\mathfrak{c}}(N_{g,i})\, p_g c\big(\pi_{\mathrm{FE}}(\phi_{\mathfrak{c}}); -l^2\big).$$

We have $c\big(\pi_{\mathrm{FE}}(\phi_{\mathfrak{c}}); -l^2\big) = 0$, if $l \notin \{0, \pm 1\}$, since ϕ is a weak Jacobi forms.

Therefore,

$$\begin{aligned}
&e_{q_1}\big(B_{N_{g,i}}[p_g\phi_{g,i}, n_{g,i}]\big) \\
&\quad = \frac{n_{g,i}}{24} \sum_{\mathfrak{c}\in\mathcal{C}(N_{g,i})} h_{\mathfrak{c}}(N_{g,i}) \left(p_g \mathrm{tc}_0(\phi_{g,i}) \cdot \left(c\left(\frac{\phi_{0,1}}{12}; 0\right) + 2c\left(\frac{\phi_{0,1}}{12}; -1\right)\right)\right. \\
&\qquad \left. + \, c(p_g \pi_{\mathrm{FE}}(\mathrm{tc}_2(\phi_{g,i})_{\mathfrak{c}}); 0) \cdot \big(c(\phi_{-2,1}; 0) + 2c(\phi_{-2,1}; -1)\big)\right),
\end{aligned}$$

which equals

$$\frac{n_{g,i}}{24} \sum_{\mathfrak{c}\in\mathcal{C}(N_{g,i})} h_{\mathfrak{c}}(N_{g,i}) p_g \mathrm{tc}_0(\phi_{g,i}).$$

A similar computation gives

$$\begin{aligned}
e_{\zeta}\big(B_{N_{g,i}}[p_g\phi_{g,i}, n_{g,i}]\big) &= e_{q_2}\big(B_{N_{g,i}}[p_g\phi_{g,i}, n_{g,i}]\big) \\
&= \frac{n_{g,i}}{2} \sum_{\mathfrak{c}\in\mathcal{C}(N_{g,i})} h_{\mathfrak{c}}(N_{g,i}) \\
&\quad \times \left(\frac{1}{12} p_g \mathrm{tc}_0(\phi_{g,i}) + p_g c(\Pi_{\mathrm{FE}}(2, N_{g,i}; \mathfrak{c}) \mathrm{tc}_2(\phi_{g,i}); 0)\right).
\end{aligned}$$

The 0th coefficient of a modular form can be easily read of from its coordinates with respect to the echelon basis. For this reasons, the above expressions can be directly evaluated using table 6.

The weight $p_g k_g$ of $\prod_i B_{N_{g,i}}[p_g\phi_{g,i}, n_{g,i}]$ can be computed along the same line. By Theorem 3.1, we have

$$\begin{aligned}
p_g k_g = \frac{1}{2} \sum_i \sum_{\mathfrak{c}\in\mathcal{C}(N_{g,i})} & \frac{h_{\mathfrak{c}}(N_{g,i})}{N_{\mathfrak{c}}(N_{g,i})} \\
& \times \left(p_g \mathrm{tc}_0(p_g\, \phi_{g,i}) + p_g c(\Pi_{\mathrm{FE}}(2, N_{g,i}; \mathfrak{c}) \mathrm{tc}_2(p_g\, \phi_{g,i}); 0)\right).
\end{aligned}$$

We have reduced all computations to straight forward linear algebra. We give details only in the case of $g = 3\mathrm{B}$, and leave all other cases to the reader.

For $g = 3\mathrm{B}$, we have

$$\begin{aligned}\phi_{3\mathrm{B},1}(\tau, z) &= -8\frac{\phi_{0,1}(\tau, z)}{12}\\ &\quad\text{with}\quad N_{3\mathrm{B},1} = 1, n_{3\mathrm{B},1} = 1;\\ \phi_{3\mathrm{B},2}(\tau, z) &= 8\frac{\phi_{0,1}(\tau, z)}{12} + \mathrm{M}_2(3)[2](\tau)\phi_{-2,1}(\tau, z)\\ &\quad\text{with}\quad N_{3\mathrm{B},1} = 3,\ n_{3\mathrm{B},1} = 1.\end{aligned}$$

