Quantum geometry of elliptic Calabi–Yau manifolds

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We study the quantum geometry of the class of Calabi–Yau three-folds, which are elliptic fibrations over a two-dimensional toric base. A holomorphic anomaly equation for the topological string free energy is proposed, which is iterative in the genus expansion as well as in the curve classes in the base. T-duality on the fibre implies that the topological string free energy also captures the BPS-invariants of $D4$-branes wrapping the elliptic fibre and a class in the base. We verify this proposal by explicit computation of the BPS invariants of 3 $D4$-branes on the rational elliptic surface.

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1. Introduction

Topological string theory on local Calabi–Yau manifolds has been a remarkable success story. It counts the open and closed instantons corrections to topological numbers, which can be seen as an extension from classical geometry to quantum geometry. By now we can solve it in very different ways, namely by localisation, by direct integration of the holomorphic anomaly equations, by the topological vertex [1] or by the matrix model techniques in the remodelled B-model [12]. Topological string theory on local Calabi–Yau manifolds gives deep insights in the interplay between large N gauge theory/string theory duality, mirror duality, the theory of modular forms and knot theory and is by geometric engineering [37] intimately related to the construction of effective $N = 2$ and $N = 1$ rigid supersymmetric theories in four dimensions.

On global, i.e., compact, Calabi–Yau manifolds, which give rise to $N = 2$ and $N = 1$ effective supergravity theories in four dimensions, the situation is

less understood. Direct integration extends the theory of modular objects to the Calabi–Yau spaces and establishes that closed topological string amplitudes can be written as polynomial in modular objects, but the boundary conditions for the integration are in contrast to the local case not completely known. As an example, on the quintic surface the closed topological string can be solved up to genus 51 [33].

In [17] mirror symmetry was made local in the decompactification limit of fibred Calabi–Yau threefolds. Here we want to do the opposite and study how the quantum geometry extends from the local to the global case, when a class of local Calabi–Yau geometries is canonically compactified by an elliptic Calabi–Yau fibration with projection $\pi : M \to B$. This easy class of local to global pairs, will be described to a large extend by complete intersections in explicit toric realizations. As we review in Section 2, if the elliptic fibration has only I_1 fibres the classical cohomology of M is completely determined by the classical intersection of the base B and the number of sections, which depends on the Mordell Weyl group of the elliptic family.

The decisive question to which extend this holds for the quantum geometry is addressed in Section 3 using mirror symmetry. The instanton numbers are counted by (quasi)-modular forms of congruence subgroups of $\mathrm{SL}(2,\mathbb{Z})$ capturing curves with a fixed degree in the base for all degrees in the fibre. The weights of the forms depend on the genus and the base class. This structure has been discovered for elliptically fibred surfaces in [39] and for elliptically fibred threefolds in [40]. We establish here a holomorphic anomaly Equation (3.9) based on the non-holomorphic modular completion of the quasi-modular forms which is iterative in the genus, as in [9], and also in the base classes generalizing [32, 57].

Our construction can be viewed also as a step to a better understanding of periods and instanton corrections in F-theory compactifications and a preliminary study using the data of [38, 41, 56] reveals that the structure at the relevant generera $g = 0, 1$ extends.

A holomorphic anomaly equation is also known to appear for generating functions of Bogomolny-Prasad-Sommerfield (BPS) invariants of higher dimensional D-branes, in particular $D4$-branes on a surface [2, 20, 49, 53, 59]. Interestingly, on elliptic Calabi–Yau fibrations, double T-duality on the elliptic fibre (or Fourier–Mukai transform) [3, 4, 7, 57] transforms $D2$-branes wrapped on base classes into $D4$-branes that also wrap the elliptic fibre and vice versa. The $D4$-brane holomorphic anomaly is therefore related to the one of Gromov–Witten theory for these geometries. Moreover, the mirror periods provide predictions for $D4$-brane BPS invariants which correspond to those of (small) black holes in supergravity.

We discuss higher dimensional branes on Calabi–Yau elliptic fibrations in Sections 5 and 6. We compare the predictions from the periods for $D4$-brane BPS invariants with existing methods in the literature for the computation of small charge BPS invariants [11, 16, 19, 22, 23, 52, 54]. The predictions of the periods are in many cases compatible with these methods. We leave a more precise study of $D4$-brane BPS states on general elliptic fibrations to future work.

Section 6 specializes to the elliptic fibration over the Hirzebruch surface $\mathbb{F}_1$. The periods of its mirror geometry provide the BPS invariants of $D4$-branes on the rational elliptic surface (also known as $\frac{1}{2}K_3$) as proposed originally by Minahan et al. [57]. We revisit and extend the verification of this proposal for ≤ 3 $D4$-branes using algebraic-geometric techniques [25, 26, 51, 53, 55, 60, 62].

2. Classical geometry of elliptically fibred Calabi–Yau spaces

In this section, we study the classical geometry of elliptically fibered Calabi–Yau threefolds M with base B and projection map $\pi : M \to B$. We provide expressions for the Chern classes as well as the construction of such Calabi–Yau three manifolds by means of toric geometry. Elliptic fibrations are locally described by a Weierstrass form

$$(2.1) \qquad y^2 = 4x^3 - xw^4 g_2(\underline{u}) - g_3(\underline{u})w^6,$$

where $\underline{u}$ are coordinates on the base B. A global description can be defined by an embedding as a hypersurface or complete intersection in an ambient space W. Explicitly, we consider cases, which allow a representation as a hypersurface or complete intersection in a toric ambient space. We restrict our attention to the case where the fibre degenerations are only of Kodaira type I_1, which means that the discriminant $\Delta = g_2^3 - 27g_3^2$ of (2.1) has only simple zeros on B, which are not simultaneously zeros of g_2 and g_3. It was observed in [38] that such tame fibrations can be constructed torically over toric bases, which are given themselves defined by reflexive polyhedra. These tame fibrations are not enough to address immediately phenomenological interesting models in F-theory, due to the lack of non-abelian gauge symmetry in the effective four-dimensional physics, which come precisely from more singular fibres in the Kodaira classification. However, we note that the examples discussed here have a particular large number of complex moduli.

Adjusting the latter and blowing up the singularities, not necessarily torically, is a more local operation, at least of co-dimension one in the base, which can be addressed in a second step.

2.1. The classical geometrical data of elliptic fibrations

Let $W \to B$ be a fibre bundle whose fibre is an $r - 1$ dimensional weighted projective space $\mathbb{P}(w_1, \dots, w_r)$ and B an almost toric Fano surface. We define elliptically fibred Calabi–Yau threefolds $M \to B$ as hypersurfaces or complete intersections in W. We consider the following choices of weights

$$(2.2) \qquad (w_1, \dots, w_r) = \{(1, 2, 3), (1, 1, 2), (1, 1, 1), (1, 1, 1, 1)\}.$$

In particular, the elliptic fibres are degree 6, 4, 3 hypersurfaces and a bidegree $(2, 2)$ complete intersection in the coordinates of the given weighted projective space. In the case of rational elliptic surfaces these fibres lead to E_8, E_7, E_6 and D_5 del Pezzo surfaces, named so as the integers cohomology lattice of the surface contains the intersection form of the Cartan-matrix of the corresponding Lie algebras. In the following we keep these names for the fibration types.

Let us discuss the first case. This leads canonically to an embedding with a single section; however, most of the discussion below applies to the other cases with minor modifications. Denote by $\alpha = c_1(\mathcal{O}(1))$ with $\mathcal{O}(1)$ the line bundle on W induced by the hyperplane class of the projective fibre and $K = -c_1$ the canonical bundle of the base.

The coordinates w, x, y are sections of $\mathcal{O}(1)$, $\mathcal{O}(1)^2 \otimes K^{-2}$ and $\mathcal{O}(1)^3 \otimes K^{-3}$ while g_2 and g_3 are sections of K^{-4} and K^{-6} respectively so that (2.1) is a section of $\mathcal{O}(1)^6 \otimes K^{-6}$. The corresponding divisors $w = 0, x = 0, y = 0$ have no intersection, i.e., $\alpha(\alpha + c_1)(\alpha + c_1) = 0$ in the cohomology ring of W and

$$(2.3) \qquad \alpha(\alpha + c_1) = 0$$

in the cohomology ring of M. Let us assume that the discriminant Δ vanishes for generic complex moduli only to first order in the coordinates of B at loci, which are not simultaneously zeros of g_2 and g_3. In this case, its class must satisfy

$$(2.4) \qquad [\Delta] = c_1(B) = -K$$

Table 1: Chern classes $\mathcal{C}_i$ of regular elliptic Calabi–Yau manifolds. Integrating α over the fibre yields a factor $a = \frac{\prod_i d_i}{\prod_i w_i}$, i.e., the number of sections $1, 2, 3, 4$ for the three fibrations in turn.

Fibre	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$
$E8$	$12\alpha c_1 + (11c_1^2 + c_2)$	$-60\alpha c_1^2 - (60c_1^3 + c_2 c_1 - c_3)$	$12\alpha c_1 (30c_1^2 + c_2)$
$E7$	$6\alpha c_1 + (5c_1^2 + c_2)$	$-18\alpha c_1^2 - (18c_1^3 + c_2 c_1 - c_3)$	$6\alpha c_1 (12c_1^2 + c_2)$
$E6$	$4\alpha c_1 + (3c_1^2 + c_2)$	$-8\alpha c_1^2 - (8c_1^3 + c_2 c_1 - c_3)$	$4\alpha c_1 (6c_1^2 + c_2)$
$D5$	$3\alpha c_1 + (2c_1^2 + c_2)$	$-4\alpha c_1^2 - (4c_1^3 + c_2 c_1 - c_3)$	$3\alpha c_1 (3c_1^2 + c_2)$

to obey the Calabi–Yau condition and the fibre over the vanishing locus of the discriminant is of Kodaira type I_1. For this generic fibration, the properties of M only depend on the properties of B.

For example, using the adjunction formula and the relation (2.3) to reduce to linear terms in α allows us to write the total Chern class as[1]

$$(2.5) \qquad \mathcal{C} = \left(1 + \sum_{i=1}^{d_M-1} c_i\right) \frac{(1+\alpha)(1 + w_2\alpha + w_2 c_1)(1 + w_3\alpha + w_3 c_1)}{1 + d\alpha + dc_1}.$$

The Chern forms $\mathcal{C}_k$ of M are the coefficients in the formal expansion of (2.5) of the degree k in terms of a and the monomials of the Chern forms c_i of base B. The formulas (2.3) and (2.5) apply for all projectivisations.

In the following, the results for various dimensions d_M are presented. For $d_M = 2$, one gets from table 1 by integrating over the fibre in all cases $\chi(M) = 12 \int_B c_1$ and $\mathbb{P}^1$ is the only admissible base. Similar for $d_M = 3$ one gets for the different projectivisations $\chi(M) = -60 \int_B c_1^2$, $\chi(M) = -36 \int_B c_1^2$, $\chi(M) = -24 \int_B c_1^2$ and $\chi(M) = -16 \int_B c_1^2$.

The following discussion extends to all dimensions but for the sake of brevity we specialize to Calabi–Yau threefolds. Let K_i, $i = 1, \ldots, b_2(B)$, span the Kähler (or ample) cone of B with intersection numbers $K_i K_j = c_{ij}$. Moreover, let C^i be a basis for the dual Kähler cone. We expand the

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[1] In the $D5$ complete intersection case $d_1 = d_2 = 2$. One has to add a factor $(1 + \alpha + c_1)$ in the numerator and a factor $(1 + 2\alpha + 2c_1)$ in the denominator.

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canonical class of B in terms of K_i and C^i as:

$$(2.6) \qquad K_B = -c_1 = -\sum_i a^i K_i = -\sum a_i C^i,$$

with a_i and a^i in $\mathbb{Z}$. We denote by $\mathcal{K}_a$, $a = 1, \ldots, h^{1,1}(M)$, the divisors of the total space of the elliptic fibration and distinguish between $\mathcal{K}_e$ the divisor dual to the elliptic fibre curve and $\mathcal{K}_i$, $i = 1, \ldots, b = b_2(B)$, which are $\pi^*(C^i)$

$$(2.7) \qquad \begin{aligned} \mathcal{K}_e^3 &= a \int_B c_1^2, \\ \mathcal{K}_e^2 \mathcal{K}_i &= a a_i, \\ \mathcal{K}_e \mathcal{K}_i \mathcal{K}_j &= a c_{ij}. \end{aligned}$$

Here, a denotes the number of sections; see table 1. The intersection with the second Chern class of the total space can be calculated using table 1 for the elliptic and other fibres as

$$(2.8) \qquad \int_M c_2 J_e = \begin{cases} \int_B (11 c_1^2 + c_2) & E_8, \\ 2 \int_B (5 c_1^2 + c_2) & E_7, \\ 3 \int_B (3 c_1^2 + c_2) & E_6, \\ 4 \int_B (2 c_1^2 + c_2) & D_5, \end{cases}$$

$$\int_M c_2 J_i = 12 a_i.$$

Here, we denoted by J_i the basis of harmonic $(1,1)$ forms dual to the $\mathcal{K}_i$.

Let us note two properties about the intersection numbers. These properties can be established using the properties of the toric almost Fano bases B and (2.7), which follows from the construction of the elliptic fibration summarized in (2.20). To start, define the matrix

$$(2.9) \qquad C_e = \begin{pmatrix} \int_B c_1^2 & a_1 & \ldots & a_b \\ a_1 & & & \\ \vdots & & c_{ij} & \\ a_b & & & \end{pmatrix},$$

then we can conclude from properties of the intersection numbers and the canonical class that

$$(2.10) \qquad\qquad \det(C_e) = 0.$$

A further property concerns a decoupling limit between base and fibre in the Kähler moduli space. Generally, we can make a linear change in the basis of Mori vectors l_i, which results in corresponding linear change of the basis in dual spaces of the Kähler moduli t^i and the divisors D_i

$$(2.11) \qquad \tilde{l}_i = \sum_j m_{ij} l_j, \quad \tilde{t}^i = \sum (m^T)_{ij} t^j.$$

To realize a decoupling between the base and the fibre, we want to find a not necessarily integer basis change, which eliminates the couplings $\tilde{\mathcal{K}}_e^2 \tilde{\mathcal{K}}_i$ and leaves the couplings $\tilde{\mathcal{K}}_e \tilde{\mathcal{K}}_i \tilde{\mathcal{K}}_j$ invariant. It follows from (2.6), (2.7) and the obvious transformation of the triple intersections that there is a unique solution

$$(2.12) \qquad m = \begin{pmatrix} 1 & \dfrac{a^1}{2} & \cdots & \dfrac{a^b}{2} \\ 0 & 1 & 0\ldots & 0 \\ \vdots & & \vdots & \\ 0 & 0 & \ldots 0 & 1 \end{pmatrix},$$

such that

$$(2.13) \qquad \begin{aligned} \tilde{\mathcal{K}}_e^3 &= a \left(\int_B c_1^2 - \frac{3}{2} a_i a^i + \frac{3}{4} c_{ij} a^i a^j \right), \\ \tilde{\mathcal{K}}_e^2 \tilde{\mathcal{K}}_i &= 0, \\ \tilde{\mathcal{K}}_e \tilde{\mathcal{K}}_i \tilde{\mathcal{K}}_j &= a c_{ij}. \end{aligned}$$

As we have seen the classical topological data of the total space of the elliptic fibration follows from simple properties of the fibre and the topology of the base. We want to extend these results in the next section to the quantum cohomology of the elliptic fibration. We focus again on the Calabi–Yau threefold case, where the instanton contributions to the quantum cohomology is richest. To actually calculate quantum cohomology we need an explicit realisation of a class of examples, which we discuss in the next subsection.
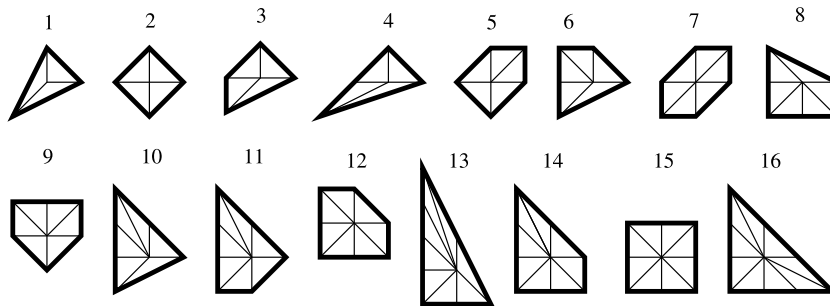
Figure 1: These are the 16 reflexive polyhedra Δ_B in two dimensions, which build 11 dual pairs (Δ_B, Δ_B^*). Polyhedron k is dual to polyhedron $17 - k$ for $k = 1, \ldots, 5$. The polyhedra $6, \ldots, 11$ are self-dual.

2.2. Realizations in toric ambient spaces

In this subsection, we discuss the toric bases B leading to the above described tame elliptically fibered Calabi–Yau d_M-folds with only I_1 fibre singularities. It was observed in examples in [38], that they can be defined over toric bases defined themselves by reflexive polyhedra [6] Δ_B in $d_M - 1$ dimensions. Here we explore a class of elliptic Calabi–Yau fibrations, which are defined from a reflexive polyhedra Δ_B as the canonical hypersurface in the toric ambient space defined by the reflexive polyhedra (2.20) following Batyrev's work [6]. Note that for each Δ_B, one has the choice of the elliptic fibre as discussed in the previous section. We provide the toric data, including a basis for the Mori cone for this class of elliptic Calabi–Yau fibrations. The construction of the Mori cone from the star triangulation and the associated secondary fan follows the discussion in [8, 30]. Throughout the subsection we assume some familiarity with the construction of the toric ambient spaces from polyhedra as described, e.g., in [21, 58].

   The toric ambient spaces, which allow for smooth Calabi–Yau hypersurfaces of complex dimension d_M as section of the canonical bundle, can be described by pairs of reflexive polyhedra (Δ, Δ^*) of real dimension $d_M - 1$. For the threefold case, the possibilities of two-dimensional reflexive base polyhedra are listed in figure 1 [6]. Together with a complete star triangulation of Δ, they define a complex family of Calabi–Yau threefolds. The mirror family is given by exchanging the role of Δ and Δ^*. A complete triangulation divides Δ in simplices of volume 1. In a star triangulation all simplices contain the unique interior lattice point of the reflexive polyhedron. Let us give

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first two examples for toric smooth ambient spaces in which the canonical hypersurface leads to the E_8 elliptic fibration over $\mathbb{P}^2$ and over the Hirzebruch surface $\mathbb{F}_1$. The polyhedron Δ for the E_8 elliptic fibration over $\mathbb{P}^2$ with $\chi(M) = -540$ is given by the following data:

(2.14)

			ν_i			$l^{(e)}$	$l^{(1)}$
D_0	1	0	0	0	0	-6	0
D_1	1	1	0	-2	-3	0	1
D_2	1	0	1	-2	-3	0	1
D_3	1	-1	-1	-2	-3	0	1
D_z	1	0	0	-2	-3	1	-3
D_x	1	0	0	1	0	2	0
D_y	1	0	0	0	1	3	0

Here, we give the relevant points ν_i of the four- dimensional convex reflexive polyhedron Δ embedded into a hyperplane in a five-dimensional space and the linear relations $l^{(i)}$ spanning the Mori cone. This model has an unique star triangulation, given in (2.22). We calculate the intersection ring as follows from (2.7) with $a = 1$

(2.15)
$$\mathcal{R} = 9J_e^3 + 3J_e^2 J_1 + J_e J_1^2.$$

The evaluation of c_2 on the basis of the Kähler cone follows from (2.8) as $\int_M c_2 J_e = 102$ and $\int_M c_2 J_1 = 36$.

The polyhedron Δ for the E_8 elliptic fibration over $\mathbb{F}_1$ with $\chi = -480$ reads

(2.16)

			ν_i			$l^{(e)}$	$l^{(1)}$	$l^{(2)}$	$l^{(e)} + l^{(2)}$	$l^{(1)} + l^{(2)}$	$-l^{(2)}$
D_0	1	0	0	0	0	-6	0	0	-6	0	0
D_1	1	1	0	-2	-3	0	0	1	1	1	-1
D_2	1	0	1	-2	-3	0	1	0	0	1	0
D_3	1	-1	-1	-2	-3	0	0	1	1	1	-1
D_4	1	0	-1	-2	-3	0	1	-1	-1	0	1
D_z	1	0	0	-2	-3	1	-2	-1	0	-3	1
D_x	1	0	0	1	0	2	0	0	2	0	0
D_y	1	0	0	0	1	3	0	0	3	0	0

This example shows that there are two Calabi–Yau phases possible over $\mathbb{F}_1$, which are related by flopping a $\mathbb{P}^1$ represented by $l^{(2)}$. This transforms the

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half K3 to a del Pezzo eight surface, which can be shrunken to a point. In the first phase, the triangulation is described by (2.22) the intersection ring and $\int_M c_2 J_i$ follows by (2.7), (2.8) as

$$(2.17) \qquad \mathcal{R} = 8J_e^3 + 3J_e^2 J_1 + J_e J_1^2 + 2J_e^2 J_2 + J_1 J_2 J_3.$$

and $\int_M c_2 J_e = 92$, $\int_M c_2 J_1 = 36$ and $\int_M c_2 J_3 = 24$. For the second phase, we flop the $\mathbb{P}^1$ that corresponds to the Mori cone element $l^{(2)}$. Generally, if we flop the curve C this changes the triple intersection of the divisors $\mathcal{K}_i \mathcal{K}_j \mathcal{K}_k$ [15] by

$$(2.18) \qquad \Delta_{ijk} = -(C \cdot \mathcal{K}_i)(C \cdot \mathcal{K}_j)(C \cdot \mathcal{K}_k).$$

Now the intersection of the curves C_i which correspond to the Mori cone vector $l^{(i)}$ with the toric divisors D_k is given by $(C_i \cdot D_k) = l_k^{(i)}$. On the other hand, the $\mathcal{K}_k$ are combinations of D_k restricted to the hypersurface so that $(\mathcal{K}^k \cdot C_i) = \delta_i^k$.

In addition, one has to change the basis in order to maintain positive intersection numbers[2] $\tilde{l}^{(e)} = l^{(e)} + l^{(2)}$, $\tilde{l}^{(1)} = l^{(1)} + l^{(2)}$ and $\tilde{l}^{(2)} = -l^{(2)}$. For the $(1,1)$ forms J_i, which transform dual to the curves, we then get the intersection ring in the new basis of the Kähler cone

$$(2.19) \qquad \begin{aligned} \mathcal{R} &= 8\tilde{J}_e^3 + 3\tilde{J}_e^2 \tilde{J}_1 + \tilde{J}_e \tilde{J}_1^2 + 9\tilde{J}_e^2 \tilde{J}_2 + 3\tilde{J}_e \tilde{J}_1 \tilde{J}_2 + \tilde{J}_1^2 \tilde{J}_2 + 9\tilde{J}_e \tilde{J}_2^2 \\ &\quad + 3\tilde{J}_1 \tilde{J}_2^2 + 9\tilde{J}_2^3. \end{aligned}$$

The intersections with c_2 are not affected by the flop, only the basis change has to be taken into account. In the second phase, the triangulation of the base is given in the the middle of figure 2 and the triangulation of Δ is specified by (2.21). In this phase, an E_8 del Pezzo surface can be shrunken to get to the elliptic fibration over $\mathbb{P}^2$. This identifies the classes of the latter example as $J_e = \tilde{J}_2$, $J_1 = \tilde{J}_1$, while the divisor dual to $\tilde{J}_e^3$ is shrunken.

We denote by Δ_B the toric polyhedron for the base and specifying by

$$\{(e_1, e_2)\} = \{(-2, -3), (-1, -2), (-1, -1)\}$$

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[2]This is one criterion that holds in a simplicial Kähler cone. The full specification is that $\int_{\mathcal{C}} J > 0$, $\int_{\mathcal{D}} J \wedge J > 0$ and $\int_M J \wedge J \wedge J > 0$ for J in the Kähler cone and $\mathcal{C}$, $\mathcal{D}$ curves and divisors. If the latter is simplicial and generated by J_i, then $J = \sum d_i J_i$ with $d_i > 0$.

