Invariants of hypergeometric groups for Calabi–Yau complete intersections in weighted projective spaces

SUSUMU TANABÉ AND KAZUSHI UEDA

Let Y be a Calabi–Yau complete intersection in a weighted projective space. We show that the space of quadratic invariants of the hypergeometric group associated with the twisted I-function is one-dimensional, and spanned by the Gram matrix of a split-generator of the derived category of coherent sheaves on Y with respect to the Euler form.

1. Introduction

Let $\boldsymbol{q} = (q_0, \ldots, q_N)$ and $\boldsymbol{d} = (d_1, \ldots, d_r)$ be sequences of positive integers such that

$$Q := q_0 + \cdots + q_N = d_1 + \cdots + d_r,$$

and consider a complete intersection Y of degree $(d_1, \ldots, d_r)$ in the weighted projective space $\mathbb{P} = \mathbb{P}(q_0, \ldots, q_N)$. If Y is smooth, then it is a Calabi–Yau manifold of dimension $n = N - r$. The derived category $D^b \operatorname{coh} \mathbb{P}$ of coherent sheaves on $\mathbb{P}$ has a full strong exceptional collection

$$(\mathcal{E}_i)_{i=1}^{Q} = (\mathcal{O}_{\mathbb{P}}, \mathcal{O}_{\mathbb{P}}(1), \ldots, \mathcal{O}_{\mathbb{P}}(Q-1))$$

of line bundles [2, 4]. Let $(\mathcal{F}_i)_{i=1}^{Q}$ be the full exceptional collection dual to $(\mathcal{E}_i)_{i=1}^{Q}$, so that

$$\chi(\mathcal{E}_{Q-i+1}, \mathcal{F}_j) = \delta_{ij},$$

where

(1.1) $$\chi(\mathcal{E}, \mathcal{F}) = \sum_k (-1)^k \dim \operatorname{Ext}^k(\mathcal{E}, \mathcal{F})$$

is the Euler form. The derived restrictions $\{\overline{\mathcal{F}}_i\}_{i=1}^{Q}$ of $\{\mathcal{F}_i\}_{i=1}^{Q}$ to Y split-generate the derived category $D^b \operatorname{coh} Y$ of coherent sheaves on Y [19, Lemma 5.4].

Following [8, Equation (4)], let us introduce the *twisted I-function*

$$(1.2) \qquad I_{\mathbb{P},Y}(t) = \sum_\alpha e^{P_\alpha \log t} t^{\rho_\alpha} \sum_{n=0}^\infty t^n \frac{\prod_{k=1}^r \prod_{\substack{b:\, \langle b\rangle = \langle \rho_\alpha d_k\rangle \\ 0 < b \leq (n+\rho_\alpha)d_k}} (d_k P_\alpha + b)}{\prod_{\nu=0}^N \prod_{\substack{b:\, \langle b\rangle = \langle \rho_\alpha q_\nu\rangle \\ 0 < b \leq (n+\rho_\alpha)q_\nu}} (q_\nu P_\alpha + b)},$$

where $\langle x\rangle = x - \lfloor x\rfloor$ is the fractional part of x. This is an element of the ring $\bigoplus_\alpha \mathbb{C}[P_\alpha]_\alpha / (P_\alpha^{\mu_\alpha})_\alpha$, which is isomorphic to the orbifold cohomology of $\mathbb{P}$ as a vector space. See Section 2 for the definition of $\rho_\alpha \in \mathbb{Q}$ and $\mu_\alpha \in \mathbb{N}$. The twisted I-function $I_{\mathbb{P},Y}$ contains the information of twisted Gromov–Witten invariant of the bundle $\mathcal{O}_\mathbb{P}(d_1) \oplus \cdots \oplus \mathcal{O}_\mathbb{P}(d_r) \to \mathbb{P}$ through the twisted J-function [6].

The components of the twisted I-function span the space of solutions of the hypergeometric differential equation

$$(1.3) \qquad \left[\prod_{\nu=0}^N \prod_{a=0}^{q_\nu-1} (q_\nu \theta_t - a) - t \prod_{k=1}^r \prod_{b=1}^{d_k} (d_k \theta_t + b) \right] I = 0,$$

where $\theta_t = t\dfrac{\partial}{\partial t}$. Let $\mathcal{H}$ be the differential operator on the left-hand side and $\mathcal{H}^{\mathrm{red}}$ be the operator obtained from $\mathcal{H}$ by removing common factors from the two summands of $\mathcal{H}$. Both $\mathcal{H}$ and $\mathcal{H}^{\mathrm{red}}$ have regular singularities at 0, ∞ and $\lambda = \prod_{\nu=0}^N q_\nu^{q_\nu} \big/ \prod_{k=1}^r d_k^{d_k}$. The local system $\mathcal{L}^{\mathrm{red}}$ defined by $\mathcal{H}^{\mathrm{red}}$ is irreducible, and its rank Q^{red} is smaller than the rank Q of the local system $\mathcal{L}$ defined by $\mathcal{H}$. The irreducible system $\mathcal{L}^{\mathrm{red}}$ supports a pure and polarized variation of Hodge structures, whose Hodge numbers are computed by Corti and Golyshev [7, Theorem 1.3].

The mirror of Y is identified by Batyrev and Borisov [3] as the family of toric complete intersections whose affine part is given by

$$(1.4) \qquad X_t = \{(x_0, \ldots, x_N) \in (\mathbb{C}^\times)^{N+1} \mid f_i(x) = 0,\ i = 0, \ldots, r\},$$

where

$$f_0(x) = x_0^{q_0} x_1^{q_1} \cdots x_N^{q_N} - t,$$
$$f_i(x) = \sum_{k \in S_i} x_k - 1, \quad i = 1, \ldots, r,$$

the variable t is the parameter of the family, and $S_1 \sqcup \cdots \sqcup S_r = \{0, 1, \ldots, N\}$ is a partition of $\{0, 1, \ldots, N\}$ into r disjoint subsets such that $d_k = \sum_{i \in S_k} q_i$. The period integral

$$(1.5) \qquad I(t) = \int_\gamma \frac{x_0^{q_0} \cdots x_N^{q_N}}{df_0 \wedge \cdots \wedge df_r} \frac{dx_0}{x_0} \wedge \cdots \wedge \frac{dx_n}{x_n}$$

for a middle-dimensional cycle $\gamma \in H_n(X_t)$ satisfies the irreducible hypergeometric differential equation $\mathcal{H}^{\mathrm{red}} I = 0$.

Define the *hypergeometric group* $H_{\boldsymbol{q}, \boldsymbol{d}}$ as the subgroup of $GL(Q, \mathbb{Z})$ generated by

$$(1.6) \qquad h_\infty = \begin{pmatrix} 0 & 0 & \cdots & 0 & -A_Q \\ 1 & 0 & \cdots & 0 & -A_{Q-1} \\ 0 & 1 & \cdots & 0 & -A_{Q-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -A_1 \end{pmatrix}$$

and

$$(1.7) \qquad h_0^{-1} = \begin{pmatrix} 0 & 0 & \cdots & 0 & -B_Q \\ 1 & 0 & \cdots & 0 & -B_{Q-1} \\ 0 & 1 & \cdots & 0 & -B_{Q-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -B_1 \end{pmatrix},$$

where

$$(1.8) \qquad \prod_{k=1}^{r} (T^{d_k} - 1) = T^Q + A_1 T^{Q-1} + A_2 T^{Q-2} + \cdots + A_Q$$

and

$$(1.9) \qquad \prod_{\nu=0}^{N} (T^{q_\nu} - 1) = T^Q + B_1 T^{Q-1} + B_2 T^{Q-2} + \cdots + B_Q$$

are the characteristic polynomials of the monodromy of (1.3) at infinity and zero, respectively. If the system is irreducible, then a result of Levelt [17] states that the monodromy group is conjugate to the hypergeometric group $H_{\boldsymbol{q}, \boldsymbol{d}}$. Although the system $\mathcal{L}$ is reducible and one cannot apply the result of Levelt directly, we can show the following:

Theorem 1.1. *For any sequences* $\boldsymbol{q} = (q_0, \ldots, q_N)$ *and* $\boldsymbol{d} = (d_1, \ldots, d_r)$ *of positive integers satisfying*

$$Q := q_0 + \cdots + q_N = d_1 + \cdots + d_r,$$

the monodromy group of (1.3) *is conjugate to the hypergeometric group* $H_{\boldsymbol{q}, \boldsymbol{d}}$

An element $h \in H_{\boldsymbol{q}, \boldsymbol{d}}$ acts naturally on the space of $Q \times Q$-matrices by

$$H_{\boldsymbol{q}, \boldsymbol{d}} \ni h : X \mapsto h \cdot X \cdot h^{\mathrm{T}},$$

where h^{T} is the transpose of h. The following is a corollary of Theorem 1.1:

Theorem 1.2. *The space of* $Q \times Q$-*matrices invariant under the action of the monodromy group* $H_{\boldsymbol{q}, \boldsymbol{d}}$ *of* (1.3) *is one-dimensional and spanned by the Gram matrix*

$$\left(\chi(\overline{\mathcal{F}}_i, \overline{\mathcal{F}}_j) \right)_{i,j=1}^{Q}$$

of the split-generator $\{\overline{\mathcal{F}}_i\}_{i=1}^{Q}$ *with respect to the Euler form.*

Theorem 1.2 is closely related to the works of Horja [14, Theorem 4.9] and Golyshev [12, Section 3.5], which goes back to Kontsevich [16]. The main difference from their works is that we work with the reducible system $\mathcal{L}$ which contains sections not coming from period integrals on the mirror manifold. In the case of the irreducible local system $\mathcal{L}^{\mathrm{red}}$, Golyshev gave a beautiful interpretation in terms of autoequivalences of the derived category of the mirror manifold Y (cf. Section 5). Although the geometric meaning of extra sections of $\mathcal{L}$ is unclear, our proof of Theorem 1.2 also use autoequivalences of the derived category of the mirror manifold Y, just as in the case of the irreducible system.

The organization of this paper is as follows: the proof of Theorem 1.1 is given in Section 2. The essential step is to show the existence of a cyclic vector for the monodromy around the origin, which satisfies additional condition with respect to the monodromy at infinity. The uniqueness of the invariant of the hypergeometric group is shown in Section 3, and the invariance of the Gram matrix of the split-generator with respect to the Euler form is shown in Section 4. In Section 5, we discuss the relationship between the Gram matrix in Theorem 1.2 and the Stokes matrix for the quantum cohomology of the weighted projective space.

