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BPS invariants of $\mathcal{N} = 4$ gauge theory on Hirzebruch surfaces

Jan Manschot

Generating functions of BPS invariants for $\mathcal{N} = 4$ $U(r)$ gauge theory on a Hirzebruch surface with $r \leq 3$ are computed. The BPS invariants provide the Betti numbers of moduli spaces of semi-stable sheaves. The generating functions for $r = 2$ are expressed in terms of higher level Appell functions for a certain polarization of the surface. The level corresponds to the self-intersection of the base curve of the Hirzebruch surface. The non-holomorphic functions are determined, which added to the holomorphic generating functions provide functions, which transform as a modular form.

1. Introduction

The study of supersymmetric spectra of field theories and supergravities is a major subject in theoretical physics and also mathematics. The BPS invariant counts the number of BPS states weighted by a sign. From a more mathematical perspective, the index corresponds to topological invariants (e.g., the Euler number or the Betti numbers) of a moduli space of objects (of an appropriate category) corresponding to the BPS states.

One of the seminal papers on BPS invariants of supersymmetric gauge theory on a Kähler surface is [33] by Vafa and Witten. They show that the topologically twisted path integral localizes on the instanton solutions, and equals the generating function of the Euler numbers of instanton moduli spaces, whose natural compactification is the moduli space of semi-stable sheaves. One of their main motivations was to test the strong-weak coupling duality [29] or S-duality, which acts by $SL(2, \mathbb{Z})$ transformations on the theory. The coupling constant g and the theta angle θ combine to the modular parameter $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$. S-duality suggests that the generating function of the BPS invariants (3.2) should exhibit modular properties if the gauge group is $SU(r)$ or $U(r)$. They tested this in various cases, for example for sheaves with rank $r = 1$ [11], and $r = 2$ on $\mathbb{P}^2$ [18, 35, 36]. The generating functions for rank 1 were found to be genuine (weakly) holomorphic modular forms. However, the generating functions for rank 2 transform only approximately as a modular form. These functions are (mixed) mock modular forms, i.e., functions which do transform as a modular form only after the addition of a non-holomorphic "completion" [39].

Vafa and Witten [33] has inspired many results in later years. In particular, for $r=2$ the dependence of the BPS invariant on the polarization $J\in H^2(S,\mathbb{Z})$ was included in the generating functions using indefinite theta functions [13]. Moreover, the reduced modular properties for $r\geq 2$ were understood physically as a "holomorphic anomaly" [1,28].

Although modularity has proven useful for various computations [13,28, 38], physical expectations for $r>2$ could never be rigorously tested since generating functions for $r>2$ were not known. This was one of the motivations for [26], which computed the generating functions of refined BPS invariants for $r=3$ on $\mathbb{P}^2$ and its blow-up $\tilde{\mathbb{P}}^2$, which is the Hirzebruch surface Σ_1. A convenient property of Σ_1 is that the BPS invariants vanish for certain choices of the first Chern class and choice of polarization. Wall-crossing and the blow-up formula [37] provide then the invariants in the other chambers and for $\mathbb{P}^2$.[1]

This paper generalizes the computation of the generating function $\mathcal{Z}_r(z,\rho,\tau;\Sigma_1,J)$ of BPS invariants for $r\leq 3$ of [26] to more general Hirzebruch surfaces Σ_ℓ, where $-\ell$ is the self-intersection number of the base curve of Σ_ℓ. The arguments $z\in\mathbb{C}$, $\rho\in H^2(\Sigma_\ell,\mathbb{C})$ and $\tau\in\mathcal{H}$ in $\mathcal{Z}_r(z,\rho,\tau;\Sigma_\ell,J)$ are generating variables for the Betti numbers of the moduli spaces, and first & second Chern classes of the sheaves, respectively.

Section 3.1 derives expressions for the generating functions with $r=2$ in terms of indefinite theta functions [13] and Appell functions of level ℓ [3,31]. The non-holomorphic but modular completed functions $\widehat{\mathcal{Z}}_2(z,\rho,\tau;\Sigma_\ell,J)$ are determined for $z\in\mathbb{C}$ (generating function of Betti numbers) as well as $z=\frac{1}{2}$ (Euler numbers). Due to the presence of these terms the action of the heat operator D_r on the generating function $\widehat{\mathcal{Z}}_r(\rho,\tau;\Sigma_\ell,J)$ (3.1) of Euler numbers does not vanish, which is known in the physics literature as a "holomorphic anomaly". A novel result of the paper is that $D_2\widehat{\mathcal{Z}}_2(\rho,\tau;\Sigma_\ell,J)$ in general consists of two terms (3.11):

$$D_2\widehat{\mathcal{Z}}_2(\rho,\tau;\Sigma_\ell,J)=C_2(\operatorname{Im}\tau,J)\,\mathcal{Z}_1(\rho,\tau,\Sigma_\ell)^2+R_2(\rho,\tau;\Sigma_\ell,J), \tag{1.1}$$

where $C_2(\operatorname{Im}\tau,J)$ is a simple function of $\operatorname{Im}\tau$ and J. The appearance of $\mathcal{Z}_1(\rho,z,\tau,\Sigma_\ell)^2$ has been conjectured and discussed in the literature before [4, 28, 33], but the additional term $R_2(\rho,z,\tau;\Sigma_\ell,J)$ is novel. Remarkably, the

[1]Kool and Weist [21, 34] computed earlier generating functions for the Euler numbers for rank 3 using different techniques.

additional term vanishes for special choices of J, in particular for $J=-K_\ell$ where K_ℓ is the canonical class of Σ_ℓ.[2]

Another important property of the non-holomorphic completion is that it renders $\mathcal{Z}_r(\rho,z,\tau;\Sigma_\ell,J)$ continuous as a function of the polarization J [24], which is expected of a physical path integral. Although a more intrinsic derivation of the anomaly in physics or algebraic geometry is desirable, this gives already important insights.

Section 3.3 presents the holomorphic generating function for $r=3$ (3.12) for $r=3$ and presents tables 1–3 with the Betti numbers for $\ell=1$. The modular properties of $\mathcal{Z}_3(\rho,z,\tau;\Sigma_\ell,J)$ are much more intricate then for $r=2$, and will be discussed elsewhere [5].

The outline of the paper is as follows. Section 2 reviews the necessary properties of sheaves and Hirzebruch surfaces, including BPS invariants and their wall-crossing. Section 3 defines the generating functions and gives explicit expressions for $r=1,2$ and 3. The non-holomorphic terms and the holomorphic anomaly are determined for $r=2$ in Section 3.2, and for $r=3$ tables with Betti numbers are presented in Section 3.3.

