Modularity of Maschke's octic and Calabi–Yau threefold

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We prove the modularity of Maschke's octic and two Calabi–Yau threefolds derived from it as double octic and quotient thereof by a suitable Heisenberg group, as conjectured by Bini and van Geemen. The proofs rely on automorphisms of the varieties and isogenies of K3 surfaces.

1. Introduction

In their paper [1], Bini and van Geemen study Maschke's octic surface S and two Calabi–Yau threefolds X and Y which arise as double covering of $\mathbf{P}^3$ branched along S and quotient thereof by a suitable Heisenberg group. Thanks to the huge automorphism group of S, Bini and van Geemen are able to decompose the cohomology of either variety into small pieces, each of dimension 1, 2 or 3. The decomposition works over $\mathbf{Q}$ (for S and Y) or $\mathbf{Q}(i)$ where $i^2 = -1$ (for X). This leads to the problem of the corresponding Galois representations. The aim of this note is to prove their modularity, as conjectured by Bini and van Geemen.

Throughout the paper we employ the notation from [1], see Section 1.4 for a brief account. Particularly, we are concerned with the following modular forms with notation from [2] indicating the level:

$$f120 \quad \text{weight 4;} \quad f15C, f24b, f120E \quad \text{weight 2}$$

(see [2] or [1, 7.2] for Fourier coefficients). The associated compatible systems of ℓ-adic Galois representations are denoted by $V_{f,\ell}$. In detail we prove the following:

Theorem 1.1. *Maschke's double octic X is modular over $\mathbf{Q}$. Its third cohomological Galois representation decomposes as*

$$H^3_{\acute{e}t}(X, \mathbf{Q}_\ell) \cong V_{f120,\ell} \oplus V_{f120E,\ell}(-1)^{32} \oplus (V_{f120E,\ell}(-1) \otimes \chi_{-1})^{18}$$

$$\oplus V_{f24B,\ell}(-1)^{36} \oplus (V_{f24B,\ell}(-1) \otimes \chi_{-1})^{18}$$
$$\oplus V_{f15C,\ell}(-1)^{27} \oplus (V_{f15C,\ell}(-1) \otimes \chi_{-1})^{18},$$

where χ_{-1} denotes the quadratic Dirichlet character of conductor 4.

Theorem 1.2. *The Calabi–Yau threefold Y is modular over $\mathbf{Q}$. Its third cohomological Galois representation decomposes as*

$$H^3_{\acute{e}t}(Y, \mathbf{Q}_\ell) \cong V_{f120,\ell} \oplus V_{f120E,\ell}(-1)^5 \oplus V_{f24B,\ell}(-1)^9.$$

Theorem 1.3. *Maschke's octic S is modular over $\mathbf{Q}$. Its transcendental Galois representation decomposes as*

$$T_{S,\ell} \cong (V_{f15,\ell})^5 \oplus (\mathrm{Sym}^2 V_{f1200,\ell} \otimes \chi_{-15})^{18} \oplus (\mathrm{Sym}^2 V_{f1200,\ell} \otimes \chi_{15})^{12}$$

for the weight 3 form $f15$ of level 15 from [11, Table 2] and the new-form $f1200 \in S_2(\Gamma_0(1200), \chi_3)$ specified in 4.12. The quadratic characters $\chi_3, \chi_{15}, \chi_{-15}$ are Legendre symbols as indicated in the subscript.

The proof of Theorem 1.2 is given in Section 2. It builds on effective (and feasible) explicit methods for 2-dimensional Galois representations, combined with the use of involutions on Y. Section 3 proceeds with the proof of Theorem 1.1, using similar techniques. A notable difference is that we first have to prove that the cohomological Galois representation decomposes into 2-dimensional pieces over $\mathbf{Q}$. Here again automorphisms of X play a crucial role. The note concludes with the proof of Theorem 1.3 in Section 4. It proceeds by relating S to a product of an elliptic $\mathbf{Q}$-curve E and one of its conjugates through various quotient maps of K3 surfaces and eventually a Shioda–Inose structure as elaborated in [3]. Then this $\mathbf{Q}$-curve is modular with associated newform $f1200$, and we verify the claimed equality. As a by-product we also deduce the modularity of the K3 surfaces involved in this construction, and of another K3 surface related to E by [3] (see Remark 4.14).

1.4. Notation and set-up. Maschke's octic is defined by the homogeneous equation

$$S = \left\{ \sum_{i=0}^3 x_i^8 + 14 \sum_{i<j} x_i^4 x_j^4 + 168 x_0^2 x_1^2 x_2^2 x_3^2 = 0 \right\} \subset \mathbf{P}^3.$$

In [1] it is checked using Groebner basis that S is smooth outside characteristics $2, 3, 5$, in particular over $\mathbf{C}$. S comes with a large group of automorphisms $\bar{G}$ of order 11520 which can be used to analyse its (middle) cohomology. Bini and van Geemen compute the Picard number $\rho(S) = 202$ and decompose the 100-dimensional transcendental part of $H^2(S)$ into 2-dimensional and 3-dimensional pieces over $\mathbf{Q}$ (as Galois representations or Hodge structures, see [1, 5.7]):

$$(1.1) \qquad T_{S,\ell} = (W_{3,\ell})^5 \oplus (W_{7,\chi_0,\ell})^{18} \oplus (W_{7,\chi_0,\ell} \otimes \chi-1)^{12}$$

The 2-dimensional piece $W_{3,\ell}$ was shown in [1] to correspond to the weight 3 form $f15$. The modularity of the 3-dimensional Galois representation $W_{7,\chi_0,\ell}$ is the subject of Theorem 1.3, see Section 4 for the proof.

The automorphism group lifts to the Calabi–Yau threefold X which arises as double cover of $\mathbf{P}^3$ branched along S:

$$X \xrightarrow{2:1} \mathbf{P}^3.$$

This double octic has quite large middle cohomology, $b_3(X) = 300$, but the automorphisms can be used to split it up completely into 2-dimensional pieces over $\mathbf{Q}(i)$. In Section 3, we show that these 2-dimensional Galois representations indeed descend to $\mathbf{Q}$, and prove their modularity (Theorem 1.1).