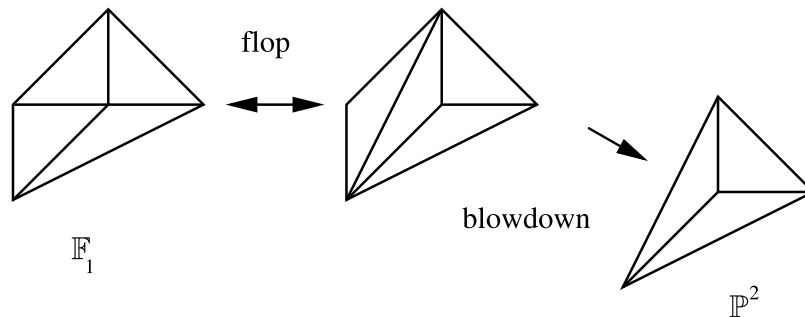860   Albrecht Klemm, Jan Manschot & Thomas Wotschke

Figure 2: The base triangulation for the flop in second example and the blowdown of an E_8 del Pezzo surface.

the toric data for the E_8, E_7, E_6 fibre respectively. It is easy to see that all toric hypersurfaces with the required fibration have the following general form of the polyhedron Δ:

$$
(2.20) \quad
\begin{array}{c|ccccc|cccc}
 & & & \nu_i & & & l^{(e)} & l^{(1)} & \dots & l^{(b)} \\
D_0 & 1 & 0 & 0 & 0 & 0 & \sum_i e_i - 1 & 0 & \dots & 0 \\
D_1 & 1 & & & e_1 & e_2 & 0 & * & \dots & * \\
\vdots & 1 & \Delta_B & \vdots & \vdots & & \vdots & * & \dots & * \\
D_r & 1 & & & e_1 & e_2 & 0 & * & \dots & * \\
D_z & 1 & 0 & 0 & e_1 & e_2 & 1 & -\sum * & \dots & -\sum * \\
D_x & 1 & 0 & 0 & 1 & 0 & -e_1 & 0 & \dots & 0 \\
D_y & 1 & 0 & 0 & 0 & 1 & -e_2 & 0 & \dots & 0
\end{array}.
$$

We note that the fibre elliptic curve is realized in a two-dimensional toric variety, which can be defined also by a reflexive two-dimensional polyhedron Δ_F. It is embedded into Δ, so that the inner of Δ_F is also the origin of Δ. Its corners are given by

$$\{(0,0,e_1,e_2),(0,0,1,0),(0,0,0,1)\}.$$

The E_6, E_7 and E_8 fibre types correspond to the polyhedra in figure 1 with numbers $1, 4$ and 10. To check the latter equivalence requires an change of coordinates in $SL(2,\mathbb{Z})$. The dual reflexive polyhedron Δ^* contains Δ_F^* embedding likewise in the coordinate plane spanned by the 3rd and 4th axes.

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A triangulation of Δ_B as in figures 1 or 2 lifts in a universal way to a star triangulation of Δ as follows. To set the conventions denote by (ν_i^B, e_1, e_2) the points of the embedded base polyhedron Δ_B and label them as the points of Δ_B starting with the positive x-axis, which points to the right in the figures, and label points of Δ_B counter clockwise from $1, \ldots, r$. The inner point in Δ_B, $(0, 0, e_1, e_2)$ is labelled z. The two remaining points of Δ; $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$ are labelled by x and y.

Denote the k-th d-dimensional simplex in Δ_B by the labels of its vertices, i.e.,

$$\operatorname{sim}_k^{(d)} := (\lambda_1^k, \ldots, \lambda_{d+1}^k)$$

and in particular denote the outer edges of Δ_B by

$$\{\operatorname{ed}_k | k = 1, \ldots, r\} := \{(1, 2), \ldots, (r, 1)\}.$$

Any triangulation of Δ_B is lifted to a star triangulation of Δ, which is spanned by the simplices containing besides the inner point $(0, 0, 0, 0)$ of Δ the points with the labels

(2.21)
$$\operatorname{Tr}_\Delta = \{(\operatorname{sim}_k^{(2)}, x), (\operatorname{sim}_k^{(2)}, y) | k = 1, \ldots, p\} \cup \{(\operatorname{ed}_k, x, y) | k = 1, \ldots r\}.$$

In particular for star triangulations of Δ_B, one has

(2.22) $\qquad \operatorname{Tr}_\Delta = \{(\operatorname{ed}_k, z, x), (\operatorname{ed}_k, z, y), (\operatorname{ed}_k, x, y) | k = 1, \ldots r\}$

and generators of the Mori cone for the elliptic phase contain the Mori cone generators $l^{(1)}, \ldots, l^{(b)}$, which correspond to a star triangulation of the base polyhedron, which is the one in figure 1. We list here the Mori cones of the first seven cases,

Δ_B	1(1)	2(2)		3(2)		4(3)		5(3)		
ν_i^B	$l^{(1)}$	$l^{(1)}$	$l^{(2)}$	$l^{(1)}$	$l^{(2)}$	$l^{(1)}$	$l^{(2)}$	$l^{(1)}$	$l^{(2)}$	$l^{(3)}$
z	-3	-2	-2	-2	-1	0	-2	-1	-1	-1
1	1	1	0	1	0	0	1	-1	1	0
2	1	0	1	0	1	1	0	1	-1	1
3	1	1	0	1	-1	-2	1	0	1	-1
4		0	1	0	1	1	0	0	0	1
5								1	0	0
6										
ex	$-$	$-$		1		$-$		4		

	6(3)				7(4)			
$l^{(1)}$	$l^{(2)}$	$l^{(3)}$	$l^{(1)}$	$l^{(2)}$	$l^{(3)}$	$l^{(4)}$	$l^{(5)}$	$l^{(6)}$
-1	-1	0	-1	-1	-1	-1	-1	-1
1	0	0	-1	1	0	0	0	1
-1	1	0	1	-1	1	0	0	0
1	-1	1	0	1	-1	1	0	0
0	1	-2	0	0	1	-1	1	0
0	0	1	0	0	0	1	-1	1
			1	0	0	0	1	-1
	3					17		

the remaining nine cases are given in Appendix A. We indicate in the brackets behind the model the number of Kähler moduli. If the latter is smaller then the number of Mori cone generators and the dual Kähler cone are non-simplicial. This is the case for the models 7, 9 and for 11–16. In the last column, we list the number of extra triangulations. The corresponding phases involve non-star triangulations of Δ and can be reached by flops. By the rules discussed above, we can find the intersection ring and the Mori cone in phases related by flops. We understand also the blowing down of one model. Non-reflexivity possesses a slight technical difficulty in providing the data for the calculation of the instantons. The fastest way to get the data for all cases is to provide for the models 15 and 16 a simplicial Kähler cone and reach all other cases[3] by flops and blowdowns. We will do this in Appendix A.

3. Quantum geometry of elliptic fibrations

The aim of this section is to describe the quantum geometry of elliptic fibrations. For this, we introduce the needed notions from topological string theory and discuss two examples with base $\mathbb{P}^2$ and $\mathbb{F}_1$. One of our key findings is the holomorphic anomaly (3.9). This section finishes with a discussion of the modular properties due to the monodromy group.

From the data provided in the last section, namely the Mori cone and the intersection numbers, follow differential equations as well as particular solutions, which allow us to calculate the instanton numbers as established mathematically for genus zero by Givental, Lian, Liu and Yau [24, 46]. These can be calculated very efficiently using the program described in [31]. In the cases

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[3]Except for 13 which is available on request.

at hand one can evaluate the genus one data using the genus zero results, the holomorphic anomaly equation for the Ray Singer Torsion, boundary conditions provided by the evaluation of $\int_M J_a c_2$ and the behaviour of the discriminant at the conifold to evaluate the instantons of the elliptic fibrations.

The higher genus curves are less systematically studied on compact threefolds. However, if the total space of the elliptic fibration over a base class is a contractable rational surface, one can shrink the latter and obtain a local model on which the modular structure of higher genus amplitudes has been intensively studied. The explicit data suggest that that this structure is maintained for all classes in the base.

We summarize in the next subsection the strategy to obtain the instanton data and based on the results we propose a general form of the instantons corrected amplitude in terms of modular forms coming from the elliptic geometry of the fibre and a simple and general holomorphic anomaly formula, which governs the all genus instanton corrected amplitudes for the above discussed class of models.

In the following subsection, we use the B-model to prove some aspects of the proposed statements. This can establish the A-model results for genus 0 and 1, since mirror symmetry is proven and the B-model techniques apply. Higher genus B-model calculations have been first extended to compact multi-moduli Calabi–Yau manifolds in [29].

3.1. Quantum cohomology, modularity and the anomaly equations

The basic object, the instanton corrected triple intersections $C_{abc}(t^a)$ are due to special geometry all derivable from the holomorphic prepotential $F^{(0)}$, which also fixes the Kähler potential $K(t^a, \bar{t}^a)$ for the metric on the vector moduli space. At the point of maximal unipotent monodromy [13, 31]

$$(3.1) \qquad F^{(0)}(t^a) = (X^0)^2 \left[-\frac{t^3}{3!} + A_{ab}t^a t^b + c_a t^a + \frac{\chi(M)\zeta(3)}{2(2\pi i)^3} \right.$$

$$\left. + \sum_{\gamma \in H_2(M,\mathbb{Z})} n_\gamma^{(0)} \mathrm{Li}_3(\exp(2\pi i \gamma_a t^a)) \right],$$

where $t^3 = d_{abc}t^a t^b t^c$ with d_{abc} the triple intersection numbers, $c_a = \frac{1}{24} \int_M c_2 J_a$ and $\chi(M)$ is the Euler number of M. By J_a, $a = 1, \ldots, h^{1,1}(M)$, we denote harmonic $(1,1)$ forms, which form a basis of the Kähler cone and

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the complexified Kähler parameter $t^a = \int_{C_a}(i\omega + b)$, where C_a is a curve class in the Mori cone dual to the Kähler cone and b is the Neveu–Schwarz $(1,1)$-form b-field. The real coefficients A_{ab} are not completely fixed. They are unphysical in the sense that the Kähler potential $K(t^a, \bar{t}^a)$ and $C_{abc}(t^a)$ do not depend on them. The upper index (0) on the $F^{(0)}$ indicates the genus of the instanton contributions. The triple couplings receive only contributions from genus 0. At the point of maximal unipotent monodromy, the classical topological data provide us with the B-model period integrals $\Pi = (F_I, X^I) = \left(\int_{B^I}\Omega, \int_{A_I}\Omega\right)^T$ over an integral symplectic basis of 3-cycles of the mirror geometry $\tilde{M}$: (A_I, B^I), $I = 0, \ldots, h^{2,1}(\tilde{M})$. This is achieved by matching the $b_3(\tilde{M})$ solutions to the Picard–Fuchs equation with various powers of $\log(z_a) \sim t^a$, with the expected form of the A-model period vector

$$(3.2) \qquad \Pi = X^0 \begin{pmatrix} 2F^{(0)} - t^a\partial_{t^a}F^{(0)} \\ \partial_{t^a}F^{(0)} \\ 1 \\ t^a \end{pmatrix}$$

$$= X^0 \begin{pmatrix} \dfrac{t^3}{3!} + c_a t^a - i\chi(M)\dfrac{\zeta(3)}{(2\pi)^3} + 2f(p) - t^a\partial_{t^a}f(p) \\ -\dfrac{d_{abc}t^b t^c}{2} + A_{ab}t^b + c_a + \partial_{t^a}f(p) \\ 1 \\ t^a \end{pmatrix},$$

where the lower case indices run from $a = 1, \ldots, h^{1,1}(M)$. In the following we will set $X^0 = 1$.

One can define a generating function for the free energy in terms of a genus expansion in the coupling g_s

$$(3.3) \qquad F(g_s, t^a) = \sum_{g=0}^{\infty} g_s^{2g-2} F^{(g)}(t^a),$$

where the upper index $F^{(g)}(t^a)$ indicates as before the genus.

According to the split of the homology $H_2(M, \mathbb{Z})$ into the base and the fibre homology, we define $p^\beta = \prod_{k=1}^{b_2(B)}\exp(2\pi i\int_\beta i\omega + b) = \exp(2\pi i\sum_{i=1}^{b_2(B)}\beta_i t^i)$, where $\beta \in H_2(M, \mathbb{Z})$ lies in the image of the map $\sigma_* : H_2(B, \mathbb{Z}) \hookrightarrow H_2(M, \mathbb{Z})$ induced by the embedding $\sigma : B \hookrightarrow M$; and we define $q = \exp(2\pi i\int_e i\omega + b) = \exp(2\pi i\tau)$, where $e \in H_2(M, \mathbb{Z})$ is the curve representing

the fibre. Now we define the following objects

$$(3.4) \qquad F_\beta^{(g)}(\tau) = \text{Coeff}(F^{(g)}(t^a), p^\beta).$$

We have the following universal sectors:

$$(3.5)$$
$$F_0^{(0)}(\tau) = \left(\int_B c_1^2 \right) \frac{t^3}{3!} + \chi(M) \frac{\zeta(3)}{2(2\pi i)^3} - \chi(M) \sum_{n=1}^{\infty} \text{Li}_3(q^n),$$
$$(3.6)$$
$$F_0^{(1)}(\tau) = \left(\frac{\int_B c_2}{24} \right) \text{Li}_1(q), \quad F_0^{(g>1)}(\tau) = (-1)^g \frac{\chi(M)}{2} \frac{|B_{2g} B_{2g-2}|}{2g(2g-2)(2g-2)!}.$$

We note that it follows from the expression for $F_0^{(0)}(q)$ that

$$(3.7) \qquad C_{\tau\tau\tau} = \int_B c_1^2 + \frac{\chi(M)}{2} \zeta(-3) - \frac{\chi(M)}{2} \zeta(-3) E_4(\tau).$$

The $F_\beta^{(g)}(\tau)$ have distinguished modular properties. The $F_\beta^{(g)}(\tau)$ can be written in the following general form [32, 40]:

$$(3.8) \qquad F_\beta^{(g)}(\tau) = \left(\frac{q^{\frac{1}{24}}}{\eta(\tau)} \right)^{12 \sum_i a_i \beta^i} P_{2g+6\sum_i a_i\beta^i-2}(E_2(\tau), E_4(\tau), E_6(\tau)),$$

where $\eta(\tau)$ is the Dedekind eta function (C.1), and $P_{2g+6\sum_i a_i\beta^i-2}(E_2, E_4, E_6)$ are (quasi)-modular forms [36] of weight $2g + 6\sum_i a_i\beta^i - 2$, where $E_{2k}(\tau)$ are the Eisenstein series of weight $2k$ (C.2). The functions P_m are quasi-modular forms since, besides being a function of the true modular forms E_4 and E_6, they are also a function of E_2 that does not transform as a true modular form.

For the sectors $\beta > 0$, which describe non-trivial dependence on the Kähler class of the base, the E_2 dependence satisfies the following recursive condition

$$(3.9) \qquad \frac{\partial F_\beta^{(g)}(\tau)}{\partial E_2} = \frac{1}{24} \sum_{h=0}^{g} \sum_{\beta'+\beta''=\beta} (\beta' \cdot \beta'') F_{\beta'}^{(h)}(\tau) F_{\beta''}^{(g-h)}(\tau)$$
$$+ \frac{1}{24} \beta \cdot (\beta - K_B) F_\beta^{(g-1)}(\tau).$$

We derive the above relations in Section 4. For the other types of elliptic fibrations E_7, E_6, & D_5, the right-hand side is divided by $a = 2, 3, 4$,

respectively. Equation (3.9) generalizes the similar Equation (1.2) in [32], to arbitrary classes in the base and types of fibres. In particular, if one restricts to elliptic fibrations over the blow up of $\mathbb{P}^2$ and the Hirzebruch surface $\mathbb{F}_1$ the rational fibre class in the base (3.9) becomes the Equation of [32] counting curves of higher genus on the E_8, E_7, E_6, & D_5 del Pezzo surfaces. The form (3.8) and its relation to [32] has been observed in [40] for the Hirzebruch surface $\mathbb{F}_0$ as base. A derivation of Equation (3.9) is given in Section 4.

3.2. The B-model approach to elliptically fibred Calabi–Yau spaces

We continue the discussion with some B-model aspects of elliptically fibred Calabi–Yau manifolds. We assume some familiarity with the formalism developed in [30, 31] and concentrate on features relevant and common to the B-model geometry of elliptic fibrations and how they emerge from the topological data of the A-model discussed in Section 2.

The vectors $l^{(i)}$ are the generators of the Mori cone, i.e., the cone dual to the Kähler cone. As such they reflect classical properties of the Kähler moduli space and the classical intersection numbers, like the Euler number and the evaluation of $\int_M c_2 \omega_a$ on the basis of Kähler forms on the elliptic fibration.

On the other hand, the differential operators

$$(3.10) \qquad \left(\prod_{l_i^{(r)}>0} \partial_{a_i}^{l_i^{(r)}} - \prod_{l_i^{(r)}<0} \partial_{a_i}^{-l_i^{(r)}} \right) \tilde{\Pi} = 0,$$

annihilate the periods $\tilde{\Pi} = \frac{1}{a_0}\Pi$ of the mirror $\tilde{M}$. Here, the a_i are the coefficients of the monomials in the equation defining $\tilde{M}$. They are related to the natural large complex structure variables of $\tilde{M}$ by

$$(3.11) \qquad z_r = (-1)^{l_0^r} \prod_i a_i^{l_i^r}.$$

Note that Π is well defined on $\tilde{M}$, while $\tilde{\Pi}$ is not an invariant definition of periods on $\tilde{M}$. However by commuting out a_0^{-1} one can rewrite Equations (3.10) so that they annihilate Π. Further, they can be expressed in the independent complex variables z_r using the gauge condition $\theta_{a_i} = \sum_r l_i^k \theta_{z_r}$, where $\theta_x = x\frac{d}{dx}$ denotes the logarithmic derivative. Equations (3.10) reflect symmetries of the holomorphic $(3,0)$ form and every positive l in the Mori cone (3.10) leads to a differential operator annihilating Π. The operators

obtained in this way are contained in the left differential ideal annihilating Π, but they do not generate this ideal. There is however a factorization procedure, basically factoring polynomials $P(\theta)$ to the left, that leads in our examples to a finite set of generators that determines linear combinations of periods as their solutions. It is referred to as a complete set of Picard–Fuchs operators. In this way, properties of the instanton corrected moduli space of M, often called the quantum Kähler moduli space are intimately related to the $l^{(r)}$ and below we will relate some of it properties to the topology of M.

In particular, the Mori generator $l^{(e)}$ determines to a large extend the geometry of the elliptic fibre modulus. As one sees from (2.20) the mixing between the base and the fibre is encoded in the z row of $l^{(i)}$, $i = 1, \ldots, h^{1,1}(B)$ and $l^{(e)}$. Let us call this the z-component of $l^{(i)}$ and the corresponding variable a_z.

Following the procedure described above, one obtains after factorizing from $l^{(e)}$ a second-order generator Picard–Fuchs operator. For the fibrations types introduced before it is given by

$$(3.12) \qquad \mathcal{L}_{\mathrm{e}}^k = \theta_{\mathrm{e}}\left(\theta_{\mathrm{e}} - \sum_i a^i \theta_i\right) - \mathcal{D}^K,$$

where $k = E8, E7, E6, D5$ refers to the fibration type and $\mathcal{D}^K$ contains the dependence on the type

$$(3.13) \qquad \begin{aligned} \mathcal{D}^{E8} &= 12(6\theta_{\mathrm{e}} - 1)(6\theta_{\mathrm{e}} - 5)z_{\mathrm{e}}, \quad \mathcal{D}^{E7} = 4(4\theta_{\mathrm{e}} - 1)(4\theta_{\mathrm{e}} - 3)z_{\mathrm{e}}, \\ \mathcal{D}^{E6} &= 3(3\theta_{\mathrm{e}} - 1)(3\theta_{\mathrm{e}} - 2)z_{\mathrm{e}}, \quad \mathcal{D}^{D5} = 4(2\theta_{\mathrm{e}} - 1)^2 z_{\mathrm{e}}. \end{aligned}$$

Formally setting $\theta_i = 0$ corresponds to the large base limit. Then Equation (3.12) becomes the Picard–Fuchs operator, which annihilates the periods over the standard holomorphic differential on the corresponding family of elliptic curves.

In the limit of large fibre one gets as local model the total space of the canonical line bundle $\mathcal{O}(K_B) \to B$ over the Fano base B. Local mirror symmetry associates to such non-compact Calabi–Yau manifolds a genus one curve with a meromorphic 1-form λ that is the limit of the holomorphic $(3,0)$-form. The local Picard–Fuchs system $\mathcal{L}_i^B$ annihilating the periods Π_{loc} of λ can be obtained as a limit of the compact Picard–Fuchs system for $l^{(i)}$, $i = 1, \ldots, h^{1,1}(B)$ by formally setting $\theta_{\mathrm{e}} = 0$. It follows directly from (3.10), since the Mori generators of the base have vanishing first entry and commuting out a_0^{-1} becomes trivial. Differently then for the elliptic curve of the fibre, these Picard–Fuchs operators do not annihilate the periods over

holomorphic differential one form of the elliptic curve, which are $\frac{1}{a_z}\Pi_{\text{loc}}$. Given the local Picard–Fuchs system the dependence on θ_e can be restored by replacing θ_{a_z} by $\theta_e - \sum_i a_i\theta_i$ instead of $-\sum_i a_i\theta_i$. Since $l^{(i)}$ is negative θ_e appears in $\mathcal{L}_b^i$ only multiplied by at least one explicit z_i^b factor.

There are important conclusions that follow already from the general form of the Picard–Fuchs system. To see them it is convenient to rescale $x_e = c_k z_e$, where $c_{E8} = 432$, $c_{E7} = 64$, $c_{E6} = 27$, $c_{D5} = 16$. It is often useful to also rescale the z_i in a similar fashion and call them x_i.

The effect of this is that the symbols of the Picard–Fuchs system become the same for all fibre types. From this, we can conclude that for all fibre types the Yukawa-couplings and the discriminants are identical in the rescaled variables.

The second conclusion is that the Picard–Fuchs equation of the compact Calabi–Yau is invariant under the $\mathbb{Z}_2$ variable transformation

$$(3.14) \qquad x_e \to (1 - x_e), \quad x_i \to \left(-\frac{x_e}{1 - x_e}\right)^{a^i} x_i.$$

This means that there is always a $\mathbb{Z}_2$ involution acting on the moduli space parameterized by (x_e, x_i), which must be divided out to obtain the truly independent values of the parameters.

Another consequence of this statement is that the discriminants $\Delta_i(x_j)$ of the base Picard–Fuchs system determine the discriminant locus of the global system apart from the fibre related $\Delta(x_e)$ components. The former contains always a conifold component $\Delta_c(x_j)$ and only that one, if there are no points on the edges of the 2d polyhedron. Points on the edges correspond to $SU(2)$ or $SU(3)$ gauge symmetry enhancement discriminants, which contain only x_i variables dual to Kähler classes, whose $a^i = 0$. They are therefore invariant under (3.14). Moreover the lowest order term in the conifold discriminant is a constant and highest terms are weighted monomials of degree $\chi(B)$ with weights for the x_i a^i or 1 if $a^i = 0$. It follows by (3.14) that the transformed conifold discriminant $\Delta_c'(x_j) \sim (1 - x_e)^{\chi(B)} + \mathcal{O}(x_i)$.