2. Monodromy of hypergeometric equation

We prove Theorem 1.1 in this section. Let h_0, h_1 and h_∞ be the global monodromy matrix of the hypergeometric differential equation (1.3) around the origin, one and infinity with respect to some basis of solutions satisfying $h_0 \cdot h_1 \cdot h_\infty = 1$. Recall that a vector $v \in \mathbb{C}^Q$ is said to be *cyclic* with respect to $h \in GL(Q, \mathbb{C})$ if the set $\{h^i \cdot v\}_{i=0}^{Q-1}$ spans $\mathbb{C}^Q$. The following lemma is used by Levelt [17] to compute the monodromy of hypergeometric functions (see also Beukers and Heckman [5, Theorem 3.5]).

Lemma 2.1. *Assume that there exists a vector satisfying*

$$(2.1) \qquad h_0^i v = h_\infty^{-i} v, \quad i = 1, \ldots, Q - 1,$$

which is cyclic with respect to h_0. Then the monodromy group of (1.3) is isomorphic to $H_{q,d}$.

Proof. The condition (2.1) shows that the action of h_0 and h_∞^{-1} with respect to the basis $\{h_\infty^{-i} v\}_{i=0}^{Q-1}$ of $\mathbb{C}^Q$ is given by

$$\begin{pmatrix} 0 & 0 & \ldots & 0 & * \\ 1 & 0 & \ldots & 0 & * \\ 0 & 1 & \ldots & 0 & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & * \end{pmatrix}.$$

The last line is determined by the characteristic equations

$$\det(T - h_0) = T^Q + A_1 T^{Q-1} + A_2 T^{Q-2} + \cdots + A_Q$$

and

$$\det(T - h_\infty^{-1}) = T^Q + B_1 T^{Q-1} + B_2 T^{Q-2} + \cdots + B_Q. \qquad \square$$

Remark 2.2. Even if there is no vector satisfying (2.1) which is cyclic with respect to h_0, one can consider the subspace generated from any vector satisfying (2.1) by the action of h_0, and the resulting matrix presentation of the monodromy action with respect to $\{h_\infty^{-i} v\}_{i=0}^{Q-1}$ will be given by (1.7) and (1.6). Since $\{h_\infty^{-i} v\}_{i=0}^{Q-1}$ is not a basis but only a generator in such a case, this matrix presentation is not unique.

Hence, the proof of Theorem 1.1 is reduced to the following:

Proposition 2.3. *There exists a vector v in the space of solutions of* (1.3) *which is cyclic with respect to h_0 and satisfies* (2.1).

The rest of this section is devoted to the proof of Proposition 2.3. Note that the existence of such v is automatic if the hypergeometric differential equation is irreducible (cf. [5, Theorem 3.5]).

The hypergeometric differential equation (1.3) has regular singularities at $t = 0, \infty$ and λ where $\lambda = \prod_{\nu=0}^{N} q_\nu^{q_\nu} \big/ \prod_{k=1}^{r} d_k^{d_k}$. To simplify notations, we introduce another variable z by $t = \lambda z$. Then the local exponents are given by

$$\frac{b}{d_k}, \quad k = 1, \ldots, r, \quad b = 1, \ldots, d_k, \qquad\qquad \text{at } z = \infty,$$

$$\frac{a}{q_\nu}, \quad \nu = 1, \ldots, N, \quad a = 0, \ldots, q_\nu - 1, \qquad \text{at } z = 0, \text{ and}$$

$$0, \ 1, \ 2, \ldots, Q-2, \ \frac{n-1}{2}, \qquad\qquad\qquad \text{at } z = 1.$$

Let

$$1 > \rho_1 > \rho_2 > \cdots > \rho_p = 0$$

be the characteristic exponents of (1.3) at $z = 0$ so that

$$\{\rho_1, \ldots, \rho_p\} = \bigcup_{0 \le \nu \le N} \left\{ 0, \frac{1}{q_\nu}, \ldots, \frac{q_\nu - 1}{q_\nu} \right\}.$$

Let further

$$\mu_\alpha = \# \left\{ (q_\nu, a) \,\middle|\, \rho_\alpha = \frac{a}{q_\nu}, \quad 0 \le a \le q_\nu - 1, \quad 0 \le \nu \le N \right\}$$

be the multiplicity of the exponent ρ_α and put

$$e_\alpha = \exp(2\pi\sqrt{-1}\rho_\alpha), \quad 1 \le \alpha \le p.$$

We remark that $\mu_p = N + 1$. For the quantity defined by

$$\nu_\alpha = \# \left\{ (d_k, b) \,\middle|\, \rho_\alpha = 1 - \frac{b}{d_k}, \quad 1 \le b \le d_k - 1, \quad 1 \le k \le r \right\}, \quad 1 \le \alpha \le p$$

the following relation holds:

$$Q^{\mathrm{red}} = \sum_{\alpha=1}^{p} (\mu_\alpha - \nu_\alpha).$$

Let us introduce the matrices

$$M_0 = \begin{pmatrix} \rho_1 \operatorname{id}_{\mu_1} + J_{\mu_1,-} & 0 & \cdots & 0 \\ 0 & \rho_2 \operatorname{id}_{\mu_2} + J_{\mu_2,-} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \rho_p \operatorname{id}_{\mu_p} + J_{\mu_p,-} \end{pmatrix},$$

$$E_0 = \begin{pmatrix} e_1 \operatorname{id}_{\mu_1} + J_{\mu_1,-} & 0 & \cdots & 0 \\ 0 & e_2 \operatorname{id}_{\mu_2} + J_{\mu_2,-} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e_p \operatorname{id}_{\mu_p} + J_{\mu_p,-} \end{pmatrix},$$

where $J_{i,\pm}$ are $i \times i$ matrices defined by

$$J_{i,+} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

and

$$J_{i,-} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

A series solution to (1.3) at the origin can be obtained by the Frobenius method:

Lemma 2.4. *A basis of solutions to (1.3) can be obtained as the coefficient of P_α^i for $\alpha = 1, \ldots, p$ and $i = 0, \ldots, \mu_\alpha - 1$ in the Γ-series in (1.2). Solutions to the irreducible equation $H^{\mathrm{red}} u = 0$ correspond to the coefficient of P_α^i for $i = 0, \ldots, \mu_\alpha - \nu_\alpha - 1$ in (1.2).*

Proof. Let

$$(2.2) \qquad H = e^{P_\alpha \log t} t^{\rho_\alpha} \sum_{n=0}^\infty t^n \frac{\prod_{k=1}^r \prod_{\substack{b:\ \langle b \rangle = \langle \rho_\alpha d_k \rangle \\ 0 < b \le (n+\rho_\alpha)d_k}} (d_k P_\alpha + b)}{\prod_{\nu=0}^N \prod_{\substack{b:\ \langle b \rangle = \langle \rho_\alpha q_\nu \rangle \\ 0 < b \le (n+\rho_\alpha)q_\nu}} (q_\nu P_\alpha + b)}$$

be the Γ-series in (1.2) considered as a formal power series in P_α, t and $\log t$. Here $e^{P_\alpha \log t}$ is considered as $\sum_{n=0}^\infty (P_\alpha \log t)^n/(n!)$, which satisfies $\theta \left(e^{P_\alpha \log t}\right) = P_\alpha e^{P_\alpha \log t}$. A direct calculation shows

$$\left[\prod_{\nu=0}^N \prod_{a=0}^{q_\nu - 1} (q_\nu \theta - a) - t \prod_{k=1}^r \prod_{b=1}^{d_k} (d_k \theta + b) \right] I$$

$$= \left(\prod_{\nu=0}^N \prod_{a=0}^{q_\nu - 1} (q_\nu P_\alpha + q_\nu \rho_\nu - a) \right) e^{P_\alpha \log t} t^{\rho_\alpha},$$

where the right-hand side is proportional to $P_\alpha^{\mu_\alpha}$, so that the coefficients of P_α^i for $i = 0, 1, \ldots, \mu_\alpha - 1$ give solutions to the hypergeometric equation (1.3). If one think of the series in (2.2) as a formal power series in t and $\log t$ with values in the μ_α-dimensional vector space $\mathbb{C}[P_\alpha]/(P_\alpha^{\mu_\alpha})$, then it is a polynomial in t^{ρ_α} and $\log t$ (this is clear from (2.2)) and a convergent power series in t (this follows either from the fact that the origin is a regular singularity of (1.3), or a direct estimate of the radius of convergence, which gives λ), which gives a multi-valued solution to (1.3).

Alternatively, one can also argue as follows: for the set of poles $\Pi_\alpha = \{w \in \mathbb{C} : |w + \rho_\alpha + n| = \epsilon, n \in \mathbb{N}, 0 < \epsilon \ll 1\}$ the Mellin–Barnes integral

$$\frac{1}{2\pi i} \int_{\Pi_\alpha} \prod_{\nu=0}^N \Gamma(q_\nu w) \prod_{k=1}^r \Gamma(1 - d_k w) s^{-w} dw,$$

with $s = (-1)^Q t$ gives us a solution that is the $P_\alpha^{\mu_\alpha - 1}$ part of (1.2). This can be seen from the following calculation:

$$\sum_{n \ge 0} \mathrm{Res}_{w=-\rho_\alpha - n} \prod_{\nu=0}^N \Gamma(q_\nu w) \prod_{k=1}^r \Gamma(1 - d_k w) s^{-w}$$

$$= \sum_{n \ge 0} \frac{1}{(\mu_\alpha - 1)!} \left(\frac{d}{dw} \right)^{\mu_\alpha - 1} ((w + \rho_\alpha + n)^{\mu_\alpha}$$

$$\times \prod_{\nu=0}^N \Gamma(q_\nu w) \prod_{k=1}^r \Gamma(1 - d_k w) s^{-w})|_{w=-\rho_\alpha - n}$$

$$= \sum_{n \geq 0} \frac{1}{(\mu_\alpha - 1)!} \left(-\frac{d}{dP} \right)^{\mu_\alpha - 1} ((-P)^{\mu_\alpha} \prod_{\nu=0}^{N} \Gamma(-q_\nu(\rho_\alpha + n + P))$$

$$\times \prod_{k=1}^{r} \Gamma(1 + d_k(\rho_\alpha + n + P)) s^{\rho_\alpha + n + P}|_{P=0}$$

$$= -\sum_{n \geq 0} \frac{(-1)^{nQ}}{(\mu_\alpha - 1)!} \sum_{\kappa=0}^{\mu_\alpha - 1} \binom{\mu_\alpha - 1}{\kappa}$$

$$\times \left(\left(\frac{d}{dP} \right)^{\mu_\alpha - 1 - \kappa} \frac{\prod_{k=1}^{r} \prod_{j=1}^{nd_k} (d_k(\rho_\alpha + P) + j)}{\prod_{\nu=0}^{N} \prod_{i=1}^{nq_\nu} (q_\nu(\rho_\alpha + P) + i)} ((-1)^{Q} t)^{\rho_\alpha + n + P} \right) |_{P=0}$$

$$\times \left(\left(\frac{d}{dP} \right)^{\kappa} P^{\mu_\alpha} \prod_{\nu=0}^{N} \Gamma(-q_\nu(\rho_\alpha + P)) \prod_{k=1}^{r} \Gamma(1 + d_k(\rho_\alpha + n + P)) \right) |_{P=0}.$$

