On Rademacher sums, the largest Mathieu group and the holographic modularity of moonshine

MIRANDA C.N. CHENG AND JOHN F.R. DUNCAN

Recently a conjecture has proposed which attaches (mock) modular forms to the largest Mathieu group. This may be compared to monstrous moonshine, in which modular functions are attached to elements of the Monster group. One of the most remarkable aspects of monstrous moonshine is the following genus zero property: the modular functions turn out to be the generators for the function fields of their invariance groups. In particular, these invariance groups define genus zero quotients of the upper half-plane. It is therefore natural to ask if there is an analogue of this property in the Mathieu case, and at first glance the answer appears to be negative since not all the discrete groups arising there have genus zero. On the other hand, in this paper we prove that each (mock) modular form appearing in the Mathieu correspondence coincides with the Rademacher sum constructed from its polar part. This property, inspired by the AdS/CFT correspondence in physics, was shown previously to be equivalent to the genus zero property of monstrous moonshine. Hence, we conclude that this "Rademacher summability" property serves as the natural analogue of the genus zero property in the Mathieu case. Our result constitutes further evidence that the Rademacher method provides a powerful framework for understanding the modularity of moonshine, and leads to interesting physical questions regarding the gravitational duals of the relevant conformal field theories.

1. Introduction

In 2010, an unexpected relation between the elliptic genus of a $K3$ surface and the sporadic group M_{24} was observed by Eguchi et al. [1]. Subsequently, the connection between the $K3$ elliptic genus twisted by a group element of M_{24} and the characters of M_{24}-representations was first explored in [2,3] and further studied in [4,5]. From the point of view of the $K3$ supercon-formal sigma model, these twisted (or twining) elliptic genera are believed to specify the way in which the $K3$ elliptic cohomology furnishes an M_{24}-representation of infinite dimension. Note that such a twisted object only depends on the conjugacy class $[g]$ to which the element g belongs. From the

$\mathcal{N} = 4$ superconformal symmetry of the sigma model, it follows that they are weak Jacobi forms, and moreover admit natural decompositions leading to a set of (mock) modular forms H_g of weight 1/2. As a result of these studies the existence of a natural relationship between representations of M_{24} and the mock modular forms H_g has been conjectured. While such a relationship is yet to be established, extensive evidence has been gathered in [2, 4, 5]. In this context, the H_g are often referred to as the McKay–Thompson series and a key prediction of the conjecture is that they exhibit particular modular behaviour under certain discrete subgroups Γ_g of $\mathrm{SL}_2(\mathbb{R})$. This may be compared to the moonshine conjecture of Conway and Norton [6], which relates genus zero subgroups of $\mathrm{SL}_2(\mathbb{R})$ to elements of the Monster group. More precisely, to each conjugacy class $[g]$ of the Monster is attached a function T_g (its McKay–Thompson series) whose Fourier coefficients are given by characters of the Monster. The genus zero property of monstrous moonshine then refers to the empirical fact (ultimately proven by Borcherds [7]) that T_g defines a function on the upper half-plane that generates the function field of its invariance group Γ_g. In particular, the (compactified) quotient of the upper half-plane by Γ_g is a Riemann surface of genus zero. This property is also expected to extend to the so-called generalized moonshine [8] which attaches modular functions to commuting pairs of elements of the Monster. Ogg's observation [9] that the normalizer of $\Gamma_0(p)$ in $\mathrm{SL}_2(\mathbb{R})$ for p a prime is genus zero if and only if p divides the order of the Monster was arguably the first hint of a relation between the Monster and genus zero groups. Assuming the validity of the M_{24} conjecture, and in particular, the validity of the descriptions of the H_g given in [2, 4, 5], a natural question to ask is whether the groups $\Gamma_g \subset SL_2(\mathbb{R})$ attached to the McKay–Thompson series H_g for M_{24} share this fundamental pattern present in monstrous moonshine: do they have a similar genus zero property? At first glance the answer is negative since not all the groups Γ_g arising from M_{24} define genus zero quotients of the upper half-plane. When a pattern breaks, we may seek to replace it with a new pattern which encompasses both the old paradigm and the new "exception". Moreover, we also want to understand why a pattern exists at all in the first place. These are exactly the goals of this paper, and these considerations lead us to recast the question in a new form: a form suggested by earlier work of I. Frenkel which reconsidered the origin, and interpretation, of the genus zero property of monstrous moonshine. The article [10] makes an important step towards a physical explanation of the genus zero property. There it is shown that a discrete group Γ has genus zero if and only if a certain regularization procedure, inspired by Rademacher's work on the j-function [11] and summarized here as "Reg", renders the Poincaré-like

series

$$(1.1) \qquad T_\Gamma(\tau) = \mathrm{Reg}\left(\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} q^{\epsilon-1}\big|_\gamma\right)$$

convergent without spoiling invariance under the group Γ. (We say a group Γ has genus zero if it defines a genus zero quotient of the upper half-plane.) In the above formula, Γ_∞ is the subgroup of Γ fixing the cusp (representative) $i\infty$, the sum is over a set of representatives γ for the cosets of Γ_∞ in Γ, and $q = e^{2\pi i\tau}$. As is turns out, the above expression, which we refer to as a *Rademacher sum*, and which can be used to define a function for any discrete subgroup Γ of $\mathrm{SL}_2(\mathbb{R})$ commensurable with $\mathrm{SL}_2(\mathbb{Z})$ (for example), is anomaly free — that is, invariant under Γ even after the regularization — if and only if the group Γ has genus zero. In the case that Γ does have genus zero the Rademacher sum (1.1) defines a generator for the field of Γ-invariant functions, and it defines precisely the generator T_g when $\Gamma = \Gamma_g$ for g in the Monster. So the genus zero property of monstrous moonshine may be reformulated as follows:

> The McKay–Thompson series T_g coincides with the Rademacher sum T_Γ attached to its invariance group $\Gamma = \Gamma_g$.

Significantly, Rademacher sums with a higher-order pole q^{-n}, which are relevant for conformal field theories (CFTs) with higher central charges, also define modular functions as long as the group Γ has genus zero. A compelling physical interpretation of the Rademacher sum is provided by the so-called anti de Sitter (AdS)/CFT correspondence [12] (also referred to as the gauge/gravity duality or the holographic duality in more general contexts), which asserts, among many other things, that the partition function of a given two-dimensional (2D) CFT "with an AdS dual" equals the partition function of another physical theory in three Euclidean dimensions with gravitational interaction and with asymptotically AdS boundary condition and toroidal conformal boundary. There is not yet a systematic understanding of the conditions a CFT has to satisfy in order for it to have an AdS dual. Nevertheless, the correspondence, when applicable, provides both deep intuitive insights and powerful computational tools for the study of the theory. From the fact that the only smooth three-manifold with asymptotically AdS torus boundary condition is a solid torus, it follows that the saddle points of such a partition function are labelled by the different possible ways to

"fill in the torus"; that is, the different choices of primitive cycle on the boundary torus which may become contractible in a solid torus that fills it [13]. These different saddle points are therefore labelled by the coset space $\Gamma_\infty \backslash \Gamma$, where $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ [14]. From a bulk, gravitational point of view, the group $\mathrm{SL}_2(\mathbb{Z})$ has an interpretation as the group of large diffeomorphisms, and Γ_∞ is the subgroup that leaves the contractible cycle invariant and therefore can be described by a mere change of coordinates. Such considerations underlie the previous use of Rademacher sums in the physics literature [14, 15, 16, 17, 18, 19, 20]. Apart from the partition function, which computes the dimension of the Hilbert space graded by the basic quantum numbers (the energy, for instance) of the theory, more interesting information can be gained in the presence of a non-trivial symmetry group acting on the Hilbert space by studying the twisted partition function (a trace over the Hilbert space with a group element inserted) which computes the group characters of the Hilbert space. In the Lagrangian formulation of quantum field theories, this twisting corresponds to a modification of the boundary condition. Now, in the study of moonshine the McKay–Thompson series attached to the group element g has a physical interpretation as the twisted partition function of a 2D CFT with a boundary condition twisted by g. This twisting procedure has a clear geometrical meaning for a 2D CFT with an AdS gravity dual: from the point of view of the gravity theory, the change of boundary condition by insertion of a group element g also changes the set of allowed saddle points, and relatedly, the allowed large diffeomorphisms is now given by a different discrete group $\Gamma_g \subset \mathrm{SL}_2(\mathbb{R})$. From this argument, assuming that the moonshine CFT (or its cousins with higher central charges) have semi-classical-like AdS descriptions in which the path integrals are controlled by their saddle points, the modular objects should admit expressions as Rademacher sums over the coset spaces $\Gamma_\infty \backslash \Gamma_g$. This picture motivates the relevance of the Rademacher summability property:

> We say a function f is *Rademacher summable* if it coincides with the Rademacher sum attached to its polar part.

As we have observed above, Rademacher summability is equivalent to the genus zero property in the monstrous case by the results of [10], where the McKay–Thompson series are modular functions. Moreover, the Rademacher summing procedure applied to Γ produces functions, which do not have singularities at cusps of Γ other than the infinite one. This is in accordance with our expectation that our theory of gravity should admit a unique low-energy description. The situation is arguably more subtle and also more interesting for M_{24}. As mentioned earlier, in this case the relevant modular objects

are weight $1/2$ (mock) modular forms, and one does not expect any clear relation between the genus of Γ_g and the Rademacher summability of H_g—indeed, not all the Γ_g attached to elements of M_{24} have genus zero—while in this work, we verify explicitly that the functions H_g are all Rademacher summable:

$$(1.2) \qquad H_g(\tau) = -2\operatorname{Reg}\left(\sum_{\gamma \in \Gamma_\infty \backslash \Gamma_g} q^{\epsilon - 1/8}\big|_\gamma\right)$$

for all $g \in M_{24}$. The precise meaning of the above equation will be elucidated in Section 4, and is in particular the content of the main theorem (Theorem 4.1) of the present paper. Closely related sums for two of the above (mock) modular forms, corresponding to taking g to be the identity or in a certain class of order 2, have been suggested previously in [21]. The function H_g for g the identity is one of the mock modular forms constructed over a decade ago by Pribitkin in [22]. Note the remarkable similarity between (1.2) and (1.1), and recall that Rademacher summability is equivalent to the genus zero property in the monstrous case. In view of this, the Rademacher summability property clearly serves as a natural analogue of the genus zero property, and as a new paradigm that applies to both the Monster and M_{24}. Moreover, Rademacher sums provide a new angle from which to view the moonshine phenomenon: is it not miraculous that the Rademacher machinery, applied with such simple inputs, uniformly produces infinite q-series that contain so much information about sporadic simple groups? The rest of the paper is organized as follows. In Section 2, we review the basic properties of the sporadic group M_{24}, focusing on its 24D permutation representation. In Section 3, we review the properties of the (mock) modular forms that have been conjectured to encode an infinite-dimensional, $\mathbb{Z}$-graded M_{24} module. In Section 4, we explain our construction of the Rademacher sums and state the main theorem (Theorem 4.1) of the present paper. The rest of the paper is devoted to the proof of Theorem 4.1. In Section 5, we discuss the convergence of the Rademacher sums and establish that these sums are indeed convergent given the convergence of certain Selberg–Kloosterman zeta functions, which will be shown in Section 9. Having established the convergence of the sums, in Section 6 we give explicit expressions for the Fourier coefficients of our Rademacher sums, and discuss their asymptotic growth. In Section 7, we study the transformation properties of the Rademacher sums and establish their (mock) modularity. Given the modularity and information about their behaviour at the cusps, in Section 8 we finally establish

that the Rademacher sums constructed in Section 4 indeed coincide with the (mock) modular forms which are the proposed McKay–Thompson series for the group M_{24} described in Section 3.

2. The largest Mathieu group

We shall start by recalling some facts about the largest Mathieu group. The group M_{24} may be characterized as the automorphism group of the unique doubly even self-dual binary code of length 24 with no words of weight 4, also known as the *(extended) binary Golay code*. In other words, there is a unique (up to permutation) set, $\mathcal{G}$ say, of length 24 binary codewords (sequences of 0's and 1's) such that any other length 24 codeword has even overlap with all the codewords of $\mathcal{G}$ if and only if this word itself is in $\mathcal{G}$, and the number of 1's in each codeword of $\mathcal{G}$ is divisible by and not equal to 4. The group of permutations of the 24 coordinates that preserves the set $\mathcal{G}$ is the sporadic group M_{24}. See, for instance, [23]. As such, M_{24} naturally admits a permutation representation of degree 24, and this allows us to assign a *cycle shape* to each of its elements. For example, to the identity element we associate the cycle shape 1^{24}; to an element of M_{24} that is a product of 12 mutually commuting transpositions we associate the cycle shape 2^{12}, and so on. More generally, any cycle shape arising from an element of M_{24} (or S_{24}, for that matter) is of the form

$$i_1{}^{\ell_1} i_2{}^{\ell_2} \cdots i i_r{}^{\ell_r}, \quad \sum_{s=1}^{r} \ell_s\, i_s = 24,$$

for some $\ell_s \in \mathbb{Z}^+$ and $1 \leq i_1 < \cdots < i_r \leq 23$ with $r \geq 1$. Clearly, the cycle shape of an element of M_{24} depends only on its conjugacy class, denoted by b, although different conjugacy classes can share the same cycle shape. For future reference we denote the character underlying this defining 24D representation of M_{24} by χ. The value of the character $\chi(g)$ equals the number of fixed points of g in the action on the set of 24 points. In particular, note that $\chi(g) = \ell_1$ in case $i_1 = 1$ and $\chi(g) = 0$ otherwise. It turns out that the cycle shapes of M_{24} have many special properties that will be important for the understanding of the modular properties of the associated McKay–Thompson series which we will discuss shortly. First, the M_{24} cycle shapes are privileged in that they are all of the so-called *balanced type* (cf. [6]), meaning that for each $g \in M_{24}$ there exists a positive integer N_g such that

if $\prod_s i_s^{\ell_s}$ is the cycle shape associated to g then

$$\prod_s i_s^{\ell_s} = \prod_s \left(\frac{N_g}{i_s} \right)^{\ell_s}.$$

We will refer to the number N_g as the *level* of the g. If g has cycle shape $i_1^{\ell_1} \cdots i_r^{\ell_r}$ then the order of g is the least common multiple of the i_s's. A second special property of $[g] \subset M_{24}$ is that the order of g coincides with the length i_r of the longest cycle in the cycle shape. Henceforth we will denote $n_g = i_r$. Finally, observe that for all $g \in M_{24}$ the level N_g defined above equals the product of the shortest and the longest cycle. Hence, we have $h_g n_g = N_g$ where h_g denotes the length of the shortest cycle in the cycle shape. Moreover, we also have the property $h_g | n_g$ and $h_g | 12$. This is very reminiscent of the monstrous moonshine [6]. We also set $k_g = \sum_{s=1}^r \ell_s / 2$ to be half of the total number of cycles and call it the *weight* of g. Of course, N_g, n_g, h_g and k_g only depend on the conjugacy class $[g]$ containing g and can be found in table 1. To each element $g \in M_{24}$ we can attach an *eta-product*, to be denoted η_g, which is the function on the upper half-plane given by

$$(2.1) \qquad \eta_g(\tau) = \prod_s \eta(i_s \tau)^{\ell_s},$$

where $\prod_s i_s^{\ell_s}$ is the cycle shape attached to g, and $\eta(\tau)$ is the Dedekind eta function satisfying $\eta(\tau) = q^{1/24} \prod_{n \in \mathbb{Z}^+} (1 - q^n)$ for $q = e(\tau)$, where for later convenience, here and everywhere else in this paper we will use the shorthand notation

$$e(x) = e^{2\pi i x}.$$

As was observed in [25, 26], the eta-product η_g associated to an element $g \in M_{24}$ (or rather, to its conjugacy class $[g]$) is a cusp form of weight k_g for the group $\Gamma_0(N_g)$, with a *Dirichlet character* ς_g that is trivial if the weight k_g is even and is otherwise defined, in terms of the Jacobi symbol $\left(\frac{n}{m} \right)$, by

$$\varsigma_g(\gamma) = \begin{cases} \left(\frac{N_g}{d} \right) (-1)^{\frac{d-1}{2}}, & d \text{ odd}, \\ \left(\frac{N_g}{d} \right), & d \text{ even}, \end{cases}$$

in case d is the lower right entry of $\gamma \in \Gamma_0(N_g)$. Let us recall that

$$\Gamma_0(N) = \left\{ \gamma \, \middle| \, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \quad c = 0 \mod N \right\}.$$

Table 1: The cycle shapes, weights (k_g), levels (N_g) and orders (n_g) of the 26 conjugacy classes of the sporadic group M_{24}. The length of the shortest cycle is $h_g = N_g/n_g$. The naming of the conjugacy classes follows the ATLAS convention (cf. [24]). We write $7AB$, for example, to indicate that the entries of the incident row are valid for both the conjugacy classes $7A$ and $7B$.

$[g]$	Cycle shape	$\eta_g(\tau)$	k_g	n_g	N_g	h_g
$1A$	1^{24}	$\eta^{24}(\tau)$	12	1	1	1
$2A$	$1^8 2^8$	$\eta^8(\tau)\eta^8(2\tau)$	8	2	2	1
$2B$	2^{12}	$\eta^{12}(2\tau)$	6	2	4	2
$3A$	$1^6 3^6$	$\eta^6(\tau)\eta^6(3\tau)$	6	3	3	1
$3B$	3^8	$\eta^8(3\tau)$	4	3	9	3
$4A$	$2^4 4^4$	$\eta^4(2\tau)\eta^4(4\tau)$	4	4	8	2
$4B$	$1^4 2^4 4^4$	$\eta^4(\tau)\eta^2(2\tau)\eta^4(4\tau)$	5	4	4	1
$4C$	4^6	$\eta^6(4\tau)$	3	4	16	4
$5A$	$1^4 5^4$	$\eta^4(\tau)\eta^4(5\tau)$	4	5	5	1
$6A$	$1^2 2^2 3^2 6^2$	$\eta^2(\tau)\eta^2(2\tau)\eta^2(3\tau)\eta^2(6\tau)$	4	6	6	1
$6B$	6^4	$\eta^4(6\tau)$	2	6	36	6
$7AB$	$1^3 7^3$	$\eta^3(\tau)\eta^3(7\tau)$	3	7	7	1
$8A$	$1^2 2^1 4^1 8^2$	$\eta^2(\tau)\eta(2\tau)\eta(4\tau)\eta^2(8\tau)$	3	8	8	1
$10A$	$2^2 10^2$	$\eta^2(2\tau)\eta^2(10\tau)$	2	10	20	2
$11A$	$1^2 11^2$	$\eta^2(\tau)\eta^2(11\tau)$	2	11	11	1
$12A$	$2^1 4^1 6^1 12^1$	$\eta(2\tau)\eta(4\tau)\eta(6\tau)\eta(12\tau)$	2	12	24	2
$12B$	12^2	$\eta^2(12\tau)$	1	12	144	12
$14AB$	$1^1 2^1 7^1 14^1$	$\eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau)$	2	14	14	1
$15AB$	$1^1 3^1 5^1 15^1$	$\eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau)$	2	15	15	1
$21AB$	$3^1 21^1$	$\eta(3\tau)\eta(21\tau)$	1	21	63	3
$23AB$	$1^1 23^1$	$\eta(\tau)\eta(23\tau)$	1	23	23	1

The eta-product η_g also defines a cusp form of weight k_g on the larger (or equal) group $\Gamma_0(n_g) \supseteq \Gamma_0(N_g)$ if we allow for a slightly more sophisticated multiplier system. We remark here that, according to our conventions, a function $\xi : \Gamma \to \mathbb{C}^*$ is called a *multiplier system for Γ of weight w* in case the identity

$$(2.2) \qquad \xi(\gamma\sigma)\,\mathrm{jac}(\gamma\sigma, \tau)^{w/2} = \xi(\gamma)\,\xi(\sigma)\,\mathrm{jac}(\gamma, \sigma\tau)^{w/2}\,\mathrm{jac}(\sigma, \tau)^{w/2}$$

holds for all $\gamma, \sigma \in \Gamma$ and $\tau \in \mathbb{H}$. Writing $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$, we have in the above formula $\gamma\tau = \frac{a\tau+b}{c\tau+d}$, and the Jacobian

$$\mathrm{jac}(\gamma, \tau) = (c\tau + d)^{-2}.$$

In detail, for $\gamma \in \Gamma$, we define the *slash operator* $|_{\xi,w}$ of weight w associated with multiplier ξ as

$$(2.3) \qquad (f|_{\xi,w}\gamma)(\tau) = \xi(\gamma) f(\gamma\tau) \mathrm{jac}(\gamma, \tau)^{\epsilon w/2},$$

and in particular a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is a modular form of weight w and multiplier ξ on Γ if and only if $(f|_{\xi,w}\gamma)(\tau) = f(\tau)$ for all $\gamma \in \Gamma$. In the present case, we have

$$(2.4) \qquad \left(\frac{1}{\eta_g} \Big|_{\xi_g,k_g} \gamma \right)(\tau) = \frac{1}{\eta_g(\tau)}, \quad \text{for all} \quad \gamma \in \Gamma_0(n_g),$$

where $\xi_g(\gamma) = \rho_{n_g|h_g}(\gamma)\varsigma_g(\gamma)$ with

$$(2.5) \qquad \rho_{n|h}(\gamma) = e\left(-\frac{1}{(\gamma\infty - \gamma 0)nh} \right) = e(-cd/nh).$$

Note that $\rho_{n_g|h_g}$ is actually a character on $\Gamma_0(n_g)$ since we have that $xy \equiv 1$ (mod h_g) implies $x \equiv y$ (mod h_g) by virtue of the fact that $h_g = N_g/n_g$ is a divisor of 24 for every $g \in M_{24}$ (cf. [6, Section 3]). Evidently the kernel of $\rho_{n|h}$ is $\Gamma_0(nh)$, and in particular, $\rho_{n|h}$ is trivial on Γ_∞. These (meromorphic) modular forms $1/\eta_g(\tau)$ have also the interpretation as the partition function of the CFT of 24 free chiral bosons, twisted by $g \in M_{24}$ that acts on the 24 bosons as an element of S_{24}. As such, they are also McKay–Thompson series whose Fourier coefficients are positive-integral linear combinations of M_{24} characters. See table 4 for the first ten M_{24}-representations appearing in $1/\eta_g(\tau)$. Following an earlier observation in [27], it was shown in [2] that they are connected to the elliptic genera of $K3$ surfaces and the moonshine for M_{24}, which will be discussed in the next section, via a lifting to Siegel modular forms and generalized Kac–Moody superalgebras. More details will be given in [28]. Observe that the behaviour of the multipliers naturally divides the conjugacy classes of M_{24} into two types: those for which the phase is trivial on $\Gamma_0(n_g)$ and those for which it is not. The former occurs just when $h_g = 1$, and the elements of $g \in M_{24}$ with $h_g > 1$ are exactly the classes which act fixed-point-freely in the defining permutation representation on 24 points. The elements with $h_g = 1$ have at least one fixed point and thus

can be located in a (maximal) subgroup of M_{24} isomorphic to the second largest Mathieu group, M_{23}.

