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Pfaffian Calabi–Yau threefolds and mirror symmetry

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The aim of this paper is to report on recent progress in understanding mirror symmetry for some non-complete intersection Calabi–Yau threefolds. We first construct four new smooth non-complete intersection Calabi–Yau threefolds with $h^{1,1} = 1$, whose existence was previously conjectured by C. van Enckevort and D. van Straten in [19]. We then compute the period integrals of candidate mirror families of F. Tonoli's degree 13 Calabi–Yau threefold and three of the new Calabi–Yau threefolds. The Picard–Fuchs equations coincide with the expected Calabi–Yau equations listed in [18,19]. Some of the mirror families turn out to have two maximally unipotent monodromy points.

1. Introduction

The aim of this paper is to report on recent progress in understanding mirror symmetry for some non-complete intersection Calabi–Yau threefolds. Throughout this paper we adopt the following definition.

Definition 1.1. A d-dimensional Calabi–Yau variety X is a normal compact variety over $\mathbb{C}$ with at worst Gorenstein canonical singularities and with trivial dualizing sheaf $\omega_X \cong \mathscr{O}_X$ such that $H^i(X, \mathscr{O}_X) = 0,\ (i = 1, \ldots, d-1)$.

Among smooth Calabi–Yau threefolds, those with 1-dimensional Kähler moduli spaces have been attracting much attention because their expected mirror partners have 1-dimensional complex moduli spaces and hence one can work on them in detail. There are around thirty known examples of topologically distinct smooth Calabi–Yau threefolds with $h^{1,1} = 1$, most of which are complete intersections of hypersurfaces in toric varieties or homogeneous spaces. Although non-complete intersection Calabi–Yau threefolds are only partially explored, they are intriguing on their own and provide important testing grounds for mirror symmetry. We hope that new non-complete intersection Calabi–Yau threefolds with $h^{1,1} = 1$ and mirror phenomena we report in this paper are of interest and will be the first step toward the future investigations. This paper is clearly influenced by E. Rødland's work [14] and we owe a lot of arguments to it. We mention it here and do not repeat it each time in the sequel.

In the following we give a brief overview of this paper. Section 2 is mainly devoted to the study of pfaffian threefolds in weighted projective spaces. Pfaffian Calabi–Yau threefolds in $\mathbb{P}^6$ was first studied by F. Tonoli in his thesis [17]. By replacing the ambient space $\mathbb{P}^6$ by weighted projective spaces, we obtain several new low degree Calabi–Yau threefolds with $h^{1,1} = 1$. We then determine their fundamental topological invariants $\int_X H^3$, $\int_X c_2(X) \cdot H$ and $\int_X c_3(X)$, which determine the diffeomorphism class of X when X is simply connected and $h^{1,1} = 1$ (Wall's classification theorem [20]). The main result of Section 2 is the following.

Theorem 1.1. *There exist pfaffian threefolds X_5, X_7, X_{10} and X_{25}, which are smooth and Calabi–Yau with the following topological invariants.*

X_i	$h^{1,1}$	$h^{1,2}$	$\int_{X_i} H^3$	$\int_{X_i} c_2(X_i) \cdot H$
X_5	1	51	5	38
X_7	1	61	7	46
X_{10}	1	59	10	52
X_{25}	1	51	25	70

The existence of Calabi–Yau threefolds with these topological invariants was previously conjectured by C. van Enckevort and D. van Straten based on the classification of Calabi–Yau equations in [18, 19].

In Sections 3 and 4, we report on mirror symmetry for these Calabi–Yau threefolds. A pfaffian Calabi–Yau threefold X_{13} of degree 13 was constructed by F. Tonoli [17] and later a candidate mirror family of X_{13} was proposed by J. Böhm from the viewpoint of tropical geometry [2]. We confirm the proposal by computing the Picard–Fuchs equation of the family. After computing the conjectural genus $g = 0, 1$ BPS invariants n_d^g $(d \in \mathbb{N})$, we heuristically determine the number of degree 1 rational curves in X_{13} and find that it coincides with n_1^0 as mirror symmetry predicts. Interestingly, the mirror family has a special point where all the indices of the Picard–Fuchs operator are $1/2$, in addition to the usual maximally unipotent monodromy point at $0 \in \mathbb{P}^1$. This observation is further discussed in comparison with E. Rødland's work [14].

Although the existence of mirror family of a given Calabi–Yau threefold is highly non-trivial, inspired by the mirror family of X_{13}, we exhibit explicit mirror families of the Calabi–Yau threefolds X_5, X_7 and X_{10}. We verify that their Picard–Fuchs equations coincide with the expected Calabi–Yau equations listed in [18, 19]. A general member of these families is quite singular and it has not been settled yet whether a general member of the families admits any crepant resolution or not.

Section 5 studies a degree 9 pfaffian Calabi–Yau threefold $X_9 \subset \mathbb{P}(1^6, 2)$, which is isomorphic to a complete intersection Calabi–Yau threefold $\mathbb{P}^5_{3^2}$. This twofold interpretation yields non-isomorphic special one-parameter families, both of which have the same Picard–Fuchs equation. These two families may bridge our pfaffian mirror construction and the conventional Batyrev–Borisov mirror construction.

It is worth mentioning a relevant work; based on the results of this paper, physicists M. Shimizu and H. Suzuki studied open mirror symmetry for our pfaffian Calabi–Yau threefolds [15].

2. Pfaffian Calabi–Yau threefolds

2.1. Pfaffian threefolds in projective spaces

Suppose that R is a regular local ring and $I \subset R$ is an ideal of height 1 or 2. J. -P. Serre proved that R/I is Gorenstein if and only if it is complete intersection. This is no longer true for height three ideals, but such Gorenstein ideals are characterized as pfaffian ideals of certain skew-symmetric matrices [5]. This observation suggests that pfaffian varieties form a reasonable class of varieties to study when we investigate non-complete intersection Gorenstein varieties. In this subsection, we review the basics of pfaffians in order to make this paper self-contained.

Throughout this paper we work over complex numbers $\mathbb{C}$. Let $\mathrm{SkewSym}(n, \mathbb{C})$ be the set of $n \times n$ skew symmetric matrices. For $N = (n_{i,j}) \in \mathrm{SkewSym}(n, \mathbb{C})$ the pfaffian $\mathrm{Pf}(N)$ is defined as

$$\mathrm{Pf}(N) = \frac{1}{r!2^r} \sum_{\sigma \in \mathfrak{S}_{2r}} \mathrm{sign}(\sigma) \prod_{i=1}^{r} n_{\sigma(i)\sigma(r+i)}$$

if $n = 2r$ is even, and $\mathrm{Pf}(N) = 0$ if n is odd. It can be check that $\mathrm{Pf}(N)^2 = \det(N)$. We define $N_{i_1,\ldots,i_l}$ as a skew-symmetric matrix obtained by removing all the i_jth rows and columns from N, and set $P_{i_1,\ldots,i_l} = \mathrm{Pf}(N_{i_1,\ldots,i_l})$. Let us next assume that n is odd, say $n = 2r + 1$. The adjoint matrix $\mathrm{adj}(N)$ of N

is the rank 1 matrix given by

$$\mathrm{adj}(N) = P \cdot P^t, \quad P = (P_1, P_2, \ldots, P_{2r+1})^t.$$

We then have $P \cdot N = \det(N) \cdot E_n = 0$ and if $\mathrm{rank}(N) = 2r$ then $P_1, \ldots, P_{2r+1}$ generate $Ker(N)$. $\mathrm{GL}(n, \mathbb{C})$ acts on $\mathrm{SkewSym}(n, \mathbb{C})$ by conjugation with a finite number of orbits $\{O_{2i}\}_{i=0}^{r}$ where the orbit O_{2i} consists of all skew-symmetric matrices of rank $2i$. The closure $\overline{O_{2i}}$ of O_{2i} is singular along its boundary $\overline{O_{2i}} \setminus O_{2i}$, which consists of the union of $\{O_{2j}\}_{j=0}^{i-1}$.

Let us first recall the construction of pfaffian varieties in $\mathbb{P}^n$ $(n > 3)$. From now on we shall always identify $H^2(\mathbb{P}^n, \mathbb{Z}) \cong \mathbb{Z}$ via the hyperplane class. Given an integer $t \in \mathbb{Z}$ and a locally free sheaf $\mathscr{E}$ of odd rank $2r+1$ on $\mathbb{P}^n$, a global section $N \in H^0(\mathbb{P}^n, \wedge^2 \mathscr{E}(t))$ defines an alternating morphism $\mathscr{E}^\vee(-t) \xrightarrow{N} \mathscr{E}$. The pfaffian complex associated to $(t, \mathscr{E}, N)$ is given by

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^n}(-t-2s) \xrightarrow{P^t} \mathscr{E}^\vee(-t-s) \xrightarrow{N} \mathscr{E}(-s) \xrightarrow{P} \mathscr{O}_{\mathbb{P}^n},$$

where $s = c_1(\mathscr{E}) + rt$ and P is defined as

$$P = \frac{1}{r!} \wedge^r N \in H^0(\mathbb{P}^n, \wedge^{2r} \mathscr{E}(rt)).$$

The first and third morphisms are given by taking the wedge product with P and P^t respectively. Once we fix a basis of sections $e_1, \ldots, e_{2r+1}$ of $\mathscr{E}$, N may be expressed as a matrix and then P is given by

$$P = \sum_{i=1}^{2r+1} \mathrm{Pf}(N_i) \bigwedge_{j \neq i} e_j.$$

Definition 2.1. A projective variety $X \subset \mathbb{P}^n$ is called the pfaffian variety associated to $(t, \mathscr{E}, N)$ if the structure sheaf $\mathscr{O}_X$ is given by $\mathrm{Coker}(P)$. The sheaf $\mathrm{Im}(P) \subset \mathscr{O}_{\mathbb{P}^n}$ is called the pfaffian ideal sheaf of X and denoted by $\mathscr{I}_X$.

The twist t is usually fixed and often omitted without harm. In case a choice of global section N is not explicitly specified, it is understood that N is a general element of $H^0(\mathbb{P}^n, \wedge^2 \mathscr{E}(t))$. In his thesis [17], F. Tonoli constructed several smooth Calabi–Yau threefolds with $h^{1,1} = 1$ in $\mathbb{P}^6$, using the following (globalized version of the classical) theorem of D. A. Buchsbaum and D. Eisenbud.

Theorem 2.1 (D. A. Buchsbaum and D. Eisenbud [5]). *Let $X \subset \mathbb{P}^n$ be a pfaffian variety associated to $(t, \mathscr{E}, N)$. X is then the degeneracy locus* *of the skew-symmetric map N and if N is generically of rank 2r it degenerates to rank 2r-2 in the expected codimension 3, in which case, the pfaffian complex gives the self-dual resolution of the ideal sheaf of X. Moreover, X is locally Gorenstein, subcanonical with $\omega_X \cong \mathscr{O}_X(t+2s-n-1)$.*

Let X be a pfaffian Calabi–Yau variety of dimension 3 in $\mathbb{P}^6$. Here we have $t+2s=7$. By applying suitable twists, we can assume that $s=3$ and henceforth we consider the pfaffian complex of the following type:

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^6}(-7) \xrightarrow{P^t} \mathscr{E}^\vee(-4) \xrightarrow{N} \mathscr{E}(-3) \xrightarrow{P} \mathscr{O}_{\mathbb{P}^n} \longrightarrow \mathscr{O}_X \longrightarrow 0.$$

It is natural to expect some bounding of topological invariants of pfaffian Calabi–Yau threefolds in $\mathbb{P}^6$. To see the range of possible degree, we reduce the question to the compact complex surface theory by taking a hyperplane section. Let S be a compact, smooth complex surface. There are two important numerical invariants of S, namely the geometric genus $p_g(S) = \dim H^0(S, K_S)$ and the self-intersection of the canonical divisor K_S^2.

Theorem 2.2 (Castelnuovo inequality). *Let S be a minimal surface of general type. If the canonical map $\Phi_{|K_S|}: S \to \mathbb{P}^n$ is birational to the image, then $K_S^2 \geq 3p_g(S) - 7$.*

We say an embedded variety $X \subset \mathbb{P}^n$ is full if X is not contained in any hyperplane $\mathbb{P}^{n-1} \subset \mathbb{P}^n$. Let S be a smooth surface obtained by taking a hyperplane section of a full Calabi–Yau threefold $X \subset \mathbb{P}^6$. Then $\deg(X) = K_S^2$ and K_S is nef since S is a canonical surface. As $X \subset \mathbb{P}^6$ is full, the short exact sequence $0 \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_X(1) \longrightarrow \mathscr{O}_S(1) \longrightarrow 0$ yields $p_g(S) = 6$. We thereby conclude that the lower bound of the degree of X is 11. Since a complete intersection Calabi–Yau threefold $\mathbb{P}^5_{3^2}$ has degree 9, we cannot remove the fullness on X.

In his paper [17], F. Tonoli constructed pfaffian Calabi–Yau threefolds of degree d in the range $11 \leq d \leq 17$. Although the Castelnuovo inequality tells us that the minimal degree d of a full Calabi–Yau threefold in $\mathbb{P}^6$ is 11, there seems no smooth degree 11 Calabi–Yau threefold in $\mathbb{P}^6$. Degree 12 pfaffian Calabi–Yau threefolds are complete intersections $\mathbb{P}^6_{2^2,3}$. Therefore, degree 13 is a good starting point to analyze.