3.2.1. Examples: elliptic fibrations over $\mathbb{P}^2$ and $\mathbb{F}_1$. Let us demonstrate the above general statements with a couple of examples. We discuss the $E8$ elliptic fibration with base $\mathbb{P}^2$ and with base $\mathbb{F}_1$. For the first example, the Mori vectors are given as

$$(3.15) \qquad \begin{aligned} l^{(e)} &= (-6, 3, 2, 1, 0, 0, 0), \\ l^{(1)} &= (0, 0, 0, -3, 1, 1, 1). \end{aligned}$$

Form this, we can derive the following set of Picard–Fuchs equations, where we denote $\theta_i = z_i \partial_{z_i}$.

$$(3.16) \quad \begin{aligned} \mathcal{L}_e &= \theta_e(\theta_e - 3\theta_2) - 12z_e(6\theta_e + 5)(6\theta_e + 1), \\ \mathcal{L}_1 &= \theta_2^3 - z_2(\theta_e - 3\theta_2)(\theta_e - 3\theta_2 - 1)(\theta_e - 3\theta_2 - 2). \end{aligned}$$

The Yukawa couplings for this example read as follows, where we use $z_e = \frac{x_e}{432}$, $z_1 = \frac{x_1}{27}$ and the discriminants $\Delta_e = 1 - 3x_e + 3x_e^2 - x_e^3 - x_e^3 x_1$ and $\Delta_1 = 1 + x_1$

$$(3.17) \quad \begin{aligned} C_{eee} &= \frac{9}{x_e^3 \Delta_e}, \\ C_{ee1} &= -\frac{3(-1 + x_e)}{x_e^2 x_1 \Delta_1}, \\ C_{e11} &= \frac{(-1 + x_e)^2}{x_e x_1^2 \Delta_e}, \\ C_{111} &= \frac{1 - 3x_e + 3x_e^2}{3x_1^2 \Delta_e \Delta_1}. \end{aligned}$$

The second example over $\mathbb{F}_1$ has the following three generators of the Mori cone

$$(3.18) \quad \begin{aligned} l^{(e)} &= (-6, \ 3, \ \ 2, \ \ 1, \ \ 0, \ \ 0, \ \ 0, \ \ 0), \\ l^{(1)} &= (\ \ 0, \ 0, \ \ 0, -1, -1, \ \ 0, \ \ 1, \ \ 1), \\ l^{(2)} &= (\ \ 0, \ 0, \ \ 0, -2, \ \ 1, \ \ 1, \ \ 0, \ \ 0), \end{aligned}$$

and gives rise to the following Picard–Fuchs equations:

$$(3.19) \quad \begin{aligned} \mathcal{L}_e &= \theta_e(\theta_e - 2\theta_2 - \theta_1) - 12z_e(6\theta_e + 5)(6\theta_e + 1), \\ \mathcal{L}_1 &= \theta_1^2 - z_1(\theta_1 - \theta_2)(2\theta_2 + \theta_1 - \theta_e), \\ \mathcal{L}_2 &= \theta_2(\theta_2 - \theta_1) - z_2(2\theta_2 + \theta_1 - \theta_e)(2\theta_2 + \theta_1 - \theta_e + 1). \end{aligned}$$

This example contains the rational elliptic surface, which we discuss in detail in Section 6. Furthermore, we focus on this example to give a proof of the holomorphic anomaly at genus zero by using mirror symmetry in Section 4.1.

3.3. Modular subgroup of monodromy group

The deeper origin of the appearance of modular forms is the monodromy group of the Calabi–Yau threefold. Candelas et al. [14] explain that in the large volume limit of $X_{18}[1, 1, 1, 6, 9]$, which corresponds to the elliptic
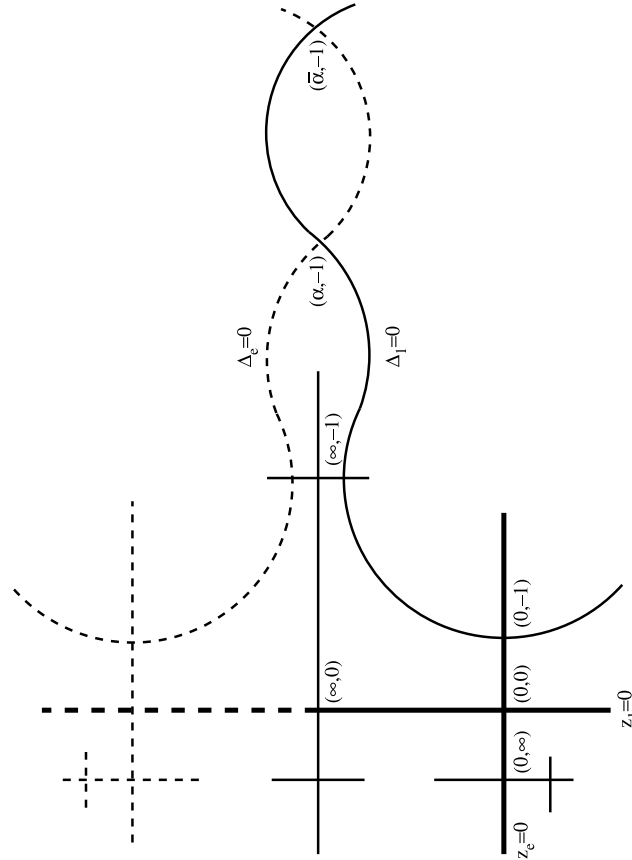
Figure 3: The moduli space for the elliptic Calabi–Yau fibration over $\mathbb{P}^2$ $X_{18}[1,1,1,6,9]$.

fibration over $\mathbb{P}^2$, the monodromy group reduces to an $SL(2,\mathbb{Z})$ monodromy group. This section recalls the appearance of this modular group and how it generalizes to other elliptic fibrations. The moduli space of $X_{18}[1,1,1,6,9]$ with the degeneration loci is portrayed in figure 3.

We continue by recalling the monodromy for the model in [14] adapted to our discussion. The fundamental solution is given by

$$(3.20) \qquad w_0(z_{\mathrm{e}}, z_1) = \sum_{m,n=0}^{\infty} \frac{(18n+6m)!}{(9n+3m)!\,(6n+2m)!\,(n!)^3\,m!}\, z_{\mathrm{e}}^{3n+m}\, z_1^m$$

$$= \sum_{k=0}^{\infty} \frac{(6k)!}{k!\,(2k)!\,(3k)!}\, z_{\mathrm{e}}^k\, U_k(z_1),$$

with

$$(3.21) \qquad U_\nu(z_1) = z_1^\nu \sum_{n=0}^\infty \frac{\nu!}{(n!)^3 \, \Gamma(\nu - 3n + 1)} \, z_1^{-3n}$$

$$= z_1^\nu \sum_{n=0}^\infty \frac{\Gamma(3n - \nu)}{\Gamma(-\nu) \, (n!)^3} \, z_1^{-3n},$$

which is a finite polynomial for positive integers ν, since $\Gamma(\nu - 3n + 1) = \infty$ for sufficiently large n. The translation to the parameters in [14] is $(z_e, z_1) = ((18\psi)^{-6}, -3\phi)$. The natural coordinates obtained from toric methods are $\tilde{z}_e = z_e z_1$ and $\tilde{z}_1 = z_1^{-3}$. Note that the second line (3.20) makes manifest the presence of the elliptic curve in the geometry. For this regime of the parameters one can easily find logarithmic solutions by taking derivatives to k and n [30]

$$(3.22) \qquad 2\pi i \, w_e^{(1)}(z_e, z_1) = \log(z_e z_1) \, w_0 + \cdots,$$

$$2\pi i \, w_1^{(1)}(z_e, z_1) = -3 \log(z_1) \, w_0 + \cdots.$$

The periods are defined by $\tau = w_e^{(1)}/w_0$ and $t_1 = w_1^{(1)}/w_0$ and $q = e^{2\pi i \tau}$, $q_1 = e^{2\pi i t_1}$.

The two monodromies which generate the modular group are

$$M_0 : \quad (z_e, z_1) \to (e^{2\pi i} z_e, z_1), \quad z_e \text{ small}, \quad z_1 \text{ large},$$

$$M_\infty : \quad (z_e, z_1) \to (e^{2\pi i} z_e, z_1), \quad z_e \text{ large}, \quad z_1 \text{ large}.$$

The monodromy around $z_e = 0$ follows directly from (3.22), it acts as

$$(3.23) \qquad \mathbf{M}_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

on $(w_e^{(1)}, w_0)^{\mathrm{T}}$. To determine the action on the periods of M_∞, we need to analytically continue w_0 and $w_1^{(1)}$ to large z_e. To this end, we write w_0 as a Barnes integral

$$(3.24) \qquad w_0(z_e, z_1) = \frac{1}{2\pi i} \int_C ds \, \frac{\Gamma(-s) \, \Gamma(6s + 1)}{\Gamma(2s + 1) \, \Gamma(3s + 1)} \, e^{\pi i s} \, z_e^s \, U_s(z_1),$$

where C is the vertical line from $-i\infty - \varepsilon$ to $i\infty - \varepsilon$. For small $|z_e|$ the contour can be deformed to the right giving back the expression in (3.20).

For large $|z_e|$ one instead obtains the expansion

$$(3.25) \quad w_0(z_e, z_1) = \frac{1}{6\pi^2} \sum_{r=1,5} \sin(\pi r/3) \sum_{k=0}^{\infty} a_r(k)\,(-z_e)^{-k-\frac{r}{6}}\, U_{-k-r/6}(z_1),$$

with

$$a_r(k) = (-1)^k \frac{\Gamma(k + r/6)\Gamma(2k + r/3)\Gamma(3k + r/2)}{\Gamma(6k + r)}.$$

The logarithmic solution $w_e^{(1)}$ is given similarly by

$$(3.26)$$

$$\begin{aligned} w_e^{(1)}(z_e, z_1) &= \frac{1}{2\pi i} \int_C ds\, \frac{\Gamma(-s)^2\,\Gamma(6s+1)\,\Gamma(s+1)}{\Gamma(2s+1)\,\Gamma(3s+1)}\, e^{2\pi i s}\, z_e^s\, U_s(z_1), \\ &= \frac{1}{6\pi^2 i} \sum_{r=1,5} e^{-\pi i r/6} \cos(\pi r/6) \sum_{k=0}^{\infty} a_r(k)\,(-z_e)^{-k-\frac{r}{6}}\, U_{-k-r/6}(z_1). \end{aligned}$$

To determine the action of M_∞, we define the basis $f_r(z_e, z_1) = \sum_{n=0}^{\infty} a_r(k)\,(-z_e)^{-k-\frac{r}{6}}\, U_{-k-r/6}(z_1)$ for $r = 1, 5$, and the matrix $\mathbf{A}$ that relates the two bases: $(w_e^{(1)}, w_1^{(1)})^T = \mathbf{A}\,(f_1, f_5)^T$. Clearly, M_∞ acts diagonally on the f_r: $\mathbf{T} = \mathrm{diag}(\alpha^{-1}, \alpha^{-5})$ with $\alpha = e^{2\pi i/6}$, which gives for $\mathbf{M}_\infty$

$$(3.27) \qquad \mathbf{M}_\infty = \mathbf{A}\mathbf{T}\mathbf{A}^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \in SL_2(\mathbb{Z}).$$

This gives for the monodromy around the conifold locus

$$(3.28) \qquad \mathbf{M}_1 = \mathbf{M}_0 \mathbf{M}_\infty^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

The generator $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ of $SL_2(\mathbb{Z})$ corresponds to $\mathbf{M}_0 \mathbf{M}_\infty^{-1}$.

The large volume limit is such that $u = q^{3/2}p \to 0$. We see that M_0 and M_∞ map small u to small u. The monodromies act on u by [14]

$$(3.29) \qquad M_0 u = -u, \quad M_\infty u = u$$

Thus we have established an action of $SL_2(\mathbb{Z})$ on the boundary of the moduli space.

The above analysis can be extended straightforwardly to the other types of fibrations using the expansions (B.2). The matrix $\mathbf{M}_0$ is for all fibre types the same. We find that $\mathbf{M}_\infty = \begin{pmatrix} 1-a & -1 \\ a & 1 \end{pmatrix} \in \Gamma_0(a)$ for $a = 2, 3$ and 4

corresponding to the fibre types E_7, E_6 and D_5. Note that $\mathbf{M}_\infty$ has order 4 and 3 for $a = 2$ and 3, respectively, while the order is infinite for $a = 4$. Generalization to other base surfaces B is also straightforward. In case of multiple 2-cycles in the base, it is natural to define parameters for each base class: $u_i = q^{a_i/2} p_i$, with $p_i = \exp(2\pi \mathrm{i}\, t^i)$, $i = 1, \ldots, b_2(B)$. This is precisely the change of parameters given by (2.12). These transform under the action of M_0 and M_∞ as:

$$(3.30) \qquad M_0 u_i = (-)^{a_i} u_i, \quad M_\infty u_i = u_i.$$

4. Derivation of the holomorphic anomaly equation

4.1. The elliptic fibration over $\mathbb{F}_1$

In this section, we discuss the holomorphic anomaly equation that arises for elliptically fibered Calabi–Yau threefolds. By using mirror symmetry the anomaly for genus 0 is proven for the base being $\mathbb{F}_1$. We further establish a derivation from the BCOV holomorphic anomaly equations.

We start by deriving the holomorphic anomaly equation at genus zero by adapting the proof which appeared in [32] for a similar geometry. For this, we determine the solution to the Picard–Fuchs equation and express the solution in terms of modular forms. After determining the mirror map, we find a recursive relation in the functions $c_m(x_\mathrm{e})$, which are related to the genus 0 topological amplitude. This can be used to prove the holomorphic anomaly equation for genus 0. Furthermore, the genus 0 topological amplitude can be expressed in terms of standard Eisenstein series of the elliptic parameter. We start by studying the Picard–Fuchs operator associated to the elliptic fiber $X_6[1, 2, 3]$ only. Denoting by $\theta_\mathrm{e} = z_\mathrm{e} \partial_{z_\mathrm{e}}$ the Picard–Fuchs operator can be written as

$$(4.1) \qquad \mathcal{L} = \theta_\mathrm{e}^2 - 12 z_\mathrm{e}(6\theta_\mathrm{e} + 5)(6\theta_\mathrm{e} + 1).$$

One can immediately write down two solutions as power series expansions around $z_\mathrm{e} = 0$. They are given by

$$(4.2) \qquad \phi(z_\mathrm{e}) = \sum_{n \geq 0} a_n z_\mathrm{e}^n, \quad \tilde{\phi}(z_\mathrm{e}) = \log(z_\mathrm{e})\phi(z_\mathrm{e}) + \sum_{n \geq 0} b_n z_\mathrm{e}^n,$$

with

$$(4.3) \qquad a_n = \frac{(6n)!}{(3n)!(2n)!n!},$$
$$b_n = a_n(6\psi(1+6n) - 3\psi(1+3n) - 2\psi(1+2n) - \psi(1+n)),$$

where $\psi(z)$ denotes the digamma function. The mirror map is thus given by

$$(4.4) \qquad 2\pi i\tau = \frac{\tilde{\phi}(z_e)}{\phi(z_e)}.$$

Using standard techniques from the Gauss–Schwarz theory for the Picard–Fuchs equation (cf. [45]) one observes

$$(4.5) \qquad j(\tau) = \frac{1}{z_e(1 - 432z_e)},$$

which can be inverted to yield

$$(4.6) \qquad z_e(\tau) = \frac{1}{864}(1 - \sqrt{1 - 1728/j(\tau)}) = q - 312q^2 + \mathcal{O}(q^3).$$

Further, the polynomial solution $\phi(x_e)$ can be expressed in terms of modular forms as

$$(4.7) \qquad \phi(z_e) = {}_2F_1\left(\frac{5}{6}, \frac{1}{6}, 1; 432z_e\right) = \sqrt[4]{E_4(\tau)},$$

from which one can conclude that

$$(4.8) \qquad \begin{aligned} E_4(\tau) &= \phi^4(z_e), \\ E_6(\tau) &= \phi^6(z_e)(1 - 864z_e), \\ \Delta(\tau) &= \phi^{12}(z_e)z_e(1 - 432z_e), \\ \frac{1}{2\pi i}\frac{dz_e}{d\tau} &= \phi^2(z_e)z_e(1 - 432z_e). \end{aligned}$$

Let us now examine the periods of the mirror geometry $\tilde{M}$ in the limit that the fibre f over the Hirzebruch surface $\mathbb{F}_1$ becomes small. At $z_e = z_1 = z_2$ we have a point of maximal unipotent monodromy. In particular, this implies that one has among the eight periods one holomorphic one, $w_0(z_e, z_1, z_2)$, one that starts with $\log(z_e)$, $w_e^{(1)}(z_e, z_1, z_2)$, and one that starts with $\log(z_1)$, $w_1^{(1)}(z_e, z_1, z_2)$. In the limit of small fibre f, i.e., $z_2 = 0$, one

obtains for these periods [32]:

$$
(4.9) \quad
\begin{aligned}
w_0(z_{\mathrm{e}}, z_1, 0) &= \phi(z_{\mathrm{e}}), \\
w_{\mathrm{e}}^{(1)}(z_{\mathrm{e}}, z_1, 0) &= \tilde{\phi}(z_{\mathrm{e}}), \\
w_1^{(1)}(z_{\mathrm{e}}, z_1, 0) &= \log(z_1)\phi(z_{\mathrm{e}}) + \xi(z_{\mathrm{e}}) + \sum_{m \geq 1}(\mathcal{L}_m \phi(z_{\mathrm{e}}))z_1^m,
\end{aligned}
$$

with

$$
(4.10) \quad \xi(z_{\mathrm{e}}) = \sum_{n \geq 0} a_n(\psi(1+n) - \psi(1))z_{\mathrm{e}}^n,
$$

and

$$
(4.11) \quad \mathcal{L}_m = \frac{(-)^m}{m(m!)}\prod_{k=1}^{m}(\theta_{z_{\mathrm{e}}} - k + 1).
$$

This can be obtained by applying the Frobenius method to derive the period integrals, see, e.g., [31]. The mirror map reads

$$
(4.12) \quad 2\pi\mathrm{i}\,t_j = \frac{w_j^{(1)}(z_{\mathrm{e}}, z_1, 0)}{w_0(z_{\mathrm{e}}, z_1, 0)}, \quad j = e, 1.
$$

Comparing this with our previous discussion about the Picard–Fuchs operator of the elliptic fibre we see that for $t_{\mathrm{e}} = \tau$ there is nothing left to discuss. Hence, let us study the mirror map associated to $t_1 = t$. We observe that by formally inverting, the inverse mirror map can be determined iteratively through the relation [32]

$$
(4.13) \quad z_1(q, p) = p\,\zeta\,e^{-\sum_{m \geq 1} c_m(z_{\mathrm{e}})z_1^m},
$$

where $\zeta = \mathrm{e}^{-\frac{\xi(z_{\mathrm{e}})}{\phi(z_{\mathrm{e}})}}$, $p = \exp(2\pi\mathrm{i}\,t)$ and

$$
(4.14) \quad c_m(z_{\mathrm{e}}) = \frac{\mathcal{L}_m \phi(z_{\mathrm{e}})}{\phi(z_{\mathrm{e}})}.
$$

Using Equation (4.8) $c_1(z_{\mathrm{e}})$ is given by

$$
(4.15) \quad
\begin{aligned}
c_1(z_{\mathrm{e}}) &= -\frac{1}{12}(f_1 - 2) - \frac{f_1}{12}\frac{E_2(\tau)}{\phi^2(z_{\mathrm{e}})} \\
&= -\frac{1}{\phi^6}\frac{f_1}{12}(E_2 E_4 - E_6),
\end{aligned}
$$

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where we introduced $f_1 = (1 - 432z_e)^{-1}$. In order to obtain the other $c_m(z_e)$ one uses

$$
(4.16) \qquad
\begin{aligned}
\theta_e f_1 &= f_1(f_1 - 1), \\
\theta_e \left(\frac{E_2}{\phi^2} \right) &= -\frac{1}{\phi^8} \frac{f_1}{12} \left(E_2^2 E_4 - 2E_2 E_6 + E_4^2 \right), \\
\theta_e \left(\frac{E_6}{\phi^6} \right) &= -\frac{1}{\phi^{12}} \frac{f_1}{12} \left(6E_4^3 - 6E_6^2 \right),
\end{aligned}
$$

and finds the following kind of structure. One can show inductively that

$$
(4.17) \qquad c_m(z_e) = \frac{1}{\phi^{6m}} \left(\frac{f_1}{12} \right)^m Q_{6m}(E_2, E_4, E_6),
$$

where Q_{6m} is a quasi-homogeneous polynomial of degree $6m$ and type $(2, 4, 6)$, i.e.,

$$
Q_{6m}(\lambda^2 z_e, \lambda^4 z_1, \lambda^6 z_2) = \lambda^{6m} Q_{6m}(z_e, z_1, z_2).
$$

Also by induction, it follows from (4.15) and (4.16) that Q_{6m} is linear in E_2. This allows us to write a second structure which is analogous to the one appearing in [32] and given by

$$
(4.18) \qquad c_m(z_e) = B_m \frac{E_2}{\phi^2} + D_m,
$$

where the coefficients B_m, D_m obey the following recursion relation:

$$
(4.19) \qquad
\begin{aligned}
B_{m+1} &= -\frac{m}{(m+1)^2} \left[(\theta_{z_e} - m)B_m + D_1 B_m - B_1 D_m \right], \\
D_{m+1} &= -\frac{m}{(m+1)^2} \left[(\theta_{z_e} - m)D_m - D_1 D_m + B_1 B_m \right],
\end{aligned}
$$

with $B_1 = -\frac{f_1}{12}$ and $D_1 = -\frac{1}{12}(f_1 - 2)$. A formal solution to the recursion relation (4.19) can be given by

$$
(4.20) \qquad
\begin{aligned}
B_m &= -\frac{f_m}{12}, \\
D_m &= \frac{1}{f_1} \left[\frac{(m+1)^2}{m} f_{m+1} + (\theta_e - m - \frac{1}{12}(f_1 - 2)) f_m \right],
\end{aligned}
$$

where we define f_m to be

$$
(4.21) \qquad f_m(z_e) = \tilde{\phi}(z_e) \mathcal{L}_m \phi(z_e) - \phi(z_e) \mathcal{L}_m \tilde{\phi}(z_e).
$$

Owing to the relations (4.16) we conclude, that the f_m as well as B_m and D_m are polynomials in f_1. Since f_1 is a rational function of z_e, it transforms well under modular transformations. Therefore, modular invariance is broken only by the E_2 term in c_m. We express this via the partial derivative of c_m

$$(4.22) \qquad \frac{\partial c_m(z_e)}{\partial E_2} = -\frac{1}{12} \frac{f_m(z_e)}{\phi^2(z_e)}.$$

In order to prove the holomorphic anomaly Equation (3.9), one first shows using the general results about the period integrals in [31] that the instanton part of the prepotential can be expressed by the functions $f_m(z_e)$. A tedious calculation reveals

$$(4.23) \qquad \frac{1}{2\pi i} \frac{\partial}{\partial t} F^{(0)}(\tau, t) = \sum_{m \geq 1} \frac{f_m(z_e)}{\phi^2(z_e)} z_1^m.$$

Using the inverse function theorem and Equations (4.22), (4.13) yields

$$(4.24) \qquad \frac{\partial z_1}{\partial E_2} = \frac{1}{12} \left(\frac{1}{2\pi i} \frac{\partial z_1}{\partial t} \right) \left(\frac{1}{2\pi i} \frac{\partial F^{(0)}}{\partial t} \right).$$

Now, we have

$$(4.25) \qquad \frac{\partial}{\partial E_2} \left(\frac{1}{2\pi i} \frac{\partial F^{(0)}}{\partial t} \right) = \frac{1}{12} \left(\frac{\partial^2 F^{(0)}}{\partial (2\pi i t)^2} \right) \left(\frac{1}{2\pi i} \frac{\partial F^{(0)}}{\partial t} \right),$$

which implies that up to a constant term in p one arrives at

$$(4.26) \qquad \frac{\partial F^{(0)}}{\partial E_2} = \frac{1}{24} \left(\frac{1}{2\pi i} \frac{\partial F^{(0)}}{\partial t} \right)^2.$$

By definition of $F_n^{(0)}$, Equation (3.4), we have $\frac{1}{2\pi i} \frac{\partial}{\partial t} F^{(0)}(\tau, t) = \sum_{m \geq 1} m \, F_m^{(0)} p^m$ and hence obtain by resummation

$$(4.27) \qquad \frac{\partial F_n^{(0)}}{\partial E_2} = \frac{1}{24} \sum_{s=1}^{n-1} s(n-s) F_s^{(0)} F_{n-s}^{(0)}.$$

This almost completes the derivation of (3.9). We still need to determine the explicit form of $F_n^{(0)}$. To achieve this we proceed inductively. Using (4.8),

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(4.23) and (4.13) one obtains

$$(4.28) \qquad F_1^{(0)} = \frac{\zeta f_1}{\phi^2} = q^{\frac{1}{2}} \frac{E_4}{\eta^{12}}.$$

Employing the structure (4.17) one can evaluate (4.23) and calculate that

$$
(4.29) \quad
\begin{aligned}
F_n^{(0)} &= \frac{\zeta^n f_1^n}{\phi^{6n}} P_{6n-2}(E_2, E_4, E_6), \\
&= \left(\frac{\zeta f_1}{\phi^2} \right)^n \frac{1}{\phi^{4n}} P_{6n-2}(E_2, E_4, E_6), \\
&= \frac{q^{\frac{n}{2}}}{\eta^{12n}} P_{6n-2}(E_2, E_4, E_6),
\end{aligned}
$$

where P_{6n-2} is of weight $6n - 2$ and is decomposed out of (parts of) Q_m's. This establishes a derivation of the holomorphic anomaly Equation (3.9) at genus zero for the elliptic fibration over Hirzebruch surface $\mathbb{F}_1$ with large fibre class. We collect some results for the other fibre types in Appendix B.