3. Mock modular forms

In the recent article [1], a remarkable observation relating M_{24} and the unique (up to scale) weak Jacobi form of weight zero and index one, here denoted $Z(\tau, z)$, was made via a decomposition of the latter object into a combination of mock theta series and mock modular forms. As is shown in [29, 30, 21], the function $Z(\tau, z) = 8 \sum_{i=2,3,4} \left(\frac{\theta_i(\tau,z)}{\theta_i(\tau,0)} \right)^2$ admits an expression

$$(3.1) \qquad Z(\tau, z) = \frac{\theta_1(\tau, z)^2}{\eta(\tau)^3} \left(a \, \mu(\tau, z) + q^{-1/8} \left(b + \sum_{k=1}^{\infty} t_k \, q^k \right) \right)$$

with $a, b, t_k \in \mathbb{Z}$ for all $k \in \mathbb{Z}^+$ where $\mu(\tau, z)$ denotes the *Appell–Lerch sum*, satisfying

$$\mu(\tau, z) = \frac{-iy^{1/2}}{\theta_1(\tau, z)} \sum_{\ell=-\infty}^{\infty} \frac{(-1)^\ell y^n q^{\ell(\ell+1)/2}}{1 - yq^\ell}$$

for $q = e(\tau)$ and $y = e(z)$. The Jacobi theta functions are given by

$$\theta_1(\tau, z) = -iq^{1/8} y^{1/2} \prod_{n=1}^{\infty} (1 - q^n)(1 - yq^n)(1 - y^{-1}q^{n-1}),$$

$$\theta_2(\tau, z) = q^{1/8} y^{1/2} \prod_{n=1}^{\infty} (1 - q^n)(1 + yq^n)(1 + y^{-1}q^{n-1}),$$

$$\theta_3(\tau, z) = \prod_{n=1}^{\infty} (1 - q^n)(1 + y \, q^{n-1/2})(1 + y^{-1} q^{n-1/2}),$$

$$(3.2) \qquad \theta_4(\tau, z) = \prod_{n=1}^{\infty} (1 - q^n)(1 - y \, q^{n-1/2})(1 - y^{-1} q^{n-1/2}).$$

By inspection, $a = 24 = \chi(1A)$, $b = -2$ and the first few t_k are

$$2 \times 45, \ 2 \times 231, \ 2 \times 770, \ 2 \times 2277, \ 2 \times 5796, \dots.$$

These positive integers t_k have the interpretation of enumerating the k-th massive representations of the $\mathcal{N} = 4$ superconformal algebra in the elliptic genus of a $K3$ surface. The surprising connection to M_{24}, beyond the fact that $a = 24$ is the dimension of the defining permutation representation of

M_{24}, is the following: the integers 45, 231, 770, 2277 and 5796, which are the $t_k/2$ for $k = 1, 2, 3, 4, 5$, are the dimensions of irreducible representations of M_{24}. It was conjectured that the entire set of values t_k for $k \in \mathbb{Z}^+$ encode the graded dimension of a naturally defined graded M_{24} module $K = \bigoplus_k K_k$ with $\dim K_k = t_k$. If the conjecture holds then we can obtain new functions by replacing t_k with $\operatorname{tr}_{K_k} g$ for $g \in M_{24}$. This idea was first investigated in [2], and independently in [3]; see also [4, 5]. It is an idea that we pursue further in this paper. Define $H(\tau)$ so that $Z(\tau, z)\eta(\tau)^3 = \theta_1(\tau, z)^2(a\mu(\tau, z) + H(\tau))$. Then

$$(3.3) \qquad\qquad H(\tau) = q^{-\frac{1}{8}}\left(-2 + \sum_{k=1}^{\infty} t_k q^k\right),$$

so that $H(\tau)$ is (essentially) a power series incorporating the t_k, and hence the graded dimension of the conjectural M_{24} module K. This function $H(\tau)$ enjoys a special relationship with the group $\mathrm{SL}_2(\mathbb{Z})$; namely, it is a *weakly holomorphic mock modular form of weight* $1/2$ *on* $\mathrm{SL}_2(\mathbb{Z})$ *with shadow* $24\,\eta(\tau)^3$ (cf. [31]), which means that $H(\tau)$ is a holomorphic function on the upper half-plane $\mathbb{H}$ with at most exponential growth as $\tau \to \alpha$ for any $\alpha \in \mathbb{Q}$, and if we define the *completion* of $H(\tau)$, to be denoted $\hat{H}(\tau)$, by setting

$$\hat{H}(\tau) = H(\tau) + 24\,(4i)^{-1/2}\int_{-\bar{\tau}}^{\infty}(z + \tau)^{-1/2}\overline{\eta(-\bar{z})^3}dz,$$

then $\hat{H}(\tau)$ transforms as a modular form of weight $1/2$ on $\mathrm{SL}_2(\mathbb{Z})$ with multiplier system conjugate to that of $\eta(\tau)^3$. In other words, we have

$$\left(\hat{H}(\tau)|_{\epsilon^{-3}, \epsilon 1/2}\gamma\right)(\tau) = \epsilon(\gamma)^{\epsilon-3}\hat{H}(\gamma\tau)\operatorname{jac}(\gamma, \tau)^{\epsilon 1/4} = \hat{H}(\tau)$$

for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, where $\epsilon : \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{C}^*$ is the multiplier system for $\eta(\tau)$ satisfying

$$\left(\eta|_{\epsilon, \epsilon 1/2}\gamma\right)(\tau) = \eta(\tau).$$

(See Section Appendix A for an explicit description of ϵ.) More generally, a holomorphic function $h(\tau)$ on $\mathbb{H}$ is called a *(weakly holomorphic) mock modular form of weight* w for a discrete group Γ (e.g., a congruence subgroup of $\mathrm{SL}_2(\mathbb{R})$) if it has at most exponential growth as $\tau \to \alpha$ for any $\alpha \in \mathbb{Q}$, and if there exists a holomorphic modular form $g(\tau)$ of weight $2 - w$ on Γ such

that $\hat{h}(\tau)$, given by

$$(3.4) \qquad \hat{h}(\tau) = h(\tau) + (4i)^{w-1} \int_{-\bar\tau}^{\infty} (z+\tau)^{-w} \overline{g(-\bar{z})} dz,$$

is a (non-holomorphic) modular form of weight w for Γ for some multiplier system ψ say. In this case, the function g is called the *shadow* of the mock modular form h and we call ψ the multiplier system of h. If the conjectural M_{24}-module K exists, apart from $H(\tau)$ there must be a family of functions $H_g(\tau)$, for each conjugacy class $[g] \subset M_{24}$, obtainable by replacing each t_k with the trace of g on K_k. These are the McKay–Thompson series given by

$$(3.5) \qquad H_g(\tau) = q^{-\frac{1}{8}} \left(-2 + \sum_{k=1}^{\infty} \mathrm{tr}_{K_k}(g) q^k \right).$$

Explicit expressions for these function $H_g(\tau)$ have been proposed in [2,3], mostly for conjugacy classes with $h_g = 1$, and completed for all $[g] \subset M_{24}$ in [4,5]. These proposals state that they are given by $H(\tau)$, the character $\chi(g)$ of g in the 24D representation, and certain weight two modular forms $\tilde{T}_g(\tau)$ for the group $\Gamma_0(N_g)$, by

$$(3.6) \qquad H_g(\tau) = \frac{\chi(g)}{24} H(\tau) - \frac{\tilde{T}_g(\tau)}{\eta(\tau)^3}.$$

These data are collected in table 2. The expression (3.6) makes manifest that the q-series $H_g(\tau)$ is a mock modular form with shadow $\chi(g)\eta^3(\tau)$, and is in particular a usual modular form when $h_g > 1$, or equivalently, $\chi(g) = 0$. Moreover, it is easy to check that $H_g(\tau)$ transforms nicely under the group $\Gamma_0(n_g)$, with the multiplier system $\psi(\gamma) = \epsilon(\gamma)^{\epsilon-3} \rho_{n_g|h_g}(\gamma)$. Note that the extra multiplier ρ that appears when $h_g \neq 1$ is the same as that of the inverse eta-products $1/\eta_g(\tau)$ which are also related to M_{24}, a fact that is in accordance with the 1/2- and 1/4-BPS spectrum of the $\mathcal{N} = 4$, $d = 4$ theory obtained by $K3 \times T^2$ compactification of the type II string theory. These mock modular forms $H_g(\tau)$ are the central objects of study in this work. Although conjecture (3.5) stating that the Fourier coefficients of $H_g(\tau)$ are all given by characters of M_{24}-module still remains to be proven, it has been checked up to the first 1000 terms. We list the first few in table 5. Together with other pieces of evidence, this makes it an extremely plausible conjecture. In this paper, we will hence assume the validity of this conjecture as a part of our motivation, while our main result (Theorem 4.1) holds independent of it.

Table 2: In this table, we collect the data that via Equation (3.6) define the weight 1/2 (mock) modular forms $H_g(\tau)$. For $N \in \mathbb{Z}_+$, we denote by Λ_N the weight 2 modular form on $\Gamma_0(N)$ given by $\Lambda_N = Nq\frac{d}{dq}\left(\log \frac{\eta(N\tau)}{\eta(\tau)}\right)$ (cf. (8.13)). For $N = 23$, there are two new forms and we choose the basis $\phi_{23,1} = \eta_{23AB}^2$ and $\phi_{23,2}$ given in (8.14).

$[g]$	$\chi(g)$	$\tilde{T}_g(\tau)$
$1A$	24	0
$2A$	8	$16\Lambda_2$
$2B$	0	$-24\Lambda_2 + 8\Lambda_4 = 2\eta(\tau)^8/\eta(2\tau)^4$
$3A$	6	$6\Lambda_3$
$3B$	0	$2\eta(\tau)^6/\eta(3\tau)^2$
$4A$	0	$4\Lambda_2 - 6\Lambda_4 + 2\Lambda_8 = 2\eta(2\tau)^8/\eta(4\tau)^4$
$4B$	4	$4(-\Lambda_2 + \Lambda_4)$
$4C$	0	$2\eta(\tau)^4\eta(2\tau)^2/\eta(4\tau)^2$
$5A$	4	$2\Lambda_5$
$6A$	2	$2(-\Lambda_2 - \Lambda_3 + \Lambda_6)$
$6B$	0	$2\eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^2/\eta(6\tau)^2$
$7AB$	3	Λ_7
$8A$	2	$-\Lambda_4 + \Lambda_8$
$10A$	0	$2\eta(\tau)^3\eta(2\tau)\eta(5\tau)/\eta(10\tau)$
$11A$	2	$2(\Lambda_{11} - 11\eta(\tau)^2\eta(11\tau)^2)/5$
$12A$	0	$2\eta(\tau)^3\eta(4\tau)^2\eta(6\tau)^3/\eta(2\tau)\eta(3\tau)\eta(12\tau)^2$
$12B$	0	$2\eta(\tau)^4\eta(4\tau)\eta(6\tau)/\eta(2\tau)\eta(12\tau)$
$14AB$	1	$(-\Lambda_2 - \Lambda_7 + \Lambda_{14} - 14\eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau))/3$
$15AB$	1	$(-\Lambda_3 - \Lambda_5 + \Lambda_{15} - 15\eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau))/4$
$21AB$	0	$(7\eta(\tau)^3\eta(7\tau)^3/\eta(3\tau)\eta(21\tau) - \eta(\tau)^6/\eta(3\tau)^2)/3$
$23AB$	1	$(\Lambda_{23} - 23\phi_{23,1} + 23\phi_{23,2})/11$

4. Rademacher sums

In this section, we will state and explain the main theorem (Theorem 4.1) of the paper, while postponing the proof till the later sections. Recall that $H(\tau)$ is a mock modular form of weight 1/2 on $\mathrm{SL}_2(\mathbb{Z})$ with a multiplier system given by $\epsilon(\gamma)^{\epsilon-3}$ and with leading term $-2q^{\epsilon-1/8}$ in its q-expansion. We may consider the problem of attaching a function with these data via Rademacher sums, and also analogous functions to discrete subgroups of

$\mathrm{SL}_2(\mathbb{R})$ other than the modular group[1] Let Γ be a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$ containing the group Γ_∞ of upper triangular matrices in $\mathrm{SL}_2(\mathbb{Z})$. In particular, we assume that Γ contains $-\mathrm{Id}$. Define $R_\Gamma(\tau)$ by setting

$$(4.1) \qquad R_\Gamma(\tau) = \lim_{K \to \infty} \sum_{\gamma \in (\Gamma_\infty \backslash \Gamma)_{<K}} \epsilon(\gamma)^{\epsilon-3} e\left(-\frac{\gamma\tau}{8}\right) \mathrm{reg}(\gamma, \tau) \, \mathrm{jac}(\gamma, \tau)^{\epsilon 1/4},$$

where $\mathrm{reg}(\gamma, \tau)$ is the *Rademacher regularization factor (of weight $1/2$ and index $-1/8$)*, defined by $\mathrm{reg}(\gamma, \tau) = 1$ in case γ is upper triangular (i.e., $\gamma \cdot \infty = \infty$) and

$$(4.2) \qquad \mathrm{reg}(\gamma, \tau) = e\left(\frac{\gamma\tau - \gamma\infty}{8}\right) e\left(\frac{\gamma\infty - \gamma\tau}{8}, \frac{1}{2}\right),$$

otherwise, where $e(x, s)$ is the following generalization of the exponential function:

$$(4.3) \qquad e(x, s) = \sum_{m \geq 0} \frac{(2\pi i x)^{m+s}}{\Gamma(m+s+1)}.$$

Note that we recover $e(x) = e^{2\pi i x}$ by taking $s = 0$ in $e(x, s)$, and for n a positive integer $e(x, n)$ is the difference between $e(x)$ and the order $n - 1$ Taylor approximation to $e(x)$. It is also closely related to the incomplete Gamma function $\Gamma(s, x) = \int_x^\infty t^{s-1} e^t dt$: applying integration by parts repeatedly, we obtain

$$(4.4) \qquad e(x, s) = \frac{e(x)}{\Gamma(s)} \int_0^{2\pi i x} t^{s-1} e^t dt.$$

Let T be the element of Γ_∞ such that $\mathrm{tr}(T) = 2$ and $T\tau = \tau + 1$. Note that two elements γ, σ of Γ are in the same right coset of Γ_∞ if and only if there is some $n \in \mathbb{Z}$ such that $\sigma = T^n\gamma$ or $-T^n\gamma$. In particular, they have the same lower rows up to multiplication by $\pm\mathrm{Id}$ and we can therefore use the lower rows to parametrize the cosets. This motivates the sum over the "rectangle" in (4.1): For any given positive K, we consider a sum over a set $\{\gamma\}$ of representatives for the cosets of Γ_∞ in Γ whose elements have lower rows

---

[1]For the purpose of establishing relations to M_{24}, it suffices to look at subgroups of $\mathrm{SL}_2(\mathbb{Z})$, whereas it is possible to generalize the analysis to $\mathrm{SL}_2(\mathbb{R})$ subgroups commensurable to $\mathrm{SL}_2(\mathbb{Z})$.

(c, d) satisfying $0 \leq c < K$ and $-K^2 < d < K^2$. Using a superscript $^\times$ to indicate a sum over coset representatives other than the trivial one, we have

$$(4.5) \qquad (\Gamma_\infty \backslash \Gamma)^\times_{<K} = \left\{ \Gamma_\infty \gamma \mid 0 < c < K, \; -K^2 < d < K^2 \right\}.$$

Here and everywhere else we use (c, d), a shorthand for $(c(\gamma), d(\gamma))$, to denote the lower row entries when writing γ as a 2×2 matrix. Complimented with a term given by the trivial coset (the one with a representative $\gamma = \mathrm{Id}$), this is the range of the sum, denoted by $(\Gamma_\infty \backslash \Gamma)_{<K}$, taken in (4.1). With a slight abuse of notation, we use $\gamma \in \Gamma_\infty \backslash \Gamma$ to denote a sum over a representative from each coset. To ensure that (4.1) is well defined, we require to check that the summands are invariant under the replacement of γ with $\pm T^n \gamma$ for $n \in \mathbb{Z}$. Obviously $\mathrm{reg}(T^n \gamma, \tau) = \mathrm{reg}(\gamma, \tau)$ for any n, and γ and $-\gamma$ act in the same way on $\mathbb{H}$ as well as on the cusp representatives so the Rademacher regularization factor has the required invariance. Left multiplication by T does not change the lower row of γ so the factor $\mathrm{jac}(\gamma, \tau)^{\epsilon 1/4}$ is invariant under the substitution of $T^n \gamma$ for γ. Using the explicit description (A.1) for ϵ we see that $\epsilon(T^n \gamma) = e\left(-\frac{n}{24}\right) \epsilon(\gamma)$, so that the product $\epsilon(\gamma)^{\epsilon - 3} e\left(-\frac{\gamma \tau}{8}\right)$, is also invariant under the substitution of $T^n \gamma$ for γ, and then the identity $\epsilon(-\gamma)^{\epsilon - 3} \mathrm{jac}(-\gamma, \tau)^{\epsilon 1/4} = \epsilon(\gamma)^{\epsilon - 3} \mathrm{jac}(\gamma, \tau)^{\epsilon 1/4}$ completes the verification that every summand of (4.1) is invariant under the replacement of γ with $\pm T^n \gamma$. While the regularization factor might seem ad hoc at the first sight, here we would like to argue that it is in fact strictly necessary and very natural. First, to motivate the appearance of a regularisation note that a series

$$(4.6) \qquad \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \psi(\gamma) e(\alpha \gamma \tau) \, \mathrm{jac}(\gamma, \tau)^{\epsilon w/2},$$

for α a real constant and ψ a (compatible) multiplier system of weight w on Γ, will generally not converge (absolutely and uniformly on compacta) unless $w > 2$. (In case it does converge it defines a modular form of weight w on Γ.) In [11] Rademacher demonstrated a method to regularize this sum for $\psi \equiv 1$, $\alpha = -1$ and $w = 0$ in the case that $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. His prescription may be described as follows: multiplying each summand by the regularization factor $\mathrm{r}^0(-1, \gamma, \tau)$ which is 1 when γ is upper triangular and

$$e(\epsilon \gamma \tau - \gamma \infty) e(\gamma \infty - \gamma \tau, 1) = e(\gamma \tau - \gamma \infty)(e(\gamma \infty - \gamma \tau) - 1)$$

otherwise, and replacing the sum over the coset space $\Gamma_\infty\backslash\Gamma$ with a limit of sums over rectangles $(\Gamma_\infty\backslash\Gamma)_{<K}$, we obtain

$$e(-\tau) + \lim_{K\to\infty} \sum_{\gamma\in(\Gamma_\infty\backslash\Gamma)^\times_{<K}} \left(e(-\gamma\tau) - e(-\gamma\infty)\right).$$

He then went on to prove (cf. loc. cit.) that this expression converges and defines an $\mathrm{SL}_2(\mathbb{Z})$ invariant function on the upper half-plane, which is nothing but the familiar j-function (up to an additive constant). More generally, in [10], for example, it is shown that for $\psi \equiv 1$ and $\alpha, w/2 \in \mathbb{Z}$, inclusion of the factor

$$\mathrm{r}^{\epsilon w/2}(\alpha, \gamma, \tau) = e(\alpha(\gamma\infty - \gamma\tau))e(\alpha(\gamma\tau - \gamma\infty), 1 - w)$$

for summands corresponding to cosets with upper triangular representatives, together with a limit of sums over rectangles, regularizes the sum (4.6) for Γ a subgroup of $\mathrm{SL}_2(\mathbb{R})$ that is commensurable with the modular group. Putting $\alpha = -1/8$ and $w = 1/2$, we recover exactly our regularization factor (4.2). It is rather surprising that this straightforward generalization of regularization scheme does its job in regularizing the sum in (4.1). To understand the modular properties of R_Γ, it is useful to consider the companion function S_Γ defined by

$$(4.7) \qquad S_\Gamma(\tau) = \lim_{K\to\infty} \sum_{\gamma\in(\Gamma_\infty\backslash\Gamma)_{<K}} \epsilon(\gamma)^3 e\left(\frac{\gamma\tau}{8}\right) \mathrm{jac}(\gamma, \tau)^{3/4}.$$

According to the philosophy of the previous paragraph S_Γ, supposing it converges, should have modular transformations of weight $3/2$ for Γ with multiplier system coinciding with that of $\eta(\tau)^3$. Also, we should have that $S_\Gamma = \mathcal{O}(q^{1/8})$ as $\tau \to \infty$. It will develop in Section 7 that R_Γ is a mock modular form and that S_Γ is its shadow. Apart from considering the Rademacher with $\psi(\gamma) = \epsilon^{-3}(\gamma)$, to construct the M_{24} McKay–Thompson series we also need to extend the above construction to the cases with a more general multiplier system ψ. In particular, we would sometimes like to consider different Rademacher sums with the same group Γ but different multipliers. Given a group morphism $\rho : \Gamma \to \mathbb{C}^*$ that is trivial on Γ_∞ we may consider the

following twists of R_Γ and S_Γ:

$$R_{\Gamma,\rho}(\tau) = \lim_{K\to\infty} \sum_{\gamma\in(\Gamma_\infty\backslash\Gamma)_{<K}} \psi(\gamma)e\left(-\frac{\gamma\tau}{8}\right) \mathrm{reg}(\gamma,\tau)\,\mathrm{jac}(\gamma,\tau)^{1/4}$$

$$(4.8) \qquad S_{\Gamma,\rho}(\tau) = \lim_{K\to\infty} \sum_{\gamma\in(\Gamma_\infty\backslash\Gamma)_{<K}} \bar\psi(\gamma)e\left(\frac{\gamma\tau}{8}\right) \mathrm{jac}(\gamma,\tau)^{3/4},$$

where we have $\psi(\gamma) = \rho(\gamma)\epsilon(\gamma)^{\epsilon-3}$. In this paper, we will take $\Gamma = \Gamma_0(n)$ for some $n \in \mathbb{Z}^+$ and $\rho = \rho_{n|h}$ given in (2.5) for h simultaneously dividing n and 24. Not surprisingly, later on we are going to take (n, h) to be the (n_g, h_g) coming from the cycle shapes of M_{24} as given by table 1. To ease notation, we write $R_{n|h}$ for $R_{\Gamma,\rho}$ when $\Gamma = \Gamma_0(n)$ and $\rho = \rho_{n|h}$, and we apply the directly analogous interpretation to the $S_{n|h}$. The main result in this paper is the following.

