COMMUNICATIONS IN
NUMBER THEORY AND PHYSICS
Volume 5, Number 4, 779–826, 2011

Geometry and arithmetic of Maschke's Calabi–Yau three-fold

Gilberto Bini and Bert van Geemen

Maschke's Calabi–Yau three-fold is the double cover of projective three space branched along Maschke's octic surface. This surface is defined by the lowest degree invariant of a certain finite group acting on a four-dimensional (4D) vector space. Using this group, we show that the middle Betti cohomology group of the three-fold decomposes into the direct sum of 150 2D Hodge substructures. We exhibit 1D families of rational curves on the three-fold and verify that the associated Abel–Jacobi map is non-trivial. By counting the number of points over finite fields, we determine the rank of the Néron–Severi group of Maschke's surface and the Galois representation on the transcendental lattice of some of its quotients. We also formulate precise conjectures on the modularity of the Galois representations associated to Maschke's three-fold (these have now been proven by M. Schütt) and to a genus 33 curve, which parametrizes rational curves in the three-fold.

The Hodge structure on the middle dimensional Betti cohomology group of a Calabi–Yau (CY) three-fold carries important information on the moduli and the one-dimensional (1D) algebraic cycles on the three-fold. However, if the three-fold is easy to define, say by one equation in a (weighted) projective space, the dimension h^3 of this vector space tends to be large. For example, a smooth quintic three-fold in $\mathbf{P}^4$ has $h^3 = 204$ and a double octic, i.e., a double cover of $\mathbf{P}^3$ branched along a smooth surface of degree 8, has $h^3 = 300$. Using automorphisms of the three-folds, one can decompose the cohomology into subrepresentations, which give rise to Hodge substructures. In this paper, we consider a double octic with a particularly large automorphism group $\overline{G}$ of order $16 \cdot (6!) = 11,520$.

In 1887, Heinrich Maschke studied the invariants of a finite group G, related to genus two theta functions, acting on a 4D complex vector space [20]. The group $\overline{G}$ is the image of G in $Aut(\mathbf{P}^3)$, it is the quotient of G by its center, which is cyclic of order four. Maschke's main result is the determination of generators of the ring of G-invariants.

In particular, the lowest degree invariant has degree 8 and is given by [20], equation (12)

$$F := \sum_{i=0}^{3} x_i^8 + 14 \sum_{i<j} x_i^4 x_j^4 + 168 x_0^2 x_1^2 x_2^2 x_3^2.$$

The octic surface S in $\mathbf{P}^3$ defined by this polynomial

$$S := \text{Zeroes}(F) \quad (\subset \mathbf{P}^3)$$

is a smooth surface, which we call Maschke's surface. This surface, and a subgroup G_8 of its automorphism group, appeared recently in [4]. The double cover X of $\mathbf{P}^3$, branched along S, is a smooth CY three-fold, which we call Maschke's double octic.

Using an easy generalization of a formula of Chênevert [8] (see Section 10) and a computer, we find in Section 6.2 that $H^3(X, \mathbf{Q})$ is a direct sum of 2D Hodge substructures. As the action of G on $\mathbf{C}^4$ is given by matrices with coefficients in the field $\mathbf{Q}(i)$, $i^2 = -1$, the étale cohomology group $H^3_{\text{ét}}(X, \mathbf{Q}_\ell)$ is decomposed into 2D $G_{\mathbf{Q}(i)} := Gal(\overline{\mathbf{Q}}/\mathbf{Q}(i))$-representations. Our numerical results, using point counting over finite fields with MAGMA [19] and the Lefschetz fixed point formula, suggest that $H^3_{\text{ét}}(X, \mathbf{Q}_\ell)$ is a direct sum of 2D $G_{\mathbf{Q}} := Gal(\overline{\mathbf{Q}}/\mathbf{Q})$-representations. For such representations there are now various modularity results, and we did succeed in matching the numerical data to explicitly given elliptic modular forms. Although we were not able to provide a proof for our conjectured decomposition given in Section 7.3, recently M. Schütt [29] succeeded in doing so.

The decomposition of $H^3(X, \mathbf{Q})$ shows that the Griffiths intermediate Jacobian of X has an abelian subvariety $J(X)_a$ of dimension 149. In Section 8 we describe a family of rational curves on X, parametrized by a curve $\widetilde{C}_+$, such that the Abel–Jacobi map $J(\widetilde{C}_+) \to J(X)_a$ is non-constant. This provides some evidence for the generalized Hodge conjecture for X. We also analyze the étale cohomology of $\widetilde{C}_+$ in Section 9. It seems that $J(\widetilde{C}_+)$ is isogeneous to a product of 33 elliptic curves, and that the image of the Abel–Jacobi map might be 'as big as possible', that is, the image might have dimension 24.

A somewhat similar study of the Abel–Jacobi map for complete intersections of four quadrics in $\mathbf{P}^7$ with an involution was done in [1]. The action of a finite group and the Galois representations associated with the quintic three-folds of the Dwork pencil were analyzed in [5]. In that case, each three-fold has a 1D family of lines, which was analyzed in [24], and

recently Candelas obtained an explicit description of this family [6]. As far as we know, the Abel–Jacobi map has not yet been described in detail. The paper [15] considers surfaces, which are products of a $\mathbf{P}^1$ and an elliptic curve E in certain CY three-folds and shows that in some cases $H^1(E, \mathbf{Q}) \cong H^3(E \times \mathbf{P}^1, \mathbf{Q})$ contributes to $H^3(X, \mathbf{Q})$. This can be viewed as an instance of an injective Abel–Jacobi map associated to a family of $\mathbf{P}^1$'s parametrized by E.

The group G also acts on Maschke's surface S and in Sections 3, 4 and 5 we use it to study the Hodge structure on $H^2(S, \mathbf{Q})$ and the Galois representation on $H^2_{\text{ét}}(S, \mathbf{Q}_\ell)$. This allows us to determine the rank of the Néron–Severi group of S, from which we conclude that the classes of lines in S generate a subgroup of finite index. We also give projective models of some quotients of S.

In the first section, we briefly recall how a finite group of automorphisms of an algebraic variety X can be used to decompose the cohomology of X. The group G is introduced in Section 2.

We are indebted to Matthias Schütt for his comments on a first draft of this paper.

1. Decomposing cohomology groups

1.1. Overview

The Betti cohomology groups $H^k(X, \mathbf{Q})$ of a smooth complex projective variety X are rational Hodge structures of weight k. A subgroup G of $Aut(X)$ induces a decomposition of these groups into Hodge substructures. Similarly, if X and each $g \in G$ are defined over $\mathbf{Q}$, one obtains a corresponding (via the comparison theorem) decomposition of the étale cohomology groups of X into $G_{\mathbf{Q}}$-representations.

We briefly recall the basics of rational Hodge structures (see for example [30], Chapter 7), representations of finite groups and their applications to algebraic geometry.

1.2. Rational Hodge structures

The Betti cohomology groups $H^k(X, \mathbf{Q})$ of a smooth complex projective variety X are rational Hodge structures of weight k. A rational Hodge structure is a finite dimensional $\mathbf{Q}$-vector space with a decomposition of its complexification $V_{\mathbf{C}} := V \otimes \mathbf{C} = \oplus V^{p,q}$ with $p, q \in \mathbf{Z}_{\geq 0}$, $p + q = k$ and $\overline{V^{p,q}} = V^{q,p}$. A morphism of Hodge structures is a $\mathbf{Q}$-linear map $f : V \to$

W such that its $\mathbf{C}$-linear extension satisfies $f(V^{p,q}) \subset W^{p,q}$ for all p, q. Morphisms between varieties induce morphisms of Hodge structures on the Betti cohomology groups. For a Hodge structure V, the (endo)morphisms of Hodge structures $V \to V$ form a $\mathbf{Q}$-algebra (so $\lambda f + \mu g$, $fg := f \circ g$ are morphisms of Hodge structures if f, g are, for any $\lambda, \mu \in \mathbf{Q}$). Thus if $G \subset Aut(X)$ is a subgroup, we get an action, by Hodge endomorphisms, of G on each $H^k(X, \mathbf{Q})$.

A subspace $W \hookrightarrow V$ is called a Hodge substructure if $W_{\mathbf{C}} = \oplus W^{p,q}$ where $W^{p,q} := W \cap V^{p,q}$, in that case W is a Hodge structure and the inclusion is a morphism of Hodge structures. The kernel and image of a morphism of Hodge structures are easily seen to be Hodge substructures.

A Hodge structure V is called simple if the only Hodge substructures of V are $\{0\}$ and V. As the Hodge structure on $H^k(X, \mathbf{Q})$ is polarized, it is a direct sum of simple Hodge structures and these are unique up to isomorphism.

An automorphism $\phi : X \to X$ induces endomorphisms of Hodge structures ϕ^* on $H^k(X, \mathbf{Q})$ for each k. For any $a_0, \ldots, a_m \in \mathbf{Q}$, one has the Hodge endomorphism $\sum_{i=0}^{m} a_i (\phi^*)^i$, the kernel and image of which are thus Hodge substructures of V.

1.3. Representations of finite groups

We recall some of the basics on representations of finite groups. Let G be a finite group, let $G = C_1 \coprod \ldots \coprod C_M$ be the partition of G in conjugacy classes (so $g, g' \in C_i$ for some i iff $g = hg'h^{-1}$ for some $h \in G$). A representation of G is a homomorphism of groups $\rho : G \to GL(V)$ where V is a finite dimensional vector space over a field K. A representation is said to be irreducible (over K) if the only subspaces $W \subset V$ with $\rho(g)W \subset W$ for all $g \in G$ are $W = \{0\}$ and $W = V$. A representation $\rho' : G \to GL(V')$ is said to be equivalent (or isomorphic) to ρ if there is a K-linear isomorphism $A : V \to V'$ such that $\rho'(g) = A\rho(g)A^{-1}$ for all $g \in G$.

The group G has M irreducible complex (i.e., $K = \mathbf{C}$) representations, we denote them by $\rho_1, \ldots, \rho_M$ on vector spaces $V_1, \ldots, V_M$. A complex representation of G (so $K = \mathbf{C}$) is isomorphic to a direct sum of the irreducible representations, $V \cong \oplus V_i^{n_i}$.

1.4. Characters

The isomorphism class of a complex representation ρ is determined by its character χ_ρ, so ρ, ρ' are isomorphic representations iff $\chi_\rho(g) = \chi_{\rho'}(g)$ for all

$g \in G$. Here the character of ρ is the function

$$\chi_\rho : G \longrightarrow \mathbf{C}, \quad \chi_\rho(g) := \mathrm{Tr}(\rho(g)),$$

where Tr is the trace of a linear map. This function is constant on the conjugacy classes. Note that if $V \cong \oplus V_i^{n_i}$, then $\chi_\rho = \sum n_i \chi_i$ where $\chi_i := \chi_{\rho_i}$ is the character of the irreducible representation ρ_i. The decomposition of a character into irreducibles is easily done using a scalar product on the space of characters for which the χ_i are an orthonormal basis. The trace of the action of an automorphism on the primitive cohomology of a hypersurface or a ramified cover of $\mathbf{P}^{n+1}$ can be found in Proposition 10.1.