4.2. Derivation from BCOV

The last section provided a derivation of the anomaly Equation (3.9) for genus 0 from the mirror geometry. More fundamental is a derivation purely within the context of moduli spaces of maps from Riemann surfaces to a Calabi–Yau manifold. This is the approach taken by Bershadsky *et al.* [9] (BCOV) to derive holomorphic anomaly equations for genus g n-point correlation function with $2g - 2 + n > 0$. The correlation functions are given by covariant derivatives to the free energies $F^{(g)}$: $C_{a_1 a_2 \dots a_n}^{(g)} = D_{a_1} \dots D_{a_n} F^g$, with D_a covariant derivatives of for sections of the bundle $\mathcal{L}^{2-2g} \otimes \mathrm{Sym}^n T$, with T the tangent bundle of the coupling constant moduli space, and $\mathcal{L}$ a line bundle over this space whose Chern class corresponds to $G_{a\bar{b}}$.

The holomorphic anomaly equation reads for the n-point functions

$$
\begin{aligned}
\bar{\partial}_a C_{a_1 \dots a_n}^{(g)} =\ & \frac{1}{2} \bar{C}_{\bar{a}\bar{b}\bar{c}} e^{2K} G^{b\bar{b}} G^{c\bar{c}} C_{bca_1 \dots a_n}^{(g-1)} + \\
& + \frac{1}{2} \bar{C}_{\bar{a}\bar{b}\bar{c}} e^{2K} G^{b\bar{b}} G^{c\bar{c}} \sum_{r=0}^{g} \sum_{s=0}^{n} \frac{1}{s!(n-s)!}
\end{aligned}
$$

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$$\times \sum_{\sigma \in S_n} F^{(r)}_{ba_{\sigma(1)}...a_{\sigma(s)}} C^{(g-r)}_{ca_{\sigma(s+1)}...a_{\sigma(n)}}$$

$$(4.30) \qquad - (2g - 2 + n - 1) \sum_{s=1}^{n} G_{a\bar{a}_s} C^{(g)}_{a_1...a_{s-1}a_{s+1}...a_n}.$$

This equation can be summarized in terms of the generating function

$$(4.31) \qquad F(g_s, t^a; x^a) = \sum_{g=0}^{\infty} \sum_{n=0}^{\infty} g_s^{2g-2} \frac{1}{n!} C^{(g)}_{a_1...a_n} x^{a_1} \dots x^{a_n}$$
$$+ \left(\frac{\chi(M)}{24} - 1 \right) \log g_s.$$

Contrary to [9], we take the terms with $2g - 2 + n \leq 0$ as given by $D_1 \dots D_n F^{(g)}$ instead of setting them to 0. Equation (4.30) implies that F satisfies

$$(4.32) \qquad \bar{\partial}_a \exp(F) = \left[\frac{g_s^2}{2} \bar{F}_{\bar{a}\bar{b}\bar{c}} e^{2K} G^{b\bar{b}} G^{c\bar{c}} \frac{\partial^2}{\partial x^b \partial x^c} \right.$$
$$\left. - G_{\bar{a}b} x^b \left(g_s \frac{\partial}{\partial g_s} + x^c \frac{\partial}{\partial x^c} \right) \right] \exp(F).$$

To relate (4.32) to the holomorphic anomaly Equation (3.9) for this geometry, we split again the Kähler parameters t^a into the fibre parameter τ and base parameters t^i. Then, we write $F(g_s, \tau, t^i; x^i)$ as a Fourier expansion instead of a Taylor expansion in x^i:

$$(4.33) \qquad F(g_s, \tau, t^i; x^i) = \sum_{g=0}^{\infty} \sum_{\beta \in H_2(B,\mathbb{Z})} g_s^{2g-2} F^{(g)}_\beta(\tau) f^{(g)}_\beta(x^i, t^i) e^{2\pi i \beta \cdot x} p^\beta$$
$$+ \left(\frac{\chi(M)}{24} - 1 \right) \log g_s,$$

with as before $p^\beta = \exp(2\pi i t^i \beta_i)$, $q = \exp(2\pi i \tau)$. Moreover, the functions $f^{(g)}_\beta(x^i, t^i)$ satisfy $D_i F|_{x=0} = \partial_{x_i} F|_{x=0}$ and $f^{(g)}_\beta(0, t^i) = 1$. In the large volume limit, the covariant derivatives D_i become flat derivatives $\frac{\partial}{\partial t^i}$ and thus $f^g_\beta(x^i, t^i) \to 1$. Therefore, to deduce (3.9) from (4.32) we can set $x^i = 0$ and replace the $\frac{\partial}{\partial x^i}$ by $\frac{\partial}{\partial t^i}$.

Equation (3.9) follows now by considering $\frac{1}{2\pi i}\partial_{\bar{\tau}}\exp(F)$ on the right-hand side of (4.32). As discussed earlier, all $\bar{\tau}$ dependence arises from completing the weight 2 Eisenstein series $\widehat{E}_2(\tau) = E_2(\tau) - \frac{3}{\pi\tau_2}$, which gives:

$$(4.34) \qquad \frac{\partial}{\partial E_2} = \frac{4\pi^2\tau_2^2}{3}\frac{\partial}{2\pi i\partial\bar{\tau}}.$$

We first discuss how the right-hand side of (3.9) can be derived from Equation (4.32) for the geometry $X_{18}[1,1,1,6,9]$. We use the basis (2.7) and choose as parameters the "base" parameter $t = B + i\mathcal{J}$ and the fibre parameter $\tau = \tau_1 + i\tau_2$. We are interested in the large volume limit $\tau \to i\infty$, $t \to i\infty$ in such a way that $\mathcal{J} \gg \tau_2$. In this limit, the Kähler potential is well approximated by the polynomial form

$$(4.35) \qquad K \approx -\log(\tfrac{4}{3}\tilde{d}_{abc}\mathcal{J}^a\mathcal{J}^b\mathcal{J}^c) = -\log(\tfrac{4}{3}(\alpha\tau_2^3 + 3\tau_2\mathcal{J}^2)),$$

with $\alpha = \tilde{\mathcal{K}}_e^3$ (2.7). This gives for the metric

$$\begin{pmatrix} G_{\tau\bar{\tau}} & G_{t\bar{\tau}} \\ G_{\tau\bar{t}} & G_{t\bar{t}} \end{pmatrix} \approx \begin{pmatrix} \dfrac{1}{4\tau_2^2} & \dfrac{\alpha\tau_2}{3\mathcal{J}^3} \\ \dfrac{\alpha\tau_2}{3\mathcal{J}^3} & \dfrac{1}{2\mathcal{J}^2} \end{pmatrix},$$

which gives for the matrix $e^K G^{i\bar{j}}$

$$(4.36) \qquad e^K G^{-1} \approx \begin{pmatrix} \dfrac{1}{\mathcal{J}^2} & -\dfrac{2\alpha\tau_2^2}{3\mathcal{J}^3} \\ -\dfrac{\alpha\tau_2^2}{3\mathcal{J}^3} & \dfrac{1}{2\tau_2} \end{pmatrix}.$$

Thus in the limit $\mathcal{J} \to \infty$, one finds that only $e^K G^{t\bar{t}} \approx \frac{1}{2\tau_2}$ does not vanish. Therefore, $\bar{C}_{\bar{\tau}\bar{b}\bar{c}}e^{2K}G^{b\bar{b}}G^{c\bar{c}}\frac{\partial^2}{\partial x^b\partial x^c} \approx \frac{1}{4\pi^2}\frac{1}{4\tau_2^2}\frac{\partial^2}{\partial x^t\partial x^t}$.[4] Using (4.34), this shows that (4.32) reduces to:

$$(4.37) \qquad \frac{\partial}{\partial E_2}\exp(F) = \frac{g_s^2}{24}\left(p\frac{\partial}{\partial p}\right)^2\exp(F).$$

Expansion of both sides in p and taking the p^n coefficient gives a holomorphic anomaly Equation as (3.9) for $g = 0$. It also gives the correct (3.9) for $g > 0$ except for the appearance of K_B. We belief that a more thorough

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[4]The factor $\frac{1}{4\pi^2}$ appears due to a factor $-2\pi i$ between the moduli in [9] and ours.

analysis of the covariant derivatives will explain this term. Assuming the form $f_\beta^{(g)}(x,t) \to 1 + x^2 \beta \cdot K_B + \cdots$ would give the shift in (3.9).

The derivation is very similar for the other types of fibres discussed in Section 2. The right hand side of Equation (4.37) is simply divided by a, in agreement with [32].

5. T-duality on the fibre

In this section, we discuss properties of T-duality on the elliptic fibre in order to relate the result from our period calculations to $D4$-brane counting. One can perform two T-dualities around the circles of the elliptic fibre. Due to the freedom in choosing the circles, this leads to an $SL(2,\mathbb{Z})$ (or a congruence subgroup) group of dualities mapping IIA branes to IIA branes. This duality group is equal to the modular subgroup of the monodromy group which leave invariant the $F_\beta^{(g)}$'s discussed in Section 3.3.

Let $D2_{e/\beta}$ be a $D2$-brane wrapped either on the elliptic fibre e or on a class β in the base. Moreover, we denote by $D4_e$ a D4-brane wrapped around the base and by $D4_\beta$ a D4-brane wrapped around the cycle β in the base and the elliptic fibre e. The double T-duality on both circles of the elliptic fibre transforms pairs of D-brane charges heuristically in the following way:

(5.1)
$$\begin{pmatrix} D6 \\ D4_e \end{pmatrix} = \gamma \begin{pmatrix} \tilde{D6} \\ \tilde{D4}_e \end{pmatrix},$$
$$\begin{pmatrix} D4_\beta \\ D2_\beta \end{pmatrix} = \gamma \begin{pmatrix} \tilde{D4}_\beta \\ \tilde{D2}_\beta \end{pmatrix},$$
$$\begin{pmatrix} D2_e \\ D0 \end{pmatrix} = \gamma \begin{pmatrix} \tilde{D2}_e \\ \tilde{D0} \end{pmatrix},$$

with γ in $SL_2(\mathbb{Z})$ or a congruence subgroup. See for more a more formal treatment of T-duality on Calabi–Yau manifolds [3,4]. In the following sections, we will always consider the element $\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. T-duality is not valid for every choice of the Kähler parameter. One way to see this is that the BPS invariants of $D2$ branes do not depend on the choice of the Kähler moduli but those of $D4$ and $D6$ branes do through wall-crossing. The choice where the two are related by T-duality is sufficiently close to the class of the elliptic fibre, this is called a suitable polarization in the literature [34]. Sufficiently close means that no wall is crossed between the fibre class and the suitable polarization.

The equality of invariants of $D0$ branes and $D2$ branes wrapping the fibre can be easily verified. The BPS invariant of an arbitrary number $n > 0$

of $D0$ branes is known to be equal to the Euler number of the Calabi–Yau [42, Section 6.5]:

$$(5.2) \qquad \Omega(\,(0,0,0,n), M) = -\chi(M).$$

One can verify that these equal the BPS invariants of n $D2$ branes wrapping the E_8 elliptic fibre of M. See for example the tables with GW-invariants for $X_{18}(9,6,1,1,1)$ and $X_{24}(12,8,2,1,1)$ in Appendix B of [30]. If the modular group is a congruence subgroup of level k then only the BPS invariant corresponding to $n = 0 \mod k$ $D2$ branes wrapping the fibre equals (5.2).

Our interest is in the $D4$-branes which can be obtained from $D2_\gamma$ with $\gamma = \beta + ne$ by T-duality. These $D4$-branes wrap classes in the base times the fibre, and have $D0$ brane charge n. $D4$-branes on Calabi–Yau manifolds correspond to black holes in four-dimensional space-time and are well studied [47]; in particular, M-theory relates the degrees of freedom of $D4$-brane black holes to those of a $\mathcal{N} = (4,0)$ conformal field theory (CFT) with left and right central charges:

$$(5.3) \qquad c_L = P^3 + \tfrac{1}{2}c_2 \cdot P, \quad c_R = P^3 + c_2 \cdot P,$$

with P the 4-cycle wrapped by the $D4$-brane, and c_2 the second Chern class of the Calabi–Yau threefold. Typically, the number of 2-cycles in the $D4$-brane is larger than the number of 2-cycles in the Calabi–Yau threefold.

In the following, we will use the notation of [49]. The homology class P naturally gives rise to a quadratic form $D_{ab} = d_{abc}P^c$ which has signature $(1, b_2(M) - 1)$. Let Λ be the lattice $\mathbb{Z}^{b_2}$ with quadratic form D_{ab}. The dual lattice with quadratic form D_{ab}^{-1} is denoted by Λ^*. The Kähler modulus $\mathcal{J} = \int_X \omega$ gives the natural projection of a vector $\mathbf{k} \in \Lambda$ to the positive-definite subspace of $\Lambda \otimes \mathbb{R}$

$$(5.4) \qquad \mathbf{k}_+ = \frac{\mathbf{k} \cdot \mathcal{J}}{\mathcal{J}^2}\, \mathcal{J},$$

with $\mathcal{J}^2 = d_{abc}P^a \mathcal{J}^b \mathcal{J}^c$.

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The supergravity partition function of $D4$-branes generically takes the following form [10, 19, 49]:

$$
(5.5) \qquad \mathcal{Z}_P(C_{\mathrm{R}}, \tau; t) = \sum_{Q_0, Q} \bar{\Omega}(\Gamma; t) \, (-1)^{P \cdot Q}
$$
$$
\times e \left(-\bar{\tau} \hat{Q}_{\bar{0}} + \tau (Q - B)_+^2 / 2 + \bar{\tau}(Q - B)_-^2 / 2 \right.
$$
$$
\left. + C_{\mathrm{R}} \cdot (Q - B/2) \right).
$$

where P, Q and Q_0 correspond to the homology classes of respectively the $D4$-, $D2$- and $D0$-branes. The parameter $C_{\mathrm{R}} \in \Lambda \otimes \mathbb{R}$ is related to the Ramond–Ramond 3-form that couples to $D2$-branes: $\mathcal{C}_{\mathrm{R},3} = C_{\mathrm{R}}^i J_i \wedge dt$ in terms of the basis J_i of (1,1)-forms on M introduced below Equation (2.8). Furthermore, τ is the modular parameter valued in the upper-half plane $\mathcal{H}$, and $t = B + \mathrm{i} \mathcal{J} \in \Lambda \otimes \mathbb{C}$ the complexified Kähler modulus. On the right hand side, the coefficients $\bar{\Omega}(\Gamma; t)$ are rational invariants and related to the integer invariants $\Omega(\Gamma; t)$ by the multi-cover formula

$$
(5.6) \qquad \bar{\Omega}(\Gamma; t) = \sum_{m | \Gamma, m \geq 1} \frac{\Omega(\Gamma/m; t)}{m^2}.
$$

Note that the multi-cover contributions come here with a factor m^{-2}, whereas the generating function $F^{(0)}(t^a)$ (3.1) of genus 0 Gromov–Witten invariants $\bar{n}_\gamma^{(0)}$ weights the Gopakumar–Vafa invariants $n_{\gamma/m,}^{(0)}$ by m^{-3}: $\bar{n}_\gamma^{(0)} = \sum_{m | \gamma, m \geq 1} n_{\gamma/m}^{(0)} / m^3$. The invariants $\Omega(\Gamma; t)$ are related to the Euler number of the appropriate moduli space $\mathcal{M}_t(\Gamma)$ by

$$
(5.7) \qquad \Omega(\Gamma; t) = (-1)^{\dim_{\mathbb{C}} \mathcal{M}_t(\Gamma)} \chi(\mathcal{M}_t(\Gamma)).
$$

Furthermore, the quantity $\hat{Q}_{\bar{0}}$ is a shorthand notation for $-Q_0 + \frac{1}{2} Q^2$. This quantity determines the horizon area of the black hole $A_{\mathrm{bh}} = 4\pi \sqrt{\frac{2}{3} c_R \hat{Q}_{\bar{0}}}$, which is the classical manifestation of the entropy of the black hole.

Physical arguments imply that $\mathcal{Z}_P(C_{\mathrm{R}}, \tau; t)$ transforms as a modular form of weight $(\frac{1}{2}, -\frac{3}{2})$. If the B-field decouples from the stability condition, $\mathcal{Z}_P(C_{\mathrm{R}}, \tau; t)$ allows a theta function decomposition:

$$
(5.8) \qquad \mathcal{Z}_P(C_{\mathrm{R}}, \tau; t) = \sum_\mu \overline{h_{P,\mu}(\tau)} \, \Theta_{P,\mu}(\tau, C_{\mathrm{R}}, B),
$$

and $h_{P,\mu}(\tau)$ is a vector valued modular form of weight $-1 - b_2(M)/2$ given by

$$(5.9) \qquad h_{P,\mu}(\tau) = \sum_{Q_0} \Omega_P(\hat{Q}_{\bar{0}})\, q^{\hat{Q}_{\bar{0}}}$$

with $\hat{Q}_{\bar{0}} = -Q_0 + \frac{1}{2}\mu^2$. This symmetry is also present in the Maldacena-Strominger-Witten (MSW) conformal field theory which arises in the near horizon geometry of a single centre $D4$-brane black hole [10]. We refer to [49] for a discussion of the relation between the supergravity partition and the CFT partition function. In terms of the central charges $c_{L/R}$ of this conformal field theory, $h_{P,\mu}(\tau)$ typically takes the form

$$(5.10) \qquad h_{P,\mu}(\tau) = \frac{f_{P,\mu}(\tau)}{\eta(\tau)^{c_R}}$$

with $f_{P,\mu}(\tau)$ a vector-valued modular form of weight $-1 - b_2(M)/2 + c_R/2$.

We continue by specializing to the elliptic Calabi–Yau fibrations. We denote the four-cycle obtained by T-duality from a curve β_i in the base by β^i or simply β. Whether β denotes a two- or four-cycle should be clear from the context. One derives from the triple intersection numbers (2.13) that these four-cycles β^i, have a vanishing triple intersection number $(\beta)^3 = 0$. These $D4$-branes correspond therefore not to large black holes, but to "small" black holes. This means that only after addition of higher derivative corrections to the supergravity effective action one finds a non-vanishing area of the horizon. It is very intriguing that we can obtain detailed knowledge about the spectrum of these black holes using mirror symmetry.

Let $\iota : \beta \to M$ be the embedding of the divisor β into M, which provides a pull back map on the second cohomology $\iota^* : H^2(M, \mathbb{Z}) \to H^2(\beta, \mathbb{Z})$. Since the divisors β are not positive, this map is not injective. One deduces from (2.13) that the rank of the quadratic form $D_{ab} = d_{abj}\beta^j$ is 2 independent of M. Therefore, the image of the map ι^* is two-dimensional, and we find consequently that the modular weight of $h_{\beta,\mu}(\tau)$ is equal to -2. More details of the modular properties of $h_{\beta,\mu}(\tau)$ can be derived. If β is primitive, i.e., $\beta/n \notin H_4(M, \mathbb{Z})$ for $n \geq 2$, one can find another divisor $\tilde{\beta}$ such that $\mathcal{K} \cdot \beta \cdot \tilde{\beta} = 1$. The quadratic form D_{ab} then takes the form:

$$(5.11) \qquad \begin{pmatrix} 0 & 1 \\ 1 & \mathcal{K} \cdot \tilde{\beta}^2 \end{pmatrix}.$$

With this information one can precisely determine the modular transformation properties of the vector-valued modular form $h_{\beta,\mu}(\tau)$. See Equation (4.17) of [48]. The elements of the modular vector are modular forms of a congruence subgroup $\Gamma(m)$.

The genus 0 free energies $F_\beta^{(0)}$ (3.8) give a prediction $h_{\beta,\mathbf{0}}^{\mathrm{pr}}(\tau)$ for $h_{\beta,\mathbf{0}}(\tau)$. Correcting the power by which non-primitive charges are weighted, we find for $h_{\beta,\mathbf{0}}^{\mathrm{pr}}(\tau)$ in terms of $F_\beta^{(0)}(\tau)$:

$$(5.12) \qquad h_{\beta,\mathbf{0}}^{\mathrm{pr}}(\tau) = \sum_{m|\beta} \frac{1}{m^2} \left(\sum_{n|m} \frac{\mu(n)}{n} \right) F_{\beta/m}^{(0)}(m\tau),$$

where $\mu(m)$ is the Möbius function, which is defined for $m \geq 1$ by $\mu(m) = (-1)^\ell$ if m is the product of ℓ distinct primes ≥ 2 and $\mu(m) = 0$ otherwise). The modular properties of $h_{P,\mathbf{0}}(\tau)$ defined this way are precisely consistent with the structure found for the genus 0 amplitudes obtained from the mirror periods, see Equation (3.8) combined with Equation (2.8). As explained in Section 3, the free energy $F_\beta^{(0)}$ is a modular form of weight -2, in agreement with the weight of $h_{\beta,\mu}(\tau)$. Due to contributions to $h_{\beta,\mathbf{0}}^{\mathrm{pr}}(\tau)$ (5.12) of $F_\beta^{(0)}(m\tau)$ with $m > 1$, $h_{\beta,\mathbf{0}}^{\mathrm{pr}}(\tau)$ is in general an element of the congruence subgroup $\Gamma(m)$ in agreement with the analysis of the modular properties of (5.5). Generically, one cannot determine uniquely from $h_{\beta,\mathbf{0}}^{\mathrm{pr}}(\tau)$ the other elements of the modular vector, but in simple examples this can be done.

Besides verifying that the modular properties of $h_{\beta,\mathbf{0}}^{\mathrm{pr}}(\tau)$ agree with $h_{\beta,\mathbf{0}}(\tau)$, it is also possible to verify the agreement for the first few coefficients, for small $D0$ and $D4$-brane charge, the BPS invariants can be computed either from the microscopic D-brane perspective or the supergravity context [5,11,16,19,22,23,25,52,54]. For example from the microscopic point of view, the moduli space of a single $D4$-brane is given by projective space $\mathbb{P}^n$. Using index theorems one can compute that $n = \frac{1}{6}P^3 + \frac{1}{12}c_2 \cdot P - 1$ [47]. Therefore, the first coefficient of $h_{P,\mathbf{0}}(\tau)$ is expected to be

$$(5.13) \qquad \Omega_P(-\tfrac{1}{24}c_R) = \tfrac{1}{6}P^3 + \tfrac{1}{12}c_2 \cdot P.$$

The second coefficient corresponds to adding a unit of (anti) $D0$-brane charge. Now the linear system for the divisor of the $D4$-brane is constrained to pass through the $D0$-brane. This gives with Equation (5.2) [22]

$$(5.14) \qquad \Omega_P(1 - \tfrac{1}{24}c_R) \cong \chi(M)(\tfrac{1}{12}c_2 \cdot P - 1).$$

Here we have written a "$\cong$" instead of "$=$" since if $1 - \frac{1}{24}c_R \geq 0$ the formula for the horizon area (given below Equation (5.7)) gives a positive value, such that the BPS states might correspond to black holes with intrinsically gravitational degrees of freedom, which are less well understood.