Definition 2.2 (F. Tonoli [17]). We define $X_{13} \subset \mathbb{P}^6$ as a pfaffian threefold associated to the locally free sheaf $\mathscr{E} = \mathscr{O}_{\mathbb{P}^6}(1) \oplus \mathscr{O}_{\mathbb{P}^6}^{\oplus 4}$.

X_{13} is indeed a smooth Calabi–Yau threefold and the geometric invariants are given by

$$h^{1,1} = 1,\ h^{1,2} = 61,\ \int_{X_{13}} H^3 = 13,\ \int_{X_{13}} c_2(X_{13}) \cdot H = 58.$$

A degree 14 pfaffian Calabi–Yau threefold X_{14} is defined as a pfaffian threefold associated to the locally free sheaf $\mathscr{E} = \mathscr{O}_{\mathbb{P}^6}^{\oplus 7}$. This is nothing but the intersection of $\mathbb{P}^6$ with $\mathrm{Pfaff}(7) \subset \mathbb{P}^{20}$, the rank 4 locus of projectivized general skew-symmetric 7×7 matrices

$$\mathbb{P}(\bigwedge^2 \mathbb{C}^7) = \mathbb{P}(\mathrm{SkewSym}(7, \mathbb{C})) \supset \mathrm{Pfaff}(7) = \{[M] \mid \mathrm{rank}(M) \leq 4\}.$$

It is verified that X_{14} and its mirror partner $\check{X}_{14}$ have rich mathematical structures in [4, 12, 14].

2.2. Pfaffian threefolds in weighted projective spaces

F. Tonoli's construction may be generalized by replacing the ambient space $\mathbb{P}^6$ by any Fano variety. Special care must be taken when the ambient space is singular. In the following we shall study the simplest case, when the ambient space is a weighted projective space $\mathbb{P}_{\mathbf{w}}$. Given an integer $t \in \mathbb{Z}$ and a locally free sheaf $\mathscr{E}$ of odd rank $2r + 1$ on $\mathbb{P}_{\mathbf{w}}$, a global section $N \in H^0(\mathbb{P}_{\mathbf{w}}, \wedge^2 \mathscr{E}(t))$ defines an alternating morphism $\mathscr{E}^\vee(-t) \xrightarrow{N} \mathscr{E}$. The pfaffian complex associated to $(t, \mathscr{E}, N)$ is given by

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}_{\mathbf{w}}}(-t - 2s) \xrightarrow{P^t} \mathscr{E}^\vee(-t - s) \xrightarrow{N} \mathscr{E}(-s) \xrightarrow{P} \mathscr{O}_{\mathbb{P}_{\mathbf{w}}},$$

where $s = c_1(\mathscr{E}) + rt$ and $P = \frac{1}{r!} \wedge^r N$ as before. The pfaffian variety $X \subset \mathbb{P}_{\mathbf{w}}$ associated to $(t, \mathscr{E}, N)$ is a variety whose structure sheaf $\mathscr{O}_X$ is given by $Coker(P)$. We define $|\mathbf{w}|$ as a sum of weights of $\mathbb{P}_{\mathbf{w}}$.

Proposition 2.1. *Let $\mathbb{P}_{\mathbf{w}}$ be a weighted projective space of dimension 6 and $(t, \mathscr{E}, N)$ as above. The pfaffian threefold X associate to $(t, \mathscr{E}, N)$ has trivial dualizing sheaf $\omega_X \cong \mathscr{O}_X$ if and only if $t + 2s = |\mathbf{w}|$.*

Proof. Apply the functor $\mathscr{H}om(-, \omega_{\mathbb{P}_{\mathbf{w}}})$ to the pfaffian resolution to compute the dualizing sheaf $\omega_X \cong \mathscr{E}xt^3(\mathscr{O}_X, \omega_{\mathbb{P}_{\mathbf{w}}})$, which is isomorphic to $\cong \mathscr{O}_X$ if and only if $t + 2s = |\mathbf{w}|$ by the definition of pfaffian variety. □

As we are interested in Calabi–Yau threefolds, we restrict ourselves to the case $t + 2s = |\mathbf{w}|$. Moreover, up to an opportune twist, we may assume $t = 1$ for $|\mathbf{w}|$ odd and $t = 0$ for $|\mathbf{w}|$ even.

Definition 2.3. Define X_i as a pfaffian threefold associated to the following locally sheaf $\mathscr{E}_i$ on $\mathbb{P}_{\mathbf{w}_i}$ for $i = 5, 7, 10$.

i	$\mathbf{w}_i$	$\mathscr{E}_i$
5	$(1^4, 2^3)$	$\mathscr{O}_{\mathbb{P}_{\mathbf{w}_5}}^{\oplus 5}(1)$
7	$(1^5, 2^2)$	$\mathscr{O}_{\mathbb{P}_{\mathbf{w}_7}}(1)^{\oplus 2} \oplus \mathscr{O}_{\mathbb{P}_{\mathbf{w}_7}}^{\oplus 3}$
10	$(1^6, 2^1)$	$\mathscr{O}_{\mathbb{P}_{\mathbf{w}_{10}}}(1)^{\oplus 4} \oplus \mathscr{O}_{\mathbb{P}_{\mathbf{w}_{10}}}$

There are many other choices of weights $\mathbf{w}$ and locally sheaves $\mathscr{E}$ on $\mathbb{P}_{\mathbf{w}}$ to produce pfaffian Calabi–Yau threefolds but it seems only three cases above yield smooth Calabi–Yau threefolds in weighted projective spaces of dimension 6.

Theorem 2.3. *For a generic choice of $N \in H^0(\mathbb{P}_{\mathbf{w}_i}, \wedge^2\mathscr{E}_i(t))$, the pfaffian varieties X_5, X_7 and X_{10} are smooth varieties.*

Proof. A generic choice of N guarantees quasi-smoothness of X_i as follows. For X_5 we have $\mathrm{Sing}(\mathbb{P}_{\mathbf{w}_5}) \cong \mathbb{P}^2$ and $X_5 \cap \mathrm{Sing}(\mathbb{P}_{\mathbf{w}_5})$ is identified with the intersection of $\mathbb{P}^2$ with the rank 2 locus of projectivized general skew-symmetric 5×5 matrices $\mathrm{Pfaff}(5) \cap \mathbb{P}^2$, which is empty. For X_7 the matrix N on $\mathrm{Sing}(\mathbb{P}_{\mathbf{w}_7}) \cong \mathbb{P}^1$ has the following form

$$N = \begin{pmatrix} 0 & 0 & g_1 & g_2 & g_3 \\ 0 & 0 & g_4 & g_5 & g_6 \\ -g_1 & -g_4 & 0 & 0 & 0 \\ -g_2 & -g_5 & 0 & 0 & 0 \\ -g_3 & -g_6 & 0 & 0 & 0 \end{pmatrix},$$

where $g_1, \dots, g_6$ are linear polynomials of x_5 and x_6. It is obvious that this has rank greater than 2 for a generic choice of $g_1, \dots, g_6$. Finally, for general X_{10} we have $P_5|_{\mathrm{Sing}(\mathbb{P}_{\mathbf{w}_{10}})} \neq 0$ while $P_i|_{\mathrm{Sing}(\mathbb{P}_{\mathbf{w}_{10}})} = 0$ for $1 \leq i \leq 4$. This completes the proof of quasi-smoothness. We henceforth assume that X_i avoids the singular locus $\mathrm{Sing}(\mathbb{P}_{\mathbf{w}_i})$.

We denote by $\mathbb{P}_{\mathbf{w}_i}^{sm}$ the smooth open subset $\mathbb{P}_{\mathbf{w}_i} \setminus \mathrm{Sing}(\mathbb{P}_{\mathbf{w}_i})$. Since $H^0(\mathbb{P}_{\mathbf{w}_i}^{sm}, \wedge^2\mathscr{E}_i(t))$ is generated by global sections, we have a surjection

$$H^0(\mathbb{P}_{\mathbf{w}_i}^{sm}, \wedge^2\mathscr{E}_i(t)) \otimes_{\mathbb{C}} \mathscr{O}_{\mathbb{P}_{\mathbf{w}_i}^{sm}} \longrightarrow \wedge^2\mathscr{E}_i(t).$$

This map induces a morphism f of $\mathbb{P}^{sm}_{\mathbf{w}_i}$-schemes of full rank everywhere

$$\begin{array}{ccc} \mathbb{P}^{sm}_{\mathbf{w}_i} \times H^0(\mathbb{P}^{sm}_{\mathbf{w}_i}, \wedge^2\mathscr{E}_i(t)) & \xrightarrow{f} & E = \mathrm{Spec}(\mathrm{Sym}(\wedge^2\mathscr{E}_i(t))) \\ & {\scriptstyle \pi_1} \searrow \quad \swarrow {\scriptstyle \pi_2} & \\ & \mathbb{P}^{sm}_{\mathbf{w}_i} & \end{array}$$

sending $(x, N) \mapsto N(x)$. Let $p : \mathbb{P}^{sm}_{\mathbf{w}_i} \times H^0(\mathbb{P}^{sm}_{\mathbf{w}_i}, \wedge^2\mathscr{E}_i(t)) \to H^0(\mathbb{P}^{sm}_{\mathbf{w}_i}, \wedge^2\mathscr{E}_i(t))$ be the second projection. Define E_2 to be the codimension 3 variety of E whose fiber over a point $x \in \mathbb{P}^{sm}_{\mathbf{w}_i}$ is

$$O_2 \subset \mathrm{SkewSym}(5, \mathbb{C}) \cong \pi_2^{-1}(x)$$

Note that O_2 is independent of the identification $\mathrm{SkewSym}(5, \mathbb{C}) \cong \pi_2^{-1}(x)$. Then $Y = f^{-1}(E_2)$ is of codimension 3 and singular along $f^{-1}(\mathrm{Sing}(E_2)) = \mathbb{P}^{sm}_{\mathbf{w}_i} \times \{0\}$. $p|_{Y\setminus(\mathrm{Sing}(Y))}$ is dominant and generic smoothness of $p|_{Y\setminus(\mathrm{Sing}(Y))}$ proves that for a generic choice of $N \in H^0(\mathbb{P}^{sm}_{\mathbf{w}_i}, \wedge^2\mathscr{E}_i(t))$

$$p|^{-1}_{Y\setminus(\mathrm{Sing}(Y))}(N) = \{(x, N) \mid \mathrm{rank}(N(x)) = 2\}$$

is smooth and of dimension 3. The quasi-smoothness of X_i then shows

$$\mathbb{P}^{sm}_{\mathbf{w}_i} \supset X_i = \pi_2(p|^{-1}_{Y\setminus(\mathrm{Sing}(Y))}(N)) \cong p|^{-1}_{Y\setminus(\mathrm{Sing}(Y))}(N).$$

We have thus proved the theorem. □

For each X_i, vanishing of $H^j(X_i, \mathscr{O}_{X_i}) = 0$ for $j = 1, 2$ readily follows from the pfaffian resolution. Therefore X_5, X_7 and X_{10} are smooth Calabi–Yau threefolds. In the following we assign to each X_i a polarization H coming from the hyperplane class of the ambient space $\mathbb{P}_{\mathbf{w}_i}$.

Lemma 2.1. *The Hilbert series $H_{X_i}(t)$ of the pfaffian Calabi–Yau threefold X_i is given by the following:*

$$H_{X_5}(t) = \frac{1 + 3t^2 + t^4}{(1-t)^4} \qquad H_{X_7}(t) = \frac{1 + t + 3t^2 + t^3 + t^4}{(1-t)^4}$$
$$H_{X_{10}}(t) = \frac{1 + 2t + 4t^2 + 2t^3 + t^4}{(1-t)^4}.$$

Proof. As we already have a resolution of the structure sheaf of X_i, the claim easily follows from the additivity of Hilbert series and the formula:

$$H_{\mathbb{P}_{\mathbf{w}_i}}(\mathscr{O}_{\mathbb{P}_{\mathbf{w}_i}}(k))(t) = \frac{t^k}{\prod_{i=0}^{n}(1-t_i^{w_i})}. \qquad \square$$

Proposition 2.2. *The degree $\int_{X_i} H^3$ of the pfaffian Calabi–Yau threefold X_i is i.*

Proof. Let d be 3! times the leading coefficient of the Hilbert polynomial $P_{X_i}(t)$, which is readily available thanks to Lemma 2.1. Since a pfaffian variety is locally a complete intersection, the triple intersection $\int_{X_i} H^3$ coincides with d. $\square$

Proposition 2.3. *$\int_{X_i} c_2(X_i)\cdot H$ is given below for $i = 5, 7, 10$.*

X_i	X_5	X_7	X_{10}
$\int_{X_i} c_2(X_i)\cdot H$	38	46	52

Proof. Since we know that X_i is a smooth Calabi–Yau threefold, the Hirzebruch–Riemann–Roch Theorem gives

$$\chi(X_i, \mathscr{O}_{X_i}(H)) = \frac{1}{6}\deg(X_i) + \frac{1}{12}\int_{X_i} c_2(X_i)\cdot H.$$

By the Kodaira vanishing theorem, $H^j(X_i, \mathscr{O}_{X_i}(H)) = 0$ except for $j = 0$ and hence we have

$$\chi(X_i, \mathscr{O}_{X_i}(H)) = \dim H^0(X_i, \mathscr{O}_{X_i}(H)) = \dim H^0(\mathbb{P}_{\mathbf{w}_i}, \mathscr{O}_{\mathbb{P}_{\mathbf{w}_i}}(H)).$$

This determines $\int_{X_i} c_2(X_i)\cdot H$. $\square$

We will also need resolutions of the powers of pfaffian ideals, which are studied, for example, in [3]. Let R be a commutative ring. We consider a free R-module E of rank $2r+1$ and a generic alternating map $N : E^\vee \to E$, then we have the pfaffian resolution

$$0 \longrightarrow R \xrightarrow{P^t} E^\vee \xrightarrow{N} E \xrightarrow{P} R \longrightarrow R/I \longrightarrow 0.$$

Let $L_\lambda E$ be the representation of $GL(E)$ corresponding to a hook Young tableau λ (we refer the reader to [3] for the precise definition of $L_\lambda E$).

