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Moonshine for M_{24} and Donaldson invariants of $\mathbb{C}\mathrm{P}^2$

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Eguchi et al. [9] recently conjectured a new *moonshine* phenomenon. They conjecture that the coefficients of a certain mock modular form $H(\tau)$, which arises from the $K3$ surface elliptic genus, are sums of dimensions of irreducible representations of the Mathieu group M_{24}. We prove that $H(\tau)$ surprisingly also plays a significant role in the theory of Donaldson invariants. We prove that the Moore–Witten [15] u-plane integrals for $H(\tau)$ are the SO(3)-Donaldson invariants of $\mathbb{C}\mathrm{P}^2$. This result then implies a moonshine phenomenon where these invariants conjecturally are expressions in the dimensions of the irreducible representations of M_{24}. Indeed, we obtain an explicit expression for the Donaldson invariant generating function $\mathrm{Z}(p, S)$ in terms of the derivatives of $H(\tau)$.

1. Introduction and statement of results

This paper concerns the deep properties of the modular forms and mock modular forms, which arise from a study of the $K3$ surface elliptic genus. To define these objects, we require Dedekind's eta-function $\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n)$ ($\tau \in \mathbb{H}$ throughout and $q := \mathrm{e}^{2\pi \mathrm{i}\tau}$), and the classical Jacobi theta function

$$\vartheta_{ab}(v|\tau) := \sum_{n\in\mathbb{Z}} q^{\frac{(2n+a)^2}{8}} \, \mathrm{e}^{\pi i\,(2n+a)(v+\frac{b}{2})},$$

where $a, b \in \{0, 1\}$ and $v \in \mathbb{C}$. We recall some standard identities.

$\vartheta_1(v\|\tau) = \vartheta_{11}(v\|\tau)$	$\vartheta_1(0\|\tau) = 0$	$\vartheta_1'(0\|\tau) = -2\pi\eta^3(\tau)$
$\vartheta_2(v\|\tau) = \vartheta_{10}(v\|\tau)$	$\vartheta_2(0\|\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{(2n+1)^2}{8}}$	$\vartheta_2'(0\|\tau) = 0$
$\vartheta_3(v\|\tau) = \vartheta_{00}(v\|\tau)$	$\vartheta_3(0\|\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{n^2}{2}}$	$\vartheta_3'(0\|\tau) = 0$
$\vartheta_4(v\|\tau) = \vartheta_{01}(v\|\tau)$	$\vartheta_4(0\|\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{n^2}{2}}$	$\vartheta_4'(0\|\tau) = 0$

Moreover, for convenience we let $\vartheta_j(\tau) := \vartheta_j(0|\tau)$ for $j = 2, 3, 4$.

The $K3$ surface elliptic genus [7] is given by

$$Z(z|\tau) = 8\left[\left(\frac{\vartheta_2(z|\tau)}{\vartheta_2(\tau)}\right)^2 + \left(\frac{\vartheta_3(z|\tau)}{\vartheta_3(\tau)}\right)^2 + \left(\frac{\vartheta_4(z|\tau)}{\vartheta_4(\tau)}\right)^2\right].$$

This expression is obtained by an orbifold calculation on $T^4/\mathbb{Z}_2$ in [6]. Its specializations at $z = 0$, $z = 1/2$ and $z = (\tau + 1)/2$ gives the classical topological invariants $\chi{=}24, \sigma{=}16$ and $\hat{A} = -2$, respectively. Here, we consider the following alternate representation obtained by Eguchi and Hikami [8] motivated by superconformal field theory:

$$Z(z|\tau) = \frac{\vartheta_1(z|\tau)^2}{\eta(\tau)^3}\Big(24\,\mu(z;\tau) + H(\tau)\Big).$$

Here $H(\tau)$ is defined by

$$H(\tau) := -8 \sum_{w\in\left\{\frac{1}{2},\frac{1+\tau}{2},\frac{\tau}{2}\right\}} \mu(w;\tau) = 2q^{-\frac{1}{8}}\left(-1 + \sum_{n=1}^{\infty} A_n\, q^n\right), \tag{1.1}$$

where $\mu(z;\tau)$ is the famous function

$$\mu(z;\tau) = \frac{i\,\mathrm{e}^{\pi \mathrm{i} z}}{\vartheta_1(z|\tau)} \sum_{n\in\mathbb{Z}} (-1)^n\, \frac{q^{\frac{1}{2}n(n+1)}\,\mathrm{e}^{2\pi \mathrm{i} nz}}{1 - q^n\,\mathrm{e}^{2\pi\mathrm{i} z}}$$

defined by Zwegers [18] in his thesis on Ramanujan's mock theta functions.

