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On multiple higher Mahler measures and Witten zeta values associated with semisimple Lie algebras

Yoshitaka Sasaki

The Witten zeta-functions associated with semisimple Lie algebras were defined by Zagier, and their special values at even positive integers were first studied by Witten in connection with quantum gauge theory. In this paper, relations between multiple higher Mahler measures for some families of polynomials and special values of Witten zeta-functions at positive integers are showed. Consequently, a geometrical interpretation of the multiple higher Mahler measure as the volume of certain moduli space is given.

1. Introduction

The Mahler measure of a Laurent polynomial $P \in \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]\backslash\{0\}$ is defined by

$$\mathrm{m}(P) := \int_0^1 \cdots \int_0^1 \log |P(\mathrm{e}^{2\pi \mathrm{i} t_1}, \dots, \mathrm{e}^{2\pi \mathrm{i} t_n})| \, dt_1 \cdots dt_n.$$

Various properties of the Mahler measure, for instance, relationships with special values of L-functions, the volumes of certain manifolds and so on, have been discovered.

Recently, the multiple higher Mahler measure was introduced by Kurokawa et al. [6] as a generalization of the Mahler measure.

Definition 1.1 The multiple higher Mahler measure [6]. Let $\mathscr{P} := \{P_j\}_{j=1}^{l}$ be a family of Laurent polynomials $P_j \in \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]\backslash\{0\}$. Then the multiple higher Mahler measure of $\mathscr{P}$ is defined by

$$\mathrm{m}(\mathscr{P}) = \mathrm{m}(P_1, \dots, P_l) := \int_0^1 \cdots \int_0^1 \prod_{j=1}^{l} \log |P_j(\mathrm{e}^{2\pi \mathrm{i} t_1}, \dots, \mathrm{e}^{2\pi \mathrm{i} t_n})| \, dt_1 \cdots dt_n.$$

In [6], many examples of multiple higher Mahler measures for some families of polynomials were calculated. For instance, they gave the following formula:

$$\mathrm{m}(1-x, 1-\mathrm{e}^{2\pi\mathrm{i}\alpha}x, 1-\mathrm{e}^{2\pi i\beta}x) = -\frac{1}{4}\Bigg(\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{\cos(2\pi((k+l)\beta - l\alpha))}{kl(k+l)} + \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{\cos(2\pi((k+l)\alpha - l\beta))}{kl(k+l)} + \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{\cos(2\pi(l\alpha + k\beta))}{kl(k+l)}\Bigg).$$

The double series appearing in the right-hand side of the above formula are a kind of Tornheim's double series. Tornheim's double series is also defined as the special value of the multi-variable Witten zeta-function associated with $\mathfrak{sl}(3)$ (or A_2, see (5.1) below).

In this paper, we first show relations between multiple higher Mahler measures for some families of polynomials and special values of the multi-variable Witten zeta-functions for semisimple Lie algebras at positive integers. Consequently, a geometrical interpretation of the multiple higher Mahler measure as the volume of certain moduli space is naturally derived from the property of the Witten zeta-function.

The Witten zeta-function associated with a semisimple Lie algebra $\mathfrak{g}$ is defined by

$$\zeta_W(s;\mathfrak{g}) := \sum_{\varphi}(\dim\varphi)^{-s}, \tag{1.1}$$

where φ runs over all finite dimension irreducible representations of $\mathfrak{g}$. The above definition is due to Zagier [16], and the values $\zeta_W(2k;\mathfrak{g})$ for positive integers k were first studied by Witten [17] in order to express the volumes of the moduli spaces of flat connections on G bundles over the compact 2-manifold, where G is the semisimple compact Lie group. Therefore, as mentioned above, our main theorem indicates that multiple higher Mahler measures for some families of polynomials can be interpreted as the volumes of such moduli spaces. Zagier noted in [16] that

$$\zeta_W(2k;\mathfrak{g}) \in \mathbb{Q}\pi^{kl} \quad (k\in\mathbb{N}),$$

where l is the number of positive roots of $\mathfrak{g}$, by using Witten's result [17]. This formula is called “*Witten's volume formula*”. Some explicit formulas for $\zeta_W(2k,\mathfrak{g})$ $(k\in\mathbb{N})$ were given by Mordell [9], Zagier [16], Subbarao and Sitaramachandrarao [12] and Gunnells and Sczech [1]. Further Matsumoto and Tsumura [8] and Komori et al. [3] introduced the multi-variable Witten zeta-functions associated with semisimple Lie algebras, and evaluated special values at positive integers of those functions, including $\zeta_W(2k;\mathfrak{g})$, for some $\mathfrak{g}$ explicitly (see [2, 4, 5, 8]).