1.5. Decomposing representations

Given a complex representation ρ of G and a conjugacy class C_j, consider the linear map

$$c_{j,\rho} := \sum_{g \in C_i} \rho(g) \quad (\in \mathrm{End}(V)).$$

By construction, $\rho(h) c_{j,\rho} \rho(h)^{-1} = c_{j,\rho}$ for all $h \in G$. By Schur's lemma, $c_{j,\rho}$ is then scalar multiplication by a complex number $\lambda_{i,j}$ on any irreducible subrepresentation $V_i \subset V$. This scalar can be easily found by computing the trace of $c_{j,\rho}$ on V_i in two ways. The first is simply $\mathrm{Tr}(c_{j,\rho}|V_i) = (\dim V_i)\lambda_{i,j}$, the other uses a character: $\mathrm{Tr}(c_{j,\rho}|V_i) = (\sharp C_j)\chi_i(g)$, for any $g \in C_i$. Thus on the subspace $V_i^{n_i}$ of V we get, for any $g \in C_i$:

$$\lambda_{i,j} := (c_{j,\rho})_{|V_i^{n_i}} = \frac{\sharp C_j}{\dim V_i} \chi_i(g), \quad \text{define} \quad \widetilde{V}_i := \cap_{j=1}^{M} \ker(c_{j,\rho} - \lambda_{i,j}).$$

Then, by construction, $\widetilde{V}_i$ is a subrepresentation of V, and we have $V_i^{n_i} \subseteq \widetilde{V}_i$. One actually has equality here, since $\mathrm{Tr}(\rho(g)_{|\widetilde{V}_i}) = n_i \chi_i(g)$, for all $g \in G$ which implies that the character of the representation $\rho(g)_{|\widetilde{V}_i}$ coincides with the character of $V_i^{n_i}$. Hence these representations are isomorphic:

$$\widetilde{V}_i \cong V_i^{n_i},$$

and this provides an explicit method to decompose V into isotypical components.

1.6. Applications

In this paper, we use the action of finite groups $G \subset Aut(X)$ of automorphisms of smooth complex varieties on the Betti cohomology groups $H^k(X, \mathbf{Q})$ of X. In particular, if ρ_i is an irreducible representation of G such that its character satisfies $\chi_i(g) \in \mathbf{Q}$ for all $g \in G$, then each $\lambda_{i,j} \in \mathbf{Q}$ and we can split

$$H^k(X, \mathbf{Q}) = \widetilde{V}_i \oplus \widetilde{V}_i' \quad \widetilde{V}_i := \cap_{j=1}^{M} \ker(c_{j,\rho} - \lambda_{i,j}) \quad (\chi_i(g) \in \mathbf{Q} \quad \forall g \in G),$$

where $\rho : G \to GL(H^k(X, \mathbf{Q}))$ is the action of G, $\rho(g) := g^*$. As observed in Section 1.2, $\widetilde{V}_i$ and $\widetilde{V}_i'$ will be rational Hodge structures.

Similarly, one can decompose the étale cohomology group $H^k_{\text{ét}}(X, \mathbf{Q}_\ell) = \widetilde{V}_{i,\ell} \oplus \widetilde{V}_{i,\ell}'$. The comparison isomorphism between étale and Betti cohomology groups implies that these decompositions coincide after embedding $\mathbf{Q}_\ell$ into $\mathbf{C}$ and tensoring by $\mathbf{C}$.

1.7. Commuting group actions

Let again $\rho : G \to GL(V)$ be a representation of the finite group G on a K-vector space V and let

$$V \cong \oplus_i \widetilde{V}_i, \quad \widetilde{V}_i \cong V_i^{n_i}$$

be the decomposition into representations V_i which are irreducible over K. More intrinsically, one has

$$\widetilde{V}_i \cong V_i \otimes W_i, \quad W_i := \mathrm{Hom}_G(V_i, V) = \mathrm{Hom}_G(V_i, \widetilde{V}_i) \cong K^{n_i},$$

the G-equivariant maps; the isomorphism is given by $v \otimes f \mapsto f(v)$. In case another group H acts linearly on V and the actions of G and H commute, one has an action of the group $G \times H$ on V. Essentially by Schur's lemma (cf. [14], Section 3.3) each W_i is then an H-representation space and the action of $G \times H$ on V is given by

$$V = \oplus_i V_i \otimes W_i, \quad (g, h)(v \otimes w) = (gv) \otimes (hw).$$

In particular, as an H-representation one has

$$V \cong \oplus_i W_i^{m_i}, \quad m_i := \dim V_i.$$

In case G acts on $V = H^k(X, \mathbf{Q})$ by morphisms of Hodge structures, then W_i is a rational Hodge structure and $\widetilde{V}_i \cong W_i^{m_i}$, where $m_i = \dim V_i$, is an isomorphism of rational Hodge structures. In fact, the rational Hodge structure on $\widetilde{V}_i$ is defined by a representation

$$h : \mathbf{C}^* \longrightarrow GL(\widetilde{V}_{i,\mathbf{R}}), \quad h(z)v_{p,q} := z^p \overline{z}^q v_{p,q},$$

where $v_{p,q} \in (\widetilde{V}_i)^{p,q}$ and the action of h is extended $\mathbf{C}$-linearly to $\widetilde{V}_{i,\mathbf{C}}$ where $\widetilde{V}_{i,K} := \widetilde{V}_i \otimes_{\mathbf{Q}} K$. As $g \in G$ preserves the Hodge structure, it must commute with $h(z)$ for all z and hence $G \times \mathbf{C}^*$ acts on $V_{i,\mathbf{R}}$. The representation h is then obtained from a representation $h' : \mathbf{C}^* \to GL(W_i)$. This representation defines the Hodge structure on W_i.

In case $X \subset \mathbf{P}^n$ is defined by equations with coefficients in a number field L and the action of any $g \in G \subset Aut(X)$ is given by maps with coefficients in L, the maps g^* will commute with the action of $G_L := Gal(\overline{L}/L)$ on $V_\ell := H^k_{\text{ét}}(X, \mathbf{Q}_\ell)$. Thus, as a $G \times G_L$-representation,

$$V_\ell \cong \oplus_i \widetilde{V}_{i,\ell} \cong \oplus_i V_{i,\ell} \otimes W_{i,\ell}.$$

In particular, the G-representation $\widetilde{V}_{i,\ell} \cong V_{i,\ell}^{n_i}$ defines a Galois representation on $W_{i,\ell}$, with $\dim W_{i,\ell} = n_i$, and as G_L-representations $\widetilde{V}_{i,\ell} \cong W_{i,\ell}^{m_i}$ where $m_i := \dim V_i$.

1.8. Cycles and motives

The splitting of the various cohomology groups comes from cycles on the product $X \times X$ and is thus motivic. The graph $\Gamma_g := \{(x, gx) : x \in X\}$ has a class $[\Gamma_g] \in H^{2n}(X \times X)$ with $n = \dim_{\mathbf{C}} X$ (where one can choose any Weil cohomology group). Using the Künneth theorem and Poincaré duality to identify $\text{End}(H^k(X))$ with $H^{2n-k}(X) \otimes H^k(X)$, the Künneth components of this class induce maps, compatible with the Hodge structure or the Galois action, which are just the g^* on $H^k(X)$:

$$g^* = [\Gamma_g]_k : H^k(X) \longrightarrow H^k(X).$$

The operators $c_{j,\rho}$ are thus induced from (the action of) certain cycles in $CH^*(X \times X)$ and will act on any Weil cohomology group of X.

2. The group G

2.1. A Heisenberg group H

For the groups in these sections we refer to [7].

The Heisenberg group we will use is defined as the set

$$H = \mu_4 \times (\mathbf{Z}/2\mathbf{Z})^2 \times (\mathbf{Z}/2\mathbf{Z})^2, \quad \mu_4 := \{z \in \mathbf{C} : z^4 = 1\},$$

so, with the obvious group structure, H would be isomorphic to $(\mathbf{Z}/4\mathbf{Z}) \times (\mathbf{Z}/2\mathbf{Z})^4$. However, we define the group operation by

$$\begin{aligned}(s, x, x^*)(t, y, y^*) &:= (st(-1)^{y^*(x)}, x + y, x^* + y^*) \quad \text{with} \\ y^*(x) &:= y_1^* x_1 + y_2^* x_2,\end{aligned}$$

where $x = (x_1, x_2), y^* = (y_1^*, y_2^*) \in (\mathbf{Z}/2\mathbf{Z})^2$. One easily verifies that

$$(s, x, x^*)^{-1} = (s^{-1}(-1)^{x^*(x)}, x, x^*)$$

and that the commutator of two elements in H is given by:

$$(s, x, x^*)(t, y, y^*)(s, x, x^*)^{-1}(t, y, y^*)^{-1} = ((-1)^{x^*(y) - y^*(x)}, 0, 0).$$

The center of H is μ_4 and the commutator defines a symplectic form E on the $\mathbf{F}_2$-vector space $H/\mu_4 \cong (\mathbf{Z}/2\mathbf{Z})^4$:

$$E((x, x^*), (y, y^*)) = y^*(x) - x^*(y) = y_1^* x_1 + y_2^* x_2 + x_1^* y_1 + x_2^* y_2,$$

with $x = (x_1, x_2)$, $x^* = (x_1^*, x_2^*)$, $y = (y_1, y_2)$ and $y^* = (y_1^*, y_2^*)$, all in $(\mathbf{Z}/2\mathbf{Z})^2$.

2.2. The Schrödinger representation

There is a (unique, faithful) representation U of the finite group H on the vector space $\mathbf{C}^4$, called the Schrödinger representation, such that $s \in \mu_4$ acts as scalar multiplication by s. Identifying $\mathbf{C}^4$ with the $\mathbf{C}$-vector space of maps $f : (\mathbf{Z}/2\mathbf{Z})^2 \to \mathbf{C}$ this representation is defined as:

$$\Big(U_{(s,x,x^*)} f\Big)(z) := s(-1)^{x^*(z)} f(x + z),$$

where $U_{(s,x,x^*)} \in GL(4, \mathbf{C})$ gives the action of $(s, x, x^*) \in H$. The δ-functions provide a basis of this vector space: $\delta_x(y) = 0$ if $x \neq y$ and $\delta_x(x) = 1$ for

$x, y \in (\mathbf{Z}/2\mathbf{Z})^2$. The Schrödinger representation is then:

$$U_{(s,x,x^*)}\delta_a := s(-1)^{x^*(x+a)}\delta_{x+a}.$$

Identifying $\{0,1\}$ with $\mathbf{Z}/2\mathbf{Z}$ and writing

$$x_0 = \delta_{00}, \quad x_1 = \delta_{01}, \quad x_2 = \delta_{10}, \quad x_3 = \delta_{11}$$

for the basis of $\mathbf{C}^4$ and writing

$$h_{abcd} := U_{(1,(a,b),(c,d))}, \qquad c := U_{(i,(0,0),(0,0))}, \quad i^2 = -1,$$

the Schrödinger representation on the four variables $x_0, \ldots, x_3$ is:

$$x := (x_0, x_1, x_2, x_3) \longmapsto \begin{cases} h_{0001}(x) &= (x_0, -x_1, x_2, -x_3) \\ h_{0010}(x) &= (x_0, x_1, -x_2, -x_3) \\ h_{0100}(x) &= (x_1, x_0, x_3, x_2) \\ h_{1000}(x) &= (x_2, x_3, x_0, x_1) \\ c(x) &= (ix_0, ix_1, ix_2, ix_3). \end{cases}$$

2.3. The group G

For convenience, we will now identify the abstract group H with its image under U in $GL(4, \mathbf{C})$. The normalizer N in $GL(4, \mathbf{C})$ of H is defined as:

$$N := \{M \in GL(4, \mathbf{C}) : MHM^{-1} = H\}.$$

An element $M \in N$ induces an automorphism of H, which we also denote by M,

$$M : H \longrightarrow H, \quad h \longmapsto h' \quad \text{if} \quad MU_hM^{-1} = U_{h'}.$$

As N acts by automorphisms on H, each $M \in N$ induces a linear map ϕ_M on the quotient of H by its center, $H/\mu_4 \cong (\mathbf{Z}/2\mathbf{Z})^4$. These linear maps are easily seen to preserve the symplectic form E:

$$E(\phi_M v, \phi_M w) = E(v, w), \quad (M \in N, \phi_M \in Aut((\mathbf{Z}/2\mathbf{Z})^4) = GL(4, \mathbf{F}_2)),$$

for all $v, w \in (\mathbf{Z}/2\mathbf{Z})^4$. Thus we get a homomorphism to a finite symplectic group:

$$N \longrightarrow Sp(4, \mathbf{F}_2), \qquad M \longmapsto \phi_M,$$

which can be shown to be surjective. The group $Sp(4, \mathbf{F}_2)$ is isomorphic to the symmetric group S_6: see, for instance, [7], Appendix C.

