Modular forms in quantum field theory

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The amplitude of a Feynman graph in Quantum Field Theory is related to the point-count over finite fields of the corresponding graph hypersurface. This article reports on an experimental study of point counts over $\mathbb{F}_q$ modulo q^3, for graphs up to loop order 10. It is found that many of them are given by Fourier coefficients of modular forms of weights ≤ 8 and levels ≤ 17.

1. Introduction

We first explain the definition of the c_2-invariant of a graph and its connection to its period. Then we turn to modularity and our results.

1.1. The c_2-invariant

Let G be a connected graph. The graph polynomial of G is defined by associating a variable x_e to every edge e of G and setting

$$(1.1) \qquad \Psi_G(x) = \sum_{T \, \text{span.tree}} \prod_{e \notin T} x_e,$$

where the sum is over all spanning trees T of G. These polynomials first appeared in Kirchhoff's work on currents in electrical networks [13].

The set of finite graphs G is filtered by the maximal degree of the set of vertices of G. We say that

$$(1.2) \qquad G \text{ is in } \phi^n \text{ theory if } \deg(v) \leq n \text{ for all vertices } v \text{ of } G$$

and we will mostly restrict to the physically meaningful case of ϕ^4.

The arithmetic content of perturbative Quantum Field Theories is given by integrals of rational functions, whose denominator is the square of the graph polynomial. This requires a convergency condition for the graphs. A

(a)   (b)   (c)   (d)   (e)
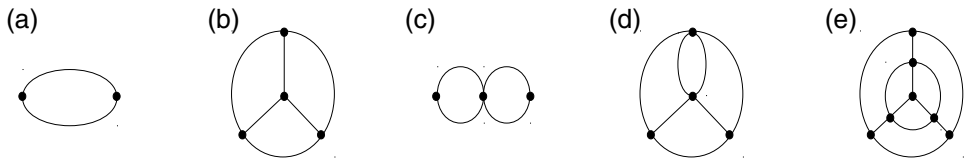

Figure 1: Graphs (a) and (b) are primitive-divergent whereas graphs (c), (d), and (e) have subdivergences.

connected graph G is called primitive-divergent (see figure 1) if

(1.3) $N_G = 2h_G,$

(1.4) $N_\gamma > 2h_\gamma$ for all non-trivial strict subgraphs $\gamma \subsetneqq G,$

where h_γ denotes the number of loops (first Betti number) and N_γ the number of edges in a graph γ. It is easy to see that primitive-divergent graphs with at least three vertices are simple: they have no multiple edges or self-loops.

   If G is primitive-divergent, the period of G is defined by the convergent integral [2,5] (which is independent of the choice of edge N_G)

(1.5) $$P(G) = \int_0^\infty \cdots \int_0^\infty \frac{\mathrm{d}x_1 \cdots \mathrm{d}x_{N_G-1}}{\Psi_G(x)^2|_{x_{N_G}=1}} \in \mathbb{R}_+.$$

In this way, $P(G)$ defines a map from the set of primitive-divergent graphs to positive real numbers. In the case of ϕ^4 theory they are renormalization-scheme independent contributions to the β-function [12].

   Since (1.1) is defined over the integers, it defines an affine scheme of finite type over $\mathrm{Spec}\,\mathbb{Z}$ which is called the graph hypersurface $X_G \subset \mathbb{A}^{N_G}$. For any field k, we can therefore consider the zero locus $X_G(k)$ of Ψ_G in k^{N_G}. If the ground field $k \cong \mathbb{F}_q$ is finite, we have the point-counting function

(1.6) $[X_G]_q : q \mapsto \#X_G(\mathbb{F}_q) \in \mathbb{N}.$

It defines a map from prime powers to non-negative integers. Inspired by the appearance of multiple zeta values in the period integral [3], Kontsevich informally conjectured in 1997 that the function $[X_G]$ might be polynomial in q for all graphs [14]. Although the conjecture is true for small graphs [23] and for certain sets of nearly complete graphs [10, 22], it is false in general. In [1] Belkale and Brosnan used Mnëv's universality theorem to prove that the $[X_G]$ for arbitrary graphs are, in a certain sense, of general type. In [5,7]

we proved the conjecture is true for some infinite families of graphs in ϕ^4 theory.

Nonetheless a connection between the point-counting function and the period (1.5) remains valid in all cases. Recent work [6, 7, 21] shows that certain information about the period is indeed detected by a small piece of the point-counting function $[X_G]_q$, called the c_2-invariant.

Proposition 1 [7]. *If G has at least three vertices there exists a quantity*

$$c_2(G) = (c_2(G)_q)_q \in \prod_q \mathbb{Z}/q\mathbb{Z},$$

where q ranges over the set of prime powers and $c_2(G)_q \in \mathbb{Z}/q\mathbb{Z}$ is defined as follows. One shows that $[X_G]_q \equiv 0 \mod q^2$, and sets

(1.7) $$c_2(G)_q \equiv [X_G]_q q^{-2} \mod q.$$

In the case when $[X_G]_q$ is a polynomial in q, the c_2-invariant is simply the reduction mod q of the coefficient of q^2 in this polynomial, and so there is an integer M such that $c_2(G)_q \equiv M \mod q$ for all q. When this happens, we say that the c_2-invariant is constant. In other words, the c_2-invariant is said to be constant if and only if it is in the image of the map

$$\mathbb{Z} \to \prod_q \mathbb{Z}/q\mathbb{Z}.$$

Any graph G with a non-constant c_2-invariant is therefore a counter-example to Kontsevich's problem. The connection between the period and the c_2-invariant is further borne out by the following conjecture, which holds in all presently known examples:

Conjecture 2. *If $P(G_1) = P(G_2)$ for primitive divergent graphs G_1, G_2, then $c_2(G_1)_q \equiv c_2(G_2)_q \mod q$ for all prime powers q.*

This conjecture is supported by [6], where it is shown that, for a large class of graphs, the c_2-invariant is related to the de Rham framing on the cohomology of the graph hypersurface given by the integrand of (1.5).

1.2. Which motives for quantum field theory?

Since graph polynomials are not polynomially countable [1, 7], and more to the point, the period (1.5) does not factorize through a category of

mixed Tate motives [6], it follows from standard conjectures in transcendence theory that the integral (1.5) will not be a multiple zeta value in general. An important question is to try to ascertain which families of periods do occur as values of (1.5), especially when one places physically meaningful restrictions on the graphs G. Due to the immense difficulty of computing the periods (1.5) directly, or even to obtain any non-trivial information about the mixed Hodge structure or motive [2] underlying (1.5) beyond 6 loops, one is forced to find new methods to probe the arithmetic content of ϕ^4 theory at high loop orders. The goal of this paper is to argue that the c_2-invariant gives an effective method to do just this, and to report on an experimental study of all c_2-invariants of primitive ϕ^4-graphs with up to and including 10 loops.

Note that a naive approach to computing the point-counting functions $[X_G]_q$, for any q, is completely impossible: at ten loops the graph polynomials Ψ_G are of degree 10 in 20 variables, and have thirty to forty thousand monomials. Furthermore, there are several thousand primitive divergent graphs. The main point is that the c_2-invariant satisfies sufficiently many combinatorial properties to reduce this to a manageable computation.

A first reduction, which uses graph-theoretical arguments together with a weakened version of Conjecture 2 (see Conjecture 14) allows us to reduce the number of relevant cases to 284 'prime ancestors' (see Theorem 18). The number of prime ancestors at a certain loop order is listed in table 2. A second, crucial, reduction is Theorem 22, which reduces the c_2-invariant of the hypersurface X_G to that of a hypersurface of much smaller dimension, for which the points can be counted for at least the first 6 primes in all 284 cases. This is enough to distinguish 145 c_2-invariants which are listed in Tab's. 7 and 8 at the end of the article. For graphs up to 9 loops we computed the c_2-invariants of their prime ancestors for the first 12 primes (table 6).

1.3. Findings

For small graphs (with ≤ 6 loops [23]) the c_2-invariant is constant and it is known that the integral (1.5) is a linear combination of multiple zeta values [3, 20]. At 7 loops, we find the first examples of c_2-invariants that are quasi-constant [11, 21]. This means that the c_2 is constant after excluding finitely many primes, or by passing to a finite field extension (Definition 23). Experimentally, we find that for graphs in ϕ^4 theory up to 10 loops, only the constants $c_2 = 0, -1$, and three other quasi-constants can occur corresponding to extending by 2nd, 3rd and 4th roots of unity (Conjecture 25). All these examples correspond to Tate motives.

The first non-Tate examples occur at 8 loops. We say that a c_2-invariant is *modular* if the point-counts $c_2(G)_p$ over finite fields $\mathbb{F}_p$ where p is prime, coincide with the pth Fourier coefficients of a modular form (possibly with a non-trivial character). In [7] we proved using modularity theorems for singular K3 surfaces that a certain graph with 8 loops is modular for all primes p. No such theorem is currently available for any other c_2-invariant. Therefore, we shall abusively say that a graph G is modular if $c_2(G)_p$ coincides with the Fourier coefficients of a modular form for small p. In practice, computing just a handful of primes is enough to fix the character of c_2 uniquely: for instance, the likelihood of a false identification after counting points over the first 11 primes is of the order of one in $|\mathbb{F}_2 \times \cdots \times \mathbb{F}_{31}| \sim 2 \times 10^{11}$. Thus we can be fairly confident that our experimentally-modular graphs (modular for at least the first 11 primes) are indeed modular for all primes.