In the next section, we describe the definition of the multi-variable Witten zeta-function associated with $\mathfrak{g}$, which is due to Komori, Matsumoto and Tsumura. In Section 3, we introduce some notation and state the main theorem as Theorem 3.2. The proof of Theorem 3.2 is given in Section 4. In the last section, some examples of Theorem 3.2 is presented.

2. The multi-variable Witten zeta-function

In this section, we describe the definition of the multi-variable Witten zeta-functions associated with semisimple Lie algebras, which is mentioned in terms of roots and weights for the corresponding root systems by using Weyl's dimension formula. As mentioned above, the following definition is due to Matsumoto and Tsumura [8] and Komori et al. [3]. Matsumoto and Tsumura [8] first introduced the $\mathfrak{sl}(l)$ case. Afterwards, Komori et al. [3] introduced the other cases (also see [2]).

Let $\mathfrak{g}$ be a semisimple Lie algebra of rank r. We denote the set of all roots of $\mathfrak{g}$ by Δ, the set of all positive roots (resp. negative roots) by Δ_+ (resp. Δ_-) and the fundamental system of Δ by Π. For any $\alpha \in \Delta$, we denote the associated coroot by $\alpha^\vee$. Let λ_α ($\alpha \in \Pi$) be the fundamental weights satisfying $\langle \beta^\vee, \lambda_\alpha \rangle = \delta_{\beta,\alpha}$ (Kronecker's delta) for $\alpha, \beta \in \Pi$. We put

$$P := \bigoplus_{\alpha\in\Pi} \mathbb{Z}\lambda_\alpha \quad \text{and} \quad P_{++} := \bigoplus_{\alpha\in\Pi} \mathbb{Z}_{\geq 1}\lambda_\alpha.$$

Then we define the multi-variable Witten zeta-function associated with $\mathfrak{g}$ by

$$\zeta_r(\mathbf{s};\mathfrak{g}) := \sum_{\lambda\in P_{++}} \prod_{\alpha\in\Delta_+} \langle \alpha^\vee, \lambda \rangle^{-s_\alpha},$$

where $\mathbf{s} = (s_\alpha)_{\alpha\in\Delta_+} \in \mathbb{C}^n$ ($n = |\Delta_+|$). The Witten zeta-function (1.1) can be expressed as

$$\zeta_W(s;\mathfrak{g}) = K(\mathfrak{g})^s \zeta_r(\underbrace{s,\ldots,s}_{n};\mathfrak{g}),$$

where $K(\mathfrak{g}) := \prod_{\beta\in\Delta_+} \langle \beta^\vee, \sum_{\alpha\in\Pi} \lambda_\alpha \rangle$. If $\mathfrak{g}$ is of type X_r ($X = A, B, C, D, E, F, G$), then we also denote $\zeta_r(\mathbf{s}; X_r)$ instead of $\zeta_r(\mathbf{s};\mathfrak{g})$.

3. The main theorem

Hereafter, we assume $\mathbf{s} = (s_\alpha)_{\alpha\in\Delta_+}$ are positive integers. For $\alpha \in \Delta_+$, we put

$$X_\alpha^{(l)} := 1 - x_\alpha^{(l)} \quad (l = 2, \ldots, s_\alpha),$$
$$Z_\alpha := 1 - \prod_{\beta\in\Delta_+\setminus\Pi} z_\beta^{C_{\beta,\alpha}} \prod_{l=2}^{s_\alpha} x_\alpha^{(l)},$$

where $x_\alpha^{(l)}$ and z_β are indeterminate elements and

$$C_{\beta,\alpha} = \begin{cases} \langle \beta^\vee, \lambda_\alpha \rangle & \text{when } \alpha \in \Pi, \\ \delta_{\beta,\alpha} & \text{when } \alpha \in \Delta_+ \setminus \Pi. \end{cases}$$

Then we denote the family of those polynomials by

$$\mathscr{P}(\mathbf{s};\mathfrak{g}) = \mathscr{P}(\mathbf{s};X_r) := \left\{ X_\alpha^{(l)}, Z_\alpha \;\middle|\; \alpha \in \Delta_+,\ l = 2, \ldots, s_\alpha \right\}. \tag{3.1}$$

Remark 3.1. When $s_\alpha = 1$, we except the corresponding polynomials $X_\alpha^{(l)}$ and understand that the corresponding products $\prod_{l=2}^{1} x_\alpha^{(l)}$ are equal to 1.