To get a solution with $P_\alpha^{\mu_\alpha - 2}$ part of (1.2) we choose $\nu_1 \in [0, N]$ such that $\rho_\alpha = \frac{a}{q_{\nu_1}}$ for some $a \in [0, q_{\nu_1} - 1]$ and calculate

$$\frac{1}{2\pi i} \int_{\Pi_\alpha} (-1)^{q_{\nu_1} w} \frac{\prod_{\nu=0, \nu \neq \nu_1}^{N} \Gamma(q_\nu w) \prod_{k=1}^{r} \Gamma(1 - d_k w) s^{-w}}{\Gamma(1 - q_{\nu_1} w)} dw.$$

In this way we increase the number of Γ-factors in the denominator. The factor $\Gamma(q_\nu w)$ multiplied by a function $\frac{\sin (\pi q_\nu w)}{\pi} (-1)^{q_\nu w}$ with period $2\pi \sqrt{-1}$ gives $\frac{(-1)^{q_\nu w}}{\Gamma(1 - q_\nu w)}$. Thus, we obtain a μ_α tuple of Mellin–Barnes integral solutions to (1.3) that are linear combinations of (1.2). To get (1.2) solutions from the Mellin–Barnes integral solutions we need only to solve a system of linear equations determined by a $\mu_\alpha \times \mu_\alpha$ upper triangle matrix with non-zero diagonal entries.

As for the statement on the solutions to the irreducible operator H^{red} we shall consider the Mellin–Barnes integrals

$$\frac{1}{2\pi \sqrt{-1}} \int_{\Pi_\alpha} \frac{\prod_{\nu=0}^{N} \Gamma(q_\nu w)}{\prod_{k=1}^{r} \Gamma(d_k w)} s^{-w} dw,$$

whose poles $w \in \Pi_\alpha$ are at most of order $\mu_\alpha - \nu_\alpha$. On calculating its residues, we obtain a subspace of solutions to (1.3) of dimension $Q^{\mathrm{red}} = \sum_{\alpha=1}^{p} (\mu_\alpha - \nu_\alpha)$. $\qquad \square$

Corollary 2.5. *There is a basis*

$$\boldsymbol{X}(z) = (X_1(z), \ldots X_Q(z))$$

of solutions to (1.3) *such that the monodromy around* $z = 0$ *is given by*

$$\boldsymbol{X}(z) \to \boldsymbol{X}(z) \cdot E_0.$$

Proof. Recall that $\lambda = \prod_{\nu=0}^{N} q_\nu^{q_\nu} \big/ \prod_{k=1}^{r} d_k^{d_k}$ and $t = \lambda z$. The monodromy of the Γ-series in (2.2) around $t = 0$ comes only from

$$(2.3) \qquad \mathrm{e}^{P_\alpha \log t} t^{\rho_\alpha} \mapsto \mathrm{e}^{P_\alpha \log(\exp(2\pi\sqrt{-1})\, t)} (\exp(2\pi\sqrt{-1})\, t)^{\rho_\alpha}$$

$$(2.4) \qquad = \mathrm{e}^{2\pi\sqrt{-1}P_\alpha} \cdot \exp(2\pi\sqrt{-1}\rho_\alpha) \cdot \mathrm{e}^{P_\alpha \log t} t^{\rho_\alpha},$$

and other terms are single-valued. Now Corollary 2.5 follows from the fact that the linear operator

$$\mathrm{e}^{2\pi\sqrt{-1}P_\alpha} \cdot \exp(2\pi\sqrt{-1}\rho_\alpha)$$
$$= \left(1 + 2\pi\sqrt{-1}P_\alpha + \cdots + \frac{1}{(\mu_\alpha - 1)!} \left(2\pi\sqrt{-1}P_\alpha\right)^{\mu_\alpha - 1} \right) \cdot e_\alpha$$

on the μ_α-dimensional vector space $\mathbb{C}[P_\alpha]/(P_\alpha^{\mu_\alpha})$ has Jordan normal form $e_\alpha \operatorname{id}_{\mu_\alpha} + J_{\mu_\alpha, -}$.

Alternatively, one can also argue as follows: this can be seen by a generalization of Frobenius method for a differential equation with multiple local exponents at $s = 0$. The residue

$$\operatorname{Res}_{w = -\rho_\alpha} \prod_{\nu=0}^{N} \Gamma(q_\nu w) \prod_{k=1}^{r} \Gamma(1 - d_k w) s^{-w}$$

admits an asymptotic expansion of the following type:

$$\sum_{j=1}^{\mu_\alpha} (\log s)^{\mu_\alpha - j} s^{\rho_\alpha} h_j(s)$$

for holomorphic functions $h_j(s)$, $j = 1, \ldots, \sigma_\alpha$ at $s = 0$. If we apply the monodromy action $s \mapsto s\mathrm{e}^{2\pi\sqrt{-1}}$ to this series μ_α times repeatedly, we get a local basis of a rank μ_α subspace of solutions. The monodromy action induced by a turn around $s = 0$ on this subspace of solutions has the Jordan normal form $e_\alpha \operatorname{id}_{\mu_\alpha} + J_{\mu_\alpha, -}$. For each α, this subspace is invariant with

respect to the monodromy action within the total solution space, we get the statement. □

Set $\sigma_0 = 0$ and $\sigma_i = \sum_{\alpha=1}^{i} \mu_\alpha$ for $i = 1, \ldots, p$.

Lemma 2.6. $X_{\sigma_i}(z)$ *is singular at $z = 1$ for any $1 \leq i \leq p$.*

Proof. Assume that $X_{\sigma_i}(z)$ is holomorphic at $z = 1$. Since $X_{\sigma_i}(z)$ is a solution to (1.3), its only possible singular points on $\mathbb{C}$ are $z = 0$ and 1, so that $z^{-\rho_i} X_{\sigma_i}(z)$ is in fact an entire function. Since (1.3) has a regular singularity at infinity, $X_{\sigma_i}(z)$ has at most polynomial growth at infinity. This implies that $z^{-\rho_i} X_{\sigma_i}(z)$ is a polynomial, which cannot be the case since the series (1.2) defining $X_{\sigma_i}(z)$ around the origin is infinite. □

Lemma 2.7. *There is a fundamental solution $\boldsymbol{Y}(z) = (Y_1(z), \ldots, Y_Q(z))$ of (1.3) around $z = 1$ such that $Y_i(z)$ is holomorphic for $i = 1, \ldots, Q - 1$.*

Proof. We prove the following stronger result; Y_i has a series expansion

$$Y_i = (z - 1)^{i-1} \sum_{m \geq 0} G_m (z - 1)^m$$

for $i = 1, \ldots, Q - 1$, and $Y_Q(z)$ has the series expansion

$$Y_Q(z) = (z - 1)^{\frac{n-1}{2}} \sum_{m \geq 0} G'_m (z - 1)^m + \sum_{m \geq 0} G''_m (z - 1)^m$$

when n is even, and

$$Y_Q(z) = (z - 1)^{\frac{n-1}{2}} \log(z - 1) \left(\sum_{m \geq 0} G'_m (z - 1)^m \right) + \sum_{m \geq 0} G''_m (z - 1)^m$$

when n is odd. Since $Q - 2$ is the largest exponent, one can find a series solution

$$Y_{Q-1} = (z - 1)^{Q-2} \sum_{m \geq 0} G_m (z - 1)^m$$

to (1.3).

Next we remove the common factor θ in two terms of (1.3) from the left to obtain a differential equation; the factor $q_0 \theta_t$ with $\nu = a = 0$ in the first term is replaced with q_0, and the factor $(d_1 \theta_t + d_1)$ with $k = 1$ and $b = d_1$ in the second term is replaced with d_1 (note that $t(d_1 \theta_t + d_1) = d_1 \theta_t t$). One

can see (cf., e.g., [5, (2.8)]) that the set of local exponents of the resulting equation at $z = 1$ is given by

$$\left\{0, 1, \ldots, Q - 3, \frac{n-1}{2}\right\}.$$

Now $Q - 3$ is the largest exponent, and one can find a series solution

$$Y_{Q-2} = (z - 1)^{Q-3} \sum_{m \geq 0} G_m (z - 1)^m$$

to this equation.

One can continue this process until the differential equation becomes irreducible of rank Q^{red} whose set of exponents is given by $\{0, 1, \ldots, Q^{\mathrm{red}} - 2, \frac{n-1}{2}\}$. This irreducible differential equation describes the period of the Calabi–Yau manifold obtained by compactifying Y_t (cf. [7, Theorem 1.1]). This Calabi–Yau manifold has an ordinary double point at $z = 1$, and the period integral along the vanishing cycle gives the singular solution $Y_Q(z)$, while integrals against classes orthogonal to the vanishing cycle give holomorphic solutions $Y_1(z), \ldots, Y_{Q^{\mathrm{red}}-1}(z)$. $\square$

Lemmas 2.6 and 2.7 imply the following:

Lemma 2.8. *One can choose a fundamental solution* $\boldsymbol{Y}(z) = (Y_1(z), \ldots, Y_Q(z))$ *around* $z = 1$ *so that the connection matrix*

(2.5) $$\boldsymbol{X}(z) = \boldsymbol{Y}(z) \cdot L_1$$

is given by

(2.6) $$L_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ c_1 & c_2 & \cdots & c_{Q-1} & 1 \end{pmatrix}$$

where $c_{\sigma_i} \neq 0$ *for any* $i = 1, \ldots, p$.