2. Sheaves on Hirzebruch surfaces

The Gieseker–Maruyama moduli space of semi-stable sheaves with rank r on S is the natural compactification of the moduli space of instantons with gauge group $U(r)$, i.e., anti-self-dual solutions for the field strength: $*F=-F$. The Chern classes of the sheaf are determined by the topological classes of the instanton:

$$c_1=\frac{\mathrm{i}}{2\pi}\mathrm{Tr}\,F,\quad c_2-\frac{1}{2}c_1^2=\frac{1}{8\pi^2}\mathrm{Tr}\,F\wedge F.$$

Most of the following is phrased in the more algebraic language of sheaves, since this setting is most suitable for explicit computations.

2.1. Sheaves and stability

The Chern character of a sheaf F on a surface S is given by $\mathrm{ch}(F)=r(F)+c_1(F)+\frac{1}{2}c_1(F)^2-c_2(F)$ in terms of the rank $r(F)$ and its Chern classes $c_1(F)$ and $c_2(F)$. The vector $\Gamma(F):=(r(F),\mathrm{ch}_1(F),\mathrm{ch}_2(F))$ summarizes the topological properties of F. Other frequently occurring quantities are the determinant $\Delta(F)=\frac{1}{r(F)}(c_2(F)-\frac{r(F)-1}{2r(F)}c_1(F)^2)$, and $\mu(F)=c_1(F)/r(F)$.

[2]Note for $\ell>2$, $-K_\ell$ does not lie in the ample cone of Σ_ℓ and is therefore not a permissible choice for J.

Let $0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = F$ be a filtration of the sheaf F. The quotients are denoted by $E_i = F_i/F_{i-1}$ with $\Gamma_i = \Gamma(E_i)$. With the above notation, the discriminant $\Delta(F)$ is given in terms of the topological quantities of E_i and F_i by

(2.1)
$$\Delta(F) = \sum_{i=1}^{s} \frac{r(E_i)}{r(F)} \Delta(E_i) - \frac{1}{2r(F)} \sum_{i=2}^{s} \frac{r(F_{i-1})\, r(F_i)}{r(E_i)} \left(\mu(F_{i-1}) - \mu(F_i)\right)^2 .$$

The notion of a moduli space for sheaves is only well defined after the introduction of a stability condition. To this end let $C(S) \in H^2(S,\mathbb{Z})$ be the ample cone of S. Given a choice $J \in C(S)$, a sheaf F is called μ-stable if for every subsheaf $F' \subset F$, $\mu(F') \cdot J < \mu(F) \cdot J$, and μ-semi-stable if $\mu(F') \cdot J \leq \mu(F) \cdot J$. A wall of marginal stability W is a (codimension 1) subspace of $C(S)$, such that $(\mu(F') - \mu(F)) \cdot J = 0$, but $(\mu(F') - \mu(F)) \cdot J \neq 0$ away from W.

Let S be a Kähler surface, whose intersection pairing on $H^2(S,\mathbb{Z})$ has signature $(1, b_2 - 1)$. Since at a wall, $(\mu_2 - \mu_1) \cdot J = 0$ and $J^2 > 0$, we have $(\mu_2 - \mu_1)^2 < 0$. Therefore, the set of semi-stable filtrations for F, with $\Delta_i \geq 0$ is finite. The ample class J provides natural projections $\mathbf{c}_\pm$ for an element $\mathbf{c} \in H^2(S,\mathbb{Z})$ to the positive and negative definite subspaces of $H^2(S,\mathbb{R})$:

$$\mathbf{c}_+ = \frac{\mathbf{c} \cdot J\, J}{J^2}, \quad \mathbf{c}_- = \mathbf{c} - \mathbf{c}_+. \tag{2.2}$$

2.2. Some properties of ruled surfaces

A ruled surface is a surface Σ together with a surjective morphism $\pi : \Sigma \to C$ to a curve C, such that the fibre Σ_y is isomorphic to $\mathbb{P}^1$ for every point $y \in C$. Let f be the fibre of π, then $H_2(\Sigma,\mathbb{Z}) = \mathbb{Z}C \oplus \mathbb{Z}f$, with intersection numbers $C^2 = -\ell < 0$, $f^2 = 0$ and $C \cdot f = 1$. The canonical class is $K_\Sigma = -2C + (2g - 2 - \ell)f$. The holomorphic Euler characteristic $\chi(\mathcal{O}_\Sigma)$ is for a ruled surface $1 - g$. An ample class is parametrized by $J_{m,n} = m(C + \ell f) + nf \in C(\Sigma)$ with $m, n > 0$. The following only considers surfaces with $g = 0$, these are known as rationally ruled surfaces or Hirzebruch surfaces. They are denoted by Σ_ℓ and furthermore K_ℓ denotes the canonical class.

To learn about the set of semi-stable sheaves on Σ_ℓ for $J \in C(S)$, it is useful to first consider the restriction of the sheaves on Σ_ℓ to f. Namely the restriction to $E_{|f}$ is stable if and only if E is μ-stable for $J = J_{0,1}$ and in the adjacent chamber [14]. However, since every bundle of rank ≥ 2 on $\mathbb{P}^1$ is a sum of line bundles, there are no stable bundles with $r \geq 2$ on $\mathbb{P}^1$. Therefore, the BPS invariant $\Omega(\Gamma, w; J)$ (defined in the next subsection) vanishes for $\Gamma = (r(F), -C - \alpha f, \mathrm{ch}_2)$ with $r(F) \geq 2$ and $\alpha = 0, 1$.

2.3. Invariants and wall-crossing

The moduli space $\mathcal{M}_J(\Gamma)$ of semi-stable sheaves (with respect to the ample class J) whose rank and Chern classes are determined by Γ has complex dimension:

$$\dim_{\mathbb{C}} \mathcal{M}_J(\Gamma) = 2r^2\Delta - r^2\chi(\mathcal{O}_S) + 1. \tag{2.3}$$

To define the refined BPS invariants $\Omega(\Gamma, w; J)$ in an informal way, let $p(X, s) = \sum_{i=0}^{2\dim_{\mathbb{C}}(X)} b_i s^i$, with b_i the Betti numbers $b_i = \dim H^2(X, \mathbb{Z})$, be the Poincaré polynomial of a compact complex manifold X. Then:

$$\Omega(\Gamma, w; J) := \frac{w^{-\dim_{\mathbb{C}} \mathcal{M}_J(\Gamma)}}{w - w^{-1}}\, p(\mathcal{M}_J(\Gamma), w). \tag{2.4}$$

The rational refined invariants are defined by [26]:

$$\bar{\Omega}(\Gamma, w; J) = \sum_{m|\Gamma} \frac{\Omega(\Gamma/m, -(-w)^m; J)}{m}.$$

See [27] for a physical motivation of these rational invariants and [19, 30] for mathematical motivations. The numerical BPS invariant $\Omega(\Gamma; J)$ follows from the $\Omega(\Gamma, w; J)$ by:

$$\Omega(\Gamma; J) = \lim_{w \to -1} (w - w^{-1})\, \Omega(\Gamma, w; J), \tag{2.5}$$

and similarly for the rational invariants $\bar{\Omega}(\Gamma; J)$.