Lemma 2.2 ([3]). *There exists a resolution of I^2 of the form*

$$0 \longrightarrow L_{(2r-1)}E \cong \Lambda^{2r-1}E \xrightarrow{\vartheta_3} L_{(2r,1)}E \xrightarrow{\vartheta_2} L_{(2r+1,1^2)}E \cong S_2E \xrightarrow{\vartheta_1} I^2 \longrightarrow 0,$$

where ϑ_1 is the second symmetric power of P and ϑ_3 and ϑ_2 are induced by the map

$$\wedge^a E \otimes_R S^b E \to \wedge^{a+1} E \otimes_R S^{b+1} E, \quad u \otimes v \mapsto \sum_{i,j+1}^{2r+1} n_{i,j} e_i \wedge u \otimes v e_j,$$

where $N = \sum_{i,j=1}^{2r+1} n_{i,j} e_i \otimes e_j$ with respect to some fixed basis $e_1, \dots e_{2r+1}$ for E.

Lemma 2.3. *There exist resolutions of the ideal sheaves $\mathscr{I}^2_{X_5}$, $\mathscr{I}^2_{X_7}$ and $\mathscr{I}^2_{X_{10}}$ of the following form.*

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}_{\mathbf{w}_5}}(-12)^{\oplus 10} \xrightarrow{\vartheta_3} \mathscr{O}_{\mathbb{P}_{\mathbf{w}_5}}(-10)^{\oplus 24} \xrightarrow{\vartheta_2} \mathscr{O}_{\mathbb{P}_{\mathbf{w}_5}}(-8)^{\oplus 15} \xrightarrow{\vartheta_1} \mathscr{I}^2_{X_5} \longrightarrow 0,$$

$$\begin{aligned} 0 \longrightarrow\ & \mathscr{O}_{\mathbb{P}_{\mathbf{w}_7}}(-12) \oplus \mathscr{O}_{\mathbb{P}_{\mathbf{w}_7}}(-11)^{\oplus 6} \oplus \mathscr{O}_{\mathbb{P}_{\mathbf{w}_7}}(-10)^{\oplus 3} \\ & \xrightarrow{\vartheta_3} \mathscr{O}_{\mathbb{P}_{\mathbf{w}_7}}(-10)^{\oplus 6} \oplus \mathscr{O}_{\mathbb{P}_{\mathbf{w}_7}}(-9)^{\oplus 12} \oplus \mathscr{O}_{\mathbb{P}_{\mathbf{w}_7}}(-8)^{\oplus 6} \\ & \xrightarrow{\vartheta_2} \mathscr{O}_{\mathbb{P}_{\mathbf{w}_7}}(-8)^{\oplus 6} \oplus \mathscr{O}_{\mathbb{P}_{\mathbf{w}_7}}(-7)^{\oplus 6} \oplus \mathscr{O}_{\mathbb{P}_{\mathbf{w}_7}}(-6)^{\oplus 3} \xrightarrow{\vartheta_1} \mathscr{I}^2_{X_7} \longrightarrow 0, \end{aligned}$$

$$\begin{aligned} 0 \longrightarrow\ & \mathscr{O}_{\mathbb{P}_{\mathbf{w}_{10}}}(-10)^{\oplus 6} \oplus \mathscr{O}_{\mathbb{P}_{\mathbf{w}_{10}}}(-9)^{\oplus 4} \\ & \xrightarrow{\vartheta_3} \mathscr{O}_{\mathbb{P}_{\mathbf{w}_{10}}}(-9)^{\oplus 4} \oplus \mathscr{O}_{\mathbb{P}_{\mathbf{w}_{10}}}(-8)^{\oplus 16} \oplus \mathscr{O}_{\mathbb{P}_{\mathbf{w}_{10}}}(-7)^{\oplus 4} \\ & \xrightarrow{\vartheta_2} \mathscr{O}_{\mathbb{P}_{\mathbf{w}_{10}}}(-8) \oplus \mathscr{O}_{\mathbb{P}_{\mathbf{w}_{10}}}(-7)^{\oplus 4} \oplus \mathscr{O}_{\mathbb{P}_{\mathbf{w}_{10}}}(-6)^{\oplus 10} \\ & \xrightarrow{\vartheta_1} \mathscr{I}^2_{X_{10}} \longrightarrow 0. \end{aligned}$$

Here each term from left to right is regarded as 5×5 skew-symmetric, general but top left being zero, and symmetric matrices and the morphisms are given by $\vartheta_3(X) = NX - (NX)_{1,1}I$, $\vartheta_2(X) = XN + (XN)^t$, $\vartheta_1(X) = P^t XP$.

Proof. Let F be a free R-module of rank 5. We may suitably identify $\wedge^3 F$ with 5×5 skew-symmetric matrices, $L_{(4,1)}F$ with general but top left being zero matrices, and S^2F with symmetric matrices. By Lemma 2.2 it is straightforward to see that the morphisms ϑ_i are of the forms described in the claim. □

Theorem 2.4. *The Hodge numbers $h^{1,1}$ and $h^{1,2}$ of the pfaffian Calabi–Yau threefold X_i are given by the following table.*

X_i	X_5	X_7	X_{10}
$h^{1,1}$	1	1	1
$h^{1,2}$	51	61	59

Proof. In this proof, we simply write $X = X_i$ and $\mathbb{P}_{\mathbf{w}} = \mathbb{P}_{\mathbf{w}_i}$ for some $i = 5, 7, 10$. Twisting the pfaffian resolution of the structure sheaf, we know that $H^i(X, \mathscr{O}_X(-j)) \cong H^{i+3}(\mathbb{P}_{\mathbf{w}}, \mathscr{O}_{\mathbb{P}_{\mathbf{w}}}(-|\mathbf{w}| - j))$ $(j = 1, 2)$, which do not vanish only for $i = 3$. Restricting the weighted analogue of the Euler sequence to X, we obtain

$$0 \longrightarrow \Omega_{\mathbb{P}_{\mathbf{w}}} \otimes_{\mathscr{O}_{\mathbb{P}_{\mathbf{w}}}} \mathscr{O}_X \longrightarrow \bigoplus_{i=0}^{6} \mathscr{O}_X(-w_i) \longrightarrow \mathscr{O}_X \longrightarrow 0.$$

Since X is a smooth Calabi–Yau threefold, the long exact sequence induced by the short exact sequence above yields $H^i(X, \Omega_{\mathbb{P}_{\mathbf{w}}} \otimes_{\mathscr{O}_{\mathbb{P}_{\mathbf{w}}}} \mathscr{O}_X) = 0$ $(i = 0, 2)$, $H^1(X, \Omega_{\mathbb{P}_{\mathbf{w}}} \otimes_{\mathscr{O}_{\mathbb{P}_{\mathbf{w}}}} \mathscr{O}_X) \cong \mathbb{C}$ and the exact sequence

$$\begin{aligned} 0 \longrightarrow H^2(X, \mathscr{O}_X) \longrightarrow H^3(X, \Omega_{\mathbb{P}_{\mathbf{w}}} \otimes_{\mathscr{O}_{\mathbb{P}_{\mathbf{w}}}} \mathscr{O}_X) \\ \longrightarrow H^3(X, \bigoplus_{i=0}^{6} \mathscr{O}_X(-w_i)) \longrightarrow H^3(X, \mathscr{O}_X) \longrightarrow 0. \end{aligned}$$

We hence have $h^3(X, \Omega_{\mathbb{P}_{\mathbf{w}}} \otimes_{\mathscr{O}_{\mathbb{P}_{\mathbf{w}}}} \mathscr{O}_X) = h^3(X, \bigoplus_{i=0}^{6} \mathscr{O}_X(-w_i)) - 1$. From the resolution of $\mathscr{F}_\bullet \to \mathscr{I}_X^2$ in Lemma 2.3 we obtain

$$h^4(\mathbb{P}_{\mathbf{w}}, \mathscr{I}_X^2) - h^5(\mathbb{P}_{\mathbf{w}}, \mathscr{I}_X^2) = \sum_{i=1}^{3} (-1)^{i+1} h^6(\mathbb{P}_{\mathbf{w}}, \mathscr{F}_i) - h^6(\mathbb{P}_{\mathbf{w}}, \mathscr{I}_X^2).$$

The pfaffian resolution gives $H^4(\mathbb{P}_{\mathbf{w}}, \mathscr{I}_X) \cong H^6(\mathbb{P}_{\mathbf{w}}, \mathscr{O}_{\mathbb{P}_{\mathbf{w}}}(-|\mathbf{w}|)) \cong \mathbb{C}$ and $H^i(X, \mathscr{I}_X) = 0$ (otherwise). Since we assume that X is smooth, we have the short exact sequence of sheaves $0 \to \mathscr{I}_X^2 \to \mathscr{I}_X \to \mathscr{N}_{X/\mathbb{P}_{\mathbf{w}}}^{\vee} \to 0$. The induced long exact sequence gives $H^i(\mathbb{P}_{\mathbf{w}}, \mathscr{N}_{X/\mathbb{P}_{\mathbf{w}}}^{\vee}) = 0$ $(0 \leq i \leq 2)$ and $H^5(\mathbb{P}_{\mathbf{w}}, \mathscr{I}_X^2) = H^6(\mathbb{P}_{\mathbf{w}}, \mathscr{I}_X^2) = 0$. Moreover, we also have the short exact sequence

$$0 \longrightarrow H^3(\mathbb{P}_{\mathbf{w}}, \mathscr{N}_{X/\mathbb{P}_{\mathbf{w}}}^{\vee}) \longrightarrow H^4(\mathbb{P}_{\mathbf{w}}, \mathscr{I}_X^2) \longrightarrow H^4(\mathbb{P}_{\mathbf{w}}, \mathscr{I}_X) \longrightarrow 0.$$

It then follows immediately that

$$h^3(\mathbb{P}_{\mathbf{w}}, \mathscr{N}^\vee_{X/\mathbb{P}_{\mathbf{w}}}) = h^3(X, \mathscr{N}^\vee_{X/\mathbb{P}_{\mathbf{w}}}) = \sum_{i=1}^{3}(-1)^{i+1}h^6(\mathbb{P}_{\mathbf{w}}, \mathscr{F}_i).$$

On the other hand, the conormal exact sequence yields the exact sequence

$$0 \longrightarrow H^2(X, \Omega_X) \longrightarrow H^3(X, \mathscr{N}^\vee_{X/\mathbb{P}_{\mathbf{w}}}) \longrightarrow H^3(X, \Omega_{\mathbb{P}_{\mathbf{w}}} \otimes_{\mathscr{O}_{\mathbb{P}_{\mathbf{w}}}} \mathscr{O}_X) \longrightarrow 0$$

and $H^1(X, \Omega_X) \cong \mathbb{C}$. We finally establish the formula

$$\begin{aligned} h^2(X, \Omega_X) &= h^3(X, \mathscr{N}^\vee_{X/\mathbb{P}_{\mathbf{w}}}) - h^3(\Omega_{\mathbb{P}_{\mathbf{w}}} \otimes_{\mathscr{O}_{\mathbb{P}_{\mathbf{w}}}} \mathscr{O}_{X_{10}}) \\ &= \sum_{i=1}^{3}(-1)^{i+1}h^6(\mathbb{P}_{\mathbf{w}}, \mathscr{F}_i) - h^3(X, \bigoplus_{i=0}^{6} \mathscr{O}_X(-w_i)). \end{aligned}$$

Therefore, $h^{1,2}$ is determined by the explicit description of $\mathscr{F}_\bullet \to \mathscr{I}_X^2$ derived in Lemma 2.3. □

The existence of smooth Calabi–Yau threefolds X_5, X_7 and X_{10} with the computed topological invariants was previously conjectured by C. van Enckevort and D. van Straten from the viewpoint of Calabi–Yau equations in [19]. Regrettably it has not been settled yet whether they are simply connected or not.

2.3. Complete intersection type

In this subsection, we study complete intersections of pfaffian varieties and hypersufaces in weighted projective spaces. The main idea is to use pfaffian varieties as codimension 3 analogue of hypersurfaces in the ambient space.

Definition 2.4. Set $t = 1$ and $\mathscr{E}_{25} = \mathscr{O}_{\mathbb{P}^9}^{\oplus 5}$. Two generic global sections $N_1, N_2 \in H^0(\mathbb{P}^9, \wedge^2\mathscr{E}_{25}(1))$ define alternating morphisms $N_1, N_2 : \mathscr{E}_{25}^\vee(-1) \to \mathscr{E}_{25}$. Define X_{25} as the common degeneracy loci of N_1 and N_2.