When n is odd, the monodromy of Y_Q around $s = 1$ is given by

$$Y_Q(z) \rightarrow Y_Q(z) + 2\pi\sqrt{-1}(z-1)^{(n-1)/2} \sum_{m=0}^{\infty} G'_m (z-1)^m.$$

The second term is holomorphic at $z = 1$ and can be expressed as a linear combination of $Y_1(z), \ldots, Y_{Q-1}(z)$. Hence, the monodromy around $z = 1$ is given by

$$\boldsymbol{Y}(z) \rightarrow \boldsymbol{Y}(z) \cdot E_1$$

where

$$E_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 & c'_1 \\ 0 & 1 & \cdots & 0 & c'_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & c'_{Q-1} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

When n is even,

$$Y_Q(z) \rightarrow -Y_Q(z) + 2 \sum_{m=0}^{\infty} G'_m (z-1)^m,$$

so that the monodromy around $z = 1$ is given by

$$\boldsymbol{Y}(z) \rightarrow \boldsymbol{Y}(z) \cdot E_1,$$

where

$$E_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 & c'_1 \\ 0 & 1 & \cdots & 0 & c'_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & c'_{Q-1} \\ 0 & 0 & \cdots & 0 & -1 \end{pmatrix}.$$

Note that the monodromy of $\boldsymbol{Y}(z)$ around $z = 0$ is given by

$$\boldsymbol{Y}(z) = \boldsymbol{X}(z) \cdot L_1^{-1}$$
$$\rightarrow \boldsymbol{X}(z) \cdot E_0 \cdot L_1^{-1} = \boldsymbol{Y}(z) \cdot L_1 \cdot E_0 \cdot L_1^{-1}.$$

By a straightforward calculation, we have the following:

Proposition 2.9. *The monodromy matrices h_0, h_1 and h_∞ around $z = 0$, 1 and ∞ with respect to the basis $\boldsymbol{Y}(z)$ of solutions of* (1.3) *are given by*

$$
h_0 = E_0 + \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ \gamma_1 & \gamma_2 & \cdots & \gamma_Q \end{pmatrix},
$$

$$
h_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 & g_1 \\ 0 & 1 & \cdots & 0 & g_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & g_{Q-1} \\ 0 & 0 & \cdots & 0 & (-1)^{n-1} \end{pmatrix},
$$

$$
h_\infty^{-1} = h_0 + \begin{pmatrix} 0 & 0 & \cdots & 0 & \delta_1 \\ 0 & 0 & \cdots & 0 & \delta_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \delta_Q \end{pmatrix}.
$$

Lemma 2.10. *Let $v = (v_1, \ldots, v_Q)^{\mathrm{T}}$ be a column vector and define a $Q \times Q$ matrix by*

$$
T = (v, h_0 \cdot v, \ldots, h_0^{Q-1} \cdot v).
$$

Then one has

$$
\det T = \pm \prod_{1 \le \alpha < \beta \le p} (e_\alpha - e_\beta)^{\mu_\alpha \cdot \mu_\beta} \cdot \prod_{\alpha=1}^p (v_{\sigma_{\alpha-1}+1})^{\mu_\alpha}.
$$

Proof. Let $T(\alpha, j) \in \mathrm{SL}_Q(\mathbb{C})$ be the block diagonal matrix defined by

$$
T(\alpha, j) = \begin{pmatrix} \mathrm{id}_{Q-j-1} & 0 \\ 0 & \mathrm{id}_{j+1} - e_\alpha \cdot J_{j+1,+} \end{pmatrix}.
$$

Then

$$
\begin{aligned}
&T \cdot T(1, Q - 1) \cdot T(1, Q - 2) \cdot \cdots \cdot T(1, Q - \mu_1) \\
&\quad \cdot T(2, Q - \mu_1 - 1) \cdot \cdots \cdot T(2, Q - \mu_1 - \mu_2) \\
&\quad \cdot T(p, Q - \sigma_{p-1} - 1) \cdot \cdots \cdot T(p, 1)
\end{aligned}
$$

is a lower-triangular matrix whose ith diagonal component for $\sigma_{\alpha-1} < i \leq \sigma_\alpha$ is given by

$$\prod_{\beta < \alpha} (e_\alpha - e_\beta)^{\mu_\beta} \cdot v_{\sigma_{\alpha-1}+1}. \qquad \square$$

Corollary 2.11. $v = (v_1, \ldots, v_Q)^\mathrm{T}$ *is a cyclic vector with respect to h_0 if and only if the condition*

$$(2.7) \qquad\qquad \prod_{\alpha=1}^{p} v_{\sigma_{\alpha-1}+1} \neq 0$$

is satisfied.

Lemma 2.12. *If $v \in \mathbb{C}^Q$ satisfies*

$$(2.8) \qquad\qquad h_\infty^{-i} \cdot v = h_0^i v, \quad i = 1, 2, \ldots, Q - 1,$$

then (2.7) holds.

Proof. Since the kernel of $h_\infty^{-1} - h_0$ is the orthogonal complement of the last coordinate vector $e_Q = (0, \ldots, 0, 1) \in \mathbb{C}^Q$, Equations (2.8) for $v = (\boldsymbol{v}, 0)$ where $\boldsymbol{v} = (v_1, \ldots, v_{Q-1})$ can be rewritten as

$$\Sigma \cdot \boldsymbol{v} = 0,$$

where Σ is a $(Q-1) \times (Q-1)$ matrix whose jth row vector is the first $Q-1$ components of the last row vector of h_0^j. Define a block diagonal $(Q-1) \times (Q-1)$ matrix by

$$S(\alpha, j) = \begin{pmatrix} \mathrm{id}_{Q-j-2} & 0 \\ 0 & S' \end{pmatrix}$$

where $S' \in \mathrm{SL}_{j+1}(\mathbb{C})$ is given by

$$S' = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ -e_\alpha & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & -e_\alpha & 1 \end{pmatrix}.$$

Then the components of the matrix

$$\widetilde{\Sigma} = S(1,1) \cdots S(1, \mu_1 - 1) \cdot S(2, \mu_1) \cdots S(2, \sigma_2 - 1) \cdot S(3, \sigma_2) \cdots$$
$$S(3, \sigma_3 - 1) \cdots S(p, \sigma_{p-1}) \cdots S(p, \sigma_p - 2) \cdot \Sigma$$

are zero below the anti-diagonal (i.e., $\widetilde{\Sigma}_{ij} = 0$ if $i + j > Q$) and the ith anti-diagonal component $\widetilde{\Sigma}_{i,Q-i-1}$ for $\sigma_{\alpha-1} < i \le \sigma_\alpha$ is given by

$$\prod_{\beta > \alpha} (e_\alpha - e_\beta)^{\mu_\beta} c_{\sigma_\alpha}.$$

The $(Q-1)$-st equation

$$(\text{const}) \cdot v_1 + \prod_{\beta > 1} (e_1 - e_\beta)^{\mu_\beta} c_{\mu_1} v_2 = 0$$

together with Lemma 2.8 implies that $v_2 = 0$ if $v_1 = 0$. By repeating this type of argument, one shows that $v_1 = 0$ implies $\boldsymbol{v} = 0$. Moreover, one can run the same argument by interchanging the role of (v_1, e_1, c_{μ_1}) with $(v_{\sigma_{\alpha-1}+1}, e_\alpha, c_{\sigma_\alpha})$ to show that $v_{\sigma_{\alpha-1}+1} = 0$ implies $\boldsymbol{v} = 0$. Hence, a non-trivial solution to (2.8) must satisfy (2.7). $\qquad \square$

This concludes the proof of Theorem 1.1.

3. Invariants of the hypergeometric group

We prove the following in this section:

Proposition 3.1. *Let $\boldsymbol{q} = (q_0, \ldots, q_N)$ and $\boldsymbol{d} = (d_1, \ldots, d_r)$ be sequences of positive integers such that $Q := \sum_{i=0}^{N} q_i = \sum_{k=1}^{r} d_r$. Then the space of $Q \times Q$ matrices invariant under the action*

$$H_{\boldsymbol{q},\boldsymbol{d}} \ni h : X \mapsto h \cdot X \cdot h^{\mathrm{T}}$$

is at most one-dimensional.

Proof. Let X be a $Q \times Q$ matrix invariant under the hypergeometric group $H_{\boldsymbol{q},\boldsymbol{d}}$, so that

$$(3.1) \qquad\qquad h \cdot X \cdot h^{\mathrm{T}} = X$$

for any $h \in H_{\boldsymbol{q},\boldsymbol{d}}$. Let $e_1 = (1, 0, \ldots, 0)^{\mathrm{T}}$ be the first coordinate vector. Since $\{(h_\infty^{\mathrm{T}})^i e_1\}_{i=0}^{Q}$ spans $\mathbb{C}^Q$, X_{ij} is determined by the $H_{\boldsymbol{q},\boldsymbol{d}}$-invariance once we

know X_{i1} for $i = 1, \ldots, Q$. Put

$$(3.2) \quad h_1 = h_0^{-1} \cdot h_\infty^{-1} = \begin{pmatrix} (-1)^r B_Q & 0 & \cdots & 0 & 0 \\ (-1)^r (B_{Q-1} - A_{Q-1}) & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^r (B_2 - A_2) & 0 & \cdots & 1 & 0 \\ (-1)^r (B_1 - A_1) & 0 & \cdots & 0 & 1 \end{pmatrix} \in H_{q,d}$$

and consider (3.1) for $h = h_1$. Since

$$\begin{aligned}
(h_1 \cdot X \cdot h_1^{\mathrm{T}})_{i1} &= \sum_{k,l=1}^{Q} (h_1)_{ik} X_{kl} (h_1)_{1l} \\
&= \sum_{k,l=1}^{Q} (h_1)_{ik} X_{kl} (-1)^{r+N+1} \delta_{1l} \\
&= \sum_{k=1}^{Q} (-1)^{N+r+1} (h_1)_{ik} X_{k1},
\end{aligned}$$

the first column of (3.1) reduces to

$$(3.3) \qquad\qquad (-1)^{N+r+1} ((h_1)_{i1} X_{11} + X_{i1}) = X_{i1}$$

for $2 \le i \le Q$. If $n = N - r$ is even, then (3.3) implies

$$X_{i1} = -\frac{1}{2} (h_1)_{i1} X_{11},$$

so that the space of $H_{q,d}$-invariants is at most one-dimensional. If $N + r$ is odd, then (3.3) gives $X_{11} = 0$. Fix $j \ne 1$ such that $(h_1)_{j1} = (-1)^r (B_{Q-j+1} -$