Any element $U_w \in H$ is obviously an element of N. Since $U_w U_v U_w^{-1} = (-1)^{E(v,w)} U_v$ we get $\phi_{\mathrm{M}} = I$ if $M = U_w \in H \subset N$. The homomorphism above fits in an exact sequence:

$$0 \longrightarrow \mathbf{C}^\times \cdot H \longrightarrow N \longrightarrow S_6 \longrightarrow 0.$$

The group G is the subgroup of N generated by the following two elements in $GL(4, \mathbf{C})$:

$$g_1 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}, \quad g_2 := \frac{1}{2} \begin{pmatrix} -1 & -i & -i & -1 \\ i & 1 & -1 & -i \\ i & -1 & 1 & -i \\ 1 & -i & -i & 1 \end{pmatrix}, \quad i := \sqrt{-1}.$$

The group G maps onto S_6 and $G \cap (\mathbf{C}^\times \cdot H) = H$. It thus has order

$$\sharp G = (\sharp H)(\sharp S_6) = 2^6 \cdot (6!) = 46,080.$$

The homomorphism $G \to S_6$ can be chosen so that the images of g_1, g_2 in S_6 are the cycles (12) and (123456), respectively. We used the computer program MAGMA [19] for computations involving H and G.

3. The second cohomology group of S

3.1. The Hodge decomposition of $H^2(S, \mathbf{Q})$

The second (singular) cohomology group with coefficients in $\mathbf{Q}$ of an octic surface in $\mathbf{P}^3$ has dimension $b_2(S) = 302$. As for any smooth surface, this rational Hodge structure splits as

$$H^2(S, \mathbf{Q}) = T_{S,\mathbf{Q}} \oplus NS(S)_{\mathbf{Q}} \quad (NS(S)_{\mathbf{Q}} := NS(S) \otimes_{\mathbf{Z}} \mathbf{Q}).$$

The Hodge substructure $NS(S)_{\mathbf{Q}}$ is the Néron–Severi group of S tensored with $\mathbf{Q}$, it is pure of type $(1,1)$. The transcendental substructure, which is the orthogonal complement of $NS(S)_{\mathbf{Q}}$ w.r.t. the intersection form on H^2,

has the Hodge decomposition

$$T_{S,\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{C} = T_S^{2,0} \oplus T_S^{1,1} \oplus T_S^{0,2}, \quad \overline{T_S^{2,0}} = T_S^{0,2}.$$

One has $h^{2,0}(S) = \dim T_S^{2,0} = \dim h^0(\omega_S) = p_g(S)$, the geometric genus of S (and ω_S is the canonical bundle of S). For an octic surface we have, by adjunction, that $\omega_S \cong \mathcal{O}_S(4)$ and thus

$$H^{2,0}(S) \cong R_4, \quad \text{where} \quad R_4 = H^0(\mathbf{P}^3, \mathcal{O}(4))$$

is the complex vector space of homogeneous polynomials of degree four in $x_0, \ldots, x_3$. Hence $h^{2,0}(S) = \dim R_4 = 35$ and

$$\begin{aligned} &b_2(S) = \dim H^2(S, \mathbf{Q}) = 302, \\ &\begin{cases} h^{2,0}(S) &= \dim H^{2,0}(S) &= 35, \\ h^{1,1}(S) &= \dim H^{1,1}(S) &= 302 - 2 \cdot 35 = 232. \end{cases} \end{aligned}$$

From this we get that $\dim NS(S)_{\mathbf{Q}} \leq 232$ and $\dim T_{S,\mathbf{Q}} \geq 70$. We need to work a little harder to actually find these dimensions, they are $\dim NS(S)_{\mathbf{Q}} = 202$, $\dim T_{S,\mathbf{Q}} = 100$, see Sections 3.3, 5.6.

3.2. The Néron–Severi group of S

To get a lower bound on the rank of the Néron–Severi group of S, we study the subgroup L_S spanned by the lines in S. In [4] it was shown that the surface S contains exactly 352 lines. Under the action of G, there are two orbits on the set of lines. The orbit of the line

$$l_3 : \quad \langle (\alpha : 1 : 0, 0),\ (0 : 0 : \alpha : 1) \rangle, \quad \alpha^4 - 2\alpha^3 + 2\alpha^2 + 2\alpha + 1 = 0,$$

($\alpha = \zeta_{12} + \zeta_{12}^2$, where ζ_{12} is a primitive 12-th root of unity) contains 160 lines, whereas the orbit of l_5 contains 192 elements with

$$l_5 : \langle (1 : a : a\sqrt{-1} : 0),\ (0 : a : -a\sqrt{-1} : 1) \rangle, \quad a := (1 + \sqrt{-1})(1 + \sqrt{5})/4.$$

Using a computer, we found that the rank of the symmetric matrix of intersection numbers $(l \cdot m)$, where l, m run over the 352 lines on S, is 202. Hence these lines span a subspace L_S of dimension 202 of $NS(S)_{\mathbf{Q}}$ ($\subset H^{1,1}(S)$). We will see later that actually $L_S = NS(S)_{\mathbf{Q}}$, but at this point we only know:

$$\dim L_S = 202 \leq \dim NS(S)_{\mathbf{Q}} \leq h^{1,1}(S) = 232.$$

3.3. The action of G on $H^2(S, \mathbf{Q})$

Using Magma, we found the character table of G. There are 59 irreducible representations. Using a formula of Chênevert, see Proposition 10.1, it is easy to compute the traces of elements of G on $H^2(S, \mathbf{Q})_{\rm pr}$, the primitive cohomology, and thus to find the decomposition of the primitive cohomology into irreducible G-representations. The decomposition of $H^2(S, \mathbf{Q})$ is then just the sum of this decomposition with the 1D trivial representation.

The primitive cohomology group $H^2(S, \mathbf{Q})_{\rm pr}$ is the direct sum of 10 distinct irreducible representations, (which we do not describe explicitly) with:

Dimension	1	5	5	10	15	15	30	45	45	45
Multiplicity	1	1	3	1	2	1	3	1	1	1

Moreover, the characters of these 10 representations take values in $\mathbf{Z}$, so we can split

$$H^2(S, \mathbf{Q})_{\rm pr} = \bigoplus_{i=1}^{10} V_i \otimes W_i,$$

where the V_i are irreducible G-representations and W_i are rational Hodge structures, and $\dim W_i$ is the multiplicity of V_i in $H^2(S, \mathbf{Q})_{\rm pr}$.

To relate this decomposition to the one in the transcendental and trivial (Néron–Severi) Hodge substructures, we used the isomorphism of G-representations $H^{2,0}(S) \cong R_4(\epsilon)$, where ϵ is the unique non-trivial 1D representation of G (recall that $G/H \cong S_6$, and ϵ factors over the sign representation of S_6).

We found that $H^{2,0}(S)$ is the direct sum of two irreducible representations, one of dimension 5 and one of dimension 30. As the 10 subrepresentations of $H^2(S, \mathbf{Q})$ have characters with values in $\mathbf{Z}$, the direct sum of these two representations is also isomorphic to $\overline{H^{2,0}} = H^{0,2}$. In particular, their multiplicity in $H^2(S, \mathbf{Q})_{\rm pr}$ is at least 2. This suffices to identify them as the V_i with $i = 3, 7$ in the order given in the table above, so:

$$H^{2,0}(S) \cong H^{0,2}(S) \cong V_3 \oplus V_7, \quad \dim V_3 = 5, \quad \dim V_7 = 30.$$

Applying the methods of Section 1.7 to the Hodge structure $V := T_{S,\mathbf{Q}}$ we get a decomposition

$$T_{S,\mathbf{Q}} \cong V_3 \otimes W_3' \oplus V_7 \otimes W_7', \quad \dim W_3',\ \dim W_7' \in \{2, 3\},$$

where W_3', W_7' are Hodge substructures of W_3, W_7 (both of which are 3D), respectively, with $\dim(W_i')^{2,0} = 1$.

To find the dimension of W_7', we consider the action of G on the subspace L_S of $NS(S)_{\mathbf{Q}}$ spanned by classes of lines. Elements of G permute the lines and using the intersection form one can find an explicit matrix which gives the action of any $g \in G$ on L_S. Thus the representation of G on L_S can be decomposed into irreducibles. We found, of course, that all representations in $H^2(S)_{\mathrm{pr}}$, which do not appear in $H^{2,0}$, do appear in L_S. The interesting thing is that V_3 appears also in L_S, hence we can conclude that $\dim W_3' = 2$, but V_7 does not appear in L_S. Thus we get the decomposition

$$H^2(S, \mathbf{Q}) = V_3 \otimes W_3' \oplus V_7 \otimes W_7 \oplus L_S.$$

Now it only remains to decide whether $\dim W_7'$ is 2 or 3 (equivalently, whether $\dim NS(S)_{\mathbf{Q}} = 202 + 30$, in this case $L_S \neq NS(S)_{\mathbf{Q}}$, or 202, respectively). We will see in Section 5.6 that

$$T_{S,\mathbf{Q}} \cong V_3 \otimes W_3' \oplus V_7 \otimes W_7, \quad \dim W_3' = 2, \quad \dim W_7 = 3,$$

hence $L_S = NS(S)_{\mathbf{Q}}$ and W_7 is a simple 3D Hodge structure.

4. Quotients of Maschke's surface S

4.1. Overview

In Section 3.3, we showed that the transcendental Hodges structure $T_{S,\mathbf{Q}} \subset H^2(S, \mathbf{Q})$ of S has a summand $V_3 \otimes W_3'$ which is isomorphic to $(W_3')^{\oplus 5}$ as a rational Hodge structure. We will now find desingularizations $\widetilde{U}$, $\widetilde{W}$ of quotients of S by subgroups of G with transcendental Hodge structure $T_{\widetilde{U},\mathbf{Q}} \cong (W_3')^{\oplus 5}$ and $T_{\widetilde{W},\mathbf{Q}} \cong W_3'$. We did not succeed in finding a model of a surface with transcendental Hodge structure isomorphic to W_7', the other Hodge substructure in $T_{S,\mathbf{Q}}$.