As explained in [8], $H(\tau)$ is the holomorphic part of a weight 1/2 harmonic Maass form, a so-called *mock modular form*. Its first few coefficients A_n are:

n	1	2	3	4	5	6	7	8	$\cdots$
A_n	45	231	770	2277	5796	13915	30843	65550	$\cdots$

Amazingly, Eguchi et al. [9] recognized these numbers as sums of dimensions of the irreducible representations of the Mathieu group M_{24}. Indeed, the dimensions of the irreducible representations are (in increasing order):

$$1, 23, 45, 45, 231, 231, 252, 253, 483, 770, 770, 990, 990, 1035, 1035, 1035,$$
$$1265, 1771, 2024, 2277, 3312, 3520, 5313, 5544, 5796, 10395.$$

One sees that A_1, A_2, A_3, A_4 and A_5 are dimensions, while

$$A_6 = 3520 + 10395 \text{ and } A_7 = 10395 + 5796 + 5544 + 5313 + 2024 + 1771.$$

We have their "moonshine" conjecture[1] — also referred to as "umbral moonshine" [4]:

Conjecture Moonshine. *The Fourier coefficients A_n of $H(\tau)$ are given as special sums*[2] *of dimensions of irreducible representations of the simple sporadic group M_{24}.*

Here we prove that the coefficients of $H(\tau)$ encode further deep information. We compute the numbers $\mathbf{D}_{m,2n}[H(\tau)]$, the Moore–Witten [15] u-plane integrals for $H(\tau)$, and we prove that they are, up to a multiplicative factor of 12, the $SO(3)$-Donaldson invariants for $\mathbb{CP}^2$. These invariants are a sequence of rational numbers which together form a diffeomorphism class invariant for $\mathbb{CP}^2$ (for background see [5, 12–14]).

Theorem 1.1. *For all $m, n \in \mathbb{N}_0$, the* SO(3)*-Donaldson invariants $\mathbf{\Phi}_{m,2n}$ for $\mathbb{CP}^2$ satisfy*

$$12\,\mathbf{\Phi}_{m,2n} = \mathbf{D}_{m,2n}[H(\tau)].$$

Remark. The u-plane integrals $\mathbf{D}_{m,2n}[H(\tau)]$ are given explicitly in terms of the coefficients of $H(\tau)$ (see 3.1). Therefore, the Eguchi–Ooguri–Tachikawa Moonshine Conjecture implies that these Donaldson invariants are given explicitly in terms of the dimensions of the irreducible representations of M_{24}. We will discuss the numerical identities implied by Theorem 1.1 in Section 3.2. We also describe the Donaldson invariant generating function in terms of derivatives of $H(\tau)$.

This paper builds upon earlier work by the authors [14] on the Moore–Witten Conjecture for $\mathbb{CP}^2$. We shall make substantial use of the results in that paper, and we will recall the main facts that we need to prove Theorem 1.1.

In Section 2 we recall basic facts about those weight 1/2 harmonic Maass forms whose shadow is the cube of Dedekind's eta-function. In Section 3, we recall and apply the main results from [14]. In particular, we

[1] This is analogous to the "Monstrous Moonshine" conjecture by Conway and Norton which related the coefficients of Klein's j-function to the representations of the Monster [1]. By work of Frenkel, Lepowsky and Meurman [10] and Borcherds [2] (among others), moonshine for $j(\tau)$ is now understood.

[2] As in the case of the Montrous Moonshine Conjecture, there are many representations of the generic A_n, and so the proper formulation of this conjecture requires a precise description of these sums [11].

recall the relationship between the u-plane integrals for such forms and the $SO(3)$-Donaldson invariants for $\mathbb{C}\mathrm{P}^2$. We then conclude with the proof of Theorem 1.1.