Since the pfaffian sixfold associated to the data $(\mathscr{E}_{25}, N_i)$ is isomorphic to $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$, X_{25} may be seen as a complete intersection of two Grassmannians embedded in two different ways $i_j : \mathrm{Gr}(2,5) \hookrightarrow \mathbb{P}^9$ $(j = 1, 2)$.

$$X_{25} = i_1(\mathrm{Gr}(2,5)) \cap i_2(\mathrm{Gr}(2,5)).$$

Lemma 2.4. *Let X be the pfaffian variety associated to $(\mathscr{E}_{25}, N_i)$. Then $\mathscr{I}_X^2$ has the following resolution.*

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^9}(-6)^{\oplus 10} \longrightarrow \mathscr{O}_{\mathbb{P}^9}(-5)^{\oplus 24} \longrightarrow \mathscr{O}_{\mathbb{P}^9}(-4)^{\oplus 15} \longrightarrow \mathscr{I}_X^2 \longrightarrow 0.$$

Proof. This may be proved in a similar fashion to Lemma 2.3. □

Proposition 2.4. *X_{25} is a smooth Calabi–Yau threefold with the following topological invariants.*

$$h^{1,1} = 1, \; h^{1,2} = 51, \quad \int_{X_{25}} H^3 = 25, \quad \int_{X_{25}} c_2(X_{25}) \cdot H = 70.$$

Proof. The basic strategy is to divide the construction of X_{25} into two steps and repeat the similar argument in the previous subsection. The Grassmannian description guarantees the smoothness of X_{25}. It is also easy to see that X_{25} is a Calabi–Yau threefold. $\int_{X_{25}} H^3$ and $\int_{X_{25}} c_2(X_{25}) \cdot H$ may be determined in the same manner as before. The only non-trivial part is the determination of the Hodge numbers and we sketch a proof.

Let X be the pfaffian sixfold associated to $(\mathscr{E}_{25}, N_1)$, which is isomorphic to $\mathrm{Gr}(2,5)$. Then $Y = X_{25}$ is the pfaffian threefold associated to $(\mathscr{O}_X^{\oplus 5}, N_2)$. A straightforward computation with Lemma 2.4 shows that $h^3(Y, \mathscr{N}_{Y/X}^{\vee}) = 75$ and there is an exact sequence

$$\begin{aligned} 0 \longrightarrow H^2(X, \Omega_X \otimes_{\mathscr{O}_{\mathbb{P}_{\mathbf{w}}}} \mathscr{O}_Y) &\longrightarrow H^3(X, \mathscr{N}_{Y/X}^{\vee} \otimes_{\mathscr{O}_{\mathbb{P}_{\mathbf{w}}}} \mathscr{O}_Y) \\ &\longrightarrow H^3(X, \Omega_{\mathbb{P}^9} \otimes_{\mathscr{O}_{\mathbb{P}_{\mathbf{w}}}} \mathscr{O}_Y) \longrightarrow H^3(X, \Omega_X \otimes_{\mathscr{O}_{\mathbb{P}_{\mathbf{w}}}} \mathscr{O}_Y) \longrightarrow 0. \end{aligned}$$

Combining this with the long exact sequence induced from the conormal sequence, we obtain

$$\begin{aligned} h^2(Y, \Omega_Y) &= h^3(Y, \mathscr{N}_{Y/X}^{\vee}) + h^2(X, \Omega_X \otimes_{\mathscr{O}_{\mathbb{P}_{\mathbf{w}}}} \mathscr{O}_Y) - h^3(X, \Omega_X \otimes_{\mathscr{O}_{\mathbb{P}_{\mathbf{w}}}} \mathscr{O}_Y) \\ &= 2h^3(Y, \mathscr{N}_{Y/X}^{\vee}) - h^3(X, \Omega_{\mathbb{P}^9} \otimes_{\mathscr{O}_{\mathbb{P}_{\mathbf{w}}}} \mathscr{O}_Y) = 51. \end{aligned}$$ □

The existence of a smooth Calabi–Yau threefold with the computed topological invariants was also predicted in [19]. This Calabi–Yau equation has two maximally unipotent monodromy points of the same type and this may be explained by the self-duality of $\mathrm{Gr}(2,5)$.

Example 2.1. A complete intersection of a pfaffian variety associated to $\mathscr{F}_{10} = \mathscr{O}_{\mathbb{P}_{\mathbb{P}_{(1^7,2)}}}^{\oplus 5}$ and a quartic hypersurface in $\mathbb{P}_{(1^7,2)}$ yields a smooth Calabi–Yau threefold Y_{10} with the following topological invariants.

$$h^{1,1} = 1,\ h^{1,2} = 101, \quad \int_{Y_{10}} H^3 = 10, \quad \int_{Y_{10}} c_2(Y_{10}) \cdot H = 64.$$

We expect this to coincides with the double covering of Fano threefold in the list of C. Borcea [19].

Example 2.2. A complete intersection of a pfaffian variety associated to $\mathscr{F}_5 = \mathscr{O}_{\mathbb{P}_{(1^6,2,3)}}^{\oplus 5}$ and a sextic hypersurface in $\mathbb{P}_{(1^6,2,3)}$ yields a Calabi–Yau threefold Y_5. Assuming it is smooth, we can compute the topological invariants of Y_5.

$$h^{1,1} = 1,\ h^{1,2} = 156, \quad \int_{Y_5} H^3 = 5, \quad \int_{Y_5} c_2(Y_5) \cdot H = 62.$$

Although we could not find a smooth example of Y_5, the existence of a Calabi–Yau threefold with the above invariants was predicted in [19].

The author is grateful to Makoto Miura for indicating the existence of X_{25}, Y_5 and Y_{10}. There are many choices for locally free sheaves $\mathscr{E}$ of odd rank and weights $\mathbf{w}$ that yield Calabi–Yau threefolds, but there does not seem to exist any other smooth example that is not previously known. There is, nevertheless, an interesting example X_9, which we will analyze in Section 5.

3. Mirror symmetry for degree 13 pfaffian

3.1. Mirror partner

Our main aim of this subsection is to explicitly construct a mirror family of X_{13}. As X_{13} is not a complete intersection Calabi–Yau threefold, the Batyrev–Borisov mirror construction is not applicable. We shall first briefly review the tropical mirror construction proposed by J. Böhm. His construction reproduces the conventional Batyrev–Borisov mirror construction for complete intersection Calabi–Yau in toric Fano varieties. For a thorough treatment of the tropical mirror construction, we refer the reader to the original paper [2].

We begin our exposition by recalling the Batyrev–Borisov mirror construction, using the standard notation in [7]. Let M and $N = \mathrm{Hom}(M, \mathbb{Z})$

dual free abelian groups of rank d, and $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$ be the scalar extension of M and N respectively. Suppose that $\mathbb{P}_\Delta$ is an n-dimensional toric variety associated with the normal fan Σ_Δ of an integral polytope $\Delta \subset M$. The Cox ring $S = \mathbb{C}[x_r | r \in \Sigma_\Delta(1)]$ of $\mathbb{P}_\Delta$ is graded by Chow group $A_{n-1}(\mathbb{P}_\Delta)$ via the presentation sequence

$$0 \longrightarrow M \xrightarrow{A} \mathbb{Z}^{\Sigma_\Delta(1)} \longrightarrow A_{n-1}(\mathbb{P}_\Delta) \longrightarrow 0.$$

Suppose that Δ is reflexive and given a nef-partition $\Delta = \Delta_1 + \cdots + \Delta_k$ or equivalently $\Sigma_\Delta(1) = I_1 \cup \cdots \cup I_k$, then a complete intersection Calabi–Yau variety $X = V(I) \subset \mathbb{P}_\Delta$ of dimension $d = n - k$ is the zero locus of a generic section $(f_i)_{i=1}^k \in H^0(\mathbb{P}_\Delta, \bigoplus_{i=1}^k \mathscr{O}_{\mathbb{P}_\Delta}(E_i))$, where $\bigotimes_{i=1}^k \mathscr{O}_{\mathbb{P}_\Delta}(E_i) \cong -K_{\mathbb{P}_\Delta}$ corresponds to the nef-partition.

Define $\nabla_i = \mathrm{Conv.}(\{0\} \cup I_i)$ and the Minkowski sum $\nabla = \nabla_1 + \cdots + \nabla_k \subset N$. Then the following holds.

$$\Delta^* = \mathrm{Conv.}(\nabla_1 \cup \cdots \cup \nabla_k), \quad \nabla^* = \mathrm{Conv.}(\Delta_1 \cup \cdots \cup \Delta_k)$$

$\nabla = \nabla_1 + \cdots + \nabla_k$ is again reflexive and this gives a nef-partition of ∇, called the dual nef-partition. We define a complete intersection Calabi–Yau variety $\check{X}$ by using $\nabla \subset N$. Choosing a maximal projective subdivision of the normal fan of Δ and ∇, we get families $\mathscr{X}$ and $\check{\mathscr{X}}$ of Calabi–Yau varieties, which are conjectured to form a mirror pair. It is important to observe that giving a nef-partition is essentially equivalent to determining a union of toric varieties $X_0 = V(I_0)$ to which a general fiber of the family $\mathscr{X}$ maximally degenerates.

Let I_0 be a reduced monomial ideal in the Cox ring S. The degree 0 homomorphisms $\mathrm{Hom}(I_0, S/I_0)_0$ form a finite dimensional vector space. The torus $T = \mathbb{C}^{\Sigma_\Delta(1)}$ acts on S and thus on $\mathrm{Hom}(I_0, S/I_0)_0$ as well. So the vector space has a basis of deformations which are characters of T. Being of degree 0, any such character ρ corresponds to an element $m_\rho \in M \cong \mathrm{Im}(A)$. Given a flat family $\mathscr{X}$ of Calabi–Yau varieties in $\mathbb{P}_\Delta$ with special fiber X_0 such that the corresponding ideal $I_0 \subset S$ is a reduced monomial ideal. We represent the complex moduli space of a generic fiber X of $\mathscr{X}$ by a one-parameter family $\mathscr{X}'$ as follows. Take a T-invariant basis $\rho_1, \ldots, \rho_l \in \mathrm{Hom}(I_0, S/I_0)_0$ of the tangent space of the component of Hilbert scheme containing $\mathscr{X}$ at X_0 and assume that the tangent vector $v = \sum_i^l a_i \rho_i$ of $\mathscr{X}'$ at X_0 satisfies $a_i \neq 0$ for all i. The elements $\rho_1, \ldots, \rho_l$ correspond to elements $m_1, \ldots, m_l \in M$ of the lattice of monomials of $\mathbb{P}_\Delta$. The construction of the first order deformation of a mirror family $\overline{\mathscr{X}'}$ comes with a natural map via the interpretation of lattice points as deformations and divisors (see also the monomial divisor map discussed in [7]). Take the convex hull ∇^* of $m_1, \ldots, m_l$ and define $\mathbb{P}_\nabla$ the toric variety associated to the normal fan of the (not necessarily integral) polytope ∇. Then the toric divisors of $\mathbb{P}_\nabla$ and the induced divisors on a prospective mirror inside will correspond to deformations of X_0 in $\mathscr{X}$. The Bergman complex of X_0 defines a special fiber $\check{X}_0 \subset \mathbb{P}_\nabla$ and the first order deformations $\overline{\check{\mathscr{X}}}$ contributing to the mirror degeneration $\check{X}_0$ are constructed by the lattice points of the support of $\mathrm{Strata}(X_0)^* \subset \Delta^*$. It is sufficient to know a given family up to first order deformation in case of complete intersection or pfaffian varieties as their deformations are unobstructed.

To relate $\overline{\check{\mathscr{X}}}$ to the initial family $\mathscr{X}$. We need to blow-down the ambient toric variety $\mathbb{P}_\nabla$ to obtain an orbifold quotient of a weighted projective space $\mathbb{P}_{\mathbf{w}}/G$, contracting divisors which do not correspond to Fermat deformation of $\mathscr{X}$ (see [2] for the Fermat deformation). This blow-down is in general not unique and we choose appropriate one on case-by-case basis. The next one-parameter family was proposed as a mirror family of the degree 13 pfaffian Calabi–Yau threefold X_{13}. This family is obtained by deforming the special monomial fiber $\check{X}_0$ over $t = 0$.