A non-trivial portion of $H^3(X)$ can be handled by considering another Calabi–Yau threefold Y which is the desingularization of the quotient of X by a suitable Heisenberg group H (see [1]). Here Y can be given as crepant resolution of the double covering of the Igusa quartic threefold Z in $\mathbf{P}^4$,

$$Z = \{y_4^4 + (y_0^2 - y_1^2 - y_2^2 - y_3^2)y_4^2 + y_1^2y_2^2 + y_1^2y_3^2 + y_2^2y_3^2 - 2y_0y_1y_2y_3 = 0\} \subset \mathbf{P}^4$$

branched along the quadric

$$y_0^2 + 3(y_1^2 + y_2^2 + y_3^2) + 6y_4^2 = 0.$$

One has $b_3(Y) = 30$, and $H^3(Y)$ is decomposed into 2-dimensional Galois representations properly over $\mathbf{Q}$ in [1], so we can attack their modularity right away. This is achieved in the next section, proving Theorem 1.2.

2. Proof of Theorem 1.2

In this section, we prove the modularity of the Calabi–Yau threefold Y over $\mathbf{Q}$. Recall that Y is the quotient of Maschke's double octic X by a suitable

Heisenberg group H. In [1] it is proved that the Galois representation of the middle cohomology of Y decomposes as

$$(2.1) \qquad H^3_{\text{ét}}(Y, \mathbf{Q}_\ell) = W_1 \oplus W_5^5 \oplus W_9^9$$

for 2-dimensional Galois representations W_1, W_5, W_9 over $\mathbf{Q}$. These Galois representations can be determined by computing sufficiently many traces of Frobenius elements (up to $p = 73$, see 2.8). In general these traces can be obtained from $\#Y(\mathbf{F}_q)$ for $q = p, p^2, p^3$ by Lefschetz' fixed point formula. However, since Y is not a hypersurface, already counting points over $\mathbf{F}_{p^2}$ soon becomes infeasible. To circumvent this, we use involutions of Y to derive all required traces from point counts over $\mathbf{F}_p$ for Y and two quadratic twists. Before going into the details we prove an auxiliary property concerning Tate twists.

2.1. Tate twist. We start by proving that W_5, W_9 are Tate twists of integral Galois representations of weight 1 (i.e., the eigenvalues have absolute value the square root of the norm). For this purpose we invoke the fact that the Newton polygon lies above the Hodge polygon (cf. [7]). Since we work with $H^3(Y)$, the Hodge polygon has slopes $0, 14 \times 1, 14 \times 2, 3$. Presently, the decisive property are the multiplicities (being greater than 1) combined with the following easy fact about 2-dimensional Galois representations:

Fact 2.2. *A 2-dimensional Galois representation $Gal(\bar{\mathbf{Q}}/\mathbf{Q}) \to GL_2(\mathbf{Q}_\ell)$ of weight k can only have Newton slopes $0, 1, \ldots, k$ and $k/2$ (if this is not already an integer).*

Proof. Since the characteristic polynomial at p splits over some quadratic extension K of $\mathbf{Q}$, the slopes are half-integers. In fact they are integers unless p ramifies in K which immediately gives slope $k/2$. $\qquad \square$

Lemma 2.3. *W_5, W_9 are Tate twists of integral weight one Galois representations U_5, U_9:*

$$W_5 = U_5(-1), \quad W_9 = U_9(-1).$$

Proof. Fact 2.2 a priori leaves the possibilities $0, 1, 3/2, 2, 3$ for the slopes of W_5, W_9 at a fixed prime p. Assuming that one Newton slope were 0, this would count with multiplicity 5 resp. 9 in $H^3_{\text{ét}}(Y, \mathbf{Q}_\ell)$. In either case, this would cause the Newton polygon of $H^3(Y)$ to sit underneath the Hodge polygon. This gives the required contradiction. By symmetry the Newton

slopes thus can only be $1, 3/2, 2$ at any prime p. This shows that W_5 and W_9 are Tate twists.   □

2.4. Traces.   Since Lemma 2.3 limits the possible traces for W_5 and W_9 (using the bounds imposed by the Weil conjectures), this in fact often enables us to compute the traces from substantially less information than point counts up to $\mathbf{F}_{p^3}$. Namely counting points over $\mathbf{F}_p$ and $\mathbf{F}_{p^2}$ we obtain a linear and a quadratic relation for the traces:

$$1 + p + p^2 + p^3 - \#Y(\mathbf{F}_p) = \mathrm{tr}_p(W_1) + 5p\,\mathrm{tr}_p(U_5) + 9p\,\mathrm{tr}_p(U_9),$$
$$1 + p^2 + p^4 + p^6 - \#Y(\mathbf{F}_{p^2}) = \mathrm{tr}_p(W_1)^2 + 5p^2\,\mathrm{tr}_p(U_5)^2$$
$$+ 9p^2\,\mathrm{tr}_p(U_9)^2 - 30p^3.$$

For $p = 7, 11, 13$ we verified that this gives a unique integer solution respecting the Weil bounds. This agrees exactly with the values from [1, 7.2]:

p	7	11	13
$\mathrm{tr}_p(W_1)$	0	4	54
$\mathrm{tr}_p(U_5)$	0	−4	6
$\mathrm{tr}_p(U_9)$	0	4	−2

2.5. Involutions.   In order to compute these traces for larger p without counting points over $\mathbf{F}_{p^2}$ we consider the involutions

$$\imath_1 : \begin{cases} x_0 \mapsto -x_0 \\ x_1 \mapsto -x_1 \end{cases} \qquad \imath_2 : x_4 \mapsto -x_4.$$

We want to compute the traces of $\imath_1^*, \imath_2^*$ on our representations W_1, W_5, W_9 inside $H^3_{\text{ét}}(Y, \mathbf{Q}_\ell)$. This could certainly be done abstractly, but we will use the composition with Frobenius at the above primes and Lefschetz' fixed point formula again. Since the involutions and Frobenius commute, these traces are directly related to the number of fixed points of their composition. Presently the fixed points of $F_p \circ \imath_j$ are defined over $\mathbf{F}_{p^2}$, but they are in 1:1-correspondence with F_p-fixed points of a quadratic twist of Y (at (x_0, x_1) resp. x_4) and thus easy to compute. At the above primes we find:

p	7	11	13
$\mathrm{tr}(F_p^* \circ \imath_1^*)$	0	180	−102
$\mathrm{tr}(F_p^* \circ \imath_2^*)$	0	−4	−210

By Lemma 2.3 we have

$$\mathrm{tr}(F_p^* \circ \imath_j^*) \equiv \pm \mathrm{tr}_p(W_1) \mod p \quad (j = 1, 2).$$

The trace at $p = 11$ or 13 thus gives the sign of $\imath_j^*$ on W_1. Then the traces of $F_p^* \circ \imath_j^*$ for $p = 11, 13$ give a linear system in the two unknown traces of $\imath_j^*$ on W_5 and W_9. The unique solution is (in the order (W_1, W_5, W_9))

$$(2.2) \qquad \mathrm{tr}\, \imath_1^* = (1, -1, 3), \quad \mathrm{tr}\, \imath_2^* = (-1, -3, -3).$$

2.6. Traces and Fourier coefficients. Thanks to the involutions $\imath_1, \imath_2$ and their traces (2.2), we can compute the traces of Frobenius on W_1, U_5, U_9 from the number of points of Y and its two quadratic twists over $\mathbf{F}_p$. For p from 7 up to 73 we find exactly the Fourier coefficients of the newforms from Theorem 1.2.