Continuing with two units of $\overline{D0}$ charge, one finds

$$(5.15) \qquad \Omega_P(2 - \tfrac{1}{24}c_R) \cong \tfrac{1}{2}\chi(M)(\chi(M) + 5)(\tfrac{1}{12}c_2 \cdot P - 2).$$

One can in principle continue along these lines, which becomes increasingly elaborate for three reasons. First effects of $D2$-branes become important, second single centre black holes contribute for $\hat{Q}_{\bar{0}} > 0$ and third the index might depend on the background moduli t.

We now briefly explain which bound states appear in the supergravity picture for small $D0/4$-brane charge. The first terms in the q-expansion cannot correspond to single centre black holes since $\hat{Q}_{\bar{0}} < 0$. The first terms correspond to bound states of $D6$ and $\overline{D6}$-branes [19]. If P is an irreducible cycle (i.e., it cannot be written as $P = P_1 + P_2$ with P_1 and P_2 effective classes) then the charges Γ_1 and Γ_2 of the constituents are

$$(5.16) \qquad \Gamma_1 = (1, P, \tfrac{1}{2}P^2 - \tfrac{c_2}{24}, \tfrac{1}{6}P^3 + \tfrac{c_2 \cdot P}{24}), \quad \Gamma_2 = (-1, 0, \tfrac{c_2}{24}, 0),$$

The index of a 2-centre bound state is given by:

$$\langle \Gamma_1, \Gamma_2 \rangle \, \Omega(\Gamma_1) \, \Omega(\Gamma_2),$$

with $\langle \Gamma_1, \Gamma_2 \rangle = -P_1^0 Q_{0,2} + P_1 \cdot Q_2 - P_2 \cdot Q^1 + P_2^0 Q_{0,1}$ the symplectic inner product. Since the constituents are single D6-branes with a non-zero flux, their index is $\Omega(\Gamma_i; t) = 1$. Therefore, $\Omega_P(-\frac{1}{24}c_R) = \langle \Gamma_1, \Gamma_2 \rangle = \frac{1}{6}P^3 + \frac{1}{12}c_2 \cdot P$, which reproduces Equation (5.13).

One can continue in a similar fashion with adding other constituents to compute indices with higher charge. For example, BPS states with charge $\Gamma = (0, 2P, 0, \frac{1}{3}P^3 + \frac{c_2 \cdot P}{12})$ corresponds to Γ_1 as in (5.16) and

$$(5.17) \qquad \Gamma_2 = (-1, P, -\tfrac{1}{2}P^2 + \tfrac{c_2}{24}, \tfrac{1}{6}P^3 + \tfrac{c_2 \cdot P}{24}).$$

One obtains then $\Omega_{2P}(-\frac{1}{24}c_R) = \frac{8}{6}P^3 + \frac{2}{12}c_2 \cdot P$. Similarly, one could also add $\overline{D0}$ charges, and find the right hand sides of Equations (5.13) to (5.15) with P replaced by $2P$.

Example: $X_{18}[1, 1, 1, 6, 9]$

We now consider the periods for $X_{18}[1, 1, 1, 6, 9]$, i.e., a elliptic fibration over $\mathbb{P}^2$ and compare with the above discussion. This Calabi–Yau has a two-dimensional Kähler cone, and lends it self well to studies of $D4$-branes. We consider D4-branes wrapping the divisor whose Poincaré dual is the hyperplane class H of the base surface $\mathbb{P}^2$. The number of wrappings is denoted by r.

As explained in Section 3, the genus 0 Gromov–Witten invariants are well-studied [14, 30]. Using (5.12), the $F_\beta^{(0)}(\tau)$ provide the following predictions for $h_{nH,\mathbf{0}}(\tau)$:

$$
\begin{aligned}
h_{H,\mathbf{0}}^{\mathrm{pr}}(\tau) &= \frac{31E_4^4 + 113E_4E_6^2}{48\eta(\tau)^{36}} \\
&= q^{-3/2}(3 - 1080\,q + 143770\,q^2 + 204071184\,q^3 + \cdots), \\
h_{2H,\mathbf{0}}^{\mathrm{pr}}(\tau) &= \frac{-196319E_4E_6^5 - 755906E_4^4E_6^3 - 208991E_4^7E_6}{221184\,\eta(\tau)^{72}} \\
&\quad - \frac{1}{24}E_2 h_{H,0}^{\mathrm{pr}}(\tau)^2 + \frac{1}{8}h_{H,0}^{\mathrm{pr}}(2\tau) \\
&= q^{-3}(-6 + 2700\,q - 574560\,q^2 + \cdots) + \frac{1}{4}h_{H,0}^{\mathrm{pr}}(2\tau), \\
h_{3H,\mathbf{0}}^{\mathrm{pr}}(\tau) &= q^{-9/2}(27 - 17280\,q + 5051970\,q^2 + \cdots) + \frac{1}{9}h_{H,0}^{\mathrm{pr}}(3\tau).
\end{aligned}
$$

We want to compare this to the expressions derived above from the point of view of $D4$-branes. For $r = 1$, we have

$$
\Omega(\Gamma; \mathcal{J}) = \frac{1}{12}c_2 \cdot H = 3, \tag{5.18}
$$

in agreement with the first coefficient of $h_H(\tau)$. The second term in the q-expansion corresponds to

$$
\Omega(1, \tfrac{1}{2}H, -1) = \chi(M)(\tfrac{1}{12}c_2 \cdot P - 1) = 1080, \tag{5.19}
$$

which is also in agreement with the periods. For two $\bar{D}0$ branes, we find a small discrepancy, one finds:

$$
\tfrac{1}{2}(\tfrac{1}{12}c_2 \cdot P - 2)\chi(M)\,(\chi(M) + 5) = 144450. \tag{5.20}
$$

This is an excess of $1080 = -2\chi(M)$ states compared to the 3rd coefficient in $h_1(\tau)$. This number is very suggestive of a bound state picture, possibly

involving $D2$ branes. Since $\hat{Q}_{\bar{0}} > 0$ one could argue that these states are due to intrinsic gravitational degrees of freedom, but it seems actually a rather generic feature if we consider other elliptic fibrations (e.g., over $\mathbb{F}_1$).

For $r = 2$, also the first two coefficients of the spectrum match with the $D4$-brane indices, and the 3rd differs by $-6\chi(M)$. Something non-trivial happens for $r = 3$. We leave an interpretation of these indices from multi-centre solutions for a future publication, and continue with the example of the local elliptic surface [57].

6. BPS invariants of the rational elliptic surface

This section continues with the comparison of the $D4$- and $D2$-brane spectra for the E_8 elliptic fibration over the Hirzebruch surface $\mathbb{F}_1$, which was first addressed by Refs. [57, 63]. Let $\sigma : \mathbb{F}_1 \to M$ be the embedding of $\mathbb{F}_1$ into the Calabi–Yau threefold. The surface $\mathbb{F}_1$ is itself a fibration $\pi : \mathbb{F}_1 \to C \cong \mathbb{P}^1$ with fibre $f \cong \mathbb{P}^1$, with intersections $C^2 = -1$, $C \cdot f = 1$ and $f^2 = 0$. The Kähler cone of M is spanned by the elliptic fibre class J_1, and the classes $J_2 = \sigma_*(C + f)$ and $J_3 = \sigma_*(f)$. The Calabi–Yau intersections and Chern classes are given by (2.17).

A few predictions from the periods for the $D4$-brane partition functions are

$$(6.1) \quad h_{C,\mathbf{0}}^{\mathrm{pr}}(\tau) = \frac{E_4(\tau)}{\eta(\tau)^{12}} = q^{-1/2}(1 + 252q + \cdots),$$

$$h_{f,\mathbf{0}}^{\mathrm{pr}}(\tau) = \frac{2E_4(\tau)E_6(\tau)}{\eta(\tau)^{24}}$$

$$= -2q^{-1} + 480 + 282888q + \cdots,$$

$$h_{2C,\mathbf{0}}^{\mathrm{pr}}(\tau) = \frac{E_2(\tau)E_4(\tau)^2 + 2E_4(\tau)E_6(\tau)}{24\eta(\tau)^{24}} + \frac{1}{8}h_{C,0}^{\mathrm{pr}}(2\tau)$$

$$= -9252\,q - 673760\,q^2 + \cdots + \frac{1}{4}h_C(2\tau),$$

$$h_{3C,\mathbf{0}}^{\mathrm{pr}}(\tau) = \frac{54E_2^2E_4^3 + 216E_2E_4^2E_6 + 109E_4^4 + 197E_4E_6^2}{15552\eta^{36}} + \frac{2}{27}h_{C,0}^{\mathrm{pr}}(3\tau)$$

$$= 848628\,q^{3/2} + 115243155\,q^{5/2} + \cdots + \frac{1}{9}h_{C,0}^{\mathrm{pr}}(3\tau).$$

Since $c_2(M) \cdot f = 24$, explicit expressions in terms of modular forms for the divisors $h_{C+nf}(\tau)$ become rather lengthy. Interestingly, one finds that for this class the first coefficients (checked up to $n = 12$), are given by $1 + 2n$ in agreement with Equation (5.13). Moreover, the second and third coefficients

are respectively given by $\chi(M)(\frac{1}{12}c_2 \cdot P - 1)$ and $\frac{1}{2}\chi(\chi + 9)(\frac{1}{12}c_2 \cdot P - 2)$ as long as the corresponding $\hat{Q}_{\bar{0}} < 0$.

Another interesting class are r $D4$ branes wrapped on the divisor C, which is however not an ample divisor since $C = J_2 - J_3$. The Euler number of this divisor is $c_2 \cdot C = 12$, it is in fact the rational elliptic surface $\mathbb{F}_9$, which is the 9-point blow-up of the projective plane $\mathbb{P}^2$, or equivalently, the 8-point blow-up of $\mathbb{F}_1$.

For r $D4$ branes we have $P = rC$. Equation (2.17) shows that the quadratic form $D_{abc}P^c$ restricted to J_1 and J_3 is

$$(6.2) \qquad\qquad r \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

The other 8 independent classes of $H^2(P, \mathbb{Z})$ are not "visible" to the computation based on periods, since these two-cycles of P do not pull back to 2-cycles of M. We continue by confirming the expressions found from the periods with a computation of the Euler numbers of the moduli spaces of semi-stable sheaves as in [57, 63]. The algebraic computations are more naturally performed in terms of Poincaré polynomials, and thus give more refined information about the moduli space [55]. Moreover, the eight independent classes which are not visible from the Calabi–Yau point of view, can be distinguished from this perspective.

One might wonder whether the extra parameter appearing with the Poincaré polynomial is related to the higher genus expansion of topological strings. However, the refined information of the genus expansion is different. Roughly speaking, the $D2$-brane moduli space is a torus fibration over a base manifold [28]. The genus expansion captures the cohomology of the torus, whereas the $D4$-brane moduli space gives naturally the cohomology of the total moduli space. For $r = 1$, [32] argues that the torus fibration is also present for moduli spaces of rank 1 sheaves on $\mathbb{F}_9$, but it is non-trivial to continue this to higher rank. Another approach to verify the Fourier–Mukai transform at a refined level is consider the refined topological string partition function with parameters ϵ_1 and ϵ_2, and then take the Nekrasov–Shatashvili limit $\epsilon_1 = 0$, $\epsilon_2 \ll 1$ instead of the topological string limit $\epsilon_1 = -\epsilon_2 = g_s$.

The structure described in Section 5 for $D4$-brane partition functions simplifies when one specializes to a (local) surface. The charge vector Γ becomes $(r, \mathrm{ch}_1, \mathrm{ch}_2)$ with r the ranks and ch_i the Chern characters of the sheaf. Other frequently occurring quantities are the determinant $\Delta = \frac{1}{r}(c_2 - \frac{r-1}{2r}c_1^2)$, and $\mu = c_1/r \in H^2(S, \mathbb{Q})$. In terms of the Poincaré polynomial $p(\mathcal{M}, w) = \sum_{i=0}^{2\dim_{\mathbb{C}}(\mathcal{M})} b_i(\mathcal{M}) w^i$ of the moduli space $\mathcal{M}$, the (refined)

BPS invariant reads

$$\Omega(\Gamma, w; \mathcal{J}) := \frac{w^{-\dim_{\mathbb{C}} \mathcal{M}_{\mathcal{J}}(\Gamma)}}{w - w^{-1}} \, p(\mathcal{M}_{\mathcal{J}}(\Gamma), w).$$

In the case of surfaces, a formula is available for the dimension of the moduli space

$$\dim_{\mathbb{C}} \mathcal{M}_{\mathcal{J}}(\Gamma) = 2r^2 \Delta - r^2 \chi(\mathcal{O}_S) + 1.$$

One can verify that the Poincaré polynomials computed later in this section are in agreement with this formula.

The rational invariant corresponding to $\Omega(\Gamma, w; \mathcal{J})$ is [51]

$$(6.3) \qquad \bar{\Omega}(\Gamma, w; \mathcal{J}) = \sum_{m | \Gamma} \frac{\Omega(\Gamma/m, -(-w)^m; \mathcal{J})}{m}.$$

The numerical BPS invariant $\Omega(\Gamma; \mathcal{J})$ follows from the $\Omega(\Gamma, w; \mathcal{J})$ by

$$(6.4) \qquad \Omega(\Gamma; \mathcal{J}) = \lim_{w \to -1} (w - w^{-1}) \, \Omega(\Gamma, w; \mathcal{J}),$$

and similarly for the rational invariants $\bar{\Omega}(\Gamma; \mathcal{J})$.

The generating function (5.5) becomes for a complex surface S

$$(6.5)$$
$$\mathcal{Z}_r(\rho, z, \tau; S, \mathcal{J}) = \sum_{c_1, c_2} \bar{\Omega}(\Gamma, w; \mathcal{J}) \, (-1)^{r c_1 \cdot K_S}$$
$$\times \bar{q}^{r\Delta(\Gamma) - \frac{r\chi(S)}{24} - \frac{1}{2r}(c_1 + rK_S/2)^2_-} q^{\frac{1}{2r}(c_1 + rK_S/2)^2_+} e^{2\pi i \rho \cdot (c_1 + rK_S/2)},$$

with $\rho \in H^2(S, \mathbb{C})$, $w = e^{2\pi i z}$ and $q = e^{2\pi i \tau}$. Twisting by a line bundle leads to an isomorphism of moduli spaces. It is therefore sufficient to determine $\Omega(\Gamma, w; \mathcal{J})$ only for $c_1 \mod r$, and it moreover implies that $\mathcal{Z}_r(\rho, z, \tau; S, \mathcal{J})$ allows a theta function decomposition as in (5.8)

$$(6.6) \qquad \mathcal{Z}_r(\rho, z, \tau; S, \mathcal{J}) = \sum_{\mu \in \Lambda^*/\Lambda} \overline{h_{r,\mu}(z, \tau; S, \mathcal{J})} \, \Theta_{r,\mu}(\rho, \tau; S),$$

with

$$(6.7) \qquad h_{r,\mu}(z, \tau; S, \mathcal{J}) = \sum_{c_2} \bar{\Omega}(\Gamma, w; \mathcal{J}) \, q^{r\Delta(\Gamma) - \frac{r\chi(S)}{24}},$$

and

$$\Theta_{r,\mu}(\rho,\tau;S) = \sum_{\mathbf{k}\in H^2(S,r\mathbb{Z})+rK_S/2+\mu} (-1)^{r\mathbf{k}\cdot K_S} q^{\mathbf{k}_+^2/2r} \bar{q}^{-\mathbf{k}_-^2/2r}\, e^{2\pi i\rho\cdot\mathbf{k}}.$$

Note that $\Theta_{r,\mu}(\rho,\tau;S)$ depends on $\mathcal{J}$ through $\mathbf{k}_\pm$ and does not depend on z.

The generating function of the numerical invariants $\Omega(\Gamma;\mathcal{J})$ follows simply from Equation (6.4)

(6.8) $$\mathcal{Z}_r(\rho,\tau;S,\mathcal{J}) = \lim_{z\to\frac{1}{2}}\ (w-w^{-1})\,\mathcal{Z}_r(z,\rho,\tau;S,\mathcal{J}).$$

Physical arguments imply that this function transforms as a multivariable Jacobi form of weight $(\frac{1}{2},-\frac{3}{2})$ [48, 59] with a non-trivial multiplier system. For rank > 1, this is only correct after the addition of a suitable non-holomorphic term [57, 59].

This section verifies the agreement of the BPS invariants obtained from the periods and vector bundles for $h_{r,c_1}(z,\tau;\mathbb{F}_9,\mathcal{J}_{m,n})$ for $r \le 3$. The results for $r \le 2$ are due to Göttsche [25] and Yoshioka [63]. The computations apply notions and techniques from algebraic geometry as Gieseker stability, Harder–Narasimhan filtrations and the blow-up formula. We refer to [51, 55] for further references and details. The most crucial difference between the computations for $\mathbb{F}_9$ and those for Hirzebruch surfaces in [51, 55] is that the lattice arising from $H^2(\mathbb{F}_9,\mathbb{Z})$ is now ten-dimensional. We continue therefore with giving a detailed description of different bases of $H^2(\mathbb{F}_9,\mathbb{Z})$, gluing vectors and theta functions.

6.1. The lattice $H^2(\mathbb{F}_9,\mathbb{Z})$

The second cohomology $H^2(\mathbb{F}_9,\mathbb{Z})$ gives naturally rise to a unimodular basis, it is in fact isomorphic to the unique unimodular lattice with signature $(1,9)$, which we denote by $\Lambda_{1,9}$. For this paper, three different bases ($\mathbf{C}$, $\mathbf{D}$ and $\mathbf{E}$) of $\Lambda_{1,9}$ are useful. The first basis is the geometric basis $\mathbf{C}$, which keeps manifest that $\mathbb{F}_9$ is the 9-point blow-up of the projective plane $\mathbb{P}^2$. The basis vectors of $\mathbf{C}$ are H (the hyperplane class of $\mathbb{P}^2$) and $\mathbf{c}_i$ (the exceptional divisors of the blow-up).[5] The quadratic form is $\mathrm{diag}(1,-1,\ldots,-1)$. The

---

[5]We will use in general boldface to parameterize vectors.

canonical class K_9 of $\mathbb{F}_9$ is given in terms of this basis by

$$(6.9) \qquad K_9 = -3H + \sum_{i=1}^{9} \mathbf{c}_i.$$

One can easily verify that $K_9^2 = 0$. Note that $-K_9$ is numerically effective but not ample.

The second basis $\mathbf{D}$ parameterizes $\Lambda_{1,9}$ as a gluing of the two non-unimodular lattices A and D. The basis $\mathbf{D}$ is given in terms of $\mathbf{C}$ by

$$(6.10) \qquad \begin{aligned} \mathbf{a}_1 &= -K_9, \quad \mathbf{a}_2 = H - \mathbf{c}_9, \\ \mathbf{d}_i &= \mathbf{c}_i - \mathbf{c}_{i+1}, \quad 1 \le i \le 7, \\ \mathbf{d}_8 &= -H + \mathbf{c}_7 + \mathbf{c}_8 + \mathbf{c}_9. \end{aligned}$$

The $\mathbf{a}_i$ are basis elements of A and $\mathbf{d}_i$ of D. Since A and D are not unimodular, integral lattice elements of $\mathbf{C}$ do not correspond to integral elements of D. For example, $\mathbf{c}_9$ is given by

$$(6.11) \qquad \mathbf{c}_9 = \frac{1}{2}\left(\mathbf{a}_1 + \mathbf{a}_2 + \sum_{i=1}^{6} i\mathbf{d}_i + 3\mathbf{d}_7 + 4\mathbf{d}_8 \right).$$

The other $\mathbf{c}_i$ are easily determined using $\mathbf{c}_9$. The quadratic form $\mathcal{Q}_A$ of the lattice A is

$$(6.12) \qquad \mathcal{Q}_A = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix},$$

and $\mathcal{Q}_D$ of the lattice D is minus the D_8 Cartan matrix

$$(6.13) \qquad \mathcal{Q}_D = -\mathcal{Q}_{D_8} = -\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix}.$$

Gluing of A and D to obtain $\Lambda_{1,9}$ corresponds to an isomorphism between A^*/A and D^*/D. This isomorphism is given by 4 gluing vectors $\boldsymbol{g}_i$, since

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the discriminants of A and D are equal to 4. We choose them to be

$$
\begin{aligned}
\boldsymbol{g}_0 &= \boldsymbol{0}, \\
\boldsymbol{g}_1 &= \tfrac{1}{2}(1,0,1,0,1,0,1,0,0,1), \\
\boldsymbol{g}_2 &= \tfrac{1}{2}(0,1,0,0,0,0,0,0,1,1), \\
\boldsymbol{g}_3 &= \tfrac{1}{2}(1,1,1,0,1,0,1,0,1,0).
\end{aligned}
$$

Theta functions which sum over D will play an essential role later in this section. The theta functions $\Theta_{rD_8,\boldsymbol{\mu}}(\tau)$ are defined by

$$
(6.14) \qquad \Theta_{rD_8,\mu}(\tau) = \sum_{\mathbf{k}=\boldsymbol{\mu} \mod r\mathbb{Z}} q^{\frac{\mathbf{k}^2}{2r}}.
$$

Such sums converge rather slowly. Therefore, we also give their expression in terms of unary theta functions $\theta_i(\tau) = \theta_i(0,\tau)$ (defined in Appendix C). For $r = 1$ and the glue vectors $\boldsymbol{g}_i$, one has

$$
\begin{aligned}
\Theta_{D_8,\boldsymbol{g}_0}(\tau) &= \tfrac{1}{2}\left(\theta_3(\tau)^8 + \theta_4(\tau)^8\right), \\
\Theta_{D_8,\boldsymbol{g}_1}(\tau) &= \tfrac{1}{2}\theta_2(\tau)^8, \\
\Theta_{D_8,\boldsymbol{g}_2}(\tau) &= \tfrac{1}{2}\left(\theta_3(\tau)^8 - \theta_4(\tau)^8\right), \\
\Theta_{D_8,\boldsymbol{g}_3}(\tau) &= \tfrac{1}{2}\theta_2(\tau)^8.
\end{aligned}
$$

For $r = 2$, the μ in the $\Theta_{2D_8,\mu}(\tau)$ take values in $D/2D$. The 2^8 elements are naturally grouped in six classes with multiplicities 1, 56, 140, 1, 56 and two depending on the corresponding theta function $\Theta_{2D_8,\mu}(\tau)$. We choose as representative for each class

$$
\begin{aligned}
\boldsymbol{d}_0 &= \boldsymbol{0}, \\
\boldsymbol{d}_1 &= (1,0,0,0,0,0,0,0), \\
\boldsymbol{d}_2 &= (1,0,1,0,0,0,0,0), \\
\boldsymbol{d}_3 &= (0,0,0,0,0,0,1,1), \\
\boldsymbol{d}_4 &= (1,0,1,0,1,0,0,0), \\
\boldsymbol{d}_5 &= (1,0,1,0,1,0,1,0).
\end{aligned}
$$

Elements $\boldsymbol{\mu} \in \boldsymbol{g}_i + D/2D$ fall similarly in conjugacy classes corresponding to their theta functions. We let $m_{i,j}$ denote the number of elements in the class represented by $\boldsymbol{g}_i + \boldsymbol{d}_j$. The non-vanishing $m_{i,j}$ are given in table 2.

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Table 2: The number of elements $m_{i,j}$ in $\boldsymbol{g}_i + D/2D$ with equal theta functions $\Theta_{2D_8,\boldsymbol{g}_i+\boldsymbol{d}_j}(\tau)$.