A crucial tool for the computation of the generating functions in Section 3 is the wall-crossing formula, which provides the change $\Delta\Omega(\Gamma; J_{\mathcal{C}} \to J_{\mathcal{C}'})$ across walls of marginal stability. Yoshioka [36] gives as criterion for his wall-crossing formula for $r = 2$ that $K_\ell \cdot J < 0$, which holds for any ℓ and $J \in C(\Sigma_\ell)$. For $r = 3$, more complicated wall-crossings appear, in particular walls where the slope of three rank 1 sheaves with different c_1 become equal. Physical arguments suggest that for these walls one could use the wall-crossing formulas of Kontsevich–Soibelman [19, 20] or Joyce and Song [16] since they are shown to hold in both supergravity and field theory [2, 6, 10].

These wall-crossing formulas are derived for Donaldson–Thomas invariants, which are defined for six-dimensional gauge theory on a Calabi–Yau three-fold [8]. The mathematical justification for the use of these wall-crossing formulas for sheaves on surfaces is therefore not well established. Joyce [15] gives as criterion for the applicability that K_S^{-1} must be numerically effective (i.e., $-K_S \cdot D \geq 0$ for any curve in $D \in H^2(S, \mathbb{Z})$). This would exclude the Hirzebruch surfaces with $\ell > 2$. The generating function (3.12) for $r = 3$ is consistent with the wall-crossing formulas for DT-invariants and in agreement with previous results in the literature for $\ell = 1$, but in view of the above requires at least for $\ell > 2$ further justification.

Keeping in mind these comments, I continue by giving the explicit change of the invariants in case of primitive wall-crossing. To this end, define the following quantities:

$$\langle \Gamma_1, \Gamma_2 \rangle = r_1 r_2 (\mu_2 - \mu_1) \cdot K_S, \quad \mathcal{I}(\Gamma_1, \Gamma_2; J) = r_1 r_2 (\mu_2 - \mu_1) \cdot J.$$

The change $\Delta\Omega(\Gamma_1 + \Gamma_2, w; J_{\mathcal{C}} \to J_{\mathcal{C}'})$ for Γ_1 and Γ_2 primitive is [6, 19, 37]

(2.6)
$$\begin{aligned} \Delta\Omega(\Gamma, w; J_{\mathcal{C}} \to J_{\mathcal{C}'}) = &-\frac{1}{2} \left(\mathrm{sgn}(\mathcal{I}(\Gamma_1, \Gamma_2; J_{\mathcal{C}'})) - \mathrm{sgn}(\mathcal{I}(\Gamma_1, \Gamma_2; J_{\mathcal{C}})) \right) \\ &\times \left(w^{\langle \Gamma_1, \Gamma_2 \rangle} - w^{-\langle \Gamma_1, \Gamma_2 \rangle} \right) \Omega(\Gamma_1, w; J)\, \Omega(\Gamma_2, w; J). \end{aligned}$$

with

$$\mathrm{sgn}(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

The subscript $W_{\mathcal{C}}$ in $J_{W_{\mathcal{C}}}$ refers to a point in $\mathcal{C}$ which is sufficiently close to the wall W, such that no wall is crossed for the constituent between the wall and $J_{W_{\mathcal{C}}}$. Note that the wall is independent of c_2.

For the computation of the invariants of rank 3, one also needs to determine the wall-crossing formula across walls of marginal stability for non-primitive charges $2\Gamma_1 + \Gamma_2$ and walls where the slope of three non-parallel charges becomes equal. These can be determined using the wall-crossing formulas [6,16,19]. The result takes a simple form in terms of rational invariants and (2.6) [25].

3. Generating functions

This section computes the generating functions of the BPS invariants $\Omega(\Gamma, w; J)$. We start by defining the generating functions and a brief discussion of their properties. The generating function $\mathcal{Z}_r(\rho, z, \tau; S, J)$ for a Kähler surface S is defined by:

$$\mathcal{Z}_r(\rho, z, \tau; S, J) = \sum_{c_1, c_2} \bar{\Omega}(\Gamma, w; J)\, (-1)^{rc_1\cdot K_S} \times \bar{q}^{r\Delta(\Gamma) - \frac{r\chi(S)}{24} - \frac{1}{2r}(c_1 + rK_S/2)_-^2}\, q^{\frac{1}{2r}(c_1+rK_S/2)_+^2}\, \mathrm{e}^{2\pi \mathrm{i}\rho\cdot(c_1+rK_S/2)},$$

with $\rho \in H^2(S, \mathbb{C})$, $w = \mathrm{e}^{2\pi \mathrm{i} z}$ and $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$. Twisting by a line bundle leads to an isomorphism of moduli spaces. It is therefore sufficient to determine $\Omega(\Gamma, w; J)$ only for $c_1 \mod r$, and it moreover implies that $\mathcal{Z}_r(\rho, z, \tau; S, J)$ allows a theta function decomposition [6, 28]:

$$\mathcal{Z}_r(\rho, z, \tau; S, J) = \sum_{\mu \in \Lambda^*/\Lambda} \overline{h_{r,\mu}(z, \tau; S, J)}\, \Theta_{r,\mu}(\rho, \tau; S), \tag{3.1}$$

where the bar over $h_{r,\mu}(z, \tau; S, J)$ denotes complex conjugation, and $h_{r,\mu}(z, \tau; S, J)$ and $\Theta_{r,\mu}(\rho, \tau; S)$ are defined by:

$$\begin{aligned} h_{r,\mu}(z, \tau; S, J) &= \sum_{c_2} \bar{\Omega}(\Gamma, w; J)\, q^{r\Delta(\Gamma) - \frac{r\chi(S)}{24}}, \\ \Theta_{r,\mu}(\rho, \tau; S) &= \sum_{\mathbf{k} \in H^2(S, r\mathbb{Z}) + rK_S/2 + \mu} (-1)^{r\mathbf{k}\cdot K_S} q^{\mathbf{k}_+^2/2r}\, \bar{q}^{-\mathbf{k}_-^2/2r} e^{2\pi i \rho\cdot \mathbf{k}}. \end{aligned} \tag{3.2}$$

Note that $\Theta_{r,\mu}(\rho, \tau; S)$ depends on J through $\mathbf{k}_\pm$ and does not depend on z.