Motivated by the 8-loop example, we searched for other modular examples of primitive-divergent graphs G in ϕ^4 theory up to and including 10 loops. The output of this search is summarized in table 1. In total 16 out of the (at least) 145 c_2-invariants of graphs with ≤ 10 loops in ϕ^4 theory are modular for (at least) the first 11 primes. The modular forms that correspond to these c_2-invariants are in heavy boxes in table 1.

One immediately notices from table 1 that the modular forms coming from ϕ^4 graphs have very low levels and never have weight 2. With our data we can rule out (assuming Conjecture 14) weight 2 modularity for all levels up to 1200 in graphs with ≤ 10 loops. This absence of weight 2 should be a general property of primitive ϕ^4-graphs (Conjecture 26). Moreover, we only found levels 7, 8, and 12 at weight 3. Remarkably, all these modular forms were already found in graphs with not more than 9 loops (the lower index in the boxes). At weight 4 we found modular forms of level 5, 6, 7, 13, and 17. At weight 5 only level 4 could be identified. At weight 6 we had modular forms of levels 3, 4, 7, 10, whereas at weights 7 and 8 levels 3 and 2, 5 could be found. We observe a tendency to higher weight for higher loop order. Otherwise we are not aware of any particular pattern in the identified modular forms.

In order to investigate the effect of the topology of the graph on the c_2-invariant, we also computed the c_2-invariants of all log-divergent graphs (graphs which satisfy (1.3) and (1.4), but which are not necessarily in ϕ^4 theory, see figure 6) up to and including 9 loops. These additional graphs give rise to the first three levels of weight 2 and fill in the gap at level 11 of weight 3. They are shown in light boxes in table 1. Moreover, already at 8 loops there exists a non-ϕ^4 graph with a quasi-constant c_2-invariant which does not occur in the set of ϕ^4 graphs with up to 10 loops. Its point-count

Table 1: Newforms of low level with rational Fourier coefficients and their first appearance in ϕ^4-theory (see main text). The lower index indicates the lowest loop order at which they occur, an upper index η indicates that the modular form is an η-product; see table 3 in Section 5. A subscript '$\phi^{>4}, 9$' denotes the modular form of a graph which does not lie in ϕ^4 theory (i.e., which has a vertex with valency greater than four) with 9 loops. The table does not include any non-ϕ^4 10 loop graphs.

weight	2	3	4	5	6	7	8
level	$\boxed{11}^\eta_{\phi^{>4},9}$	$\boxed{7}^\eta_8$	$\boxed{5}^\eta_8$	$\boxed{4}^\eta_9$	$\boxed{3}^\eta_8$	$\boxed{3}_9$	$\boxed{2}^\eta_{10}$
	$\boxed{14}^\eta_{\phi^{>4},9}$	$\boxed{8}^\eta_8$	$\boxed{6}^\eta_9$	7	$\boxed{4}^\eta_9$	7	3
	$\boxed{15}^\eta_{\phi^{>4},9}$	$\boxed{11}_{\phi^{>4},9}$	$\boxed{7}_{10}$	8	5	8	$\boxed{5}_{10}$
	17	$\boxed{12}^\eta_9$	8	11	6	11	6
	19	15	9	12	$\boxed{7}_9$	15	7
	20	15	10	15	8	15	8
	21	16	12	15	9	16	8
	24	19	$\boxed{13}_9$	19	$\boxed{10}_{10}$	19	9
	26	20	$\vdots$	20	10	20	10
	26	20	$\boxed{17}_{10}$	20	10	20	12

(5.5) is given by the number of zeros of $x^2 + x - 1$, which is not cyclotomic. In conclusion, it seems that the point-counting functions of 'physical graphs' (i.e. in ϕ^4) are highly constrained compared to the set of all graphs.

   Besides the modular and (quasi-)constant examples, there are many c_2-invariants that we were unable to identify. Up to 8 loops ϕ^4 theory is fully quasi-constant or modular, but at 9 loops there are 10 c_2-invariants which are neither quasi-constant nor modular of low level. Their sequences are listed for the first 12 primes in table 6. At 10 loops we have another 114 unidentified sequences which are listed for the first 6 primes in table 8. In one case it is possible to reduce an unidentified c_2-invariant (i_{101} in table 8) to the (affine) point-count of a 4-fold which is the projective zero locus of the degree 6 polynomial (5.4). The i_{101} c_2-invariant is listed for the first 100 primes in table 5, and the c_2-invariants of 7 other accessible cases are listed for the first 50 primes.

   A surprising consequence of our findings is the following trichotomy for graphs in ϕ^4 theory. Up to 10 loops they fall into three categories:

(1) Graphs G with c_2-invariant equal to -1. These appear to have a unique prime ancestor: namely the wheel with three spokes (whose completion is the graph K_5). In other words, this entire class is generated *by a single graph* by completion and double-triangle operations.

(2) Graphs that have c_2-invariant equal to 0. These graphs are expected to have weight-drop and therefore contribute to the perturbative expansion in a quite different way from the previous class.

(3) Graphs with non-constant c_2-invariants. These start at 7 loops and are all counter-examples to Kontsevich's conjecture. This class contains the modular graphs described above.

The last set of columns (loop order ℓ) in table 7 suggests that the wheel with three spokes is the only graph in ϕ^4 theory which plays a distinguished role. In particular, there are many prime ancestors with vanishing c_2-invariant (the smallest of which is the $K_{3,4}$ graph at loop order 6). In [9], several families of graphs with vanishing c_2-invariant were constructed.

2. Equivalence classes of graphs

2.1. Completed primitive graphs

The map (1.5) from graphs to periods satisfies various identities which we review here. Recall that a graph is 4-regular if every vertex has degree 4.

Definition 3. *Let Γ be a connected 4-regular graph with at least three vertices. We say that G is completed primitive if every 4-edge cut of Γ is either connected, or has a component consisting of a single vertex.*

The simplest completed primitive graph is a 3-cycle of double edges. There exists no completed primitive graph with four vertices but there is a unique completed primitive graph with five vertices, the complete graph K_5, and a unique one with six vertices, the octahedron O_3 (see figure 4).

Let G be primitive-divergent in ϕ^4. From Euler's relation for a connected graph: $N_G - V_G = h_G - 1$, combined with (1.3), one easily shows that G has four vertices of degree 3 or two vertices of degree 2. Its *completion* Γ is defined to be the graph obtained by adding a new vertex to G, and connecting it via a single edge to every 3-valent vertex in G, or a double edge to every 2-valent vertex of G. One shows that the graph Γ is completed primitive. The following lemma implies that every completed primitive graph arises in this way (note that two non-isomorphic graphs can have the same completion).

Lemma 4. *Let Γ be a completed primitive graph. Then for any vertex $v \in \Gamma$, the graph $\Gamma \backslash v$ is primitive-divergent in ϕ^4 theory.*

Proof. See the proof of proposition 2.6 in [20]. □

An immediate consequence of the lemma is that completed primitive graphs with at least five vertices are simple. The following proposition states that the period of a primitive divergent graph in ϕ^4 theory only depends on its completion.

Proposition 5. *Let Γ be a completed primitive graph, and let $G = \Gamma \backslash v$. Then the integral (1.5) corresponding to the graph G converges, and the value of the integral is independent of the choice of vertex v.*

Proof. See proposition 2.6 in [20] and Lemma 5.1 and Proposition 5.2 in [2] for the convergence. Independence of v follows from Theorem 2.7 (5) in [20]. □

Let Γ be a completed primitive graph. By the previous proposition we can define the period of Γ to be the real number

$$(2.1) \qquad\qquad P_\Gamma = P(\Gamma \backslash v)$$

for any vertex $v \in \Gamma$. Abusively, we denote the loop order ℓ_Γ of a completed primitive graph Γ to be the number of independent cycles in any of its primitive-divergent graphs $\Gamma \backslash v$:

$$(2.2) \qquad\qquad \ell_\Gamma = h_{\Gamma \backslash v} = V_\Gamma - 2.$$

The period of the 3-cycle of double edges is 1, $P_{K_5} = 6\zeta(3)$, and $P_{O_3} = 20\zeta(5)$. For a list of all completed primitive graphs up to loop order 8 and their known periods see [20].

2.2. The product identity

We say that a completed primitive graph Γ is *reducible* if there exists a set of three vertices v_1, v_2, v_3 in Γ such that $\Gamma \backslash \{v_1, v_2, v_3\}$ has more than one connected component; see figure 2. A well-known feature of the period is the following factorization property.