$A_{Q-j+1}) \neq 0$. Since

$$
\begin{aligned}
(h_1 \cdot X \cdot h_1^{\mathrm{T}})_{ij} &= \sum_{k,l=1}^{Q} (h_1)_{ik} X_{kl} (h_1)_{jl} \\
&= \sum_{k=1}^{Q} (h_1)_{ik} (X_{k1}(h_1)_{j1} + X_{kj}(h_1)_{jj}) \\
&= \sum_{k=1}^{Q} (h_1)_{ik} (X_{k1}(h_1)_{j1} + X_{kj}) \\
&= (h_1)_{i1} (X_{11}(h_1)_{j1} + X_{1j}) + (X_{i1}(h_1)_{j1} + X_{ij}) \\
&= (h_1)_{i1} X_{1j} + X_{i1}(h_1)_{j1} + X_{ij},
\end{aligned}
$$

the jth column of (3.1) gives

$$
(h_1)_{i1} X_{1j} + (h_1)_{j1} X_{i1} = 0
$$

for $2 \leq i \leq Q$. Since $(h_1)_{j1} \neq 0$, one obtains

$$
X_{i1} = -\frac{(h_1)_{1i}}{(h_1)_{j1}} X_{1j}
$$

for $2 \leq i \leq Q$, so that the space of H-invariants is at most one-dimensional also in this case. □

4. Coherent sheaves on Calabi–Yau complete intersections in weighted projective spaces

We prove the $H_{\boldsymbol{q},\boldsymbol{d}}$-invariance of the Gram matrix in Theorem 1.2 in this section. The proof is closely related to the discussion of Golyshev [12, Section 1], although the use of the right dual collection $(\mathcal{F}_i)_{i=1}^{Q}$ seems to be new.

Let Y be a smooth complete intersection of degree $(d_1, \ldots, d_r)$ in the weighted projective space $\mathbb{P} = \mathbb{P}(q_0, \ldots, q_N)$. We use the Koszul resolution

$$(4.1) \quad 0 \longrightarrow \mathcal{O}_{\mathbb{P}}(-d_1 - \cdots - d_r) \longrightarrow \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}}(-d_1 - \cdots - \widehat{d_i} - \cdots - d_r)$$

$$\longrightarrow \cdots \longrightarrow \bigoplus_{1 \le i < j \le r} \mathcal{O}_{\mathbb{P}}(-d_i - d_j) \longrightarrow \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}}(-d_i)$$

$$\longrightarrow \mathcal{O}_{\mathbb{P}} \longrightarrow \iota_* \mathcal{O}_Y \longrightarrow 0$$

of the structure sheaf $\mathcal{O}_Y$ of Y to compute the derived restriction

$$\mathbb{L}\iota^*(-) := (-) \overset{\mathbb{L}}{\underset{\iota^{-1}\mathcal{O}_{\mathbb{P}}}{\bigotimes}} \mathcal{O}_Y : D^b \operatorname{coh} \mathbb{P} \to D^b \operatorname{coh} Y,$$

where $\iota : Y \hookrightarrow \mathbb{P}$ is the inclusion.

Let $(\mathcal{E}_i)_{i=1}^{Q}$ be the full strong exceptional collection on $D^b \operatorname{coh} \mathbb{P}$ given as

$$(\mathcal{E}_1, \ldots, \mathcal{E}_Q) = (\mathcal{O}, \ldots, \mathcal{O}(Q-1)),$$

and $(\mathcal{F}_1, \ldots, \mathcal{F}_Q)$ be its right dual exceptional collection characterized by the condition

$$\operatorname{Ext}^k(\mathcal{E}_{Q-i+1}, \mathcal{F}_j) = \begin{cases} \mathbb{C} & i = j, \text{ and } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\mathcal{F}_1 = \mathcal{O}_{\mathbb{P}}(-1)[N]$ and $\mathcal{F}_Q = \mathcal{E}_1 = \mathcal{O}_{\mathbb{P}}$. The Euler form on the Grothendieck group $K(\mathbb{P})$ defined by (1.1) is neither symmetric nor anti-symmetric, whereas that on $K(Y)$ is either symmetric or anti-symmetric depending on the dimension of Y. The bases $\{[\mathcal{E}_i]\}_{i=1}^{Q}$ and $\{[\mathcal{F}_i]\}_{i=1}^{Q}$ of $K(\mathbb{P})$ are dual to each other in the sense that

$$(4.2) \qquad\qquad \chi(\mathcal{E}_{Q-i+1}, \mathcal{F}_j) = \delta_{ij}.$$

We will write the derived restrictions of $\mathcal{E}_i$ and $\mathcal{F}_i$ to Y as $\overline{\mathcal{E}}_i$ and $\overline{\mathcal{F}}_i$ respectively. Unlike $\{[\mathcal{E}_i]\}_{i=1}^{Q}$ and $\{[\mathcal{F}_i]\}_{i=1}^{Q}$, $\{[\overline{\mathcal{E}}_i]\}_{i=1}^{Q}$ and $\{[\overline{\mathcal{F}}_i]\}_{i=1}^{Q}$ are not

bases of $K(Y)$, and their images in the numerical Grothendieck group are linearly dependent. Put

$$\overline{X}_{ij} = \chi([\overline{\mathcal{F}}_i], [\overline{\mathcal{F}}_j])$$

and let $(a_{ij})_{i,j=1}^Q$ be the transformation matrix between two bases $\{[\mathcal{E}_i]\}_{i=1}^Q$ and $\{[\mathcal{F}_i]\}_{i=1}^Q$ so that

$$[\mathcal{F}_i] = \sum_{j=1}^{Q} [\mathcal{E}_j] a_{ji}.$$

We prove the following in this section:

Proposition 4.1. $\overline{X}$ is an invariant of the hypergeometric group $H_{\boldsymbol{q},\boldsymbol{d}}$.

We divide the proof into three steps.

Lemma 4.2. *Let Φ be an autoequivalence of $D^b \operatorname{coh} Y$ such that its action on $\{[\overline{\mathcal{F}}_i]\}_{i=1}^Q$ is given by*

$$[\overline{\mathcal{F}}_i] \mapsto \sum_{j=1}^{Q} h_{ij} [\overline{\mathcal{F}}_j].$$

Then $\overline{X}$ is invariant under the action of $h = (h_{ij})_{i,j=1}^Q$;

$$\overline{X} = h \cdot \overline{X} \cdot h^{\mathrm{T}}.$$

Proof. Since an autoequivalence Φ induces an isometry of $K(Y)$, one has

$$
\begin{aligned}
\overline{X}_{ij} &= \chi([\overline{\mathcal{F}}_i], [\overline{\mathcal{F}}_j]) \\
&= \chi([\Phi(\overline{\mathcal{F}}_i)], [\Phi(\overline{\mathcal{F}}_j)]) \\
&= \sum_{k,l=1}^{Q} h_{ik} \chi([\overline{\mathcal{F}}_k], [\overline{\mathcal{F}}_l]) h_{jl} \\
&= \sum_{k,l=1}^{Q} h_{ik} \overline{X}_{kl} h_{jl}
\end{aligned}
$$

for any $1 \le i, j \le Q$. $\qquad\square$

Remark 4.3. Since $\{[\overline{\mathcal{F}}_i]\}_{i=1}^Q$ are not linearly independent, the choice of h in Lemma 4.2 is not unique.

Lemma 4.4. *The action of the autoequivalence of $D^b \operatorname{coh} Y$ defined by the tensor product with $\mathcal{O}_Y(-1)$ on $\{\overline{\mathcal{F}}_i\}_{i=1}^Q$ is given by h_0;*

$$[\overline{\mathcal{F}}_i \otimes \mathcal{O}_Y(-1)] = \sum_{j=1}^Q (h_0)_{ij} [\overline{\mathcal{F}}_j].$$

Proof. Since tensor product with $\mathcal{O}(-1)$ commutes with restriction, it suffices to show

$$[\mathcal{F}_i \otimes \mathcal{O}_{\mathbb{P}}(-1)] = \sum_{j=1}^Q (h_0)_{ij} [\mathcal{F}_j].$$

Since $\{[\mathcal{E}_{Q-i+1}]\}_{i=1}^Q$ and $\{[\mathcal{F}_i]\}_{i=1}^Q$ are dual bases, this is equivalent to

(4.3) $$[\mathcal{E}_{Q-i+1} \otimes \mathcal{O}(-1)] = \sum_{j=1}^Q [\mathcal{E}_{Q-j+1}] (h_0^{-1})_{ji}.$$

Recall from (1.7) that

$$(h_0^{-1})_{ji} = \delta_{j,i+1} - \delta_{i,Q} B_{Q-j+1}.$$

Since $\mathcal{E}_i = \mathcal{O}(i-1)$, Equation (4.3) for $i \neq Q$ gives

$$\mathcal{O}_{\mathbb{P}}(Q-i) \otimes \mathcal{O}_{\mathbb{P}}(-1) = \mathcal{O}_{\mathbb{P}}(Q-i-1),$$

which is obvious. Equation (4.3) for $i = Q$ gives

$$[\mathcal{O}_{\mathbb{P}}(-1)] + \sum_{j=1}^Q B_{Q-j+1} [\mathcal{O}_{\mathbb{P}}(Q-j)] = 0,$$

which is $\mathcal{O}_{\mathbb{P}}(-1)$ times the relation

$$[\mathcal{O}_{\mathbb{P}}] + B_1 [\mathcal{O}_{\mathbb{P}}(1)] + \cdots + B_{Q-1} [\mathcal{O}_{\mathbb{P}}(Q-1)] + B_Q [\mathcal{O}_{\mathbb{P}}(Q)] = 0$$

coming from the exact sequence

(4.4)
$$0 \to \mathcal{O}_{\mathbb{P}} \to \bigoplus_{i=0}^N \mathcal{O}_{\mathbb{P}}(q_i) \to \bigoplus_{0 \leq i < j \leq N} \mathcal{O}_{\mathbb{P}}(q_i + q_j) \to \cdots \to \mathcal{O}_{\mathbb{P}}\left(\sum_{i=0}^N q_i\right) \to 0$$

obtained by sheafifying the Koszul resolution

$$0 \to \Lambda^N V \otimes \operatorname{Sym}^* V^* \to \cdots \to \Lambda^2 V \otimes \operatorname{Sym}^* V^*$$
$$\to V \otimes \operatorname{Sym}^* V^* \to \operatorname{Sym}^* V^* \to \mathbb{C} \to 0,$$

where V is a graded vector space such that $\mathbb{P} = \operatorname{Proj}(\operatorname{Sym}^* V^*)$. $\qquad \square$