2. Certain harmonic Maass forms

We let $M(\tau)$ be a weight 1/2 harmonic Maass form[3] (for definitions see [3,16,17]) for $\Gamma(2) \cap \Gamma_0(4)$ whose shadow[4] is $\eta(\tau)^3$. Namely, we have that

$$\sqrt{2}i \, \frac{d}{d\bar{\tau}} \, M(\tau) = \frac{1}{\sqrt{\mathrm{Im}\tau}} \, \overline{\eta^3(\tau)} \, . \tag{2.1}$$

For such $M(\tau)$, we write $M(\tau) = M^+(\tau) + M^-(\tau)$, where the *holomorphic part*, a *mock modular form*, is $M^+(\tau) = q^{-1/8} \sum_{n \geq 0} H_n \, q^{n/2}$. The *non-holomorphic part* $M^-(\tau)$ is

$$M^-(\tau) = -\frac{2i}{\sqrt{\pi}} \sum_{l \geq 0} (-1)^l \, \Gamma\left(\frac{1}{2}, \pi \, \frac{(2l+1)^2}{2} \, \mathrm{Im}\tau\right) \, q^{-\frac{(2l+1)^2}{8}},$$

where $\Gamma(1/2, t)$ is the incomplete Gamma function. This follows from Jacobi's identity

$$\eta(\tau)^3 = q^{\frac{1}{8}} \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{\frac{n^2+n}{2}}.$$

Remark. Note that the non-holomorphic part $M^-(\tau)$ is the same for every weight 1/2 harmonic Maass form with shadow $\eta^3(\tau)$ since this part is obtained as the "Eichler–Zagier" integral of the shadow. However, the holomorphic part is not uniquely determined. It is unique up to the addition of a *weakly holomorphic modular form*, a form whose poles (if any) are supported at cusps.

The next result gives families of modular forms from such an $M(\tau)$ using Cohen brackets. To make this precise, we recall the two Eisenstein series

$$E_2(\tau) := 1 - 24 \sum_{n=1}^{\infty} \sum_{d|n} d q^n \quad \text{and} \quad \widehat{E}_2(\tau) := E_2(\tau) - \frac{3}{\pi \mathrm{Im}\tau}.$$

The authors proved the following lemma in [14].

[3]These forms were first defined by Bruinier and Funke [3] in their work on geometric theta lifts.
[4]The term *shadow* was coined by Zagier in [17].

Lemma 2.1. [Lemma 4.10 of [14]] *Assuming the hypotheses above, we have that*

$$\mathcal{E}^k_{\frac{1}{2}}\left[M(\tau)\right] := \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+j\right)} 2^{2j}\, 3^j\, E_2^{k-j}(\tau) \left(q\frac{d}{dq}\right)^j M(\tau)$$

is modular of weight $2k+1/2$ *for* $\Gamma(2)\cap\Gamma_0(4)$, *and it satisfies*

$$\sqrt{2}i\,\frac{d}{d\bar{\tau}}\,\mathcal{E}^k_{\frac{1}{2}}\left[M(\tau)\right] = \frac{1}{\sqrt{\mathrm{Im}\tau}}\,\widehat{E}_2^k(\tau)\,\overline{\eta^3(\tau)}.$$

This lemma implies the following corollary:

Corollary 2.2. *If* $M(\tau)$ *and* $\widetilde{M}(\tau)$ *are weight* $\frac{1}{2}$ *harmonic Maass forms on* $\Gamma(2)\cap\Gamma_0(4)$ *whose shadow is* $\eta(\tau)^3$, *then*

$$\mathcal{E}^k_{\frac{1}{2}}\left[M(\tau)\right] - \mathcal{E}^k_{\frac{1}{2}}\left[\widetilde{M}(\tau)\right] = \mathcal{E}^k_{\frac{1}{2}}\left[M(\tau)-\widetilde{M}(\tau)\right] = \mathcal{E}^k_{\frac{1}{2}}\left[M^+(\tau)-\widetilde{M}^+(\tau)\right]$$

is a weakly holomorphic modular form of weight $2k+1/2$.