Theorem 4.1. *Let $g \in M_{24}$. Then $H_g = -2R_{n|h}$ when $n = n_g$ and $h = h_g$.*

The rest of the paper is devoted to the proof of the above theorem.

Remark. There exists an alternative approach to the Rademacher sums of (holomorphic) mock modular forms. Note that the completion $\hat{h}(\tau)$ (cf. (3.4)) of the mock modular form $h(\tau)$ is necessarily annihilated by the differential operator

$$\frac{\partial}{\partial\tau}(\mathrm{Im}\,\tau)^w\frac{\partial}{\partial\bar\tau}$$

and hence is a weak harmonic Maaß form of weight w. Using this relation to weak harmonic Maaß forms, if one's goal is to obtain an exact expression for the coefficients of the mock modular form coinciding with our Rademacher sum $R_{\Gamma,\rho}$, an alternative way to achieve it is to first obtain a Poincaré series expression for the corresponding harmonic Maaß forms and then take its holomorphic part. Exactly such a route was first taken by Bringmann and Ono for the cases of mock modular forms with non-negative weight (and with conditional results for $w = 1/2$) in [?, ?]. Using their result, the Poincaré series for the weak harmonic Maaß forms corresponding to R_Γ for $\Gamma = SL_2(\mathbb{Z})$ and $\Gamma = \Gamma_0(2)$ have been investigated by Eguchi and Hikami in [21]. More precisely, one can show that the non-holomorphic completion of

$R_\Gamma(\tau)$ also has a Rademacher form given by

$$(4.9) \qquad \hat{R}_\Gamma(\tau) = \lim_{K \to \infty} \sum_{\gamma \in (\Gamma_\infty \backslash \Gamma)_{<K}} \epsilon(\gamma)^{\epsilon-3} e\left(-\frac{\gamma\tau}{8}\right) \mathrm{com}(\gamma\tau) \, \mathrm{jac}(\gamma, \tau)^{1/4},$$

where $\mathrm{com}(\tau)$ is the *non-holomorphic completion factor* given in terms of the incomplete Gamma function, $\Gamma(\alpha, x) = \int_x^\infty t^{\alpha-1} e^t \mathrm{d}t$, by

$$\mathrm{com}(\tau) = 1 - \Gamma(1/2, \pi\Im(\tau)/2)/\sqrt{\pi}.$$

5. Convergence

After defining the Rademacher sums formally in the last section, in this section we would like to discuss the convergence of the sums. A proof of the convergence will constitute an integral step towards Theorem 4.1. It is a rather subtle issue and in this section we will establish it conditional upon a hypothesis whose validity will be established in Section 9. To be more precise, in this section we assume that the Selberg–Kloosterman zeta function $Z_{0,k}(s, \psi)$ (5.7) with k a positive integer attached to $\psi = \rho\epsilon^{\epsilon-3}$ and $\Gamma = \Gamma_0(n)$ converges at $s = 3/4$ for some character ρ on Γ, an assumption that will be proven in Theorem 9.1. The aim in this section is to demonstrate that under this assumption, the limits (4.1) and (4.7) defining the Rademacher sums $R_{\Gamma,\rho}$ and $S_{\Gamma,\rho}$, respectively, define holomorphic functions on the upper half-plane. Our arguments in this section owe a lot to the work of Niebur in [32]. We cannot apply his results verbatim since he worked with (mock) modular forms of negative weight and ours have positive weight, but many of the procedures we employ have a counterpart in loc. cit. Let $\Gamma = \Gamma_0(n)$ for some positive integer n and $\psi(\gamma) = \rho(\gamma)\epsilon(\gamma)^{\epsilon-3}$ for some character ρ on Γ satisfying $\rho(\gamma) = \rho(T\gamma) = \rho(\gamma T)$. According to the definitions of $R_{\Gamma,\rho}$ and $S_{\Gamma,\rho}$, we can separate the contribution from the trivial coset and the rest in the following way:

$$R_{\Gamma,\rho}(\tau) = q^{-1/8} + \lim_{K \to \infty} \sum_{\gamma \in (\Gamma_\infty \backslash \Gamma)^\times_{<K}} R_\gamma(\tau),$$

$$S_{\Gamma,\rho}(\tau) = q^{1/8} + \lim_{K \to \infty} \sum_{\gamma \in (\Gamma_\infty \backslash \Gamma)^\times_{<K}} S_\gamma(\tau).$$

Using the expression of the (generalized) exponentials as infinite sums and isolating the first term in the sum, we obtain $R_\gamma(\tau) = R^0_\gamma(\tau) + R^+_\gamma(\tau)$ and

$S_\gamma(\tau) = S_\gamma^0(\tau) + S_\gamma^+(\tau)$, where

$$
(5.1) \qquad R_\gamma^0 = e\left(\frac{1}{8}\right) \psi(\gamma) e\left(-\frac{\gamma\infty}{8}\right) c^{\epsilon-3/2}\left(\tau - \gamma^{-1}\infty\right)^{-1},
$$

$$
S_\gamma^0 = \bar{\psi}(\gamma) e\left(\frac{\gamma\infty}{8}\right) c^{\epsilon-3/2}\left(\tau - \gamma^{-1}\infty\right)^{-3/2}
$$

and

$$
(5.2) \quad R_\gamma^+ = \psi(\gamma) e\left(-\frac{\gamma\infty}{8}\right) \sum_{m=1}^\infty \frac{c^{-2m-3/2}(\tau - \gamma^{-1}\infty)^{-m-1}(2\pi i/8)^{m+1/2}}{\Gamma(m + 3/2)},
$$

$$
S_\gamma^+ = \bar{\psi}(\gamma) e\left(\frac{\gamma\infty}{8}\right) \sum_{m=1}^\infty \frac{c^{-2m-3/2}(\tau - \gamma^{-1}\infty)^{-m-3/2}(-2\pi i/8)^m}{m!}.
$$

We have also applied the identities

$$
\mathrm{jac}(\gamma, \tau) = c^{-2}(\tau - \gamma^{-1}\infty)^{-2}, \quad \gamma\infty - \gamma\tau = c^{-2}(\tau - \gamma^{-1}\infty)^{-1},
$$

in rewriting the above expressions. With these definitions, we have the following:

Lemma 5.1. *The sums $\sum_{\epsilon\gamma\in(\Gamma_\infty\backslash\Gamma)^\times} R_\gamma^+(\tau)$ and $\sum_{\epsilon\gamma\in(\Gamma_\infty\backslash\Gamma)^\times} S_\gamma^+(\tau)$ are absolutely convergent, locally uniformly for $\tau \in \mathbb{H}$.*

Proof. We give details for the first sum. A very similar argument takes care of the second one. By (5.2) we have that $R_\gamma^+(\tau) = \mathcal{O}(c^{\epsilon-7/2} |\tau - \gamma^{-1}\infty|^{-2})$ as $c \to \infty$. For $\Gamma = \Gamma_0(n)$, the representatives of the non-trivial cosets in the coset space $\Gamma_\infty\backslash\Gamma$ are indexed by their lower rows which are tuples of two integers (c, d) with $c > 0$ and $n|c$, $\gcd(c, d) = 1$. A procedure we will repeatedly use is to split the sum over d into the following double sum:

$$
(5.3) \qquad\qquad \sum_{\gamma\in(\Gamma_\infty\backslash\Gamma)^\times} f(c, d) = \sum_{\substack{0<c \\ n|c}} \sum_{\substack{0\le d'<c \\ (c,d')=1}} \sum_{\ell\in\mathbb{Z}} f(c, d' + \ell c),
$$

for an arbitrary function $f(x, y)$. Note that the above sum over the tuple (c, d') can be viewed as the sum over a complete and irredundant set of representatives of a double coset in $\Gamma_\infty\backslash\Gamma/\Gamma_\infty$. Taking $f(x, y) = x^{-7/2}|\tau + y/x|^{-2}$

we find that

$$\sum_{\gamma\in(\Gamma_\infty\backslash\Gamma)^\times}|R_\gamma^+(\tau)| \ll \sum_{\substack{0<c\\n|c}}c^{-7/2}\sum_{\substack{0\le d'<c\\(c,d')=1}}\sum_{\ell\in\mathbb{Z}}|\tau+d'/c+\ell|^{-2}.$$

Since there are not more than c invertible elements in the ring $\mathbb{Z}/c$, we conclude that the sum $\sum_{\epsilon\gamma\in(\Gamma_\infty\backslash\Gamma)^\times}R_\gamma^+(\tau)$ is absolutely convergent, locally uniformly for $\tau\in\mathbb{H}$, as required. $\qquad\square$

Hence, in order to show the convergence of the sums defining $R_{\Gamma,\rho}(\tau)$ and $S_{\Gamma,\rho}(\tau)$ it now suffices to establish the convergence of the limits

$$(5.4)\qquad \lim_{K\to\infty}\sum_{\gamma\in(\Gamma_\infty\backslash\Gamma)_{<K}^\times}R_\gamma^0(\tau),\quad \lim_{K\to\infty}\sum_{\gamma\in(\Gamma_\infty\backslash\Gamma)_{<K}^\times}S_\gamma^0(\tau),$$

and apply Lemma 5.1. After carefully examining the limit and employing the Lipschitz summation formula in the form of Lemma Appendix C.1, the first of the above sums can be rewritten as

$$(5.5)\quad \lim_{K\to\infty}\sum_{\epsilon\gamma\in(\Gamma_\infty\backslash\Gamma)_{<K}^\times}R_\gamma^0(\tau)$$
$$= 2\pi e\left(-\frac{1}{8}\right)\sum_{k=1}^\infty e\left(\left(k-\frac{1}{8}\right)\tau\right)\lim_{K\to\infty}\sum_{\substack{0<c<K\\n|c}}c^{-3/2}S(0,k,c,\psi),$$

where we have used the *generalized Kloosterman sum* (cf. [33]) for a given group Γ:

$$(5.6)\qquad S(m,\ell,c,\xi)=\sum_{\substack{\gamma\in\Gamma_\infty\backslash\Gamma/\Gamma_\infty\\c(\gamma)=c}}\xi(\gamma)e((m-\alpha)\gamma\infty)e(-(\ell-\alpha)\gamma^{-1}\infty),$$

for $\ell\in\mathbb{Z}$, where for $m,\ell,c\in\mathbb{Z}$ with $0<c$ and ξ a multiplier system on Γ, the value $0\le\alpha<1$ is determined by ξ according to the requirement that $\xi(T)=e(\alpha)$. An important object closely related to the generalized Kloosterman sum is the *Selberg–Kloosterman zeta function* of Γ (cf. loc. cit.), defined as

$$(5.7)\qquad Z_{m,\ell}(s,\psi)=\sum_{c>0}\frac{S(m,\ell,c,\psi)}{c^{2s}}.$$

The details of the manipulation leading to (5.5) are given in Appendix D. Hence, as promised, if we assume the convergence of $Z_{0,k}(s, \rho\epsilon^{-3})$ at $s = 3/4$ and the estimate $Z_{0,k}(3/4, \rho\epsilon^{-3}) = \mathcal{O}(k)$ we will have shown the convergence of the first sum in (5.4), and hence $R_{\Gamma,\rho}$ upon using Lemma (5.1). The convergence of the Selberg–Kloosterman zeta function and the aforementioned growth estimate will be demonstrated in Section 9. Finally, a similar but less subtle argument also establishes the convergence of the second sum in (5.4). In particular, in the case of S_γ^0 the application of the Lipschitz summation formula is less delicate: we can apply (C.1) with $s = 3/2, \alpha = 7/8$ and the error term E_K is absent. Taken together we have the following:

Proposition 5.2. *The expression (4.1) defining $R_{\Gamma,\rho}(\tau)$ converges, locally uniformly for $\tau \in \mathbb{H}$, thus defining a holomorphic function on $\mathbb{H}$. Also, the expression (4.7) defining $S_{\Gamma,\rho}(\tau)$ converges, locally uniformly for $\tau \in \mathbb{H}$, thus defining a holomorphic function on $\mathbb{H}$.*

In practice, it is useful to know that there is some flexibility in the sum over "rectangles" in the definition of $R_{\Gamma,\rho}$ and $S_{\Gamma,\rho}$ (cf. (4.1) and (4.7)). For example, given $\sigma \in \Gamma$ we may consider the sums

$$(5.8) \qquad \lim_{K\to\infty} \sum_{\gamma \in (\Gamma_\infty \backslash \Gamma)_{<K}\sigma} R_\gamma(\tau), \quad \lim_{K\to\infty} \sum_{\gamma \in (\Gamma_\infty \backslash \Gamma)_{<K}\sigma} S_\gamma(\tau),$$

where $(\Gamma_\infty \backslash \Gamma)_{<K}\sigma$ denotes the "parallelogram" that is the image of $(\Gamma_\infty \backslash \Gamma)_{<K}$ under right multiplication by σ. The following result can be proved by using the technique employed in Lemma 4.3 of [32] to extend the convergence arguments appearing already in this section. We suppress the details.

Lemma 5.3. *Let $\sigma \in \Gamma$, and let $R_\gamma(\tau)$ and $S_\gamma(\tau)$ be as in (5.1) and (5.2), respectively. Then the expressions in (5.8) converge uniformly for $\tau \in \mathbb{H}$, thus defining holomorphic functions on $\mathbb{H}$. Moreover these functions coincide with $R_{\Gamma,\rho}(\tau)$ and $S_{\Gamma,\rho}(\tau)$, respectively.*

6. Coefficients

A common motivation to study Rademacher sums is to obtain an exact expression for the Fourier coefficients of a given infinite q-series. In this section, we will give such expressions for the functions $R_{\Gamma,\rho}$ and $S_{\Gamma,\rho}$. They are useful, for instance, in the physical study of the growth (6.2) of the twisted indices of supersymmetric states — the (twisted) entropy — as the energy of

the states gets large. In the last section, we have seen that, by splitting the sum over representatives of cosets $\Gamma_\infty\backslash\Gamma$ into a double sum over those of the double cosets $\Gamma_\infty\backslash\Gamma/\Gamma_\infty$ and over all integers $\ell \in \mathbb{Z}$, and applying the Lipschitz summation formula (C.2), we obtain expressions (5.5) for the objects (5.4) as an infinite q-series. In this section, we would like to show that the same can be done for the full $R_{\Gamma,\rho}, S_{\Gamma,\rho}$, and write down explicit expressions for the infinite q-series that the infinite sums (4.8) converge to. Following similar steps as delineated in the last section, this time with the help of Equation (C.1), it is not difficult to prove that the functions $R_{\Gamma,\rho}, S_{\Gamma,\rho}$ can be written as q-series with the following Fourier coefficients:

$$c_{\Gamma,\rho}\left(k - \frac{1}{8}\right) = \frac{2\pi\, e\left(-\frac{1}{8}\right)}{(8k-1)^{1/4}} \sum_{\substack{0<c \\ n|c}} \frac{1}{c} I_{1/2}\left(\frac{\pi}{2c}(8k-1)^{1/2}\right) S(0,k,c,\psi),$$

$$c^*_{\Gamma,\rho}\left(k + \frac{1}{8}\right) = e\left(-\frac{3}{8}\right) 2\pi(8k+1)^{1/4} \sum_{\substack{0<c \\ n|c}} \frac{1}{c} J_{1/2}\left(\frac{\pi}{2c}(8k+1)^{1/2}\right)$$

$$(6.1) \qquad \times\; S(1, k+1, c, \bar{\psi}),$$

where $S(m, \ell, c, \psi)$ again denotes the generalized Kloosterman sum (5.6) and $\psi = \rho\epsilon^{-3}$. First note that the q-series with the above coefficients indeed define holomorphic functions on $\mathbb{H}$. To start with, again assuming the convergence of the Selberg–Kloosterman zeta function $Z_{0,k}\left(\frac{3}{4}, \psi\right)$, we will show that the sums (over c) in (6.1) converge: observe that since $c^{-1}I_{1/2}$ $\left(\frac{\pi}{2c}\sqrt{8k-1}\right)$ is bounded by a constant times $c^{\epsilon-3/2}$ for c sufficiently large (cf. (B.5)), the convergence of the right-hand side of (6.1) is guaranteed by the convergence of $Z_{0,k}\left(\frac{3}{4}, \psi\right)$. We establish the convergence of $Z_{0,k}\left(\frac{3}{4}, \psi\right)$ for $k \in \mathbb{Z}$ in Section 9 for the case that $\Gamma = \Gamma_0(n)$ and $\rho = \rho_{n|h}$ for some h dividing both n and 24. From the above argument we see that this result implies the convergence of both the $c_{\Gamma,\rho}\left(k - \frac{1}{8}\right)$ for $k > 0$ and the $c^*_{\Gamma,\rho}\left(k + \frac{1}{8}\right)$ for $k \geq 0$. Once armed with the convergence of the terms $c_{\Gamma,\rho}\left(k - \frac{1}{8}\right)$ and $c^*_{\Gamma,\rho}\left(k + \frac{1}{8}\right)$ we may consider the following series:

$$\sum_{k\geq 1} c_{\Gamma,\rho}(k - \frac{1}{8})q^{k-1/8}, \quad \sum_{k\geq 0} c^*_{\Gamma,\rho}(k + \frac{1}{8})q^{k+1/8}.$$

The Bessel function $I_{1/2}(x)$ is asymptotic to $e^x/\sqrt{2\pi x}$ for large x (cf. Section Appendix B), so $c_{\Gamma,\rho}(k - \frac{1}{8})$ is dominated by the first term in the summation over c in (6.1) for n sufficiently large, and similarly for $c^*_{\Gamma,\rho}(k + \frac{1}{8})$;

in the case that $\Gamma = \Gamma_0(n)$ this is the term with $c = n$. We then have

$$(6.2) \qquad c_{\Gamma,\rho}\left(k - \frac{1}{8}\right) = \mathcal{O}\left(\frac{e^{\pi\sqrt{8k-1}/2n}}{\sqrt{8k-1}}\right)$$

as $k \to \infty$ and a similar estimate holds for $c^*_{\Gamma,\rho}\left(k + \frac{1}{8}\right)$. In particular, we may conclude that the series $\Phi_{\Gamma,\rho}$ and $\Phi^*_{\Gamma,\rho}$, defined by

$$(6.3) \qquad \Phi_{\Gamma,\rho}(\tau) = q^{-1/8} + \sum_{k \geq 1} c_{\Gamma,\rho}\left(k - \frac{1}{8}\right) q^{k-1/8},$$

$$(6.4) \qquad \Phi^*_{\Gamma,\rho}(\tau) = q^{1/8} + \sum_{k \geq 0} c^*_{\Gamma,\rho}\left(k + \frac{1}{8}\right) q^{k+1/8},$$

converge absolutely and locally uniformly for $\tau \in \mathbb{H}$ upon identifying $q = e(\tau)$. We summarize the discussion of the previous two paragraphs in the following lemma.