$m_{i,j}$	0	1	2	3	4	5
0	1	56	140	1	56	2
1	128			128		
2	16	112	112		16	
3	128			128		

The corresponding theta functions are given by

$$
\begin{aligned}
(6.15) \quad
&\Theta_{2D_8,\boldsymbol{d}_0}(\tau) = \tfrac{1}{2}\left(\theta_3(2\tau)^8 + \theta_4(2\tau)^8\right), \\
&\Theta_{2D_8,\boldsymbol{d}_1}(\tau) = \tfrac{1}{16}\left(\theta_3(\tau)^8 - \theta_4(\tau)^8\right) - \tfrac{1}{2}\theta_2(2\tau)^6\theta_3(2\tau)^2, \\
&\Theta_{2D_8,\boldsymbol{d}_2}(\tau) = \tfrac{1}{32}\theta_2(\tau)^8, \\
&\Theta_{2D_8,\boldsymbol{d}_3}(\tau) = \tfrac{1}{2}\left(\theta_3(2\tau)^8 - \theta_4(2\tau)^8\right), \\
&\Theta_{2D_8,\boldsymbol{d}_4}(\tau) = \tfrac{1}{2}\theta_2(2\tau)^6\theta_3(2\tau)^2, \\
&\Theta_{2D_8,\boldsymbol{d}_5}(\tau) = \tfrac{1}{2}\theta_2(2\tau)^8,
\end{aligned}
$$

For $\boldsymbol{g}_1$

$$
\begin{aligned}
&\Theta_{2D_8,\boldsymbol{g}_1}(\tau) = \tfrac{1}{8}\theta_2(\tau)^4\left(\theta_3(2\tau)^4 - \tfrac{1}{2}\theta_4(2\tau)^4\right), \\
&\Theta_{2D_8,\boldsymbol{g}_1+\boldsymbol{d}_3}(\tau) = \Theta_{2D_8,\boldsymbol{d}_2}(\tau),
\end{aligned}
$$

for $\boldsymbol{g}_2$

$$
\begin{aligned}
(6.16) \quad
&\Theta_{2D_8,\boldsymbol{g}_2}(\tau) = \tfrac{1}{4}\theta_2(\tau)^2\,\theta_3(2\tau)^6, \\
&\Theta_{2D_8,\boldsymbol{g}_2+\boldsymbol{d}_1}(\tau) = \tfrac{1}{16}\theta_2(\tau)^6\,\theta_3(2\tau)^2, \\
&\Theta_{2D_8,\boldsymbol{g}_2+\boldsymbol{d}_2}(\tau) = \tfrac{1}{16}\theta_2(\tau)^6\left(\theta_3(2\tau)^2 - \theta_4(\tau)^2\right), \\
&\Theta_{2D_8,\boldsymbol{g}_2+\boldsymbol{d}_4}(\tau) = \tfrac{1}{4}\theta_2(2\tau)^6\theta_2(\tau)^2,
\end{aligned}
$$

and for $\boldsymbol{g}_3$:

$$
\begin{aligned}
&\Theta_{2D_8,\boldsymbol{g}_3}(\tau) = \Theta_{2D_8,\boldsymbol{g}_1}(\tau), \\
&\Theta_{2D_8,\boldsymbol{g}_3+\boldsymbol{d}_3}(\tau) = \Theta_{2D_8,\boldsymbol{g}_1+\boldsymbol{d}_1}(\tau).
\end{aligned}
$$

The third basis is basis $\mathbf{E}$ corresponding to the representation of $\Lambda_{1,9}$ as the direct sum of the two lattices B and E, whose basis vectors $\mathbf{b}_i$

and $\mathbf{e}_i$ are

$$
\begin{aligned}
\mathbf{b}_1 &= -K_9, \quad \mathbf{b}_2 = \mathbf{c}_9, \\
\mathbf{e}_i &= \mathbf{c}_i - \mathbf{c}_{i+1}, \quad 1 \le i \le 7, \\
\mathbf{e}_8 &= -H + \mathbf{c}_6 + \mathbf{c}_7 + \mathbf{c}_8.
\end{aligned}
$$
(6.17)

The element H of basis $\mathbf{C}$ is in terms of this basis: $H = (3, 3, 3, 6, 9, 12, 15, 10, 5, 2)$. The intersection numbers for $\mathbf{b}_i$ are $\mathbf{b}_1^2 = 0$, $\mathbf{b}_2^2 = -1$ and $\mathbf{b}_1 \cdot \mathbf{b}_2 = 1$. The quadratic form $\mathcal{Q}_E$ for E is minus the E_8 Cartan matrix, which is given by

(6.18)
$$
\begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 2
\end{pmatrix},
$$

The 256 elements in $E/2E$ fall in 3 inequivalent Weil orbits with vectors of length 0, 2 and 4, which have multiplicities $m_0 = 1$, $m_1 = 120$ and $m_2 = 135$ respectively. We choose as representatives

$$
\begin{aligned}
\boldsymbol{e}_0 &= \mathbf{0}, \\
\boldsymbol{e}_1 &= (1, 0, 0, 0, 0, 0, 0, 0), \\
\boldsymbol{e}_2 &= (1, 0, 1, 0, 0, 0, 0, 0).
\end{aligned}
$$

The corresponding theta functions $\Theta_{rE_8, \boldsymbol{e}_0}$ are for $r = 1, 2$:

$$
\begin{aligned}
\Theta_{E_8, \boldsymbol{e}_0}(\tau) &= E_4(\tau), \\
\Theta_{2E_8, \boldsymbol{e}_0}(\tau) &= E_4(2\tau), \\
\Theta_{2E_8, \boldsymbol{e}_1}(\tau) &= \frac{1}{240} \left(E_4(\tau/2) - E_4(\tau/2 + 1/2) \right), \\
\Theta_{2E_8, \boldsymbol{e}_2}(\tau) &= \frac{1}{15} \left(E_4(\tau) - E_4(2\tau) \right).
\end{aligned}
$$

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6.2. BPS invariants for $r \leq 3$

Rank 1   The results from the periods for $h_{C,\mathbf{0}}^{\mathrm{pr}}(\tau)$ is (6.1)

$$(6.19) \qquad\qquad h_{C,\mathbf{0}}^{\mathrm{pr}}(\tau) = \frac{E_4(\tau)}{\eta(\tau)^{12}}.$$

This can easily be verified with the results for sheaves on surfaces. The result for $r = 1$ and a complex, simply connected surface S is [25]

$$(6.20) \qquad\qquad h_{1,c_1}(z, \tau; S) = \frac{i}{\theta_1(2z, \tau)\, \eta(\tau)^{b_2(S)-1}}.$$

The dependence on $\mathcal{J}$ can be omitted for $r = 1$ since all rank 1 sheaves are stable. If we specialize to $S = \mathbb{F}_9$, take the limit $w \to -1$, and sum over all $c_1 \in E = H^2(\mathbb{F}_9, \mathbb{Z})/\iota^* H^2(M, \mathbb{Z})$ one obtains $h_{C,\mathbf{0}}(\tau) = E_4(\tau)/\eta(\tau)^{12}$ in agreement with Equation (6.19).

Rank 2   The prediction by the periods for $r = 2$ is given by $h_{2C,\mathbf{0}}^{\mathrm{pr}}(\tau)$ in (6.1). This is a sum over all BPS invariants for $c_1 \cdot \mathbf{a}_i = 0$, $i = 1, 2$. In order to verify this result, it is useful to decompose $h_{2C,\mathbf{0}}^{\mathrm{pr}}(\tau)$ according to the three conjugacy classes of $E/2E \in H_2(\mathbb{F}_9)/(\iota^* H_2(M, \mathbb{Z}))$. One obtains $h_{C,\mathbf{0}}^{\mathrm{pr}}(\tau) = \sum_{i=0,1,2} m_i\, h_{2,\mathbf{e}_i}^{\mathrm{pr}}(\tau; \mathbb{F}_9, \mathbf{a}_1)\, \Theta_{2E_8, \mathbf{e}_i}(\tau)$ with [57]:

(6.21)
$$
h_{2,\mathbf{e}_0}^{\mathrm{pr}}(\tau; \mathbb{F}_9, \mathbf{a}_1) = \frac{1}{24\, \eta(\tau)^{24}} \left[E_2(\tau)\, \Theta_{2E_8, \mathbf{e}_0}(\tau) + \left(\theta_3(\tau)^4 \theta_4(\tau)^4 - \tfrac{1}{8}\theta_2(\tau)^8 \right) \right.
$$
$$
\left. \times \left(\theta_3(\tau)^4 + \theta_4(\tau)^4 \right) \right] + \frac{1}{8} h_{1,\mathbf{e}_0}^{\mathrm{pr}}(2\tau; \mathbb{F}_9),
$$
$$
h_{2,\mathbf{e}_1}^{\mathrm{pr}}(\tau; \mathbb{F}_9, \mathbf{a}_1) = \frac{1}{24\, \eta(\tau)^{24}} \left[E_2(\tau)\, \Theta_{2E_8, \mathbf{e}_1}(\tau) - \tfrac{1}{8} E_4(\tau)\, \theta_2(\tau)^4 \right],
$$
$$
h_{2,\mathbf{e}_2}^{\mathrm{pr}}(\tau; \mathbb{F}_9, \mathbf{a}_1) = \frac{1}{24\, \eta(\tau)^{24}} \left[E_2(\tau)\, \Theta_{2E_8, \mathbf{e}_2}(\tau) - \tfrac{1}{8} \theta_2(\tau)^8 \left(\theta_3(\tau)^4 + \theta_4(\tau)^4 \right) \right].
$$

In the following we will omit the argument "$\mathbb{F}_9$" in the from the generating functions.

Verification of the expressions (6.21) is much more elaborate then for $r = 1$. We will use the approach of [60, 61, 63]. The main issues are the determination of the BPS invariants for a polarization close to the class $\mathbf{a}_2$

(a suitable polarization) and wall-crossing from the suitable polarization to $\mathcal{J} = -K_9 = \mathbf{a}_1$. These issues are dealt with for the Hirzebruch surfaces [60, 61], and for $\mathbb{F}_9$ in [63]. The main difficulty for $\mathbb{F}_9$ compared to the Hirzebruch surfaces is that that the class f and K_9 span the lattice A, which is related to $\Lambda_{1,9}$ by a non-trivial gluing with the lattice D.

Before turning to the explicit expressions, we briefly outline the computation; we refer for more details about the used techniques to [55]. The polarization $\mathcal{J}$ is parameterized by $\mathcal{J}_{m,n} = m\,\mathbf{a}_1 + n\,\mathbf{a}_2$. In order to determine the BPS invariants for the suitable polarization $\mathcal{J}_{\varepsilon,1}$, view $\mathbb{F}_9$ as the 8-point blow-up of the Hirzebruch surface $\mathbb{F}_1$: $\phi : \mathbb{F}_9 \to \mathbb{F}_1$. We choose to perform this blow-up for the polarization $\mathcal{J}_{\mathbb{F}_1} = f$, with f the fibre class of the Hirzebruch surface. The pull back of this class to $\mathbb{F}_9$ is $\phi^* f = \mathcal{J}_{0,1}$. The generating function of the BPS invariants for this choice takes a relatively simple form: it either vanishes or equals a product of eta and theta functions [55, 61] depending on the Chern classes. This function represents the sheaves whose restriction to the rational curve $\mathbf{a}_2$ is semi-stable. The generating function $h_{r,c_1}(z, \tau; \mathbb{F}_9, \mathcal{J}_{0,1})$ is therefore this product formula multiplied by the factors due to blowing-up the 8 points. To obtain the BPS invariants from this function, one has to change $\mathcal{J}_{0,1}$ to $\mathcal{J}_{\varepsilon,1}$ and subtract the contribution due to sheaves which became (Gieseker) unstable due to this change [55]. Consequently, we can determine the BPS invariants for any other choice of $\mathcal{J}$ by the wall-crossing formula [35, 42, 62]. In particular, we determine the invariants for $\mathcal{J}_{1,0} = -K_9$ and change to the basis $\mathbf{E}$ in order to compare with the expression from the periods.

We continue with determining the BPS invariants for $\mathcal{J} = \mathcal{J}_{0,1}$. The BPS invariants vanish for $c_1 \cdot \mathbf{a}_2 = 1 \mod 2$

$$(6.22) \qquad h_{2,c_1}(z, \tau; \mathcal{J}_{\varepsilon,1}) = 0, \quad c_1 \cdot \mathbf{a}_2 = 1 \mod 2.$$

Since BPS invariants depend on $c_1 \mod 2\Lambda_{1,9}$, we distinguish further $c_1 \cdot \mathbf{a}_2 = 0 \mod 4$ and $c_1 \cdot \mathbf{a}_2 = 2 \mod 4$. For these cases, we continue as in [55] using the (extended) Harder-Narasimhan filtration. A sheaf F which is unstable for $\mathcal{J}_{\varepsilon,1}$ but semi-stable for $\mathcal{J}_{0,1}$, can be described as a HN-filtration of length 2 whose quotients we denote by E_i, $i = 1, 2$. If we parameterize the first Chern class of E_2 by $\mathbf{k} = (\,\mathbf{k}_A, \mathbf{k}_D\,)$, then the discriminant $\Delta(F)$ is given by

$$(6.23) \qquad 2\Delta(F) = \Delta(E_1) + \Delta(E_2) - \frac{1}{4}(2\mathbf{k}_A - c_1|_A)^2 - \frac{1}{4}(2\mathbf{k}_D - c_1|_D)^2.$$

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The choice of $\mathbf{k}_D$ does not have any effect on the stability of F as long as $\mathcal{J}$ is spanned by $\mathcal{J}_{0,1}$ and $\mathcal{J}_{1,0}$. Therefore (6.23) shows that the sum over $\mathbf{k}_D$ gives rise to the theta functions $\Theta_{2D_8,\mu}(\tau)$. The condition for semi-stability for $\mathcal{J}_{0,1}$ but unstable for $\mathcal{J}_{\varepsilon,1}$ implies $(c_1(E_1) - c_1(E_2)) \cdot \mathbf{a}_2 = 0$. This combined with $c_1 \cdot \mathbf{a}_2 = 0 \mod 4$ gives for $c_1(E_i) = 0 \mod 2$, which shows that $c_1(E_i) = \boldsymbol{g}_j \mod 2\Lambda_{1,9}$ only for $j = 0, 2$. One obtains after a detailed analysis for $c_1 \cdot \mathbf{a}_2 = 0 \mod 4$

$$(6.24) \qquad h_{2,c_1}(z,\tau; \mathcal{J}_{\varepsilon,1}) = \frac{-i\,\eta(\tau)}{\theta_1(2z,\tau)^2\,\theta_1(4z,\tau)} \prod_{i=1}^{8} B_{2,\ell_i}(z,\tau)$$

$$+ \left(\frac{w^{4\{(\frac{1}{2}\boldsymbol{g}_0 - \frac{1}{4}c_1)\cdot\mathbf{a}_1\}}}{1 - w^4} - \frac{1}{2}\delta_{0,\{(\frac{1}{2}\boldsymbol{g}_0 - \frac{1}{4}c_1)\cdot\mathbf{a}_1\}} \right)$$

$$\times \Theta_{2D_8, c_1 - 2\boldsymbol{g}_0}(\tau)\, h_{1,\mathbf{0}}(z,\tau)^2$$

$$+ \left(\frac{w^{4\{(\frac{1}{2}\boldsymbol{g}_2 - \frac{1}{4}c_1)\cdot\mathbf{a}_1\}}}{1 - w^4} - \frac{1}{2}\delta_{0,\{(\frac{1}{2}\boldsymbol{g}_2 - \frac{1}{4}c_1)\cdot\mathbf{a}_1\}} \right)$$

$$\times \Theta_{2D_8, c_1 - 2\boldsymbol{g}_2}(\tau)\, h_{1,\mathbf{0}}(z,\tau)^2,$$

where $\{\lambda\} = \lambda - \lfloor\lambda\rfloor$ and $\ell_i = c_1 \cdot \mathbf{c}_i$. The right-hand side on the first line correspond to the sheaves whose restriction to $\mathbf{a}_2$ are semi-stable. The functions $B_{2,\ell}(z,\tau) = \sum_{n\in\mathbb{Z}+\ell/2} q^{n^2} w^n / \eta(\tau)^2$ are due to the blow-up formula [27,43,55,62]. The second and third line are the subtractions due to sheaves that are unstable for $\mathcal{J}_{\varepsilon,1}$.

Similarly, one obtains for $c_1 \cdot \mathbf{a}_2 = 2 \mod 4$

$$(6.25) \qquad h_{2,c_1}(z,\tau; \mathcal{J}_{\varepsilon,1}) = \frac{-i\,\eta(\tau)}{\theta_1(2z,\tau)^2\,\theta_1(4z,\tau)} \prod_{i=1}^{8} B_{2,\ell_i}(z,\tau)$$

$$+ \left(\frac{w^{4\{(\frac{1}{2}\boldsymbol{g}_1 - \frac{1}{4}c_1)\cdot\mathbf{a}_1\}}}{1 - w^4} - \frac{1}{2}\delta_{0,\{(\frac{1}{2}\boldsymbol{g}_1 - \frac{1}{4}c_1)\cdot\mathbf{a}_1\}} \right)$$

$$\times \Theta_{2D_8, c_1 - 2\boldsymbol{g}_1}(\tau)\, h_{1,\mathbf{0}}(z,\tau)^2$$

$$+ \left(\frac{w^{4\{(\frac{1}{2}\boldsymbol{g}_3 - \frac{1}{4}c_1)\cdot\mathbf{a}_1\}}}{1 - w^4} - \frac{1}{2}\delta_{0,\{(\frac{1}{2}\boldsymbol{g}_3 - \frac{1}{4}c_1)\cdot\mathbf{a}_1\}} \right)$$

$$\times \Theta_{2D_8, c_1 - 2\boldsymbol{g}_3}(\tau)\, h_{1,\mathbf{0}}(z,\tau)^2.$$

What remains is to change the polarization $\mathcal{J}$ from $\mathcal{J}_{\varepsilon,1}$ to $\mathcal{J}_{1,0}$ and determine the change of the invariants using wall-crossing formulas. For

Table 3: The Betti numbers b_n (with $n \leq \dim_{\mathbb{C}} \mathcal{M}$) and Euler numbers χ of the moduli spaces of semi-stable sheaves on $\mathbb{F}_9$ with $r = 2$, $c_1 = \boldsymbol{e}_i$, and $1 \leq c_2 \leq 4$ for $\mathcal{J} = \mathcal{J}_{1,0}$.

c_1	c_2	b_0	b_2	b_4	b_6	b_8	b_{10}	b_{12}	b_{14}	b_{16}	χ
$\boldsymbol{e}_0$	2	1	10	55							132
	3	1	11	76	396	1356					3680
	4	1	11	78	428	1969	7449	20124			60120
	5	1	11	78	430	2012	8316	30506	95498	221132	715968
$\boldsymbol{e}_1$	1	1	9								20
	2	1	11	75	309						792
	3	1	11	78	426	1843	5525				15768
	4	1	11	78	430	2010	8150	27777	68967		214848
$\boldsymbol{e}_2$	1	1									2
	2	1	11	60							144
	3	1	11	78	404	1386					3760
	4	1	11	78	430	1981	7495	20244			60480

$\mathcal{J} = (m, n, \boldsymbol{0}) \in A \oplus D$, we obtain the following expression:

$$(6.26) \qquad h_{2,c_1}(z, \tau; \mathcal{J}_{m,n}) = \frac{-i\,\eta(\tau)}{\theta_1(2z,\tau)^2\,\theta_1(4z,\tau)} \prod_{i=1}^{8} B_{2,\ell_i}(z,\tau)$$
$$+ \sum_{j=0,\ldots,3} h_{2,c_1-2\boldsymbol{g}_j}^A(z,\tau; \mathcal{J}_{m,n})\,\Theta_{2D,c_1-2\boldsymbol{g}_j}(\tau),$$

with

$$h_{2,c_1}^A(z,\tau; \mathcal{J}_{m,n}) = h_{2,c_1}^A(z,\tau; \mathcal{J}_{\varepsilon,1}) + \tfrac{1}{2} \sum_{(a_1,a_2)\in A+c_1}$$
$$\tfrac{1}{2}\left(\,\mathrm{sgn}(a_1 n + a_2 m) - \mathrm{sgn}(a_1 + a_2\varepsilon)\,\right)$$
$$\times (w^{4a_2} - w^{-4a_2})\, q^{-4a_1 a_2} h_{1,\boldsymbol{0}}(z,\tau)^2.$$

The functions $h_{2,c_1}^A(z,\tau; \mathcal{J}_{\varepsilon,1})$ are rational functions in w multiplied by $h_{1,\boldsymbol{0}}(z,\tau)^2$, which can easily be read off from Equation (6.24). For $\mathcal{J} = \mathcal{J}_{1,0}$ the functions can be expressed in terms of modular functions.

Table 3 presents the BPS invariants for $\mathcal{J} = \mathcal{J}_{1,0}$. As expected, the Euler numbers are indeed in agreement with the predictions (6.21). One can also verify that for increasing c_2, the Betti numbers asymptote to those of $r = 1$ or equivalently the Hilbert scheme of points of $\mathbb{F}_9$.

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We define the functions $h_{2,c_1}^A(z,\tau) := h_{2,c_1}^A(z,\tau; \mathcal{J}_{1,0})$, which only depend on $c_1|_A = \alpha_1 \mathbf{a}_1 + \alpha_2 \mathbf{a}_2$ with $\alpha_1, \alpha_2 \in 0, \frac{1}{2}, 1, \frac{3}{2}$. One finds for $\alpha_2 = 0 \mod 4$

$$(6.27) \qquad \frac{h_{2,c_1}^A(z,\tau)}{h_{1,0}(z,\tau)^2} = -\frac{1}{8}\frac{1}{2\pi i}\frac{\partial}{\partial z}\ln\left(\theta_1(4\tau, 4z + 2\alpha_1)\,\theta_1(4\tau, 4z - 2\alpha_1)\right),$$

and for $\alpha_2 \neq 0 \mod 4$ using (C.3)

$$(6.28) \qquad \frac{h_{2,c_1}^A(z,\tau)}{h_{1,0}(z,\tau)^2} = \frac{i}{2}\frac{q^{-\alpha_1\alpha_2}\,\eta(4\tau)^3}{\theta_1(4\tau, 2\alpha_2\tau)}\left(\frac{w^{-2\alpha_2}\,\theta_1(4\tau, 4z + 2(\alpha_1 - \alpha_2)\tau)}{\theta_1(4\tau, 4z + 2\alpha_1\tau)}\right.$$
$$\left. - \frac{w^{2\alpha_2}\,\theta_1(4\tau, -4z + 2(\alpha_1 - \alpha_2)\tau)}{\theta_1(4\tau, 4z + 2\alpha_1\tau)}\right).$$

To prove the agreement of the Euler numbers with the periods, we specialize to $w = -1$. Let $D_k = \frac{1}{2\pi i}\frac{\partial}{\partial \tau} - \frac{k}{12}E_2(\tau)$ be the differential operator which maps weight k modular forms to modular forms of weight $k+2$. Then one can write $h_{2,c_1}(\tau; \mathcal{J}_{1,0})$ as

$$(6.29) \qquad h_{2,c_1}(\tau; \mathcal{J}_{1,0}) = \frac{1}{\eta(\tau)^{24}}\left(\frac{1}{2}\delta_{c_1 \cdot \mathbf{a}_2,0}D_4(\theta_3(2\tau)^m\theta_2(2\tau)^{8-m})\right.$$
$$\left. + \sum_{i=0,\ldots,3} f_{c_1-2\boldsymbol{g}_j}^A(\tau)\,\Theta_{2D,c_1-2\boldsymbol{g}_j}(\tau)\right),$$

with

$$f_{0,0}^A(\tau) = \tfrac{1}{8}\theta_3(2\tau)^4 + \tfrac{1}{24}E_2(\tau),$$
$$f_{-\frac{1}{2},\frac{1}{2}}^A(\tau) = \tfrac{1}{2}\theta_2(2\tau)\,\theta_3(2\tau)^3,$$
$$f_{\frac{1}{2},\frac{1}{2}}^A(\tau) = \tfrac{1}{2}\theta_2(2\tau)^3\,\theta_3(2\tau),$$
$$f_{1,0}^A(\tau) = \tfrac{1}{12}\theta_2(2\tau)^4 - \tfrac{1}{24}\theta_3(2\tau)^4 + \tfrac{1}{24}E_2(\tau),$$
$$f_{0,1}^A(\tau) = \tfrac{1}{24}\theta_2(2\tau)^4 - \tfrac{1}{12}\theta_3(2\tau)^4,$$
$$f_{1,1}^A(\tau) = -\tfrac{1}{8}\theta_2(2\tau)^4.$$

If $c_1|_B = 0$, this reproduces the functions in [57, 63] depending on whether the classes in lattice E are even or odd.