4.2. K3 surfaces and W_3'

The Heisenberg group H acts through $\overline{H} := H/\mu_4 \cong (\mathbf{Z}/2\mathbf{Z})^4$, an abelian group, on $\mathbf{P}^3$ and thus on S. This gives a natural splitting

$$T_{S,\mathbf{Q}} = T_{S,\mathbf{Q},1} \oplus (\oplus_{\chi \neq 1} T_{S,\mathbf{Q},\chi}),$$

where the first summand is the subspace of H-invariants and the sum is over the 15 non-trivial characters of H, which factor over $\overline{H}$. The character table of G shows that H acts trivially in the G-representation V_3, so we conclude that $V_3 \otimes W_3' \cong T_{S,\mathbf{Q},1}$ and this also identifies $V_7 \otimes W_7' \subset T_{S,\mathbf{Q}}$:

$$T^H_{S,\mathbf{Q}} = T_{S,\mathbf{Q},1} \cong V_3 \otimes W_3', \quad \oplus_{\chi \neq 1} T_{S,\mathbf{Q},\chi} \cong V_7 \otimes W_7'.$$

The desingularization $\widetilde{U}$ of the quotient surface $U := S/H$ will have transcendental Hodge substructure $T^H_{S,\mathbf{Q}}$, hence

$$T_{\widetilde{U},\mathbf{Q}} \cong V_3 \otimes W_3' \cong (W_3')^{\oplus 5}.$$

A further quotient of U by an involution ι_U will give us a surface W whose desingularization $\widetilde{W}$ is a K3 surface with $T_{\widetilde{W}} \cong W_3'$.

$$S \longrightarrow U := S/H \ (\subset \mathbf{P}^4) \longrightarrow W := U/\iota_U \ (\subset \mathbf{P}^3).$$

4.3. The surface $U = S/H$

To find a projective model of the quotient surface $U = S/H$, we consider the subring of H-invariant polynomials in $\mathbf{C}[x_0, x_1, x_2, x_3]$. It is generated by the following 5 polynomials of degree 4:

$$p_0 := x_0^4 + x_1^4 + x_2^4 + x_3^4, \quad p_1 := 2(x_0^2x_1^2 + x_2^2x_3^2), \quad p_2 := 2(x_0^2x_2^2 + x_1^2x_3^2),$$
$$p_3 := 2(x_0^2x_3^2 + x_1^2x_2^2), \quad p_4 := 4x_0x_1x_2x_3,$$

and we have

$$\mathbf{C}[x_0, x_1, x_2, x_3]^H = \mathbf{C}[p_0, \ldots, p_4] \cong \mathbf{C}[y_0, \ldots, y_4]/(G_{\mathrm{I}}),$$

where the isomorphism is given by $y_i \mapsto p_i$, with kernel the quartic polynomial

$$G_{\mathrm{I}} := y_4^4 + (y_0^2 - y_1^2 - y_2^2 - y_3^2)y_4^2 + y_1^2y_2^2 + y_1^2y_3^2 + y_2^2y_3^2 - 2y_0y_1y_2y_3.$$

Thus the image of the map $\mathbf{P}^3 \to \mathbf{P}^4$, $x \mapsto (\ldots : p_i(x) : \ldots)$ is the (singular) quotient variety $\mathbf{P}^3/H$, which is known as the Igusa quartic, and it is defined by $G_{\mathrm{I}} = 0$.

The image U of Maschke's surface S is the intersection of the Igusa quartic with the quadric with defining equation

$$G_{\mathrm{M}} := y_0^2 + 3(y_1^2 + y_2^2 + y_3^2) + 6y_4^2,$$

so $G_{\mathrm{M}}(p_0, \ldots, p_4) = F$, the defining equation of S. In fact, the group $G/H = S_6$ acts on $\mathbf{P}^4 = \mathbf{P}(\mathbf{C}^5)$ and the representation on $\mathbf{C}^5$ is irreducible with unique quadratic invariant G_{M}. The singular locus of $U := S/H$ consists of 30 nodes which are the images of the fixed points of H in S. By adjunction, we get an isomorphism $H^{2,0}_{\widetilde{U}} \cong H^0(\mathbf{P}^4, \mathcal{O}(1))$, the vector space of linear forms on $\mathbf{P}^4$.

4.4. The surface $W = U/\iota_U$

We define an involution ι_U on U and we give an explicit projective model of the surface $W = U/\iota_U$ as a 12 nodal quartic surface in $\mathbf{P}^3$.

The matrix $\mathrm{diag}(-1, 1, 1, 1)$ lies in G and it induces the map $\mathrm{diag}(1, 1, 1, 1, -1)$ on the basis $p_0, \ldots, p_4$ of the space of H-invariants of degree 4, and hence on $\mathbf{P}^4$. In any case, it fixes the two defining equations G_{I} and G_{M} of U and hence it induces an involution ι_U on U.

The quotient W of U by ι_U is the surface in $\mathbf{P}^3$, with coordinates $y_0, \ldots, y_3$, whose defining equation is obtained by substituting $y_4^2 = -(y_0^2 + 3(y_1^2 + y_2^2 + y_3^2)/6$ in G_{I}. Thus

$$U \longrightarrow W := U/\langle \mathrm{diag}(1, 1, 1, 1, -1)\rangle, \quad W = (H = 0) \quad (\subset \mathbf{P}^3),$$

where the quartic polynomial H is given by:

$$H := 5y_0^4 + 6y_0^2(y_1^2 + y_2^2 + y_3^2) - 27(y_1^4 + y_2^4 + y_3^4) - 90(y_1^2y_2^2 + y_1^2y_3^2 + y_2^2y_3^2) + 72y_0y_1y_2y_3.$$

The automorphism group of W contains a group G_{24} of 24 elements generated by a subgroup isomorphic to S_3, given by permutations of the variables y_1, y_2, y_3, and a subgroup isomorphic to $(\mathbf{Z}/2\mathbf{Z})^2$, given by changing an even number of signs of these three variables. The point $p := (3 : 3 : \sqrt{-3} : \sqrt{-3})$ is a singular point, a node, of W. Its orbit under G_{24} consists of 12 singular points and this set is the singular locus of W.

The surface W contains the line m spanned by

$$m : \langle (3 : 0 : \sqrt{-3} : 0),\ (0 : 3 : 0 : \sqrt{-3})\rangle.$$

The orbit of this line under G_{24} consists of 12 lines. Another line m' on W is given by

$$m': \quad \langle(\omega-1:-\omega:1:0),\ (-\omega-2:\omega+1:0:1)\rangle, \quad \omega:=(-1+\sqrt{-3})/2,$$

so ω is a primitive cube root of unity.

The desingularization $\widetilde{W}$ of W is a K3 surface. The subgroup of its Néron–Severi group $NS(\widetilde{W})$ generated by the classes of the 12 rational curves over the nodes, the strict transforms of the 12 lines in the G_{24}-orbit of m and m' has rank 20, as an explicit computation of the intersection matrix of these curves shows. As this is also the maximal value that the rank can have, we conclude that $\operatorname{rank}(NS(\widetilde{W}))=20$. This agrees with the fact that $\dim H^2(\widetilde{W},\mathbf{Q})=22$, $T_{\widetilde{W}}\cong W_3'$ and $\dim W_3'=2$ (cf. Section 3.3).

In Section 5.3 we will determine the Galois representation on $H^2_{\text{ét}}(\widetilde{W},\mathbf{Q}_\ell)$.

5. Galois representations and Maschke's surface

5.1. Decomposing $H^2_{\text{ét}}(S,\mathbf{Q}_\ell)$

Maschke's surface S is defined by a polynomial with integer coefficients and thus there is a representation of the Galois group $G_{\mathbf{Q}}:=Gal(\overline{\mathbf{Q}}/\mathbf{Q})$ on the étale cohomology groups $H^2_{\text{ét}}(S,\mathbf{Q}_\ell)$ (where now $S=S_{\overline{\mathbf{Q}}}$ is considered as a surface over $\overline{\mathbf{Q}}$). The ideal in $\mathbf{Z}[x,y,z]$ generated by the partial derivatives of $F(1,x,y,z)$ contains the integer $2^{15}3^25$ (according to Magma). Hence, using the invariance of F under permutations of the four variables, the surface S has good reduction at all primes p with $p>5$.

There is a decomposition of $G_{\mathbf{Q}}$-representations:

$$H^2_{\text{ét}}(S,\mathbf{Q}_\ell)=T_{S,\ell}\oplus NS(S)_\ell,$$

where $NS(S)_\ell$ is the subspace spanned by divisor classes in $S_{\overline{\mathbf{Q}}}$, so $\dim NS(S)_\ell=\dim NS(S)_{\mathbf{Q}}$, and $T_{S,\ell}=NS(S)_\ell^{\perp}$.

The action of $G\subset Aut(S)$ is defined by matrices with entries in $\mathbf{Q}(i)$. Thus, the actions of G and $G_{\mathbf{Q}(i)}:=Gal(\overline{\mathbf{Q}}/\mathbf{Q}(i))\subset G_{\mathbf{Q}}$ on $T_{S,\ell}$ commute. Using the results from Section 3.3 and the comparison theorem, we find the splitting, as $G\times G_{\mathbf{Q}(i)}$-representations:

$$T_{S,\mathbf{Q}_\ell}\cong V_{3,\ell}\otimes W'_{3,\ell}\oplus V_{7,\ell}\otimes W'_{7,\ell}, \quad \dim W'_{3,\ell}=2, \quad \dim W'_7\in\{2,3\},$$

we will see that $\dim W'_7=3$ in Section 5.6.

As the representations V_3, V_7 of G are irreducible and have distinct dimensions, the two summands are preserved by the $G_{\mathbf{Q}}$-action on $T_{S,\mathbf{Q}_\ell}$. The G-representation on $V_3 \otimes W_3'$ factors over S_6 and is then isomorphic to the S_6-representation on $T_{\widetilde{U},\mathbf{Q}}$. As S_6 acts by matrices with coefficients in $\mathbf{Q}$ on $U \subset \mathbf{P}^4$, we conclude that $W_{3,\ell}'$ is in fact a $G_{\mathbf{Q}}$-representation.

Since $W_3' \cong T_{\widetilde{W}}$, as rational Hodge structures, we get by a similar argument that $W_{3,\ell}'$ is the Galois representation on the orthogonal complement of $NS(\widetilde{W})_\ell$ in $H^2_{\text{ét}}(\widetilde{W}, \mathbf{Q}_\ell)$. We will describe it explicitly in Section 5.3.

5.2. The Galois representation on L_S

The 202D subspace L_S of $NS(S)_{\mathbf{Q}}$ is spanned by the classes of the lines in S. These lines are defined over the field $\mathbf{Q}(\zeta_{12}, \sqrt{5})$, a subfield of $\mathbf{Q}(\zeta_{60})$, where ζ_k is a primitive k-th root of unity. The Galois representation on L_S, given by permutations of the classes of the lines, thus factors over $Gal(\mathbf{Q}(\zeta_{12}, \sqrt{5})/\mathbf{Q}) \cong (\mathbf{Z}/2\mathbf{Z})^3$. This group has 8 1D representations, which extend to $G_{\mathbf{Q}}$-representations:

$$\sigma_{a,b,c} := \sigma_{1,0,0}^a \sigma_{0,1,0}^b \sigma_{0,0,1}^c : G_{\mathbf{Q}} \longrightarrow \{\pm 1\}$$

for $a, b, c \in \{0, 1\}$ where we define the following non-trivial Galois representations by their kernel, which has index two in $G_{\mathbf{Q}}$:

$$\ker(\sigma_{1,0,0}) = G_{\mathbf{Q}(\sqrt{-1})}, \quad \ker(\sigma_{0,1,0}) = G_{\mathbf{Q}(\sqrt{-3})}, \quad \ker(\sigma_{0,0,1}) = G_{\mathbf{Q}(\sqrt{5})}.$$

Thus for a Frobenius element $F_p \in G_{\mathbf{Q}}$ at the prime p one has: $\sigma_{1,0,0}(F_p) = -1$ iff $p \equiv 3 \bmod 4$, $\sigma_{0,1,0}(F_p) = -1$ iff $p \equiv 2 \bmod 3$ and $\sigma_{0,0,1}(F_p) = -1$ iff $p \equiv 2, 3 \bmod 5$.