Lemma 6.1. *The expressions (6.1) defining $c_{\Gamma,\rho}\left(k - \frac{1}{8}\right)$ and $c^*_{\Gamma,\rho}\left(k + \frac{1}{8}\right)$ converge. Further, the generating series (6.3) and (6.4) for the $c_{\Gamma,\rho}\left(k - \frac{1}{8}\right)$ and $c^*_{\Gamma,\rho}\left(k + \frac{1}{8}\right)$ converge absolutely and locally uniformly for $\tau \in \mathbb{H}$, thus defining holomorphic functions $\Phi_{\Gamma,\rho}(\tau)$ and $\Phi^*_{\Gamma,\rho}(\tau)$ on $\mathbb{H}$. These functions coincide with $R_{\Gamma,\rho}(\tau)$ and $S_{\Gamma,\rho}(\tau)$, respectively.*

7. Variance

After establishing the convergence of the Rademachers sums $R_{\Gamma,\rho}$ and $S_{\Gamma,\rho}$, in this section, we wish to determine how they transform under the action of Γ. In particular, we will show that $S_{\Gamma,\rho}$ is a weight $3/2$ cusp form and $R_{\Gamma,\rho}$ is a weight $1/2$ mock modular form on Γ. Moreover, the shadow of $R_{\Gamma,\rho}$ is given by $S_{\Gamma,\rho}$. As before, we have $\Gamma = \Gamma_0(n)$ for some positive integer n, and let $\rho = \rho_{n|h}$ (cf. (2.5)) for some h dividing both n and 24 and set $\psi = \rho\epsilon^{-3}$. The variance of the Rademacher sum $S_{\Gamma,\rho}$ can be easily established, and we have

Proposition 7.1. *The function $S_{\Gamma,\rho}(\tau)$ is a cusp form of weight $3/2$ for Γ with multiplier system $\bar{\psi}$.*

Proof. Since ϵ^3 is the multiplier of the cusp form $\eta^3(\tau)$ and ρ is a group character satisfying $\rho(\gamma)\rho(\sigma) = \rho(\gamma\sigma)$ for all $\gamma, \sigma \in \Gamma_0(n)$, we see that $\bar{\psi} = \rho^{-1}\epsilon^3$

is a weight $3/2$ multiplier system, which leads immediately to

$$S_\gamma|_{\overline{\psi}, \epsilon^3/2}\sigma = S_{\gamma\sigma}, \tag{7.1}$$

for $\sigma \in \Gamma$, where S_γ is as defined in (5.1) and (5.2). Moreover, when $h = 1$ and hence ρ is trivial, this also holds more generally for $\sigma \in \mathrm{SL}_2(\mathbb{Z})$. Applying Lemma 5.3 to $S_{\Gamma,\rho} = \lim_{\epsilon K \to \infty} \sum_{\gamma \in (\Gamma_\infty \backslash \Gamma)_{\epsilon < K}} S_\gamma(\tau)$ and using (7.1), we immediately have $S_{\Gamma,\rho}|_{\overline{\psi}, \epsilon^3/2}\sigma = S_{\Gamma,\rho}$ for $\sigma \in \Gamma$. This establishes that $S_{\Gamma,\rho}$ is a modular form of weight $3/2$ for Γ with multiplier system $\overline{\psi}$. We also need to show that $S_{\Gamma,\rho}$ vanishes at all cusps. For the cusp at ∞, this is clear from Lemma 6.1. We will now show that it is also true for the other cusps of Γ. An important property of a given cusp is its width. Suppose $\kappa \in \mathbb{Q}$ is a representative of a cusp of $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ that is related to the infinite cusp by $\sigma\infty = \kappa$, $\sigma \in \mathrm{SL}_2(\mathbb{Z})$. We define the *width* u of the cusp κ to be the positive integer such that the subgroup of Γ stabilizing κ is given by

$$\{\gamma \in \Gamma | \gamma\kappa = \kappa\} = \sigma\langle \pm T^u \rangle \sigma^{-1} = \sigma\Gamma_\infty^u \sigma^{-1}. \tag{7.2}$$

Alternatively, we can rephrase the above definition using the concept of a 'scaling element' which will become important later. Consider an element $\tilde{\sigma} \in \mathrm{SL}_2(\mathbb{R})$, to be called a *scaling element* of Γ at cusp κ, such that $\kappa = \tilde{\sigma}\infty$ and the subgroup of Γ stabilizing κ is given by $\tilde{\sigma}\Gamma_\infty\tilde{\sigma}^{-1}$ (Cf. [10, Section 2.6]). Such a scaling element necessarily has the form

$$\tilde{\sigma} = \sigma U T^\beta, \quad \text{with} \quad \sigma \in \mathrm{SL}_2(\mathbb{Z}), \quad \beta \in \mathbb{Q}, \tag{7.3}$$

where $U = \begin{pmatrix} \sqrt{u} & 0 \\ 0 & 1/\sqrt{u} \end{pmatrix}$ maps $\tau \to u\tau$ and $T^\beta = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$. Given a cusp representative $\kappa = \sigma\infty$ of width v of $\Gamma_0(n)$, in order to study the behaviour of $S_{n|h}$ at κ, it is convenient to consider the function $S_{n|h}^{(\kappa)}(\tau)$, which we will now define. For a given scaling element $\tilde{\sigma}$, for $h = 1$ we define

$$S_{n|1}^{(\kappa)}(\tau) = S_{n|1}\big|_{\epsilon^3, \epsilon^3/2}\sigma(v(\tau + \beta)) = \epsilon^3(\sigma)\,\mathrm{jac}(\sigma, v(\tau + \beta))^{\epsilon^3/4} S_{n|h}(\tilde{\sigma}\tau).$$

From (7.1) we obtain

$$S_{n|1}^{(\kappa)}(\tau) = \lim_{\epsilon K \to \infty} \sum_{\gamma \in (\Gamma_\infty \backslash \Gamma)_{\epsilon < K}\sigma} S_\gamma(v(\tau + \beta)).$$

Splitting the above sum into a sum over the double coset $(\Gamma_\infty \backslash \Gamma)_{\epsilon < K}\sigma/\Gamma_\infty^v$ and applying the Lipschitz summation formula as before, we obtain that $S_{n|1}^{(\kappa)}(\tau)$ takes the form of an infinite series $\sum_{k=0}^\infty c_{\Gamma,\rho}^{*(\kappa)}(k+\nu)\,e((k+\nu)\tau)$

where $0 < \nu \le 1$ differs from $\frac{v}{8}$ by an integer. This proves that $S_{n|1}$ vanishes at κ. More generally, for $h \ne 1$ we again consider

(7.4)
$$S_{n|h}\big|_{\epsilon^3, \epsilon 3/2}\sigma(\tau) = \epsilon^3(\sigma)\,\mathrm{jac}(\sigma, \tau)^{\epsilon 3/4} S_{n|h}(\sigma\tau)$$
$$= \lim_{\epsilon K \to \infty} \sum_{\gamma \in (\Gamma_\infty \backslash \Gamma)_{\epsilon < K}\sigma} \rho^{\epsilon-1}(\gamma\sigma^{-1})\epsilon^3(\gamma)\,\mathrm{jac}(\gamma, \tau)^{\epsilon 3/4} e(\frac{\gamma\tau}{8}).$$

If κ has width v in $\Gamma_0(n)$ and has width u in $\Gamma_0(nh)$, we can define for a given scaling element $\tilde{\sigma}$

$$S_{n|h}^{(\kappa)}(\tau) = S_{n|h}\big|_{\epsilon^3, \epsilon 3/2}\sigma\,(u(\tau + \beta)).$$

To obtain its expression as an infinite series, define $0 \le m < h$ such that $\rho(\sigma T^v \sigma^{-1}) = e\left(\frac{m}{h}\right)$ and proceed as before, ultimately obtain in the general expression:

(7.5) $$S_{n|h}^{(\kappa)}(\tau) = \sum_{k=0}^{\infty} c_{\Gamma,\rho}^{*(\kappa)}(k + \nu)e\left((k + \nu)\frac{u}{v}\tau\right), \quad 0 < \nu \le 1.$$

More precisely, $\nu = \{\{\frac{m}{h} + \frac{v}{8}\}\}$ is given by the sawtooth function $\{\{x\}\}$ defined as the difference between a real number and the largest integer that is smaller than it:

$$\{\{x\}\} = \begin{cases} x - \lfloor x \rfloor & x \in \mathbb{R}\backslash\mathbb{Z}, \\ 1 & x \in \mathbb{Z}, \end{cases}$$

where the floor function $\lfloor x \rfloor$ denotes the largest integer not exceeding x. This shows that $S_{n|h}(\tau)$ vanishes at all cusps of $\Gamma_0(n)$ and thereby finishes the proof. $\square$

Our next task is to establish the variance of the Rademacher sum $R_{\Gamma,\rho}$.

Proposition 7.2. *The function $R_{\Gamma,\rho}$ is a mock modular form of weight $1/2$ on Γ with multiplier system ψ and with shadow $S_{\Gamma,\rho}$.*

Proof. First observe that $R_\gamma = \psi(\gamma)e\left(-\frac{\gamma\tau}{8}\right)\mathrm{jac}(\gamma, \tau)^{1/4}\,\mathrm{reg}(\gamma, \tau)$ would transform nicely if it were not for the regularization factor. Therefore, it

would be useful to separate from R_γ the "unregularized" part

$$(7.6) \qquad\qquad P_\gamma(\tau) = \psi(\gamma)e(-\gamma\tau/8)\,\mathrm{jac}(\gamma,\tau)^{1/4},$$

for $\gamma \in \Gamma$, and consider

$$R_\gamma(\tau) - P_\gamma(\tau) = \psi(\gamma)e\left(-\frac{\gamma\infty}{8}\right)\mathrm{jac}(\gamma,\tau)^{1/4}(e(X,1/2) - e(X)),$$

with $X = \frac{\gamma\infty - \gamma\tau}{8}$. Using the definition (4.4), we have

$$e(X,s) - e(X) = -\frac{e(X)}{\Gamma(s)}\int_{2\pi i X}^{\infty} \mathrm{e}^{-t}t^{s-1}\,dt.$$

Setting $t = \frac{2\pi i}{8}(\gamma z - \gamma\tau)$ in the integral and using the identity

$$(7.7) \qquad\qquad \mathrm{jac}(\gamma,z)\,\mathrm{jac}(\gamma,\tau)(z-\tau)^2 = (\gamma z - \gamma\tau)^2$$

we get

$$\begin{aligned} e(X,1/2) - e(X) = \frac{1}{2}e\left(\frac{1}{8}\right)e\left(\frac{\gamma\infty}{8}\right)\mathrm{jac}(\gamma,\tau)^{\epsilon-1/4} \\ \times \int_{\gamma^{-1}\infty}^{\infty} e\left(-\frac{\gamma z}{8}\right)\mathrm{jac}^{\epsilon 3/4}(\gamma,z)\,(z-\tau)^{\epsilon-1/2}\,dz. \end{aligned}$$

It will be convenient to introduce an integral operator J_x for x in the extended upper half-plane $(\mathbb{H} \cup \mathbb{Q} \cup \infty)$, which acts on a holomorphic function g on $\mathbb{H}$ (with sufficiently rapid decay as $\Im(z) \to \infty$) as

$$(7.8) \qquad\qquad (J_x g)(\tau) = \frac{e\left(\frac{1}{8}\right)}{2}\int_{\bar{x}}^{\infty} \bar{g}(z)(z-\tau)^{-1/2}dz.$$

From definition (3.4), we see that we have a weight $1/2$ mock modular form with shadow g if $\hat{h} = h + J_\tau g$ transforms as a weight $1/2$ modular form. Comparing with the definition of the integral operator J_x, we have

$$R_\gamma(\tau) - P_\gamma(\tau) = \left(J_{\gamma^{-1}\infty}S_\gamma\right)(\tau).$$

Note in particular that this is trivially true when γ is in the trivial coset given by upper triangular matrices, since both sides of the equation vanish. By the fact that ψ is a multiplier system for Γ of weight $1/2$, we immediately

have for an element σ of Γ

$$P_\gamma|_{\psi,\epsilon1/2}\sigma = P_{\gamma\sigma}, \text{ and } (J_{\gamma^{-1}\infty}S_\gamma)|_{\psi,\epsilon1/2}\sigma = (J_{\sigma^{-1}\gamma^{-1}\infty} - J_{\sigma^{-1}\infty})S_{\gamma\sigma}.$$

Upon summation over the representatives of the rectangle $(\Gamma_\infty\backslash\Gamma)_{<K}, K \to \infty$ and evoking Lemma 5.3, we obtain

$$(7.9) \qquad\qquad R_{\Gamma,\rho}|_{\psi,\epsilon1/2}\,\sigma = R_{\Gamma,\rho} - J_{\sigma^{-1}\infty}S_{\Gamma,\rho}.$$

Now it is an easy calculation to show that

$$(R_{\Gamma,r} + J_\tau S_{\Gamma,\rho})|_{\psi,\epsilon1/2}\,\sigma = R_{\Gamma,r} + J_\tau S_{\Gamma,\rho},$$

which by definition (3.4) proves the claim.   □

8. Coincidence

Recall from Section 4 that we write $R_{n|h}$ for $R_{\Gamma,\rho}$ when $\Gamma = \Gamma_0(n)$ and $\rho = \rho_{n|h}$, and similarly for $S_{n|h}$. In Section 7, we have established the transformation property of $R_{n|h}$ and $S_{n|h}$ under the group $\Gamma_0(n)$. In this section, we will explicitly identify them. More precisely, we will first prove in Section 8.1 that

$$(8.1) \qquad\qquad S_{n|h} = \begin{cases} \lambda_n\eta^3, \lambda_n \in \mathbb{C} & h = 1, \\ 0 & h > 1, \end{cases}$$

for the $(n, h) = (n_g, h_g)$ that appear in the cycle shapes of $[g] \subset M_{24}$, (cf. table 1), and later in Section 8.2 complete the proof our main theorem 4.1 by showing $H_g = -2R_{n|h}$, assuming the result of Section 9.

8.1. Determining S

Let n and h be positive integers such that h divides both n and 24 and consider the function

$$F_{n|h}(\tau) = \frac{S_{n|h}(\tau)}{\eta(\tau)^3}.$$

Since $S_{n|h}$ has the same multiplier system as η^3 on $\Gamma_0(nh)$ and η does not vanish on the upper half-plane, $F_{n|h}$ is a modular function for $\Gamma_0(nh)$ that

satisfies the transformation rule

$$(8.2) \qquad\qquad F_{n|h}(\gamma\tau) = \rho(\gamma)F_{n|h}(\tau),$$

for $\gamma \in \Gamma_0(n)$ where $\rho = \rho_{n|h}$ is given explicitly by (2.5). In particular, $F_{n|h}$ induces a morphism, which we denote by $f_{n|h}$, of Riemann surfaces $X_0(nh) \to \mathbb{P}^1$ where $X_0(N)$ denotes the *modular curve* of level N and is formed by taking the quotient of the *extended upper half-plane*

$$\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$$

by the action of the group $\Gamma_0(N)$. We will denote the corresponding canonical map by

$$(8.3) \qquad\qquad \varphi_N : \mathbb{H}^* \to X_0(N).$$

Note that there are no non-constant maps $X_0(nh) \to \mathbb{C}$, so either $F_{n|h}$ has a pole at some cusp of $\Gamma_0(nh)$, or $S_{n|h} = \lambda_{n|h}\eta^3$ for some $\lambda_{n|h} \in \mathbb{C}$. Of course, $S_{n|h}$ and η^3 have different automorphy for $\Gamma_0(n)$ when $h > 1$, and an identity $S_{n|h} = \lambda_{n|h}\eta^3$ implies that $\lambda_{n|h}$ and hence $S_{n|h}$ are identically zero. Our main strategy to show (8.1) is to show that $F_{n|h}$ is bounded at all cusps of $\Gamma_0(nh)$ for all the pairs (n, h) under consideration. Since the Fourier expansion of both $S_{n|h}$ and η^3 around the infinite cusp starts with a term $q^{1/8}$ (cf. (6.1)), we see that $F_{n|h}(\tau)$ is bounded as $\tau \to i\infty$ and can only have poles at the cusps of $\Gamma_0(nh)$ that are not represented by infinity. Furthermore, from the transformation (8.2) of $F_{n|h}$ under $\Gamma_0(n)$, we see that $f_{n|h}$ has a pole at a given cusp of $\Gamma_0(nh)$ if and only if it has a pole of the same order at every cusp that is an image of it under $\Gamma_0(n)$. Therefore, to prove (8.1) it is sufficient to show that $F_{n|h}$ are bounded at all cusps of $\Gamma_0(n)$ other than the infinite cusp. In order to analyse the behaviour of $F_{n|h}$ at the cusp representative $\kappa \in \mathbb{Q}$, as before we will now consider its transformation under a given scaling element (cf. (7.3))

$$\tilde\sigma = \sigma \begin{pmatrix} \sqrt{u} & 0 \\ 0 & 1/\sqrt{u} \end{pmatrix} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}, \quad \sigma \in \mathrm{SL}_2(\mathbb{Z}), \quad \beta \in \mathbb{R},$$

for $\Gamma_0(nh)$ at $\kappa = \tilde\sigma\infty$:

$$(8.4) \qquad F_{n|h}^{(\kappa)}(\tau) = F_{n|h}(\tilde\sigma\tau) = \frac{S_{n|h}(\tilde\sigma\tau)}{\eta^3(\tilde\sigma\tau)} = \frac{S_{n|h}^{(\kappa)}(\tau)}{\eta^3(u(\tau + \beta))}.$$

When $\tilde{\sigma}$ normalizes $\Gamma_0(nh)$, $F_{n|h}^{(\kappa)}(\tau)$ again defines a modular function for $\Gamma_0(nh)$. The order of poles or zeros of the function $F_{n|h}^{(\kappa)}(\tau)$ at $\tau \to i\infty$ gives the order of poles or zeros of the function $f_{n|h} : X_0(nh) \to \mathbb{P}^1$ at the point x given by the image of the κ under the canonical map φ_{nh}. In the proofs, we will be especially interested in the scaling elements that are elements of the so-called Atkin–Lehner involutions W_u, which are defined for every exact divisor u of n as the set of matrices of the form

$$\frac{1}{\sqrt{u}} \begin{pmatrix} au & b \\ cn & du \end{pmatrix}, \quad a, b, c, d \in \mathbb{Z}$$

with determinant 1. We say u is an *exact divisor* of n if $(n/u, u) = 1$. An important element of W_n is the *Fricke involution*, given by

$$(8.5) \qquad\qquad w_n = \begin{pmatrix} 0 & -1/\sqrt{n} \\ \sqrt{n} & 0 \end{pmatrix}.$$

From the expansion of $S_{n|h}^{(\kappa)}(\tau)$ recorded in (7.5), we see that $F_{n|h}^{(\kappa)}(\tau)$ has the expansion:

$$(8.6) \qquad F_{n|h}^{(\kappa)}(\tau) = \sum_{k=0}^{\infty} C_{n|h}(k)\, q^{\ell + \frac{u}{v}k}, \quad \ell = \frac{u}{v} \{\{\frac{m}{h} + \frac{v}{8}\}\} - \frac{u}{8}.$$

As before, u denotes the width of the cusp in $\Gamma_0(nh)$, v denotes the width of the cusp in $\Gamma_0(n)$, and $e\left(\frac{m}{h}\right) = \rho(\sigma T^v \sigma^{-1})$ as defined in (7.2). Moreover, since $F_{n|h}^{(\kappa)}(\tau)$ is invariant under $\langle \pm T \rangle \subset \tilde{\sigma}^{-1}\Gamma_0(nh)\tilde{\sigma}$, its Fourier expansion can only involve integral powers of q and it follows immediately that

$$(8.7) \qquad\qquad F_{n|h}^{(\kappa)}(\tau) = \mathcal{O}(q^\ell), \quad \ell \geq 1 + \lfloor -\frac{u}{8} \rfloor \quad, \quad \ell \in \mathbb{Z}.$$

More generally, from the above formula it is also clear that, to prove that $f_{n|h}$ is bounded at κ it is sufficient to prove that

$$(8.8) \qquad\qquad \{\{\frac{m}{h} + \frac{v}{8}\}\} \geq \frac{v}{8}.$$

In the remainder of this section, we will assume knowledge of various facts about $\Gamma_0(n)$, such as its genus, its cusps, and the widths of those cusps. For any given value of n the required computations are standard, if not at times laborious, and the reader may refer to e.g., [34] or [35] for the necessary background. Let us consider the cases $n \leq 8$ when there are no cusps of

$\Gamma_0(n)$ with width greater than 8. For these cases, we have the following Lemma:

Lemma 8.1. *Consider the group $\Gamma_0(n)$ with $1 \leq n \leq 8$ and multiplier $\rho_{n|h}$ defined in (2.5) with h dividing both n and 24. The pole-free condition (8.8) is satisfied for all cusps of $\Gamma_0(n)$ if and only if $(n, h) = (n_g, h_g)$ is one of the 13 pairs that correspond to the cycle shapes of M_{24}, as collected in table 1.*

Proof. First consider the case $h = 1$. In this case, since there are no cusps with $v > 8$, we get from (8.7) that $f_{n|h}$ has no pole for all $1 \leq n \leq 8$, $h = 1$. For $h > 1$, we need more refined information about the order of zeros of $S_{n|h}^{(\kappa)}$ to show that, for a given $n < 9$, among all the possible values of h that divide both n and 24, the condition (8.8) is satisfied if and only if $(n, h) = (n_g, h_g)$ is one of the 13 pairs that correspond to the cycle shapes of M_{24}. For the cusp with a representative $1/q$ and width v, taking $\sigma = \begin{pmatrix} 1 & 1 \\ q & q+1 \end{pmatrix}$ we get

$$(8.9) \qquad m = \frac{q^2 v}{n}(1 + qv) \mod h.$$

In particular, we have $m = q$ and $v = n/q$ if q is an exact divisor of n. For the zero cusp ($q = 1$, note that 0 and 1 are related by $T \in \Gamma_0(n)$), therefore, we need h to satisfy $\{\{\frac{1}{h} + \frac{n}{8}\}\} \geq \frac{n}{8}$. It is straightforward to check that this condition is satisfied if and only if $(n, h) = (n_g, h_g)$ is one of the 13 pairs that show up in table 1. It remains to be shown that this condition is also satisfied at cusps other than 0 or ∞. First, we check that $\{\{\frac{q}{h} + \frac{n/q}{8}\}\} \geq \frac{n/q}{8}$ is indeed satisfied for $n = h = 6$, $q = 2$ and $q = 3$. Finally, for $n = 4$, we have $q = 2$, $v = 1$ and $m = 3$ and we need to check $\{\{\frac{3}{h} + \frac{1}{8}\}\} \geq \frac{1}{8}$, which is indeed the case for $h = 4$. $\square$

Corollary 8.2. *For $(n, h) = (n_g, h_g)$ being one of the 13 pairs with $1 \leq n \leq 8$ that arise from in the cycle shapes of M_{24}, (cf. table 1), we have $S_{n|1} = \lambda_n \eta^3$ and $S_{n|h} = 0$ for $h > 1$.*

For $n > 8$, a direct proof via (8.8) is less straightforward and instead we would like to use arguments involving the topology of $X_0(n)$. When $X_0(nh)$ has genus one, we have the following two lemmas:

Lemma 8.3. *If $n \in \{11, 14, 15\}$ then $S_{n|1} = \lambda_n \eta^3$ for some $\lambda_n \in \mathbb{C}$.*

Proof. For these values of n there is just one cusp of $\Gamma_0(n)$ represented by 0 which has width $v = n > 8$. Hence, from (8.7) we see that the function

$F_{n|1}^{(\kappa=0)}$ either has a simple pole at 0 or is a constant. Suppose the former were true, then $f_{n|1}$ defines a degree one function from $X_0(n)$ to $\mathbb{P}^1$, implying that $X_0(n)$ has genus zero as a Riemann surface. For $n \in \{11, 14, 15\}$ however, the genus of $X_0(n)$ is one, so we conclude that $f_{n|1}$ is constant in these cases. □

Lemma 8.4. *Suppose that $n \in \{10, 12\}$ and $h = 2$. Then $S_{n|h} = 0$.*

Proof. The only cusp of $\Gamma_0(20)$ that has width greater than 8 is that represented by 0, and similarly for $\Gamma_0(24)$, so in either case the only possible pole of $f_{n|h}$ is at the image under $\varphi_{nh} : \mathbb{H}^* \to X_0(nh)$ of the zero cusp. The degree of the pole, if the pole exists, is given by (8.6) with $2v = u = 2n$ and $m = 1$ equals 1, and so must be 1. In this case, $f_{n|h} : X_0(nh) \to \mathbb{P}^1$ is a degree one map, implying that $X_0(nh)$ has genus zero. However, this is not the case for $nh = 20$ or $nh = 24$. We conclude that $f_{n|h}$ has no poles and is thus constant. The claim follows. □

For the remaining three classes with $(n, h) = (23, 1), (21, 3), (12, 12)$, the modular curves $X = X_0(nh)$ have higher genera and we will use the Riemann–Roch theorem:

$$(8.10) \qquad \dim K_X(D) = \dim \Omega_X(D) + \deg(D) + 1 - \text{genus}(X)$$

to gain control over the allowed properties of the function $f_{n|h}$. Here D denotes a divisor on the Riemann surface X, $K_X(D)$ denotes the vector space of meromorphic functions f with the divisor (f) given by its zeros and poles satisfying $(f) + D \geq 0$, and $\Omega_X(D)$ denotes the space of holomorphic differentials ω on X satisfying $(\omega) - D \geq 0$.