Lemma 4.5. *The action of the autoequivalence of $D^b \operatorname{coh} Y$ given by the dual spherical twist $T^\vee_{\overline{\mathcal{F}}_1}$ along $\overline{\mathcal{F}}_1$ is given on $\{\overline{\mathcal{F}}_i\}^Q_{i=1}$ by h_1;*

$$[T^\vee_{\overline{\mathcal{F}}_1}(\overline{\mathcal{F}}_i)] = \sum_{i=1}^Q (h_1)_{ij}[\overline{\mathcal{F}}_j].$$

Proof. Recall that for a spherical object $\mathcal{E}$ and an object $\mathcal{F}$, the dual spherical twist $T^\vee_\mathcal{E} \mathcal{F}$ of $\mathcal{F}$ along $\mathcal{E}$ is defined as the mapping cone

$$T^\vee_\mathcal{E} \mathcal{F} = \{\mathcal{F} \to \hom(\mathcal{F}, \mathcal{E})^\vee \otimes \mathcal{F}\}$$

of the dual evaluation map. Since the induced action on the Grothendieck group is given by the reflection

$$[T^\vee_\mathcal{E}(\mathcal{F})] = [\mathcal{F}] - \chi(\mathcal{F}, \mathcal{E})[\mathcal{E}],$$

it suffices to show that

(4.5) $$(h_1)_{ij} = \delta_{ij} - \overline{X}_{i1}\delta_{j1}.$$

Recall from (3.2) that

$$(h_1)_{ij} = \begin{cases} (-1)^r B_Q & i = j = 1, \\ \delta_{ij} & j \neq 1, \\ (-1)^r (B_{Q-i+1} - A_{Q-i+1}) & i \neq 1 \text{ and } j = 1. \end{cases}$$

Equation (4.5) for $j \neq 1$ is obvious, and that for $i = j = 1$ follows from

$$(-1)^r B_Q = (-1)^{r+N+1} = (-1)^{N-r+1} = (-1)^{n+1}$$

and

$$\overline{X}_{11} = \begin{cases} 0 & n \text{ is odd,} \\ 2 & n \text{ is even.} \end{cases}$$

To prove (4.5) for $i \neq 1$ and $j = 1$, one can use

$$\chi(\mathcal{F}_i(1)) = \sum_{j=1}^{Q} \chi((h_0^{-1})_{ij}\mathcal{F}_j) = \sum_{j=1}^{Q} (h_0^{-1})_{ij}\chi(\mathcal{E}_1, \mathcal{F}_j) = (h_\infty^{-1})_{iQ} = -B_{Q-i+1}$$

and

$$\chi(\mathcal{F}_i(j)) = \chi(\mathcal{O}(-j), \mathcal{F}_i) = \chi(\mathcal{E}_{-j+1}, \mathcal{F}_i) = \delta_{Q+j,i}$$

for $-Q + 1 \leq j \leq 0$ to show

$$\begin{aligned}
(-1)^r \overline{X}_{i1} &= (-1)^r \chi(\overline{\mathcal{F}}_i, \overline{\mathcal{F}}_1) \\
&= (-1)^N \chi(\overline{\mathcal{F}}_1, \overline{\mathcal{F}}_i) \\
&= (-1)^N \chi(\mathcal{O}_Y(-1)[N], \overline{\mathcal{F}}_i) \\
&= \chi(\mathcal{O}_Y(-1), \overline{\mathcal{F}}_i) \\
&= \chi(\overline{\mathcal{F}}_i(1)) \\
&= \chi(\mathcal{F}_i(1)) - \sum_{k=1}^{r} \chi(\mathcal{F}_i(1 - d_k)) + \sum_{1 \leq k < l \leq r} \chi(\mathcal{F}_i(1 - d_k - d_l)) \\
&\quad - \cdots + (-1)^r \chi(\mathcal{F}_i(1 - d_1 - \cdots - d_r)) \\
&= -B_{Q-i+1} - \sum_{k=1}^{r} \delta_{Q-d_k+1,i} + \sum_{1 \leq k < l \leq r} \delta_{Q-d_k-d_l+1,i} \\
&\quad - \cdots + (-1)^r \delta_{Q-d_1-\cdots-d_r+1,i} \\
&= -B_{Q-i+1} - \sum_{k=1}^{r} \delta_{Q-i+1,d_k} + \sum_{1 \leq k < l \leq r} \delta_{Q-i+1,d_k+d_l} \\
&\quad - \cdots + (-1)^r \delta_{Q-i+1,d_1+\cdots+d_r} \\
&= -B_{Q-i+1} + A_{Q-i+1},
\end{aligned}$$

where we have used (1.8) in the last equality.   □

5. Mirror manifolds and Stokes matrices

In this section, we discuss the relation between the Gram matrix in Theorem 1.2 in the case when Y is a hypersurface and the Stokes matrix for the quantum cohomology of the weighted projective space. By [8, Corollary 1.8], the quantum differential equation for the small J-function of $\mathbb{P}$ is given

by

$$(5.1) \qquad \prod_{i=0}^{n} \prod_{k=0}^{q_i-1} \left(q_i z \frac{\partial}{\partial t_1} - kz \right) J_{\mathbb{P}} = \mathrm{e}^{t_1} J_{\mathbb{P}},$$

where t_1 is the flat coordinate associated with the positive generator of $H^2(\mathbb{P}; \mathbb{Z}) \subset H^*_{\mathrm{orb}}(\mathbb{P}; \mathbb{C})$ and z is the quantization parameter. It follows that the stationary-phase integrals

$$(5.2) \qquad J_i(t_1; z) = \int_{\Gamma_i} \mathrm{e}^{f/z} \Omega$$

span the identity component of the space of flat sections of the first structure connection, where f is the function $f(x) = \sum_{i=0}^{N} q_i x_i$ on

$$\mathbb{T} = \{ (x_0, \ldots, x_N) \in (\mathbb{C}^\times)^{N+1} \mid x_0^{q_0} \cdots x_N^{q_N} = \mathrm{e}^{t_1} \},$$

Ω is the holomorphic volume form $\Omega = dx_0 \wedge \cdots \wedge dx_N / d(x_0^{q_0} \cdots x_N^{q_N})$ on $\mathbb{T}$, and $\{\Gamma_i\}_{i=1}^{Q}$ is a basis of flat sections of the local system whose fiber is the relative homology group $H_N(\mathbb{T}, \, \mathfrak{Re}(f/z) \ll 0; \mathbb{Z})$.

The function f has Q critical points

$$p_i = \mathrm{e}^{(t_1 - 2i\pi\sqrt{-1})/Q} \cdot (1, \ldots, 1), \quad i = 1, \ldots, Q$$

with critical values

$$f(p_i) = Q \mathrm{e}^{(t_1 - 2i\pi\sqrt{-1})/Q},$$

where the minus sign comes from the clockwise order on the distinguished set $(c_i)_{i=1}^{Q}$ of vanishing paths, which we choose as straight line segments from the origin to the critical values as in Fig. 1. See e.g. [1] for vanishing cycles and the Picard–Lefschetz formula. Let $(\gamma_i)_{i=1}^{Q}$ be the corresponding distinguished basis of vanishing cycles in $H_{N-1}(f^{-1}(0); \mathbb{Z})$. We choose Lefschetz thimbles $(\Gamma_i)_{i=1}^{Q}$ as in Fig. 2, which gives a basis of the relative homology group $H_N(\mathbb{T}, \, \mathfrak{Re}(f/z) \ll 0; \mathbb{Z})$ for $\arg(z) > 0$. They conjecturally correspond to the full exceptional collection $(\mathcal{F}_i)_{i=1}^{Q}$ in the derived category of coherent sheaves on $\mathbb{P}$ under homological mirror symmetry [16], and Theorem 5.1 below gives an evidence for this conjecture. The thimbles $(\Gamma'_i)_{i=1}^{Q}$ shown in dotted lines are the dual Lefschetz thimbles, which is a basis of $H_N(\mathbb{T}, \, \mathfrak{Re}(f/z) \ll 0; \mathbb{Z})$ for $\arg(z) < 0$ and should correspond to the dual exceptional collection $(\mathcal{E}_i)_{i=1}^{Q}$ under homological mirror symmetry.
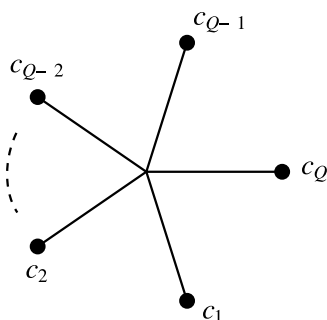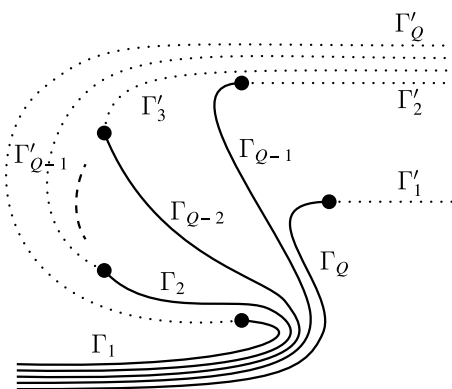
Figure 1: Vanishing paths.

Figure 2: Lefschetz thimbles.