Since $N_{3\mathrm{B},i} \,|\, 3$ for $i \in \{1, 2\}$, we only need to consider forms of level 1 and 3. The cusp data for $\Gamma_0(3)$ (see table 4) is:

$$h_\infty(3) = 1,\ N_\infty(3) = 1; \quad h_0(3) = 4,\ N_0(3) = 3.$$

By definition, $\Pi_{\mathrm{FE}}(2, N; \infty)$ is the identity for all N. In addition, we need $\Pi_{\mathrm{FE}}(2, 3; 0) = (-\frac{1}{3})$. We consider the third power of Φ_g corresponding to $p_{3\mathrm{B}} = 3$. Putting everything together, we obtain

$$\begin{aligned}\mathcal{E}\big(B_1[p_{3\mathrm{B}}\,\phi_{3\mathrm{B},1}, 1]\big)(1) &= \frac{h_\infty(1)}{N_\infty(1)}\big(3\cdot(-8),\ 0\big) = (-24, 0),\\ \mathcal{E}\big(B_3[p_{3\mathrm{B}}\,\phi_{3\mathrm{B},2}, 1]\big)(1) &= \frac{h_\infty(3)}{N_\infty(3)}\big(3\cdot 8,\ 3\cdot \mathrm{M}_2(3)[2]\big) = \big(24,\ \mathrm{M}_2(3)[6]\big),\\ \mathcal{E}\big(B_3[p_{3\mathrm{B}}\,\phi_{3\mathrm{B},2}, 1]\big)(3) &= \frac{h_\infty(3)}{N_\infty(3)}\big(3\cdot 8,\ 3\cdot(-\tfrac{1}{3})\,\mathrm{M}_2(3)[2]\big) = \big(24, \mathrm{M}_2(3)[-2]\big).\end{aligned}$$

Comparing this with

$$p_{3\mathrm{B}}\,\mathcal{E}(\Phi_{3\mathrm{B}})(1) = (0,\ \mathrm{M}_2(3)[6]), \quad p_{3\mathrm{B}}\,\mathcal{E}(\Phi_{3\mathrm{B}})(3) = (24,\ \mathrm{M}_2(3)[-2]),$$

we establish

$$p_{3\mathrm{B}}\,\mathcal{E}(\Phi_{3\mathrm{B}}) = \mathcal{E}\big(B_1[p_{3\mathrm{B}}\,\phi_{3\mathrm{B},1}, 1]\big) + \mathcal{E}\big(B_3[p_{3\mathrm{B}}\,\phi_{3\mathrm{B},2}, 1]\big).$$

Next, we find that

$$\begin{aligned}e_{q_1}\big(B_1[p_{3\mathrm{B}}\,\phi_{3\mathrm{B},1}, 1]\big) &= \frac{1}{24}h_\infty(1)\cdot 3\cdot(-8) = -1,\\ e_{q_1}\big(B_3[p_{3\mathrm{B}}\,\phi_{3\mathrm{B},2}, 1]\big) &= \frac{1}{24}(h_\infty(3) + h_0(3))\cdot 3\cdot 8 = 4,\end{aligned}$$

and

$$\begin{aligned}
e_\zeta\big(B_1[p_{3\mathrm{B}}\,\phi_{3\mathrm{B},1},1]\big) &= \frac{1}{2}h_\infty(1)\cdot 3\cdot\left(-\frac{8}{12}\right) = -1,\\
e_\zeta\big(B_3[p_{3\mathrm{B}}\,\phi_{3\mathrm{B},2},1]\big) &= \frac{1}{2}\left(h_\infty(3)\left(3\cdot\frac{8}{12}+c\big(3\mathrm{M}_2(3)[2];0\big)\right)\right.\\
&\qquad\left.+\,h_0(3)\left(3\cdot\frac{8}{12}+c\left(\Pi_{\mathrm{FE}}(2,3;0)\,3\mathrm{M}_2(3)[2];\,0\right)\right)\right)\\
&= \frac{1}{2}\left(\left(3\cdot\frac{8}{12}+6\right)+3\left(3\cdot\frac{8}{12}+\left(-\frac{1}{3}\right)6\right)\right) = 4,
\end{aligned}$$

while

$$3e_{q_1}(\Phi_{3\mathrm{B}}) = 3e_\zeta(\Phi_{3\mathrm{B}}) = 3e_{q_2}(\Phi_{3\mathrm{B}}) = 3.$$

This proves that $\Phi_{3\mathrm{B}}^{p_{3\mathrm{B}}}$ is modular. □

6. Conclusion

We have shown that M_{24}-twisted product expansions that arise form second quantized elliptic genera of K3 are modular, if the twisting element has non-composite order. As opposed to results that were obtained so far, this modularity does not directly come from Borcherds products. Instead, we have illustrated how products of rescaled Borcherds products enter the picture. This is a new type of modularity that has not yet been observed in string theory. It is, however, a natural generalization of the concept of eta products to Siegel modular forms.