Definition 3.1 (J. Böhm [2]). Define $\check{\mathscr{X}} = \{\check{X}_t\}_{t \in \mathbb{P}^1}$ as the one-parameter flat family of the pfaffian Calabi–Yau threefolds associated to the following special skew-symmetric 5×5 matrix N_t parametrized by $t \in \mathbb{P}^1$.

$$N_t = \begin{pmatrix} 0 & tx_0^2 & x_5x_6 & x_3x_4 & tx_2^2 \\ -tx_0^2 & 0 & t(x_3+x_4) & x_2 & x_1 \\ -x_5x_6 & -t(x_3+x_4) & 0 & tx_1 & x_0 \\ -x_3x_4 & -x_2 & -tx_1 & 0 & t(x_5+x_6) \\ -tx_2^2 & -x_1 & -x_0 & -t(x_5+x_6) & 0 \end{pmatrix}.$$

The family $\check{\mathscr{X}}$ is nothing but a special one-parameter family of degree 13 pfaffian Calabi–Yau threefolds. More explicitly, the pfaffian ideal sheaf $\mathscr{I}_{\check{X}}$ of $\check{\mathscr{X}}$ is generated by

$$\begin{aligned} P_1 &= x_0x_2 - tx_1^2 - t^2(x_3+x_4)(x_5+x_6), \\ P_2 &= x_0x_3x_4 - tx_5x_6(x_5+x_6) - t^2x_1x_2^2, \\ P_3 &= x_1x_3x_4 - tx_2^3 - t^2x_0^2(x_5+x_6), \\ P_4 &= x_1x_5x_6 - tx_0^3 - t^2x_2^2(x_3+x_4), \\ P_5 &= x_2x_5x_6 - tx_3x_4(x_3+x_4) - t^2x_0^2x_1. \end{aligned}$$

Since $\check{X}_t$ is originally contained in the toric variety $\mathbb{P}^6/\mathbb{Z}_{13}$, $\mathbb{Z}_{13}$ acts on $\check{X}_t$ as

$$\zeta_{13} \cdot [x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6] \\ = [x_0 : \zeta_{13}^4 x_1 : \zeta_{13}^8 x_2 : \zeta_{13}^{10} x_3 : \zeta_{13}^{10} x_4 : \zeta_{13}^{11} x_5 : \zeta_{13}^{11} x_6],$$

where $\zeta_{13} = \mathrm{e}^{\frac{2\pi \mathrm{i}}{13}}$. The fixed locus of the $\mathbb{Z}_{13}$-action consists of six points. Four of them $p_i = \{x_i \neq 0, x_j = 0 \ (j \neq i)\}$ $(i = 3, 4, 5, 6)$ are singular and and other two $p_{i,i+1} = \{x_i + x_{i+1} = 0, \ x_i \neq 0 \ x_j = 0 \ (j \neq i, i+1)\}$ $(i = 3, 5)$ are smooth.

Proposition 3.1. *For a generic choice of parameter $t \in \mathbb{P}^1$, the singular locus of $\check{X}_t$ consists of four points p_3, p_4, p_5 and p_6, each of which has multiplicity 12.*

Proof. Let us first work on the singular point p_3. In a neighborhood of p_3, since $P_{1,4,5} \neq 0$, $\check{X}_t$ is defined by the complete intersection of P_1, P_4 and P_5. Then it is easily seen that the germ of this singularity is isomorphic to a compound Du Val singularity given by the equation

$$f(x, y, z, w) = x^2 + y^3 + z^5 + zw^2 = 0, \ (x, y, z, w) \in \mathbb{C}^4.$$

Here the action of $\mathbb{Z}_{13}$ is given by $\zeta_{13} \cdot (x, y, z, w) = (\zeta_{13}^{11} x, \zeta_{13}^3 y, \zeta_{13}^7 z, \zeta_{13} w)$. The Milnor number of this singularity turns out to be 12. On the other hand, the Jacobian ideal of $\mathscr{I}_{\check{X}}$ has dimension 0 and degree 48 [1]. Due to symmetry, other singular points are of multiplicity 12 as well and hence we conclude the singular points are only $\{p_i\}_{i=3}^6$. □

Now we have a family of Calabi–Yau threefolds $\check{\mathscr{X}} = \{\check{X}_t\}_{t \in \mathbb{P}^1}$ parametrized by $t \in \mathbb{P}^1$. However, this is not an effective family because $\check{X}_t \cong \check{X}_{\zeta_7 t}$ for $\zeta_7 = \mathrm{e}^{\frac{2\pi \mathrm{i}}{7}}$ via the map

$$[x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6] \mapsto [x_0 : \zeta_7^3 x_1 : x_2, \zeta_7^6 x_3 : \zeta_7^6 x_4 : \zeta_7^6 x_5 : \zeta_7^6 x_6].$$

It is proved in [13] that the cD_4-singularity above does not admit any crepant resolution. However it turns out that the quotient $\{f(x, y, z, w) = 0\}/\mathbb{Z}_{13} \subset \mathbb{C}^4/\mathbb{Z}_{13}$ admits a crepant resolution; in her Ph.D. thesis [9],

[1]This is done by Macaulay2 [10].

I. Fausk found a crepant resolution $\widetilde{\check{X}_t/\mathbb{Z}_{13}}$ of $\check{X}_t/\mathbb{Z}_{13}$ (for a generic choice of parameter $t \in \mathbb{P}^1$) and verified the relation

$$\chi(\widetilde{\check{X}_t/\mathbb{Z}_{13}}) = 120 = -\chi(X_{13})$$

as mirror symmetry predicts. The definition of the family $\check{\mathscr{X}} = \{\check{X}_t\}_{t\in\mathbb{P}^1}$ shall also be justified by calculating its Picard–Fuchs equation in the following subsection.

3.2. Period map and Picard–Fuchs equation

Since X_{13} is a smooth Calabi–Yau threefold, it has a nowhere vanishing holomorphic 3-form up to multiplication with a non-zero constant. Although a pfaffian variety is in general not a complete intersection and there is no way of explicitly getting one in general, there is an analogous way of obtaining a global section of $\omega_{X_{13}} \cong \Omega^3_{X_{13}}$. For the sake of simplicity we restrict ourselves to the degree 13 pfaffian Calabi–Yau threefold X_{13}, but generalization to other pfaffian Calabi–Yau threefolds is straightforward.

Proposition 3.2 (E. Rødland [14]). *Let $\sigma \in \mathfrak{S}_5$ be an element of the symmetric group of degree 5. We have a nowhere vanishing global section of $\Omega^3_{X_{13}} \cong \mathscr{O}_{X_{13}}(-7) \otimes_{\mathscr{O}_{X_{13}}} \bigwedge^3 \mathscr{N}_{X_{13}/\mathbb{P}^6}$ given by*

$$\alpha = C_\sigma \operatorname{Res}_X \frac{P_{\sigma(1),\sigma(2),\sigma(3)}\Omega_0}{P_{\sigma(1)}P_{\sigma(2)}P_{\sigma(3)}},$$

where $C_\sigma \in \mathbb{C}^\times$ is some constant and

$$\Omega_0 = \frac{1}{(2\pi i)^6}\sum_{i=0}^{6}(-1)^i x_i dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_6.$$

This expression is independent of the choice of σ so long as the constant C_σ is chosen appropriately.

Proof. First of all, the invariance of the integrand under scaling of the coordinates can be checked for each σ and thus it is well defined as a rational 6-form on $\mathbb{P}^6$. On the affine open set $U_{\sigma(4),\sigma(5)} = \{P_{\sigma(1),\sigma(2),\sigma(3)} \neq 0\}$, $\{P_{\sigma(i)}\}_{i=1}^3$ forms a complete intersection and $P_{\sigma(1)}$, $P_{\sigma(2)}$ and $P_{\sigma(3)}$ can be seen as a part of the local coordinate. We may therefore assume that $\{P_{\sigma(1)}$, $P_{\sigma(2)}$,

$P_{\sigma(3)}$, x_4, x_5, x_6, x_7 form the coordinate of $\mathbb{A}^7$, i.e., $\frac{\partial(P_{\sigma(1)},P_{\sigma(2)},P_{\sigma(3)})}{\partial(x_1,x_2,x_3)} \neq 0$. Since we have

$$dP_{\sigma(1)} \wedge dP_{\sigma(2)} \wedge dP_{\sigma(3)} = \sum_{i<j<k} \frac{\partial(P_{\sigma(1)}, P_{\sigma(2)}, P_{\sigma(3)})}{\partial(x_i, x_j, x_k)} dx_i \wedge dx_j \wedge dx_k,$$

the residue theorem provides the following holomorphic 3-form on $X_{13}|_{U_{\sigma(4),\sigma(5)}}$.

$$\alpha = C_\sigma \frac{P_{\sigma(1),\sigma(2),\sigma(3)}}{(2\pi i)^3 \frac{\partial(P_{\sigma(1)},P_{\sigma(2)},P_{\sigma(3)})}{\partial(x_0,x_1,x_2)}} \sum_{i=4}^{7} (-1)^i x_i dx_4 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_7.$$

On the other hand, $P_{\sigma(1),\sigma(2),\sigma(3)}$ vanishes if and only if $\{P_{\sigma(i)}\}_{i=1}^3$ does not form a complete intersection. Therefore the Jacobian $\frac{\partial(P_{\sigma(1)},P_{\sigma(2)},P_{\sigma(3)})}{\partial(x_1,x_2,x_3)}$ divides $P_{\sigma(1),\sigma(2),\sigma(3)}$, and α is globally defined. Furthermore, the local description of α shows that it is nowhere vanishing on X_{13}. Since X_{13} is Calabi–Yau threefold, $\Omega^3_{X_{13}}$ is trivial and the expression of α for each ν is different merely by a constant. □

Although a general member $\check{X}_t$ of the family is singular, a nowhere vanishing holomorphic 3-form α may be defined on the non-singular locus $\check{X}_t \setminus \mathrm{Sing}(\check{X}_t)$. Then the period integral of the family $\check{\mathscr{X}}$ is defined as usual since integration can be performed on 3-cycle away from the singular locus. For the sake of convenience we shall work with the singular threefold $\check{X}_t$ in $\mathbb{P}^6$ instead of a crepant resolution $\widetilde{\check{X}_t/\mathbb{Z}_{13}}$. Note that Picard–Fuchs equations invariant under resolution of singularities. It is also preserved under taking the quotient of the threefold by a finite group under which the 3-from α is invariant. More precisely, we can perform the integration on $\check{X}_t$ and obtain the genuine period integral by dividing by 13.

At the origin $t = 0$ the threefold $\check{X}_t$ decomposes into thirteen 3-dimensional planes and hence the origin is a good candidate for a maximally unipotent monodromy point of the one-parameter family $\check{\mathscr{X}}$. The fundamental period integral $\Phi_0(t)$ (defined up to multiplication by a non-zero scalar) can be obtained by integrating a holomorphic 3-form on a torus cycle that vanishes at the origin $t = 0$. In what follows, we always assume that the fundamental period integral is normalized by setting $\Phi_0(0) = 1$. Fix a 3-dimensional plane H defined by $H = \{x_1 = x_2 = x_3 = 0\}$. On the domain $H \setminus (\{x_4 = 0\} \cup \{x_5 = 0\} \cup \{x_6 = 0\})$, there is a cycle given by $|\frac{x_4}{x_0}| = |\frac{x_5}{x_0}| = |\frac{x_6}{x_0}| = \epsilon$, which extends to a 3-dimensional torus cycle $\gamma \in H_3(\check{X}_t, \mathbb{C})$ for $|t| \ll 1$.

Theorem 3.1. *Let $\Phi_0(t) = \int_\gamma \alpha$ be the fundamental period integral of the one-parameter family $\{\check{X}_t\}_{t\in\mathbb{P}^1}$. Then $\Phi_0(t)$ has the following expansion near the origin $t=0$.*

$$\Phi_0(t) = \sum_{n=0}^{\infty} \binom{2n}{n}^2 \sum_{k=0}^{n} \binom{2n+k}{n} \binom{n}{k}^2 t^{7n}.$$

It can be observed that $\phi = t^7 \in \mathbb{P}^1$ is the genuine moduli parameter of $\{\check{X}_t\}_{t\in\mathbb{P}^1}$. We can thus write $\Phi_0(\phi)$ and $\{\check{X}_\phi\}_{\phi\in\mathbb{P}^1}$. Moreover, the Picard–Fuchs operator $\mathscr{D}$ of the family is

$$\begin{aligned}\mathscr{D} = {} & 13^2\Theta^4 - \phi(59397\Theta^4 + 117546\Theta^3 + 86827\Theta^2 + 28054\Theta + 3380) \\ & + 2^4\phi^2(6386\Theta^4 - 1774\Theta^3 - 17898\Theta^2 - 11596\Theta - 2119) \\ & + 2^8\phi^3(67\Theta^4 + 1248\Theta^3 + 1091\Theta^2 + 312\Theta + 26) - 2^{12}\phi^4(2\Theta+1)^4,\end{aligned}$$

where Θ is the Euler operator $\phi\frac{\partial}{\partial\phi}$.