With a computer we found that

$$L_S \cong \sigma_{0,0,0}^{44} \oplus \sigma_{0,0,1}^{28} \oplus \sigma_{0,1,0}^{28} \oplus \sigma_{1,0,0}^{42} \oplus \sigma_{1,0,1}^{33} \oplus \sigma_{1,1,0}^{27},$$

note that $\dim L_S = 202 = 44 + 28 + 28 + 42 + 33 + 27$. The $G_{\mathbf{Q}}$-representation on $L_{S,\ell} \subset H^2_{\text{ét}}(S, \mathbf{Q}_\ell)$ is then the tensor product of the representation on L_S above with $\mathbf{Q}_\ell(-1)$. In particular, for $p > 5$ a Frobenius element $F_p \in G_{\mathbf{Q}}$ at the prime p will act as $\sigma_{a,b,c}(F_p)p$ on the summand $\sigma_{a,b,c} \otimes \mathbf{Q}_\ell(-1)$. This allows one to compute the trace of F_p on $L_{S,\ell}$ explicitly.

5.3. The Galois representation on $H^2_{\text{ét}}(\widetilde{W}, \mathbf{Q}_\ell)$

The surface $\widetilde{W}$, defined in Section 4.4, has good reduction at primes $p > 5$. The Galois representation on $H^2_{\text{ét}}(\widetilde{W}, \mathbf{Q}_\ell)$ splits into the 20D representation on $NS(\widetilde{W})_\ell$ and a 2D representation on the orthogonal complement $T_{\widetilde{W},\ell} \cong W'_{3,\ell}$. Using the explicit description of the generators of $NS(\widetilde{W})_{\mathbf{Q}}$ one finds that all generators are rational over $\mathbf{Q}(\sqrt{-3})$ and that the nontrivial element in the Galois group $Gal(\mathbf{Q}(\sqrt{-3})/\mathbf{Q})$ has 10 eigenvalues $+1$ and 10 eigenvalues -1 on this sublattice. Hence

$$NS(\widetilde{W})_\ell \cong \mathbf{Q}_\ell(-1)^{10} \oplus \quad \sigma_{0,1,0} \otimes \mathbf{Q}_\ell(-1)^{10}.$$

Moreover, we found that the determinant of the intersection matrix on a basis of the sublattice spanned by the rational curves given previously is, up to a square, equal to -15. The theory of the arithmetic of singular K3 surfaces then asserts that the representation on $T_{\widetilde{W}_\ell}$ is determined by a Hecke character of the imaginary quadratic field $L := \mathbf{Q}(\sqrt{-15})$.

To determine the Hecke character, a computer counted the number of points $\sharp W(\mathbf{F}_p)$ on W in $\mathbf{P}^3(\mathbf{F}_q)$ for many small prime powers q. If $q \equiv 1 \bmod 3$, the 12 nodes are rational over $\mathbf{F}_q$ and thus $\sharp\widetilde{W}(\mathbf{F}_q) = \sharp W(\mathbf{F}_q) + 12p$, else $\sharp\widetilde{W}(\mathbf{F}_q) = \sharp W(\mathbf{F}_q)$. Then Lefschetz's fixed point formula implies:

$$\begin{aligned} &\sharp\widetilde{W}(\mathbf{F}_q) = 1 + 10(1 + \sigma_{0,1,0}(F_q))q + a_q + q^2, \\ &a_q := Trace(F_p^k | T_{\widetilde{W},\ell}), \qquad \text{where} \quad q = p^k. \end{aligned}$$

The values of various a_p are listed below.

Let $\mathcal{O}_L := \mathbf{Z}[\alpha]$, $\alpha := (1 + \sqrt{-15})/2$ be the ring of integers of L. Its class number is 2, hence the square of any of its ideals is principal. As the units in $\mathcal{O}_L$ are ± 1, a generator of a principal ideal is unique up to sign. As the minimal polynomial of α is $f := x^2 - x + 4$, and $f \equiv x^2 + 2x + 1 \equiv (x+1)^2 \bmod 3$, there is a surjective homomorphism

$$\phi_3 : \mathcal{O}_L := \mathbf{Z}[\alpha] \longrightarrow \mathbf{Z}/3\mathbf{Z}, \quad a + b\alpha \longmapsto a - b \bmod 3$$

with $\phi_3(-1) = -1$. Thus any principal ideal $I = \beta\mathcal{O}_L$, prime with $3\mathcal{O}_L$, has a unique generator β such that $\phi_3(\beta) = +1$. We define a Hecke character χ on $\mathcal{O}_L$ by defining

$$\chi(I) = \beta, \quad \text{where} \quad I^2 = (\beta), \quad \phi_3(\beta) = 1.$$

To be explicit, if $p = \wp\overline{\wp}$ is a prime which splits in $\mathcal{O}_L$, then $\wp^2 = (a + b\alpha)$ for some integers a, b, with $a, b \neq 0$, which can be found by considering the norm:

$$p^2 = N(a + b\alpha) := (a + b\alpha)\overline{(a + b\alpha)} = a^2 - ab + 4b^2.$$

Changing signs, if necessary, to obtain $a - b \equiv 1 \bmod 3$, one thus has $\chi(\wp) = a + b\alpha$ and $\chi(\wp) + \overline{\chi(\wp)} = 2a + b$. For example, if $p = 17$ and $\wp_{17}$ is one of the primes over 17, then

$$\begin{aligned} &11^2 - 11 \cdot 8 + 8^2 = 17^2, \quad \phi_3(11 - 8\alpha) = 19 \equiv 1 \bmod 3, \\ &\text{so } \chi(\wp_{17}) = 11 - 8\alpha. \end{aligned}$$

This Hecke character determines a 1D ℓ-adic Galois representation of $G_L := Gal(\overline{\mathbf{Q}}/L)$ which induces a 2D Galois ℓ-adic representation of $G_{\mathbf{Q}}$. This is the representation of $G_{\mathbf{Q}}$ on $T_{\widetilde{W}_\ell} = W'_{3,\ell}$ which we found. In particular, the trace a_p of F_p in this representation is:

$$a_p = \begin{cases} \chi(\wp) + \overline{\chi(\wp)}, & \text{if } p \text{ splits,} \\ 0, & \text{if } p\mathcal{O}_L \text{ is prime.} \end{cases}$$

For example, $\chi(\wp_{17}) = 11 - 8\alpha$, so we get $a_{17} = 2 \cdot 11 + (-8) = 14$, below are some other values of a_p.

p	7	11	13	17	19	23	29	31	...	79	83	89	97
a_p	0	0	0	14	−22	−34	0	2	...	98	−154	0	0

The a_p are also the Fourier coefficients of an elliptic modular form f of weight three (cf. [28]). In this case f is one of the two newforms with Dirichlet character $p \mapsto \left(\frac{-15}{p}\right)$, it has level $N = 15$.

Before continuing with the study of $H^2_{\text{ét}}(S, \mathbf{Q}_\ell)$ in Section 5.6, we make a digression to discuss various surfaces related to S.

5.4. Remark

The polynomial F defining S is a quartic in the x_i^2, hence S is a ramified cover of the surface $\overline{S}$ defined by

$$\overline{S} : \sum_{i=0}^{3} x_i^4 + 14 \sum_{i<j} x_i^2 x_j^2 + 168 x_0 x_1 x_2 x_3.$$

The surface $\overline{S}$ is smooth and hence it is a K3 surface. It is the quotient of S by the subgroup $K \cong (\mathbf{Z}/2\mathbf{Z})^4$ of diagonal matrices with entries ± 1 of G and thus $H^2(\overline{S}, \mathbf{Q}) \cong H^2(S, \mathbf{Q})^H$.

The Galois representation on the transcendental lattice $T_{\overline{S},\ell}$ is the same as the one on $T_{\widetilde{W},\ell}$:

$$\sharp\overline{S}(\mathbf{F}_p) = 1 + n_p p + a_p + p^2,$$

where $n_p p$ is the trace of the Frobenius F_p on the Néron–Severi group of $\overline{S}$, with

$$n_p := (4\sigma_{0,0,0} \oplus 4\sigma_{0,1,0} \oplus 6\sigma_{1,0,0} \oplus 3\sigma_{1,0,1} \oplus 3\sigma_{1,1,0})(F_p).$$

5.5. The Galois representation on $H^2_{\text{ét}}(\widetilde{U}, \mathbf{Q}_\ell)$

Using the action of S_6 on $H^2(\widetilde{U}, \mathbf{Q})$, we already concluded that $T_{\widetilde{U}} \cong W_3'^5$. As the automorphisms of $\widetilde{U}$ corresponding to the elements of S_6 are defined over $\mathbf{Q}$ this implies that the $G_{\mathbf{Q}}$-representation on $H^2_{\text{ét}}(\widetilde{U}, \mathbf{Q}_\ell)$ splits into $T_{\widetilde{U},\ell} \cong T^5_{\widetilde{W},\ell}$ and a direct sum of $\mathbf{Q}_\ell(-1)$'s twisted by Dirichlet characters, unramified for primes $p > 5$.

As U is a complete intersection of type $(2, 4)$ in $\mathbf{P}^4$ with only nodes as singularities, the Euler characteristic $\chi(\widetilde{U})$ of $\widetilde{U}$ is the same as the one of a smooth complete intersection of type $(2, 4)$, so $\chi(\widetilde{U}) = 64$. As the cohomology in odd degree is zero, we get $\dim H^2(\widetilde{U}, \mathbf{Z}) = 62$ and thus $\dim NS(\widetilde{U})_{\mathbf{Q}} = 52$.

Note that U has 30 nodes, defined over $\mathbf{Q}(\sqrt{-3})$, hence $\sharp\widetilde{U}(\mathbf{F}_p) = \sharp U(\mathbf{F}_p)$ $+ 30p$ if $p \equiv 1 \mod 3$ and else $\sharp\widetilde{U}(\mathbf{F}_p) = \sharp U(\mathbf{F}_p)$, for $p > 5$. Using a computer and the Lefschetz fixed point formula one finds, for many small primes p with $p > 5$,

$$\sharp\widetilde{U}(\mathbf{F}_p) = 1 + (26 + 25\sigma_{0,1,0}(F_p) + \sigma_{0,0,1}(F_p))p + 5a_p + p^2,$$

hence this determines the $G_{\mathbf{Q}}$-representation on $H^2_{\text{ét}}(\widetilde{U}, \mathbf{Q}_\ell)$.

5.6. The Galois representation $W_{7,\ell}$

From Sections 3.3, 5.1 we have the following decomposition of $H^2_{\text{ét}}(S, \mathbf{Q}_\ell)$ as $G \times G_{\mathbf{Q}(i)}$-representations:

$$H^2_{\text{ét}}(S, \mathbf{Q}_\ell) = V_{3,\ell} \otimes W'_{3,\ell} \oplus V_{7,\ell} \otimes W_{7,\ell} \oplus L_{S,\ell}$$

where $W_{7,\ell}$ is 3D. As $\dim V_{3,\ell} = 5$, we have $\mathrm{Tr}(F_q|V_{3,\ell} \otimes W'_{3,\ell}) = 5a_q$ with $a_q := Tr(F_q|T_{\widetilde{W},\ell})$ as in Section 5.3. Using that $\dim V_{7,\ell} = 30$ and taking prime numbers $q \equiv 1 \bmod 4$, the Lefschetz fixed point formula gives:

$$\sharp S(\mathbf{F}_q) = 1 + 5a_q + 30b_q + \mathrm{tr}(F_q|L_{S,\ell}) + q^2, \qquad b_q := Tr(F_q|W_{7,\ell}).$$

Some b_q, for primes q, are listed in Section 5.7 below.

The $G_{\mathbf{Q}(i)}$-representation on $W_{7,\ell}$ is compatible with the non-degenerate quadratic form induced by the intersection form on $H^2_{\text{ét}}(S, \mathbf{Q}_\ell)$. If $W_{7,\ell}$ is reducible, then it must be the direct sum of a one and a 2D representation, which is compatible with a non-degenerate quadratic form. The orthogonal group $O(2)$ has a normal abelian subgroup of finite index, and therefore the 2D representation is of CM type, in particular the eigenvalues of the Frobenius elements must be in a fixed CM field of degree at most two over $\mathbf{Q}(i)$. However, we checked that this is not the case, using that the characteristic polynomial for F_p with p a prime which is 1 mod 4, is given by $x^3 - b_p x^2 + \epsilon_p b_p p x - \epsilon_p p^3$ with $\epsilon_p \in \{\pm 1\}$ (cf. Section 5.7).