By using the above notation, we have

Theorem 3.2. *For a semisimple Lie algebra $\mathfrak{g}$ (type of X_r) such that $\Delta_+ \setminus \Pi \neq \emptyset$ and positive integers $\mathbf{s} = (s_\alpha)_{\alpha\in\Delta_+}$, we have*

$$\mathrm{m}(\mathscr{P}(\mathbf{s};\mathfrak{g})) = \mathrm{m}(\mathscr{P}(\mathbf{s};X_r)) = \frac{(-1)^{S_n}}{2^{S_n}} \sum_{w\in W} \zeta_r(w\mathbf{s};\mathfrak{g}), \tag{3.2}$$

where $S_n := \sum_{\alpha\in\Pi} s_\alpha$ $(n = |\Delta_+|)$, W is the Weyl group of X_r and $w\mathbf{s}$ implies the action of Weyl group to the index defined by $w\mathbf{s} = (s_{|w^{-1}(\alpha)|})_{\alpha\in\Delta_+}$.

Remark 3.3. For $\alpha \in \Delta$, we have set $|\alpha| = \alpha$ if $\alpha \in \Delta_+$ and $|\alpha| = -\alpha$ if $\alpha \in \Delta_-$ in Theorem 3.2.

4. The proof of Theorem 3.2

We put $\mathbf{e}(\theta) := \mathrm{e}^{2\pi i\theta}$ and

$$\mathcal{L}_k(\theta) := \sum_{n=1}^{\infty} \frac{\cos(2\pi n\theta)}{n^k} \quad (0 < \theta < 1, k \geq 1). \tag{4.1}$$

Note that $\mathcal{L}_1(\theta) = -\log|1 - \mathbf{e}(\theta)|$. From the definition of the multiple higher Mahler measure and the above notation, we have

$$\begin{aligned} \mathrm{m}(\mathscr{P}(\mathbf{s};\mathfrak{g})) = (-1)^{S_n} \int_0^1 \cdots \int_0^1 \prod_{\alpha\in\Delta_+} \mathcal{L}_1\Bigg(\sum_{l=2}^{s_\alpha} t_\alpha^{(l)} + \sum_{\beta\in\Delta_+\setminus\Pi} C_{\beta,\alpha} v_\beta\Bigg) \\ \times \prod_{l=2}^{s_\alpha} \mathcal{L}_1(t_\alpha^{(l)})\, d\boldsymbol{T}\, d\boldsymbol{V}, \end{aligned} \tag{4.2}$$

where

$$d\boldsymbol{T}\, d\boldsymbol{V} = \prod_{\alpha\in\Delta_+} \prod_{l=2}^{s_\alpha} dt_\alpha^{(l)} \prod_{\beta\in\Delta_+} dv_\beta$$

and we have put $x_\alpha^{(l)} = \mathbf{e}(t_\alpha^{(l)})$ and $z_\beta = \mathbf{e}(v_\beta)$. The change of the order of integration and summation can be justified by the following way: for a given positive integer $\lambda \geq 2$, there exists $\delta(= \delta(\lambda))$ such that

$$|\log(2 - 2\cos(2\pi x))| < x^{-1/\lambda}$$

for any $x \in (0, \delta)$. Therefore, for any $x \in (0, \delta)$ and sufficiently large R, we see that

$$\left|\sum_{l=1}^{R} \frac{\cos(2\pi l x)}{l}\right| < |\log(2 - 2\cos(2\pi x))| + 1 < x^{-1/\lambda} + 1. \tag{4.3}$$