Proposition 6. *Every reducible completed primitive graph Γ is isomorphic to a graph obtained by gluing two completed primitive graphs Γ_1 and Γ_2 on*
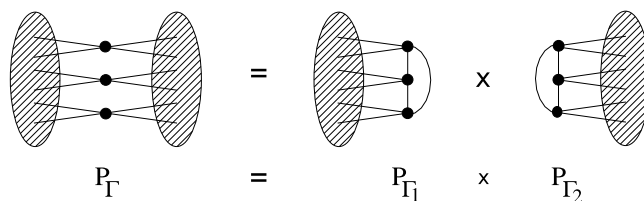
Figure 2: Vertex-connectivity 3 leads to products of periods.

triangle faces followed by the removal of the triangle edges (see figure 2). The period of Γ is the product of the periods of Γ_1 and Γ_2,

$$(2.3) \qquad\qquad P_\Gamma = P_{\Gamma_1} P_{\Gamma_2}.$$

Proof. See Theorem 2.10 in [20] or Section 3.4 in [5]. □

Note that the above gluing operation is not defined for all pairs of graphs, since not all completed primitive graphs contain triangles. Let us define

$$\mathcal{G} = \langle \Gamma, \cup \rangle$$

to be the free commutative monoid generated by completed primitive graphs Γ (with multiplication denoted by $\cup$), and let

$$\mathcal{G}_0 = \mathcal{G}/(\Gamma \text{ red.} \sim \Gamma_1 \cup \Gamma_2)$$

be the quotient by the equivalence relation generated by identifying a reducible completed primitive graph Γ with the union of its components.

Corollary 7. *The period gives a well-defined multiplicative map*

$$P : \mathcal{G}_0 \longrightarrow \mathbb{R}_+.$$

Remark 8. *There exist two other known identities on periods: the twist identity [20] and the (rather rare) Fourier identity [3]. It turns out that these are often subsumed by the double-triangle relation (below) at low loop orders and therefore we shall not include them in the present set-up.*

There are no presently known identities on periods which relate a non-trivial linear combination of graphs.
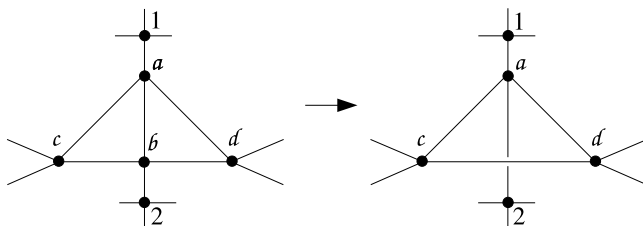
Figure 3: Double-triangle reduction: replace a joint vertex of two attached triangles by a crossing.
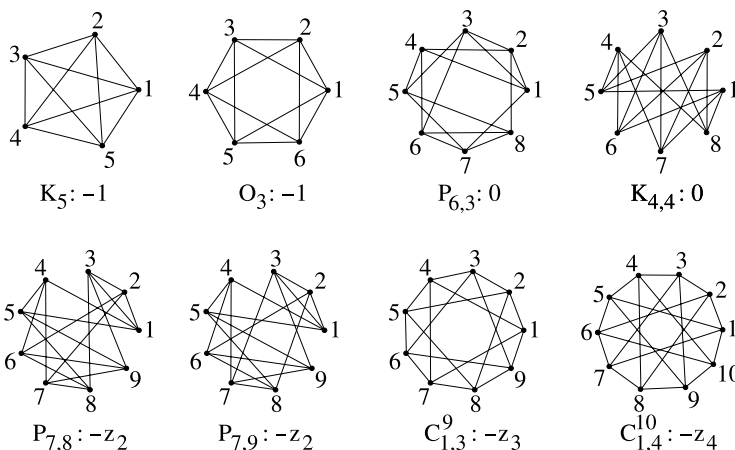
Figure 4: Quasi-constant ϕ^4-graphs with their c_2-invariants. The ancestor of O_3 is K_5, the ancestor of $P_{6,3}$ is K_5^2. The graphs K_5, $K_{4,4}$, $P_{7,8}$, $P_{7,9}$, $C_{1,3}^9$, $C_{1,4}^{10}$ are the smallest examples of ancestors with their c_2-invariant. The names of the graphs are taken from [20] and $z_\bullet$ is defined in (5.1).

2.3. Double-triangle reduction

The double-triangle reduction is defined as follows.

Definition 9. *Assume a graph Γ has an edge ab that is the common edge of (exactly) two triangles (abc) and (abd), $c \neq d$. The* double-triangle reduced graph *is defined to be the graph in which the vertex b is replaced by a crossing with edge cd as depicted in figure 3.*

A double-triangle reduced graph is completed primitive if and only if the original graph is completed primitive (the non-trivial direction of this

statement is Proposition 2.19 in [20]). Let us define

$$\mathcal{G}_1 = \mathcal{G}_0/\sim_{dt},$$

where $\sim_{dt}$ is the equivalence relation generated by $\Gamma_1 \sim_{dt} \Gamma_2$ if Γ_1 and Γ_2 are linked by a double-triangle reduction.

Note that double-triangle reduction does *not* preserve the period, but does respect the c_2-invariant, as we shall see below. We define the *family* of a completed primitive graph Γ to be its equivalence class in $\mathcal{G}_1$.

2.4. The ancestor of a family

Every family has a unique smallest member in $\mathcal{G}_0$, which we call its ancestor.

Definition 10. *A disjoint union of completed primitive graphs is an ancestor if none of its components can be reduced by the product or double triangle. An ancestor is prime if it is connected.*

Theorem 11. *Every family has a unique ancestor.*

Proof. Proposition 2.21 and Proposition 2.22 of [20]. □

The graph O_3 is in the family of the prime ancestor K_5.

Remark 12. *By combining the results of [6, 9], we deduce that the piece of maximal (generic) Hodge-theoretic weight of the cohomology of the graph hypersurface is invariant under double- triangle reduction. In particular, a graph has 'weight drop' in the sense of [9] if and only if its double-triangle reductions do also.*

3. Counting points over finite fields

Let $q = p^n$, where p is a prime number, and let $\mathbb{F}_q$ denote the finite field with q elements. If G is a graph with N_G edges, let

(3.1) $$[X_G]_q = |\{x \in \mathbb{F}_q^{N_G} : \Psi_G(x) = 0\}|$$

denote the number of points of the affine graph hypersurface $X_G(\mathbb{F}_q)$. If G has at least three vertices then

$$0 \equiv [X_G]_q \mod q^2$$

by ([7], Proposition-Definition 18). This motivates the following definition

Definition 13. *For a graph G with at least three vertices, the c_2-invariant is the map which associates to every prime power q the following element of $\mathbb{Z}/q\mathbb{Z}$:*

$$(3.2) \qquad c_2(G)_q \equiv [X_G]_q/q^2 \mod q.$$

The c_2-invariant should be thought of as a combinatorial version of the period of a graph, and carries the salient qualitative information about the arithmetic nature of the period (see Conjecture 2).

3.1. Properties of the c_2-invariant

Because graphs with isomorphic completions are proved to have the same period, Conjecture 2 implies that the c_2-invariant only depends on the completion class of a graph.

Conjecture 14 ([7], Conjectures 4 and 35). *The c_2-invariants of primitive-divergent graphs which have isomorphic completions are equal modulo q.*

By proposition 5, Conjecture 14 follows from Conjecture 2. Conjecture 14 has been verified for many families, but although it seems much more tractable than Conjecture 2 it is still unproved at present. Nonetheless, we shall assume Conjecture 14 to be true throughout the remainder of this paper. Thus, we shall assume that the c_2-invariant lifts to the completed primitive graph Γ and we will use the notation

$$(3.3) \qquad c_2[\Gamma]_q \equiv c_2(\Gamma - v)_q \mod q$$

to avoid any possible confusion with $c_2(\Gamma)_q$ (which we shall never consider for completed graphs in this paper).

Remark 15. *Using the techniques of [8, 9] and performing double-triangle reductions 'at infinity', one can presumably show that the previous conjecture is true for a family if and only if it is true for the prime ancestor of that family.*

Every completed primitive graph with at least five vertices has a c_2-invariant and the first two examples are

$$(3.4) \qquad c_2[K_5]_q \equiv c_2[O_3]_q \equiv -1 \mod q.$$

If G is uncompleted then in general $c_2(G)_q$ vanishes mod q for graphs G with vertex-connectivity ≤ 2 (Proposition 31 in [7]). Reducible completed primitive graphs have vertex-connectivity 3. Hence, the removal of one of the three connecting vertices provides a primitive graph with vanishing c_2-invariant.

Proposition 16 assuming Conjecture 14. *The c_2-invariants of reducible completed primitive graphs vanish modulo q.*

The c_2-invariant is invariant under double-triangle reductions.

Proposition 17 assuming Conjecture 14. *If a completed primitive graph Γ_1 can be double triangle reduced to Γ_2 then $c_2[\Gamma_1]_q \equiv c_2[\Gamma_2]_q \mod q$.*

Proof. Because of (3.4) we can assume that Γ_1 has at least seven vertices. Hence Γ_1 has a vertex that is not involved in the double-triangle reduction. We remove this vertex to obtain a primitive-divergent graph G which can be double triangle reduced. By Cor. 34 in [7] the c_2-invariant is unaffected by the double-triangle reduction. Upon completion of the reduced graph we obtain Γ_2. $\qquad\square$

These two propositions prove that the c_2-invariant factors through $\mathcal{G}_1$.

Theorem 18 assuming Conjecture 14. *All completed primitive graphs in a family have the same c_2-invariant. The c_2-invariant vanishes modulo q if the ancestor of the family is not prime.*

Thus, assuming Conjecture 14, the number of c_2-invariants is at most the number of prime ancestors at a given loop order.

Table 2 shows that there are a total of 284 prime ancestors up to loop order 10. These give a complete list of c_2-invariants from ϕ^4 theory up to loop order 10 (assuming conjecture 14).