Therefore the $G_{\mathbf{Q}(i)}$-representation $W_{7,\ell}$ is irreducible and hence $L_S = NS(S)_{\mathbf{Q}}$ as $\mathbf{Q}$-vector spaces. This again implies that W_7 is a simple Hodge structure of dimension three.

In the recent paper [27] another approach to solve a similar problem is given.

5.7. The Galois representation on $H^2_{\text{ét}}(S, \mathbf{Q}_\ell)$

We conclude our study of the arithmetic of S by determining the $G_{\mathbf{Q}}$-representation on $H^2_{\text{ét}}(S, \mathbf{Q}_\ell)$ in terms of known representations and a 3D $G_{\mathbf{Q}}$-representation which restricts to the $G_{\mathbf{Q}(i)}$-representation $W_{7,\ell}$.

Upon restriction to the Heisenberg group H, the irreducible 30D G-representation V_7 splits into a direct sum

$$V_7 \cong \oplus_{\chi \neq 1} V_{7,\chi}, \quad \dim V_{7,\chi} = 2,$$

where the sum is over the 15 non-trivial 1D representations of H. As H is defined over $\mathbf{Q}$, we get a $H \times G_{\mathbf{Q}}$-subrepresentation

$$\oplus_{\chi \neq 1}\, V_{7,\chi,\ell} \otimes W_{7,\chi,\ell} \subset H^2_{\text{ét}}(S, \mathbf{Q}_\ell),$$

where the $W_{7,\chi,\ell}$ are 3D $G_{\mathbf{Q}}$-representations.

The group G acts on H and it permutes the 15 non-trivial χ transitively. Thus, the $G_{\mathbf{Q}}$-representations $W_{7,\chi,\ell}$ are all isomorphic when restricted to

the subgroup $G_{\mathbf{Q}(i)}$, and in fact they are all isomorphic to $W_{7,\ell}$. Fix one non-trivial χ_0, then given a non-trivial χ we have that either

$$W_{7,\chi,\ell} \cong W_{7,\chi_0,\ell} \quad \text{or} \quad W_{7,\chi,\ell} \cong W_{7,\chi_0,\ell} \otimes \sigma_{1,0,0}.$$

As $\dim V_{7,\chi,\ell} = 2$, we get a decomposition, as $G_{\mathbf{Q}}$-representations:

$$H^2_{\text{ét}}(S, \mathbf{Q}_\ell) = (W'_{3,\ell})^{\oplus 5} \oplus (W_{7,\chi_0,\ell})^{\oplus 2a} \oplus (W_{7,\chi_0,\ell} \otimes \sigma_{1,0,0})^{\oplus 2b} \oplus L_{S,\ell},$$
$$a + b = 15,$$

for some integers a, b and a certain 3D $G_{\mathbf{Q}}$-representation $W_{7,\chi_0,\ell}$. Replacing $W_{7,\chi_0,\ell}$ by $W_{7,\chi_0,\ell} \otimes \sigma_{1,0,0}$ if necessary, we may assume that $a > b$. This determines the $G_{\mathbf{Q}}$-representation $W_{7,\chi_0,\ell}$ uniquely.

To determine a, b, we use the Lefschetz fixed point formula:

$$\sharp S(\mathbf{F}_q) = 1 + 5a_q + (2a + 2b\sigma_{1,0,0}(F_q))b_p + R(F_q, L_{S,\ell}) + q^2,$$
$$b_q := Tr(F_q | W_{7,\chi_0,\ell}).$$

In case $q \equiv 1 \bmod 4$, one has $\sigma_{1,0,0}(F_q) = 1$, so we recover the formula from Section 5.6. In case $q \equiv 3 \bmod 4$, $\sigma_{1,0,0}(F_q) = -1$ and the computer finds $2(a-b)b_q$. For small primes q, we found that the g.c.d. of these integers is 6. Hence

$$a - b = 1 \quad \text{or} \quad a - b = 3.$$

We now exclude the case $a - b = 1$. For $q = 11$ we found $2(a-b)b_{11} = -78$, hence $b_{11} = -39$ if $a - b = 1$. But $|b_{11}| \leq 3q = 33$ since b_{11} is the sum of the three eigenvalues of F_q on $W_{7,\chi_0,\ell} \subset H^2_{\text{ét}}(S, \mathbf{Q}_\ell)$. Therefore, we must have $a - b = 3$ and thus

$$a = 9, \quad b = 6.$$

The following is a table of some of the b_p for primes p:

p	7	11	13	17	19	23	29	31	37	41	43	47	53
b_p	-7	-13	-11	5	7	13	-21	19	13	-9	29	-11	-55

To determine the characteristic polynomial of F_p on $W_{7,\chi_0,\ell}$, knowing only b_p, we proceed as follows. First of all, if α is an eigenvalue of F_q on $W_{7,\chi_0,\ell}$ then so is $q^2/\alpha = \bar{\alpha}$. As $\dim W'_{7,\chi_0,\ell} = 3$ this implies that one of the eigenvalues is $\pm q$. Let $\alpha, \bar{\alpha}, \epsilon_p p$ be the eigenvalues of F_p, with $\epsilon_p \in \{\pm 1\}$, then $\det(F_p) = \epsilon_p p^3$. As the determinant is also a $G_{\mathbf{Q}}$-representation, unramified

for $p > 5$, this allows us to determine it explicitly once we know it for small primes.

To find ϵ_p for small primes we used the following identity involving b_p, b_{p^2}, which we determine by counting points, and ϵ_p:

$$b_p = \alpha + \bar{\alpha} + \epsilon_p p, \quad b_{p^2} = \alpha^2 + \bar{\alpha}^2 + p^2 = b_p^2 - 2p(b_p - \epsilon_p p) - 2p^2.$$

We found that

$$\epsilon_p = \sigma_{1,0,1}(F_p),$$

hence the determinant of the $G_{\mathbf{Q}}$-representation on $W_{7,\chi_0,\ell}$ is $\sigma_{1,0,1}$. The characteristic polynomial f_p of F_p on $W_{7,\chi_0,\ell}$ is thus determined by b_p:

$$f_p = x^3 - b_p x^2 + \epsilon_p b_p p x - \epsilon_p p^3 = (x^2 - (b_p - \epsilon_p p)x + p^2)(x - \epsilon_p p).$$

5.8. Remark

M. Schütt pointed out that there is a K3 surface $\mathcal{X}_{3,2/3}$, defined over $\mathbf{Q}$, whose $G_{\mathbf{Q}}$-representation on $H^2_{\text{ét}}$ has a rank three summand $T_{3,2/3,\ell}$ which has the same characteristic polynomial of F_p for primes p with $7 \leq p \leq 131$ as $W_{7,\chi_0,\ell}$. This surface has an elliptic fibration given by:

$$\mathcal{X}_{3,r} : y^2 = x^3 - t^2(r^2 t - 1 - 2r)x^2 - 2(t+1)t^3 r(rt-1)x - (t+1)^2 t^5 r^2,$$

with $r = 2/3$. This family of elliptic surfaces is studied in [12], Section 6, where it is shown that the $\mathcal{X}_{3,r}$ are birationally isomorphic to quotients of products of two isogeneous elliptic curves. For $r = 2/3$ there is an elliptic curve over $\mathbf{Q}(\sqrt{-5}, \sqrt{-15})$, that is 2-isogeneous to its $Gal(\mathbf{Q}(\sqrt{-5})/\mathbf{Q})$ conjugate and 3-isogeneous to its $Gal(\mathbf{Q}(\sqrt{-15})/\mathbf{Q})$ conjugate, which produces the K3 surface. It is the curve $C^{(a)}$ in the family parametrized by $X^*(6)$ with parameter value $a = -16/5$ in [26], p.312.

If there is indeed an isomorphism of $G_{\mathbf{Q}}$-representations $T_{3,2/3,\ell} \cong W_{7,\chi_0,\ell}$, then the Tate conjecture predicts a correspondence, defined over $\mathbf{Q}$, between S and $\mathcal{X}_{3,2/3}$. Recently, M. Schütt [29] provided an explicit correspondence.

6. Maschke's double octic

6.1. The cohomology of X

Maschke's double octic is the CY three-fold which is the double cover

$$\pi : X \longrightarrow \mathbf{P}^3$$

branched along Maschke's (smooth) degree 8 surface $S \subset \mathbf{P}^3$. As the surface S has Euler–Poincaré characteristic $\chi_{\text{top}} = 304$, X has Euler–Poincaré characteristic

$$\chi_{\text{top}}(X) = 2 \cdot \chi_{\text{top}}(\mathbf{P}^3) - \chi_{\text{top}}(S) = 8 - 304 = -296.$$

Results of Lazarsfeld ([17], Thm. 1, Prop. 3.1) imply that

$$H^i(\mathbf{P}^3, \mathbf{Z}) \xrightarrow{\cong} H^i(X, \mathbf{Z}), \qquad i = 0, 1, 2.$$

Recall that $H^i(\mathbf{P}^3, \mathbf{Z}) = 0$ if $i = 1$ and $H^i(\mathbf{P}^3, \mathbf{Z}) = \mathbf{Z}$ if $i = 0, 2$, and that X is a CY three-fold, hence we get:

$$h^3(X) = 300, \quad h^{3,0}(X) = 1, \quad h^{2,1}(X) = 149.$$

As X is a hypersurface in weighted projective space $WP(1, 1, 1, 1, 4)$, one can also use toric methods (cf. [9], 4.1.3) to compute the h^i.

6.2. The action of G on $H^3(X, \mathbf{Q})$

Using our generalization of Chênevert's formula, cf. Proposition 10.1, we found that $H^3(X, \mathbf{Q})$ is the direct sum of seven distinct irreducible G-representations, each with multiplicity two and of

$$\text{dimension} \qquad 1, \quad 5, \quad 9, \quad 15, \quad 30, \quad 45, \quad 45.$$

The characters of these representations are integer valued, thus, labeling these irreducible G-representations by their dimension, we have (cf. Section 1.7)

$$\begin{aligned} H^3(X, \mathbf{Q}) \cong\ & V_1 \otimes W_1 \oplus V_5 \otimes W_5 \oplus V_9 \otimes W_9 \oplus V_{15} \otimes W_{15} \oplus V_{30} \otimes W_{30} \\ & \oplus V_{45} \otimes W_{45} \oplus V'_{45} \otimes W'_{45}, \end{aligned}$$

where the W_i are rational Hodge structures of dimension two. Obviously, $H^{3,0}(X)$ and $H^{0,3}(X)$ are 1D subrepresentations in $H^3(X, \mathbf{C})$ and the G-action on V_1 factors over the sign representation of S_6, hence $W_1^{3,0} \cong H^{3,0}(X)$. The other six W_i thus have $W_i^{3,0} = 0$, $\dim W_i^{2,1} = 1$, so they are Tate twists of Hodge structures of weight one.