2.1. The $\mathcal{Q}(q)$ series

Here, we recall one explicit example of a harmonic Maass form which plays the role of $M(\tau)$ in the previous subsection. To this end, we define modular forms $\mathcal{A}(\tau)$ and $\mathcal{B}(\tau)$ by

$$\begin{aligned}\mathcal{A}(\tau) &:= A(8\tau) = \sum_{n=-1}^{\infty} a(n)q^n := \frac{\eta(4\tau)^8}{\eta(8\tau)^7} = q^{-1} - 8q^3 + 27q^7 - \cdots,\\ \mathcal{B}(\tau) &:= B(8\tau) = \sum_{n=-1}^{\infty} b(n)q^n := \frac{\eta(8\tau)^5}{\eta(16\tau)^4} = q^{-1} - 5q^7 + 9q^{15} - \cdots.\end{aligned}$$

We sieve on the Fourier expansion of $\mathcal{A}(\tau)$ to define the modular forms

$$\begin{aligned}\mathcal{A}_{3,8}(\tau) &:= A_{3,8}(8\tau) = \sum_{n\equiv 3 \pmod 8} a(n)q^n = -8q^3 - 56q^{11} + \cdots,\\ \mathcal{A}_{7,8}(\tau) &:= A_{7,8}(8\tau) = \sum_{n\equiv 7 \pmod 8} a(n)q^n = q^{-1} + 27q^7 + 105q^{15} + \cdots.\end{aligned}$$

We also recall the definition of the following mock theta function

$$\mathcal{M}(q) := q^{-1} \sum_{n=0}^{\infty} \frac{(-1)^{n+1} q^{8(n+1)^2} \prod_{k=1}^{n} (1-q^{16k-8})}{\prod_{k=1}^{n+1} (1+q^{16k-8})^2}$$
$$= -q^7 + 2q^{15} - 3q^{23} + \cdots .$$

We define

$$\mathcal{Q}^+(q) = \mathcal{Q}^+(\tau) := -\frac{7}{2}\mathcal{A}_{3,8}(\tau) + \frac{3}{2}\mathcal{A}_{7,8}(\tau) - \frac{1}{2}\mathcal{B}(\tau) + 4\mathcal{M}(q),$$

and so we have that

$$\mathcal{Q}^+(\tau/8) = \frac{1}{q^{\frac{1}{8}}} \left(1 + 28\, q^{\frac{1}{2}} + 39\, q + 196\, q^{\frac{3}{2}} + 161\, q^2 + \cdots \right). \tag{2.2}$$

In terms of this q-series, the authors proved the following theorem in [14].

Theorem 2.3. [Theorem 7.2 of [14]] *The function* $\mathcal{Q}^+(\tau/8)$ *is the holomorphic part of a weight 1/2 harmonic Maass form on* $\Gamma(2) \cap \Gamma_0(4)$ *whose shadow is* $\eta(\tau)^3$.

3. u-plane integrals, Donaldson invariants and the proof of Theorem 1.1

Suppose again that $M(\tau)$ is a weight 1/2 harmonic Maass form on $\Gamma(2) \cap \Gamma_0(4)$ whose shadow is $\eta(\tau)^3$. For $m, n \in \mathbb{N}_0$, the authors proved that the quantities

$$\mathbf{D}_{m,2n}[M^+(\tau)] := \sum_{k=0}^{n} \frac{(-1)^{k+1}}{2^{n-1}\, 3^n} \, \frac{(2n)!}{(n-k)!\, k!} \times \left[\frac{\vartheta_4^9(\tau)\, \left[\vartheta_2^4(\tau) + \vartheta_3^4(\tau)\right]^{m+n-k}}{\left[\vartheta_2(\tau)\, \vartheta_3(\tau)\right]^{2m+2n+3}} \, \mathcal{E}_{\frac{1}{2}}^{k} \left[M^+(\tau)\right] \right]_{q^0}, \tag{3.1}$$

where $[.]_{q^0}$ denotes the constant coefficient term, are the Moore–Witten *u-plane integrals for* $M(\tau)$ (cf. [14]). Note that if $\widetilde{M}(\tau)$ is another such form, then

$$\mathbf{D}_{m,2n}[M^+(\tau)] - \mathbf{D}_{m,2n}[\widetilde{M}^+(\tau)] = \mathbf{D}_{m,2n}[M^+(\tau) - \widetilde{M}^+(\tau)] \,. \tag{3.2}$$

In their seminal paper [15], Moore and Witten essentially conjectured that the u-plane integrals in (3.1) for a suitable $M^+(\tau)$ should give the

SO(3)-Donaldson invariants of $\mathbb{CP}^2$. These invariants are an infinite sequence of rational numbers $\mathbf{\Phi}_{m,2n}$ labeled by integers $m, n \in \mathbb{N}$ that can be assembled in a generating function in the two formal variables p, S:

$$\mathbf{Z}(p, S) = \sum_{m,n \geq 0} \mathbf{\Phi}_{m,2n} \ \frac{p^m}{m!} \frac{S^{2n}}{(2n)!}.$$

This power series is a diffeomorphism invariant for $\mathbb{CP}^2$. The main theorem in [14] proved this conjecture for $\mathcal{Q}^+(\tau/8)$.