3.2. Constant c_2-invariants

There are various families of (uncompleted) graphs for which the c_2-invariant is known. In [7] it was proven that any graph of vertex-width at most 3 has a polynomial point-count. In particular,

Theorem 19 [7]. *Let G be a graph of vertex-width ≤ 3. Then the c_2-invariant is constant: $c_2(G)_q \equiv c \mod q$ for some constant $c \in \mathbb{Z}$.*

Table 2: The number of completed primitive graphs and prime ancestors up to 10 loops.

Loop order	Comp. prim. graphs	Prime ancestors
3	1	1
4	1	0
5	2	0
6	5	1
7	14	4
8	49	10
9	227	37
10	1 354	231

In [8] it was shown that any graph in ϕ^4 theory which is not primitive, i.e., containing a non-trivial subdivergence, has vanishing c_2-invariant.

Theorem 20. *Let G be a log-divergent graph in ϕ^4 theory. Then if G has a strict subgraph γ such that $2h_\gamma \geq N_\gamma$, it satisfies $c_2(G)_q \equiv 0 \mod q$.*

Due to renormalization, and more specifically, the addition of counter-terms, the connection between the c_2-invariant and the period is more complicated in the subdivergent case. Even in 'symmetric' cases where one can associate canonical periods to graphs with subdivergences it is not proved that their periods have weight drop. However, examples (see e.g. Equation (5.12) in [17]) and some indirect arguments suggest that suitably defined periods of subdivergent graphs should have a double weight drop.

There are several other combinatorial criteria for a graph to have vanishing c_2-invariant which are discussed in [9].

3.3. Modularity

We are mainly interested in graphs whose c_2-invariant is congruent modulo p to the Fourier coefficients of a modular form.

Definition 21. *A completed primitive graph Γ is modular if there exists a normalized Hecke eigenform f for a congruence subgroup of $SL_2(\mathbb{Z})$, possibly with a non-trivial Dirichlet-character, with an integral Fourier expansion*

$$(3.5) \qquad f(\tau) = \sum_{k=0}^{\infty} a_k q^k, \quad q = \exp(2\pi i\tau), \quad a_k \in \mathbb{Z},$$

such that the c_2-invariant modulo p satisfies

$$(3.6) \qquad\qquad c_2[\Gamma]_p \equiv -a_p \mod p$$

for all primes p.

For simplicity, we only consider point counts over fields with prime numbers of elements p, rather than the more general case of prime powers q, because the latter can involve quadratic residue symbols, which vanish modulo p (this is the case for the graph $P_{8,37}$, whose modularity was proved in [7]). Note that it is expected, but not known in general, whether a modular form of the type considered in Definition 21 is uniquely defined by its coefficients a_p modulo p for all primes p.

4. Denominator reduction

Although the c_2-invariant appears to be the most complicated part of the graph hypersurface point-count, the method of denominator reduction provides a tool to access precisely this part of the point-count [7].

4.1. Matrix representation

We recall some basic results from [5]. We will use the following matrix representation for the graph polynomial.

Choose an orientation of the edges of G, and for every edge e and vertex v of G, define the incidence matrix:

$$(4.1) \quad (\mathcal{E}_G)_{e,v} = \begin{cases} 1, & \text{if the edge } e \text{ begins at } v \text{ and does not end at } v, \\ -1, & \text{if the edge } e \text{ ends at } v \text{ and does not begin at } v, \\ 0, & \text{otherwise.} \end{cases}$$

Let A be the diagonal matrix with entries x_e, for $e \in E(G)$, the set of edges of G, and set

$$(4.2) \qquad\qquad \widetilde{M}_G = \left(\begin{array}{c|c} A & \mathcal{E}_G \\ \hline -\mathcal{E}_G^T & 0 \end{array} \right)$$

where the first e_G rows and columns are indexed by $E(G)$, and the remaining v_G rows and columns are indexed by the set of vertices of G, in some order. The matrix $\widetilde{M}_G$ has corank ≥ 1. Choose any vertex of G and let M_G

denote the square $(N_G + V_G - 1) \times (N_G + V_G - 1)$ matrix obtained from it by deleting the row and column indexed by this vertex.

It follows from the matrix-tree theorem that the graph polynomial satisfies

$$(4.3) \qquad \Psi_G = \det(M_G).$$

4.2. The five-invariant

Let I, J, K be subsets of the set of edges of G which satisfy $|I| = |J|$. Let $M_G(I, J)_K$ denote the matrix obtained from M_G by removing the rows (resp. columns) indexed by the set I (resp. J) and setting $x_e = 0$ for all $e \in K$. Let

$$(4.4) \qquad \Psi_{G,K}^{I,J} = \det M_G(I, J)_K.$$

Now let i, j, k, l, m denote any five distinct edges in a graph G. The five-invariant of these edges, denoted $^5\Psi_G(i, j, k, l, m)$ is defined up to a sign, and is given by the determinant

$$(4.5) \qquad {}^5\Psi_G(i, j, k, l, m) = \pm \det \begin{pmatrix} \Psi_{G,m}^{ij,kl} & \Psi_{G,m}^{ik,jl} \\ \Psi_G^{ijm,klm} & \Psi_G^{ikm,jlm} \end{pmatrix}.$$

It is well-defined, i.e., permuting the five indices i, j, k, l, m only modifies the right-hand side by a sign. In general, the 5-invariant is irreducible of degree 2 in each variable x_e. However, in many cases it factorizes into a product of polynomials each of which is linear in a variable x_6. In this case, denominator reduction allows us to further eliminate variables by taking resultants.

4.3. Reduction algorithm

Given a graph G and an ordering $e_1, \ldots, e_{N_G}$ on its edges, we can extract a sequence of higher invariants (each defined up to a sign) as follows. Define $D_G^5(e_1, \ldots, e_5) = {}^5\Psi_G(e_1, \ldots, e_5)$. Let $n \geq 5$ and suppose that we have defined $D_G^n(e_1, \ldots, e_n)$. Suppose furthermore that $D_G^n(e_1, \ldots, e_n)$ factorizes into a product of linear factors in x_{n+1}, i.e., it is of the form $(ax_{n+1} + b)(cx_{n+1} + d)$. Then we define

$$(4.6) \qquad D_G^{n+1}(e_1, \ldots, e_{n+1}) = \pm(ad - bc),$$

to be the resultant of the two factors of $D_G^n(e_1, \ldots, e_n)$ with respect to x_{n+1}. A graph G for which the polynomials $D_G^n(e_1, \ldots, e_n)$ can be defined for all n is called *denominator-reducible*.

One can prove, as for the 5-invariant, that $D_G^n(e_1, \ldots, e_n)$ does not depend on the order of reduction of the variables, although it may happen that the intermediate terms $D_G^k(e_{i_1}, \ldots, e_{i_k})$ may factorize for some choices of orderings and not others.

Denominator reduction is connected to the c_2-invariant by the following theorem (Theorem 29 in [7] or Equation (2.33) in [21]):

Theorem 22. *Let G be a connected graph with $N_G \geq 5$ edges and $h_G \leq N_G/2$ independent cycles. Suppose that $D_G^n(e_1, \ldots, e_n)$ is the result of the denominator reduction after $n < N_G$ steps. Then*

$$(4.7) \qquad c_2(G)_q \equiv (-1)^n [D_G^n(e_1, \ldots, e_n)]_q \mod q.$$

For general graphs above a certain loop order and any ordering on their edges, there will come a point where $D_G^n(e_1, \ldots, e_n)$ is irreducible (typically for $n = 5$). Thus, the generic graph is not denominator reducible. The c_2-invariant of a graph G vanishes if $h_G < N_G/2$ (Cor. 2.10 in [21]).

5. Results

Up to loop order 10 we have a total of 284 prime ancestors (see table 2). With the first 6 primes (whose product is 30030) we can distinguish 145 c_2-invariants, which are listed in tables 7 and 8 at the end of this section. For the 16 prime ancestors up to loop order 8 we have determined the c_2-invariant for at least the first 26 primes; for the 37 graphs at loop order 9 the c_2-invariant is known for at least the first 12 primes (see table 6). At loop order 10 we evaluated the c_2-invariant for many graphs for $p = 17$ and beyond without resolving any more c_2-invariants. Therefore we expect that the total number of c_2-invariants up to loop order 10 does not significantly exceed 145. Although a few of the many identities among c_2-invariants are explained by twist and Fourier identities (see Section 2) most do not seem to follow from any known identities.

We found three types of c_2-invariants: quasi-constant, modular, and unidentified sequences.

5.1. Quasi-constant graphs

Definition 23. *A completed primitive graph G is quasi-constant (quasi c) if there exists an $m \in \mathbb{N}_+$ such that $c_2(G)_{p^{mn}} \equiv c \mod p^{mn}$ for almost all primes p and all $n \in \mathbb{N}_+$.*

For $k \geq 2$, consider the following residue symbol, denoted z_k, which depends on the number of kth roots of unity in $\mathbb{F}_q$,

$$(5.1) \qquad z_k(q) = \begin{cases} 1 & \text{if } \#\{x \in \mathbb{F}_q : x^k = 1\} = k, \\ 0 & \text{if } \gcd(k, q) > 1, \\ -1 & \text{otherwise.} \end{cases}$$

It is a quasi-constant (quasi 1). There is an abundance of quasi -1 c_2-invariants, which is partly explained by denominator reduction, Theorem 22.