The rational Hodge structure $H^3(X, \mathbf{Q})$ is thus isomorphic to $\oplus W_i^{m_i}$, where $m_i = \dim V_i$, and all seven W_i have dimension two. We define (the isogeny class of) the elliptic curve E_i by $E_i = W_i^{p,q}/\Lambda_i$, where $p < q$, $\Lambda_i \subset W_i$, $\Lambda_i \cong \mathbf{Z}^2$ and $\Lambda_i \otimes_{\mathbf{Z}} \mathbf{Q} = W_i$. There are natural isomorphisms of $\mathbf{Q}$-vector spaces

$$H^1(E_i, \mathbf{Q}) \longrightarrow W_i, \quad (E_i = W_i^{p,q}/\Lambda_i),$$

which, for $i \neq 1$, are morphisms of rational Hodge structures with $H^{p,q}(E_i) \xrightarrow{\cong} W_i^{p+1,q+1}$. For $i = 1$ there cannot exist a morphism of Hodge structures between $H^1(E_1, \mathbf{Q})$ and W_1 because $W_1^{2,1} = 0$. In particular, the Griffiths intermediate Jacobian $J(X)$ of X, a complex torus associated to $H^3(X, \mathbf{Q})$ (cf. Section 8.5), is isogeneous to a product $\prod E_i^{m_i}$ of elliptic curves.

6.3. The action of H on $H^3(X, \mathbf{Q})$

The Heisenberg group $H \subset G$ acts through its abelian quotient $\overline{H} \cong (\mathbf{Z}/2\mathbf{Z})^4$ on X and thus on the cohomology groups $H^i(X, \mathbf{Q})$. We found that

$$H^3(X, \mathbf{Q}) = H^3(X, \mathbf{Q})^H \oplus \left(\oplus_{\chi \neq 1} H^3(X, \mathbf{Q})_\chi\right),$$

where the sum is over the 15 non-trivial characters of H and

$$\dim H^3(X, \mathbf{Q})^H = 30, \quad \dim H^3(X, \mathbf{Q})_\chi = 18.$$

Comparing this to the G-decomposition above, we find that the H-invariants are:

$$H^3(X, \mathbf{Q})^H \cong V_1 \otimes W_1 \ \oplus \ V_5 \otimes W_5 \oplus V_9 \otimes W_9.$$

In the next section, we will consider the Heisenberg quotient $Y := X/H$ of X and its desingularization $\widetilde{Y}$.

7. The arithmetic of X

7.1. The Heisenberg quotient Y of X

The CY three-fold X is the double cover of $\mathbf{P}^3$ branched along Maschke's surface S. The Heisenberg group H acts on X, through its action on $\mathbf{P}^3$.

The quotient of $\mathbf{P}^3$ by H is the Igusa quartic three-fold Z in $\mathbf{P}^4$, cf. Section 4.3, so there is a double cover

$$Y = X/H \longrightarrow Z \cong \mathbf{P}^3/H,$$

which is branched along the image U of S in Z, which is the intersection of Z with a quadric in $\mathbf{P}^4$.

The variety Z is singular along 15 lines, defined over $\mathbf{Q}$, which are the images of the fixed lines in $\mathbf{P}^3$ of elements in H. These 15 lines in $\mathrm{Sing}(Z)$ intersect, three at the time, in 15 points. A desingularization $\widetilde{Z}$ is obtained by blowing up the singular locus (cf. [18]). The fiber of $\widetilde{Z} \to Z$ over a point $p \in \mathrm{Sing}(Z)$ which lies on exactly one line is a $\mathbf{P}^1$, but the fiber over a point in three lines consists of three $\mathbf{P}^1$'s meeting in one point.

The fixed point set in $\mathbf{P}^3$ of a non-trivial element $h \in H$ is the union of two lines and the stabilizer in H of a general point in such a line is the subgroup generated by h. Locally the action of h is given by the action of $diag(1,-1,-1)$ on $\mathbf{C}^3$, which has quotient $\mathbf{C} \times A_1$ where $A_1 \cong Spec(\mathbf{C}[u,v,w]/(uw-v^2))$. This gives the local description of the singular locus of $Z = \mathbf{P}^3/H$. It follows, using adjunction, that the strict transform in $\widetilde{Z}$ of a (linear) hyperplane section of Z is an anti-canonical divisor in $\widetilde{Z}$.

The variety $\widetilde{Y}$ is the double cover of $\widetilde{Z}$ branched along the strict transform of the image U of S in Z, which is isomorphic to $\widetilde{U}$. As the smooth surface $\widetilde{U}$ is a divisor in $|-2K_{\widetilde{Z}}|$, the three-fold $\widetilde{Y}$ has trivial canonical bundle.

The variety $\widetilde{Y}$ is birationally isomorphic to the crepant resolution $\tilde{Y}'$ of X/H, which is a CY three-fold. Thus the Hodge numbers of $\widetilde{Y}$ and $\widetilde{Y}'$ are the same (see [2]) and we determined them using orbifold cohomology (see [10]):

$$\begin{aligned} &h^1(\widetilde{Y}) = 0, \quad h^2(\widetilde{Y}) = h^{1,1}(\widetilde{Y}) = 16, \quad h^3(\widetilde{Y}) = 30, \\ &h^{3,0}(\widetilde{Y}) = 1, \quad h^{2,1}(\widetilde{Y}) = 14. \end{aligned}$$

Comparing this with the H-invariants in $H^3(X, \mathbf{Q}_\ell)$, we conclude that

$$H^3_{\text{ét}}(\widetilde{Y}, \mathbf{Q}_\ell) \cong H^3_{\text{ét}}(X, \mathbf{Q}_\ell)^H \cong V_{1,\ell} \otimes W_{1,\ell} \ \oplus \ V_{5,\ell} \otimes W_{5,\ell} \oplus V_{9,\ell} \otimes W_{9,\ell},$$

where the $W_{i,\ell}$ are 2D $\mathbf{Q}_\ell$ vector spaces.

As the S_6-representation on $H^3_{\text{ét}}(\widetilde{Y}, \mathbf{Q}_\ell)$ is induced by the action of S_6 on $\mathbf{P}^4$, where it acts by matrices with rational coefficients, the $W_{i,\ell}$ are 2D

$G_{\mathbf{Q}}$-representations:

$$\sigma_i : G_{\mathbf{Q}} := Gal(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow GL(W_{i,\ell}) \cong GL(2, \mathbf{Q}_\ell)$$

for $i = 1, 5, 9$.

Recent modularity results [13], [16], [11] imply that the Galois representation σ_1 corresponds to a newform of weight 4 on $\Gamma_0(N)$ for an integer N which is divisible only by primes where $\widetilde{Y}$ has bad reduction. The representations σ_5, σ_9 are expected to be Tate twists of Galois representations associated to elliptic curves defined over $\mathbf{Q}$, these curves should become isogeneous, over $\mathbf{C}$, to the curves E_5, E_9 from Section 6.2. This is now proven in [29]. Hence, by Wiles' theorem, these Galois representations correspond to newforms of weight two on $\Gamma_0(N_i)$ $(i = 5, 9)$ for certain integers N_i divisible only by primes where $\widetilde{Y}$ has bad reduction.

7.2. The Galois representation on $H^3_{\text{ét}}(\widetilde{Y}, \mathbf{Q}_\ell)$

From the description of $\widetilde{Z}$ as the blow up of Z, it follows in particular that $H^2_{\text{ét}}(\widetilde{Z}, \mathbf{Q}_\ell)$ is generated by classes of divisors defined over $\mathbf{Q}$, hence the Galois representation on $H^2_{\text{ét}}$ (and by duality, also the one on $H^4_{\text{ét}}$) are direct sums of the Tate representations $\mathbf{Q}_\ell(-1)$ (and $\mathbf{Q}_\ell(-2)$ respectively). In particular:

$$\sharp\widetilde{Z}(\mathbf{F}_q) = \sharp Z(\mathbf{F}_q) + 15(q^2 + q).$$

Using the 2:1 map $\widetilde{Y} \to \widetilde{Z}$ it then follows that also

$$\sharp\widetilde{Y}(\mathbf{F}_q) = \sharp Y(\mathbf{F}_q) + 15(q^2 + q).$$

(Use that over each singular line ($\cong \mathbf{P}^1$ in Z) there is a $\mathbf{P}^1$-bundle over $\mathbf{P}^1$ in $\hat{Z}$ and after taking the double cover, its preimage is a $\mathbf{P}^1$-bundle over a $\mathbf{P}^1$ in $\widetilde{Y}$.)

From the Lefschetz fixed point formula one has:

$$\sharp\widetilde{Y}(\mathbf{F}_q) = \sum_{i=0}^{6} (-1)^i \mathrm{tr}(F_q | H^i_{\text{ét}}(\widetilde{Y}, \mathbf{Q}_\ell)).$$

The double cover $\widetilde{Y} \to \widetilde{Z}$ induces an isomorphism on H^2 and H^4 and this map is defined over $\mathbf{Q}$, so we get:

$$\sum_{i=0}^{6} (-1)^i \mathrm{tr}(F_q | H^i_{\text{ét}}(\widetilde{Y}, \mathbf{Q}_\ell)) = 1 + 16(q + q^2) + q^3 - \mathrm{tr}(F_q | H^3_{\text{ét}}(\widetilde{Y}, \mathbf{Q}_\ell)).$$

The trace of the Frobenius F_q on $H^3_{\text{ét}}(\widetilde{Y})$ can thus be determined by counting points on the (singular) variety $Y \subset \mathbf{P}^5$:

$$\operatorname{tr}(F_q | H^3_{\text{ét}}(\widetilde{Y}, \mathbf{Q}_\ell)) = -\sharp Y(\mathbf{F}_q) + 1 + q + q^2 + q^3.$$

Computer computations lead to the following table:

q	7	11	13	17	19	23	29	7^2	...	19^2
$\operatorname{tr}(F_q \mid H^3)$	0	180	210	−90	−1020	−1560	1410	−10290	...	−122970

The table below lists the Fourier coefficients of some newforms on $\Gamma_0(N)$ of weight k, the names of the weight two forms are those from Table 3 in [23].

	p	7	11	13	17	19	23	29	31	...	79	83	89	97
$f120, k=4$	a_p	0	4	54	114	44	96	134	−272	...	688	1188	−694	−1726
$f24B, k=2$	b_p	0	4	−2	2	−4	−8	6	8	...	−8	−4	−6	2
$f120E, k=2$	c_p	0	−4	6	−6	−4	0	−2	−8	...	−8	−12	10	2
$f15C, k=2$	d_p	0	−4	−2	2	4	0	−2	0	...	0	12	−6	2

Then one can verify that for $7 \leq p \leq 97$ one also has:

$$\operatorname{tr}(F_p | H^3_{\text{ét}}(\widetilde{Y}, \mathbf{Q}_\ell)) = a_p + p(9b_p + 5c_p).$$

This leads us to conjecture that

$$W_{1,\ell} \overset{?}{\cong} V_{f120,\ell}, \quad W_{5,\ell} \overset{?}{\cong} V_{f120E,\ell}(-1), \quad W_{9,\ell} \overset{?}{\cong} V_{f24B,\ell}(-1),$$

where $V_{g,\ell}$ denotes the ℓ-adic Galois representation associated to the newform g. The conjecture was recently proved by M. Schütt [29].

To find the a_p, we assumed that σ_5 and σ_9 are Tate twists of Galois representations, in particular that $\operatorname{tr}(F_p | W_{i,\ell})$ is a multiple of p for $i = 5, 9$. Thus, by counting points, one can find $a_p \bmod p$. Comparing with the Fourier coefficients of weight four newforms of level $N = 2^a 3^b 5^c$ for small a, b, c (using Magma) we found that those of $f120$ match perfectly.