The analog of the Weyl denominator formula for Kac–Moody algebras remains to be found. Since its multiplicative part is more complicated than being just one Borcherds products, we speculate that the additive side might also be more complicated. It is possible that it is the sum of rescaled additive lifts. We have computed the weight of the twisted product expansions Φ_g in all cases that we could resolve. In the cases $g = 11\mathrm{A}$ and $g = 23\mathrm{AB}$ the inequality $k_g \le 0$ implies that Φ_g is meromorphic. Therefore, it will be necessary to take the regularized additive lift [7] into consideration, while searching for the additive side of the attached Weyl denominator formula. A physical interpretation for polar divisors of Siegel modular forms that occur would be of certain interest.

Finally, the composite levels N_g are of seemingly different nature. The easiest case is $g = 6\mathrm{A}$ in which case the predicted level is $N_{6\mathrm{A}} = 6$. We have tried — without success — to represent $\Phi_{6\mathrm{A}}$ as a product of rescaled Borcherds products coming from weakly holomorphic Jacobi forms with pole order less than 4. Clearly, novel ideas are needed to make progress in this direction.

Appendix A. How we found the data in table 2

In order to express some power of Φ_g as products of rescaled Borcherds products, we solved the system of linear equations associated to the ansatz

$$\mathcal{E}(\Phi_g) = \sum_{\substack{nN \mid N'_g \\ N \le N_{\max}}} \mathcal{E}(B_N(\phi_{N,n}, n)), \qquad \text{(Appendix A.1)}$$

where $\phi_{N,n} \in \mathrm{J}^{!}_{k,m}(\Gamma_0(N))$ with maximal pole order $0 \le o \in \mathbb{Z}$ at all cusps. This is a finite-dimensional system of linear equations, since $\Delta^o \phi_{N,n} \in \mathrm{J}^{(!)}_{12o,1}(\Gamma_0(N))$, and $\dim \mathrm{J}^{(!)}_{12o,1}(\Gamma_0(N)) < \infty$. Note that for all cases that we have successfully solved, we have $N'_g = N_g$ and $o = 0$, while $N_{\max}$ equals the level of $\widetilde{T}_g$.

We used Sage [32] (mind the version of Sage, which might be relevant to build and run our implementation) to build the matrix attached to (Appendix A.1) and solve it. The code that we have used can be downloaded at [30]. We briefly describe how to invoke relevant functions in order to reprove our results. At this point, we mention that, at [30], also the results of can be downloaded in machine readable form.

As a first step, download the code for this paper at [30]. Extract "cheng_products.tar.gz", and run the following commands in terminal:

```
cd eta_products
sage -sh
sage --python compile.py
exit
```

The basic solving procedure is invoked by `cheng_solve_case`, whose declaration looks as follows.

```
cheng_solve_case(g, conjectured_Ng = None, max_phi_level = None,
                 order = 0, minimize_scaling = False)
```

The first argument is the label of a conjugacy class that one wants to let compute. The second argument, which is optional, is the conjectured level N'_g of Φ_g. If this parameter is not specified it will be replaced by the levels N_g conjectured in [10]. The third parameter corresponds to $N_{\max}$. Clearly, we must have $N_{\max} \mid N'_g$. The default value in this case is the level of $\widetilde{T}_g$. The next argument corresponds to o in (Appendix A.1). The default value equals 0. The last parameter determines whether or not the returned solution should involve as few rescaled Borcherds products as possible.

The next lines of code compute the solution in the case $g = 11\mathrm{A}$.

```
sage:
sage: cheng_solve_case('11A', minimize_scaling = True)
('11A',
 11,
 11,
 (0, 2, 11/6, 0, 0),
 [(1, 1, 0), (11, 1, 0), (11, 1, 1), (11, 1, 2), (1, 11, 0)])
sage: compute_borcherds_lift_data(_)
(0, (1, 1, 1), {(11, 1, 0): (2, 0), (11, 1, Infinity): (-2, 2)})
```

The return value of the first command is $(g, N'_g, N_{\max}, v, l)$, where v is a vector whose coordinates correspond to the elements of the list l of triples (N, n, i). The first and second components of this triple are the same as in (Appendix A.1). The third component refers to the $(i+1)$th basis vector in the echelon basis of $\mathrm{M}_0^{!(o)}(N)$, if i is less than its dimension, and to the $(i + 1 - \dim \mathrm{M}_0^{!(o)}(N))$th echelon basis vector of $\mathrm{M}_2^{!(o)}(N)$, otherwise. Here we write $\mathrm{M}_k^{!(o)}(N)$ for weakly holomorphic modular forms of weight k and level N whose pole order at all cusps is bounded by o. In the given example, only $N = 11$, $n = 1$ contributes to the solution, and we have

$$\phi_{11,1} \cong \big(\mathrm{M}_0^{!(0)}(N)[2],\, \mathrm{M}_2^{!(0)}(N)[\tfrac{11}{6}]\big)$$

via Isomorphism (2.2).