Theorem 3.1. [Theorem 1.1 of [14]] *For $m, n \in \mathbb{N}_0$ we have that*

$$\mathbf{\Phi}_{m,2n} = \mathbf{D}_{m,2n}[\mathcal{Q}^+(\tau/8)].$$

Using the work in [14], we prove the following important theorem.

Theorem 3.2. *Let $M(\tau)$ be as above, then for all $m, n \in \mathbb{N}_0$ we have:*

$$\mathbf{D}_{m,2n}[\mathcal{Q}^+(\tau/8)] - \mathbf{D}_{m,2n}[M^+(\tau)] = \mathbf{D}_{m,2n}[\mathcal{Q}^+(\tau/8) - M^+(\tau)] = 0.$$

Proof. We prove that the constant terms vanish in expressions of the form

$$\sum_{k=0}^{n} \frac{(-1)^{k+1}}{2^{n-1}\, 3^n} \ \frac{(2n)!}{(n-k)!\, k!} \ \frac{\vartheta_4^9(\tau) \ \left[\vartheta_2^4(\tau) + \vartheta_3^4(\tau)\right]^{m+n-k}}{\left[\vartheta_2(\tau)\, \vartheta_3(\tau)\right]^{2m+2n+3}} \times \mathcal{E}_{\frac{1}{2}}^{k} \left[\mathcal{Q}^+(\tau/8) - M(\tau)\right].$$

It is sufficient to show that this is the case for each summand. Therefore, after rescaling $\tau \to 8\tau$ and $q \to q^8$ it is enough to show that the constant vanishes in

$$\begin{aligned} &\frac{\Theta_4^9(\tau) \ \left[16\, \Theta_2^4(\tau) + \Theta_3^4(\tau)\right]^{m+n-k}}{\left[\Theta_2(\tau)\, \Theta_3(\tau)\right]^{2m+2n+3}} \ \mathcal{E}_{\frac{1}{2}}^{k} \left[\mathcal{Q}^+(\tau) - M(8\tau)\right] \\ &\quad = \frac{\Theta_4^9(\tau)}{\Theta_2(\tau)\Theta_3(\tau)\eta(8\tau)^3} \ \frac{\left[16\, \Theta_2^4(\tau) + \Theta_3^4(\tau)\right]^{m+n-k}}{\left[\Theta_2(\tau)\, \Theta_3(\tau)\right]^{2m+2n-2k}} \\ &\quad\quad \times \frac{\eta(8\tau)^3}{(\Theta_2(\tau)\Theta_3(\tau))^{2k+2}} \ \mathcal{E}_{\frac{1}{2}}^{k} \left[\mathcal{Q}^+(\tau) - M(8\tau)\right]. \end{aligned} \tag{3.3}$$

Here, the classical theta functions are defined by

$$\Theta_2(\tau) := \frac{\eta(16\tau)^2}{\eta(8\tau)} = \sum_{n=0}^{\infty} q^{(2n+1)^2} = q + q^9 + q^{25} + \cdots,$$
$$\Theta_3(\tau) := \frac{\eta(8\tau)^5}{\eta(4\tau)^2\eta(16\tau)^2} = 1 + 2\sum_{n=1}^{\infty} q^{4n^2} = 1 + 2q^4 + 2q^{16} + 2q^{36} + \cdots,$$
$$\Theta_4(\tau) := \frac{\eta(4\tau)^2}{\eta(8\tau)} = 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{4n^2} = 1 - 2q^4 + 2q^{16} - 2q^{36} + \cdots.$$

These are related to the theta functions $\vartheta_2(\tau), \vartheta_3(\tau)$ and $\vartheta_4(\tau)$ by

$$\vartheta_2(\tau) = 2\Theta_2\left(\frac{\tau}{8}\right), \quad \vartheta_3(\tau) = \Theta_3\left(\frac{\tau}{8}\right), \quad \vartheta_4(\tau) = \Theta_4\left(\frac{\tau}{8}\right).$$