Lemma 24. *Let G be a graph with $N_G = 2h_G$, which is denominator reducible up to the penultimate stage: i.e., there exists a sequence on the edges of G such that $D_G^{N_G-2}$ is defined. Then $c_2(G)_q$ is either 0 or quasi -1.*

Proof. Theorem 22 is proved in [7] by iteratively taking resultants which (if non-zero) factorize into terms which are linear in the next variable. At each step the degree of the resultant goes down by one. Because we start with D_G^5 which has degree $N_G - 5$ we observe that D_G^n is either zero or of degree $N_G - n$. If $D_G^{N_G-2} = 0$ then the c_2-invariant vanishes mod q. Otherwise $D_G^{N_G-2}$ is a homogeneous quadratic polynomial in two variables with integer coefficients and defines a hypersurface V of degree two in affine space $\mathbb{A}^2$. Let V_q denote its reduction in $\mathbb{F}_q^2$. If $D_G^{N_G-2}$ is a perfect square then V is a double line and V_q is either isomorphic to $\mathbb{F}_q$ or — if $D_G^{N_G-2}$ vanishes mod q — to $\mathbb{F}_q^2$. In both cases the c_2-invariant vanishes mod q. Now we assume that $D_G^{N_G-2}$ is not a perfect square. Let $\Delta \in \mathbb{Z}^\times$ be the discriminant of $D_G^{N_G-2}$. For every prime p not dividing Δ and every $n \in \mathbb{N}_+$, Δ is a non-zero square in $\mathbb{F}_{p^{2n}}$ and $V_{p^{2n}}$ is isomorphic to a union of two lines meeting at a point. Thus $|V_{p^{2n}}| = |\mathbb{F}_{p^{2n}}| + |\mathbb{F}_{p^{2n}}| - |pt| = 2p^{2n} - 1$, which makes the c_2-invariant quasi -1. $\square$

The graphs satisfying the conditions of the lemma have the property that the maximal weight-graded piece of the graph cohomology is one-dimensional and spanned by the Feynman differential form (see [6]).

For ϕ^4 graphs we observe that up to loop order 6 all c_2-invariants are in fact constant, equal to 0 or -1. At loop order 7 all new c_2-invariants

are quasi-constant, equal to $-z_2$ or $-z_3$. The only other quasi-constant c_2-invariant first appears at loop order 8, and equals $-z_4$. No new quasi-constant c_2-invariants were found at loop orders 9 and 10, which leads us to conjecture that this list is complete. Note that the cases z_2, z_3, z_4 become quasi-constant after excluding the prime 2, or by adjoining 3rd and 4th roots of unity, respectively. These are precisely the extensions of $\mathbb{Q}$ by roots of unity which are trivial or quadratic. Thus the conjecture is partly supported by the previous lemma, which only involves quadratic extensions.

Note, however, that for many quasi-constant graphs an edge ordering that leads to $D_G^{N_G-2}$ is not known. Furthermore, there is not a single graph with $c_2 = -z_3$ or $-z_4$ whose period is known. What we do have is the conjectured period of two completed primitive graphs at loop order 7 (graphs $P_{7,8}$ and $P_{7,9}$ in [20]) with c_2-invariant $-z_2$. They are weight 11 multiple zeta values, namely [4]

$$(5.2) \quad P_{7,8} = \frac{22383}{20}\zeta(11) - \frac{4572}{5}[\zeta(3)\zeta(3,5) - \zeta(3,5,3)] - 700\zeta(3)^2\zeta(5)$$
$$+ 1792\zeta(3)\left[\frac{27}{80}\zeta(3,5) + \frac{45}{64}\zeta(5)\zeta(3) - \frac{261}{320}\zeta(8)\right] \text{ and}$$

$$(5.3) \quad P_{7,9} = \frac{92943}{160}\zeta(11) - \frac{3381}{20}[\zeta(3)\zeta(3,5) - \zeta(3,5,3)] - \frac{1155}{4}\zeta(3)^2\zeta(5)$$
$$+ 896\zeta(3)\left[\frac{27}{80}\zeta(3,5) + \frac{45}{64}\zeta(5)\zeta(3) - \frac{261}{320}\zeta(8)\right].$$

To summarize the results for quasi-constant graphs, the 10 loop data are consistent with the following, rather surprising, conjecture.

Conjecture 25. *If a completed primitive graph Γ is quasi-constant then its c_2-invariant is 0, -1, $-z_2$, $-z_3$, or $-z_4$ [see (5.1)].*
If $c_2[\Gamma]_q \equiv -1 \mod q$ then the ancestor of Γ is K_5.

The second part of the conjecture broadly states that primitive ϕ^4 graphs fall into three categories: the weight-drop graphs with c_2-invariant 0, complicated graphs, which have non-polynomial point counts (whose prime ancestors have at least 7 loops), and a single family of graphs whose ancestor is K_5. The last category has many distinguished combinatorial properties.

5.2. Modular graphs

The first modular graphs (see Defn. 21) appear at loop order 8. In fact all four non quasi-constant graphs at loop order 8 are modular with respect to
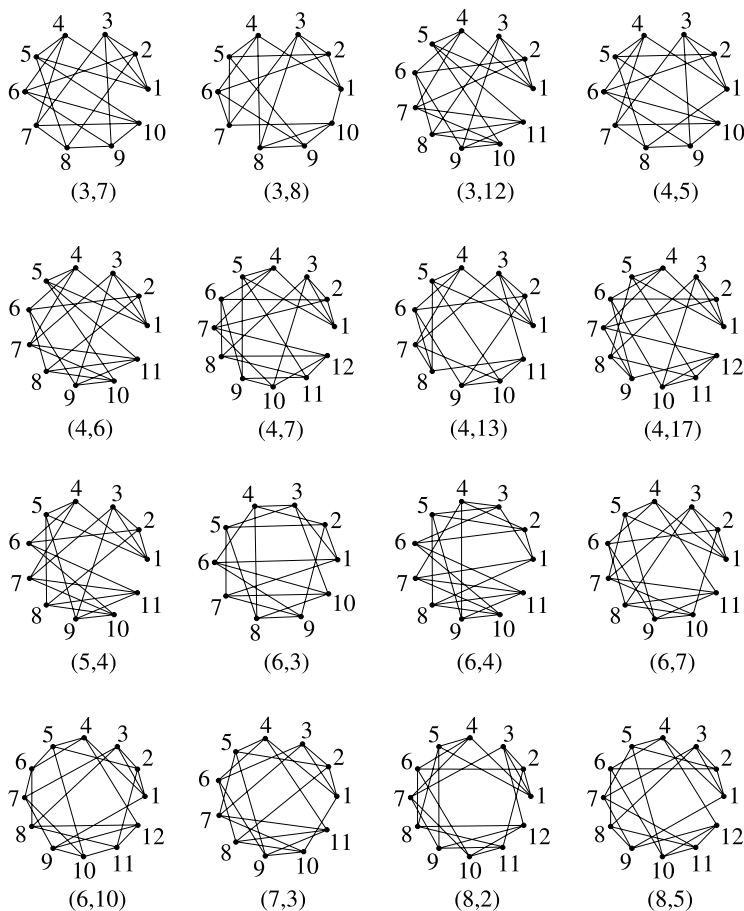
Figure 5: The 16 modular ϕ^4 graphs up to loop order 10. The numbers beneath the figures refer to the (weight, level) of the corresponding modular form.

the newforms with weight and level equal to $(3, 7)$, $(3, 8)$, $(4, 5)$ and $(6, 3)$. The graphs are depicted in figure 5. At loop order 9 we are able to identify another seven modular graphs whereas at loop order 10 we find only five new modular sequences. All modular graphs were found to have very small levels (≤ 17, see table 1). A search for much higher levels did not provide any new fits. Nine out of the 16 modular graphs correspond to products of the Dedekind η-function (see table 3 and [24]). All modular c_2-invariants are confirmed for at least the first 11 primes (whose product is $\sim 2 \times 10^{11}$).

Table 3: Newforms $f(z)$ from table 1 which are expressible as products of the Dedekind η-function.

Weight	Level	Modular form
2	11	$\eta(z)^2\eta(11z)^2$
2	14	$\eta(z)\eta(2z)\eta(7z)\eta(14z)$
2	15	$\eta(z)\eta(3z)\eta(5z)\eta(15z)$
3	7	$\eta(z)^3\eta(7z)^3$
3	8	$\eta(z)^2\eta(2z)\eta(4z)\eta(8z)^2$
3	12	$\eta(2z)^3\eta(6z)^3$
4	5	$\eta(z)^4\eta(5z)^4$
4	6	$\eta(z)^2\eta(2z)^2\eta(3z)^2\eta(6z)^2$
5	4	$\eta(z)^4\eta(2z)^2\eta(4z)^4$
6	3	$\eta(z)^6\eta(3z)^6$
6	4	$\eta(2z)^{12}$
8	2	$\eta(z)^8\eta(2z)^8$

An unexpected observation is the absence of weight 2.

Conjecture 26. *If a completed primitive graph is modular with respect to the modular form f then the weight of f is ≥ 3.*

In particular, ϕ^4 point-counts are conjecturally not of general type (compare the main theorem of [1]). In support of this conjecture, we find by computation that any counter-example Γ with ≤ 10 loops, would have to correspond to a modular form of weight 2 and level ≥ 1200.