2.7. Galois representations: even traces. We are now in the position to prove that the Galois representations are isomorphic up to semi-simplification. There is an effective method due to Faltings–Serre–Livné that applies to 2-adic Galois representations with even trace [6]. In order to see that this is applicable, we use a standard argument that we only sketch below.

Note that any Frobenius element F_p with odd trace would imply that the kernel of the mod 2-reduction of the Galois representation would correspond to a Galois extension of $\mathbf{Q}$ with Galois group C_3 or S_3. Here F_p represents the conjugacy class of an element of order 3. Going through the list of all such extension of $\mathbf{Q}$ which are unramified outside the set of bad primes $\{2, 3, 5\}$ (see [5]), we find that each has an order 3 Galois element represented by F_p for $p \in \{7, 11, 13, 19\}$. Thus the fact that the traces are even at these four primes implies that the Galois representations have already even trace.

2.8. Galois representations: isomorphic semi-simplifications. In case of even traces (and equal determinant as assured presently) the theorem of Faltings–Serre–Livné states that it suffices to compare traces at a non-cubic set of primes for the Galois group of the compositum of all quadratic extensions of $\mathbf{Q}$ which are unramified outside the bad primes [6, Thm. 4.3]. Presently this extension is $K = \mathbf{Q}(i, \sqrt{2}, \sqrt{-3}, \sqrt{5})$. The Galois group $\mathrm{Gal}(K/\mathbf{Q})$ is represented by the Frobenius elements at the primes

$$(2.3) \qquad \{7, 11, 13, 17, 19, 23, 29, 31, 41, 43, 53, 61, 71, 73, 83, 241\}.$$

In the non-cubic set we can omit the primes $83, 241$ from the full set of representatives (see [6, Prop. 4.11]). It follows that equality of the traces

at primes up to $p = 73$ suffices to deduce that the Galois representations in question are isomorphic (cf. Section 2.6). Here we use that 2-dimensional integral Galois representations of odd weight are automatically semi-simple. This completes the proof of Theorem 1.2.   □

3. Proof of Theorem 1.1

With the modularity of Y at our disposal, we continue to prove the modularity of the double octic X, branched along Maschke's octic S. Recall from [1] that the automorphism group of S, interpreted on X, splits the third cohomological Galois representation of X over $\mathbf{Q}(i)$ as follows:

$$(3.1) \quad H^3_{\text{ét}}(X, \mathbf{Q}_\ell) \cong H^3_{\text{ét}}(Y, \mathbf{Q}_\ell) \oplus W^{15}_{15,\ell} \oplus W^{30}_{30,\ell} \oplus W^{45}_{45,\ell} \oplus (W'_{45,\ell})^{45}.$$

3.1. Tate twist.   As in 2.1 we find that each summand W in (3.1) is a Tate twist of an integral Galois representation of weight 1. The argument a priori applies to the 2-dimensional Galois representations over $\mathbf{Q}(i)$, but naturally descends to the possibly 4-dimensional Galois representations over $\mathbf{Q}$.

3.2. Traces at primes $p \equiv 1 \mod 4$.   Using automorphisms of X, we can compute the traces on the Galois representations from (3.1) at any F_q for $q \equiv 1 \mod 4$ from point counts over $\mathbf{F}_q$ on X and different twists. In detail one can use the following automorphisms and their traces on the single representations V_i:

		$\operatorname{tr} \imath(V_i)$						
	$\imath$	V_1	V_5	V_9	V_{15}	V_{30}	V_{45}	V'_{45}
$\imath_1$	$(x_2, x_3) \mapsto (-x_2, -x_3)$	1	5	9	-1	-2	-3	-3
$\imath_2$	$(x_1, x_2, x_3) \mapsto (-x_1, x_3, x_2)$	1	1	1	3	2	-3	1
$\imath_3$	$(x_1, x_2, x_3) \mapsto (ix_1, -ix_2, -x_3)$	-1	1	-3	-3	-2	1	5

Then compose these automorphisms with F_q. The fixed point sets can be derived from $X(\mathbf{F}_q)$ and the $\mathbf{F}_q$-rational points of specific quadratic or quartic twists of X. Through Lefschetz' fixed point formula, these quantities relate to the trace on $H^3_{\text{ét}}(X, \mathbf{Q}_\ell)$. Since F_q and the automorphisms commute by choice of q, this trace can be computed as

$$\operatorname{tr} (F_q \circ \imath)^*(H^3_{\text{ét}}(X, \mathbf{Q}_\ell)) = \sum_i \operatorname{tr} F_q^*(W_{i,\ell}) \operatorname{tr} \imath^*(V_i).$$

Note that the 4×4-matrix comprising the traces of the automorphisms (including the identity) on the $V_i (i = 15, 30, 45, 45')$ is invertible of determinant 960. Hence we obtain the traces of F_q^* on the W_i from the known Galois representation of Y and point counts over $\mathbf{F}_q$ on X and the three twists of X as sketched above.

In particular, for all $p \equiv 1 \mod 4$ up to $p = 97$ we can easily verify that

- $W_{15,\ell}$ and $W_{30,\ell}$ have the same trace (matching the modular form $f15C$),

- $W_{9,\ell}$ and $W_{45,\ell}$ have the same trace (matching the modular form $f24B$),

- $W_{5,\ell}$ and $W'_{45,\ell}$ have the same trace (matching the modular form $f120E$).

Note that this agrees perfectly with Theorem 1.1. It remains to deal with (a few) primes $p \equiv 3 \mod 4$.