We define a weakly holomorphic modular function by

$$\widehat{Z_0}(q) = \widehat{Z_0}(\tau) := \frac{E^*(4\tau)}{\Theta_2(\tau)^2\Theta_3(\tau)^2}, \tag{3.4}$$

where $E^*(4\tau)$ is the weight 2 Eisenstein series with

$$E^*(4\tau) = 16\Theta_2(\tau)^4 + \Theta_3(\tau)^4 = 1 + 24q^4 + 24q^2 + \cdots,$$

and $\widehat{Z_0}(\tau/8)$ is a Hauptmodul for $\Gamma_0(4)$. A calculation shows that

$$q\frac{d}{dq}\widehat{Z_0}(q) = \frac{\Theta_4(\tau)^9}{\Theta_2(\tau)\Theta_3(\tau)\eta(8\tau)^3}.$$

Equation (3.3) becomes

$$q\frac{d}{dq}\widehat{Z_0}(q) \cdot \widehat{Z_0}(q)^{m+n-k} \cdot \mathcal{H}_k(q), \tag{3.5}$$

where

$$\mathcal{H}_k(q) := \frac{\eta(8\tau)^3}{(\Theta_2(\tau)\Theta_3(\tau))^{2k+2}}\, \mathcal{E}^k_{\frac{1}{2}}\left[\mathcal{Q}^+(\tau) - M(8\tau)\right]. \tag{3.6}$$

To prove the theorem, it suffices to show that the constant term in (3.5) vanishes. Hence, it is enough to show that $\mathcal{H}_k(q)$ is a polynomial in $\widehat{Z_0}(q)$. To this end, we define $M_0^*(\Gamma_0(8))$ to be the space of modular function on $\Gamma_0(8)$ which are holomorphic away from infinity, and is a subspace of $\mathbb{C}((q^2))$.

One can easily verify that $M_0^*(\Gamma_0(8))$ is precisely the set of polynomials in $\widehat{Z_0}(q)$. From Corollary 2.2, we can observe that $\mathcal{H}_k(q)$ is modular with weight 0. A calculation shows that $(\Theta_2(\tau)\Theta_3(\tau))^{-2} = q^{-2} f(q^4)$ is holomorphic away from infinity, and $f(q) \in \mathbb{Z}[[q]]$. We also have $\eta(8\tau)^3 = q\, g(q^8)$ and $\mathcal{E}^k_{\frac{1}{2}}\,[\mathcal{Q}^+(\tau) - M(8\tau)] = q^{-1}\, h(q^4)$, where $g(q), h(q) \in \mathbb{Z}[[q]]$. Hence, $\mathcal{H}_k(q) \in \mathbb{C}((q^2))$ is modular of weight 0 on $M_0^*(\Gamma_0(8))$, and so is a polynomial in $\widehat{Z}_0(\tau)$. □

3.1. Proof of Theorem 1.1

Since $H(\tau)$ is the mock modular part of a weight 1/2 harmonic Maass form on $\Gamma(2) \cap \Gamma_0(4)$ whose shadow is the $8 \times 3 \cdot \eta(\tau)^3/2 = 12\,\eta(\tau)^3$, it follows from Theorems 3.1 and 3.2 that for $m, n \in \mathbb{N}_0$ we have:

$$\begin{aligned}\mathbf{\Phi}_{m,2n} &= \mathbf{D}_{m,2n}[\mathcal{Q}^+(\tau/8)] = \mathbf{D}_{m,2n}[H(\tau)/12] + \mathbf{D}_{m,2n}\Big[\mathcal{Q}^+(\tau/8) - H(\tau)/12\Big]\\ &= \mathbf{D}_{m,2n}[H(\tau)/12] = \frac{1}{12}\,\mathbf{D}_{m,2n}[H(\tau)].\end{aligned}$$

3.2. Discussion of the identities implied by Theorem 1.1

In the table, below we list the first non-vanishing SO(3)-Donaldson invariants $\mathbf{\Phi}_{m,2n}$ of $\mathbb{C}\mathrm{P}^2$ as well as the coefficients $\mathbf{D}_{m,2n}[M^+(\tau)]$ when the mock modular form is given as $M^+(\tau) = q^{-1/8} \sum_{k\geq 0} H_k\, q^{k/2}$. In general, $\mathbf{D}_{m,2n}[M^+(\tau)]$ is non-vanishing for $m + n \equiv 0 \pmod 2$ and a rational linear combination of the first $(m + n)/2 + 1$ coefficients of $M^+(\tau)$.