From the symmetric property, the left-hand side of the above inequality is estimated by $(1 - x)^{-1/\lambda} + 1$ for any $x \in (1 - \delta, 1)$. Needless to say, the left-hand side of the above inequality is bounded for any $x \in [\delta, 1 - \delta]$. Hence we see that

$$\int_0^1 \left|\sum_{l=1}^{R} \frac{\cos(2\pi l(x + \theta))}{l} \sum_{m=1}^{R} \frac{\cos(2\pi m x)}{m}\right| dx < \infty,$$

for any $\theta \in [0,1]$. Therefore, by Lebesgue's convergence theorem, we may integrate term-by-term and obtain

$$\begin{aligned}
&\int_0^1 \sum_{l=1}^{\infty} \frac{\cos(2\pi l(x+\theta))}{l} \sum_{m=1}^{\infty} \frac{\cos(2\pi m x)}{m}\, dx \\
&\quad = \int_0^1 \lim_{R\to\infty} \sum_{l=1}^{R} \frac{\cos(2\pi l(x+\theta))}{l} \sum_{m=1}^{R} \frac{\cos(2\pi m x)}{m}\, dx \\
&\quad = \lim_{R\to\infty} \sum_{l=1}^{R} \sum_{m=1}^{R} \frac{1}{lm} \int_0^1 \cos(2\pi l(x+\theta)) \cos(2\pi m x)\, dx = \frac{1}{2} \sum_{l=1}^{\infty} \frac{\cos(2\pi l\theta)}{l^2}.
\end{aligned}$$

Here, we have used

$$\int_0^1 \cos(2\pi((l-m)x+\theta))\, dx = \begin{cases} \cos 2\pi\theta & \text{if } l=m, \\ 0 & \text{otherwise.} \end{cases}$$

Repeating this manner, we have

$$\mathrm{m}(\mathscr{P}(\mathbf{s};\mathfrak{g})) = \frac{(-1)^{S_n}}{2^{S_n-n}} \int_0^1 \cdots \int_0^1 \prod_{\alpha\in\Delta_+} \mathcal{L}_{s_\alpha}\Bigg(\sum_{\beta\in\Delta_+\setminus\Pi} C_{\beta,\alpha} v_\beta \Bigg)\, d\boldsymbol{V}. \tag{4.4}$$

From the definition of $C_{\beta,\alpha}$, the integrand of the above formula can be rewritten as

$$\prod_{\alpha\in\Pi} \mathcal{L}_{s_\alpha}\Bigg(\sum_{\beta\in\Delta_+\setminus\Pi} \langle \beta^\vee, \lambda_\alpha\rangle v_\beta \Bigg) \prod_{\beta\in\Delta_+\setminus\Pi} \mathcal{L}_{s_\alpha}(v_\beta).$$

Therefore, performing the integration in (4.4), we have

$$\begin{aligned}
\mathrm{m}(\mathscr{P}(\mathbf{s};\mathfrak{g})) = \frac{(-1)^{S_n}}{2^{S_n}} \sum_{\boldsymbol{\sigma}\in\mathfrak{S}} \sum_{\substack{m_\alpha>0\ (\alpha\in\Pi) \\ \sigma_\beta\langle\beta^\vee, \sum_{\alpha\in\Pi}\sigma_\alpha m_\alpha\lambda_\alpha\rangle>0 \\ \text{for } {}^\forall\beta\in\Delta_+\setminus\Pi}} \cdots \sum \prod_{\alpha\in\Pi} \frac{1}{m_\alpha^{s_\alpha}} \\
\times \prod_{\beta\in\Delta_+\setminus\Pi} \frac{1}{(\sigma_\beta\langle\beta^\vee, \sum_{\alpha\in\Pi}\sigma_\alpha m_\alpha\lambda_\alpha\rangle)^{s_\beta}}
\end{aligned} \tag{4.5}$$

$$= \frac{(-1)^{S_n}}{2^{S_n}} \sum_{\boldsymbol{\sigma}\in\mathfrak{S}} \sum_{\substack{\sigma_\alpha m_\alpha>0 \\ \sigma_\beta\langle\beta^\vee,\sum_{\alpha\in\Pi} m_\alpha\lambda_\alpha\rangle>0 \\ \text{for } {}^\forall\beta\in\Delta_+\setminus\Pi}} \cdots \sum_{(\alpha\in\Pi)} \prod_{\alpha\in\Pi} \frac{1}{|m_\alpha|^{s_\alpha}} \times \prod_{\beta\in\Delta_+\setminus\Pi} \frac{1}{|\langle\beta^\vee,\sum_{\alpha\in\Pi} m_\alpha\lambda_\alpha\rangle|^{s_\beta}}.$$