3.3. 2-dimensional Galois representations. We shall now prove that all the W_i actually descend to 2-dimensional Galois representations over $\mathbf{Q}$. Thus we have to exclude that some form an irreducible Galois representation of dimension 4 over $\mathbf{Q}$. Note that dimension 4 would imply that Frobenius eigenvalues are duplicated at $q \equiv 1 \mod 4$ and appear with opposite signs at $q \equiv 3 \mod 4$ (so that the 4-dimensional representation would have zero trace as it equals its twist by χ_{-1}).

First we use the duplication observation. Note that none of the above three cases have the same traces at both $p = 13$ and 29 (cf. [1, 7.2]). Hence there cannot be two representations of multiplicity 45 and dimension two over $\mathbf{Q}(i)$, forming the restriction of a single 4-dimensional Galois representation over $\mathbf{Q}$. Hence, for multiplicities 15 and 45, we deduce from the odd parity of the multiplicity over $\mathbf{Q}(i)$ that the representations do in fact come from 2-dimensional Galois representations over $\mathbf{Q}$. Thus only $W_{30,\ell}$ may a priori stem from a 4-dimensional Galois representation W_ℓ over $\mathbf{Q}$. In order to establish a contradiction, we assume that it actually does:

Assumption 3.4. *$W_{30,\ell}$ comes from an irreducible 4-dimensional Galois representation W_ℓ over $\mathbf{Q}$.*

A trivial, but crucial fact now is that for $p \equiv 3 \mod 4$ not only F_p^* has zero trace on W_ℓ, but also its composition with any automorphism of X. We will use this to derive a contradiction from a detailed trace analysis at a few primes.

3.5. Galois-trace. The 2-dimensional Galois representations W_i ($= W_{15}$, W_{45}, W_{45}') over $\mathbf{Q}(i)$ can descend in two ways to $\mathbf{Q}$, distinguished by the quadratic twist by χ_{-1}. In accordance, we split the representations V_i as

$$V_i = V_i' \oplus V_i'', \quad \dim(V_i') + \dim(V_i'') = 15 \ \text{resp.} \ 45$$

such that the corresponding Galois representations over $\mathbf{Q}$ fulfil

$$(W_i')_\ell = W_{i,\ell,\mathbf{Q}}, \quad (W_i'')_\ell = W_{i,\ell,\mathbf{Q}} \otimes \chi_{-1}$$

for a fixed 2-dimensional Galois representation $W_{i,\ell,\mathbf{Q}}$ (unique up to twist by χ_{-1}). These quadratic twists interfere with the action of the automorphisms from Section 3.2. Our first goal is to compute the trace of the automorphisms weighted by the twists; for simplicity we shall call this the Galois-trace of the automorphism $\imath$:

$$\mathrm{tr}^{\mathrm{Gal}}\, \imath(V_i) = \mathrm{tr}\, \imath^*(V_i') - \mathrm{tr}\, \imath^*(V_i'').$$

Here is the motivation of the definition: for any prime $p \equiv 3 \mod 4$ we have

$$(3.2) \qquad \mathrm{tr}(F_p \circ \imath)^*(V_i \otimes W_i)_\ell = (\mathrm{tr}^{\mathrm{Gal}}\, \imath(V_i))(\mathrm{tr}\, F_p^*(W_{i,\ell,\mathbf{Q}})).$$

Recall that the sum over expressions on the left-hand side can be computed from the Lefschetz' fixed point formula (thus point counting on a twist over $\mathbf{F}_p$) and the knowledge of the action on $H^3_{\text{ét}}(Y, \mathbf{Q}_\ell)$. We shall now compute the Galois traces, still assuming 3.4, and derive a contradiction.

3.6. Trace analysis for small primes. Consider the primes $p = 11, 19, 23, 31$ which are all congruent to $3 \mod 4$. Let p be one of them and $q = p^2$. Using the automorphisms from 3.2 composed with F_q, we can compute the traces of F_q^* on the $W_{i,\ell}$ from four point counts over $\mathbf{F}_q$. We verify that the traces satisfy the natural relation with the coefficients A_p of the modular forms as specified above:

$$\mathrm{tr}\, F_q^*(W_{i,\ell}) = p^2(A_p^2 - 2p).$$

Conversely this gives us the possible traces of F_p^* on the $W_{i,\ell}$ up to sign. Once and for all and without loss of generality, we choose the signs at $p = 11$ matching the modular forms as in Section 3.2.

3.7. Involution $\imath_2$.   For any sign choice of $\operatorname{tr} F_p^*(W_{i,\ell,\mathbf{Q}})$ for $p = 19, 23, 31$, summation over (3.2) yields a linear system of equations with Galois traces as unknowns. With signs chosen to match the modular forms, one can easily check that the only integral solution respecting the trivial bounds

$$|\operatorname{tr}^{\mathrm{Gal}}\imath_2(V_i)| \leq \dim(V_i)$$

occurs for

$$\operatorname{tr}^{\mathrm{Gal}}\imath_2(V_{15}) = \operatorname{tr}^{\mathrm{Gal}}\imath_2(V_{45}) = 1, \quad \operatorname{tr}^{\mathrm{Gal}}\imath_2(V_{45}') = -3.$$

In fact, we also find unique traces of F_p on the 2-dimensional Galois representations $W_{i,\ell,\mathbf{Q}}$. As desired these match the modular forms as in 3.2. With the traces of F_p at hand, we can compute all other Galois traces from the corresponding system of linear equations (after (3.2)):

$\imath$	V_{15}	V_{45}	V_{45}'
	$\operatorname{tr}^{\mathrm{Gal}}\imath(V_i)$		
id	9	9	9
$\imath_1$	9	9	9
$\imath_2$	1	1	-3
$\imath_3$	-1	5	1

3.8. Contradiction to Assumption 3.4.   We shall use the fact that id and $\imath_1$ have the same Galois traces on V_{15} to establish a contradiction. From $\operatorname{tr}^{\mathrm{Gal}}\mathrm{id}(V_{15}) = 9$, we directly infer that

$$(3.3) \qquad\qquad \dim V_{15}' = 12, \quad \dim V_{15}'' = 3.$$

By Section 3.2 we have

$$\operatorname{tr} \imath_1(V_{15}) = -1,$$

i.e., $\imath_1^*$ has 7 eigenvalues $+1$ and 8 eigenvalues -1 on the total representation V_{15}. Combining these eigenvalues with those of the character χ_{-1} reflected in (3.3), we find at least five products evaluating as -1. That is, $\operatorname{tr}^{\mathrm{Gal}}\imath_1(V_{15}) \leq 10 \cdot (+1) + 5 \cdot (-1) = 5 < 9$, contradiction. Thus our original Assumption 3.4 is seen to be false, and we have proved:

Proposition 3.9. $H_{\acute{e}t}^3(X, \mathbf{Q}_\ell)$ *decomposes into 2-dimensional Galois representations over* $\mathbf{Q}$.