Lemma 8.5. *If $n = 23$ then $S_{n|1} = \lambda_n \eta^3$ for some $\lambda_n \in \mathbb{C}$.*

Proof. The only cusp of $\Gamma_0(23)$ with width greater than 8 is the one with width 23 represented by 0, and hence the function $f_{23|1}$ can possibly have a pole only at $x \in X$ for x the image of 0 under φ_{23} (cf. (8.3)). More specifically, from (8.7) we see that $f_{23|1}$ is in the vector space $K_X(D)$ for $D = 2x$. Since the genus of $X_0(23)$ is 2, according to the Riemann–Roch formula (8.10) we have $\dim K_X(D) = 1 + \dim \Omega_X(D)$. We would like to prove the claim by showing that $\dim \Omega_X(D) = 0$ and hence $\dim K_X(D) = 1$, and therefore a function in $X(D)$ must be a constant function. To show there is no holomorphic differential with a double zero at x, let us consider first the

divisor $D' = x$. We know that $K_{D'}(X) \cong \mathbb{C}$, or else there would be a mero-morphic function on X with a simple pole at x and no other poles, hence an isomorphism between X and $\mathbb{P}^1$, which is impossible given that the genus of X is 2. From this we conclude from the Riemann–Roch formula that $\dim \Omega_X(D') = \dim K_{D'}(X) = 1$ and there is a 1D space of holomorphic dif-ferentials ω with $(\omega) - x \geq 0$. Now such a holomorphic differential is given by the weight 2 cusp form $\eta^2(\tau)\eta^2(23\tau)$, which is a holomorphic differen-tial with a simple zero at x and a simple zero at the image of the infinite cusp under the canonical map $\mathbb{H}^* \to X_0(23)$ and no other zeros. From this we conclude $\dim \Omega_X(D) = 0$ and hence $\dim K_X(D) = 1$. This completes the proof. $\square$

Lemma 8.6. *Suppose that $n = 21$ and $h = 3$. Then $S_{n|h} = 0$.*

Proof. For $n = 21$, by direct calculation one can check that the only cusp for which (8.8) does not hold is the cusp represented by 0. It corresponds to a cusp with width 63 of $\Gamma_0(63)$ and there are no other cusps of $\Gamma_0(63)$ at which $f_{21|3}$ can have a pole. Moreover, from (8.6) for the zero cusp, with

$$\frac{63}{21}\left\{\left\{\frac{21}{8} + \frac{1}{3}\right\}\right\} - \frac{63}{8} = -5.$$

Equation (8.6) shows that $f_{n|h}$ is in $K_D(X = X_0(63))$ with $D = 5x$, $x = \varphi_{63}(0)$. The genus of $X_0(63)$ is 5 so according to the Riemann–Roch theorem (cf. (8.10)) we have $\dim K_X(D) = \dim \Omega_X(D) + 1$, where $\Omega_X(D)$ denotes the space of holomorphic differentials on X vanishing to order at least 5 at x. We claim that $\dim \Omega_X(D) = 0$. To show this, suppose $g \in S_2(\Gamma_0(63))$ corresponds to an element of Ω_X with a zero of order 5 at x. Since the Fricke involution w_{63} normalizes $\Gamma_0(63)$, it induces a linear automorphism of $S_2(\Gamma_0(63))$. Then $g|_{1,2}\sigma$ is an element of $S_2(\Gamma_0(63))$ that is $\mathcal{O}(q^6)$ as $\tau \to i\infty$. By inspection there is no such cusp form of weight 2 for $\Gamma_0(63)$. We conclude that $\dim K_X(D) = 1$. Since $K_X(D)$ includes constant functions, our $f_{n|h}$ is constant, and the desired result follows. $\square$

Lemma 8.7. *Suppose that $n = h = 12$. Then $S_{n|h} = 0$.*

Proof. The group $\Gamma_0(nh) = \Gamma_0(144)$ has nine cusps with widths exceeding 8: there are four with of width 9, having representatives 1/4, 3/4, 1/8 and 1/16, there are three of width 16, having representatives 1/3, 2/3 and 1/9, and there is one of width 36, with representative 1/2, and one of width 144, with representative 0. The four cusps of width 9 fuse under the action of

$\Gamma_0(n) = \Gamma_0(12)$, as do the three cusps of width 16. First, we would like to see at which of these cusps of $\Gamma_0(144)$ the function $f_{n|h}$ could possibly have a pole. From the above data, we only need to look at the cusps of $\Gamma_0(12)$. From formula (8.9) we see that $f_{n|h}$ is either constant or has a pole at the image of the zero cusp, as the pole-free condition (8.8) is satisfied at the all other cusps of $\Gamma_0(12)$. Moreover, putting $v = h = 12$, $m = 1$, $u = 144$ in Equation (8.6), we see that $f_{n|h}$ can only have a pole of order 11 if it is not a constant. To show that a pole of order 11 cannot happen, first we will compute $\dim K_X(D) = 2$ for $X = X_0(144)$ and $D = 11x$ with $x = \varphi_{144}(0)$. The genus of $X_0(144)$ is 13 so according to the Riemann–Roch theorem (cf. (8.10)) we have $\dim K_X(D) = \dim \Omega_X(D) - 1$ where $\Omega_X(D)$ denotes the space of holomorphic differentials on X vanishing to order at least 11 at x. As in the proof of Lemma 8.6 we observe that elements of $\Omega_X(D)$ are in correspondence with cusp forms of weight 2 on $\Gamma_0(144)$ that are $\mathcal{O}(q^{12})$ as $\tau \to i\infty$. By inspection there is a 3D space of such cusp forms so $\dim K_X(D) = 2$. Next set $D' = 10x$. We claim that then $K_X(D) = K_X(D')$. For certainly $K_X(D')$ is a subspace of $K_X(D)$ and their dimensions coincide since the Riemann–Roch theorem implies $\dim K_X(D') = \dim \Omega_X(D') - 2$ and $\dim \Omega_X(D') = 4$ by inspection of $S_2(\Gamma_0(144))$. We conclude that if $f_{n|h}$ is in $K_X(D)$ then it is also in $K_X(D')$ and then we must have $C_{n|h}(k = 0) = 0$ in (8.6), so that $f_{n|h}$ has no poles at any cusp and is therefore constant. This completes the proof. □

Remark. It is fascinating that the space $S_{3/2,\epsilon^3}(\Gamma_0(n))$ of cusp forms of weight 3/2 on $\Gamma_0(n)$ with multiplier system coinciding with that of η^3 is 1D and spanned by η^3, whenever n is the order of an element of M_{23}. This is demonstrated by the above Lemmas taken together, whereas the same is certainly not true for a generic positive integer n. For $n = 9$ and $n = 10$, for instance, it is possible that $F_{n|1} = S_{n|1}/\eta^3$ could define a degree one map $X_0(n) \to \mathbb{P}^1$, but this is no contradiction since $X_0(n)$ has genus zero for $n = 9$ and $n = 10$. Indeed, numerical approximations suggest that $S_{9|1}(\tau) = \eta(\tau)^3 + 3\eta(9\tau)^3$ and

$$S_{10|1}(\tau) = C \left(2\eta(\tau)^3 + 7\frac{\eta(2\tau)\eta(10\tau)^3}{\eta(5\tau)} \right)$$

for some $C \in \mathbb{C}$. Observe that the functions $\eta(\tau)^3/\eta(9\tau)^3$ and $\eta(\tau)^3\eta(5\tau)/\eta(2\tau)\eta(10\tau)^3$ are hauptmoduln for $\Gamma_0(9)$ and $\Gamma_0(10)$, respectively, and so $S_{3/2,\epsilon^3}(\Gamma_0(n))$ is not 1D for $n = 9$ or $n = 10$. Indeed, there are no M_{24} classes corresponding to $(n, h) = (9, 1)$ or $(10, 1)$. Also in the case that $h > 1$, it is extremely non-trivial that for all the pairs (n, h) arising from M_{24}, just the

correct combinations of genus, cusp widths and multipliers conspire to force the vanishing of $S_{n|h}$.

8.2. Determining R

First, we will establish the identity $H_g = -2R_{n|1}$, for $n = n_g$ and for those g with $h_g = 1$. Recall from the discussion in Section 2 that these are the M_{24} classes whose action on the natural permutation representation has at least one fixed point (the M_{23} classes). To do this, we consider the function

$$(8.11) \qquad G_{n|1} = \eta^3(\chi(g)\, R_{n|1} - \lambda_n H_g)$$

with g being an M_{24} class with $n_g|h_g = n|1$. By construction, it has $G_{n|1} = (\chi(g) + 2\lambda_n) + \mathcal{O}(q)$ near the infinite cusp. In Section 3, we saw that H_g is a weight $1/2$ mock modular form for $\Gamma_0(n)$ with shadow $\chi(g)\eta^3$ and the same multiplier as η^{-3}. On the other hand, in Section 7 we have proven that the Rademacher sums $R_{n|1}$ are also weight $1/2$ mock modular forms for $\Gamma_0(n)$ with the same multiplier. Moreover, it has as its shadow $S_{n|1}$, which was proven to be given by $\lambda_n\eta^3$ for some $\lambda_n \in \mathbb{C}$ in the previous subsection. Taken together, we see that the function $G_{n|1} : \mathbb{H}^* \to \mathbb{P}^1$ is a weight 2 modular form of $\Gamma_0(n)$ with trivial multiplier. Using the properties of such weight 2 modular forms, we will be able to show that $G_{n|1} = 0$ for the values of n of interest to us. In particular, we have $\chi(g) + 2\lambda_n = 0$ as the constant term in the expansion near the infinite cusp. Since $\chi(g) \neq 0$ when $h_g = 1$, this implies $H_g = -2R_{n|1}$ for $g \in M_{24}$ with $h_g = 1$.

Lemma 8.8. *Let g be the identity element of M_{24} and set $n = n_g = 1$ and $h = h_g = 1$. Then $H = H_g = -2R_{n|h}$.*

Proof. From the above discussion we know the function

$$G_{n|1} = \eta^3 \left(24R_{n|1} - \lambda_n H_g\right)$$

is a modular form of weight 2 for $\mathrm{SL}_2(\mathbb{Z})$ with trivial multiplier. There are no non-zero modular forms of weight 2 for $\mathrm{SL}_2(\mathbb{Z})$ so $G_{n|1}$ vanishes identically and by the above argument this proves the lemma. $\qquad\square$

To prove $G_{n|1} = 0$ for $n > 1$ we need to study its behaviour near the other cusp representatives of $\Gamma_0(n)$. For a cusp $\kappa = \sigma\infty$, $\sigma \in \mathrm{SL}_2(\mathbb{Z})$, with width v in $\Gamma_0(n)$, we shall again consider its transformation under a scaling

element $\tilde{\sigma}$ (cf. (7.3))

$$G_{n|1}^{(\kappa)}(\tau) = G_{n|1}\big|_{1,2}\sigma\left(v(\tau + \beta)\right).$$

To be more precise, we would like to know the form of the Fourier expansion of $G_{n|1}^{(\kappa)}(\tau)$. As will be shown explicitly in Appendix E, we have

$$G_{n|1}^{(\kappa)}(\tau) = \mathcal{O}(q^{v/8 + \Delta})$$

as $\tau \to i\infty$ with $\Delta \geq 0$, which corresponds to the fact that both $R_{n|1}$ and H_g have no pole at any cusp other than the infinite cusp. By construction, $G_{n|1}^{(\kappa)}$ is invariant under $\tau \to \tau + 1$ and hence the above argument shows

(8.12) $G_{n|1}^{(\kappa)}(\tau) = \mathcal{O}(q^{\lceil v/8 \rceil}),$

where the ceiling function $\lceil x \rceil$ gives the smallest integer not less than $x \in \mathbb{R}$.

Lemma 8.9. *Suppose that $g \in M_{24}$ has prime order $n = n_g$ and $h = h_g = 1$. Then $H_g = -2R_{n|h}$.*

Proof. The primes dividing the order of M_{24} are exactly those primes p for which $(p + 1)|24$ so $n \in \{2, 3, 5, 7, 11, 23\}$. For these n, the only non-infinite cusp of $\Gamma_0(n)$ is the one represented by 0, with width n. Take the Fricke involution w_n to be the scaling element. From (8.12) we conclude that $G_{n|1}^{(\kappa=0)}$ is $\mathcal{O}(q)$ for $n \in \{2, 3, 5, 7\}$ and $\mathcal{O}(q^2)$ for $n = 11$ and $\mathcal{O}(q^3)$ for $n = 23$. Since the Fricke involution w_n for $\Gamma_0(n)$ normalizes $\Gamma_0(n)$, $G_{n|1}^{(\kappa=0)}$ also belongs to $M_2(\Gamma_0(n))$. The space $M_2(\Gamma_0(n))$ of such weight 2 modular forms contains the space $S_2(\Gamma_0(n))$ of weight 2 cusp forms. The dimension of the latter is given by the genus of the modular curve $X_0(n)$ and the dimension of the former is the genus of $X_0(n)$ plus $m - 1$, where m is the number of cusps of $\Gamma_0(n)$ (cf. [34, Section 2.6]). It can be checked then (cf. e.g., [36, Section 7] or [35, Section 5]) that a complement to $S_2(\Gamma_0(n))$ in $M_2(\Gamma_0(n))$ is spanned by $\{\Lambda_m \mid m > 1, m|n\}$ where

$$\Lambda_m(\tau) = m\, q \frac{d}{dq} \log(\eta(n\tau)/\eta(\tau)) = \frac{m}{24}\left(m\, E_2(m\tau) - E_2(\tau)\right)$$

(8.13)
$$= \frac{m(m-1)}{24} + \sum_{k=1}^{\infty} m\sigma(k)(mq^{mk} - q^k),$$

$\sigma(k)$ is the sum of the divisors of k, and E_2 is the holomorphic quasi-modular Eisenstein series of weight 2. When n is prime $\Gamma_0(n)$ has two cusps and a

complement in $M_2(\Gamma_0(n))$ to $S_2(\Gamma_0(n))$ is 1D and spanned by Λ_n. Since the genus of $X_0(n)$ is zero for $n \in \{2, 3, 5, 7\}$, the space $M_2(\Gamma_0(n))$ is thus 1D, and spanned by Λ_n. Now, with Λ_n having non-vanishing constant term, the statement $G_{n|1}^{(\kappa=0)}(\tau) = \mathcal{O}(q)$ implies that $G_{n|1}^{(\kappa=0)}(\tau)$ vanishes identically. If $n = 11$ then $M_2(\Gamma_0(n))$ is 2D, spanned by Λ_n and the cusp form $\eta_g(\tau) = \eta(\tau)^2\eta(11\tau)^2$, and since Λ_n and η_g have non-vanishing coefficients of q^0 and q^1, respectively, we conclude that $G_{n|1}^{(\kappa=0)}(\tau) = \mathcal{O}(q^2)$ implies the vanishing of $G_{n|1}^{(\kappa=0)}$ in this case also. When $n = 23$, the space $M_2(\Gamma_0(n))$ is spanned by Λ_n, a cusp form $\phi_{23,1} = \eta_g(\tau)^2 = \eta(\tau)^2\eta(23\tau)^2$ and a further cusp form

(8.14)
$$\phi_{23,2}(\tau) = \frac{\eta(\tau)^3\eta(23\tau)^3}{\eta(2\tau)\eta(46\tau)} + 4\eta(\tau)\eta(2\tau)\eta(23\tau)\eta(46\tau) + 4\eta(2\tau)^2\eta(46\tau)^2.$$

Inspecting the Fourier expansions of Λ_n, $\phi_{23,1}$ and $\phi_{23,2}$ we see that there is no non-zero linear combination that is $\mathcal{O}(q^3)$. We conclude that $G_{n|1}^{(\kappa=0)}(\tau)$ vanishes identically for all the values of n in question. So $G_{n|1}$ also vanishes and the desired result follows. □

Lemma 8.10. *Suppose that $g \in M_{24}$ has order $n = n_g$ a product of two distinct primes and $h = h_g = 1$. Then $H_g = -2R_{n|h}$.*

Proof. According to Table 1 the values of n in question are 6, 14 and 15. From the discussion before (8.13) we see that the complement to $S_2(\Gamma_0(n))$ in $M_2(\Gamma_0(n))$ is 3D and spanned by the weight two modular forms $\Lambda_{e_1}, \Lambda_{e_2}$ and Λ_{e_3}, where $e_1 < e_2 < e_3 = n$ are the three (exact) divisors of n that are greater than 1. Therefore, we have

$$G_{n|1} = \alpha_1\Lambda_{e_1} + \alpha_2\Lambda_{e_2} + \alpha_3\Lambda_{e_3} + \phi$$

with $\phi \in S_2(\Gamma_0(n))$. In other words, ϕ vanishes when $n = 6$ and is a multiple of η_g when $n = 14, 15$, in accordance with the fact that the genera of $X_0(6)$, $X_0(14)$, $X_0(15)$ are 0, 1, 1, respectively. For each e_i there is a corresponding cusp of $\Gamma_0(n)$ with width e_i, which is represented by $\kappa_i = \frac{e_i}{n}$. These are the only cusps of $\Gamma_0(n)$ apart from the infinite cusp, as listed in table 3. As discussed before (cf. (8.12)), the corresponding function $G_{n|1}^{(\kappa_i)}(\tau)$ has the asymptotic behaviour

$$G_{n|1}^{(\kappa_i)}(\tau) = \mathcal{O}\left(q^{\Delta + \frac{e_i}{8}}\right).$$

Table 3: In this table, we list the cusps other than the one at $i\infty$, which has width 1, of the groups $\Gamma_0(n)$ for $n = n_g$ in table 1. Note that, in these cases we have an extra (non-infinite) cusp, represented by $\frac{1}{q}$, for every divisor q of n, and its width is given by the smallest positive integer v such that $n | qv^2$. Moreover, the corresponding multiplier $\rho(\sigma T^v \sigma^{-1}) = e(\frac{m}{h})$ is given by $m = \frac{q^2 v}{n}(1 + qv) \bmod h$.

n	1	2	3	4	5	6	7	8	10	11	12	14	15	21	23
Cusps	/	0	0	$0, \frac{1}{2}$	0	$0, \frac{1}{2}, \frac{1}{3}$	0	$0, \frac{1}{2}, \frac{1}{4}$	$0, \frac{1}{2}, \frac{1}{5}$	0	$0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}$	$0, \frac{1}{2}, \frac{1}{7}$	$0, \frac{1}{3}, \frac{1}{5}$	$0, \frac{1}{3}, \frac{1}{7}$	0
Widths	/	2	3	4,1	5	6,3,2	7	8,2,1	10,5,2	11	12,3,4,3,1	14,7,2	15,5,3	21,7,3	23

Consider a scaling element which is an Atkin–Lehner involution w_{e_i} with $\kappa_i = w_{e_i}\infty$ (note that it is different from the Fricke involution when $e_i \neq n$). With help from the identity

$$(8.15) \qquad \frac{1}{e}\Lambda_e|_{1,2}w_f = \frac{1}{e*f}\Lambda_{e*f} - \frac{1}{f}\Lambda_f$$

where $e * f = ef/(e,f)^2$, we obtain three equations on α_i from the vanishing of the constant terms in $G_{n|1}^{(\kappa_i)}(\tau)$. Solving the resulting linear system we quickly deduce that $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and hence $G_{n|1}(\tau)$ is given by the cusp form ϕ of $\Gamma_0(n)$. Since $X_0(6)$ has genus zero we conclude that $G_{n|1}(\tau) = 0$ for $n = 6$. For $n = 14, 15$, to prove that $G_{n|1}(\tau) = \phi = 0$ let us focus on the 0 cusp with width n. Take the scaling element to be the Fricke involution w_n. From the fact that the cusp form η_g is an eigenfunction (with eigenvalue -1) of w_n we get $G_{n|1}^{(\kappa)}(\tau) = C\eta_g = -Cq + \mathcal{O}(q^2)$ for some $C \in \mathbb{C}$. But from (8.12) we see that $G_{n|1}^{(\kappa)}(\tau) = \mathcal{O}(q^2)$ for $v = n = 14, 15$. Hence, we conclude $G_{n|1} = C = 0$. $\qquad \square$

Lemma 8.11. *Suppose that $g \in M_{24}$ is such that $h = h_g = 1$ and $n = n_g$ is 4 or 8. Then $H_g = -2R_{n|h}$.*

Proof. If $n = 4$ then $\dim M_2(\Gamma_0(n)) = 2$ and $M_2(\Gamma_0(n))$ is spanned by the modular forms Λ_2 and Λ_4 (cf. (8.13)). If $n = 8$ then $\dim M_2(\Gamma_0(n)) = 3$ and $M_2(\Gamma_0(n))$ is spanned by Λ_2, Λ_4 and Λ_8. In both cases $\dim S_2(\Gamma_0(n)) = 0$. Consider the case that $n = 4$, where we have $G_{n|1} = \alpha\Lambda_2 + \beta\Lambda_4$. There are two non-infinite cusps: one represented by 0 with width 4 and the other represented by 1/2 with width 1. Taking as their scaling elements w_4 and $T^{1/2}w_4$ respectively, we get two equations on the coefficients α, β from the requirement that both $G_{n|1}^{(\kappa=0)}$ and $G_{n|1}^{(\kappa=1/2)}$ are $\mathcal{O}(q)$, which force both α

and β and hence $G_{n|1}$ to vanish. In arriving at these equations, we have used the identity (8.15) and

$$E_2(\tau + 1/2) = -E_2(\tau) + 6E_2(2\tau) - 4E_2(4\tau).$$

The argument for $n = 8$ is very similar. We have $F = \alpha \Lambda_2 + \beta \Lambda_4 + \gamma \Lambda_8$ for some $\alpha, \beta, \gamma \in \mathbb{C}$. We may take 0, $1/2$ and $1/4$ as representatives for the three non-infinite cusps of $\Gamma_0(8)$. We may take w_8, $T^{1/2} w_8$ and $w_8 T^{1/2} w_8$, respectively, as scaling elements for these cusp representatives. Using the identities given above we compute expressions for the constant term at each non-infinite cusp as linear equations in α, β and γ. From the vanishing of each constant term we deduce that $\alpha = \beta = \gamma = 0$ and hence $G_{n|1} = 0$ also. Then the required identity $H_g = -2R_{n|h}$ follows as before. $\qquad \square$

The above Lemmas 8.8 to 8.11, when taken together, show that $H_g = -2R_{n_g|1}$ for $g \in M_{24}$ with $h_g = 1$ Now we will continue to show the identity for the remaining M_{24} classes with $n_g = n$, $h_g = h > 1$.