The generating function of the numerical invariants $\Omega(\Gamma; J)$ follows simply from Equation (2.5):

$$\mathcal{Z}_r(\rho, \tau; S, J) = \lim_{z \to \frac{1}{2}} (w - w^{-1})\, \mathcal{Z}_r(z, \rho, \tau; S, J).$$

Physical arguments imply that this function transforms as a multivariable Jacobi form of weight $(\frac{1}{2}, -\frac{3}{2})$ [6, 23, 28, 33] with a non-trivial multiplier system. For rank > 1 this is only correct after the addition of a suitable non-holomorphic term [28, 33]. This is explained for $r = 2$ in Sections 3.1 and 3.2.

The functions $h_{r,c_1}(z,\tau)$ and $h_{r,c_1}(\tau)$ contain a factor which depends only on the rank r and $b_2(S)$. It is therefore useful to define

$$f_{r,c_1}(z,\tau) = \left(\frac{\mathrm{i}}{\theta_1(2z,\tau)\eta(\tau)^{b_2(S)-1}}\right)^{-r} h_{r,c_1}(z,\tau),$$
$$f_{r,c_1}(\tau) = \left(\frac{1}{\eta(\tau)^{\chi(S)}}\right)^{-r} h_{r,c_1}(\tau),$$

with $\theta_1(z,\tau)$ and $\eta(\tau)$ defined by (A.1). The function $f_{r,c_1}(\tau)$ follows from $f_{r,c_1}(z,\tau)$ by

$$f_{r,c_1}(\tau) = \frac{(-1)^{r-1}}{2^{r-1}(r-1)!}\,\frac{1}{(2\pi i)^{r-1}}\partial_z^{r-1} f_{r,c_1}(z,\tau)|_{z=\frac{1}{2}}. \tag{3.3}$$

Note that the terms of degree $< r-1$ in the Taylor expansion with respect to z of $f_{r,\mu}(z,\tau)$ vanish.

A useful relation is the "blow-up formula" which relates the generating function of a surface S with that of its blow-up $\phi : \tilde{S} \to S$ at a non-singular point. Let C_1 be the exceptional divisor of ϕ, and take $J \in C(S)$, r, and c_1 such that $\gcd(r, c_1 \cdot J) = 1$. The generating functions $h_{r,c_1}(z,\tau;S,J)$ and $h_{r,c_1}(z,\tau;\tilde{S},J)$ are then related by [13, 22, 33, 35, 37]:

$$h_{r,\phi^*c_1-kC_1}(z,\tau;\tilde{S},J) = B_{r,k}(z,\tau)\,h_{r,c_1}(z,\tau;S,J), \tag{3.4}$$

with

$$B_{r,k}(z,\tau) = \frac{1}{\eta(\tau)^r}\sum_{\substack{\sum_{i=1}^r a_i=0\\ a_i\in\mathbb{Z}+\frac{k}{r}}} q^{\frac{1}{2}\sum_{i=1}^r a_i^2} w^{\sum_{i<j} a_i - a_j}.$$

3.1. Rank 1 and 2

This subsection presents explicit expressions for $h_{r,c_1}(z,\tau;\Sigma_\ell,J_{m,n})$. The result for $r=1$ and $S=\Sigma_\ell$ is simply [11]:

$$f_{1,c_1}(z,\tau;\Sigma_\ell) = 1.$$

Note that the dependence on J could be omitted here since all rank 1 sheaves are stable. Moreover, there is also no dependence on ℓ.

To compute the generating functions for $r \geq 2$, we use wall-crossing together with the fact that $\Omega(\Gamma,w;J_{0,1}) = 0$ for $c_1 = -C + \alpha f$ and $r \geq 2$. In the following, $c_1(E_2)$ is parametrized by $bC - af$. The walls are then at

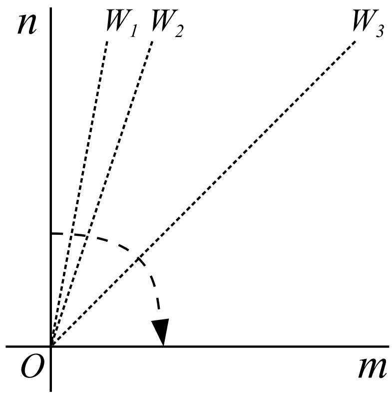

Figure 1: The ample cone of Σ_1, together with the three walls for $\Gamma = (2, -C - f, 2)$, namely for $(a, b) = (1, 0)$, $(2, 0)$, $(3, 0)$.

$\frac{m}{n} = \frac{2b-\beta}{2a-\alpha}$ for $r = 2$, with $m, n \geq 0$. See figure 1 for the walls for $\Delta(F) = \frac{9}{4}$, $r(F) = 2$. One finds [13, 26] using Equation (2.1):

$$
\begin{aligned}
(3.5) \qquad & f_{2,C-\alpha f}(z, \tau; \Sigma_\ell, J_{m,n}) \\
&= -\frac{1}{2} \sum_{a,b \in \mathbb{Z}} \frac{1}{2} (\,\mathrm{sgn}((2b+1)n - (2a-\alpha)m) - \mathrm{sgn}(2b+1)\,) \\
&\quad \times \left(w^{(\ell-2)(2b+1)+2(2a-\alpha)} - w^{-(\ell-2)(2b+1)-2(2a-\alpha)} \right) \\
&\quad \times\ q^{\frac{\ell}{4}2b+1)^2 + \frac{1}{2}(2b+1)(2a-\alpha)}.
\end{aligned}
$$

These functions are indefinite theta functions [12], which are sums over a subset of the positive definite sublattice of an indefinite lattice. Since the sum is only over a subset of the lattice, they transform as a modular form only after addition of a suitable non-holomorphic term (depending on $\bar{\tau}$ and $\bar{z}$) [39].