The alert reader might wonder why we nonetheless have found the right traces in the above argument. The reason is that the representations of multiplicities 15 and 30 do in fact contribute many zero traces to the cohomology (at primes $p \equiv 3 \mod 4$), but this is not fully supported on the representation of multiplicity 30 when we take automorphisms into account.

3.10. Galois-traces revisited. We reconsider the Galois traces of the involutions, now also taking the representation V_{30} into account. At the primes $p = 11, 19, 23, 31$, we find the same traces (up to sign) for $W_{15,\ell}$ and $W_{30,\ell}$. Thus we can compute the combined Galois trace, assuming without loss of generality that $W_{15,\ell,\mathbf{Q}}$ and $W_{30,\ell,\mathbf{Q}}$ take the same sign. In fact, the argument using $\imath_2$ in 3.7 goes through just the same with the dimension bound for V_{15} replaced by

$$|\mathrm{tr}^{\mathrm{Gal}}\, \imath_2(V_{15} \oplus V_{30})| \leq 45.$$

In particular, we find the same Galois trace as before, but now this is supported on the larger representation $V_{15} \oplus V_{30}$ (so that no elementary sign argument would cause a contradiction).

3.11. Continuation of proof. Eventually, we want to apply the Faltings–Serre–Livné method from 2.8 again. In view of Sections 3.2 and 3.10, only the traces at $p = 43, 71$ are missing from the non-cubic set (2.3). One way to proceed would be to first verify that $W_{15,\ell}$ and $W_{30,\ell}$ have the same trace (up to sign) at these two primes and then continue as above. This could be achieved by computing the full set of traces of F_{p^2} as in 3.2 and 3.6. However, since these computations tend to become fairly involved, we decided to pursue another line of argument that we hope to be of independent interest. Our ideas rely on elementary linear algebra, the Weil bounds for the traces of Frobenius, and the following divisibility property at the primes congruent to 3 modulo 4:

Lemma 3.12. *The Galois representations $W_{i,\ell}$ have four-divisible traces at primes $p \equiv 3 \mod 4, p > 3$.*

In view of the traces computed thus far, the lemma will follow immediately from the following general criterion:

Proposition 3.13. *Let $\rho : \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \to GL_2(\mathbf{Q}_\ell)$ denote an integral Galois representation of odd weight which is unramified outside $\{2, 3, 5\}$.*

Assume that

$$\operatorname{tr}\rho(Fr_{13}) \equiv 0 \mod 2, \quad \operatorname{tr}\rho(Fr_p) \equiv 0 \mod 4 \quad (p = 7, 11, 19).$$

Then ρ has even trace and, moreover,

$$\operatorname{tr}\rho(Fr_p) \equiv 0 \mod 4 \quad \forall\, p \equiv 3 \mod 4, p > 3.$$

Proof. By 2.7 ρ has even trace. In particular the mod 2-reduction ρ_2 has image consisting of 1 or 2 elements. Now consider the lifts to $\operatorname{GL}(\mathbf{Z}/4\mathbf{Z})$, i.e. the possible images G of the mod 4-reduction ρ_4. One easily checks that $\#G \leq 32$ with every element of order $1, 2$ or 4. Integrality and odd weight k imply that

$$\det\rho = \chi_\ell^k$$

for the ℓ-adic cyclotomic character χ_ℓ. In particular, we find that

$$\det\rho_4(\operatorname{Fr}_p) = p$$

specializing to -1 in $\mathbf{Z}/4\mathbf{Z}$ for all $p \equiv 3 \mod 4$. A case by case-analysis teaches us that the trace of all determinant -1 elements in G is

- 0 for lifts of the identity;

- 0 for lifts of other elements in $\operatorname{GL}(\mathbf{Z}/2\mathbf{Z})$ iff the order is 2;

- 2 for lifts of other elements in $\operatorname{GL}(\mathbf{Z}/2\mathbf{Z})$ iff the order is 4.

Hence if we assume that there is a prime $p \equiv 3 \mod 4$ with $\operatorname{tr}\rho(\operatorname{Fr}_p) \not\equiv 0 \mod 4$, then this corresponds to a Galois extension $K/\mathbf{Q}$ unramified outside $\{2, 3, 5\}$ with Galois group C_4 or D_4 where F_p induces an element of order 4. Going through the list of all such extensions from [5] we find that

- either $p = 7, 11$ or 19 gives a Galois element of order 4

- or the extension $K/\mathbf{Q}$ is the Galois closure of a cyclic polynomial $x^4 - a$.

One easily sees that in the latter case any prime $p \equiv 3 \mod 4$ will induce a Galois element of order 2. So this case does only produce 4-divisible traces at the primes in question while the first case is ruled out by the assumptions. This concludes the proof. $\qquad\square$

The proposition immediately implies Lemma 3.12. This little extra information will prove crucial for deriving the missing traces:

Lemma 3.14. *The characteristic polynomials of the Galois representations factor as follows at $p = 43, 71$:*

$$
\begin{aligned}
(V_{15} \otimes W_{15,\ell}(1)) \oplus (V_{30} \otimes W_{30,\ell}(1)) : \quad & \chi(T) = (T^2 - d_p T + p)^{27}(T^2 + d_p T + p)^{18} \\
V_{45} \otimes W_{45,\ell}(1) : \quad & \chi(T) = (T^2 - b_p T + p)^{27}(T^2 + b_p T + p)^{18} \\
V'_{45} \otimes W'_{45,\ell}(1) : \quad & \chi(T) = (T^2 - c_p T + p)^{27}(T^2 + c_p T + p)^{18}.
\end{aligned}
$$

Proof. We return to the concept of Galois traces. By Section 3.10 the Galois traces of id, $\iota_1, \iota_2, \iota_3$ combine to a 4×4 matrix

$$
M = \begin{pmatrix}
u_1 & 9 - u_1 & 9 & 9 \\
u_2 & 9 - u_2 & 9 & 9 \\
u_3 & 1 - u_3 & 1 & -3 \\
u_4 & -1 - u_4 & 5 & 1
\end{pmatrix}.
$$

Here all u_i range through odd numbers in $\{-15, \ldots, 15\}$. Recall that M can be used to compute a (not necessarily unique) solution to the trace problem from point counts over $\mathbf{F}_p$ and Lefschetz' fixed point formula by (3.2). One easily checks that the vector $v_0 = (d_p, d_p, b_p, c_p)$ always gives such a solution at $p = 43, 71$. It remains to prove its uniqueness.