The stationary-phase integral (5.2) is the Laplace transform

$$(5.3) \qquad J_i(t_1; z) = \int_{\ell_i} e^{s/z} \widetilde{I}_i(t_1; s) ds$$

of the period integral

$$\widetilde{I}_i(t_1; s) = \int_{\gamma_i \subset f^{-1}(s)} \Omega/df,$$

where ℓ_i is a path on the s-plane starting from a critical value underlying the Lefschetz thimble Γ_i and Ω/df is the Gelfand–Leray form on $f^{-1}(s)$.

   The Stokes matrix $(S_{ij})_{i,j=1}^{Q}$ is a part of the monodromy data for the stationary-phase integrals in (5.2), which is related to intersection numbers of vanishing cycles as follows (cf. e.g. [10, Section 4.1] or [21, Section 5]): Let $(\Gamma_j^+)_{j=1}^{Q}$ be a basis of $H_N(\mathbb{T}, \, \mathfrak{Re}(f/z) \ll 0; \mathbb{Z})$ for $\arg(z) > 0$, which is

obtained from the basis $(\Gamma'_{Q+1-i})_{i=1}^{Q}$ of $H_N(\mathbb{T}, \mathfrak{Re}(f/z) \ll 0; \mathbb{Z})$ for $\arg(z) < 0$ by parallel transport along a path in the upper half plane $\{z \in \mathbb{C}^{\times} \mid \mathfrak{Im}\, z \geq 0\}$ with respect to the Gauss–Manin connection on the relative homology bundle. Then the Stokes matrix is given by

$$\Gamma_j^+ = \sum_{i=1}^{Q} S_{ij} \Gamma_i.$$

On the other hand, Picard–Lefschetz formula (see, e.g., [1, 11, 18]) gives

$$\Gamma_j^+ - \Gamma_j = \sum_{i<j} (\gamma_i, \gamma_j) \Gamma_i,$$

where $(\gamma_i, \gamma_j) = (-1)^{N(N+1)/2}(\gamma_i \circ \gamma_j)$ is $(-1)^{N(N+1)/2}$ times the intersection number of vanishing cycles γ_i and γ_j. This shows that

$$S_{ij} = \begin{cases} (\gamma_i, \gamma_j) & i < j, \\ \delta_{ij} & \text{otherwise.} \end{cases}$$

Simultaneous multiplication $x_i \mapsto \alpha x_i$ by a constant $\alpha \in \mathbb{C}^{\times}$ induces an isomorphism from $f^{-1}(s)$ at $e^{t_1} = a$ to $f^{-1}(\alpha s)$ at $e^{t_1} = \alpha^Q a$, so that the period integral $\widetilde{I}_i(s; t_1)$ depends only on the ratio of s^Q and e^{t_1};

$$\widetilde{I}_i(s; t_1) = I_i(t), \quad t = \lambda Q^Q e^{t_1}/s^Q, \quad \lambda = \prod_{\nu=0}^{N} q_\nu^{q_\nu} \Big/ Q^Q.$$

Here, the factor Q^Q is chosen so that the critical values $s = f(p_i)$ go to $t = \lambda$. For any fixed value of t_1, the function $\widetilde{I}_i(s; t_1)$ is holomorphic at $s = 0$ and has singularities at $s = \infty$ and the critical values $s = f(p_i)$. On the other hand, the function $I_i(t)$ satisfies the irreducible hypergeometric differential equation $\mathcal{H}^{\mathrm{red}} I_i(t) = 0$ and has singularities at $t = 0$, λ and ∞. The singularities of $I_i(t)$ at $t = 0$ and λ come from those of $\widetilde{I}_i(s; t_1)$, whereas the singularity at $t = \infty$ comes from the Q-fold Kummer covering $t \sim 1/s^Q$.

The irreducible local system $\mathcal{L}^{\mathrm{red}}$ on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ associated with $\mathcal{H}^{\mathrm{red}} I = 0$ is described by Golyshev [12] as follows: Let Y be the anticanonical hypersurface in $\mathbb{P}$ and $\iota : Y \hookrightarrow \mathbb{P}$ be the inclusion. Let further $K(\mathbb{P})$ be the Grothendieck group of $\mathbb{P}$ and K be the subgroup of $K(\mathbb{P})$ generated by

$\{[\iota_*\mathcal{O}_Y(i)]\}_{i\in\mathbb{Z}}$. Set

$$\varphi_0(x) = \prod_{\nu=0}^{N}(x^{q_\nu} - 1), \quad \varphi_\infty(x) = x^Q - 1, \quad \eta(x) = \gcd\left(\varphi_0(x), \varphi_\infty(x)\right),$$

and

$$\overline{\varphi}_\infty(x) = \frac{\varphi_\infty(x)}{\eta(x)}, \quad \overline{\varphi}_0(x) = \frac{\varphi_0(x)}{\eta(x)},$$

so that $Q = \deg(\varphi_0(x)) = \deg(\varphi_\infty(x))$ and $Q^{\mathrm{red}} = Q - \deg(\eta(x))$. Since $K(\mathbb{P})$ is generated by $\{\mathcal{O}_\mathbb{P}(i)\}_{i\in\mathbb{Z}}$ with relations

$$\sum_{k=0}^{N+1}(-1)^k \sum_{0\le j_1 < \cdots < j_k \le N}\left[\mathcal{O}_\mathbb{P}\left(i - \sum_{\ell=1}^{k}q_{j_\ell}\right)\right] = 0$$

coming from the exact sequence (4.4), one has a ring isomorphism

(5.4) $$K(\mathbb{P}) \cong \mathbb{Z}[x, x^{-1}]/\varphi_0(x)$$

sending x^i to $[\mathcal{O}_\mathbb{P}(i)]$. It follows from (4.1) that

(5.5) $$[\iota_*\mathcal{O}_Y(i)] = [\mathcal{O}_\mathbb{P}(i)] - [\mathcal{O}_\mathbb{P}(i - Q)],$$

which goes to $x^i(1 - x^{-Q})$ under the isomorphism (5.4). This shows that the group K is isomorphic to the subgroup of $\mathbb{Z}[x, x^{-1}]/\varphi_0(x)$ generated by $x^i(1 - x^Q)$ for $i \in \mathbb{Z}$, and one has a short exact sequence

$$0 \longrightarrow K \longrightarrow K(\mathbb{P}) \longrightarrow K(\mathbb{P})/K \longrightarrow 0,$$

where

$$K \cong \varphi_\infty(x)\mathbb{Z}[x, x^{-1}]/(\varphi_0(x)) \cong \mathbb{Z}[x, x^{-1}]/(\overline{\varphi}_0(x))$$

and

$$K(\mathbb{P})/K \cong \mathbb{Z}[x, x^{-1}]/(\varphi_0(x), \varphi_\infty(x)) = \mathbb{Z}[x, x^{-1}]/(\eta(x)).$$

Let Φ_0 be an autoequivalence of $D^b \operatorname{coh} \mathbb{P}$ defined by $\Phi_0(-) = \mathcal{O}_\mathbb{P}(-1) \otimes (-)$. The induced map $\phi_0 : K(\mathbb{P}) \to K(\mathbb{P})$ is given by multiplication by x^{-1}, which clearly has $\varphi_0(x)$ as the characteristic polynomial. The autoequivalence Φ_0 induces an autoequivalence $\overline{\Phi}_0$ of $D^b \operatorname{coh} Y$ given by $\overline{\Phi}_0(-) = \mathcal{O}_Y(-1) \otimes (-)$.

The induced map $\overline{\phi}_0 : K \to K$ fits into the commutative diagram

$$
\begin{array}{ccc}
K & \hookrightarrow & K(\mathbb{P}) \\
\overline{\phi}_0 \downarrow & & \downarrow \phi_0 \\
K & \hookrightarrow & K(\mathbb{P})
\end{array}
$$

since

$$
\begin{aligned}
\iota_*(\mathcal{O}_Y(-1) \otimes (-)) &\cong \iota_*(\iota^* \mathcal{O}_{\mathbb{P}}(-1) \otimes (-)) \\
&\cong \mathcal{O}_{\mathbb{P}}(-1) \otimes \iota_*(-),
\end{aligned}
$$

and the characteristic polynomial of $\overline{\phi}_0$ is given by $\overline{\varphi}_0(x)$.

Let $\overline{\Phi}_1 = T^{\vee}_{\overline{\mathcal{F}}_1}$ be the autoequivalence of $D^b \operatorname{coh} Y$ appearing in Lemma 4.5. The induced map will be denoted by $\overline{\phi}_1 : K \to K$. Define an autoequivalence $\overline{\Phi}_{\infty}$ of $D^b \operatorname{coh} Y$ by

$$
\overline{\Phi}_{\infty} = \left(\overline{\Phi}_0 \circ \overline{\Phi}_1 \right)^{-1}.
$$

The induced map $\overline{\phi}_{\infty} : K \to K$ is given by

$$
\overline{\phi}_{\infty} = \left(\overline{\phi}_0 \circ \overline{\phi}_1 \right)^{-1},
$$

which is presented by the matrix $h_{\infty} = (h_1 \circ h_0)^{-1}$ by Lemmas 4.4 and 4.5. It follows that $\overline{\phi}_{\infty}$ acts on $\{[\overline{\mathcal{F}}_i]\}_{i=1}^Q$ by cyclic permutation

$$
[\overline{\mathcal{F}}_i] \mapsto \sum_{j=1}^Q (h_{\infty})_{ij} [\overline{\mathcal{F}}_j] = [\overline{\mathcal{F}}_{i-1}].
$$

Since $\{[\mathcal{F}_i]\}_{i=1}^Q$ is a free $\mathbb{Z}$-basis of $K(\mathbb{P})$, one can define a linear automorphism $\phi_{\infty} : K(\mathbb{P}) \to K(\mathbb{P})$ by

$$
(5.6) \qquad\qquad \phi_{\infty}([\mathcal{F}_i]) = [\mathcal{F}_{i-1}].
$$

We have a commutative diagram

$$
(5.7)
\begin{array}{ccccc}
K & \hookrightarrow & K(\mathbb{P}) & \twoheadrightarrow & K(\mathbb{P})/K \\
\overline{\phi}_{\infty} \downarrow & & \downarrow \phi_{\infty} & & \vdots \\
K & \hookrightarrow & K(\mathbb{P}) & \twoheadrightarrow & K(\mathbb{P})/K,
\end{array}
$$

which induces the dotted vertical arrow on the right. By taking the transpose of the diagram (5.7), one obtains the commutative diagram

$$
\begin{array}{ccccc}
K^\vee & \longleftarrow & K(\mathbb{P})^\vee & \lhook\joinrel\longrightarrow & (K(\mathbb{P})/K)^\vee \\
\overline{\phi}{}^T_\infty \downarrow & & \downarrow \phi^T_\infty & & \downarrow \phi^T_\infty|_{(K(\mathbb{P})/K)^\vee} \\
K^\vee & \longleftarrow & K(\mathbb{P})^\vee & \lhook\joinrel\longrightarrow & (K(\mathbb{P})/K)^\vee.
\end{array}
$$