To ensure that (Appendix A.1) corresponds to an equality of

$$\Phi_g^p = \prod_{\substack{nN \,|\, N'_g \\ N \le N_{\max}}} \mathcal{E}(B_N(p\phi_{N,n}, n)),$$

the second command computes the weight k of the right-hand side $(e_{q_1}, e_\zeta, e_{q_2})$, and a dictionary mapping $(N, n, \mathfrak{c})$ to the pair of Fourier coefficients

$$\big(c((\phi_{N,n})_{\mathfrak{c}}; 0),\, c((\phi_{N,n})_{\mathfrak{c}}; -1)\big).$$

Note that this function is currently restricted to the case of $o = 0$, but it would be easy to implement a general form.

Appendix B. Projection matrices $\Pi_{\mathbf{FE}}$

The matrices $\Pi_{\mathrm{FE}}(k, N, \mathfrak{c})$ given in table 6 have been computed using methods in [31], which are algebraic. We provide numerical double checks, to verify their correctness. Follow the initial steps described in the previous section. Install “nose” for Sage by typing

```
sage -i nose
```

Then run the corresponding tests via

```
sage -sh
nosetests -v modular_form_transformations__test.py
```

This will automatically perform all tests, which the reader can inspect by viewing the file “modular_ form_transformations_ test.py”.

Appendix C. Tables

Table 3: Cheng’s and Duncan’s initial data $\chi(g)$ and $\tilde{T}_g$, expressed in terms of the echelon bases.

g	$\chi(g)$	$\tilde{T}_g$
1A	24	0
2A	8	$\mathrm{M}_2(2)[\frac{4}{3}]$
2B	0	$\mathrm{M}_2(4)[2, -16]$
3A	6	$\mathrm{M}_2(3)[\frac{3}{2}]$
3B	0	$\mathrm{M}_2(3)[2]$
4A	0	$\mathrm{M}_2(8)[2, 0, -16]$
4B	4	$\mathrm{M}_2(4)[\frac{5}{3}, 8]$
4C	0	$\mathrm{M}_2(4)[2, -8]$
5A	4	$\mathrm{M}_2(5)[\frac{5}{3}]$
7AB	3	$\mathrm{M}_2(7)[\frac{7}{4}]$
8A	2	$\mathrm{M}_2(8)[\frac{11}{6}, 4, 12]$
11A	2	$\mathrm{M}_2(11)[\frac{11}{6}, 0]$
23AB	1	$\mathrm{M}_2(23)[\frac{23}{12}, \frac{46}{11}, -\frac{23}{11}]$

Table 4: Cusp data for $\Gamma_0(N)$, used in Section 3.

N	$\mathfrak{c}$	$h_\mathfrak{c}$	$N_\mathfrak{c}$	$\mathfrak{c}$	$h_\mathfrak{c}$	$N_\mathfrak{c}$
2	∞	1	1	0	2	2
3	∞	1	1	0	3	3
4	∞	1	1	0	4	4
	$\frac{1}{2}$	1	2			
5	∞	1	1	0	5	5
7	∞	1	1	0	7	7
8	∞	1	1	0	8	8
	$\frac{1}{2}$	2	4	$\frac{1}{4}$	1	2
11	∞	1	1	0	11	11
23	∞	1	1	0	23	23

Table 5: Values of $\mathcal{E}(\Phi_g)$ which are used in the course of the proof of Theorem 5.2.