The determinant of M is $216(u_1 - u_2)$, so uniqueness follows at once if $u_1 \neq u_2$. In the sequel we thus assume $u_1 = u_2$, so that M has rank 3. The sign argument from Section 3.8 shows that $|u_1| < 9$. This narrows down the possible matrices M enough to compute each one's kernel which can be used to translate the base solution v_0. Recall that our solution is guaranteed to be integral. In fact, Lemma 3.12 leads us to consider a minimal vector $v_{\min}$ with entries in $4\mathbf{Z}$ spanning the kernel of M. Then we compare its entries against the Weil bounds for the traces of a 2-dimensional Galois representation which (after the Tate twist indicated in the lemma) give for the translation vector at $p = 71$ an absolute upper bound

$$
|v_{\min}|_{\max} \leq 32 = \lfloor 4\sqrt{71} \rfloor.
$$

This global estimate rules out all but nine possible translation vectors $v_{\min}$. For each remaining vector, one then checks at $p = 43, 71$ that both $v_0 + v_{\min}$ and $v_0 - v_{\min}$ would have an entry exceeding the Weil bound $2\sqrt{p}$ for the traces. Thus v_0 is the unique (integral 4-divisible) solution respecting the Weil bounds regardless of the choice of M. The claimed traces follow.   □

3.15. Conclusion of the proof of Theorem 1.1.   We have verified that the Galois representations from Theorem 1.1 have even traces, and in fact the

same traces at any prime from the non-cubic set 2.8 (as sketched in 3.2 for $p \equiv 1 \mod 4$, but with the appropriate quadratic twists for $p \equiv 3 \mod 4$). Hence the Galois representations are isomorphic (being semi-simple).   □

Remark 3.16. In particular, the above proof shows that the 2-dimensional Galois representations $W_{15,\ell,\mathbf{Q}}$ and $W_{30,\ell,\mathbf{Q}}$ are isomorphic as conjectured in [1, 7.3].

4. Proof of Theorem 1.3

The proof of Theorem 1.3 proceeds along considerably different lines. Namely we pursue a genuinely geometric approach in order to work out a correspondence over $\bar{\mathbf{Q}}$ between a suitable K3 quotient of S on the one hand and the product of an elliptic $\mathbf{Q}$-curve and one of its conjugates on the other hand. In fact, we shall use several isogenous K3 surfaces following Inose's notion [4] since K3 surfaces connected by a dominant rational map over $\mathbf{Q}$ share the same Picard number, $\mathbf{Q}$-Hodge structure and transcendental Galois representation.

4.1. K3 quotient.

Maschke's octic S comes with a natural action of the dihedral group D_4 of order 8, for instance generated by the involutions

$$(x_2 \mapsto -x_2), \quad (x_3 \mapsto -x_3) \quad \text{and} \quad (x_2, x_3) \mapsto (x_3, x_2).$$

The quotient surface S_1 has a simple model in weighted projective space $\mathbf{P}[1,1,2,4]$ given by the invariants $x_2^2 + x_3^2$ and $x_2^2 x_3^2$. This turns out to be K3, but for now we only need the geometric genus $p_g(S_1) = 1$ computed from the invariant two-forms on S. From the analysis of Maschke's octic we know that the transcendental motive of S_1 will be either $W'_{3,\ell}$, $W_{7,\chi_0,\ell}$ or $W_{7,\chi_0,\ell} \otimes \chi_{-1}$ see (1.1). Counting points over $\mathbf{F}_{11}$, for instance, we obtain the trace of F^*_{11} modulo 11 which agrees exactly with the last Galois representation. Hence

$$(4.1) \qquad\qquad T(S_1)_\ell = W_{7,\chi_0,\ell} \otimes \chi_{-1}.$$

Since all steps in the sequel will be performed over $\mathbf{Q}$, the same equality holds for all K3 surfaces derived from S_1 in the following. Throughout we will therefore only refer to the transcendental Galois representation T_ℓ.

4.2. Elliptic fibration.

In fact, S_1 comes with a natural elliptic fibration where the fibre is given as a double covering of $\mathbf{P}^1$ ramified in four points in terms of the above weighted coordinates. Completing the square we find a model with a section over $\mathbf{Q}(\sqrt{-3})$. Its jacobian S_2 over $\mathbf{Q}$ can be given in the standard coordinates by

$$S_2 : y^2 = x(x^2 + 21(t^2+1)^2 x + 144(t^4 + 2t^3 + 2t^2 - 2t + 1)$$
$$\times (t^4 - 2t^3 + 2t^2 + 2t + 1)).$$

This is visibly an elliptic K3 surface with two-torsion section $(0,0)$ and 8 fibres of Kodaira type I_2. In the following we exhibit various isogenous K3 surfaces in order to reach one fitting into a Shioda–Inose structure with a product of elliptic curves (cf. [8]). In detail we will derive an isogenous elliptic K3 surface with singular fibres of Kodaira type III^*, I_2, I_1 twice each and a two-torsion section such that the formulas from [3] will be applicable.

4.3. Strategy

A word about the strategy to find a suitable chain of isogenies might be in order. Generally, it seems advisable to reduce the complexity of the elliptic fibrations such that as much of the Néron–Severi group as possible is supported on the singular fibres (i.e., the Mordell–Weil rank is lowered). This not only facilitates explicit calculations, but also is in accordance with the low MW-ranks of the known elliptic fibrations involved in Shioda–Inose structures. In particular, this eases the analysis of the configuration of (-2)-curves on the K3 surface in order to single out divisors of the Kodaira types in question. The linear system of such a divisor then induces the desired elliptic fibration. Working out an explicit basis of such a linear system (consisting of a constant and the so-called elliptic parameter u) amounts to linear algebra in the function field of the K3 surface, but can be tricky nonetheless. This gives yet another reason to consider auxiliary elliptic fibrations (see Section 4.7).

   In the sequel we first lower the MW-rank to one (to be achieved in Section 4.6) and then aim for an elliptic fibration of the specified shape which we will reach in Section 4.8. Then we use the Shioda–Inose structure to determine the shape of the transcendental Galois representation T_ℓ and thus prove Theorem 1.3.