(m,n)	$\mathbf{\Phi}_{m,2n}$	$\mathbf{D}_{m,2n}[M^+(\tau)]$
$(0,0)$	-1	$-\frac{1}{4}H_1 + 6H_0$
$(0,2)$	$-\frac{3}{16}$	$-\frac{49}{64}H_2 + \frac{9}{4}H_1 - \frac{2133}{64}H_0$
$(1,1)$	$-\frac{5}{16}$	$-\frac{7}{64}H_2 + \frac{1}{4}H_1 - \frac{195}{64}H_0$
$(2,0)$	$-\frac{19}{16}$	$-\frac{1}{64}H_2 - \frac{1}{4}H_1 + \frac{411}{64}H_0$
$(0,4)$	$-\frac{232}{256}$	$-\frac{14641}{1024}H_3 + \frac{2401}{128}H_2 + \frac{44631}{1024}H_1 + \frac{108741}{128}H_0$
$(1,3)$	$-\frac{152}{256}$	$-\frac{1331}{1024}H_3 - \frac{49}{128}H_2 + \frac{10341}{1024}H_1 - \frac{1749}{128}H_0$
$(2,2)$	$-\frac{136}{256}$	$-\frac{121}{1024}H_3 - \frac{91}{128}H_2 + \frac{2895}{1024}H_1 - \frac{3687}{128}H_0$
$(3,1)$	$-\frac{184}{256}$	$-\frac{11}{1024}H_3 - \frac{29}{128}H_2 + \frac{589}{1024}H_1 - \frac{753}{128}H_0$
$(4,0)$	$-\frac{680}{256}$	$-\frac{1}{1024}H_3 - \frac{7}{128}H_2 - \frac{505}{1024}H_1 + \frac{1725}{128}H_0$

Theorem 3.1 states that choosing $M^+(\tau) = \mathcal{Q}^+(\tau/8)$ from (2.2), we find equality of the Donaldson invariants $\mathbf{\Phi}_{m,2n}$ and the u-plane integral $\mathbf{D}_{m,2n}[M^+(\tau)]$. In fact, setting $H_0 = 1, H_1 = 28, H_2 = 39, H_3 = 196$ in the third column of the table above gives the Donaldson invariants of the second column.

On the other hand, the choice $M^+(\tau) = H(\tau)/12$ from (1.1) implies that $H_0 = -1/6$, $H_{2k} = A_k/6$, $H_{2k+1} = 0$ for $k \in \mathbb{N}$. Theorem 1.1 states that choosing $M^+(\tau) = H(\tau)/12$ we still find equality of the Donaldson invariants $\mathbf{\Phi}_{m,2n}$ and the u-plane integral $\mathbf{D}_{m,2n}[M^+(\tau)]$. In fact, setting $H_0 = -1/6, H_1 = 0, H_2 = 45/6, H_3 = 0$ in the third column of the table gives the Donaldson invariants of the second column as well.

The proof of Theorem 1.1 implies the following form for the generating function $\mathbf{Z}(p, S)$ of the SO(3)-Donaldson invariants of $\mathbb{C}\mathrm{P}^2$ in terms of the mock modular form $H(\tau)$:

$$\mathbf{Z}(p,s) = -\sum_{m,n\geq 0} \frac{p^m\, S^{2n}}{2^{2m+3n+4}\cdot 3^{n+1}\cdot m!\cdot n!} \times \left[q\frac{d}{dq}\widehat{Z_0}(q)\, \sum_{k=0}^{n}(-1)^k \binom{n}{k}\, \widehat{Z_0}(q)^{m+n-k}\, \widehat{\mathcal{E}^k}[H(8\tau)] \right]_{q^0}, \tag{3.7}$$

where $\widehat{Z_0}(q)$ was defined in (3.4) and we have set

$$\widehat{\mathcal{E}^k}[H(8\tau)] = \frac{\eta(8\tau)^3}{(\Theta_2(\tau)\Theta_3(\tau))^{2k+2}}\, \mathcal{E}^k_{\frac{1}{2}}\left[H(8\tau)\right].$$

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