g	d	$\mathcal{E}(\Phi_g)(d)$	d	$\mathcal{E}(\Phi_g)(d)$
1A	1	$(24,\, 0)$		
2A	1	$(8,\, \mathrm{M}_2(2)[\frac{4}{3}])$	2	$(8,\, \mathrm{M}_2(2)[-\frac{2}{3}])$
2B	1	$(0,\, \mathrm{M}_2(4)[2,-16])$	2	$(12,\, \mathrm{M}_2(4)[-1,8])$
3A	1	$(6,\, \mathrm{M}_2(3)[\frac{3}{2}])$	3	$(6,\, \mathrm{M}_2(3)[-\frac{1}{2}])$
3B	1	$(0,\, \mathrm{M}_2(3)[2])$	3	$(8,\, \mathrm{M}_2(3)[-\frac{2}{3}])$
4A	1	$(0,\, \mathrm{M}_2(8)[2,0,-16])$	2	$(4,\, \mathrm{M}_2(8)[-\frac{1}{3},16,24])$
	4	$(4,\, \mathrm{M}_2(8)[-\frac{1}{3},-8,-8])$		
4B	1	$(4,\, \mathrm{M}_2(4)[\frac{5}{3},8])$	2	$(2,\, \mathrm{M}_2(4)[-\frac{1}{6},12])$
	4	$(4,\, \mathrm{M}_2(4)[-\frac{1}{3},-8])$		
4C	1	$(0,\, \mathrm{M}_2(4)[2,-8])$	2	$(0,\, \mathrm{M}_2(4)[0,-4])$
	4	$(6,\, \mathrm{M}_2(4)[-\frac{1}{2},4])$		
5A	1	$(4,\, \mathrm{M}_2(5)[\frac{5}{3}])$	5	$(4,\, \mathrm{M}_2(5)[-\frac{1}{3}])$
7AB	1	$(3,\, \mathrm{M}_2(7)[\frac{7}{4}])$	7	$(3,\, \mathrm{M}_2(7)[-\frac{1}{4}])$
8A	1	$(2,\, \mathrm{M}_2(8)[\frac{11}{6},4,12])$	2	$(-1,\, \mathrm{M}_2(8)[\frac{1}{12},-2,-14])$
	4	$(2,\, \mathrm{M}_2(8)[-\frac{1}{6},8,12])$	8	$(2,\, \mathrm{M}_2(8)[-\frac{1}{6},-4,-4])$
11A	1	$(2,\, \mathrm{M}_2(11)[\frac{11}{6},0])$	11	$(2,\, \mathrm{M}_2(11)[-\frac{1}{6},0])$
23AB	1	$(1,\, \mathrm{M}_2(23)[\frac{23}{12},\frac{46}{11},-\frac{23}{11}])$	23	$(1,\, \mathrm{M}_2(23)[-\frac{1}{12},-\frac{2}{11},\frac{1}{11}])$

Table 6: Matrices $\Pi_{\mathrm{FE}}(k, N, \mathfrak{c})$ associated to $\pi_{\mathrm{FE}}(\,\cdot\,_{\mathfrak{c}}) : \mathrm{M}_k(N) \to \mathrm{M}_k(N)$; see Section 2.1.

k	N	$\mathfrak{c}$	$\Pi_{\mathrm{FE}}(k, N, \mathfrak{c})$	$\mathfrak{c}$	$\Pi_{\mathrm{FE}}(k, N, \mathfrak{c})$
2	2	0	$\begin{pmatrix} -\frac{1}{2} \end{pmatrix}$		
2	3	0	$\begin{pmatrix} -\frac{1}{3} \end{pmatrix}$		
2	4	0	$\begin{pmatrix} -\frac{1}{8} & -\frac{1}{64} \\ -3 & -\frac{3}{8} \end{pmatrix}$	$\frac{1}{2}$	$\begin{pmatrix} -\frac{1}{2} & \frac{1}{16} \\ 12 & \frac{1}{2} \end{pmatrix}$
2	5	0	$\begin{pmatrix} -\frac{1}{5} \end{pmatrix}$		
2	7	0	$\begin{pmatrix} -\frac{1}{7} \end{pmatrix}$		
2	8	0	$\begin{pmatrix} -\frac{1}{32} & -\frac{1}{64} & -\frac{1}{256} \\ -\frac{3}{4} & -\frac{3}{8} & -\frac{3}{32} \\ -\frac{3}{4} & -\frac{3}{8} & -\frac{3}{32} \end{pmatrix}$	$\frac{1}{2}$	$\begin{pmatrix} -\frac{1}{8} & \frac{1}{16} & -\frac{1}{64} \\ 3 & \frac{1}{2} & \frac{3}{8} \\ -3 & \frac{3}{2} & -\frac{3}{8} \end{pmatrix}$
		$\frac{1}{4}$	$\begin{pmatrix} -\frac{1}{2} & 0 & \frac{1}{16} \\ 0 & 1 & 0 \\ 12 & 0 & \frac{1}{2} \end{pmatrix}$		
2	11	0	$\begin{pmatrix} -\frac{1}{11} & 0 \\ 0 & -\frac{1}{11} \end{pmatrix}$		
2	23	0	$\begin{pmatrix} -\frac{1}{23} & 0 & 0 \\ 0 & -\frac{1}{23} & 0 \\ 0 & 0 & -\frac{1}{23} \end{pmatrix}$		

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