Proof. Let us work on the affine open subset $U_0 = \{x_0 = 1\}$. Fix a permutation, say $\nu = (2,4,5,1,3)$. For the sake of convenience we define $a_{i,j}$ to be

$$(a_{i,j}) = \begin{pmatrix} \frac{x_5^2x_6}{x_3x_4}t & \frac{x_5x_6^2}{x_3x_4}t & \frac{x_1x_2^2}{x_3x_4}t^2 \\ \frac{1}{x_1x_5x_6}t & \frac{x_2^2x_3}{x_1x_5x_6}t^2 & \frac{x_2^2x_4}{x_1x_5x_6}t^2 \\ \frac{x_3^2x_4}{x_2x_5x_6}t & \frac{x_3x_4^2}{x_2x_5x_6}t & \frac{x_1}{x_2x_5x_6}t^2 \end{pmatrix}.$$

Then, near the origin, the period integral is described as

$$\Phi_0(t) = \int_\gamma \mathrm{Res}_X \frac{P_{2,4,5}}{P_2P_4P_5}\Omega_0 = \int_\Gamma \frac{1}{\prod_{i=1,3,4}(1-\sum_{j=1}^3 a_{i,j})} \cdot \bigwedge_{k=1}^{6} \frac{dx_k}{2\pi i x_k},$$

where $\Gamma = \{|x_i| = \epsilon \ (i=1,\ldots,6)\}$. We then expand the denominator of the integrand as a power series in terms of $a_{i,j}$. The only terms that contribute the period integral is the products $\prod a_{i,j}^{n_{i,j}}$ that is independent of x_i. Suppose $\prod a_{i,j}^{n_{i,j}}$ ($n_{i,j} \in \mathbb{Z}_{\geq 0}$) does not contain any x_i, then $\prod a_{i,j}^{n_{i,j}}$ is a product of

$$\begin{aligned} t_1 &= a_{1,1}a_{1,2}a_{2,3}a_{3,1}a_{3,3} = t^7 = \phi, \\ t_2 &= a_{1,1}a_{1,2}a_{2,2}a_{3,2}a_{3,3} = t^7 = \phi, \\ t_3 &= a_{1,1}a_{1,2}a_{1,3}a_{2,1}a_{3,1}a_{3,2} = t^7 = \phi \end{aligned}$$

and it is easily checked that this expression is unique. Therefore the period integral $\Phi_0(t)$ is essentially a function of $\phi = t^7$, and henceforth we write

$\Phi_0(\phi)$. Note that this is compatible with the observation that $\check{X}_t \cong \check{X}_{\zeta_7 t}$. Since

$$t_1^a t_2^b t_3^c = a_{1,1}^{a+b+c} a_{1,2}^{a+b+c} a_{1,3}^c a_{3,1}^c a_{3,2}^b a_{3,3}^a a_{4,1}^{a+c} a_{4,2}^{b+c} a_{4,3}^{a+c}$$

and the coefficient of $\prod a_{i,j}^{n_{i,j}}$ appearing as an integrand of $\Phi_0(\phi)$ is given by $\prod_{i=1,3,4} \binom{n_{i,1}+n_{i,2}+n_{i,3}}{n_{i,1},n_{i,2},n_{i,3}}$, the period integral $\Phi_0(\phi)$ can be summarized as

$$\begin{aligned}
\Phi_0(\phi) &= \sum_{n=0}^{\infty} \sum_{a+b+c=n} \binom{2a+2b+3c}{a+b+c,a+b+c,c} \\
&\quad \times \binom{a+b+c}{c,b,a} \binom{2a+2b+2c}{a+c,b+c,a+b} \phi^n \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{l=0}^{n-k} \binom{2n+k}{n}\binom{n+k}{n}\binom{n}{k}\binom{n-k}{l}\binom{2n}{n-l}\binom{n+l}{k+l}\phi^n \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{2n+k}{n}\binom{n+k}{n}\binom{n}{k} \sum_{l=0}^{n-k}\binom{n-k}{l}\binom{2n}{n+k}\binom{n+k}{n-l}\phi^n \\
&= \sum_{n=0}^{\infty} \binom{2n}{n} \sum_{k=0}^{n} \binom{n+k}{n}\binom{2n}{n}^2 \sum_{l=0}^{n-k}\binom{n-k}{l}\binom{n+k}{n-l}\phi^n \\
&= \sum_{n=0}^{\infty} \binom{2n}{n}^2 \sum_{k=0}^{n} \binom{2n+k}{n}\binom{n}{k}^2 \phi^n,
\end{aligned}$$

where we used relations

$$\binom{2n}{n-l}\binom{n+l}{k+l} = \binom{2n}{n+k}\binom{n+k}{n-l}, \quad \binom{n+k}{n}\binom{2n}{n+k} = \binom{2n}{n}\binom{n}{k}$$
$$= \sum_{l=0}^{n-k}\binom{n-k}{l}\binom{n+k}{n-l} = \binom{2n}{k}.$$

The period integral $\Phi_0(\phi)$ coincides with the power series solution of the Calabi–Yau Equation of No. (99) in [18], whose Picard–Fuchs equation is exactly what we are looking for. □

Corollary 3.1. *Let α_1, α_2 be the roots of $256\phi^2 + 349\phi - 1 = 0$. Then Riemann's P-Scheme of $\mathscr{D}$ is given by the following.*

$$\left\{ \begin{array}{c|ccccc} \phi & 0 & \alpha_1 & \alpha_2 & 13/16 & \infty \\ \hline \rho_1 & 0 & 0 & 0 & 0 & 1/2 \\ \hline \rho_2 & 0 & 1 & 1 & 1 & 1/2 \\ \hline \rho_3 & 0 & 1 & 1 & 3 & 1/2 \\ \hline \rho_4 & 0 & 2 & 2 & 4 & 1/2 \end{array} \right\}$$

The conifold points are α_1 and α_2.

$\mathscr{D}$ has a maximally unipotent monodromy point at the origin $\phi = 0 \in \mathbb{P}^1$ as expected. Observe that ∞ is not a maximally unipotent monodromy point in the usual sense but very similar to that. This point will be further discussed later.

3.3. Picard–Fuchs equation around 0 and curve counting

We now briefly review Gromov–Witten and BPS invariants. Let X be a Calabi–Yau threefold. We define $N^g_\beta(X) = \int_{[\overline{M}_{g,0}(X,\beta)]^{vir}} 1$ as the 0-point genus g Gromov–Witten invariant of X in the curve class $\beta \in H_2(X, \mathbb{Z})$. Here $[\overline{M}_{g,0}(X,\beta)]^{vir}$ is the virtual fundamental class of the coarse moduli space of stable maps $\overline{M}_{g,0}(X,\beta)$ of expected complex dimension $(1-g)(\dim X - 3) + \int_\beta c_1(X) = 0$.

Definition 3.2. Define BPS invariants $n^g_\beta(X)$ by the formula

$$\sum_{\beta\neq 0}\sum_{g\geq 0} N^g_\beta(X)\lambda^{2g-2}q^\beta = \sum_{\beta\neq 0}\sum_{g\geq 0} n^g_\beta(X)\sum_{k>0}\frac{1}{k}\left(2\sin\left(\frac{kt}{2}\right)\right)^{2g-2} q^{k\beta}.$$

LHS is the generating function of Gromov–Witten invariants of $N^g_\beta(X)$ of X in all genera and all nonzero curve classes. Matching the coefficients of the two series yields equations determining $n^g_\beta(X)$ recursively in terms $N^g_\beta(X)$.

As the Picard–Fuchs operator $\mathscr{D}$ of $\check{\mathscr{X}} = \{\check{X}_\phi\}_{\phi\in\mathbb{P}^1}$ has a maximally unipotent monodromy point at $\phi = 0$, we can define the mirror map $q(\phi)$ there and calculate the conjectural genus g BPS invariants n^g_d $(d \in \mathbb{N})$ of X_{13}. In what follows, we will work on the case $g = 0, 1$ for simplicity. Since it is a routine work to calculate the mirror map and the Yukawa couplings, we omit the detail of those computations below. For a complete description, see for example [6, 7].

A good integral basis of $H_3(\check{X}_\phi, \mathbb{Z})$, which corresponds to the normalized solutions of $\mathscr{D}$ below, determines a canonical coordinate q of the complexified Kähler moduli of X_{13}. At the origin $\phi = 0$ we have two normalized solutions of $\mathscr{D}$, $\Phi_0(\phi)$ and $\Phi_1(\phi)$. $\Phi_0(\phi)$ is the fundamental period integral normalized by setting $\Phi_0(0) = 1$. The other period integral $\Phi_1(\phi)$ is of the form

$$\Phi_1(\phi) = (\log(\phi))\Phi_0(\phi) + \Psi(\phi),$$

where $\Psi(\phi)$ is regular at $\phi = 0$ and $\Psi(0) = 0$. The mirror map is then defined by $q(\phi) = \exp(\frac{\Phi_1(\phi)}{\Phi_0(\phi)})$ and can be expanded as

$$q(\phi) = \phi + 86\phi^2 + 12901\phi^3 + 2460318\phi^4 + 536898026\phi^5 + \cdots$$

Let us recall the definition of the quantum corrected Yukawa coupling

$$K_{ttt}(q) = \int_{X_{13}} H^3 + \left(q\frac{d}{dq}\right)^3 \sum_{d \geq 1} N_d(X_{13})q^d \in \mathbb{Q}[[q]].$$

Using the mirror map $q(\phi)$, we may compute the quantum corrected Yukawa coupling

$$K_{ttt}(q) = 13 + 647q + 129975q^2 + 25451198q^3 + 5134100919q^4 + \cdots .$$

We shall also apply the following BCOV formula [1] for genus $g = 1$ Gromov–Witten potential $F_1(\phi)$ to X_{13}.

$$F_1(\phi) = \frac{1}{2}\log\left\{\frac{\Phi_0(\phi)^{\frac{\chi(X_{13})}{12}-3-h^{1,1}}(q\frac{d\phi}{dq})}{\mathrm{disc}(\phi)^{\frac{1}{6}}\phi^{\frac{\int_{X_{13}} c_2(X_{13})\cdot H}{12}+1}}\right\}.$$

Here we assumed that the exponent of the discriminant is 1/6 as usual. The genus $g = 0, 1$ BPS invariants are given in the following table.

d	n_d^0	n_d^1
1	647	0
2	16166	0
3	942613	176
4	80218296	164696
5	8418215008	78309518

Since we may write down explicit equations defining a degree 13 Calabi–Yau threefold X_{13}, we may in principle count the number of degree d rational curves on general X_{13} and check that it coincides with n_d^0 as follows. Let us write a map $\mathbb{P}^1 \to \mathbb{P}^6$ as

$$[u:v] \mapsto \left[\sum_{i=0}^{d} a_i u^i v^{d-i} : \sum_{i=0}^{d} b_i u^i v^{d-i} : \cdots : \sum_{i=0}^{d} g_i u^i v^{d-i} \right].$$

Then the image of this map is contained in X_{13} if and only if $P_i(\mathbf{x}(u,v)) = 0$ $(i = 1, \ldots, 5)$ for all $[u:v] \in \mathbb{P}^1$. This containment condition yields dependent equations in the variables $a_0, a_1, \ldots, g_{d-1}, g_d$. Since what we want to count is not maps from $\mathbb{P}^1$ to X_{13} but rational curves in X_{13}, we must kill $\mathrm{Aut}(\mathbb{P}^1)$ by normalizing the map. As the proportional polynomials also define the same map, the number of the independent parameters turns out to be $7(d+1) - 3 - 1$. We predict that the ideal generated by the dependent equations has dimension 0 and the degree n_d^0.

When $d = 1$, we have nineteen dependent equations in ten variables. For a generic choice of N, we may suitably normalize the map and compute the degree of the ideal to get the answer 647, as mirror symmetry predicts[2]. Explicit calculation is available upon request.

3.4. Picard–Fuchs equation around ∞

In his thesis [14], E. Rødland constructed a mirror family for the degree 14 pfaffian Calabi–Yau threefolds $X_{14} = \mathrm{Pfaff}(7) \cap \mathbb{P}^6$ by orbifolding the initial threefolds. His work is notable from two aspects. Firstly, this is the first example of mirror symmetry for a non-complete intersection Calabi–Yau threefold with $h^{1,1} = 1$. Secondly, the Picard–Fuchs equation of the mirror family $\check{X}_{14}$ has two maximally unipotent monodromy points; $\infty \in \mathbb{P}^1$ corresponds to the initial X_{14} and $0 \in \mathbb{P}^1$ corresponds to $\mathrm{Gr}(2,7) \cap \check{\mathbb{P}}^{13} \subset \check{\mathbb{P}}^{20}$, which is the projective dual of $\mathrm{Pfaff}(7) \cap \mathbb{P}^6 \subset \mathbb{P}^{20}$. In fact, this pair is the first example of a derived equivalence between non-birational Calabi–Yau threefolds [4]. K. Hori and D. Tong presented how to describe these Calabi–Yau threefolds with GLSM using a non-abelian gauge group in two dimensions [11]. The link between the pfaffian X_{14} and the Grassmannian sections $\mathrm{Gr}(2,7)_{1^7}$ was further studied in [12], in which a thought-provoking phenomenon in the higher genus Gromov–Witten invariants is discovered.

[2]This is done by Macaulay2 [10].