Lemma 8.12. *Let g be an element of M_{24} that has $h_g > 1$ and $n_g \neq 21$. Then $H_g = -2R_{n|h}$.*

Proof. To prove the identity $-2R_{n|h} = H_g$ for the remaining classes $[g]$ with $h_g > 1$, we consider the following function:

$$K_{n|h} = \frac{R_{n|h}}{H_g}.$$

Since both $R_{n|h}$ and $H_g = -\tilde{T}_g/\eta^3$ are known to be weight $1/2$ modular forms which transform with a multiplier $\rho_{n|h}$ on $\Gamma_0(n)$, we conclude that $K_{n|h}$ is a modular function on $\Gamma_0(n)$. As in the previous subsection, such a modular function has to be either constant or have a pole. For $g \in M_{24}$ with $h_g > 1$ and $n_g \neq 21$, the McKay–Thompson series H_g are all given by certain η-quotients (cf. table 2). Since η does not have a zero in $\mathbb{H}$, we only have to check that $K_{n|h}$ has no pole at any cusp of $\Gamma_0(n)$. At the infinite cusp we have $R_{n|h} = q^{-1/8} + \mathcal{O}(q^{7/8})$ and $H_g = -2q^{-1/8} + \mathcal{O}(q^{7/8})$ and hence $K_{n|h}$ has no pole at the infinite cusp. To see that it is also bounded near a non-infinite cusp $\kappa = \sigma\infty = \tilde{\sigma}\infty$, $\sigma \in \mathrm{SL}_2(\mathbb{Z})$, with a scaling element $\tilde{\sigma}$, we consider the function

$$G_{n|h}^{(\kappa)} = G_{n|h}^{(\kappa)}(\sigma\,(v(\tau + \beta))) = \frac{R_{n|h}^{(\kappa)}(\tau)}{\tilde{T}_g/\eta^3|_{\epsilon^{-3},1/2}\,\sigma(v(\tau + \beta))},$$

where $R_{n|h}^{(\kappa)}(\tau)$ has been shown in Appendix E to have an expansion

$$(8.16) \qquad R_{n|h}^{(\kappa)}(\tau) = R_{n|h}^{(\kappa)}|_{\epsilon^3,1/2}\sigma\left(v(\tau+\beta)\right) = \sum_{k=1}^{\infty} c_{\Gamma,\rho}^{(\kappa)}(k-\nu)e(\tau(k-\nu)),$$

where $\nu = \{\{\frac{m}{h} + \frac{v}{8}\}\}$ as explained in (7.5). With direct calculation, one can check that also

$$\frac{\tilde{T}_g}{\eta^3}\Big|_{\epsilon^{-3},1/2}\sigma(v(\tau+\beta)) = a_g\, q^{1-\nu} + \mathcal{O}(q^{2-\nu})$$

at every non-infinite cusp κ of $\Gamma_0(n)$ with some $a_g \in \mathbb{C}$, $a_g \neq 0$. This shows that $K_{n|h}$ is a modular function with no pole and hence constant, which is $-1/2$ from its value at the infinite cusp. $\qquad\square$

Lemma 8.13. *Suppose that $g \in M_{24}$ is such that $n = n_g = 21$ and $h = h_g = 3$. Then $H_g = -2R_{n|h}$.*

Proof. Set $\quad \phi_{21,1}(\tau) = \eta(7\tau)^3/\eta(3\tau)\eta(21\tau) \quad$ and $\quad \phi_{21,2}(\tau) = \eta(\tau)^3/\eta(3\tau)^2$. Then both $\phi_{21,1}$ and $\phi_{21,2}$ are (meromorphic) modular forms of weight $1/2$ for $\Gamma_0(21)$ with multiplier system $\rho\epsilon^{-3}$ where $\rho = \rho_{21|3}$, so that

$$G = \frac{AR_{n|h} + B\phi_{21,1}}{\phi_{21,2}}, \quad A, B \in \mathbb{C}$$

is a modular function for $\Gamma_0(21)$ whose only poles are at the non-infinite cusps of $\Gamma_0(21)$. Since $R_{n|h}, \phi_{21,2}, \phi_{21,2}$ all have the expansion $q^{-1/8} + \mathcal{O}(q^{7/8})$ near the infinite cusp, we see that G has no pole at the infinite cusp. For $e_{1,2,3}$ with $e_1 < e_2 < e_3 = 21$ the exact divisors of 21 that are greater than 1 let w_{e_i} be an element of the Atkin–Lehner involution of $\Gamma_0(21)$ associated to e_i. Then the elements w_{e_i} furnish scaling elements at the respective representatives $\kappa_i = w_{e_i}\infty$ for the three non-infinite cusps of $\Gamma_0(21)$, represented by $\frac{e_i}{n}$ having width e_i. Using the fact that the slash operator $|_{1/2,\epsilon}w_{e_i}$ maps $\eta(e_j\tau)$ to $\eta(e_i * e_j\tau)$ (up to a non-vanishing scale factor), we see that the function $G^{(\kappa_i)} = G(w_{e_i}\tau)$ remains bounded as $\tau \to i\infty$ except possibly in the case that $e_3 = 21$, in which case we have

$$G^{(\kappa=0)}(\tau) = G(w_{21}\tau) = \left(A'R_{n|h}^{(\kappa=0)}(\tau) + B'\frac{\eta(3\tau)^3}{\eta(7\tau)\eta(\tau)}\right)\frac{\eta(7\tau)^2}{\eta(21\tau)^3}, \; A', B' \in \mathbb{C}.$$

From (8.16), we see that $G^{(\kappa=0)}(\tau)$ has a Fourier expansion of the form $Cq^{-2} + \mathcal{O}(q^{-1})$. Setting $D = 2x$ where x is the image of $w_{21}\infty = 0$ under the

natural map $\varphi_{21} : \mathbb{H}^* \to X = X_0(21)$ we see that f must belong to the space $K_X(D)$ of meromorphic functions on $X = X_0(21)$ having a pole of order at most 2 at x and no other poles. The genus of $X_0(21)$ is 1 so according to the Riemann–Roch theorem (cf. (8.10)) we have $\dim K_X(D) = \dim \Omega_X(D) + 2$ where $\Omega_X(D)$ denotes the space of holomorphic differentials on X vanishing to order at least 2 at x. The full space Ω_X of holomorphic differentials on X is in correspondence with $S_2(\Gamma_0(21))$, and this 1D space is in turn spanned by the L-series for the elliptic curve $y^2 + xy = x^3 + x$. This cusp form has a Fourier expansion of the form $q + \mathcal{O}(q^2)$ and is an eigenform for w_{21} so we conclude that it does not belong to $\Omega_X(D)$, and thus $\dim \Omega_X(D) = 0$ and $\dim K_X(D) = 2$. Now $K_X(D)$ includes constant functions on X and it also includes the non-constant function $\phi_{21,1}/\phi_{21,2}$, so it is spanned by these functions, and we may conclude that $R_{n|h} = C_1 \phi_{21,1} + C_2 \phi_{21,2}$ for some $C_1, C_2 \in \mathbb{C}$. By comparison of polar terms in the Fourier expansions of $R_{n|h}$, $\phi_{21,1}$ and $\phi_{21,2}$ we have $C_1 + C_2 = 1$. Now consider the expansion of $R_{n|h}$ at the cusp represented by $w_7 \infty$ (which is also represented by 1/3). We have $(\phi_{21,1}|w_7)(\tau) = D_1 q^{-21/24} + \mathcal{O}(q^{3/24})$ and $(\phi_{21,2}|w_7)(\tau) = D_2 q^{-21/24} + \mathcal{O}(q^{3/24})$ and the fact that $R_{n|h}$ has no poles away from the infinite cusp implies that $C_1 + 7C_2 = 0$. We conclude that $6R_{n|h} = 7\phi_{21,1} - \phi_{21,2}$. From table 2, we have the explicit expression $3H_g = \phi_{21,2} - 7\phi_{21,1}$. This proves the lemma. $\qquad \square$

9. Spectral theory

In this section, we would like to demonstrate the convergence of the Selberg–Kloosterman zeta function $Z_{0,k}(s, \rho \epsilon^{\epsilon-3})$, defined in (5.6) and (5.7), of $\Gamma = \Gamma_0(n)$ for n a positive integer. As before, we take $\psi = \epsilon^{\epsilon-3} \rho_{n|h}$ for some h dividing both 24 and n (cf. (2.5)). Combined with the discussions in Sections 5 and 6, Theorem 9.1 completes the proof of the convergence of the Rademacher sum (4.8). Let w be a positive real number and let ψ be a multiplier system of weight w for Γ. We say a function $g(z)$ on $\mathbb{H}$ is *automorphic for Γ of weight w with respect to ψ* if

$$\psi(\gamma)g(\gamma z) \exp(iw \arg(\mathrm{jac}(\gamma, z))/2) = g(z)$$

for all $\gamma \in \Gamma$, where $\arg(\cdot)$ is defined so that $-\pi < \arg(z) \leq \pi$ for $z \neq 0$, and write $\mathcal{H}_{\psi,w}$ for the Hilbert space consisting of those automorphic functions which also satisfy the growth condition

$$\iint_{\Gamma \backslash \mathbb{H}} |g(\tau)|^2 \frac{\mathrm{d}x \mathrm{d}y}{y^2} < \infty,$$

where $x = \Re(\tau)$ and $y = \Im(\tau)$ for $z \in \mathbb{H}$. We consider the operator Δ_w on $\mathcal{H}_{\psi,w}$ given by

$$\Delta_w = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - \mathrm{i}wy\frac{\partial}{\partial x}.$$

According to [37] if ψ is non-trivial on every parabolic subgroup of Γ then Δ_w has a discrete spectrum of eigenfunctions in $\mathcal{H}_{\psi,w}$ and there exists a complete orthonormal system $\{u_j(z)\}$ of eigenfunctions satisfying

$$\Delta_w u_j(z) + \lambda_j u_j(z) = 0,$$

for a set $\{\lambda_j\}$ of eigenvalues which are all real. (Roelcke proves in [38] that Δ_w has a unique self-adjoint extension to $\mathcal{H}_{\psi,w}$.) Further, the λ_j are all positive except possibly for a finite number which are of the form

(9.1) $$(w/2 - l)(1 + l - w/2),$$

where l is a non-negative integer less than $w/2$. If there is an eigenvalue of the form (9.1) then, negative or otherwise, it's multiplicity is equal to the dimension of the space $S_{w-2l}(\Gamma, \psi)$ of (holomorphic) cusp forms for Γ of weight $w - 2l$ with respect to ψ. By the trivial estimate $|S(m, k, c, \psi)| < c$ the series (5.7) defining the Selberg–Kloosterman zeta function $Z_{m,k}(s, \psi)$ converges absolutely and locally uniformly for $\Re(s) > 1$ and thus defines a holomorphic function in this domain. Selberg demonstrates in [37] that this function admits an analytic continuation to a function meromorphic in the entire s-plane that is holomorphic for $\Re(s) > 1/2$ except possibly for a finite number of simple poles lying on the real segment $1/2 < s < 1$. Poles on this segment can only occur at points of the form $1/2 + \sqrt{1/4 - \lambda_j}$ where $0 < \lambda_j < 1/4$ and λ_j is an eigenvalue of Δ_w, and such a pole does occur if λ_j is not of the form (9.1) and the coefficients of $e((m - \alpha)x)$ and $e((n - \alpha)x)$ in the Fourier expansion (with respect to x) of the corresponding eigenfunction $u_j(z)$ are both non-zero. If λ_j is of the form (9.1) with $l = \lfloor w/2 \rfloor$ then there is a pole at $s = 1 + \lfloor w/2 \rfloor - k/2$ just in the case that there is a cusp form $f(\tau) \in S_{w-2\lfloor w/2 \rfloor}(\Gamma, \psi)$ for which the coefficients of $q^{m-\alpha}$ and $q^{n-\alpha}$ (in its Fourier expansion with respect to τ) are both non-zero. For the case of relevance for us we have $w = 1/2$ and so all the λ_j will be positive. Roelcke proves in [38] that the minimal value of a λ_j is $3/16$ in case $w = 1/2$ (see [39] for a nice exposition) so we conclude that there are no poles for $Z_{0,k}(s, \rho\epsilon^{\epsilon-3})$ in $\Re(s) > 3/4$. The only possible value for l in (9.1) is $l = 0$, so that the occurrence of a pole at $s = 1 + \lfloor w/2 \rfloor - w/2 = 3/4$ depends upon the existence of a cusp form $f \in S_w(\Gamma, \psi)$ for which the coefficients

of $q^{0-\alpha}$ and $q^{k-\alpha}$ are both non-zero. We have $-\alpha = -1/8$ so there is no such cusp form and we conclude that $Z_{0,k}(s, \rho\epsilon^{\epsilon-3})$ is in fact analytic in $\Re(s) > 3/4 - \delta$ for some $\delta > 0$ and has finitely many simple poles lying on the line segment $1/2 < s < 3/4$. It remains to show that the value of the analytically continued function $Z_{0,k}(s, \rho\epsilon^{\epsilon-3})$ at $s = 3/4$ is indeed the sum

$$\sum_{c>0} \frac{S(0, k, c, \rho\epsilon^{\epsilon-3})}{c^{3/2}}$$

that we expect it is. To establish this we adopt the approach presented in Section 2 of [40][2] which is an adaptation of a technique commonly used in proving the prime number theorem (cf. [42]). (For a very crisp application of what is essentially the same argument, see the proof of Theorem 2 in [33].) Fix a group Γ, a multiplier $\psi = \rho\epsilon^{\epsilon-3}$, and an integer k. We will write $Z(s) = Z_{0,k}(s, \rho\epsilon^{\epsilon-3})$ to ease notation. Define

$$\Sigma(x) = \sum_{0<c<x} \frac{S(0, k, c, \rho\epsilon^{\epsilon-3})}{c^{3/2}}$$

for $x > 0$. We require to show that $\lim_{x\to\infty} \Sigma(x) = Z(3/4)$. Observe that for $a > 0$ and $x \neq c$ the integral

$$\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \left(\frac{x}{c}\right)^t \frac{\mathrm{d}t}{t}$$

is 1 or 0 according as $x > c$ or $x < c$. Consequently, we may rewrite $\Sigma(x)$ as

$$\Sigma(x) = \sum_{c>0} \frac{S(0, k, c, \rho\epsilon^{\epsilon-3})}{c^{3/2}} \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \left(\frac{x}{c}\right)^t \frac{\mathrm{d}t}{t},$$

for non-integral values of x. Choose $\varepsilon > 0$ and set $a = 1/2 + \varepsilon$ so that $Z(t/2 + 3/4)$ converges absolutely for $\Re(t) = a$. For $T > 0$ define $\Sigma_T(x)$ by setting

$$\Sigma_T(x) = \frac{1}{2\pi i} \int_{a-iT}^{a+iT} Z(t/2 + 3/4)x^t \frac{\mathrm{d}t}{t}.$$

Then for the difference $\Sigma(x) - \Sigma_T(x)$ we have the upper bound

$$(9.2) \qquad |\Sigma(x) - \Sigma_T(x)| < x^a \sum_{c>0} c^{-1-\varepsilon} \min\left(1, \frac{1}{T|\log(x/c)|}\right)$$

---

[2]If you consult this reference please see also Section II of [41] for some important corrections.

thanks to the following estimate, which may be found in Section 17 of [42].

$$\frac{1}{2\pi} \left| \int_{a-i\infty}^{a+i\infty} \left(\frac{x}{c}\right)^t \frac{\mathrm{d}t}{t} - \int_{a-iT}^{a+iT} \left(\frac{x}{c}\right)^t \frac{\mathrm{d}t}{t} \right| < \left(\frac{x}{c}\right)^a \min\left(1, \frac{1}{T|\log(x/c)|}\right)$$

Following exactly the argument of the last paragraph of p. 171 of [40] we can replace the summation in (9.2) with the more explicit bound

$$(9.3) \qquad |\Sigma(x) - \Sigma_T(x)| < K\frac{x^{1/2}}{T}(x^\varepsilon + \log x),$$

where K is some constant (that does not depend on our choice of k). Consider now the integral

$$\frac{1}{2\pi\mathrm{i}} \int_C Z(t/2 + 3/4)\, x^t \, \frac{\mathrm{d}t}{t},$$

where $C = C_0 + C_1 + C_2 + C_3$ is the positively oriented rectangle with corners $(\pm 1/2) + \varepsilon \pm iT$. For concreteness let us say that C_0 is the right most vertical component and C_1 is the upper most horizontal component. Then the contribution of C_0 is exactly $\Sigma_T(x)$. We denote the contribution of C_i by I_i for $i \neq 0$. By the residue theorem we have

$$(9.4) \qquad \Sigma_T(x) + I_1 + I_2 + I_3 = Z(3/4) + \sum_{j \in J} \varrho_j x^{2s_j - 3/2},$$

where $\{s_j\}_{j \in J}$ is an enumeration of the (finite) set of poles of $Z(s)$ in the interval $1/2 + \varepsilon/2 < s < 3/4$ (we may choose $\varepsilon > 0$ so that $s = 1/2 + \varepsilon/2$ is not a pole) and ϱ_j is the residue of $Z(s)$ at $s = s_j$ divided by $2s_j - 3/2$. We wish to determine what happens when $x \to \infty$ in (9.4). Observe, for a start, that the sum over j vanishes in this limit since $2s_j < 3/2$ for all $j \in J$. We claim that the integrals I_i also vanish in this limit. To estimate the I_i we utilize the growth condition

$$|Z(s)| = \mathcal{O}\left(\frac{k|s|^{1/2}}{\Re(s) - 1/2}\right)$$

as $|\Im(s)| \to \infty$, valid for $\Re(s) > 1/2$. Such a result was first established by Goldfeld–Sarnak in [33]. We have applied the generalization found in [43] due to Pribitkin. Since $Z(s)$ converges absolutely for $\Re(s) > 1$ we have that $|Z(t/2 + 3/4)| = \mathcal{O}(1)$, for a constant independent of k, as $|\Im(t)| \to \infty$ when $\Re(t) = 1/2 + \varepsilon$, and thus an application of the Phragmén–Lindelöf theorem

(cf. [44], Thm. 14) yields

$$(9.5) \qquad |Z(t/2 + 3/4)| < kK_\varepsilon \, |\Im(t)|^{1/4 - \Re(t)/2 + \varepsilon/2},$$

for all t such that $-1/2 + \varepsilon \leq \Re(t) \leq 1/2 + \varepsilon$ and $|\Im(t)|$ is sufficiently large, for some constant K_ε depending only on Γ, $\psi = \rho\epsilon^{-3}$ and ε. Applying (9.5) to the integrals I_i we obtain that $I_2 = \mathcal{O}(kT^{1/2}x^{-1/2+\varepsilon})$, and both I_1 and I_3 are $\mathcal{O}(kT^{-1}x^{1/2+\varepsilon})$ so long as $T < x^2$. Setting $T = x^{2/3}$, letting $x \to \infty$ in (9.4) and applying (9.3) gives us the desired convergence: $\lim_{x\to\infty} \Sigma(x) = Z(3/4)$, and also the estimate $Z(3/4) = \mathcal{O}(k)$ as $k \to \infty$. In particular, we have the following theorem:

Theorem 9.1. *Let $\Gamma = \Gamma_0(n)$ for some positive integer n, let h be a divisor of $\gcd(n, 24)$ and set $\rho = \rho_{n|h}$. Then for any positive integer k the Selberg–Kloosterman zeta function $Z_{0,k}(s, \rho\epsilon^{-3})$ converges at $s = 3/4$ and we have $Z_{0,k}(3/4, \rho\epsilon^{-3}) = \mathcal{O}(k)$ where the implied constant depends only on n and h.*