Here we have put $\boldsymbol{\sigma} = ((\sigma_\alpha)_{\alpha\in\Delta_+}) \in \mathfrak{S} := \{\pm 1\}^n$. By performing the summation for $\boldsymbol{\sigma} \in \mathfrak{S}$, the inner series in the above formula can be rewritten as follows:

$$\begin{aligned} \mathrm{m}(\mathscr{P}(\mathbf{s};\mathfrak{g})) &= \frac{(-1)^{S_n}}{2^{S_n}} \sum_{\substack{m_\alpha\neq 0 \\ \langle\beta^\vee,\sum_{\alpha\in\Pi} m_\alpha\lambda_\alpha\rangle\neq 0 \\ \text{for } {}^\forall\beta\in\Delta_+\setminus\Pi}} \cdots \sum_{(\alpha\in\Pi)} \prod_{\alpha\in\Pi} \frac{1}{|m_\alpha|^{s_\alpha}} \\ &\quad \times \prod_{\beta\in\Delta_+\setminus\Pi} \frac{1}{|\langle\beta^\vee,\sum_{\alpha\in\Pi} m_\alpha\lambda_\alpha\rangle|^{s_\beta}} \\ &= \frac{(-1)^{S_n}}{2^{S_n}} \sum_{\substack{\lambda\in P \\ \langle\alpha^\vee,\lambda\rangle\neq 0 \text{ for } {}^\forall\alpha\in\Delta_+}} \prod_{\alpha\in\Delta_+} \frac{1}{|\langle\alpha^\vee,\lambda\rangle|^{s_\alpha}}. \end{aligned}$$

The last formula implies that the sum of Witten zeta values for the action of the corresponding Weyl group. Thus, we have Theorem 3.2. □

5. Examples

5.1. A_2 case

The multi-variable Witten zeta-function associated with A_2 ($\mathfrak{sl}(3)$) is

$$\zeta_2(s_1,s_2,s_3;A_2) = \sum_{m,n=1}^{\infty} \frac{1}{m^{s_1}n^{s_2}(m+n)^{s_3}}. \tag{5.1}$$

The above series was already introduced by Tornheim [13] in 1950. Therefore, (5.1) is so-called "Tornheim's double series". Further Mordell [9] treated the special case such that $s_1 = s_2 = s_3$ and evaluated special values at even positive integers in 1958.

For $\mathbf{s} = (s_1, s_2, s_3) \in \mathbb{N}^3$, we put

$$X_j^{(l)} = 1 - x_j^{(l)}, \qquad \text{for } j = 1,2,3 \text{ and } l = 2, \dots, s_j,$$
$$Z_j = 1 - z\prod_{l=2}^{s_j} x_j^{(l)}, \qquad \text{for } j = 1,2,3.$$

Then the family of polynomials (3.1) for A_2 can be rewritten as

$$\mathscr{P}(\mathbf{s}; A_2) = \{X_j^{(l)}, Z_j \mid j = 1,2,3,\ l = 2, \dots, s_j\}$$

and Theorem 3.2 for A_2 can be described as

Theorem 5.1. *For any positive integers* $\mathbf{s} = (s_1, s_2, s_3)$, *we have*

$$\text{(5.2)} \qquad \mathrm{m}(\mathscr{P}(\mathbf{s}; A_2)) = \frac{(-1)^{S_3}}{2^{S_3-1}}\big(\zeta_2(s_1, s_2, s_3; A_2) + \zeta_2(s_2, s_3, s_1; A_2) + \zeta_2(s_3, s_1, s_2; A_2)\big),$$

where $S_3 = \sum_{j=1}^{3} s_j$.

In some case, we can evaluate the right-hand side of (5.2) in terms of Riemann zeta values or multiple zeta values. For instance, Nakamura [10] showed the following formula:

$$\text{(5.3)} \qquad \zeta_2(a, b, s; A_2) + (-1)^b\zeta_2(b, s, a; A_2) + (-1)^a\zeta_2(s, a, b; A_2)$$
$$= \frac{2}{a!b!}\sum_{k=0}^{\max(a,b)/2}\left\{a\binom{b}{2k} + b\binom{a}{2k}\right\}(a+b-2k-1)!(2k)!\zeta(2k) \times \zeta(a+b+s-2k),$$

which holds for all $a, b \in \mathbb{N}$ and $s \in \mathbb{C}$ except for the singular points of each side of this formula.