In our case, ∞ is apparently an interesting point of $\mathscr{D}$ and it seems worthwhile to analyze it in detail[3]. We first see that it makes sense to call ∞ a maximally unipotent monodromy point and calculate *virtual invariants* there. Changing the coordinate from ϕ to $1/\phi$ and transforming the gauge by $\sqrt{\phi}$ amount to the change, $\Theta \to -\Theta \to -\Theta - 1/2$, in the Euler operator. Let us also change the variable from ϕ to $-\phi$ for later use. Then the Picard–Fuchs operator becomes

$$\begin{aligned}\mathscr{D}' = {} & 2^{20}\Theta^4 - 2^8\phi(1072\Theta^4 - 17824\Theta^3 - 10888\Theta^2 - 1976\Theta - 145)\\ & + 2^5\phi^2(51088\Theta^4 + 116368\Theta^3 - 45264\Theta^2 - 14228\Theta - 1397)\\ & + 13\phi^3(73104\Theta^4 + 1536\Theta^3 - 488\Theta^2 + 384\Theta + 97) + 13^2\phi^4(2\Theta+1)^4.\end{aligned}$$

Although $\mathscr{D}'$ has a maximally unipotent monodromy point at $\phi = 0$, the integrality of mirror symmetry breaks. It is observed that there is a preferable choice of variable $\tilde{\phi} = \phi/2^{16}$, with which the integrality of the normalized period, the mirror map and *virtual BPS invariants* (see the next paragraph) still holds[4]. The Picard–Fuchs operator $\tilde{\mathscr{D}}$ with respect to this new variable is

$$\begin{aligned}\tilde{\mathscr{D}} = {} & \tilde{\Theta}^4 - 2^4\tilde{\phi}(1072\tilde{\Theta}^4 - 17824\tilde{\Theta}^3 - 10888\tilde{\Theta}^2 - 1976\tilde{\Theta} - 145)\\ & + 2^{17}\tilde{\phi}^2(51088\tilde{\Theta}^4 + 116368\tilde{\Theta}^3 - 45264\tilde{\Theta}^2 - 14228\tilde{\Theta} - 1397)\\ & + 13\cdot 2^{28}\tilde{\phi}^3(73104\tilde{\Theta}^4 + 1536\tilde{\Theta}^3 - 488\tilde{\Theta}^2 + 384\tilde{\Theta} + 97)\\ & + 13^2 2^{44}\tilde{\phi}^4(2\tilde{\Theta}+1)^4.\end{aligned}$$

This is the Calabi–Yau Equation of No. (225) in [19]. However, the positive Euler number corresponding to this equation [18] excludes a geometric interpretation by a Calabi–Yau threefold with $h^{1,1} = 1$.

Since the new operator $\tilde{\mathscr{D}}$ has a maximally unipotent monodromy point at the origin $\tilde{\phi} = 0$, it makes sense to speak of *virtual BPS invariants* $\tilde{n}_d^0$ ($d \in \mathbb{N}$) corresponding to the origin,

[3]This type of special point also appears when we consider, for instance, a Calabi–Yau threefold $\mathbb{P}^7_{2^4}$. The quantum differential equation of $\mathbb{P}^7_{2^4}$ is $\theta^4 - 2^4q(2\theta+1)^4$.

[4]S. Hosono pointed out that the change of the sign and the coefficient $1/2^{16}$ can be justified by the analytic continuation of the local solutions about 0 to ∞.

d	$\tilde{n}^0_d$
1	$70944a$
2	$107300032a$
3	$3707752060576a$
4	$66327758316665792a$
5	$1970671594871618215520a$

where a is supposed to be *the degree of virtual geometry* at the origin[5]. We hope to understand the meaning of this sequence of numbers, which may not come from the conventional Calabi–Yau geometry. It would also be interesting to extend the Hori–Tong GLSM description [11] to our pfaffian Calabi–Yau threefolds.

3.5. Conclusion

It is classically known that the monodromy matrix of the quantum differential equation with respect to an appropriate basis is expressed in terms of the geometric invariants of the underlying Calabi–Yau threefold with one dimensional moduli. In what follows, we assume that the origin is a maximally unipotent monodromy point. Then $\int_X H^3$ and $\int_X c_2(X) \cdot H$ can be read off from the monodromy around the origin and the conifold point. After it is analytically continued to the origin, the conifold-period $z_2(t)$ has the form

$$z_2(t) = \frac{\int_X H^3}{6} t^3 + \frac{\int_X c_2(X) \cdot H}{24} t + \frac{\int_X c_3(X)}{(2\pi i)^3} \zeta(3) + \sum_{d=1}^{\infty} N^0_d(X) q^d,$$

where $q = \mathrm{e}^{2\pi \mathrm{i} t}$. So we obtain $\int_X c_3(X)$ as well and have consistency check of $\int_X c_2(X) \cdot H$. It was numerically verified in [19] that the invariants computed from the differential equation $\mathscr{D}$ coincides with the fundamental geometric invariants $\int_{X_{13}} H^3$, $\int_{X_{13}} c_2(X_{13}) \cdot H$ and $\int_{X_{13}} c_3(X_{13})$. Our claim that $\widetilde{X_t/\mathbb{Z}_{13}}$ is a mirror partner of X_{13} is based on the coincidence the fundamental geometric invariants mentioned above. An alternative and preferable approach to the verification of mirror symmetry is direct computation of the Gromov–Witten invariants of X_{13}, such as [16].

Conjecture 1. *The BPS invariants of the degree 13 pfaffian Calabi–Yau threefold X_{13} coincides with the numbers n^g_d ($d \in \mathbb{N}$) we calculated above, as mirror symmetry predicts.*

[5] a is expected to be 1 in [19].

4. Mirror symmetry for degree 5, 7, 10 pfaffians

4.1. Mirror partners

Inspired by the mirror symmetry for X_{13}, we will construct candidate mirror families of the Calabi–Yau threefolds we obtained in Section 2, except the degree 25 case. Since we do not know a systematic way of finding an appropriate family, we omit the finding procedure of the families in this paper. Some computation on singularities in this section are carried out with the aid of Macaulay 2. See also Appendix for the conjectural BPS invariants computed by the special families of Calabi–Yau threefolds in this section.

Definition 4.1. Define $\check{\mathscr{X}}_5 = \{\check{X}_{5,t}\}_{t\in\mathbb{P}^1}$ as the one-parameter family of degree 5 pfaffian Calabi–Yau threefolds $\check{X}_{5,t}$ associated to the following special skew-symmetric 5×5 matrix $N_{5,t}$ parametrized by $t\in\mathbb{P}^1$.

$$N_{5,t} = \begin{pmatrix} 0 & tx_6 & x_4 & x_0x_1 & tx_5 \\ -tx_6 & 0 & t(x_0^2+x_1^2) & x_5 & x_2x_3 \\ -x_4 & -t(x_0^2+x_1^2) & 0 & t(x_2^2+x_3^2) & x_6 \\ -x_0x_1 & -x_5 & -t(x_2^2+x_3^2) & 0 & tx_4 \\ -tx_5 & -x_2x_3 & -x_6 & -tx_4 & 0 \end{pmatrix}.$$

This is our candidate mirror family of the degree 5 pfaffian Calabi–Yau threefold X_5. Observe that the family degenerates to a union of toric varieties with normal crossings at the origin $t=0$. In fact, we choose $\check{X}_{5,0}$ as a candidate of the fiber over a maximally unipotent monodromy point and deform it so that the first order deformation resembles a Fermat variety. Then the deformation automatically extends to higher orders, so long as it is a pfaffian Calabi–Yau threefold.

$\mathbb{Z}_{10}$ acts on $\check{X}_{5,t}$ as

$$\begin{aligned}\zeta_{10}\cdot[x_0:x_1:x_2:x_3:x_4:x_5:x_6] \\ = [x_0:x_1:\zeta_{10}x_2:\zeta_{10}x_3:\zeta_{10}^4x_4:\zeta_{10}^6x_5:\zeta_{10}^8x_6],\end{aligned}$$

where $\zeta_{10}=\mathrm{e}^{\frac{2\pi\mathrm{i}}{10}}$. There are four singular points $p_i^{\pm}=\{x_i\pm\sqrt{-1}x_{i+1}=0,\ x_i\neq 0\ x_j=0\ (j\neq i,i+1)\}$ $(i=0,2)$ with multiplicity 2. Each singularity is locally isomorphic to

$$f_5(x,y,z,w)=x^3+y^2+zw=0,\ (x,y,z,w)\in\mathbb{C}^4,$$

where the action of $\mathbb{Z}_{10}$ is given by $\zeta_{10} \cdot (x, y, z, w) = (\zeta_{10}^4 x, \zeta_{10}^6 y, \zeta_{10} z, \zeta_{10} w)$. Since $\dim(\mathrm{Sing}(\check{X}_{5,t})) = 0$ and $\deg(\mathrm{Sing}(\check{X}_{5,t})) = 8$, there is no more singular point on $\check{X}_{5,t}$.

It is observed that $\check{\mathscr{X}}_5 = \{\check{X}_{5,t}\}_{t\in\mathbb{P}^1}$ is not an effective family, as $\check{X}_{5,t} \cong \check{X}_{5,\zeta_{10}t}$ for $\zeta_{10} = \mathrm{e}^{\frac{2\pi \mathrm{i}}{10}}$ via the map

$$[x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6] \mapsto [x_0 : x_1 : x_2 : \zeta_{10}^5 x_3 : \zeta_{10}^4 x_4 : \zeta_{10}^7 x_5 : \zeta_{10}^9 x_6].$$

Proposition 4.1. *The fundamental period integral of the family $\{\check{X}_{5,t}\}_{t\in\mathbb{P}^1}$ around $t = 0$ is given by*

$$\Phi_0(\phi) = \sum_{n=0}^{\infty} \binom{2n}{n} \sum_{k}^{n} \binom{n}{k} \binom{n+k}{n} \binom{2n+2k}{n+k} \binom{2n+k}{2n-k} \phi^n$$

and the Picard–Fuchs operator $\mathscr{D}_5$ is given by

$$\begin{aligned}\mathscr{D}_5 = \Theta^4 &- 2^2\phi(500\Theta^4 + 976\Theta^3 + 677\Theta^2 + 189\Theta + 19)\\ &+ 2^4\phi^2(3968\Theta^4 + 3968\Theta^3 - 1336\Theta^2 - 1164\Theta - 177)\\ &- 2^{10}\phi^3(500\Theta^4 + 24\Theta^3 - 37\Theta^2 + 6\Theta + 3) + 2^{12}\phi^4(2\Theta+1)^4,\end{aligned}$$

where we put $\phi = t^{10}$.

Proof. The computation is almost identical to the degree 13 pfaffian case. □

This Picard–Fuchs equation $\mathscr{D}_5$ is the Calabi–Yau Equation of No. (302) listed in [18]. The topological invariants computed from $\mathscr{D}_5$ coincide with those of X_5 as expected.

Corollary 4.1. *Let α_1, α_2 be the roots of $256\phi^2 - 1968\phi + 1 = 0$, then Riemann's P-Scheme of $\mathscr{D}_5$ is given by the following.*

ϕ	0	α_1	α_2	$1/16$	∞
ρ_1	0	0	0	0	1/2
ρ_2	0	1	1	1	1/2
ρ_3	0	1	1	3	1/2
ρ_4	0	2	2	4	1/2

Interestingly enough, the Picard–Fuchs operator $\mathscr{D}_5$ has two special points, namely 0 and ∞. There is again a preferable new variable $\tilde{\phi} = \phi/2^8$ and the Picard–Fuchs equation around ∞ with respect to the new variable is identical to the initial one. Therefore, it seems that both 0 and ∞ correspond to the degree 5 Calabi–Yau threefold X_5 in this case.

Definition 4.2. Define $\check{\mathscr{X}}_7 = \{\check{X}_{7,t}\}_{t\in\mathbb{P}^1}$ as the one-parameter family of degree 7 pfaffian Calabi–Yau threefolds $\check{X}_{7,t}$ associated to the following special skew-symmetric 5×5 matrix $N_{7,t}$ parametrized by $t \in \mathbb{P}^1$.

$$N_{7,t} = \begin{pmatrix} 0 & tx_2^3 & x_0x_1 & x_5 & tx_6 \\ -tx_2^3 & 0 & tx_5 & x_6 & x_3x_4 \\ -x_0x_1 & -tx_5 & 0 & t(x_3+x_4) & x_2 \\ -x_5 & -x_6 & -t(x_3+x_4) & 0 & t(x_0+x_1) \\ -tx_6 & -x_3x_4 & -x_2 & -t(x_0+x_1) & 0 \end{pmatrix}$$

$\mathbb{Z}_7$ acts on $\check{X}_{7,t}$ as

$$\zeta_7 \cdot [x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6] = [x_0 : x_1 : \zeta_7^4 x_2 : \zeta_7 x_3 : \zeta_7 x_4 : \zeta_7^3 x_5 : \zeta_7^6 x_6],$$

where $\zeta_7 = \mathrm{e}^{\frac{2\pi\mathrm{i}}{7}}$. There are six fixed points, independent of the value of parameter t. We have $\dim(\mathrm{Sing}(\check{X}_{7,t})) = 1$ and $\deg(\mathrm{Sing}(\check{X}_{7,t})) = 4$. $\mathrm{Sing}(\check{X}_{7,t})$ passes through two of the above fixed points, namely $p_{i,i+1} = \{x_i + x_{i+1} = 0,\ x_i \neq 0\ x_j = 0\ (j \neq i, i+1)\}$ $(i = 0, 3)$.