10. Conclusion and discussion

To summarize, motivated by the AdS/CFT correspondence in physics, we propose to replace the genus zero property by the more general property of Rademacher summability as the organizing principle of the modular properties of moonshine phenomena. In monstrous moonshine, where the McKay–Thompson series are modular functions, the latter coincides with the former and is hence automatically satisfied, as shown in [10]. The two conditions decouple in the case of the recently conjectured relation between mock modular forms and M_{24}: on the one hand, it is neither necessary nor sufficient that a group Γ have genus zero for it to arise as Γ_g for some g in M_{24}, and on the other hand, we verify the Rademacher summability property in Theorem 4.1. In view of the above, the Rademacher summability property may replace the genus zero property, and applies to both monstrous moonshine and M_{24}. This also suggests that AdS/CFT considerations will lead to further elucidation of the moonshine phenomena, and we expect the powerful Rademacher machinery to prove useful in further study of the connection between modular objects and finite groups in general. This result raises many interesting physical questions. In particular, what do we know about the gravity duals, if they exist, of 2D CFTs with sporadic symmetries? Let us first concentrate on the sigma models with target spaces involving $K3$ surfaces, as they are relevant for the structures studied in the present

paper. Recall that one of the best known examples of the AdS/CFT correspondence conjectures that the 2D CFT describing the Higgs branch of the system of Q_1 D1-branes bound to Q_5 D5-branes wrapping a $K3$ surface is dual to the type IIB string theory in the background $AdS_3 \times S^3 \times K3(Q)$ [12]. Here the radius of curvature of AdS_3 and S^3 is given by $(Q_1 Q_5)^{1/4}$ in the 6D unit, and $K3(Q)$ is the "attractor K3" whose moduli are partially fixed by the requirement of minimizing the BPS mass of the given charges, or equivalently, by the so-called attractor mechanism of supergravity. Note that not all $K3$ surfaces can be such an attractor $K3$ for a given set of charges. On the CFT side, the Higgs branch is described by a 2d $\mathcal{N} = (4, 4)$ SCFT with central charge $c = 6k$, $k = Q_1 Q_5 + 1$, which is a deformation of the sigma model on the symmetric product $S^k K3$, as can be most easily understood by going to the D0–D4 duality frame [45]. This AdS/CFT dictionary makes it clear that a "semi-classical-like" AdS gravity description is only guaranteed to exist when we take the central charge $c = 6k$, or equivalently the AdS radius, to be very large. For this reason, it is in fact surprising that the Rademacher machinery works so well when applied to the partition functions for the $K3$ sigma model with $k = 1$, as demonstrated in the present paper. Two comments on this issue are in order here. First, analogous Rademacher formulas also exist for the symmetric product sigma model with higher k and we plan to report on this in more detail in the future [28]. In other words, the Rademacher machinery also works when we have very strong reasons to expect them to work. Second, recently there has been important progress in developing the localization techniques for computing gravity path integrals (see, for example, [46]), and we have seen that in some cases the highly quantum path integral actually localizes into the same form as if coming from a weakly coupled, semi-classical gravitational theory. For instance, very recently the Rademacher sum for the modular form $1/\eta(\tau)^{24}$, which is the partition function of a CFT with central charge as small as $c = 24$, has been reproduced from a localized gravity path integral [47]. This is very surprising since the corresponding black holes have zero macroscopic entropy (i.e., the Bekenstein–Hawking entropy grows only linearly with the charges) and hence a priori we have no reason to expect the gravity path integral to take this "semi-classical" form. These results show that the Rademacher formulas are not really the monopoly of a large radius, semi-classical gravity description, although the former is most easily motivated in the semi-classical limit as we reviewed in Section 1, and suggests that it is not unlikely that a localization calculation could similarly explain the Rademacher sum formulas derived in the present paper from a gravity point of view. In the case of monstrous

moonshine, on the other hand, the question of whether a semi-classical theory of gravity dual to the moonshine module CFTs, or possibly their higher central charge cousins, is an interesting open puzzle [48]. We end with a few more comments concerning the Rademacher formulas studied in this paper. First, it is remarkable that the Rademacher sums with such a simple polar term $-2q^{-1/8}$ compute not only the M_{24}-modules underlying the massive states in the $K3$ CFT but also the twisted Euler characteristics of $K3$ surfaces. The latter statement comes from the fact that the Rademacher machinery automatically computes the shadows of the resulting mock modular forms, which are in this case given by the twisted $K3$ Euler characteristics multiplied by the modular form $\eta(\tau)^3$. Finally, we would like to mention that none of the twisted elliptic genera which arise from the extra, non-M_{24} symmetries discussed in [49] lead to (mock) modular forms admitting analogous Rademacher sum expressions. Another related but logically independent observation is the following. If we were to twist the partition function counting the $1/4$-BPS index of type II theory compactified on $K3 \times T^2$ with one of these non-M_{24} symmetries, the resulting candidate 4d twisted partition function would have a system of poles and zeros that is more intricate and complicated than the ones coming from twisting with M_{24} symmetries. This adds to the distinction between the M_{24} symmetries and the extra non-M_{24} discrete symmetries, which are present in, for instance, the Gepner models, whose role in moonshine is still obscure.

Acknowledgments

Appendix A. Dedekind Eta function

The *Dedekind eta function*, denoted $\eta(\tau)$, is a holomorphic function on the upper half-plane defined by the infinite product

$$\eta(\tau) = q^{1/24} \prod_{n \geq 1} (1 - q^n),$$

where $q = e(\tau) = e^{2\pi i \tau}$. It is a modular form of weight $1/2$ for the modular group $\mathrm{SL}_2(\mathbb{Z})$ with multiplier $\epsilon : \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{C}^*$, which means that

$$\epsilon(\gamma) \eta(\gamma\tau) \, \mathrm{jac}(\gamma, \tau)^{1/4} = \eta(\tau)$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{SL}_2(\mathbb{Z})$, where $\mathrm{jac}(\gamma, \tau) = (c\tau + d)^{-2}$. The *multiplier system* ϵ may be described explicitly as

$$(\text{A.1}) \quad \epsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} e(-b/24), & c = 0, \, d = 1, \\ e(-(a+d)/24c + s(d,c)/2 + 1/8), & c > 0, \end{cases}$$

where $s(d,c) = \sum_{m=1}^{c-1} (d/c)((md/c))$ and $((x))$ is 0 for $x \in \mathbb{Z}$ and $x - \lfloor x \rfloor - 1/2$ otherwise. We can deduce the values $\epsilon(a, b, c, d)$ for $c < 0$, or for $c = 0$ and $d = -1$, by observing that $\epsilon(-\gamma) = \epsilon(\gamma) e(1/4)$ for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Let T denote the element of $\mathrm{SL}_2(\mathbb{Z})$ such that $\mathrm{tr}(T) = 2$ and $T\tau = \tau + 1$ for $\tau \in \mathbb{H}$. Observe that

$$\epsilon(T^m \gamma) = \epsilon(\gamma T^m) = e(-m/24)\epsilon(\gamma)$$

for $m \in \mathbb{Z}$.

Appendix B. Bessel function

The *Bessel function of the first kind*, denoted $J_\alpha(x)$, may be defined by the series expression

$$(\text{B.1}) \qquad J_\alpha(x) = \sum_{m \geq 0} (-1)^m \frac{(x/2)^{2m+\alpha}}{\Gamma(m + 1 + \alpha)m!}.$$

The function $J_\alpha(x)$ is asymptotic to $e^x \cos(x - \alpha\pi/4 - \pi/4)/\sqrt{2\pi x}$ for x large, and in the special case that $\alpha = 1/2$ this fact is borne out by the

identity

(B.2)
$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin(x).$$

The *modified Bessel function of the first kind*, denoted $I_\alpha(x)$, may be defined by the series expression

(B.3)
$$I_\alpha(x) = \sum_{m \geq 0} \frac{(x/2)^{2m+\alpha}}{\Gamma(m+1+\alpha)m!}.$$

The function $I_\alpha(x)$ is asymptotic to $e^x/\sqrt{2\pi x}$ for x large, for any α. In the special case that $\alpha = 1/2$ this fact is borne out by the identity

(B.4)
$$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sinh(x).$$

As x approaches 0 the functions $J_{1/2}(x)$ and $I_{1/2}(x)$ both tend towards $(x/2)^\alpha/\Gamma(1+\alpha)$. Since $\Gamma(3/2) = \sqrt{\pi}/2$, we have

(B.5)
$$J_{1/2}(x) \approx \sqrt{\frac{2x}{\pi}}, \quad I_{1/2}(x) \approx \sqrt{\frac{2x}{\pi}}$$

as $x \to 0$ in the special case that $\alpha = 1/2$.

Appendix C. Lipschitz summation

The *Lipschitz summation formula* is the identity

(C.1)
$$\frac{(-2\pi\mathrm{i})^s}{\Gamma(s)} \sum_{k=1}^{\infty} (k-\alpha)^{s-1} e((k-\alpha)\tau) = \sum_{\ell \in \mathbb{Z}} e(\alpha\ell)(\tau+\ell)^{-s},$$

valid for $\Re(s) > 1$ and $0 \leq \alpha < 1$, where $e(x) = e^{2\pi\mathrm{i}x}$. A nice proof of this using Poisson summation appears in [50]. Observe that both sides of (C.1) converge absolutely and uniformly in τ on compact subsets of $\mathbb{H}$. For applications to Rademacher sums of weight less than 1 we require an extension of (C.1) to $s = 1$. Absolute convergence on the right-hand side breaks down at this point but we can get by with the following useful analogue.

Lemma Appendix C.1. *For $0 < \alpha < 1$, we have*

$$(C.2) \qquad \sum_{k=1}^{\infty} e((k - \alpha)\tau) = \sum_{-K < \ell < K} e(\alpha\ell)(-2\pi i)^{-1}(\tau + \ell)^{-1} + E_K(\tau),$$

where $E_K(\tau) = \mathcal{O}(1/K^2)$, locally uniformly for $\tau \in \mathbb{H}$.

Proof. Our proof follows that of Lemma 4.1 in [32]. For $\tau \in \mathbb{H}$ define a function $f(z)$ by setting

$$(C.3) \qquad\qquad f(z) = \frac{1}{2\pi i z} \frac{e(\alpha(z - \tau))}{(e(z - \tau) - 1)}.$$

Then f has poles at $\tau + n$ for each $n \in \mathbb{Z}$ and also has a pole at $z = 0$. The residue at $z = 0$ is

$$\frac{1}{2\pi i} \frac{e(-\alpha\tau)}{e(-\tau) - 1} = \frac{1}{2\pi i} \sum_{k=1}^{\infty} e((k - \alpha)\tau),$$

which is $(2\pi i)^{-1}$ times the left-hand side of (C.2), and the residue at $z = \tau + n$ is $e(\alpha n)(2\pi i)^{-2}(\tau + n)^{-1}$. Suppose $\tau = \sigma + it$. Let K be a positive integer and let $\mathcal{C}$ be the positively oriented boundary of a rectangle with corners $\sigma \pm (K + 1/2) \pm iL$ where $L > \Im(\tau) = t$ and $K + 1/2 > \Re(\tau) = \sigma$. Then by the residue theorem

$$\int_{\mathcal{C}} f(z) dz = \sum_{k=1}^{\infty} e((k - \alpha)\tau) - \sum_{-K < \ell < K} e(\alpha\ell)(-2\pi i)^{-1}(\tau + \ell)^{-1}.$$

On the other hand, the integrals over the horizontal portions of $\mathcal{C}$ tend to 0 as $L \to \infty$ since $|f(z)|$ decays exponentially as $\Im(z) \to \infty$. The residues are independent of L for L sufficiently large so taking the limit as $L \to \infty$ we obtain

$$\sum_{k=1}^{\infty} e((k - \alpha)\tau) - \sum_{-K < \ell < K} e(\alpha\ell)(-2\pi i)^{-1}(\tau + \ell)^{-1} = E_K(\tau),$$

where $E_K(\tau)$ is the limit as $L \to \infty$ of the contributions to $\int_{\mathcal{C}} f(z) dz$ coming from the vertical portions of $\mathcal{C}$.

$$E_K(\tau) = i \int_{-\infty}^{\infty} (f(\sigma + K + 1/2 + iy) - f(\sigma - K - 1/2 + iy)) dy.$$

For convenience set $Q = K + 1/2$. After replacing y with $y + t$ we obtain

$$(C.4) \quad E_K(\tau) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{e(\alpha Q)}{Q + iy + \tau} - \frac{e(-\alpha Q)}{-Q + iy + \tau} \right) \frac{e(\alpha iy)}{1 + e(iy)} dy,$$

since $e(Q) = e(-Q) = -1$ for K an integer. Observe that the integrand in (C.4) is bounded by C/Q^2, for some constant C, and decays exponentially as $y \to \pm\infty$ for $0 < \alpha < 1$. Also, the constant C holds locally uniformly in τ. We conclude that $E_K(\tau) = \mathcal{O}(1/K^2)$ locally uniformly in τ, as required. $\square$

For $\alpha = 0$ we have the identity

$$\lim_{K \to \infty} \sum_{\ell=-K}^{K} \frac{1}{\tau + \ell} = \pi \cot \pi\tau = -i\pi \left(1 + 2 \sum_{m=1}^{\infty} q^m \right).$$

Appendix D. Some details on convergence

In this appendix, we will give the steps leading to expression (5.5), whose convergence we demonstrated in the main text. Using the explicit definition (4.5), (5.1) and again split the sum over d into a double sum, we obtain

(D.1)

$$\lim_{K \to \infty} \sum_{\gamma \in (\Gamma_\infty \backslash \Gamma)^\times_{<K}} R_\gamma^0(\tau)$$

$$= e\left(\frac{1}{8}\right) \sum_{\substack{0<c<K \\ n|c}} c^{-3/2} \sum_{\substack{-K^2<d<K^2 \\ (c,d)=1}} e\left(-\frac{\gamma\infty}{8}\right) \psi(\gamma)(\tau + d/c)^{-1},$$

$$= e\left(\frac{1}{8}\right) \sum_{\substack{0<c<K \\ n|c}} c^{-3/2} \sum_{\substack{0<d<c \\ (c,d)=1}} e\left(-\frac{\gamma\infty}{8}\right) \psi(\gamma) \sum_{\substack{\ell \in \mathbb{Z} \\ |d+\ell c|<K^2}} \frac{e(\frac{\ell}{8})}{(\tau + d/c + \ell)},$$

where we have used $\psi(\gamma T) = e(\frac{1}{8})\psi(\gamma)$ (see Appendix A) in deriving the last line. Consider the following two expressions:

$$(D.2) \quad \sum_{\substack{\ell \in \mathbb{Z} \\ |d+\ell c|<K^2}} e\left(\frac{\ell}{8}\right)(\tau + d/c + \ell)^{-1}, \quad \sum_{\substack{\ell \in \mathbb{Z} \\ |\ell|<K^2/c}} e\left(\frac{\ell}{8}\right)(\tau + d/c + \ell)^{-1},$$

we will now argue that we can use either of the two in the limit $K \to \infty$. The left most of (D.2) may include a term where $d + \ell c = x - K^2$ for at most

one $0 \le x < d$ and this will not appear in the right-hand expression, and the right most of (D.2) may include a term where $d + \ell c = x + K^2$ for at most one $0 < x \le d$ and these will not appear in the left-hand expression. We conclude that the difference is $\mathcal{O}(c/K^2)$ for sufficiently large K, uniformly in τ, and since $\sum_{0<c<K} c^{1/2}$ is $\mathcal{O}(K^{3/2})$, we have

$$\sum_{\substack{0<c<K \\ n|c}} c^{1/2} \sum_{\substack{0 \le d < c \\ (c,d)=1}} \mathcal{O}(c/K^2) = \mathcal{O}(1/K^{1/2}),$$

which vanishes as $K \to \infty$. Therefore, we have

(D.3)

$$\lim_{K \to \infty} \sum_{\epsilon\gamma \in (\Gamma_\infty \backslash \Gamma)^\times_{<K}} R^0_\gamma(\tau) = e\left(\frac{1}{8}\right) \lim_{K \to \infty} \sum_{\substack{0<c<K \\ n|c}} c^{-3/2} \sum_{\substack{0 \le d < c \\ (c,d)=1}} e\left(-\frac{\gamma\infty}{8}\right) \psi(\gamma)$$

$$\times \sum_{\substack{\ell \in \mathbb{Z} \\ |d| < K^2/c}} \frac{e(\frac{\ell}{8})}{(\tau + d/c + \ell)}.$$

Upon applying the Lipschitz summation formula in the form of Lemma Appendix C.1 we obtain

$$\lim_{K \to \infty} \sum_{\epsilon\gamma \in (\Gamma_\infty \backslash \Gamma)^\times_{<K}} R^0_\gamma(\tau) = 2\pi e\left(-\frac{1}{8}\right) \lim_{K \to \infty} \sum_{\substack{0<c<K \\ n|c}} c^{-3/2} \sum_{\substack{0 \le d < c \\ (c,d)=1}} e\left(-\frac{\gamma\infty}{8}\right) \psi(\gamma)$$

$$\times \left(-E_{\lfloor K^2/c \rfloor}(\tau) + \sum_{k=1}^\infty e\left(\left(k - \frac{1}{8}\right)\left(\tau + \frac{d}{c}\right)\right)\right).$$

For K large the error terms $E_{\lfloor K^2/c \rfloor}(\tau)$ are bounded by $E_K(\tau)$ and according to Lemma Appendix C.1 we have $E_K = \mathcal{O}(1/K^2)$. It follows that the term involving $-E_{\lfloor K^2/c \rfloor}(\tau)$ is $\mathcal{O}(1/K^{3/2})$ and thus tends to 0 as $K \to \infty$. Observe that the sum over k in the above formula is absolutely convergent, uniformly in τ on compact subsets of $\mathbb{H}$. Moving the summation over k past the others and using the explicit expressions for $\psi(\gamma)$, we readily obtain (5.5).

Appendix E. Rademacher sums at other cusps

To identify $R_{\Gamma,\rho}$, we need to study its behaviour at the other cusps of $\Gamma = \Gamma_0(n)$. Focusing on a given cusp representative $\kappa = \sigma\infty$, $\sigma \in \mathrm{SL}_2(\mathbb{Z})$, with

width v in $\Gamma_0(n)$, we need to study $R_{\Gamma,\rho}|_{\epsilon^{-3},1/2}\sigma$. For this purpose, let us introduce

(E.1) $$R'_\gamma = \epsilon^{\epsilon-3}(\gamma)\, e\left(-\frac{\gamma\tau}{8}\right)\, \mathrm{reg}(\gamma,\tau)\, \mathrm{jac}^{\epsilon 1/4}(\gamma,\tau),$$

(E.2) $$S'_\gamma = \epsilon^{\epsilon 3}(\gamma)\, e\left(\frac{\gamma\tau}{8}\right)\, \mathrm{jac}^{\epsilon 3/4}(\gamma,\tau),$$

analogous to the definitions in Section 5, so that

$$R_{\Gamma,\rho} = \lim_{K\to\infty} \sum_{\gamma\in(\Gamma_\infty\backslash\Gamma)_{<K}} \rho(\gamma)\, R'_\gamma, \quad S_{\Gamma,\rho} = \lim_{K\to\infty} \sum_{\gamma\in(\Gamma_\infty\backslash\Gamma)_{<K}} \rho^{\epsilon-1}(\gamma)\, S'_\gamma.$$

A straightforward calculation similar to the one in Section 7 shows

(E.3) $$R'_\gamma|_{\epsilon^{-3},1/2}\sigma = R'_{\gamma\sigma} + cJ_{\sigma^{-1}\infty}S'_{\gamma\sigma}, \quad c = -\frac{e(\frac{1}{4})}{\sqrt{8}},$$
$$S'_\gamma|_{\epsilon^3,3/2}\sigma = S'_{\gamma\sigma}.$$

Using $S_{n|1} = \lambda_n\eta^3$ and $S_{n|h} = 0$ for $h > 1$ as shown in Section 8.1, we get

(E.4) $$R_{n|1}|_{\epsilon^{-3},1/2}\sigma = \lim_{K\to\infty} \sum_{\gamma\in(\Gamma_\infty\backslash\Gamma)_{<K}\sigma} R'_\gamma + c\lambda_n J_{\sigma^{-1}\infty}\eta^3,$$

(E.5) $$R_{n|h}|_{\epsilon^{-3},1/2}\sigma = \lim_{K\to\infty} \sum_{\gamma\in(\Gamma_\infty\backslash\Gamma)_{<K}\sigma} \rho(\gamma\sigma^{\epsilon-1})\, R'_\gamma \quad \text{for } h > 1.$$

Splitting the sum into a sum over the double coset space $(\Gamma_\infty\backslash\Gamma)_{\epsilon<K}\sigma/\Gamma_\infty^v$ and applying the Lipschitz summation formula as before, we obtain that the Fourier expansion of the above sum is

(E.6) $$\lim_{K\to\infty} \sum_{\gamma\in(\Gamma_\infty\backslash\Gamma)_{<K}\sigma} \rho(\gamma\sigma^{\epsilon-1})\, R'_\gamma = \sum_{k=1}^{\infty} c_k\, e\left((k-\nu)\frac{\tau}{v}\right).$$

As before we have $0 < \nu \le 1$ given by $\nu = \{\{\frac{m}{h} + \frac{v}{8}\}\}$, where $\rho(\sigma T^v\sigma^{-1}) = e(\frac{m}{h})$ (cf. (7.5)). In particular, for $h = 1$ we have

(E.7) $$\lim_{K\to\infty} \sum_{\gamma\in(\Gamma_\infty\backslash\Gamma)_{<K}\sigma} R'_\gamma = \sum_{k=1}^{\infty} c_k\, e\left(\left(k - \left\{\left\{\frac{v}{8}\right\}\right\}\right)\frac{\tau}{v}\right).$$

Using Lemma 8.8 and Equation (3.6), we can rewrite the function $G_{n|1}$ as

$$G_{n|1} = \lambda_n\tilde{T}_g + \chi(g)\eta^3\left(R_{n|1} + \frac{\lambda_n}{12}R_{1|1}\right).$$

From (E.4) we deduce that

$$
\left(R_{n|1} + \frac{\lambda_n}{12} R_{1|1} \right) \big|_{\epsilon^{-3},1/2} \sigma = \lim_{K \to \infty} \sum_{\gamma \in (\Gamma_\infty \backslash \Gamma)_{<K} \sigma} R'_\gamma + \frac{\lambda_n}{12} R_{1|1}
$$

and hence

$$
G_{n|1}\big|_{1,2} \sigma = \eta^3 \left(-\lambda_n \left(\frac{\chi(g)}{24} H - \frac{\tilde{T}_g|_{1,2}\, \sigma}{\eta^3} \right) + \chi(g) \lim_{K \to \infty} \sum_{\gamma \in (\Gamma_\infty \backslash \Gamma)_{<K} \sigma} R'_\gamma \right).
$$

One can check from the explicit expressions for $\tilde{T}_g$ that $\frac{\chi(g)}{24} H - \frac{\tilde{T}_g|_{1,2}\, \sigma}{\eta^3} = \mathcal{O}(1)$, so that the only pole of H_g is at the infinite cusp. Together with the (E.7) we obtain expansion (8.12).