Remark 5.2. The similar formula was proved by Tsumura [15] before Nakamura showed the above formula. However that of Nakamura is simpler than that of Tsumura.

By using the above formula, we have

Corollary 5.3. *For positive even integers s_1 and s_2, we have*

$$\begin{aligned} \mathrm{m}(\mathscr{P}(\mathbf{s};A_2)) =& \frac{(-1)^{S_3}}{2^{S_3-2}s_1!s_2!} \sum_{k=0}^{\max(s_1,s_2)/2} \left\{ s_1\binom{s_2}{2k} + s_2\binom{s_1}{2k} \right\} \\ &\times (s_1+s_2-2k-1)!(2k)!\zeta(2k)\zeta(S_3-2k). \end{aligned}$$

Example 5.4. (1) $(s_1,s_2,s_3)=(2,1,1)$,

$$\begin{aligned} \mathrm{m}(1-xz,1-x,1-z,1-z) &= \frac{1}{8}\Big(2\zeta_2(2,1,1;A_2)+\zeta_2(1,1,2;A_2)\Big) \\ &= \frac{3}{8}\zeta(4) = \frac{\pi^4}{240}. \end{aligned}$$

(2) $(s_1,s_2,s_3)=(2,2,2)$ (Witten's volume formula)

$$\begin{aligned} &\mathrm{m}(1-x_1z,1-x_1,1-x_2z,1-x_2,1-x_3z,1-x_3) \\ &= \frac{3}{32}\zeta_2(2,2,2;A_2) = \frac{\pi^6}{30240}. \end{aligned}$$

(3) $(s_1,s_2,s_3)=(2,2,3)$,

$$\begin{aligned} &\mathrm{m}(1-x_1z,1-x_1,1-x_2z,1-x_2,1-x_3^{(2)}x_3^{(3)}z,1-x_3^{(2)},1-x_3^{(3)}) \\ &= -\frac{1}{64}\Big(2\zeta_2(3,2,2;A_2)+\zeta_2(2,2,3;A_2)\Big) = -\frac{1}{32}\Big(-3\zeta(7)+2\zeta(2)\zeta(5)\Big). \end{aligned}$$

5.2. B_2 and C_2 cases

The multi-variable Witten zeta-function associated with B_2 and C_2 is

$$\zeta_2(s_1,s_2,s_3,s_4;B_2) = \sum_{m,n=1}^{\infty} \frac{1}{m^{s_1}n^{s_2}(m+n)^{s_3}(m+2n)^{s_4}}.$$

Matsumoto [7] first introduced the above function as $\zeta_{\mathfrak{so}(5)}(s_1,s_2,s_3,s_4)$ and showed the meromorphic continuation of it to the whole $\mathbb{C}^4$ space.

For $\mathbf{s}=(s_1,s_2,s_3,s_4)\in\mathbb{N}^4$, we put

$$\begin{aligned} X_j^{(l)} &= 1-x_j^{(l)}, && \text{for } j=1,2,3,4 \text{ and } l=2,\ldots,s_j, \\ Z_j &= 1-\prod_{i=1}^{2} z_i^{\mathcal{C}_{i,j}} \prod_{l=2}^{s_j} x_j^{(l)}, && \text{for } j=1,2,3,4, \end{aligned}$$

where

$$\mathcal{C}_{i,j} = \begin{cases} 2 & \text{if } (i,j) = (2,2), \\ 0 & \text{if } (i,j) = (2,3),(1,4), \\ 1 & \text{otherwise.} \end{cases}$$

Then the family of polynomials (3.1) for B_2 (C_2) can be rewritten as

$$\mathscr{P}(\mathbf{s}; B_2) = \{X_j^{(l)}, Z_j \mid j = 1,2,3,4,\ l = 2,\dots,s_j\}$$

and Theorem 3.2 for B_2 (C_2) can be described as

Theorem 5.5. *For any positive integers* $\mathbf{s} = (s_1, s_2, s_3, s_4)$, *we have*

$$\text{(5.4)} \quad \mathrm{m}(\mathscr{P}(\mathbf{s}; B_2)) = \frac{(-1)^{S_4}}{2^{S_4-1}} \Big(\zeta_2(s_1,s_2,s_3,s_4;B_2) + \zeta_2(s_1,s_3,s_2,s_4;B_2) + \zeta_2(s_4,s_3,s_2,s_1;B_2) + \zeta_2(s_4,s_2,s_3,s_1;B_2) \Big),$$

where $S_4 = \sum_{j=1}^4 s_j$.