The computation of the invariants for $c_1 = -\alpha f$ is much more involved since strictly semi-stable sheaves do exist for $J_{0,1}$ or if $\alpha = 0$ for every $J \in C(\tilde{\mathbb{P}}^2)$. We will circumvent this computation by determining the functions $f_{2,-\alpha f}(z, \tau; \Sigma_\ell, J_{1,0})$ from modular transformations of $f_{2,C-\alpha f}(z, \tau; \Sigma_\ell, J_{1,0})$. One can consequently determine the invariants for arbitrary $J_{m,n}$ by application of the wall-crossing formula.

We continue by writing $f_{2,C-\alpha f}(z, \tau; \Sigma_\ell, J_{m,n})$ in terms of two new functions $A_{\ell,(\alpha,\beta)}(z, \tau)$ and $\vartheta^{m,n}_{\alpha,\beta}(z, \tau)$:

$$
f_{2,\beta C-\alpha f}(z, \tau; \Sigma_\ell, J_{m,n}) = A_{\ell,(\alpha,\beta)}(z, \tau) + \vartheta^{m,n}_{\alpha,\beta}(z, \tau), \quad \alpha, \beta \in \{0, 1\}.
$$

with

$$\vartheta^{m,n}_{\alpha,\beta}(z,\tau) = \sum_{a,b\in\mathbb{Z}} \frac{1}{2}\left(\operatorname{sgn}(-(2a-\alpha)) - \operatorname{sgn}((2b-\beta)n-(2a-\alpha)m)\right) \times w^{(\ell-2)(2b-\beta)+2(2a-\alpha)}\, q^{\frac{\ell}{4}(2b-\beta)^2+\frac{1}{2}(2b-\beta)(2a-\alpha)}.$$

Then Equation (3.5) gives for $A_{\ell,C-\alpha f}(z,\tau)$ for $\ell \geq 1$ after performing a geometric sum:[3]

(3.6)

$$A_{\ell,(1,1)}(z,\tau) = q^{\frac{\ell+2}{4}} w^{\ell} \sum_{n\in\mathbb{Z}} \frac{q^{\ell n(n+1)+n} w^{2(\ell-2)n}}{1-q^{2n+1}w^4},$$

$$A_{\ell,(0,1)}(z,\tau) = -\frac{1}{2}\sum_{n\in\mathbb{Z}} q^{\frac{\ell}{4}(2n+1)^2} w^{(\ell-2)(2n+1)} + q^{\frac{\ell}{4}} w^{\ell-2} \sum_{n\in\mathbb{Z}} \frac{q^{\ell n(n+1)} w^{2(\ell-2)n}}{1-q^{2n+1}w^4}.$$

The functions in Equation (3.6) are specializations of higher level Appell functions [3, 40], whose definition is recalled in Appendix A. These functions appeared earlier in mathematical physics in the theory of characters of superconformal algebras [9, 17, 31]. See [32] for a recent discussion. This might not be accidental since $\mathcal{N}=4$ Yang–Mills is well known to be related related to 2D conformal field theory by M-theory [28]. Deriving these functions explicitly from a two-dimensional perspective is an interesting direction for future research.

Analogously to the indefinite theta functions, the Appell functions only transform as a modular (or Jacobi) form after addition of a non-holomorphic term. Equation (A.3) gives the exact expression obtained Zwegers [40]. Application of this to our case of interest gives for the completion $\widehat{A}_{\ell,(\alpha,\beta=1)}(z,\tau)$:

(3.7)

$$\widehat{A}_{\ell,(\alpha,\beta)}(z,\tau) = A_{\ell,(\alpha,\beta)}(z,\tau) + \frac{1}{2}\sum_{k=0}^{\ell-1}\left(\sum_{\substack{n_1=2k+\beta\ell+\alpha \\ \mathrm{mod}\ 2\ell}} w^{\frac{\ell-2}{\ell}n_1} q^{\frac{n_1^2}{4\ell}}\right) \times \sum_{\substack{n_2=-2k-\alpha \\ \mathrm{mod}\ 2\ell}} \left(\operatorname{sgn}(n_2) - E\left((n_2+2(\ell+2)\operatorname{Im} z/y)\sqrt{y/\ell}\right)\right) \times w^{-\frac{\ell+2}{\ell}n_2} q^{-\frac{n_2^2}{4\ell}},$$

[3]Note that for $\Sigma_{\ell=0} = \mathbb{P}^1 \otimes \mathbb{P}^1$, the function $A_{0,(0,1)}(z,\tau)$ is undefined while $A_{0,(1,1)}(z,\tau) = 0$.

with $y = \operatorname{Im}\tau$ and $E(x) = 2\int_0^x \mathrm{e}^{-\pi u^2}\,du$. The four functions $\widehat{A}_{\ell,(\alpha,\beta)}$ transform as a vector-valued Jacobi form of weight 1 and index -8 of $SL(2,\mathbb{Z})$ [31, 40]. One finds for the action of the generators S and T:

(3.8)
$$\begin{aligned} S:\widehat{A}_{\ell,(\alpha,\beta)}\left(\frac{z}{\tau},\frac{-1}{\tau}\right) &= \frac{\tau}{2}\mathrm{e}^{2\pi\mathrm{i}(-\frac{8\,z^2}{\tau})}\sum_{\tilde\alpha,\tilde\beta\in\{0,1\}}(-1)^{\ell\beta\tilde\beta+\alpha\tilde\beta+\beta\tilde\alpha}\widehat{A}_{\ell,(\tilde\alpha,\tilde\beta)}(z,\tau),\\ T:\widehat{A}_{\ell,(\alpha,\beta)}\left(z,\tau+1\right) &= \mathrm{e}^{2\pi\mathrm{i}\frac{\beta^2+2\alpha\beta}{4}}\widehat{A}_{\ell,(\alpha,\beta)}\left(z,\tau\right). \end{aligned}$$

The modular transformations (3.8) together with the single pole in z of the refined invariants (2.4) do fix the functions $A_{\ell,(\alpha,0)}(z,\tau)$ to be:

$$\begin{aligned} A_{\ell,(1,0)}(z,\tau) &= w^2\sum_{n\in\mathbb{Z}}\frac{q^{\ell n^2+n}w^{2(\ell-2)n}}{1-q^{2n}w^4}+\frac{\mathrm{i}\,\eta(\tau)^3}{\theta_1(4z,\tau)},\\ A_{\ell,(0,0)}(z,\tau) &= -\frac{1}{2}\sum_{n\in\mathbb{Z}}q^{\ell n^2}w^{2(\ell-2)n}+\sum_{n\in\mathbb{Z}}\frac{q^{\ell n^2}w^{2(\ell-2)n}}{1-q^{2n}w^4}+\frac{\mathrm{i}\,\eta(\tau)^3}{\theta_1(4z,\tau)}. \end{aligned}$$

This agrees for $c_1 = f$ with the generating function in [36] (Corollary 3.4). The completion of these functions is given by Equation (3.7).