It is observed that $\check{\mathscr{X}}_7 = \{\check{X}_{7,t}\}_{t\in\mathbb{P}^1}$ is not an effective family, as $\check{X}_{7,t} \cong \check{X}_{7,\zeta_9 t}$ for $\zeta_9 = \mathrm{e}^{\frac{2\pi\mathrm{i}}{9}}$ via the map

$$[x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6] \mapsto [x_0 : x_1 : \zeta_9^2 x_2 : x_3 : x_4 : \zeta_9^8 x_5 : \zeta_9^8 x_6].$$

Proposition 4.2. *The fundamental period integral of the family $\{\check{X}_{7,t}\}_{t\in\mathbb{P}^1}$ around $t = 0$ is given by*

$$\Phi_0(\phi) = \sum_{n=0}^{\infty} \binom{2n}{n} \sum_{k=0}^{2n} \binom{n+k}{k} \binom{2n}{k}^2 \phi^n$$

and the Picard–Fuchs operator $\mathscr{D}_7$ is given by

$$\begin{aligned} \mathscr{D}_7 = {} & 7^2\Theta^4 - 2\cdot 3\cdot 7\phi(1272\Theta^4 + 2508\Theta^3 + 1779\Theta^2 + 525\Theta + 56) \\ & + 2^2 3\phi^2(43704\Theta^4 + 38088\Theta^3 - 25757\Theta^2 - 20608\Theta - 3360) \\ & - 2^4 3^3\phi^3(2736\Theta^4 - 1512\Theta^3 - 1672\Theta^2 - 357\Theta - 14) \\ & - 2^6 3^5 \phi^4 (2\Theta+1)^2(3\Theta+1)(3\Theta+2), \end{aligned}$$

where we put $\phi = t^9$.

Proof. The computation is almost identical to the degree 13 pfaffian case. □

This Picard–Fuchs equation $\mathscr{D}_7$ is the Calabi–Yau Equation of No.(109) listed in [18]. The topological invariants computed from $\mathscr{D}_7$ coincide with those of X_7 as expected.

Corollary 4.2. *Let α_1, α_2 be the roots of $432\phi^2 + 1080\phi - 1 = 0$, then Riemann's P-Scheme of $\mathscr{D}_7$ is given by the following.*

$$\left\{\begin{array}{c|ccccc} \phi & 0 & \alpha_1 & \alpha_2 & 7/36 & \infty \\ \hline \rho_1 & 0 & 0 & 0 & 0 & 1/2 \\ \hline \rho_2 & 0 & 1 & 1 & 1 & 1/2 \\ \hline \rho_3 & 0 & 1 & 1 & 3 & 1/3 \\ \hline \rho_4 & 0 & 2 & 2 & 4 & 2/3 \end{array}\right\}$$

0 *is the only maximally unipotent monodromy point of $\mathscr{D}_7$.*

Definition 4.3. Define $\check{\mathscr{X}}_{10} = \{\check{X}_{10,t}\}_{t\in\mathbb{P}^1}$ as the one-parameter family of degree 10 pfaffian Calabi–Yau threefolds $\check{X}_{10,t}$ associated to the following special skew-symmetric 5×5 matrix $N_{10,t}$ parametrized by $t \in \mathbb{P}^1$.

$$N_{10,t} = \begin{pmatrix} 0 & tx_4^2 & x_0x_1 & x_6 & t(x_2+x_3) \\ -tx_4^2 & 0 & tx_6 & x_2x_3 & x_5 \\ -x_0x_1 & -tx_6 & 0 & tx_5^2 & x_4 \\ -x_6 & -x_2x_3 & -tx_5^2 & 0 & t(x_0+x_1) \\ -t(x_2+x_3) & -x_5 & -x_4 & -t(x_0+x_1) & 0 \end{pmatrix}$$

$\mathbb{Z}_{10}$ acts on $\check{X}_{10,t}$ as

$$\begin{aligned} &\zeta_{10}\cdot[x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6] \\ &\qquad = [x_0 : x_1 : \zeta_{10}^6 x_2 : \zeta_{10}^6 x_3 : \zeta_{10}^9 x_4 : \zeta_{10}^7 x_5 : \zeta_{10}^1 x_6], \end{aligned}$$

where $\zeta_{10} = \mathrm{e}^{\frac{2\pi \mathrm{i}}{10}}$. There are six singular points under the $\mathbb{Z}_{10}$-action. Four of them $p_i = \{x_i \neq 0, x_j = 0\ (j \neq i)\}\ (i = 0,1,2,3)$ appear with multiplicity 12 and other two $p_{i,i+1} = \{x_i + x_{i+1} = 0,\ x_i \neq 0\ x_j = 0\ (j \neq i, i+1)\}\ (i = 0,2)$ with multiplicity 7. The singularity at p_i is locally isomorphic to

$$f_{10}(x,y,z,w) = x^4 + y^2 + z^2w + zw^2 = 0\ (x,y,z,w) \in \mathbb{C}^4,$$

where the action of $\mathbb{Z}_{10}$ is given by $\zeta_{10} \cdot (x, y, z, w) = (\zeta_{10}^7 x, \zeta_{10}^9 y, \zeta_{10}^6 z, \zeta_{10}^6 w)$. The singularity at $p_{i,i+1}$ is locally isomorphic to

$$g_{10}(x, y, z, w) = x^8 + y^2 + zw = 0,$$

where $\zeta_{10} \cdot (x, y, z, w) = (\zeta_{10}^9 x, \zeta_{10} y, \zeta_{10}^6 z, \zeta_{10}^6 w)$. Since $\dim(\mathrm{Sing}(\check{X}_{10})) = 0$ and $\deg(\mathrm{Sing}(\check{X}_{10})) = 62$, there is no more singular point on $\check{X}_{10,t}$.

It is observed that $\check{\mathscr{X}}_{10} = \{\check{X}_{10,t}\}_{t\in\mathbb{P}^1}$ is not an effective family, as $\check{X}_{10,t} \cong X_{10,\zeta_{16}^2 t}$ for $\zeta_{16} = \mathrm{e}^{\frac{2\pi i}{16}}$ via the map

$$[x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6] \mapsto [x_0 : x_1 : x_2 : x_3 : \zeta_{16}^3 x_4 : \zeta_{16}^3 x_5 : \zeta_{16}^7 x_6].$$

Proposition 4.3. *The fundamental period integral of the family $\{\check{X}_{10,t}\}_{t\in\mathbb{P}^1}$ around $t = 0$ is given by*

$$\Phi_0(\phi) = \sum_{n=0}^{\infty} \binom{2n}{n} \sum_{k=0}^{2n} (-1)^{k+n} \binom{2n}{k}^4 \phi^n$$

and the Picard–Fuchs operator $\mathscr{D}_{10}$ is given by

$$\begin{aligned}\mathscr{D}_{10} = {} & 5^2\Theta^4 - 2^2 5\phi(688\Theta^4 + 1352\Theta^3 + 981\Theta^2 + 305\Theta + 35) \\ & + 2^4\phi^2(5856\Theta^4 + 7008\Theta^3 + 96\Theta^2 - 1260\Theta - 265) \\ & - 2^{10}\phi^3(176\Theta^4 + 120\Theta^3 + 69\Theta^2 + 30\Theta + 5) + 2^{12}\phi^4(2\Theta + 1)^4,\end{aligned}$$

where we put $\phi = t^8$.

Proof. The computation is almost identical to the degree 13 pfaffian case. □

This Picard–Fuchs equation $\mathscr{D}_{10}$ is the Calabi–Yau Equation of No. (263) listed in [18]. The topological invariants computed from $\mathscr{D}_{10}$ coincide with those of X_{10} as expected.

Corollary 4.3. *Let α_1, α_2 be the roots of $256\phi^2 - 544\phi + 1 = 0$, then Riemann's P-Scheme of $\mathscr{D}_{10}$ is*

$$\left\{\begin{array}{c|ccccc} \phi & 0 & \alpha_1 & \alpha_2 & 5/16 & \infty \\ \hline \rho_1 & 0 & 0 & 0 & 0 & 1/2 \\ \hline \rho_2 & 0 & 1 & 1 & 1 & 1/2 \\ \hline \rho_3 & 0 & 1 & 1 & 3 & 1/2 \\ \hline \rho_4 & 0 & 2 & 2 & 4 & 1/2 \end{array}\right\}.$$

The Picard–Fuchs operator $\mathscr{D}_{10}$ has two special points 0 and ∞. The Picard–Fuchs operator $\tilde{\mathscr{D}}_{10}$ around ∞ with respect to the new variable $\tilde{\phi} = 1/(\phi 2^{12})$ is

$$\begin{aligned}\tilde{\mathscr{D}}_{10} = {} & \tilde{\Theta}^4 - 2^4\tilde{\phi}(704\tilde{\Theta}^4 + 928\tilde{\Theta}^3 + 612\tilde{\Theta}^2 + 148\tilde{\Theta} + 13) \\ & + 2^{12}\tilde{\phi}^2(5856\tilde{\Theta}^4 + 4704\tilde{\Theta}^3 - 1632\tilde{\Theta}^2 - 972\tilde{\Theta} - 121) \\ & - 2^{20}5\tilde{\phi}^3(2752\tilde{\Theta}^4 + 96\tilde{\Theta}^3 - 60\tilde{\Theta}^2 + 24\tilde{\Theta} + 7) + 2^{28}5^2\tilde{\phi}^4(2\tilde{\Theta} + 1)^4.\end{aligned}$$

This is the Calabi–Yau Equation of No. (271) listed in [18]. It is unknown whether or not there exists a Calabi–Yau threefold with topological invariants predicted in [19].

A general member of the one-parameter families constructed in this section is quite singular just as the degree 13 case. It is still unsettled whether or not a general member admits any crepant resolution. Hence, our verification of mirror phenomena is again based on the monodromy calculation of the Picard–Fuchs equation of the our special one-parameter family [19].

Conjecture 2. *The BPS invariants of the pfaffian Calabi–Yau threefold* X_i $(i = 5, 7, 10)$ *coincides with the numbers* n_d^g $(d \in \mathbb{N})$ *listed in Appendix as mirror symmetry predicts.*

5. Another example

Although we could not find any more (new) smooth pfaffian Calabi–Yau threefolds in weighted projective spaces, there is an interesting example X_9 defined as follows.

Definition 5.1. Define a degree 9 Calabi–Yau threefold $X_9 \subset \mathbb{P}_{(1^6,2)}$ as a pfaffian variety associated to the locally free sheaf $\mathscr{E}_9 = \mathscr{O}_{\mathbb{P}_{(1^6,2)}}(2) \oplus \mathscr{O}_{\mathbb{P}_{(1^6,2)}}(1)^{\oplus 2} \oplus \mathscr{O}_{\mathbb{P}_{(1^6,2)}}^{\oplus 2}$.

X_9 turns out to be isomorphic to a complete intersection Calabi–Yau threefold $\mathbb{P}^5_{3^2}$. Therefore, X_9 admits a twofold interpretation. If we regard X_9 as a pfaffian Calabi–Yau threefold, we can apply to it the orbifold mirror construction we studied in the preceding sections.

Definition 5.2. Define $\check{\mathscr{X}}_9 = \{\check{X}_{9,t}\}_{t \in \mathbb{P}^1}$ as the one-parameter family of degree 9 pfaffian Calabi–Yau threefolds $\check{X}_{9,t}$ associated to the following special skew-symmetric 5×5 matrix $N_{9,t}$ parametrized by $t \in \mathbb{P}^1$.

$$N_{9,t} = \begin{pmatrix} 0 & x_0x_1x_2 & 0 & tx_6 & x_3x_4 \\ -x_0x_1x_2 & 0 & x_6 & t(x_3+x_4) & tx_5 \\ 0 & -x_6 & 0 & x_5 & t(x_0+x_1+x_2) \\ -tx_6 & -t(x_3+x_4) & -x_5 & 0 & 1 \\ x_3x_4 & -tx_5 & -(x_0+x_1+x_2) & -1 & 0 \end{pmatrix}$$

$\check{X}_{9,t}$ is actually a complete intersection Calabi–Yau threefold defined by the quadric P_0 and the two cubics P_1 and P_2. This one-parameter family $\check{\mathscr{X}}_9$ is not isomorphic to the conventional mirror family of $\mathbb{P}^5_{3^2}$ defined by

$$\begin{gathered} x_0x_1x_2 + t(x_3^3+x_4^3+x_5^3) \\ x_3x_4x_5 + t(x_0^3+x_1^3+x_2^3). \end{gathered}$$

A general member of this family is a smooth Calabi–Yau threefold, while a general member of $\check{\mathscr{X}}_9$ is singular along a curve. It is, however, observed that the two families share the same normalized period integral and Picard–Fuchs operator

$$\Phi_0(\phi) = \sum_{n=0}^{\infty} \binom{3n}{n}^2 \binom{2n}{n}^2 \phi^n, \quad \mathscr{D}_9 = \Theta^4 - 3^2\phi(3\Theta+1)^2(3\Theta+2)^2,$$

where we put $\phi = t^8$. These two families may bridge the two mirror constructions.

It is classically known that a mirror family for a given family Calabi–Yau threefolds can be constructed by taking special loci of the initial family, which are not necessarily on the Fermat points emphasized by the initial construction inspired by the conformal field theories. For more details we refer the reader to [8] and the reference therein.

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Appendix

X_5		
d	n_d^0	n_d^1
1	2220	0
2	285520	460
3	95254820	873240
4	47164553340	1498922677
5	28906372957040	2306959237408

X_7		
d	n_d^0	n_d^1
1	1434	0
2	103026	26
3	18676572	53076
4	4988009280	65171063
5	1646787631350	63899034076

X_{10}		
d	n_d^0	n_d^1
1	888	0
2	33084	1
3	3003816	2496
4	399931068	2089393
5	65736977760	1210006912
d	$\tilde{n}_d^0$	$\tilde{n}_d^1$
1	$2400a$	40
2	$1829880a$	138040
3	$2956977632a$	687719624
4	$7117422755016a$	3822563543952
5	$21319886408804640a$	21893828822263288

[6]

[6] a is expected to be 2 in [19].

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