It is known that certain Witten zeta values associated with B_2 (C_2) can be expressed by using Riemann zeta values. For instance,

Theorem 5.6 Theorem in [14], Proposition 5.2 in [11]. *Suppose that* $s_j \in \mathbb{N} \cup \{0\}$ *with* $s_1 \geq 1$, $s_1+s_2+s_3 > 1$, $s_1+s_2+s_4 > 1$, $s_2+s_3+s_4 > 1$, $\sum_{j=1}^4 s_j > 2$, *and that* $\sum_{j=1}^4 s_j$ *is odd. Then* $\zeta_2(s_1,s_2,s_3,s_4;B_2)$ *can be expressed as a rational linear combination of products of Riemann zeta values at positive integers.*

Therefore $m(\mathscr{P}(\mathbf{s}; B_2))$ with $\sum_{j=1}^4 s_j \equiv 1 \pmod 2$ can be expressed in terms of Riemann zeta values. Further Nakamura showed an explicit formula which is similar to the right-hand side of (5.4).

Theorem 5.7 Theorem 5.5 in [11]. *The following formula holds for* $a, b, c \in \mathbb{N}$ *and* $s \in \mathbb{C}$ *except for singular points of each side of the formula:*

(5.5)

$$\zeta_2(a,b,s,c;B_2) + (-1)^b \zeta_2(c,b,s,a;B_2) + (-1)^a \zeta_2(a,s,b,c;B_2) + (-1)^{b+c} \zeta_2(c,s,b,a;B_2) = 2 \sum_{d=0}^{a} \binom{a+b-d-1}{a-d} \sum_{j=0}^{\max(c,d)/2}$$

$$\times \left\{ \binom{c+d-2j-1}{c-2j} + \binom{c+d-2j-1}{d-2j} \right\} 2^{2j-c-d}\zeta(2j)$$
$$\times \zeta(a+b+c+s-2j) + 2\sum_{d=0}^{b}(-1)^d \binom{a+b-d-1}{b-d} \sum_{j=0}^{\max(c,d)/2}$$
$$\times \left\{ \binom{c+d-2j-1}{c-2j} + \binom{c+d-2j-1}{d-2j} \right\} \zeta(2j)\zeta(a+b+c+s-2j).$$

By using Theorem 5.7, we can concretely evaluate the right-hand side of (5.4) in some cases.

Example 5.8. (1) $(s_1, s_2, s_3, s_4) = (1, 2, 1, 1)$

$$\begin{aligned} &\mathrm{m}(1 - z_1z_2, 1 - x_2z_1z_2^2, 1 - x_2, 1 - z_1, 1 - z_2) \\ &\quad = -\frac{1}{2^4}\{2\zeta_2(1, 2, 1, 1; B_2) + 2\zeta_2(1, 1, 2, 1; B_2)\} \\ &\quad = -\frac{1}{2^4}\left\{-3\zeta(5) + \frac{\pi^2}{3}\zeta(3)\right\}, \end{aligned}$$

since we see that

$$\zeta_2(1, 2, 1, 1; B_2) = -\frac{13}{8}\zeta(5) + \frac{\pi^2}{6}\zeta(3), \qquad \zeta_2(1, 1, 2, 1; B_2) = \frac{1}{8}\zeta(5)$$

from the list in [14, p. 151].

(2) $(s_1, s_2, s_3, s_4) = (2, 2, 2, 2)$ (Witten's volume formula)

$$\begin{aligned} &\mathrm{m}(1 - x_1z_1z_2, 1 - x_2z_1z_2^2, 1 - x_3z_1, 1 - x_4z_2, 1 - x_1, 1 - x_2, 1 - x_3, 1 - x_4) \\ &\quad = \frac{\zeta_2(2, 2, 2, 2; B_2)}{2^5} = \frac{\pi^8}{9676800}, \end{aligned}$$

since $\zeta_2(2, 2, 2, 2; B_2) = \pi^8/302400$ (see [4, (264)]).