5.3. Unidentified graphs

The first unidentified c_2-invariants appear at loop order 9 where 10 sequences cannot be associated to quasi-constants or modular forms. At loop order 10, there are 114 as yet unidentified sequences out of 231 prime ancestors. Because modular c_2-invariants appear to have low levels it is reasonable to expect that the unidentified sequences are not modular with respect to congruence subgroups of $SL_2(\mathbb{Z})$.

Table 4: All non-zero c_2-invariants of 'log divergent' graphs (graphs for which the number of edges equals twice the number of loops) up to 9 loops. The functions $z_\bullet$ are defined in (5.1), y_5 is defined in (5.5), (weight, level) refer to modular forms, and $i_\bullet$ refers to an unidentified sequence listed in table 8. $i_{\phi>4}$ refers to unidentified sequences in non-ϕ^4 graphs.

$-c_2$	loop order $\ell =$							$-c_2$	$\ell =$		$-c_2$	$\ell =$
	3	4	5	6	7	8	9		8	9		9
1	1	1	2	7	42	393	4698	(3,12)	0	8	i_{53}	11
z_2	0	0	0	0	5	49	763	(4,5)	3	73	i_{81}	6
z_3	0	0	0	0	1	19	375	(4,6)	0	36	i_{87}	5
z_4	0	0	0	0	0	1	15	(4,7)	0	5	i_{104}	1
y_5	0	0	0	0	0	1	34	(4,13)	0	8	i_{106}	6
(2,11)	0	0	0	0	0	0	6	(4,17)	0	2	i_{110}	10
(2,14)	0	0	0	0	0	0	4	(5,4)	0	29	i_{121}	6
(2,15)	0	0	0	0	0	0	2	(6,3)	1	7	i_{125}	6
(3,7)	0	0	0	0	0	6	133	(6,4)	0	3	i_{132}	11
(3,8)	0	0	0	0	0	4	81	(6,7)	0	10	i_{141}	6
(3,11)	0	0	0	0	0	0	3	(7,3)	0	3	$i_{\phi>4}$	15

The most accessible unidentified sequences are those for which the denominator reduction algorithm continues for as long as possible, and provides homogeneous polynomials of degree 7 in seven variables. In table 5 we list $-c_2$ for the first 50 primes for the eight graphs which fall into this class. The graphs are depicted in figure 6.

Using the rescaling technique presented in [21] we were able to reduce the c_2-invariant of the unidentified sequence i_{101} one step further to a projective 4-fold of degree 6. It is given explicitly by the homogeneous polynomial

$$(5.4) \qquad\qquad AB + x_2^2(x_4 + x_5)^2 x_5 x_6 + C x_5^2 x_6,$$

where

$$A = (x_2 + x_5)(x_3 + x_6)x_4 + x_2 x_3 x_5 + x_2 x_3 x_6 + x_2 x_5 x_6 + x_3 x_5 x_6,$$
$$B = x_1(x_1 x_3 + x_3 x_4 + x_3 x_6 - x_2 x_4 - x_2 x_5 - x_5 x_6),$$
$$C = (x_4 + x_6)(x_2 x_4 + x_2 x_5 - x_2 x_3 - x_3 x_4 - x_3 x_6 - x_4 x_6).$$

The c_2-invariant for this sequence is listed as i_{101} in table 5 for the first 100 primes. Remarkably, $i_{101}(p)$ is a square in $\mathbb{F}_p$ if and only if $p \not\equiv -1 \mod 12$.

Table 5: The 8 most accessible unidentified sequences (see figure 6). We list $-c_2[\Gamma]_p$ for the first 50 primes and for the first 100 primes in case of i_{101}.

p	i_{53}	i_{70}	i_{81}	i_{101}	i_{110}	i_{121}	i_{132}	i_{145}	p	i_{101}
2	0	0	0	1	1	1	1	1	233	232
3	1	1	2	0	0	1	2	2	239	179
5	1	4	2	1	4	3	1	4	241	196
7	4	2	2	4	6	3	4	6	251	183
11	6	1	6	6	0	10	1	9	257	135
13	6	0	5	9	0	8	0	10	263	94
17	14	7	14	16	9	14	4	8	269	268
19	5	1	10	9	3	1	11	1	271	242
23	15	1	1	17	6	14	6	13	277	210
29	11	10	27	28	24	3	23	17	281	261
31	21	30	16	1	28	9	2	22	283	9
37	23	5	12	30	4	24	13	20	293	240
41	1	28	6	8	17	19	12	4	307	289
43	33	31	15	6	36	10	13	21	311	51
47	33	14	0	11	36	11	6	8	313	108
53	47	8	48	13	46	19	44	14	317	113
59	25	25	11	44	16	8	29	30	331	43
61	40	5	17	42	46	26	14	23	337	100
67	60	22	10	33	24	11	35	54	347	97
71	53	59	30	65	29	33	15	27	349	273
73	46	55	36	64	22	14	2	67	353	334
79	25	68	50	38	72	35	43	14	359	335
83	51	64	8	74	2	15	23	30	367	137
89	65	12	60	88	1	40	0	44	373	179
97	33	78	53	47	89	91	7	10	379	263
101	1	80	34	52	31	48	76	83	383	310
103	33	63	25	60	58	16	71	76	389	176
107	45	12	50	17	49	21	45	83	397	136
109	42	38	34	82	56	5	24	43	401	36
113	40	38	72	102	56	39	1	0	409	377
127	62	19	61	115	30	45	43	18	419	314
131	8	96	69	37	3	104	99	97	421	51

(*Continued*)

Table 5: Continued.

p	i_{53}	i_{70}	i_{81}	i_{101}	i_{110}	i_{121}	i_{132}	i_{145}	p	i_{101}
137	125	72	0	136	61	10	7	43	431	370
139	65	48	61	89	113	20	137	29	433	104
149	86	102	131	37	70	148	132	108	439	182
151	24	9	127	40	79	30	59	21	443	76
157	155	124	107	109	120	135	100	70	449	161
163	99	34	60	38	88	46	39	158	457	238
167	22	117	134	95	60	60	2	37	461	39
173	56	85	75	157	21	73	23	55	463	388
179	4	37	122	32	173	46	132	42	467	234
181	91	33	23	75	1	24	65	90	479	325
191	53	46	45	55	136	115	4	170	487	412
193	92	31	175	98	155	126	29	156	491	343
197	39	170	68	97	185	132	23	56	499	431
199	174	126	121	178	158	133	5	173	503	5
211	8	33	182	53	189	176	146	90	509	429
223	1	119	110	175	37	216	41	184	521	169
227	183	199	105	52	224	110	70	37	523	263
229	115	20	63	9	225	94	181	71	541	412
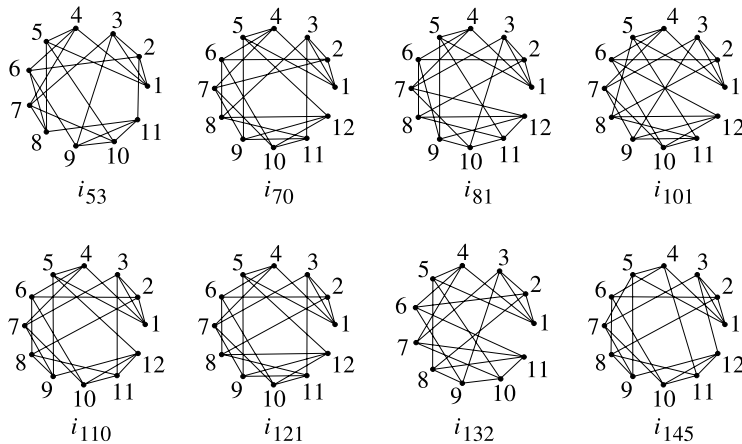
Figure 6: Eight graphs with readily accessible but as-yet unidentified c_2-invariants. Their point-counts mod p for the first 50 primes are listed in table 5
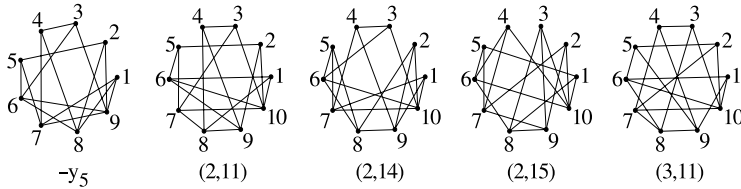
Figure 7: Non-ϕ^4 graphs that are quasi-constant ($-y_5$, see Eq. (5.5)) or modular with subscript (weight, level). Note that all graphs have a non-ϕ^4 (5-valent) vertex.

5.4. Non-ϕ^4 graphs

Since denominator reduction Theorem 22 applies to any connected graph G satisfying $N_G = 2h_G$, we can also study the c_2-invariants of graphs which are not necessarily in ϕ^4 theory (i.e., which have a vertex of degree ≥ 5). The condition $N_G = 2h_G$ means that these graphs are superficially log divergent. If, furthermore, they are free of subdivergences (1.4), they are primitive divergent and have a convergent period (1.5). Note that there is no completion conjecture for the c_2-invariants of non-ϕ^4 graphs.

Because the presence of a vertex of degree ≤ 2 causes the c_2-invariant to vanish by Lemma 17 (10) in [7] we may restrict ourselves to graphs with minimum vertex degree ≥ 3. All 23 graphs of this type up to loop order 6 have ≤ 12 edges and hence constant c_2-invariant by the work of Stembridge [23]. We find that 11 of the c_2-invariants are -1 whereas all others are 0.