Modularity Electric-magnetic duality of $\mathcal{N} = 4$ $U(r)$ Yang-Mills theory implies modular properties for its partition function [59]. Determination

of the modular properties gives therefore insight about the realization of electric–magnetic duality at the quantum level.

The expression in Equation (6.26) does not transform as a modular form for generic choices of $\mathcal{J}$. However, using the theory of indefinite theta functions [64], the functions can be completed to a function $\widehat{h}_{2,c_1}(z,\tau;\mathcal{J})$ by addition of a non-holomorphic term, such that $\widehat{h}_{2,c_1}(z,\tau;\mathcal{J})$ does transform as a modular form [53]. Interestingly, Equation (6.29) shows that $h_{2,c_1}(z,\tau;\mathcal{J})$ becomes a quasi-modular form for $\lim_{\mathcal{J}\to\mathcal{J}_{1,0}}\widehat{h}_{2,c_1}(z,\tau;\mathcal{J})$, i.e., it can be expressed in terms of modular forms and Eisenstein series of weight 2. In some cases it becomes even a true modular form. This is due to the special form of $\mathcal{Q}_A$.

The transition from mock modular to quasi-modular can be made precise. Owing to the gluing vectors, the function $f_{2,c_1}(z,\tau;\mathcal{J}) = h_{2,c_1}(z,\tau;\mathcal{J})/h_{1,c_1}(z,\tau)^2$ takes the form

$$(6.30) \qquad f_{2,c_1}(z,\tau;\mathcal{J}_{m,n}) = \sum_{\mu} f^{A}_{2,(c_1-2\mu)_A}(z,\tau;\mathcal{J}_{m,n})\,\Theta_{2D,(2\mu-c_1)_D}(\tau)$$
$$+\,\delta_{c_1\cdot\mathbf{a}_2,0}\,\frac{i\,\eta(\tau)^3}{\theta_1(\tau,4z)}\theta_3(2\tau,2z)^k\theta_2(2\tau,2z)^{8-k},$$

where k is the number of $c_1\cdot\mathbf{c}_i = 1\ \mathrm{mod}\ 2$ for $1\le i\le 8$. The completed generating function $\widehat{f}_{2,c_1}(z,\tau;\mathcal{J})$ is a slight generalization of the equations in Section 3.2 in [53]:

$$(6.31) \qquad \widehat{f}_{2,c_1}(\tau;\mathcal{J}) = f_{2,c_1}(\tau;\mathcal{J}) + \sum_{\substack{\mathbf{c}\in -c_1 \\ +H^2(\Sigma_9,2\mathbb{Z})}} \left(\frac{K_9\cdot\mathcal{J}}{4\pi\sqrt{\mathcal{J}^2\,\tau_2}}\,\mathrm{e}^{-\pi\tau_2\mathbf{c}_+^2} \right.$$
$$\left. -\,\frac{1}{4}K_9\cdot\mathbf{c}\,\mathrm{sgn}(\mathbf{c}\cdot\mathcal{J})\,\beta_{\frac{1}{2}}(\mathbf{c}_+^2\,\tau_2) \right) (-1)^{K_9\cdot\mathbf{c}}q^{-\mathbf{c}^2/4},$$

where $\tau_2 = \mathrm{Im}(\tau)$ and $\beta_\nu(x) = \int_x^\infty u^{-\nu}\,\mathrm{e}^{-\pi u}du$. We parameterize $\mathcal{J}$ by $b_1 + v\,b_2$ and carefully study the limit $v\to 0$ (this corresponds to the limit $R\to\infty$ in [57]). In this limit, $\mathcal{J}$ approaches $-K_9$. Moreover, $\mathcal{J}\cdot K_9 = -v$ and $\mathcal{J}^2 = v(2-v)$. If one parameterizes $\mathbf{c}$ by $(n_0,n_1,\mathbf{c}_\perp)$, only terms with $n_1=0$ contribute to the sum in the limit $v\to 0$. Therefore the term with $\beta_{\frac{1}{2}}(\mathbf{c}_+^2\,\tau_2)$ does not contribute to the anomaly. After a Poisson resummation on n_0, one finds that the limit is finite and given by

$$(6.32) \qquad \widehat{f}_{2,c_1}(\tau;\mathcal{J}_{1,0}) = f_{2,c_1}(\tau;\mathcal{J}_{1,0}) - \frac{\delta_{c_1\cdot\mathbf{a}_1,0}}{8\pi\,\tau_2} \sum_{\substack{\mathbf{c}\in -c_{1|E} \\ +2E}} q^{-\mathbf{c}_\perp^2/4}.$$

This is in good agreement with Equation (6.21) if $c_1 \cdot \mathbf{a}_1 = 0$. The lattice sum over $\mathbf{c}$ gives precisely the theta functions $\Theta_{2E_8,\mathbf{e}_i}(\tau)$. Recalling the modular completion of the weight 2 Eisenstein series: $\widehat{E}_2(\tau) = E_2(\tau) - \frac{3}{\pi\tau_2}$, we see that the non-holomorphic term implies that in the holomorphic part of $\widehat{f}_{2,c_1}(\tau; \mathcal{J}_{1,0})$, $E_2(\tau)$ is multiplied by the $\Theta_{2E_8,\mathbf{e}_i}(\tau)/24$ as in Equation (6.21). We have thus verified that the non-holomorphic dependence of $D4$-brane partition functions is indeed consistent with (6.21) and therefore with (3.9) for topological strings as implied by T-duality. Note that for $c_1 \cdot \mathbf{a}_1 = 1$ mod 2, the non-holomorphic dependence of $f_{2,c_1}(\tau; \mathcal{J})$ vanishes in the limit $\mathcal{J} \to \mathcal{J}_{1,0}$, in agreement with (6.29).

Rank 3   Similarly as for $r = 2$, [57] also decomposes $h_{3C,\mathbf{0}}^{\mathrm{pr}}(\tau)$ into different Weyl orbits. We will restrict in the following to the $\mathbf{e}_0 = \mathbf{0}$ orbit in $E/3E$ since the expressions become rather lengthy. In order to present $h_{3,\mathbf{e}_0}^{\mathrm{pr}}(\tau; \mathbf{a}_1)$, define

$$(6.33) \qquad b_{3,\ell}(\tau) = \sum_{m,n \in \mathbb{Z} + \ell/3} q^{m^2 + n^2 + mn}.$$

Then $h_{3,\mathbf{e}_0}^{\mathrm{pr}}(\tau; \mathbf{a}_1)$ is given by [57]

$$(6.34) \quad h_{3,\mathbf{e}_0}^{\mathrm{pr}}(\tau; \mathbf{a}_1) = \frac{1}{2592\,\eta^{36}} \left[\left(51\,b_{3,0}^{12} - 184\,b_{3,0}^9 b_{3,1}^3 + 336\,b_{3,0}^6 b_{3,1}^6 \right. \right.$$
$$+\ 288\,b_{3,0}^3 b_{3,1}^9 + 32\,b_{3,1}^{12} \Big)$$
$$+\ E_2 b_{3,0} \left(36\,b_{3,0}^9 - 112\,b_{3,0}^6 b_{3,1}^3 + 32\,b_{3,0}^3 b_{3,1}^3 - 64\,b_{3,1}^9 \right)$$
$$\left. +\ E_2^2 b_{3,0}^2 \left(9\,b_{3,0}^6 - 16\,b_{3,0}^3 b_{3,1}^3 + 16\,b_{3,0}^6 \right) \right].$$

In order to verify this expression, we extend the analysis for $r = 2$ to $r = 3$. For $c_1 \cdot \mathbf{a}_2 = \pm 1 \mod 3$, the BPS invariants vanish for a suitable polarization

$$(6.35) \qquad\qquad h_{3,c_1}(z, \tau; \mathcal{J}_{\varepsilon,1}) = 0.$$

The HN-filtrations for the sheaves which are unstable for $\mathcal{J}_{\varepsilon,1}$ but semistable for $\mathcal{J}_{0,1}$ have length 2 or 3. From those of length 2, one obtains rational functions in w multiplied by $h_{1,0}(z,\tau)\, h_{2,\boldsymbol{\mu}}(z,\tau)\, \Theta_{2D_8,\boldsymbol{\mu}}(\tau)$, with $\boldsymbol{\mu} = \mathbf{0}$, $\mathbf{a}_2$, $\boldsymbol{d}_i$ and $\boldsymbol{d}_i + \mathbf{a}_2$. The theta function arising from the sum over the D_8 lattice is more involved for filtrations of length 3. Instead of a direct sum, a "twisted"

sum of 2 D_8-lattices appears; we will denote this lattice by D_8^{t}

$$\Theta_{2D_8^{\mathrm{t}};\boldsymbol{\mu}_1,\boldsymbol{\mu}_2}(\tau) = \sum_{\mathbf{k}_i \in D_8 + \boldsymbol{\mu}_i, i=1,2} q^{\mathbf{k}_1^2 + \mathbf{k}_1 \cdot \mathbf{k}_2 + \mathbf{k}_2^2} \tag{6.36}$$

$$= \sum_i m_i\, \Theta_{2D_8,\boldsymbol{\mu}_1+\boldsymbol{\mu}_2+\boldsymbol{d}_i}(\tau)\, \Theta_{2D_8,\boldsymbol{\mu}_1-\boldsymbol{\mu}_2+\boldsymbol{d}_i}(3\tau) \tag{6.37}$$

where m_i are the multiplicities of the theta characteristics $\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2 + \boldsymbol{d}_i$, thus for $\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2 \in D$, $i = 1, \ldots, 6$, and for $\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2 \in D/2$, $i = 1, \ldots, 4$. For numerical computations the second line is considerably faster than the first line. We obtain after a careful analysis

(6.38)

$$h_{3,0}(z,\tau;\mathcal{J}_{\varepsilon,1})$$
$$= \frac{i\eta(\tau)^3}{\theta_1(2z,\tau)^2\,\theta_1(4z,\tau)^2\,\theta_1(6z,\tau)} B_{3,0}(z,\tau)^8$$
$$+ 2\left(\frac{1}{1-w^{12}} - \frac{1}{2}\right) h_{1,0}(z,\tau) \sum_{i=0,3} h_{2,(0,0,\boldsymbol{d}_i)}(z,\tau;\mathcal{J}_{\varepsilon,1})\Theta_{2D_8,\boldsymbol{d}_i}(3\tau)$$
$$+ 2\left(\frac{w^6}{1-w^{12}}\right) h_{1,0}(z,\tau) \sum_{i=0,3} h_{2,(0,1,\boldsymbol{d}_i)}(z,\tau;\mathcal{J}_{\varepsilon,1})\Theta_{2D_8,\boldsymbol{d}_i}(3\tau)$$
$$+ 2\left(\frac{1}{1-w^6} - \frac{1}{2}\right) h_{1,0}(z,\tau) \sum_{i=1,2,4,5} m_{0,i}\, h_{2,(0,0,\boldsymbol{d}_i)}(z,\tau;\mathcal{J}_{\varepsilon,1})\Theta_{2D_8,\boldsymbol{d}_i}(3\tau)$$
$$+ 2\left(\frac{w^3}{1-w^6}\right) h_{1,0}(z,\tau) \sum_{i=0,1,2,4} m_{2,i}\, h_{2,\boldsymbol{g}_2+\boldsymbol{d}_i}(z,\tau;\mathcal{J}_{\varepsilon,1})\Theta_{2D_8,\boldsymbol{g}_2+\boldsymbol{d}_i}(3\tau)$$
$$- \left(\frac{1+w^{12}}{(1-w^8)(1-w^{12})} - \frac{1}{1-w^{12}} + \frac{1}{6}\right) h_{1,0}(z,\tau)^3\, \Theta_{2D_8^{\mathrm{t}};0,0}(\tau)$$
$$- 2\left(\frac{w^6}{(1-w^4)(1-w^{12})} - \frac{w^6}{1-w^{12}}\right) h_{1,0}(z,\tau)^3\, \Theta_{2D_8^{\mathrm{t}};\boldsymbol{g}_2,0}(\tau)$$
$$- \left(\frac{w^4+w^{16}}{(1-w^8)(1-w^{12})}\right) h_{1,0}(z,\tau)^3\, \Theta_{2D_8^{\mathrm{t}};0,0}(\tau).$$

The functions due to the blowing-up of eight points are now given by $B_{3,k}(z,\tau) = \sum_{m,n\in\mathbb{Z}+k/3} q^{m^2+n^2+mn} w^{4m+2n}/\eta(\tau)^3$. We have used in (6.38) that $h_{2,c_1}(z,\tau;\mathcal{J}_{m,n})$ only depends on the conjugacy class of c_1 in $D/2D$, and moreover that $h_{2,c_1}(z,\tau;\mathcal{J}_{m,n}) = h_{2,c_1'}(z,\tau;\mathcal{J}_{m,n})$ if $c_1 = (0,0,\boldsymbol{d}_i)$ and $c_1' = (0,1,\boldsymbol{d}_i)$ for $i = 1,2,4,5$ (but not for $i = 0,3$) and $c_1 = (0,0,\boldsymbol{d}_i) + \boldsymbol{g}_2$ and $c_1' = (0,1,\boldsymbol{d}_i) + \boldsymbol{g}_2$.

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Table 4: The Betti numbers b_n (with $n \leq \dim_{\mathbb{C}} \mathcal{M}$) and the Euler number χ of the moduli spaces of semi-stable sheaves on $\mathbb{F}_9$ with $r = 3$, $c_1 = 0$, and $3 \leq c_2 \leq 5$ for $\mathcal{J} = \mathcal{J}_{1,\varepsilon}$.

c_2	b_0	b_2	b_4	b_6	b_8	b_{10}	b_{12}
3	1	10	65	320	1025	1226	
4	1	11	77	417	1902	7372	23962
5	1	11	78	429	2002	8260	30710

c_2	b_{14}	b_{16}	b_{18}	b_{20}	b_{22}	χ
3						4068
4	57452	68847				251235
5	103867	316586	836221	1706023	2029416	8037792

Having determined $h_{3,0}(z, \tau; \mathcal{J}_{\varepsilon,1})$, what rests is to perform the wall-crossing from $\mathcal{J}_{\varepsilon,1}$ to $\mathcal{J}_{1,0}$. To this end, we define

(6.39)
$$
h_{3,c_1}^A(z, \tau; \mathcal{J}) = \sum_{\mathbf{a} = c_1|_A \mod 2A} \tfrac{1}{2} \left(\mathrm{sgn}(a_1 n + a_2 m) - \mathrm{sgn}(a_1 + a_2 \varepsilon) \right)
$$
$$
\left(w^{6a_2} - w^{-6a_2} \right) q^{-3a_1 a_2} h_{2,(\mathbf{a}, c_1|_D)}(z, \tau; \mathcal{J}_{|a_1|, |a_2|}) h_{1,0}(z, \tau),
$$

with $\mathbf{a} = (a_1, a_2)$. Then $h_{3,0}(z, \tau; \mathcal{J})$ is given by [50, 51]

$$
h_{3,0}(z, \tau; \mathcal{J}) = h_{3,0}(z, \tau; \mathcal{J}_{\varepsilon,1})
$$
$$
+ \sum_{\mathbf{a} \in 2A/A} m_{i,j} \, h_{3, \mathbf{a} + \boldsymbol{g}_i + \boldsymbol{d}_j}^A(z, \tau; \mathcal{J}) \, \Theta_{2D_8, \boldsymbol{g}_i + \boldsymbol{d}_j}(3\tau).
$$

The Betti numbers for $\mathcal{J} = \mathcal{J}_{1,0}$ and small c_2 are presented in table 4, and indeed agree with the Euler numbers computed from the periods.

One might wonder how to derive the modular properties $h_{3,0}(z, \tau; \mathcal{J})$. The completion takes in general a very complicated form due to the quadratic condition on the lattice points [51]. One can show however that for $\mathcal{J} = \mathcal{J}_{1,0}$ the quadratic condition disappears from the generating function due to a special symmetry of the lattice A, and therefore one again obtains quasi-modular forms at this point.[6]

---

[6]We thank S. Zwegers for providing this argument.

Appendix A. Toric data for the elliptic hypersurfaces

Here, we collect the toric data necessary to treat all models discussed. We list the Mori cones in the star triangulation for the bases of model 8–15 of figure 1

Δ_B ν_i^B		8(4)				9(4)			
	$l^{(1)}$	$l^{(2)}$	$l^{(3)}$	$l^{(4)}$	$l^{(1)}$	$l^{(1)}$	$l^{(2)}$	$l^{(1)}$	$l^{(2)}$
z	0	−1	0	−1	−1	−1	0	−1	−1
1	0	0	0	1	−1	1	0	0	0
2	1	0	0	0	1	−1	1	0	0
3	−2	1	0	0	0	1	−2	1	0
4	1	−1	1	0	0	0	1	−1	1
5	0	1	−2	1	0	0	0	1	−1
6	0	0	1	−1	1	0	0	0	1
ex		7				12			

		10(4)					11(5)			
	$l^{(1)}$	$l^{(2)}$	$l^{(3)}$	$l^{(4)}$	$l^{(1)}$	$l^{(2)}$	$l^{(3)}$	$l^{(4)}$	$l^{(5)}$	$l^{(6)}$
	0	−1	0	0	0	−1	0	0	−1	−1
	1	0	0	0	1	0	0	0	0	1
	−2	1	0	0	−2	1	0	0	0	0
	1	−1	1	0	1	−1	1	0	0	0
	0	1	−2	1	0	1	−2	1	0	0
	0	0	1	−2	0	0	1	−2	1	0
	0	0	0	1	0	0	0	1	−1	1
					0	0	0	0	1	−1
		4					16			

Δ_B ν_i^B			12(5)							13(6)				
	$l^{(1)}$	$l^{(2)}$	$l^{(3)}$	$l^{(4)}$	$l^{(5)}$	$l^{(6)}$	$l^{(7)}$	$l^{(1)}$	$l^{(2)}$	$l^{(3)}$	$l^{(4)}$	$l^{(5)}$	$l^{(6)}$	$l^{(7)}$
z	−1	−1	−1	0	−1	0	−1	0	−1	0	0	0	−1	0
1	1	1	0	0	0	0	0	−2	1	0	0	0	0	0
2	0	−1	1	0	0	0	0	1	−1	1	0	0	0	0
3	0	1	−1	1	0	0	0	0	1	−2	1	0	0	0
4	0	0	1	−2	1	0	0	0	0	1	−2	1	0	0
5	0	0	0	1	−1	1	0	0	0	0	1	−2	1	0
6	1	0	0	0	1	−2	−2	0	0	0	0	1	−1	1
7	−1	0	0	0	0	1	1	0	0	0	0	0	1	−2
8								1	0	0	0	0	0	1
ex				29							20			

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14(6)

$l^{(1)}$	$l^{(2)}$	$l^{(3)}$	$l^{(4)}$	$l^{(5)}$	$l^{(6)}$	$l^{(7)}$	$l^{(8)}$
−1	0	−1	0	0	−1	0	−1
−1	1	0	0	0	0	0	1
1	−2	1	0	0	0	0	0
0	1	−1	1	0	0	0	0
0	0	1	−2	1	0	0	0
0	0	0	1	−1	1	0	0
0	0	0	0	1	−1	1	0
0	0	0	0	0	1	−2	1
1	0	0	0	0	0	1	−1
							43

15(5)

Δ_B ν_i^B	$l^{(1)}$	$l^{(2)}$	$l^{(3)}$	$l^{(4)}$	$l^{(5)}$	$l^{(6)}$	$l^{(7)}$	$l^{(8)}$
z	0	−1	0	−1	0	−1	0	−1
1	−2	1	0	0	0	0	0	1
2	1	−1	1	0	0	0	0	0
3	0	1	−2	1	0	0	0	0
4	0	0	1	−1	1	0	0	0
5	0	0	0	1	−2	1	0	0
6	0	0	0	0	1	−1	1	0
7	0	0	0	0	0	1	−2	1
8	1	0	0	0	0	0	1	−1
9								
ex								53

16(7)

$l^{(1)}$	$l^{(2)}$	$l^{(3)}$	$l^{(4)}$	$l^{(5)}$	$l^{(6)}$	$l^{(7)}$	$l^{(8)}$	$l^{(9)}$
0	0	−1	0	0	−1	0	0	−1
−2	1	0	0	0	0	0	0	1
1	−2	1	0	0	0	0	0	0
0	1	−1	1	0	0	0	0	0
0	0	1	−2	1	0	0	0	0
0	0	0	1	−2	1	0	0	0
0	0	0	0	1	−1	1	0	0
0	0	0	0	0	1	−2	1	0
0	0	0	0	0	0	1	−2	1
1	0	0	0	0	0	0	1	−1
							59	

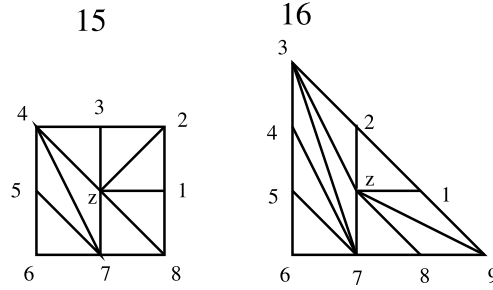Quantum geometry of elliptic Calabi–Yau manifolds   907

Figure 4: Non-star triangulations of the basis of model 15 and 16, which lead to simplicial Kähler cone for the Calabi–Yau space.