5.3. G_2 case

The multi-variable Witten zeta-function associated with G_2 is

$$\begin{aligned} &\zeta_2(s_1, s_2, s_3, s_4, s_5, s_6; G_2) \\ &\quad = \sum_{m,n=1}^{\infty} \frac{1}{m^{s_1}n^{s_2}(m+n)^{s_3}(m+2n)^{s_4}(m+3n)^{s_5}(2m+3n)^{s_6}}. \end{aligned}$$

For $\mathbf{s} = (s_1, \ldots, s_6) \in \mathbb{N}^6$, we put

$$X_j^{(l)} = 1 - x_j^{(l)}, \qquad \text{for } j = 1, \ldots, 6 \text{ and } l = 2, \ldots, s_j,$$
$$Z_j = 1 - \prod_{i=1}^{4} z_i^{\mathcal{C}_{i,j}} \prod_{l=2}^{s_j} x_j^{(l)}, \qquad \text{for } j = 1, \ldots, 6,$$

where

$$\mathcal{C}_{i,j} = \begin{cases} 3 & \text{if } (i,j) = (3,2), (4,2), \\ 2 & \text{if } (i,j) = (2,2), (4,1), \\ 1 & \text{if } (i,j) = (1,1), (1,2), (2,1), (3,1), \\ 0 & \text{otherwise.} \end{cases}$$

Then the family of polynomials (3.1) for G_2 can be rewritten as

$$\mathscr{P}(\mathbf{s}; G_2) = \{X_j^{(l)}, Z_j \mid j = 1, \ldots, 6,\ l = 2, \ldots, s_j\}$$

and Theorem 3.2 for G_2 can be described as

Theorem 5.9. *For any positive integers* $\mathbf{s} = (s_1, \ldots, s_6)$*, we have*

(5.6)
$$\begin{aligned} \mathrm{m}(\mathscr{P}(\mathbf{s}; G_2)) = \frac{(-1)^{S_6}}{2^{S_6 - 1}} \big\{ &\zeta_2(s_1, s_2, s_3, s_4, s_5, s_6; G_2) \\ &+ \zeta_2(s_6, s_3, s_4, s_2, s_1, s_5; G_2) + \zeta_2(s_6, s_4, s_3, s_2, s_5, s_1; G_2) \\ &+ \zeta_2(s_5, s_4, s_2, s_3, s_6, s_1; G_2) + \zeta_2(s_5, s_2, s_4, s_3, s_1, s_6; G_2) \\ &+ \zeta_2(s_1, s_3, s_2, s_4, s_6, s_5; G_2) \big\}. \end{aligned}$$

In [5], Komori et al. considered a "Weyl group symmetric" linear combination of the Witten zeta-function associated with G_2 which is similar to the right-hand side of (5.6) and evaluated special values at positive integers of such function (see Section 2 in [5]). From their theorem, we have

Corollary 5.10. *For even positive integers* $(s_1, \ldots, s_6)$*, we have*

$$\mathrm{m}(\mathscr{P}(\mathbf{s}; G_2)) = \frac{(-1)^{S_6}}{2^{S_6 - 1}} \left(\prod_{\alpha \in \Delta_+(G_2)} \frac{(2\pi i)^{s_\alpha}}{s_\alpha !} \right) P(\mathbf{s}; 0; G_2),$$

where $P(\mathbf{s}; 0; G_2)$ *are certain rational numbers depend on* $\mathbf{s}$ *and the root system* G_2*.*

Remark 5.11. Numbers $P(\mathbf{s};0;G_2)$ can be calculated. Indeed, $P(\mathbf{s};0;G_2)$ are defined as coefficients of the Taylor expansion of a function $F(\mathbf{t};\mathbf{y};G_2)$ introduced in [5].

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Osaka University of Health and Sport Sciences,
Asashirodai 1-1
Kumatori-cho
Sennan-gun Osaka 590-0496
Japan
E-mail address: `ysasaki@ouhs.ac.jp`

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