At loop order 7 we have 133 graphs. There are no new c_2-invariants in the sense that they already occur in ϕ^4.

Among the 1352 graphs at 8 loops there is the first non-ϕ^4 c_2-invariant. Its graph G_5 is drawn in figure 7. It satisfies $c_2(G_5)_q \equiv -y_5 \mod q$, where

$$(5.5) \qquad y_5 = \#\{x \in \mathbb{F}_q : x^2 + x - 1 = 0\} - 1$$

and has been verified for the first 50 primes.

At loop order 9, we find modular forms of type (weight, level) equal to $(2, 11)$, $(2, 14)$, $(2, 15)$ (see table 1) which do not occur in ϕ^4 theory. We also obtain a newform of weight 3 and level 11 which is absent in ϕ^4-theory up to loop order 10. Moreover, we find the two modular forms of weight 4, level 7 and weight 4, level 17 which occur in ϕ^4 at 10 loops. These modular c_2-invariants are confirmed for at least 11 primes. Finally, the c_2-invariant of 15 graphs give 7 new unidentified sequences which we do not list here.

Table 6: List of all ϕ^4 prime ancestors up to loop order 9 and their c_2-invariants for the first 12 primes. The names of the graphs are taken from [20], the graphs of loop order 9 are only available in [19]. $-c_2$ is either 1, 0 z_2, z_3, z_4, (see Eq. (5.1)), modular with (weight, level), or an unidentified sequence from table 8.

graph	$-c_2(p)$	$p =$											
		2	3	5	7	11	13	17	19	23	29	31	37
P_3	1	1	1	1	1	1	1	1	1	1	1	1	1
$P_{6,4}$	0	0	0	0	0	0	0	0	0	0	0	0	0
$P_{7,8}$	z_2	0	1	1	1	1	1	1	1	1	1	1	1
$P_{7,9}$	z_2	0	1	1	1	1	1	1	1	1	1	1	1
$P_{7,10}$	0	0	0	0	0	0	0	0	0	0	0	0	0
$P_{7,11}$	z_3	1	0	4	1	10	1	16	1	22	28	1	1
$P_{8,32}$	0	0	0	0	0	0	0	0	0	0	0	0	0
$P_{8,33}$	z_3	1	0	4	1	10	1	16	1	22	28	1	1
$P_{8,34}$	0	0	0	0	0	0	0	0	0	0	0	0	0
$P_{8,35}$	z_2	0	1	1	1	1	1	1	1	1	1	1	1
$P_{8,36}$	z_3	1	0	4	1	10	1	16	1	22	28	1	1
$P_{8,37}$	$(3, 7)$	1	0	0	0	5	0	0	0	18	4	0	36
$P_{8,38}$	$(4, 5)$	0	2	0	6	10	1	9	5	14	8	16	7
$P_{8,39}$	$(3, 8)$	0	1	0	0	3	0	2	4	0	0	0	0
$P_{8,40}$	z_4	0	2	1	6	10	1	1	18	22	1	30	1
$P_{8,41}$	$(6, 3)$	0	0	1	2	8	1	15	14	11	27	29	32
$P_{9,154}$	z_2	0	1	1	1	1	1	1	1	1	1	1	1
$P_{9,155}$	$(3, 7)$	1	0	0	0	5	0	0	0	18	4	0	36
$P_{9,156}$	z_2	0	1	1	1	1	1	1	1	1	1	1	1
$P_{9,157}$	$(3, 12)$	0	0	0	2	0	4	0	7	0	0	16	26
$P_{9,158}$	$(3, 8)$	0	1	0	0	3	0	2	4	0	0	0	0
$P_{9,159}$	z_2	0	1	1	1	1	1	1	1	1	1	1	1
$P_{9,160}$	$(3, 8)$	0	1	0	0	3	0	2	4	0	0	0	0
$P_{9,161}$	$(5, 4)$	0	0	1	0	0	9	16	0	0	24	0	16
$P_{9,162}$	$(4, 5)$	0	2	0	6	10	1	9	5	14	8	16	7
$P_{9,163}$	$(4, 5)$	0	2	0	6	10	1	9	5	14	8	16	7
$P_{9,164}$	$(3, 7)$	1	0	0	0	5	0	0	0	18	4	0	36
$P_{9,165}$	i_{132}	1	2	1	4	1	0	4	11	6	23	2	13

(*Continued*)

Table 6: Continued.

graph	$-c_2(p)$	$p =$											
		2	3	5	7	11	13	17	19	23	29	31	37
$P_{9,166}$	$(4, 6)$	0	0	1	5	1	12	10	1	7	1	5	32
$P_{9,167}$	$(4, 13)$	1	2	3	1	7	0	9	7	19	5	10	17
$P_{9,168}$	$(3, 8)$	0	1	0	0	3	0	2	4	0	0	0	0
$P_{9,169}$	i_{53}	0	1	1	4	6	6	14	5	15	11	21	23
$P_{9,170}$	$(5, 4)$	0	0	1	0	0	9	16	0	0	24	0	16
$P_{9,171}$	$(4, 6)$	0	0	1	5	1	12	10	1	7	1	5	32
$P_{9,172}$	$(4, 6)$	0	0	1	5	1	12	10	1	7	1	5	32
$P_{9,173}$	$(6, 7)$	0	1	4	0	1	3	12	16	7	25	13	36
$P_{9,174}$	$(4, 6)$	0	0	1	5	1	12	10	1	7	1	5	32
$P_{9,175}$	$(4, 5)$	0	2	0	6	10	1	9	5	14	8	16	7
$P_{9,176}$	$(6, 7)$	0	1	4	0	1	3	12	16	7	25	13	36
$P_{9,177}$	i_{141}	1	2	3	3	0	9	0	1	4	6	18	15
$P_{9,178}$	i_{125}	1	2	0	2	6	5	14	12	20	18	6	12
$P_{9,179}$	$(6, 3)$	0	0	1	2	8	1	15	14	11	27	29	32
$P_{9,180}$	i_{121}	1	1	3	3	10	8	14	1	14	3	9	24
$P_{9,181}$	i_{106}	1	0	3	3	8	8	13	11	6	11	25	33
$P_{9,182}$	i_{81}	0	2	2	2	6	5	14	10	1	27	16	12
$P_{9,183}$	$(5, 4)$	0	0	1	0	0	9	16	0	0	24	0	16
$P_{9,184}$	i_{110}	1	0	4	6	0	0	9	3	6	24	28	4
$P_{9,185}$	$(5, 4)$	0	0	1	0	0	9	16	0	0	24	0	16
$P_{9,186}$	i_{87}	0	2	3	1	1	8	9	15	12	1	18	26
$P_{9,187}$	i_{110}	1	0	4	6	0	0	9	3	6	24	28	4
$P_{9,188}$	$(7, 3)$	0	0	0	1	0	12	0	1	0	0	4	1
$P_{9,189}$	$(6, 4)$	0	0	4	3	1	11	16	0	13	15	9	35
$P_{9,190}$	i_{104}	1	0	2	6	1	9	2	9	8	9	28	18

6. Methods

We used B.D. McKay's *nauty* to generate graphs [16], and Sage [18] to generate modular forms (supplemented by comprehensive tables in weights 2, 4, 6 which can be found in [15]). The first reduction to prime ancestors reduces the total number of graphs to be considered from 1354 to 231 in the case of ϕ^4 theory at 10 loops. This assumes the completion conjecture 14.

Table 7: List of identified ϕ^4 c_2-invariants up to loop order 10 together with the number of their prime ancestors. The first five c_2-invariants are quasi-constant, i_6 to i_{21} are modular with (weight, level). There are no prime ancestors with 1, 2, 4, or 5 loops.

no.	$-c_2(p)$	$p=$ 2	3	5	7	11	13	loop order $\ell=$ 3	6	7	8	9	10
i_1	1	1	1	1	1	1	1	1	0	0	0	0	0
i_2	0	0	0	0	0	0	0	0	1	1	2	0	3
i_3	z_2	0	1	1	1	1	1	0	0	2	1	3	2
i_4	z_3	1	0	4	1	10	1	0	0	1	2	0	1
i_5	z_4	0	2	1	6	10	1	0	0	0	1	0	3
i_6	(3,7)	1	0	0	0	5	0	0	0	0	1	2	0
i_7	(3,8)	0	1	0	0	3	0	0	0	0	1	3	5
i_8	(3,12)	0	0	0	2	0	4	0	0	0	0	1	1
i_9	(4,5)	0	2	0	6	10	1	0	0	0	1	3	4
i_{10}	(4,6)	0	0	1	5	1	12	0	0	0	0	4	2
i_{11}	(4,7)	1	1	1	0	3	2	0	0	0	0	0	3
i_{12}	(4,13)	1	2	3	1	7	0	0	0	0	0	1	1
i_{13}	(4,17)	1	1	1	0	9	7	0	0	0	0	0	4
i_{14}	(5,4)	0	0	1	0	0	9	0	0	0	0	4	6
i_{15}	(6,3)	0	0	1	2	8	1	0	0	0	1	1	6
i_{16}	(6,4)	0	0	4	3	1	11	0	0	0	0	1	7
i_{17}	(6,7)	0	1	4	0	1	3	0	0	0	0	2	4
i_{18}	(6,10)	0	1	0	6	2	6	0	0	0	0	0	1
i_{19}	(7,3)	0	0	0	1	0	12	0	0	0	0	1	1
i_{20}	(8,2)	0	0	0	1	3	4	0	0	0	0	0	6
i_{21}	(8,5)	0	0	0	1	7	1	0	0	0	0	0	1