4.4. First isogeny

The parameter $u = x/(t^4 - 2t^3 + 2t^2 + 2t + 1)$ extracts an alternative elliptic fibration on S_2 with I_0^* fibres at $u = 0, \infty$. This fibration comes with 6 further singular fibres of type I_2; its jacobian has full two-torsion in MW and an involution induced from the base by $u \mapsto 12^2/u$.

4.5. Second isogeny

The involution on the base combines with the hyperelliptic involution for a symplectic involution on the jacobian of S_2 whose quotient K3 surface S_3 comes with the elliptic fibration

$$S_3: \quad y^2 = x(x^2 + 6(4t+7)(t-2)(t+2)x + 9(2t+5)(6t+13)$$
$$\times (t-2)^2(t+2)^2).$$

This has singular fibres of type I_0^* at ∞ and twice I_2, I_1^* each at the zeroes of the rightmost polynomial as well as a two-torsion section $(0,0)$.

4.6. Third isogeny

The two-isogeny results in an elliptic K3 surface S_4 with twice I_1, I_2^* on top of I_0^* (so that $\rho(S_4) \geq 18$ would follow from the Shioda–Tate formula and $\rho(S_4) = 19$ is equivalent to MW rank one):

$$S_4: \quad y^2 = x(x^2 - 3(4t+7)(t-2)(t+2)x + 9(t-2)^3(t+2)^3).$$

Recall that eventually we aim at relating S to an elliptic K3 surface with reducible fibres of type III^*, I_2 twice each. In fact this can easily be achieved on S_4 through an auxiliary fibration of root type $A_1 + E_7 + D_8$. However, it turns out that the final fibration, albeit being jacobian, does not come with a two-torsion section, so the results from [3] do not apply to it. Instead we shall go through a non-jacobian elliptic fibration on S_4 with the right fibre configuration.

4.7. Auxiliary fibrations

In order to extract the above singular fibres on S_4 (in a suitable way), we consider an auxiliary fibration with singular fibres I_2^* twice and I_2 four times

and full two-torsion. This can be given by the parameter $u = x/((t-2)(t+2)^2)$ which results in the Weierstrass form

$$S_4 : \quad y^2 = x(x - 3t(t-12))(x - t^2(4t-45)).$$

The fibre configuration might lead the reader to suspect that this fibration occurs on a Kummer surface of product type, confer the classification by Oguiso in [9]. However, there is one subtlety: the torsion sections do not meet the I_2^* fibres in symmetric components. Hence the fibration cannot be a base change of a rational elliptic surface, and thus does not coincide with a member of the Kummer fibration.

    We continue to extract two singular fibres of type III^* and I_2 each. This is achieved by the elliptic parameter $u = x/(t+90)$ which gives a non-jacobian elliptic fibration on S_4.

4.8. Fourth isogeny

The jacobian of this fibration gives another elliptic K3 surface S_5. Its Weierstrass form can be given as

$$S_5 : \quad y^2 = x(x^2 - 21t^2x - t^3(t^2 - 126t + 81)).$$

A quadratic twist over $\mathbf{Q}(\sqrt{-21})$ takes this to the form of [3, (7)] with I_2 fibres at $a, b = -\frac{1}{7} \pm \frac{4}{49}\sqrt{3}$. Through a Shioda–Inose structure as in [3, Section 4] this fibration is related to the product of elliptic curves over $K = \mathbf{Q}(\sqrt{3}, \sqrt{-5})$ with j-invariants

$$j(E) = 192632 - 111328\sqrt{3} - 145824\sqrt{-5} + 84280\sqrt{-15}$$

and its conjugate fixing $\sqrt{3}$. One directly verifies that these two curves do not have CM, but they are six-isogenous over $\bar{\mathbf{Q}}$ (by substituting into the corresponding modular polynomial). This implies that $\rho(S_5) = \rho(S_1) = 19$, and the transcendental lattice has rank 3 as required. In fact, the specified elliptic curve E is also two-isogenous to its conjugate fixing $\sqrt{-5}$, and thus three-isogenous to its conjugate fixing $\sqrt{-15}$. Comparing j-invariants we find that E is the elliptic $\mathbf{Q}$-curve $E_{-16/5}$ from [10, p. 312].

4.9. Galois representations of the K3 surfaces

Since E is $\bar{\mathbf{Q}}$-isogenous to its conjugates, the transcendental motive T_ℓ of S_5 (and thus of S_1) equals $\mathrm{Sym}^2 H^1_{\mathrm{ét}}(E, \mathbf{Q}_\ell)$ over some finite extension of $\mathbf{Q}$.

It remains to determine the precise shape of T_ℓ over $\mathbf{Q}$. The same applies to the K3 surface $X = X_{2/3}$ in the family $\mathcal{X}_3$ from [3]: the family is parameterized by the modular curve $X^*(6)$, and the given special member was shown to correspond to $E = E_{-16/5}$ by a similar Shioda–Inose structure. Thus we can right away compare the 3-dimensional Galois representations T_ℓ of S_1 and $T(X)_\ell$ of X over $\mathbf{Q}$. Since they agree over some extension while being integral over $\mathbf{Q}$, the eigenvalues of Frobenius can only differ by signs. These signs are encoded in quadratic characters which we compute explicitly below (higher-order characters can only occur in the CM case for the fields $\mathbf{Q}(\sqrt{-1}), \mathbf{Q}(\sqrt{-3})$).

4.10. Determinants

First we compare the determinants of T_ℓ and $T(X)_\ell$. By [1, 5.7] combined with (4.1), T_ℓ has determinant

$$\det(T_\ell) = \chi_5 \chi_\ell^3.$$

For X, we use point counting and Lefschetz' fixed point formula. Note that 19 out of 20 eigenvalues $\pm p$ are determined by the Néron–Severi group of X over $\bar{\mathbf{Q}}$ via reduction. For some primes, the Weil bounds then predict the 20th eigenvalue. In detail we find

p	11	53	107	127	139	179
sign	$-$	$-$	$+$	$+$	$-$	$-$

The signs at $11, 53, 107, 139$ determine the quadratic character unramified outside $\{2, 3, 5\}$ uniquely as χ_{-5}. Since the 20th eigenvalue exactly gives the determinant of $T(X)_\ell$, we deduce

$$\det(T(X)_\ell) = \chi_{-5} \chi_\ell^3.$$

4.11. Quadratic twist

We are now in the position to compare the signs of the remaining 2 eigenvalues of Frobenius. For this purpose we define a quadratic character on all primes $p \neq 2, 3, 5$ which are not supersingular for E (so for instance $p \neq 7$):

$$\phi: \quad p \mapsto \frac{\operatorname{tr} F_p(T_\ell) - \chi_5(p)\, p}{\operatorname{tr} F_p(T(X)_\ell) - \chi_{-5}(p)\, p}.$$