One can show the following relation between $A_{1,(\alpha,0)}(z,\tau)$ and $A_{1,(\alpha,1)}(z,\tau)$ using the quasi-periodicity formula (A.4):

(3.9)
$$\begin{aligned} A_{1,(1,0)}(z,\tau) &= \frac{\theta_2(2z,2\tau)}{\theta_3(2z,2\tau)}\,A_{1,(1,1)}(z,\tau),\\ A_{1,(0,0)}(z,\tau) &= \frac{\theta_3(2z,2\tau)}{\theta_2(2z,2\tau)}\,A_{1,(0,1)}(z,\tau). \end{aligned}$$

This relation is understood in algebraic geometry by the blow-up formula (3.4), which relates the functions $h_{2,c_1}(z,\tau;\Sigma_1,J_{1,0})$ with $c_1 = C - \alpha f$ to those with $c_1 = -\alpha f$. For $h_{2,c_1}(z,\tau;\mathbb{P}^2)$ one recovers the result of [4]. The multiplicative relation (3.9) does not hold for $\ell > 1$, since $\Sigma_{\ell>1}$ is the blow-up of the weighted projective plane $(1,1,\ell)$ at its *singular* point [7], and the blow-up formula is thus not applicable.

What remains is to complete the indefinite theta functions $\vartheta^{m,n}_{\alpha,\beta}(z,\tau)$. One finds using [39]:

$$\begin{aligned}(3.10)\qquad \widehat{\vartheta}^{m,n}_{\alpha,\beta}(z,\tau) = \sum_{a,b\in\mathbb{Z}} \frac{1}{2}\Big[&E\left((-2a+\alpha+2(\ell+2)\mathrm{Im} z/y)\sqrt{y/\ell}\right)\\ &- E\Big(((2b-\beta)n-(2a-\alpha)m\\ &+\ 2(2n+(\ell+2)m)\mathrm{Im}\, z/y)\sqrt{y/J^2_{m,n}}\Big)\Big]\\ &\times w^{(\ell-2)(2b-\beta)+2(2a-\alpha)}\, q^{\frac{\ell}{4}(2b-\beta)^2+\frac{1}{2}(2b-\beta)(2a-\alpha)}\end{aligned}$$

with $J^2_{m,n} = m(\ell m+2n)$. The completion for $f_{2,\beta C-\alpha f}$ follows directly from $\widehat{f}_{2,\beta C-\alpha f} = \widehat{A}_{\ell,(\alpha,\beta)} + \widehat{\vartheta}^{m,n}_{\alpha,\beta}$. The non-holomorphic term of the first line in Equation (3.10) is cancelled by the non-holomorphic term of $\widehat{A}_{\ell,(\alpha,\beta)}(z,\tau)$. Thus for the completion of $f_{2,\beta C-\alpha f}$ (and therefore also of $h_{2,\beta C-\alpha f}$) the non-holomorphic part of the second line in Equation (3.10) suffices. We define $\widehat{\mathcal{Z}}_r(\rho,z,\tau;S,J) := \sum_{\mu\in H^2(\Sigma_\ell,\mathbb{Z}/r\mathbb{Z})} \widehat{h}_{r,\mu}(z,\tau;S,J)\,\Theta_{r,\mu}(\rho,\tau;S)$.

3.2. Holomorphic anomaly for rank 2

This subsection derives $D_r\widehat{\mathcal{Z}}_r(\rho,\tau;\Sigma_\ell,J)$ for $D_r = \partial_\tau + \frac{i}{4\pi r}\partial^2_{\rho_+}$ and $r=2$. Since $D_r\Theta_{r,\mu}(\rho,\tau;\Sigma_\ell)=0$ for any r, it suffices to determine $\partial_{\bar\tau}\widehat{f}_{r,c_1}(\tau;\Sigma_\ell,J)$. For a clear exposition, the generating functions are given in this subsection in terms of K_ℓ, J etc. instead of the explicit integers ℓ, m and n, etc.

We determine first the completion $\widehat{f}_{2,c_1}(\tau;\Sigma_\ell,J)$ from the generating functions in the previous subsection. The result follows from the following three steps:

- use Equation (3.3) after replacing the functions with their completions;
- use that $E(z) = 2\int_0^z e^{-\pi u^2}du = \mathrm{sgn}(z)(1-\beta_{\frac{1}{2}}(z^2))$ with $z\in\mathbb{R}$ and

$$\beta_\nu(x) = \int_x^\infty u^{-\nu}\,\mathrm{e}^{-\pi u}\,du;$$

- and finally use

$$\beta_{\frac{3}{2}}(x) = 2x^{-\frac{1}{2}}\,\mathrm{e}^{-\pi x} - 2\pi\beta_{\frac{1}{2}}(x).$$

One obtains:

$$\begin{aligned}\widehat{f}_{2,c_1}(\tau;\Sigma_\ell,J) = f_{2,c_1}(\tau;\Sigma_\ell,J) \\ + \sum_{\substack{\mathbf{c}\in -c_1 \\ +H^2(\Sigma_\ell,2\mathbb{Z})}} \Bigg(\frac{K_\ell\cdot J\,|\mathbf{c}\cdot J|}{8\pi J^2}\,\beta_{\frac{3}{2}}(\mathbf{c}_+^2\,y) \\ -\frac{1}{4}K_\ell\cdot \mathbf{c}_-\,\mathrm{sgn}(\mathbf{c}\cdot J)\,\beta_{\frac{1}{2}}(\mathbf{c}_+^2\,y)\Bigg) \\ \times (-1)^{K_\ell\cdot\mathbf{c}} q^{-\mathbf{c}^2},\end{aligned}$$

where $\mathbf{c}_\pm$ are given by Equation (2.2). It is now straightforward to compute $\partial_{\bar\tau}\widehat{f}_{r,c_1}(\tau;\Sigma_\ell,J)$:

$$\begin{aligned}\partial_{\bar\tau}\widehat{f}_{r,c_1}(\tau;\Sigma_\ell,J) = \frac{\mathrm{i}\,K_\ell\cdot J}{16\pi\sqrt{J^2}\,y^{\frac{3}{2}}}(-1)^{K_\ell\cdot c_1}\,\overline{\Theta_{2,-c_1-K_\ell}(0,\tau;\Sigma_\ell)} \\ -\frac{\mathrm{i}}{8\sqrt{y}}\sum_{\substack{\mathbf{c}\in -c_1 \\ +H^2(\Sigma_\ell,2\mathbb{Z})}} K_\ell\cdot\mathbf{c}_-\,\frac{\mathbf{c}\cdot J}{\sqrt{J^2}}\,(-1)^{K_\ell\cdot\mathbf{c}}\,q^{-\mathbf{c}_-^2/4}\bar{q}^{\mathbf{c}_+^2/4}.\end{aligned}$$