The denominator reduction then reduces the c_2-invariant of each (uncompleted) graph to the point-count of a hypersurface of smaller degree, as follows. Primitive-divergent ϕ^4 graphs with at least five vertices have at least two three-valent vertices with no common edge joining them. If we choose the first 6 edges $e_1, \ldots, e_6$ to contain these two three-valent vertices then one can show [5] that D^5 and D^6 necessarily factorize. Therefore for ϕ^4 graphs we can always apply denominator reduction twice to reduce the number of variables by 7. Since D^7 is homogeneous, we can eliminate a further variable with the effect that at 10 loops we have to count the zeros of a polynomial in at most 12 variables. A further variable can effectively be eliminated using the fact that D^7 is quadratic in every variable and by computing tables of

Table 8: List of $-c_2$ for all unidentified c_2-invariants up to loop order 10 with the number of their prime ancestors.

no.	$p=$ 2	3	5	7	11	13	$\ell=$ 9	10	no.	$p=$ 2	3	5	7	11	13	$\ell=$ 9	10
i_{22}	0	0	0	1	9	2	0	1	i_{54}	0	1	1	5	6	3	0	1
i_{23}	0	0	0	2	6	7	0	1	i_{55}	0	1	1	6	0	5	0	1
i_{24}	0	0	0	4	5	3	0	1	i_{56}	0	1	1	6	2	0	0	1
i_{25}	0	0	1	0	0	0	0	1	i_{57}	0	1	2	1	9	2	0	1
i_{26}	0	0	1	0	7	7	0	1	i_{58}	0	1	2	2	9	12	0	1
i_{27}	0	0	1	2	3	2	0	1	i_{59}	0	1	2	3	4	10	0	1
i_{28}	0	0	1	2	6	5	0	3	i_{60}	0	1	2	4	7	1	0	1
i_{29}	0	0	1	4	6	10	0	1	i_{61}	0	1	3	1	0	11	0	1
i_{30}	0	0	1	5	6	5	0	3	i_{62}	0	1	3	1	1	9	0	1
i_{31}	0	0	2	2	5	12	0	1	i_{63}	0	1	3	2	9	1	0	1
i_{32}	0	0	2	5	2	9	0	1	i_{64}	0	1	3	3	2	10	0	1
i_{33}	0	0	2	5	9	11	0	1	i_{65}	0	1	3	4	8	4	0	4
i_{34}	0	0	2	6	4	0	0	1	i_{66}	0	1	3	5	0	12	0	1
i_{35}	0	0	2	6	10	0	0	1	i_{67}	0	1	3	5	6	9	0	1
i_{36}	0	0	3	0	9	3	0	2	i_{68}	0	1	3	5	8	9	0	1
i_{37}	0	0	3	2	1	2	0	2	i_{69}	0	1	4	1	4	0	0	1
i_{38}	0	0	3	2	8	11	0	1	i_{70}	0	1	4	2	1	0	0	3
i_{39}	0	0	3	3	2	10	0	1	i_{71}	0	1	4	2	7	11	0	1
i_{40}	0	0	4	1	0	9	0	2	i_{72}	0	1	4	3	0	8	0	1
i_{41}	0	0	4	2	2	8	0	1	i_{73}	0	1	4	5	5	7	0	1
i_{42}	0	0	4	2	9	11	0	1	i_{74}	0	2	0	0	1	8	0	1
i_{43}	0	0	4	3	0	11	0	1	i_{75}	0	2	0	0	6	4	0	1
i_{44}	0	1	0	0	2	4	0	1	i_{76}	0	2	0	0	9	2	0	1
i_{45}	0	1	0	1	4	5	0	1	i_{77}	0	2	0	5	2	5	0	1
i_{46}	0	1	0	2	1	9	0	1	i_{78}	0	2	0	6	2	0	0	1
i_{47}	0	1	0	2	6	9	0	1	i_{79}	0	2	0	6	3	9	0	3
i_{48}	0	1	0	4	0	0	0	2	i_{80}	0	2	2	2	0	9	0	1
i_{49}	0	1	0	4	4	0	0	1	i_{81}	0	2	2	2	6	5	1	2
i_{50}	0	1	1	1	10	0	0	1	i_{82}	0	2	2	3	1	2	0	1
i_{51}	0	1	1	2	0	9	0	1	i_{83}	0	2	2	3	7	2	0	1
i_{52}	0	1	1	2	10	9	0	1	i_{84}	0	2	2	5	6	4	0	1
i_{53}	0	1	1	4	6	6	1	0	i_{85}	0	2	2	6	0	6	0	1

(*Continued*)

Table 8: Continued.

no.	$p=$ 2	3	5	7	11	$\ell=$ 13	9	10	no.	$p=$ 2	3	5	7	11	$\ell=$ 13	9	10
i_{86}	0	2	3	0	6	5	0	1	i_{116}	1	1	1	5	2	8	0	1
i_{87}	0	2	3	1	1	8	1	3	i_{117}	1	1	2	2	9	7	0	1
i_{88}	0	2	3	3	2	0	0	1	i_{118}	1	1	2	4	0	12	0	3
i_{89}	0	2	4	0	3	3	0	1	i_{119}	1	1	3	1	3	6	0	1
i_{90}	0	2	4	3	4	7	0	3	i_{120}	1	1	3	2	1	12	0	2
i_{91}	0	2	4	4	1	8	0	1	i_{121}	1	1	3	3	10	8	1	5
i_{92}	0	2	4	4	10	12	0	1	i_{122}	1	1	3	6	8	6	0	1
i_{93}	0	2	4	5	1	5	0	1	i_{123}	1	1	4	2	7	10	0	1
i_{94}	0	2	4	6	1	8	0	1	i_{124}	1	1	4	6	0	8	0	1
i_{95}	1	0	0	1	9	7	0	1	i_{125}	1	2	0	2	6	5	1	1
i_{96}	1	0	0	3	6	2	0	1	i_{126}	1	2	0	5	2	0	0	1
i_{97}	1	0	1	2	1	11	0	1	i_{127}	1	2	0	5	5	8	0	1
i_{98}	1	0	1	2	4	9	0	1	i_{128}	1	2	0	5	7	0	0	1
i_{99}	1	0	1	2	8	5	0	1	i_{129}	1	2	1	0	5	10	0	1
i_{100}	1	0	1	2	9	8	0	1	i_{130}	1	2	1	0	10	6	0	1
i_{101}	1	0	1	4	6	9	0	1	i_{131}	1	2	1	2	5	1	0	1
i_{102}	1	0	2	1	10	5	0	1	i_{132}	1	2	1	4	1	0	1	2
i_{103}	1	0	2	3	5	2	0	1	i_{133}	1	2	1	4	3	1	0	2
i_{104}	1	0	2	6	1	9	1	1	i_{134}	1	2	1	6	5	2	0	1
i_{105}	1	0	3	2	2	12	0	4	i_{135}	1	2	1	6	10	8	0	1
i_{106}	1	0	3	3	8	8	1	5	i_{136}	1	2	2	0	0	4	0	3
i_{107}	1	0	4	2	4	12	0	1	i_{137}	1	2	2	3	1	8	0	1
i_{108}	1	0	4	4	1	11	0	1	i_{138}	1	2	2	5	5	2	0	1
i_{109}	1	0	4	5	4	8	0	1	i_{139}	1	2	2	5	6	4	0	1
i_{110}	1	0	4	6	0	0	2	5	i_{140}	1	2	3	3	0	3	0	1
i_{111}	1	1	0	2	9	11	0	1	i_{141}	1	2	3	3	0	9	1	1
i_{112}	1	1	0	6	2	9	0	1	i_{142}	1	2	4	1	3	2	0	1
i_{113}	1	1	0	6	4	1	0	1	i_{143}	1	2	4	5	1	6	0	1
i_{114}	1	1	1	1	1	8	0	5	i_{144}	1	2	4	6	7	9	0	1
i_{115}	1	1	1	4	8	6	0	1	i_{145}	1	2	4	6	9	10	0	2

quadratic residues modulo p. Altogether for the prime 13 we need at 10 loops a maximum of $13^{11} \approx 1.8 \times 10^{12}$ evaluations of typically $\sim$30 000 monomials to determine $c_2(G)_{13}$. For only 10 out of 231 prime 10 loop ancestors we did not manage to find more than the minimum number of reductions. The point-counts for these 10 graphs with primes $p = 11$ and $p = 13$ were

performed at the Erlanger RRZE Cluster using a total computation time of 18 core years. The results are i_{20} (twice), i_{24}, i_{34}, i_{49}, i_{60}, i_{64}, i_{75}, i_{78}, i_{85} in table 8. A further 16 core years were invested in the sequence i_{49} at prime 17 to rule out an accidental coincidence with a weight 2 modular form of level 624 for the first 6 primes. Additionally the point-count of the weight 8 level 5 modular graph for the primes 29 and 31 consumed 1 core year on the RRZE Cluster. For the 9 loop graph $P_{9,190}$ (see table 6) we evaluated the point counts of the primes 29, 31, and 37 on the RRZE Cluster in altogether 3 core years. All other computations were performed using an office PC.

Acknowledgments

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