Numerator and denominator (non-zero by assumption) encode the traces of the other two eigenvalues, thus ϕ gives their sign difference. As a quadratic character unramified outside $\{2, 3, 5\}$, ϕ is determined by its values at, say, $\{11, 13, 17, 19\}$. Counting points we find that $\phi = \chi_{-1}$. Comparison with the determinants yields that all signs of the eigenvalues of T_ℓ and $T(X)_\ell$ differ by this character:

$$(4.2) \qquad T(X)_\ell = T_\ell \otimes \chi_{-1} = W_{7, \chi_0, \ell}.$$

4.12. Modularity of E

As the elliptic $\mathbf{Q}$-curve has an isogeny class defined over $\mathbf{Q}$, it comes attached with a Galois representation over $\mathbf{Q}$. To work this out, we fix the Weierstrass model

$$E: \quad y^2 = x^3 - \frac{648}{25} \left(3 + \sqrt{-15}\right)^2 \left(-11 + 3\sqrt{-15} + 8\sqrt{-5} - \frac{24}{5}\sqrt{3}\right) x$$
$$- \frac{5184}{125} \left(3 + \sqrt{-15}\right)^3 \sqrt{3} \left(1 + \frac{4}{5}\sqrt{-5}\right)$$
$$\times \left(-\frac{77}{5} + 9\sqrt{-15} + \frac{56}{5}\sqrt{-5} - \frac{72}{5}\sqrt{3}\right).$$

This is the quadratic twist of the curve from [10, p. 312] evaluated at $a = -16/5$ over $K\left(\sqrt{\frac{3+\sqrt{-15}}{2}}\right)$ that makes E two-isogenous over K to its conjugate E^σ where $\sigma \in \mathrm{Gal}(K/\mathbf{Q})$ fixes $\sqrt{-5}$. Attached to E there is a newform $f1200 \in S_2(\Gamma_0(1200), \chi_3)$ with character χ_3 and coefficient field $F = \mathbf{Q}(\sqrt{-2}, \sqrt{-3})$. Here $f1200$ is determined uniquely up to inner twists (accounting for the conjugations in F) by the Hecke eigenvalues

p	7	11	13	17	19	23	29	31	37	41	43	47	53
b_p	0	$2\sqrt{6}$	$2\sqrt{6}$	$2\sqrt{-3}$	$-2\sqrt{-3}$	-6	$-2\sqrt{-2}$	$-2\sqrt{-3}$	$2\sqrt{6}$	$4\sqrt{-2}$	$6\sqrt{-2}$	6	$6\sqrt{-3}$

Comparing coefficients and traces at the primes from 2.8, one directly verifies that

$$H^1(E, \mathbf{Q}_\ell) \cong \mathrm{Res}_{\mathbf{Q}}^K V_{f1200, \ell}$$

as $\mathrm{Gal}(\bar{K}/K)$-representations. In particular, this also implies that the symmetric squares of these Galois representations are isomorphic over K.

4.13. Proof of Theorem 1.3

We have seen that T_ℓ and $\mathrm{Sym}^2 V_{f1200,\ell}$ agree over some extension. Hence we can proceed as in Section 4.10 and 4.11 to determine the precise relation. By construction, we have

$$\det \mathrm{Sym}^2 V_{f1200,\ell} = \chi_3 \, \chi_\ell^3.$$

Then one checks that also the irrational eigenvalues of T_ℓ and $\mathrm{Sym}^2 V_{f1200,\ell}$ are related by the character χ_{15} (as they should be since we have two absolutely irreducible Galois representations over $\mathbf{Q}$). Thus we deduce

(4.3) $T_\ell \cong (\mathrm{Sym}^2 V_{f1200,\ell}) \otimes \chi_{15}.$

By (1.1) [$\sim$l.90/91] and (4.1), we know that

$$T_{S,\ell} = (V_{f15,\ell})^5 \oplus (T_\ell \otimes \chi_{-1})^{18} \oplus T_\ell^{12}.$$

Together with (4.3) this combines to the statement of Theorem 1.3. $\square$

Remark 4.14.   Note that the above argument proves in particular the modularity of the K3 surfaces $S_1, \ldots, S_5$ and X over $\mathbf{Q}$. For instance we obtain

$$T(X)_\ell \cong (\mathrm{Sym}^2 V_{f1200,\ell}) \otimes \chi_{-15}.$$

Acknowledgments

References

[1] G. Bini and B. van Geemen, *Geometry and arithmetic of Maschke's Calabi–Yau threefold*, to appear in Commun. Number Theory Phys., this issue.

[2] B. J. Birch and W. Kuyk (eds.), *Modular functions of one variable IV*, Springer LNM **476**, 1975.

[3] A. Garbagnati and M. Schütt, *Enriques surfaces — Brauer groups and Kummer structures*, to appear in Michigan Math. J., 2010, arXiv:1006.4952.

[4] H. Inose, *Defining equations of singular K3 surfaces and a notion of isogeny*, Proceedings of the International Symposium on Algebraic Geometry (Kyoto University, Kyoto, 1977), Kinokuniya Book Store, Tokyo, 1978, 495–502.

[5] J. Jones, *Tables of number fields with prescribed ramification*, available at http://hobbes.la.asu.edu/numberfields-old.

[6] R. Livné, *Cubic exponential sums and Galois representations*, in 'Current trends in arithmetical algebraic geometry', (Arcata, CA, 1985)', Contemp. Math. **67**, ed. K. Ribet Amer. Math. Soc., Providence RI, 1987, 247–261.

[7] B. Mazur, *Frobenius and the Hodge filtration*, Bull. AMS **78** (1972), 653–667.

[8] D. R. Morrison, *On K3 surfaces with large Picard number*, Invent. Math. **75** (1) (1984), 105–121.

[9] K. Oguiso ,*On Jacobian fibrations on the Kummer surfaces of the product of nonisogenous elliptic curves*, J. Math. Soc. Japan **41** (4) (1989), 651–680.

[10] J. Quer, *Q-curves and abelian varieties of* GL$_2$*-type*, Proc. London Math. Soc. **81** (2000), 285–317.

[11] M. Schütt, *CM newforms with rational coefficients*, Ramanujan J. **19** (2009), 187–205.

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