Since the Euler form is non-degenerate by (4.2), it gives an identification $K \cong K(\mathbb{P})^\vee := \operatorname{Hom}(K(\mathbb{P}), \mathbb{Z})$ with the dual group. The transpose ϕ^T_∞ of (5.6) is given by the cyclic permutation $\phi^T_\infty([\mathcal{E}_i]) = [\mathcal{E}_{i+1}]$ of $\{[\mathcal{E}_i]\}^Q_{i=1}$, which goes to the cyclic permutation of $\{x^i\}^{Q-1}_{i=0}$ under the isomorphism $K(\mathbb{P}) \cong \mathbb{Z}[x, x^{-1}]/(\varphi_0(x))$. Note that the Euler form satisfies

$$
\begin{aligned}
\chi(x^j, x^i \cdot x^k) &= \chi(\mathcal{O}_\mathbb{P}(j), \mathcal{O}_\mathbb{P}(i+k)) = \chi(\mathcal{O}_\mathbb{P}(i-j+k)) \\
&= \chi(\mathcal{O}_\mathbb{P}(-i+j), \mathcal{O}_\mathbb{P}(k)) = \chi(x^{-i} \cdot x^j, x^k),
\end{aligned}
$$

so that the transpose of multiplication by x^i is given by multiplication by x^{-i}. One has

$$
(K(\mathbb{P})/K)^\vee \cong K^\perp \subset K(\mathbb{P})
$$

with respect the the Euler form, so that

$$
\begin{aligned}
(K(\mathbb{P})/K)^\vee &\cong \{\alpha \in K(\mathbb{P}) \mid \chi(\alpha, \beta) = 0 \text{ for any } \beta \in K\} \\
&= \{\alpha \in K(\mathbb{P}) \mid \chi(\alpha, \varphi_\infty(x)\gamma) = 0 \text{ for any } \gamma \in K(\mathbb{P})\} \\
&= \{\alpha \in K(\mathbb{P}) \mid \chi(\varphi_\infty(x^{-1})\alpha, \gamma) = 0 \text{ for any } \gamma \in K(\mathbb{P})\} \\
&= \operatorname{Ker}\left(\varphi_\infty(x^{-1}) : K(\mathbb{P}) \to K(\mathbb{P})\right) \\
&= \operatorname{Ker}\left(\varphi_\infty(x) : K(\mathbb{P}) \to K(\mathbb{P})\right).
\end{aligned}
$$

By (5.4) one has

$$
\operatorname{Ker}\left(\varphi_\infty(x) : K(\mathbb{P}) \to K(\mathbb{P})\right) \cong \mathbb{Z}[x, x^{-1}]/(\eta(x)).
$$

Since ϕ^T_∞ acts on a basis as a cyclic permutation $\phi^T_\infty([\mathcal{E}_i]) = [\mathcal{E}_{i+1}]$ that is the transpose of (5.6), the characteristic polynomial of ϕ^T_∞ is given by $\varphi_\infty(x) = x^Q - 1$. The characteristic polynomial of the induced action on $(K(\mathbb{P})/K)^\vee \cong \mathbb{Z}[x, x^{-1}]/(\eta(x))$ is given by $\eta(x)$, so that the characteristic polynomial of $\overline{\phi}{}^T_\infty : K^\vee \to K^\vee$ is given by $\overline{\varphi}_\infty(x) = \varphi_\infty(x)/\eta(x)$. Since transposition does not change the eigenvalues of a linear map, the characteristic polynomial of $\overline{\phi}_\infty$ is also given by $\overline{\varphi}_\infty(x)$.

Let $\mathcal{K}$ be the local system associated with $K \otimes_{\mathbb{Z}} \mathbb{C}$ such that the monodromy at the origin acts by $\overline{\varphi}_0 : [\iota_* \mathcal{O}_Y(i)] \mapsto [\iota_* \mathcal{O}_Y(i-1)]$ and the monodromy at λ acts by $\overline{\varphi}_1 : [\iota_* \mathcal{O}_Y(i)] \mapsto [\iota_*(T^{\vee}_{\overline{\mathcal{F}}_1} \mathcal{O}_Y(i))]$. From the explicit description of ϕ_0 and ϕ_∞ in terms of the isomorphism $K \cong \mathbb{Z}[x, x^{-1}]/(\overline{\varphi}_0(x))$, one can easily see that $\overline{\phi}_0$ and $\overline{\phi}_\infty$ have no non-trivial simultaneous invariant subspace, so that the local system $\mathcal{K}$ is irreducible.

The discussion so far shows that the irreducible local system $\mathcal{K}$ satisfies the conditions that

- the characteristic polynomial of the monodromy at 0 is given by $\overline{\varphi}_0(x)$,

- the characteristic polynomial of the monodromy at ∞ is given by $\overline{\varphi}_\infty(x)$, and

- the monodromy at λ is a pseudo-reflection.

According to [5, Theorem 3.5], an irreducible local system on $\mathbb{P}^1 \setminus \{0, \lambda, \infty\}$ satisfying the above conditions is unique up to isomorphism. One can see from [5, Section 2] that the irreducible local system $\mathcal{L}^{\mathrm{red}}$ defined in Section 1 satisfies these conditions, so that one has an isomorphism of the local systems $\mathcal{L}^{\mathrm{red}}$ and $\mathcal{K}$. The local section of $\mathcal{L}^{\mathrm{red}}$ coming from the integration along the vanishing cycle γ_1 is characterized as the unique section which is holomorphic at λ, and as such corresponds to the section $[\overline{\mathcal{F}}_1]$ in the local system $K \otimes_{\mathbb{Z}} \mathbb{C}$. Since the monodromy at infinity $\overline{\phi}_\infty$ acts by cyclic permutation

$$[\overline{\mathcal{F}}_i] \mapsto \sum_{j=1}^{Q} (h_\infty)_{ij} [\overline{\mathcal{F}}_j] = [\overline{\mathcal{F}}_{i-1}],$$

vanishing cycles $(\gamma_i)_{i=1}^{Q}$ correspond to $([\overline{\mathcal{F}}_i])_{i=1}^{Q}$ under this isomorphism. Since the intersection form on vanishing cycles is monodromy-invariant, the Gram matrix $(\gamma_i, \gamma_j)_{ij}$ is an invariant of $H_{\boldsymbol{q}, \boldsymbol{d}}$ so that it must be proportional to $(\chi(\overline{\mathcal{F}}_i, \overline{\mathcal{F}}_j))_{i,j}$ by Theorem 1.2. The multiplicative constant can be fixed from the fact that the monodromy of $\mathcal{L}^{\mathrm{red}}$ at λ is the pseudo-reflection by γ_1 and the monodromy of $\mathcal{K}$ at λ is the pseudo-reflection by $[\overline{\mathcal{F}}_1]$. If n is even, then the multiplicative constant can also be fixed by noting that $(\gamma_1, \gamma_1) = 2 = \chi(\overline{\mathcal{F}}_1, \overline{\mathcal{F}}_1)$. It follows that the Stokes matrix is given by

$$S_{ij} = (\gamma_i, \gamma_j) = \chi(\overline{\mathcal{F}}_i, \overline{\mathcal{F}}_j) = \chi(\mathcal{F}_i, \mathcal{F}_j) + (-1)^n \chi(\mathcal{F}_j, \mathcal{F}_i) = \chi(\mathcal{F}_i, \mathcal{F}_j)$$

for $i < j$, so that we have the following:

Theorem 5.1. *The Stokes matrix $(S_{ij})_{i,j=1}^{Q}$ for the quantum cohomology of the weighted projective space is given by the Gram matrix of the full exceptional collection $(\mathcal{F}_i)_{i=1}^{Q}$ with respect to the Euler form;*

$$(5.8) \qquad\qquad S_{ij} = \chi(\mathcal{F}_i, \mathcal{F}_j).$$

This generalizes the case of the projective space proved in [13, 20] and surely known to experts (see e.g. [15, Remark 4.13]). The relation between Stokes matrices and exceptional collections originates from [9] and was developed by Kontsevich [16], Zaslow [22] and Dubrovin [10].

The rank Q of the quantum differential equation (5.1) is the rank of the relative homology group $H_N(\mathbb{T}, \ \mathfrak{Re}(f/z) \ll 0; \mathbb{Z})$, whereas the rank Q^{red} of the irreducible local system $\mathcal{L}^{\mathrm{red}}$ is the rank of the homology group $H_{N-1}(\overline{f^{-1}(s)}; \mathbb{Z})$ of a smooth compactification $\overline{f^{-1}(s)}$ of the fiber $f^{-1}(s)$. As a result, sections of the reducible local system $\mathcal{L}$ not coming from sections of $\mathcal{L}^{\mathrm{red}}$ are not period integrals on the mirror manifold $\overline{f^{-1}(s)}$. Since the stationary-phase integrals J_i are Laplace transform of the period integrals $\widetilde{I}_i$ as described in (5.3), one only needs the monodromy of the irreducible local system $\mathcal{L}^{\mathrm{red}}$ for the proof of Theorem 5.1.

Acknowledgments

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DEPARTMENT OF MATHEMATICS
GALATASARAY UNIVERSITY
ÇIRAĞAN CAD. 36
BEŞIKTAŞ
ISTANBUL 34357, TURKEY
E-mail address: `tanabesusumu@hotmail.com, tanabe@gsu.edu.tr`

DEPARTMENT OF MATHEMATICS
GRADUATE SCHOOL OF SCIENCE
OSAKA UNIVERSITY
MACHIKANEYAMA 1-1
TOYONAKA
OSAKA 560-0043, JAPAN
E-mail address: `kazushi@math.sci.osaka-u.ac.jp`