After combining this result with $\Theta_{2,c_1}(\rho,\tau;\Sigma_\ell)$ as in (3.1) and manipulation of the lattice sums, one obtains for $D_2\widehat{\mathcal{Z}}_2(\rho,\tau;\Sigma_\ell,J)$:

$$\begin{aligned}D_2\widehat{\mathcal{Z}}_2(\rho,\tau;\Sigma_\ell,J) = \frac{-\mathrm{i}\,K_\ell\cdot J}{16\pi\sqrt{J^2}\,y^{\frac{3}{2}}}\,\mathcal{Z}_1(\rho,\tau;\Sigma_\ell,J)^2 \\ +\frac{\mathrm{i}}{8\sqrt{y}}\,\overline{h_{1,0}(\tau;\Sigma_\ell)}^2 \\ \times\sum_{c_1\in H^2(\Sigma_\ell,\mathbb{Z}/2\mathbb{Z})}\Upsilon_{c_1}(\tau,\Sigma_\ell)\,\Theta_{2,c_1}(\rho,\tau;\Sigma_\ell),\end{aligned} \tag{3.11}$$

where[4]

$$\Upsilon_{c_1}(\tau,\Sigma_\ell) = \sum_{\substack{\mathbf{c}\in -c_1 \\ +H^2(\Sigma_\ell,2\mathbb{Z})}} K_\ell\cdot\mathbf{c}_-\,\frac{\mathbf{c}\cdot J}{\sqrt{J^2}}\,(-1)^{K_\ell\cdot\mathbf{c}}\,q^{\mathbf{c}_+^2/4}\bar{q}^{-\mathbf{c}_-^2/4}.$$

Interestingly, Equation (3.11) differs from the conjectured form of the anomaly [1, 28, 33]. The first line has the expected factorized form, which is attributed to reducible connections or polystable sheaves [33] or multiple M5-branes [28]. However, the novel second line does not factorize and is less easily interpreted. It does vanish for special values of J, in particular

[4] A similar function appeared in [24].

for $J = -K_\ell$ since then $K_\ell \cdot \mathbf{c}_- = 0$. But for $\ell \geq 2$, K_ℓ lies outside $C(S)$ and is thus not a permissible choice for J. Viewing the surface as part of a local Calabi–Yau three-fold geometry, $J = -K_S$ corresponds to the attractor point from the point of view of supergravity [24]. It is therefore rather interesting that $\widehat{\mathcal{Z}}(\rho, \tau; \Sigma_\ell, J)$ simplifies at this point.

The function $\Upsilon_{c_1}(\tau, \Sigma_\ell)$ vanishes also for $\ell = 1$ and $J = C + f$ [4], which is not equal to $-K_1$. For this choice, the blow-up formula gives the generating function for $\mathbb{P}^2$, where $J = -K_{\mathbb{P}^2}$ is satisfied automatically. It is thus in agreement with these examples to conjecture that generically for a Kähler surface S, $D_2\widehat{\mathcal{Z}}_2(\rho, \tau; S, J) = \frac{-i\sqrt{K_S^2}}{16\pi y^{\frac{3}{2}}}\, \mathcal{Z}_1(\rho, \tau; S, -K_s)^2$ if $K_S \in C(S)$. Of course, a more intrinsic explanation based on gauge theory or algebraic geometry is desirable.

3.3. Rank 3

This subsection presents the generating functions $h_{3,\beta C - \alpha f}(z, \tau; \Sigma_\ell, J)$ with $\beta \neq 0 \mod 3$. This condition on β ensures that $h_{3,\beta C-\alpha f}(z, \tau; \Sigma_\ell, J) = 0$ for $J = J_{0,1}$ analogously to $r = 2$. The computation of $h_{3,\beta C-\alpha f}(z, \tau; \Sigma_\ell, J)$ therefore reduces again to application of the wall-crossing formula. This is for $r = 3$ more complicated than for $r = 2$ since:

- The functions $h_{2,c_1}(z, \tau; \Sigma_\ell, J)$ do themselves depend on J, and need to be determined sufficiently close to the appropriate wall.
- The total charge Γ can be of a sum of three charges $\sum_{i=1}^3 \Gamma_i$ such that at a wall W the slopes of these three constituents might be equal. This in particular happens for "semi-primitive wall-crossing" where $\Gamma(F) = 2\Gamma_1 + \Gamma_2$.

Nevertheless, the wall-crossing formulas [16, 19] imply a relatively simple form for the generating functions [25, 26]. One obtains for $\ell \geq 1$:

$$
\begin{aligned}
f_{3,\beta C-\alpha f}(z, \tau; \Sigma_\ell, J_{m,n}) = &- \sum_{a,b\in\mathbb{Z}} \frac{1}{2}\big(\operatorname{sgn}((3b-2\beta)n - (3a-2\alpha)m)\\
&- \operatorname{sgn}(3b-2\beta)\big)\Big(w^{(\ell-2)(3b-2\beta)+2(3a-2\alpha)}\\
&- w^{-(\ell-2)(3b-2\beta)-2(3a-2\alpha)}\Big)\\
&\times q^{\frac{\ell}{12}(3b-2\beta)^2+\frac{1}{6}(3b-2\beta)(3a-2\alpha)}\\
&\times f_{2,bC-af}(z, \tau; \Sigma_\ell, J_{|3b-2\beta|,|3a-2\alpha|}),
\end{aligned} \tag{3.12}
$$

Table 1: The Betti numbers b_n (with $n \leq \dim_{\mathbb{C}} \mathcal{M}$) and the Euler numbers χ of the moduli spaces of stable sheaves on Σ_1 with $r = 3$, $c_1 = -C$, and $2 \leq c_2 \leq 6$ for $J = (1, \varepsilon)$.