Appendix F. Modular forms and representations

Table 4: (i) The first ten Fourier coefficients of the McKay–Thompson series $Z_g(\tau) = \sum_{k=0}^{\infty} q^{k-1}(\text{Tr}\,\mathcal{H}_k\, g) = 1/\eta_g(\tau)$. (ii) The decomposition of the first ten M_{24}-modules $\mathcal{H}_k$ into irreducible representations $\mathcal{H}_k$ (see the character table E).

$1/\eta_g(\tau)$	1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B	7A	7B	8A	10A	11A	12A	12B	14A	14B	15A	15B	21A	21B	23A	23B
q^0	24	8	0	6	0	0	4	0	4	2	0	3	3	2	2	2	0	0	1	1	1	1	0	0	1	1
q^1	324	52	12	27	0	4	16	0	14	7	0	9	9	6	0	5	1	0	3	3	2	2	0	0	2	2
q^2	3200	256	0	104	8	0	48	0	40	16	0	22	22	12	2	10	0	0	4	4	4	4	1	1	3	3
q^3	25650	1122	90	351	0	18	142	6	105	39	0	51	51	28	5	20	3	0	9	9	6	6	0	0	5	5
q^4	176256	4352	0	1080	0	0	368	0	256	80	0	108	108	52	0	36	0	0	12	12	10	10	0	0	7	7
q^5	1073720	15640	520	3107	44	56	928	0	590	175	4	221	221	104	10	65	5	0	23	23	17	17	2	2	11	11
q^6	5930496	52224	0	8424	0	0	2176	0	1296	336	0	432	432	184	0	110	0	0	32	32	24	24	0	0	15	15
q^7	30178575	165087	2535	21762	192	175	4979	27	2740	666	0	819	819	341	20	185	10	0	55	55	37	37	0	0	22	22
q^8	143184000	495872	0	53976	0	0	10864	0	5600	1232	0	1506	1506	580	0	300	0	0	76	76	56	56	3	3	30	30
q^9	639249300	1428612	10908	129141	0	468	23184	0	11130	2289	0	2706	2706	1010	38	481	15	0	122	122	81	81	0	0	42	42

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}	χ_{17}	χ_{18}	χ_{19}	χ_{20}	χ_{21}	χ_{22}	χ_{23}	χ_{24}	χ_{25}	χ_{26}
$\mathcal{H}_1$	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathcal{H}_2$	3	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathcal{H}_3$	6	8	0	0	0	0	1	2	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
$\mathcal{H}_4$	14	20	0	0	0	0	3	6	4	1	1	0	0	1	0	0	3	0	0	0	1	0	0	0	0	0
$\mathcal{H}_5$	27	48	0	0	0	0	12	22	15	4	4	0	0	3	3	3	15	1	0	3	6	16	3	0	0	3
$\mathcal{H}_6$	59	110	0	0	0	0	33	61	51	15	15	0	0	3	3	3	54	10	9	17	34	69	25	6	9	26
$\mathcal{H}_7$	114	249	0	0	6	6	97	174	161	51	51	3	3	19	3	3	190	45	47	88	158	276	147	59	71	194
$\mathcal{H}_8$	235	552	0	0	32	32	255	457	498	40	40	39	39	70	39	39	633	220	269	393	694	1042	758	418	490	1088
$\mathcal{H}_9$	460	1217	1	1	169	169	687	1235	1504	255	255	296	296	301	294	294	994	994	1252	1730	2850	3870	3528	2354	2656	5544
$\mathcal{H}_{10}$	924	2677	24	24	731	731	4754	3294	4575	1425	1425	1675	1675	4329	1699	1699	7207	4391	5592	7131	11460	14340	15393	11758	13026	25565

Table 5: The first few Fourier coefficients of the terms $q^{-\frac{1}{8}+n}$ in the q-series $H_g(\tau)$ and the corresponding representation K_n (cf. table **??**).

ℓ \ [g]	1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B	7A	7B	8A	10A	11A	12A	12B	14A	14B	15A	15B	21A	21B	23A	23B
0	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
1	90	-6	10	0	6	-6	2	2	0	0	-2	-1	-2	0	2	2	0	2	1	1	0	0	-1	-1	-2	-2
2	462	14	-18	-6	0	-2	-2	6	2	2	2	0	0	2	0	0	-2	2	0	0	-1	-1	0	0	2	2
3	1540	-28	20	10	-14	4	-4	-4	-6	2	2	0	0	0	0	0	-2	0	0	0	0	0	-2	-2	-1	-1
4	4554	42	-38	0	12	-6	2	-6	0	2	4	4	0	-2	2	-2	0	0	0	0	-2	-2	0	0	0	0
5	11592	-56	72	-18	0	-8	8	0	2	-2	0	0	2	0	2	-2	-2	0	2	2	0	0	0	0	0	0
6	27830	86	-90	20	-16	6	-2	6	0	-4	0	-2	0	2	0	-2	0	0	2	2	0	0	-2	-2	0	0
7	61686	-138	118	0	30	6	-10	-2	6	0	-2	2	2	-2	-2	-2	2	-2	2	2	0	0	2	2	0	0
8	131100	188	-180	-30	0	-4	4	-12	0	2	2	-3	-3	0	0	2	-2	0	-1	-1	0	0	0	0	0	0
9	265650	-238	258	42	-42	-14	10	10	-10	2	6	0	0	-2	-2	0	-2	-2	0	0	2	2	0	0	0	0

ℓ \ χ	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}	χ_{17}	χ_{18}	χ_{19}	χ_{20}	χ_{21}	χ_{22}	χ_{23}	χ_{24}	χ_{25}	χ_{26}
0	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	1	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0	2	0
6	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	2	2	2	2	2	2	2	2	2	2
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	1	0	0	2	2	2	0	4	2	2	6
8	0	0	0	0	0	0	0	2	2	2	2	0	0	2	2	2	2	2	0	0	4	4	4	2	2	2
9	0	0	0	0	0	0	0	2	2	2	2	0	0	2	2	2	0	2	2	2	4	4	4	8	8	10

Table 6: Character table of M_{24}. See [24]. We adopt the naming system of [24] and use the notation $e_n = \frac{1}{2}(-1 + \sqrt{-n})$.

classes	1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B	7A	7B	8A	10A	11A	12A	12B	14A	14B	15A	15B	21A	21B	23A	23B
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	23	7	-1	5	-1	-1	3	-1	3	1	-1	2	2	1	-1	1	-1	-1	0	0	-1	-1	-1	-1	0	0
χ_3	45	-3	5	0	3	1	-3	1	0	0	-1	e_7	$\bar{e}_7$	-1	0	1	0	1	$-e_7$	$-\bar{e}_7$	0	0	e_7	$\bar{e}_7$	e_{23}	$\bar{e}_{23}$
χ_4	45	-3	5	0	3	1	-3	1	0	0	-1	$\bar{e}_7$	e_7	-1	0	1	0	1	$-\bar{e}_7$	$-e_7$	0	0	$\bar{e}_7$	e_7	$\bar{e}_{23}$	e_{23}
χ_5	231	7	-9	6	0	-1	-1	3	1	2	0	0	0	-1	-1	0	0	0	0	0	e_{15}	$\bar{e}_{15}$	0	0	1	1
χ_6	231	7	-9	6	0	-1	-1	3	1	2	0	0	0	-1	-1	0	0	0	0	0	$\bar{e}_{15}$	e_{15}	0	0	1	1
χ_7	252	28	12	9	0	4	4	0	2	1	0	0	0	0	2	-1	1	0	0	0	-1	-1	0	0	-1	-1
χ_8	253	13	-11	10	1	1	-3	1	3	-2	1	1	1	-1	-1	0	1	-1	-1	-1	0	0	1	1	0	0
χ_9	483	35	3	6	0	3	3	-1	-2	2	2	0	0	1	-2	-1	0	0	0	0	1	1	0	0	0	0
χ_{10}	770	-14	10	5	-7	-2	2	-2	0	1	1	0	0	0	0	0	-1	-1	0	0	0	0	0	0	e_{23}	$\bar{e}_{23}$
χ_{11}	770	-14	10	5	-7	-2	2	-2	0	1	1	0	0	0	0	0	-1	-1	0	0	0	0	0	0	$\bar{e}_{23}$	e_{23}
χ_{12}	990	-18	-10	0	3	2	-2	2	0	0	-1	e_7	$\bar{e}_7$	0	0	0	0	1	e_7	$\bar{e}_7$	0	0	e_7	$\bar{e}_7$	1	1
χ_{13}	990	-18	-10	0	3	2	-2	2	0	0	-1	$\bar{e}_7$	e_7	0	0	0	0	1	$\bar{e}_7$	e_7	0	0	$\bar{e}_7$	e_7	1	1
χ_{14}	1035	27	35	0	6	3	3	3	0	0	2	-1	-1	1	0	1	0	0	1	1	0	0	-1	-1	0	0
χ_{15}	1035	-21	-5	0	-3	3	3	-1	0	0	1	$2e_7$	$2\bar{e}_7$	-1	0	1	0	-1	$-e_7$	$-\bar{e}_7$	0	0	$-e_7$	$-\bar{e}_7$	0	0
χ_{16}	1035	-21	-5	0	-3	3	3	-1	0	0	1	$2\bar{e}_7$	$2e_7$	-1	0	1	0	-1	$-\bar{e}_7$	$-e_7$	0	0	$-\bar{e}_7$	$-e_7$	0	0
χ_{17}	1265	49	-15	5	8	-7	1	-3	0	-2	0	-2	-2	-1	0	0	1	0	1	1	0	0	1	1	0	0
χ_{18}	1771	-21	11	16	7	3	-5	-1	1	0	-1	0	0	-1	1	0	-1	-1	0	0	1	1	0	0	0	0
χ_{19}	2024	8	24	-1	8	8	0	0	-1	-1	1	1	1	0	-1	0	-1	0	1	1	-1	-1	1	1	0	0
χ_{20}	2277	21	-19	0	6	-3	1	1	-3	0	2	2	2	-1	1	0	0	0	-1	-1	0	0	-1	-1	0	0
χ_{21}	3312	48	16	0	-6	0	0	0	-3	0	-2	1	1	0	1	1	-1	-1	1	1	0	0	1	1	0	0
χ_{22}	3520	64	0	10	-8	0	0	0	0	-2	0	-1	-1	0	0	0	1	1	-1	-1	0	0	-1	-1	1	1
χ_{23}	5313	49	9	-15	0	1	-3	1	3	1	0	0	0	-1	-1	0	0	0	0	0	-1	-1	0	0	0	0
χ_{24}	5544	-56	24	9	0	-8	0	0	-1	1	1	0	0	0	-1	0	1	1	0	0	1	1	0	0	0	0
χ_{25}	5796	-28	36	-9	0	-4	4	0	1	-1	-1	0	0	0	1	-1	-1	-1	0	0	1	1	0	0	0	0
χ_{26}	10395	-21	-45	0	0	3	-1	3	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	-1	-1

References

[1] T. Eguchi, H. Ooguri and Y. Tachikawa, *Notes on the K3 surface and the Mathieu group M_{24}*, http://arxiv.org/abs/1004.09561004.0956.

[2] M. C. N. Cheng, *K3 Surfaces, $N = 4$ Dyons, and the Mathieu Group M_{24}*, http://arxiv.org/abs/1005.54151005.5415.

[3] M. R. Gaberdiel, S. Hohenegger and R. Volpato, *Mathieu twining characters for K3*, http://arxiv.org/abs/1006.02211006.0221.

[4] M. R. Gaberdiel, S. Hohenegger and R. Volpato, *Mathieu moonshine in the elliptic genus of K3*, http://arxiv.org/abs/1008.37781008.3778.

[5] T. Eguchi and K. Hikami, *Note on twisted elliptic genus of K3 surface*, http://arxiv.org/abs/1008.49241008.4924.

[6] J. H. Conway and S. P. Norton, *Monstrous moonshine*, Bull. Lond. Math. Soc. **11** (1979), 308–339.

[7] R. E. Borcherds, *Monstrous moonshine and monstrous Lie superalgebras*, Invent. Math. **109**(2) (1992), 405–444.

[8] S. Norton, *Generalized Moonshine*, Proc. Symp. Pure Math **47** (1987), 208–209.

[9] A. P. Ogg, *Automorphismes des Courbes Modulaires*, Seminaire Delange-Pisot (1974).

[10] J. F. R. Duncan and I. B. Frenkel, *Rademacher sums, moonshine and gravity*, http://arxiv.org/abs/0907.45290907.4529.

[11] H. Rademacher, *The Fourier series and the functional equation of the absolute modular invariant $J(\tau)$*, Amer. J. Math. **61**(1) (1939), 237–248.

[12] J. M. Maldacena, *The large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. **2** (1998), 231–252, http://arxiv.org/abs/hep-th/9711200hep-th/9711200.

[13] J. Maldacena and A. Strominger, *AdS3 Black Holes and a stringy exclusion principle*, J. High Energy Phys. **12** (1998), 005, http://arxiv.org/abs/hep-th/9804085hep-th/9804085.

[14] R. Dijkgraaf, J. Maldacena, G. Moore and E. Verlinde, *A Black Hole Farey Tail*, `http://arxiv.org/abs/hep-th/0005003` `hep-th/0005003`.

[15] G. W. Moore, *Les Houches Lectures on Strings and Arithmetic*, `http://arxiv.org/abs/hep-th/0401049hep-th/0401049`.

[16] J. de Boer, M. C. N. Cheng, R. Dijkgraaf, J. Manschot and E. Verlinde, *A Farey Tail for attractor Black Holes*, J. High Energy Phys. **11** (2006), 024, `http://arxiv.org/abs/hep-th/0608059hep-th/0608059`.

[17] P. Kraus and F. Larsen, *Partition functions and elliptic genera from supergravity*, J. High Energy Phys. **01** (2007), 002, `http://arxiv.org/abs/hep-th/0607138hep-th/0607138`.

[18] F. Denef and G. W. Moore, *Split states, entropy enigmas, Holes and Halos*, `http://arxiv.org/abs/hep-th/0702146hep-th/0702146`.

[19] J. Manschot and G. W. Moore, *A modern Fareytail*, Commun. Num. Theor. Phys. **4** (2010), 103–159 `http://arxiv.org/abs/0712.05730712.0573`.

[20] S. Murthy and B. Pioline, *A Farey tale for $N = 4$ dyons*, JHEP **0909** (2009), 022 [`arXiv:0904.4253 [hep-th]`].

[21] T. Eguchi and K. Hikami, *Superconformal Algebras and Mock Theta Functions 2. Rademacher Expansion for K3 Surface*, Commun. Num. Theor. Phys. **3** (2009), 531–554 `http://arxiv.org/abs/0904.09110904.0911`.

[22] W. d. A. Pribitkin, *The Fourier coefficients of modular forms and Niebur modular integrals having small positive weight. II*, Acta Arith. **93**(4) (2000), 343–358.

[23] J. H. Conway and N. J. A. Sloane, *Sphere packings, lattices and groups.* Springer-Verlag, 1999.

[24] J. Conway, R. Curtis, S. Norton, R. Parker and R. Wilson, *Atlas of finite groups. Maximal subgroups and ordinary characters for simple groups. With comput. assist. from J. G. Thackray.* Oxford, Clarendon Press, 1985.

[25] G. Mason, M_{24} *and Certain Automorphic Forms*, Contemporary Mathematics **45** (1985), 223–244.

[26] D. Dummit, H. Kisilevsky and J. McKay, *Multiplicative products of eta-functions.*, Contemp. Math. (Finite groups—coming of age, Proc. CMS Conf., Montreal/Que.) **45** (1982), (1985).

[27] S. Govindarajan and K. G. Krishna, *BKM Lie superalgebras from dyon spectra in Z_N-CHL orbifolds for composite N*, `http://arxiv.org/abs/0907.14100907.1410`.

[28] M. C. N. Cheng and J. F. R. Duncan, *The largest Mathieu group and (Mock) automorphic forms*, `arXiv:1201.4140 [math.RT]`.

[29] T. Eguchi, H. Ooguri, A. Taormina, and S.-K. Yang, *Superconformal algebras and string compactification on manifolds with $SU(N)$ holonomy*, Nucl. Phys. B **315** (1989), 193, `http://dx.doi.org/10.1016/0550-3213(89)90454-9`.

[30] T. Eguchi and K. Hikami, *Superconformal algebras and mock theta functions*, J. Phys. A **42** (2009), 304010, `http://arxiv.org/abs/0812.11510812.1151`.

[31] A. Dabholkar, S. Murthy and D. Zagier, *Quantum Black Holes, wall crossing, and Mock modular forms*, `arXiv:1208.4074 [hep-th]`.

[32] D. Niebur, *Construction of automorphic forms and integrals*, Trans. Amer. Math. Soc. **191** (1974), 373–385.

[33] D. Goldfeld and P. Sarnak, *Sums of Kloosterman sums*, Invent. Math. **71**(2) (1983), 243–250, `http://dx.doi.org/10.1007/BF01389098`.

[34] G. Shimura, *Introduction to the arithmetic theory of automorphic functions.* Publications of the Mathematical Society of Japan, No. 11. Iwanami Shoten, Publishers, Tokyo, (1971). Kanô Memorial Lectures, No. 1.

[35] W. Stein, *Modular forms, a computational approach*, vol. 79 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 2007. With an appendix by Paul E. Gunnells.

[36] T. Miyake, *Modular forms.* Springer-Verlag, Berlin 1989. Translated from the Japanese by Yoshitaka Maeda.

[37] A. Selberg, *On the estimation of Fourier coefficients of modular forms*, in Proc. Sympos. Pure Math., VIII 1–15. Amer. Math. Soc., Providence, R.I., (1965).

[38] W. Roelcke, *Das Eigenwertproblem der automorphen Formen in der hyperbolischen Ebene. I, II*, Math. Ann. **167** (1966), 292–337; Math. Ann. **168** (1966), 261–324.

[39] P. Sarnak, *Additive number theory and Maass forms*, in Number theory (New York,) 1982, Lecture Notes in Mathematics **1052**, 286–309. Springer: Berlin, 1984. `http://dx.doi.org/10.1007/BFb0071548`.

[40] M. I. Knopp, *On the Fourier coefficients of small positive powers of $\theta(\tau)$*, Invent. Math. **851** (1986), 165–183. `http://dx.doi.org/10.1007/BF01388796`.

[41] M. I. Knopp, *On the Fourier coefficients of cusp forms having small positive weight*, in Theta functions—Bowdoin 1987, Part 2 (Brunswick, ME, 1987), Proc. Symp. Pure Math. **49**, 111–127. Amer. Math. Soc., Providence, RI, 1989.

[42] H. Davenport, *Multiplicative number theory*, vol. 74 of *Graduate Texts in Mathematics*. Springer-Verlag: New York, 3rd ed., 2000. Revised and with a preface by Hugh L. Montgomery.

[43] W. d. A. Pribitkin, *A generalization of the Goldfeld–Sarnak estimate on Selberg's Kloosterman zeta-function*, Forum Math. **12**(4) (2000), 449–459, `http://dx.doi.org/10.1515/form.2000.014`.

[44] G. H. Hardy and M. Riesz, *The general theory of Dirichlet's series*. Cambridge Tracts in Mathematics and Mathematical Physics **18**. Stechert–Hafner, Inc., New York, 1964.

[45] C. Vafa, *Gas of D-branes and Hagedorn density of BPS states*, Nucl. Phys. B **463** (1996), 415–419.

[46] A. Dabholkar, J. Gomes and S. Murthy, *Quantum black holes, localization and the topological string*, `http://dx.doi.org/10.1007/JHEP06(2011)019`, J. High Energy Phys. **06** (2011) 019, `http://arxiv.org/abs/1012.0265arXiv:1012.0265 [hep-th]`.

[47] A. Dabholkar, J. Gomes and S. Murthy, *Localization and exact holography*, `arXiv:1111.1161 [hep-th]`.

[48] E. Witten, *Three-dimensional gravity revisited*, `http://arxiv.org/abs/0706.33590706.3359`.

[49] M. R. Gaberdiel, S. Hohenegger and R. Volpato, *Symmetries of K3 sigma models*, `http://arxiv.org/abs/1106.43151106.4315`.

[50] M. Knopp and S. Robins, *Easy proofs of Riemann's functional equation for $\zeta(s)$ and of Lipschitz summation*, Proc. Amer. Math. Soc. **129**(7) (2001), 1915–1922 (electronic).

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY
CAMBRIDGE
MA 02138
USA
E-mail address: `chengm@math.jussieu.fr`

DEPARTMENT OF PHYSICS, HARVARD UNIVERSITY
CAMBRIDGE
MA 02138
USA
E-mail address: `chengm@math.jussieu.fr`

DEPARTMENT OF MATHEMATICS, CASE WESTERN RESERVE UNIVERSITY
CLEVELAND
OH 44106
USA
E-mail address: `john.f.r.duncan@gmail.com`