The simplicial Mori cone for the model 15 and 16 occur, e.g., for the triangulation depicted in figure 4. For the model 15, the Mori cone reads

(A.1)
$$l^{(e)} = (-6, 0, 0, 0, 0, 1, -1, 1, 0, 0, 2, 3), \quad l^{(1)} = (0, -2, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0)$$
$$l^{(2)} = (0, 1, -1, 1, 0, 0, 0, 0, 0, -1, 0, 0), \quad l^{(3)} = (0, 0, 1, -2, 1, 0, 0, 0, 0, 0, 0, 0),$$
$$l^{(4)} = (0, 0, 0, 1, -1, 1, 0, 0, 0, -1, 0, 0), \quad l^{(5)} = (0, 0, 0, 0, 0, -1, 0, 1, -1, 1, 0, 0),$$
$$l^{(6)} = (0, 0, 0, 0, 0, 0, 1, -2, 1, 0, 0, 0).$$

This yields the intersection numbers

(A.2)
$$\begin{aligned}
\mathcal{R} = {} & 4J_e^3 + 2J_e^2 J_2 + 4J_e^2 J_3 + J_e J_2 J_3 + 2J_e J_3^2 + 3J_e^2 J_4 + J_e J_2 J_4 \\
& + 2J_e J_3 J_4 + J_e J_4^2 + 2J_e^2 J_5 + J_e J_2 J_5 + 2J_e J_3 J_5 + J_e J_4 J_5 + 6J_e^2 J_6 \\
& + 2J_e J_2 J_6 + 4J_e J_3 J_6 + J_2 J_3 J_6 + 2J_3^2 J_6 + 3J_e J_4 J_6 + J_2 J_4 J_6 + 2J_3 J_4 J_6 \\
& + J_4^2 J_6 + 2J_e J_5 J_6 + J_2 J_5 J_6 + 2J_3 J_5 J_6 + J_4 J_5 J_6 + 6J_e J_6^2 + 2J_2 J_6^2 \\
& + 4J_3 J_6^2 + 3J_4 J_6^2 + 2J_5 J_6^2 + 6J_6^3 + 5J_e^2 J_7 + 2J_e J_2 J_7 + 4J_e J_3 J_7 \\
& + J_2 J_3 J_7 + 2J_3^2 J_7 + 3J_e J_4 J_7 + J_2 J_4 J_7 + 2J_3 J_4 J_7 + J_4^2 J_7 + 2J_e J_5 J_7 \\
& + J_2 J_5 J_7 + 2J_3 J_5 J_7 + J_4 J_5 J_7 + 6J_e J_6 J_7 + 2J_2 J_6 J_7 + 4J_3 J_6 J_7 \\
& + 3J_4 J_6 J_7 + 2J_5 J_6 J_7 + 6J_6^2 J_7 + 5J_e J_7^2 + 2J_2 J_7^2 + 4J_3 J_7^2 + 3J_4 J_7^2 \\
& + 2J_5 J_7^2 + 6J_6 J_7^2 + 5J_7^3
\end{aligned}$$

and the evaluation of c_2 on the basis J_i

(A.3)
$$\begin{aligned}
& c_2 J_e = 52, \quad c_2 J_1 = 24, \quad c_2 J_2 = 48, \quad c_2 J_3 = 36, \\
& c_2 J_4 = 24, \quad c_2 J_5 = 72, \quad c_2 J_6 = 62.
\end{aligned}$$

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The same data for the model 16

$$
\begin{aligned}
l^{(e)} &= (-6,0,0,0,0,1,-1,1,0,0,0,2,3), \\
l^{(1)} &= (0,-2,1,0,0,0,0,0,0,1,0,0,0), \\
l^{(2)} &= (0,1,-2,1,0,0,0,0,0,0,0,0,0), \\
l^{(3)} &= (0,0,0,-1,1,0,0,-1,0,0,1,0,0), \\
l^{(4)} &= (0,0,0,1,-2,1,0,0,0,0,0,0,0), \\
l^{(5)} &= (0,0,0,0,1,-2,1,0,0,0,0,0,0), \\
l^{(6)} &= (0,0,0,0,0,0,0,1,-2,1,0,0,0), \\
l^{(7)} &= (0,1,0,0,0,0,0,0,1,-1,-1,0,0),
\end{aligned}
\tag{A.4}
$$

and the intersection is given by

$$
\begin{aligned}
\mathcal{R} = {}& 3J_e^3 + 4J_e^2 J_2 + 2J_e J_2^2 + 2J_e^2 J_3 + J_e J_2 J_3 + 6J_e^2 J_4 \\
&+ 4J_e J_2 J_4 + 2J_2^2 J_4 + 2J_e J_3 J_4 + J_2 J_3 J_4 + 6J_e J_4^2 + 4J_2 J_4^2 \\
&+ 2J_3 J_4^2 + 6J_4^3 + 5J_e^2 J_5 + 4J_e J_2 J_5 + 2J_2^2 J_5 + 2J_e J_3 J_5 + J_2 J_3 J_5 \\
&+ 6J_e J_4 J_5 + 4J_2 J_4 J_5 + 2J_3 J_4 J_5 + 6J_4^2 J_5 + 5J_e J_5^2 + 4J_2 J_5^2 \\
&+ 2J_3 J_5^2 + 6J_4 J_5^2 + 5J_5^3 + 4J_e^2 J_6 + 4J_e J_2 J_6 + 2J_2^2 J_6 + 2J_e J_3 J_6 \\
&+ J_2 J_3 J_6 + 6J_e J_4 J_6 + 4J_2 J_4 J_6 + 2J_3 J_4 J_6 + 6J_4^2 J_6 + 5J_e J_5 J_6 \\
&+ 4J_2 J_5 J_6 + 2J_3 J_5 J_6 + 6J_4 J_5 J_6 + 5J_5^2 J_6 + 4J_e J_6^2 \\
&+ 4J_2 J_6^2 + 2J_3 J_6^2 +
\end{aligned}
\tag{A.5}
$$

$$
\begin{aligned}
& 6J_4 J_6^2 + 5J_5 J_6^2 + 4J_6^3 + 3J_e^2 J_7 + 2J_e J_2 J_7 + J_e J_3 J_7 + 3J_e J_4 J_7 \\
&+ 2J_2 J_4 J_7 + J_3 J_4 J_7 + 3J_4^2 J_7 + 3J_e J_5 J_7 + 2J_2 J_5 J_7 + J_3 J_5 J_7 \\
&+ 3J_4 J_5 J_7 + 3J_5^2 J_7 + 3J_e J_6 J_7 + 2J_2 J_6 J_7 + J_3 J_6 J_7 + 3J_4 J_6 J_7 \\
&+ 3J_5 J_6 J_7 + 3J_6^2 J_7 + J_e J_7^2 + J_4 J_7^2 + J_5 J_7^2 + J_6 J_7^2 + 6J_e^2 J_8 \\
&+ 4J_e J_2 J_8 + 2J_e J_3 J_8 + 6J_e J_4 J_8 + 4J_2 J_4 J_8 + 2J_3 J_4 J_8 \\
&+ 6J_4^2 J_8 + 6J_e J_5 J_8 + 4J_2 J_5 J_8 + 2J_3 J_5 J_8 + 6J_4 J_5 J_8 + 6J_5^2 J_8 \\
&+ 6J_e J_6 J_8 + 4J_2 J_6 J_8 + 2J_3 J_6 J_8 + 6J_4 J_6 J_8 + 6J_5 J_6 J_8 + 6J_6^2 J_8 \\
&+ 3J_e J_7 J_8 + 3J_4 J_7 J_8 + 3J_5 J_7 J_8 + 3J_6 J_7 J_8 + 6J_e J_8^2 + 6J_4 J_8^2 \\
&+ 6J_5 J_8^2 + 6J_6 J_8^2
\end{aligned}
\tag{A.6}
$$

and the evaluation on c_2 is

(A.7)
$$
\begin{aligned}
&c_2 J_e = 42, \quad c_2 J_1 = 48, \quad c_2 J_2 = 24, \quad c_2 J_3 = 72, \\
&c_2 J_4 = 62, \quad c_2 J_5 = 52, \quad c_2 J_6 = 36, \quad c_2 J_7 = 72.
\end{aligned}
$$

Appendix B. Results for the other fibre types with $\mathbb{F}_1$ base

We give some results of the periods for the different fibre types with base $\mathbb{F}_1$. The corresponding Picard–Fuchs operators read [45]

(B.1)
$$
\begin{aligned}
\mathcal{L}_{E7} &= \theta^2 - 4z(4\theta + 3)(4\theta + 1), \\
\mathcal{L}_{E6} &= \theta^2 - 3z(3\theta + 2)(3\theta + 1) \\
\mathcal{L}_{D5} &= \theta^2 - 4z(2\theta + 1)^2.
\end{aligned}
$$

The solutions read as follows:

(B.2)
$$
\phi_{E7} = \sum_{n\geq 0}^{\infty} \frac{(4n)!}{(n!)^2(2n)!} z^n = {}_2F_1\left(\frac{3}{4}, \frac{1}{4}, 1, 64z\right),
$$

$$
\phi_{E6} = \sum_{n\geq 0}^{\infty} \frac{(3n)!}{(n!)^3} z^n = {}_2F_1\left(\frac{2}{3}, \frac{1}{3}, 1, 27z\right),
$$

$$
\phi_{D5} = \sum_{n\geq 0}^{\infty} \frac{(2n)!^2}{(n!)^4} z^n = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1, 16z\right),
$$

with

(B.3)
$$
{}_2F_1(a, b, c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!},
$$

where $(a)_n = a(a+1)\ldots(a+n-1)$ denotes the Pochhammer symbol.

The j-functions read for these read

(B.4)
$$
\begin{aligned}
1728 j_{E7} &= \frac{(1 + 192z)^3}{z(1 - 64z)^2}, \\
1728 j_{E6} &= \frac{(1 + 216z)^3}{z(1 - 27z)^3}, \\
1728 j_{D5} &= \frac{(1 + 244z + 256z^2)}{z(-1 + 16z)^4}.
\end{aligned}
$$

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We collect the expressions for the solutions in terms of modular forms

$$
\begin{aligned}
\phi_{E7}(z(q))^2 &= 1 + 24q + 24q^2 + 96q^3 + \cdots = -E_2(\tau) + 2E_2(2\tau), \\
\phi_{E6}(z(q)) &= 1 + 6q + 6q^3 + \cdots = \sum_{m,n\in\mathbb{Z}} q^{m^2+n^2+mn} \\
&= \theta_2(\tau)\theta_2(3\tau) + \theta_3(\tau)\theta_3(3\tau), \\
\phi_{D5}(z(q)) &= 1 + 4q + 4q^2 + \cdots = \theta_3(2\tau)^2.
\end{aligned}
$$

(B.5)

Following analogous steps presented in Section 3.2, one can again proof the holomorphic anomaly equation for genus 0.

Appendix C. Modular functions

This appendix lists various modular functions, which appear in the generating functions in the main text. Define $q := \mathrm{e}^{2\pi\mathrm{i}\tau}$ and $w := \mathrm{e}^{2\pi\mathrm{i}z}$, with $\tau \in \mathbb{H}$ and $z \in \mathbb{C}$. The Dedekind eta and Jacobi theta functions are defined by

$$
\begin{aligned}
\eta(\tau) &:= q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n), \\
\theta_1(z,\tau) &:= i \sum_{r\in\mathbb{Z}+\frac{1}{2}} (-1)^{r-\frac{1}{2}} q^{\frac{r^2}{2}} w^r, \\
\theta_2(z,\tau) &:= \sum_{r\in\mathbb{Z}+\frac{1}{2}} q^{r^2/2} w^r, \\
\theta_3(z,\tau) &:= \sum_{n\in\mathbb{Z}} q^{n^2/2} w^n.
\end{aligned}
$$

(C.1)

The Eisenstein series $E_{2k}(\tau)$ of weight $2k$ are given by:

$$
E_{2k}(\tau) = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n)\, q^n,
$$

(C.2)

with B_{2k} the Bernoulli numbers and $\sigma_k(n) = \sum_{m|n,m>0} m^k$ the divisor function.

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We define the indefinite theta function $F(\tau, u, v)$ for $0 < -\operatorname{Im} u / \operatorname{Im} \tau < 1$ and $0 < \operatorname{Im} v / \operatorname{Im} \tau < 1$ [26]

$$(C.3) \quad F(\tau, u, v) = \sum_{n \geq 0, m > 0} q^{mn} \, \mathrm{e}^{2\pi i u n + 2\pi i v m} - \sum_{n > 0, m \geq 0} q^{mn} \, \mathrm{e}^{-2\pi i u n - 2\pi i v m}$$
$$= \sum_{n \geq 0, m > 0} - \sum_{n < 0, m \leq 0} q^{nm} \, \mathrm{e}^{2\pi i u n + 2\pi i v m}.$$

Analytic extension of this function gives

$$(C.4) \qquad\qquad F(\tau, u, v) = -i \frac{\eta(\tau)^3 \, \theta_1(\tau, u + v)}{\theta_1(\tau, u) \, \theta_1(\tau, v)}.$$

Acknowledgments

References

[1] M. Aganagic, A. Klemm, M. Marino and C. Vafa, *The Topological vertex*, Commun. Math. Phys. **254** (2005), 425 [arXiv:hep-th/0305132].

[2] M. Alim, B. Haghighat, M. Hecht, A. Klemm, M. Rauch and T. Wotschke, *Wall-crossing holomorphic anomaly and mock modularity of multiple M5-branes*, arXiv:1012.1608 [hep-th].

[3] B. Andreas, G. Curio, D.H. Ruiperez and S.-T. Yau, *Fourier–Mukai transform and mirror symmetry for D-branes on elliptic Calabi–Yau*, arXiv:math/0012196 [math-ag].

[4] B. Andreas, G. Curio, D. Hernandez Ruiperez and S.-T. Yau, *Fiber wise T duality for D-branes on elliptic Calabi–Yau*, J. High Energy Phys. **03** (2001) 020, [arXiv:hep-th/0101129].

[5] E. Andriyash, F. Denef, D.L. Jafferis and G.W. Moore, *Bound state transformation walls*, arXiv:1008.3555 [hep-th].

[6]   V. Batyrev, *Dual polyhedra and mirror symmetry for Calabi–Yau hypersurfaces in toric varieties*, J. Algebr. Geom. 3 (1994), 493–535.

[7]   I. Bena, D.-E. Diaconescu and B. Florea, *Black string entropy and Fourier–Mukai transform*, J. High Energy Phys. **04** (2007) 045, [`arXiv:hep-th/0610068`].

[8]   P. Berglund, S.H. Katz and A. Klemm, *Mirror symmetry and the moduli space for generic hypersurfaces in toric varieties*, Nucl. Phys. B **456** (1995), 153, [`arXiv:hep-th/9506091`].

[9]   M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Kodaira–Spencer theory of gravity and exact results for quantum string amplitudes*, Commun. Math. Phys. **165** (1994), 311 [`arXiv:hep-th/9309140`].

[10] J. de Boer, M.C.N. Cheng, R. Dijkgraaf, J. Manschot and E. Verlinde, *A farey tail for attractor black holes*, J. High Energy Phys. **11** (2006) 024, [hep-th/0608059].

[11] J. de Boer, S. El-Showk, I. Messamah and D. Van den Bleeken, *Quantizing $N = 2$ multicenter solutions*, J. High Energy Phys. **05** (2009) 002 [`arXiv:arXiv:0807.4556` [hep-th]].

[12] V. Bouchard, A. Klemm, M. Marino and S. Pasquetti, *Remodeling the B-model*, Commun. Math. Phys. **287** (2009), 117 [`arXiv:0709.1453` [hep-th]].

[13] P. Candelas, X.C. De La Ossa, P.S. Green and L. Parkes, *A pair of Calabi–Yau manifolds as an exactly soluble superconformal theory*, Nucl. Phys. B **359** (1991), 21.

[14] P. Candelas, A. Font, S.H. Katz and D.R. Morrison, *Mirror symmetry for two parameter models. 2.*, Nucl. Phys. B **429** (1994), 626 [`arXiv:hep-th/9403187`].

[15] E. Witten, *Phase transitions in M theory and F theory*, Nucl. Phys. B **471** (1996), 195, [hep-th/9603150].

[16] A. Collinucci and T. Wyder, *The elliptic genus from split flows and Donaldson–Thomas invariants*, J. High Energy Phys. **05** (2010) 081 [`arXiv:0810.4301` [hep-th]].

[17] T.M. Chiang, A. Klemm, S.-T. Yau and E. Zaslow, *Local mirror symmetry: calculations and interpretations*, Adv. Theor. Math. Phys. **3** (1999), 495 [`arXiv:hep-th/9903053`].

Quantum geometry of elliptic Calabi–Yau manifolds   913

[18] F. Denef, *Supergravity flows and D-brane stability*, J. High Energy Phys. **08** (2000) 050 [`arXiv:hep-th/0005049`].

[19] F. Denef and G.W. Moore, *Split states, entropy enigmas, holes and halos*, J. High Energy Phys. **11** (2011) 129 [`arXiv:hep-th/0702146` [HEP-TH]].

[20] E.-D. Diaconescu and G.W. Moore, *Crossing the wall: branes versus bundles*, Adv. Theor. Math. Phys. **14** (2010), 1621–1650 [`arXiv:0706.3193` [hep-th]].

[21] W. Fulton, *Introduction to toric varieties*, Princeton University Press, 1993.

[22] D. Gaiotto, A. Strominger and X. Yin, *The M5-brane elliptic genus: modularity and BPS states*, J. High Energy Phys. **08** (2007) 070, [`arXiv:hep-th/0607010`].

[23] D. Gaiotto and X. Yin, *Examples of M5-brane elliptic genera*, J. High Energy Phys. **11** (2007) 004 [`arXiv:hep-th/0702012` [HEP-TH]].

[24] A. Givental, *Equivariant Gromov-Witten invariants*, Internat. Math. Res. Notices **13** (1996), 613–663 [`arXiv:alg-geom/9603021`].

[25] L. Göttsche, *The Betti numbers of the Hilbert scheme of points on a smooth projective surface*, Math. Ann. **286** (1990), 193.

[26] L. Göttsche and D. Zagier, *Jacobi forms and the structure of Donaldson invariants for 4-manifolds with $b_+ = 1$*, Selecta Math. New Ser. **4** (1998), 69, [`arXiv:alg-geom/9612020`].

[27] L. Göttsche, *Theta functions and Hodge numbers of moduli spaces of sheaves on rational surfaces*, Comm. Math. Physics **206** (1999), 105, [`arXiv:math.AG/9808007`].

[28] R. Gopakumar and C. Vafa, *M-theory and topological strings. 2.*, `arXiv:hep-th/9812127`.

[29] B. Haghighat and A. Klemm, *Solving the topological string on K3 fibrations*, J. High Energy Phys. **01** (2010) 009, [`arXiv:0908.0336` [hep-th]].

[30] S. Hosono, A. Klemm, S. Theisen and S.-T. Yau, *Mirror symmetry, mirror map and applications to Calabi–Yau hypersurfaces*, Commun. Math. Phys. **167** (1995), 301 [`arXiv:hep-th/9308122`].

914   Albrecht Klemm, Jan Manschot & Thomas Wotschke

[31] S. Hosono, A. Klemm, S. Theisen, S.-T. Yau, *Mirror symmetry, mirror map and applications to complete intersection Calabi–Yau spaces*, Nucl. Phys. B **433** (1995), 501–554, [`arXiv:hep-th/9406055`].

[32] S. Hosono, M.H. Saito and A. Takahashi, *Holomorphic anomaly equation and BPS state counting of rational elliptic surface*, Adv. Theor. Math. Phys. **3** (1999), 177–208, [`arXiv:hep-th/9901151`].

[33] M.-X. Huang, A. Klemm and S. Quackenbush, *Topological string theory on compact Calabi–Yau: Modularity and boundary conditions*, Lect. Notes Phys. **757** (2009), 45 [`arXiv:hep-th/0612125`].

[34] D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves*, 1996.

[35] D. Joyce and Y. Song, *A theory of generalized Donaldson–Thomas invariants*, `arXiv:0810.5645` [math.AG].

[36] M. Kaneko and D. Zagier, A generalized Jacobi theta function and quasi modular forms. in *The moduli space of curves*, Progr. Math. **129** 165–172, Birkhäuser Boston 1995.

[37] S.H. Katz, A. Klemm and C. Vafa, *Geometric engineering of quantum field theories*, Nucl. Phys. B **497** (1997), 173 [`arXiv:hep-th/9609239`].

[38] A. Klemm, B. Lian, S.S. Roan and S.-T. Yau, *Calabi–Yau fourfolds for M theory and F theory compactifications*, Nucl. Phys. B **518** (1998), 515 [`arXiv:hep-th/9701023`].

[39] A. Klemm, P. Mayr and C. Vafa, *BPS states of exceptional noncritical strings* [`arXiv:hep-th/9607139`].

[40] A. Klemm, M. Kreuzer, E. Riegler and E. Scheidegger, *Topological string amplitudes, complete intersection Calabi–Yau spaces and threshold corrections*, J. High Energy Phys. **05** (2005) 023 [`arXiv:hep-th/0410018`].

[41] A. Klemm and R. Pandharipande, *Enumerative geometry of Calabi–Yau 4-folds*, Commun. Math. Phys. **281** (2008), 621, [math/0702189].

[42] M. Kontsevich and Y. Soibelman, *Stability structures, motivic Donaldson–Thomas invariants and cluster transformations* [`arXiv:0811.2435` [math.AG]].

Quantum geometry of elliptic Calabi–Yau manifolds   915

[43] W.-P. Li and Z. Qin, *On blowup formulae for the S-duality conjecture of Vafa and Witten*, Invent. Math. **136** (1999), 451–482 [`arXiv:math.AG/9808007`].

[44] W. Lerche, P. Mayr and N.P. Warner, *Noncritical strings, Del Pezzo singularities and Seiberg–Witten curves*, Nucl. Phys. B **499** (1997), 125–148 [`arXiv:hep-th/9612085`].

[45] B.H. Lian and S.-T. Yau, *Arithmetic properties of mirror map and quantum coupling*, Commun. Math. Phys. **176** (1996), 163–192 [`arXiv:hep-th/9411234`] ———, *Mirror maps, modular relations and hypergeometric series 1* [`arXiv:hep-th/9507151`] ———, *Mirror maps, modular relations and hypergeometric series. 2.*, Nucl. Phys. Proc. Suppl. **46** (1996), 248–262 [`arXiv:hep-th/9507153`].

[46] B. Lian, K. Liu and S.T. Yau, *Mirror Principle I*, Asian J. Math. **1** (1997), 729–763 [`arXiv:alg-geom/9712011`].

[47] J.M. Maldacena, A. Strominger and E. Witten, *Black hole entropy in M theory*, J. High Energy Phys. **12** (1997) 002 [`arXiv:hep-th/9711053`].

[48] J. Manschot, *On the space of elliptic genera*, Commun. Num. Theor. Phys. **2** (2008), 803–833 [`arXiv:0805.4333` [hep-th]].

[49] J. Manschot, *Stability and duality in $\mathcal{N} = 2$ supergravity*, Commun. Math. Phys. **299** (2010), 651–676, `arXiv:0906.1767` [hep-th].

[50] J. Manschot, *Wall-crossing of D4-branes using flow trees*, Adv. Theor. Math. Phys. **15** (2011), 1 [`arXiv:1003.1570` [hep-th]].

[51] J. Manschot, *The Betti numbers of the moduli space of stable sheaves of rank 3 on $\mathbb{P}^2$*, Lett. Math. Phys. **98** (2011), 65 [`arXiv:1009.1775` [math-ph]].

[52] J. Manschot, B. Pioline and A. Sen, *Wall Crossing from Boltzmann black hole halos*, J. High Energy Phys. **07** (2011), 059 [`arXiv:1011.1258` [hep-th]].

[53] J. Manschot, *BPS invariants of $N = 4$ gauge theory on a surface*, Commun. Num. Theor. Phys. **6** (2012), 497–516 [`arXiv:1103.0012` [math-ph]].

[54] J. Manschot, B. Pioline and A. Sen, *A Fixed point formula for the index of multi-centered N=2 black holes*, J. High Energy Phys. **05** (2011) 057 [`arXiv:1103.1887` [hep-th]].

[55] J. Manschot, *BPS invariants of semi-stable sheaves on rational surfaces* [`arXiv:1109.4861` [math-ph]].

[56] P. Mayr, *Mirror symmetry, $N = 1$ superpotentials and tensionless strings on Calabi–Yau four folds*, Nucl. Phys. B **494** (1997), 489 [`arXiv:hep-th/9610162`].

[57] J.A. Minahan, D. Nemeschansky, C. Vafa and N.P. Warner, *E-strings and $\mathcal{N} = 4$ topological Yang-Mills theories*, Nucl. Phys. B **527** (1998), 581 [`arXiv:hep-th/9802168`].

[58] T. Oda, *Convex bodies and algebraic geometry: an introduction to the theory of toric varieties*, Springer, Berlin, 1988.

[59] C. Vafa and E. Witten, *A strong coupling test of S duality*, Nucl. Phys. B **431** (1994), 3 [`arXiv:hep-th/9408074`].

[60] K. Yoshioka, *The Betti numbers of the moduli space of stable sheaves of rank 2 on $\mathbb{P}^2$*, J. Reine. Angew. Math. **453** (1994), 193–220.

[61] K. Yoshioka, *The Betti numbers of the moduli space of stable sheaves of rank 2 on a ruled surface*, Math. Ann. **302** (1995), 519–540.

[62] K. Yoshioka, *The chamber structure of polarizations and the moduli of stable sheaves on a ruled surface*, Int. J. Math. **7** (1996), 411–431 [`arXiv:alg-geom/9409008`].

[63] K. Yoshioka, *Euler characteristics of SU(2) instanton moduli spaces on rational elliptic surfaces*, Commun. Math. Phys. **205** (1999), 501 [`arXiv:math/9805003`].

[64] S.P. Zwegers, *Mock theta functions*, Dissertation, University of Utrecht, 2002.

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