c_2	b_0	b_2	b_4	b_6	b_8	b_{10}	b_{12}	b_{14}	b_{16}	b_{18}	b_{20}	b_{22}	b_{24}	χ
2	1	2	4	4										18
3	1	3	9	20	37	53	59							305
4	1	3	10	25	59	119	218	338	450	490				2936
5	1	3	10	26	64	141	294	562	997	1602	2301	2886	3117	20891

Table 2: The Betti numbers b_n (with $n \leq \dim_{\mathbb{C}} \mathcal{M}$) and the Euler numbers χ of the moduli spaces of stable sheaves on Σ_1 with $r = 3$, $c_1 = -C - f$, and $2 \leq c_2 \leq 6$.

c_2	b_0	b_2	b_4	b_6	b_8	b_{10}	b_{12}	b_{14}	b_{16}	b_{18}	b_{20}	b_{22}	b_{24}	b_{26}	χ
2	1	1													3
3	1	3	8	14	17										69
4	1	3	10	24	53	93	136	152							792
5	1	3	10	26	63	135	268	470	725	950	1043				6345
6	1	3	10	26	65	145	310	612	1144	1970	3113	4391	5462	5873	40377

Table 3: The Betti numbers b_n (with $n \leq \dim_{\mathbb{C}} \mathcal{M}$) and the Euler numbers χ of the moduli spaces of stable sheaves on Σ_1 with $r = 3$, $c_1 = -C - 2f$, and $3 \leq c_2 \leq 6$.

c_2	b_0	b_2	b_4	b_6	b_8	b_{10}	b_{12}	b_{14}	b_{16}	b_{18}	b_{20}	b_{22}	χ
3	1	2	3										9
4	1	3	9	19	31	36							162
5	1	3	10	25	58	113	192	264	297				1629
6	1	3	10	26	64	140	288	536	907	1348	1733	1885	11997

for $\beta = 1, 2 \mod 3$ and $\alpha \in \mathbb{Z}$. Writing out the lattice sums in Equation (3.12), one finds a novel indefinite theta function. It has signature $(2, 2)$ and the condition which determines whether or not a lattice point contributes depends quadratically on the lattice vector, whereas previously described indefinite theta functions have signature $(n, 1)$ and the condition depends linearly on the lattice vector [12, 39]. A detailed discussion of the (mock) modular properties of $h_{3,c_1}(z, \tau; \Sigma_\ell, J)$ will appear in a future article [5].

Tables 1–3 list Betti numbers for $c_1 = -C - \alpha f$ with $\alpha = 1, 2, 3$ and $\ell = 1$, which are in agreement with the expected dimension (2.3). One can relate the Betti numbers for $c_1 = -2C - \alpha f$ to these by using $h_{r,c_1} = h_{r,-c_1}$, and $h_{r,c_1+\mathbf{k}} = h_{r,c_1}$ for $\mathbf{k} \in H_2(S, r\mathbb{Z})$. With a little more work, one can verify that $h_{3,c_1}(z,\tau;\Sigma_1,J_{1,0})$ satisfies the relations implied by the blow-up formula (3.4).

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Appendix A. Modular functions

Define $q := \mathrm{e}^{2\pi\mathrm{i}\tau}$, $w := \mathrm{e}^{2\pi\mathrm{i}z}$, with $\tau \in \mathbb{H}$ and $z \in \mathbb{C}$. The Dedekind eta and Jacobi theta functions are defined by:

$$\begin{aligned}
\eta(\tau) \quad &:= q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n), \\
\theta_1(z,\tau) &:= \mathrm{i} \sum_{r\in\mathbb{Z}+\frac{1}{2}} (-1)^{r-\frac{1}{2}} q^{\frac{r^2}{2}} w^r, \\
\theta_2(z,\tau) &:= \sum_{r\in\mathbb{Z}+\frac{1}{2}} q^{r^2/2} w^r, \\
\theta_3(z,\tau) &:= \sum_{n\in\mathbb{Z}} q^{n^2/2} w^n.
\end{aligned} \tag{A.1}$$

The Appell function at level ℓ is defined by:

$$A_\ell(u,v,\tau) = a^{\ell/2} \sum_{n\in\mathbb{Z}} \frac{(-1)^{\ell n} q^{\ell n(n+1)/2} b^n}{1-aq^n}, \tag{A.2}$$

with $a = \mathrm{e}^{2\pi\mathrm{i}u}$ and $b = \mathrm{e}^{2\pi\mathrm{i}v}$. In order to give the completion $\widehat{A}_\ell(u,v,\tau)$, define

$$\begin{aligned}
R(u,\tau) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} &\left(\mathrm{sgn}(r) - E\left((r + \mathrm{Im}\, u/y)\sqrt{2y}\right)\right) \\
&\times (-1)^{r-\frac{1}{2}} a^{-r} q^{-r^2/2},
\end{aligned}$$

with $E(x) = 2\int_0^x \mathrm{e}^{-\pi u^2}\,du$. The completion $\widehat{A}_\ell(u,v,\tau)$ is then given by [40]

$$\text{(A.3)}\qquad \begin{aligned}\widehat{A}_\ell(u,v,\tau) = A_\ell(u,v,\tau) + \frac{\mathrm{i}}{2}\sum_{k=0}^{\ell-1} a^k\,\theta_1(v+k\tau+(\ell-1)/2,\ell\tau)\\ \times R(\ell u - v - k\tau - (\ell-1)/2,\ell\tau),\end{aligned}$$

and transforms as a multivariable Jacobi form of weight 1 and index $\frac{1}{2}\left(\begin{smallmatrix}-\ell & 1\\ 1 & 0\end{smallmatrix}\right)$. The Appell function for $\ell = 1$ is related to the Lerch–Appell function: $\mu(u,v,\tau) = A_1(u,v,\tau)/\theta_1(v)$, which satisfies the quasi-periodicity property [39]:

$$\text{(A.4)}\quad \mu(u+z,v+z,\tau) - \mu(u,v,\tau) = \frac{\eta(\tau)^3\,\theta_1(u+v+z,\tau)\,\theta_1(z,\tau)}{\theta(u,\tau)\,\theta(v,\tau)\,\theta(u+z,\tau)\,\theta(v+z,\tau)},$$

for $u, v, u+z, v+z \notin \mathbb{Z}\tau + \mathbb{Z}$.

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Max Planck Institute for Mathematics
Vivatsgasse 7
53111 Bonn
Germany

Bethe Center for Theoretical Physics
Bonn Universtity
Nussallee 12
53115 Bonn
Germany
E-mail address: `manschot@uni-bonn.de`

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