Assuming that the a_p are determined correctly, one can determine b_p, c_p by counting points over $\mathbf{F}_p$ and $\mathbf{F}_{p^2}$. In fact, let $\beta, \bar{\beta}, \gamma, \bar{\gamma} \in \mathbf{C}$ be the eigenvalues of the Frobenius F_p in the 2D weight two representations, so

$$T^2 - b_p T + p = (T - \beta)(T - \bar{\beta}), \quad T^2 - c_p T + p = (T - \gamma)(T - \bar{\gamma}).$$

Then the traces of F_p^2 are given by:

$$b_{p^2} = \beta^2 + \bar{\beta}^2 = b_p^2 - 2p, \quad c_{p^2} = \gamma^2 + \bar{\gamma}^2 = c_p^2 - 2p,$$

which allows one to find a degree two polynomial whose zeroes are b_p and c_p, respectively. The polynomials turn out to have a double zero or a unique zero which is an integer for the case $p = 7, 11, 13, 17, 19$ thus allowing one to determine b_p, c_p for these primes. Comparing with the Fourier coefficients of weight two newforms, we found the modular forms $f24B$ and $f120E$.

7.3. The Galois representation on $H^3_{\text{ét}}(X, \mathbf{Q}_\ell)$

The $G_{\mathbf{Q}}$-representation on $H^3_{\text{ét}}(X, \mathbf{Q}_\ell)$ has a summand which is $H^3_{\text{ét}}(X, \mathbf{Q}_\ell)^H \cong H^3_{\text{ét}}(\widetilde{Y}, \mathbf{Q}_\ell)$, which we discussed in Section 7.2, we will use Schütt's results [29] on its modularity.

Its orthogonal complement $H^3_{\text{ét}}(X, \mathbf{Q}_\ell)_c$ w.r.t. the intersection form was decomposed as a sum of four irreducible G-representations of dimensions $15, 30, 45, 45$ and each has multiplicity two (cf. Section 6.2):

$$H^3_{\text{ét}}(X, \mathbf{Q}_\ell)_c \cong V_{15,\ell} \otimes W_{15,\ell} \oplus V_{30,\ell} \otimes W_{30,\ell} \oplus V_{45,\ell} \otimes W_{45,\ell} \oplus V'_{45,\ell} \otimes W'_{45,\ell}.$$

Each $W_{i,\ell}$ is a $G_{\mathbf{Q}(i)}$-representation (but not a $G_{\mathbf{Q}}$-representation in general, as the actions of G and $G_{\mathbf{Q}}$ do not commute). In particular, for $q \equiv 1 \bmod 4$, the trace of the Frobenius F_q on $H^3_{\text{ét}}(X, \mathbf{Q}_\ell)_c$ should be divisible by 15 in $\mathbf{Z}$. Using a computer we found that

$$\begin{aligned} \operatorname{tr}(F_q | H^3_{\text{ét}}(X, \mathbf{Q}_\ell)_c) &= \operatorname{tr}(F_q | H^3_{\text{ét}}(X, \mathbf{Q}_\ell)) - \operatorname{tr}(F_q | H^3_{\text{ét}}(\widetilde{Y}, \mathbf{Q}_\ell)) \\ &= \sharp X(\mathbf{F}_q) - (1 + q + q^2 + q^3) - \operatorname{tr}(F_q | H^3_{\text{ét}}(\widetilde{Y}, \mathbf{Q}_\ell)) \end{aligned}$$

is divisible by 45 for all such small q. This leads us to conjecture that $W_{15,\ell} \cong W_{30,\ell}$ as $G_{\mathbf{Q}(i)}$-representations. Thus we conjecture (and recently M. Schütt

[29] provided a proof):

$$\begin{array}{lll} H^3_{\text{ét}}(X,\mathbf{Q}_\ell)_c & \cong & (V_{15,\ell}\oplus V_{30,\ell})\otimes W_{15,\ell}\oplus V_{45,\ell}\otimes W_{45,\ell}\oplus V'_{45,\ell}\otimes W'_{45,\ell} \\ & \overset{?}{\cong}_{G_{\mathbf{Q}(i)}} & \left(W_{15,\ell}\oplus W_{45,\ell}\oplus W'_{45,\ell}\right)^{\oplus 45}. \end{array}$$

To determine the characteristic polynomial of F_p on $W_{15,\ell}\oplus W_{45,\ell}\oplus W'_{45,\ell}$ for $p\equiv 1 \bmod 4$, we would have to compute points on X over $\mathbf{F}_{p^k}$, $k=1,2,3$. This took too much time, but we could compute the number of points over $\mathbf{F}_p$ and $\mathbf{F}_{p^2}$ for some small primes. The characteristic polynomial must be of the form

$$f_p: \quad X^6 - s_1X^5 + s_2X^4 - s_3X^3 + ps_2X^2 - p^2s_1X + p^3,$$

with coefficients:

$$s_1 := t_p, \quad s_2 := (1/2)t_p^2 - (1/2)t_{p^2}, \quad s_3 := (1/6)t_p^3 - (1/2)t_{p^2}t_p + (1/3)t_{p^3},$$

where,

$$t_{p^k} := \operatorname{tr}(F_p^k | H^3_{\text{ét}}(X,\mathbf{Q}_\ell)_c(1)), \qquad |t_{p^k}| < 6\sqrt{p^k}$$

by Weil's estimate on the eigenvalues of F_{p^k}. For all the primes 1 mod 4 from 13 to 41 we computed t_p, t_{p^2} and we found a unique integer n, with $|n| < 6\sqrt{p^3}$, such that the polynomial f_p had three quadratic factors upon substituting $t_{p^3} := n$. Moreover, comparing the coefficients of the quadratic factors (of the type $X^2 - m_pX + p$ for an integer m_p), we found three elliptic modular forms of weight two whose Fourier coefficients were equal to the coefficients m_p. These forms are $f15C$ (notation as in [23]) and the forms $f24B$ and $f120E$ as in Section 7.2. Some Fourier coefficients, denoted by d_p, of $f15C$ are given in the table there.

Using this, we next tried to understand the $G_{\mathbf{Q}}$-representation on $H^3_{\text{ét}}(X,\mathbf{Q}_\ell)_c(1))$, so to find $\operatorname{tr}(F_q|H^3_{\text{ét}})$ also for the $q\equiv 3 \bmod 4$. For small p,k we found

$$\operatorname{tr}(F_{p^k}|H^3_{\text{ét}}(X,\mathbf{Q}_\ell)_c(1)) = \begin{cases} 45(b_p+c_p+d_p), & \text{if} \quad p^k\equiv 1 \bmod 4, \\ 9(b_p+c_p+d_p), & \text{if} \quad p^k\equiv 3 \bmod 4. \end{cases}$$

This leads us to conjecture (and this is now proved in [29])

$$H^3_{\text{ét}}(X,\mathbf{Q}_\ell)_c(1)) \overset{?}{\cong}_{G_{\mathbf{Q}}} W_{c,\ell}^{\oplus 27}\oplus(\sigma_{1,0,0}\otimes W_{c,\ell})^{\oplus 18},$$

with the 6D $G_{\mathbf{Q}}$-representation

$$W_{c,\ell} := V_{f15C,\ell} \oplus V_{f24B,\ell} \oplus V_{f120E,\ell},$$

where the $V_{g,\ell}$ are the ℓ-adic $G_{\mathbf{Q}}$-representations corresponding to the weight two new forms $f15C$, $f24B$ and $f120E$, respectively.

Putting all conjectures together, and given the proofs of them in [29], we have the following formula:

$$\sharp X(\mathbf{F}_p) = 1 + p + p^2 + p^3 - \begin{cases} (a_p + p(54b_p + 50c_p + 45d_p), & \text{if} \quad p \equiv 1 \bmod 4, \\ (a_p + p(18b_p + 23c_p + 9d_p), & \text{if} \quad p \equiv 3 \bmod 4. \end{cases}$$

8. Rational curves and the Abel–Jacobi map for Maschke's CY three-fold

8.1. Outline

We show in Section 8.2 that Maschke's CY three-fold X contains 1D families of rational curves. One such family is denoted by $\mathbb{L} \to \widetilde{C}_+$, it is parametrized by a curve $\widetilde{C}_+$ of genus 33. Other families can be obtained by applying the action of $g \in G$ to this family.

From this family one obtains a morphism of Hodge structures

$$\phi : H^1(\widetilde{C}_+, \mathbf{Z}) \longrightarrow H^3(X, \mathbf{Z}).$$

Similarly, one has maps $\phi_\ell : H^1_{\text{ét}}(\widetilde{C}, \mathbf{Q}_\ell) \to H^3_{\text{ét}}(X, \mathbf{Q}_\ell)$ which are maps of Galois representations (up to Tate twist). To understand ϕ and thus also the ϕ_ℓ, we use the Abel–Jacobi map in Section 8.5.

In Section 9 we study the Galois representation on $H^1_{\text{ét}}(\widetilde{C}_+, \mathbf{Q}_\ell)$ and relate it, via the map ϕ_ℓ, to the Galois representation on $H^3_{\text{ét}}(X, \mathbf{Q}_\ell)$.

8.2. Four-tangent lines to Maschke's octic surface

To obtain families of rational curves in X, we consider curves in X which map isomorphically to lines in $\mathbf{P}^3$ under the 2:1 map $\pi : X \to \mathbf{P}^3$. A line in $\mathbf{P}^3$ intersects the branch locus S of π in a divisor D of degree 8, so in general the inverse image of a line is a (hyperelliptic) genus three curve. However, if the line is four tangent to S, so $D = 2(p_1 + \cdots + p_4)$, then the inverse image will split into two rational curves, each of which maps isomorphically to the line.

It is a classical result that for a general surface of degree 8 in $\mathbf{P}^3$ there are 14752 lines which are four-tangent to it ([21], p. 261). However, S is rather special and in fact it does have positive dimensional families of four-tangent lines, as we show with some explicit computations.

For $c = (x, y) \in \mathbb{A}^2$ we consider the line $l_c \subset \mathbb{P}^3$, with parameter t:

$$l_c : \quad (x_0, x_1, x_2, x_3) := (x, 1, ty, t) \quad (c = (x, y) \in \mathbb{A}^2,\ t \in \mathbf{C}).$$

The intersection of l_c with the surface S is defined by the polynomial f_c:

$$l_c \cap S : \quad f_c(t) := F(x, 1, ty, t) = 0.$$

The element $g_1 \in G$ acts as $(x_0, x_1, x_2, x_3) \mapsto (x_0, x_1, ix_2, ix_3)$, with $i^2 = -1$, and thus (as is also easy to verify directly) $F(x_0, x_1, x_2, x_3) = F(x_0, x_1, ix_2, ix_3)$. Therefore, we have $f_c(t) = f_c(it)$ and $f_c(t)$ is actually a polynomial in t^4 of degree 2. An explicit computation shows that

$$f_c(t) = At^8 + Bt^4 + C$$

with

$$A := y^8 + 14y^4 + 1, \quad B := 14(x^4y^4 + x^4 + y^4 + 12x^2y^2 + 1),$$
$$C := x^8 + 14x^4 + 1.$$

Now we impose that l_c is four-tangent to the surface S by requiring that this polynomial of degree two in t^4 has a double zero. So we restrict the point (x, y) to the algebraic subset of $\mathbb{A}^2$ defined by $\Delta = B^2 - 4AC = 0$, where

$$\Delta := (14x^4y^4 + 14x^4 + 168x^2y^2 + 14y^4 + 14)^2 - 4(x^8 + 14x^4 + 1)(y^8 + 14y^4 + 1).$$

For each $c \in \mathbb{A}^2$ with $\Delta(c) = 0$ we have $f_c(t) = (\lambda t^4 + \mu)^2$ for some $\lambda, \mu \in \mathbf{C}$ and thus l_c is a four tangent line. Therefore, S has a positive dimensional family of four-tangent lines.

To understand the locus $\Delta = 0$ better, we observe that Δ is reducible:

$$\Delta = 48 g_+ g_- \quad (\in \mathbf{Z}[x, y])$$

with polynomials

$$g_+(x, y) := (2y^4 + y^2 + 2)x^4 - (y^4 - 24y^2 + 1)x^2 + 2y^4 + y^2 + 2$$

and

$$g_-(x,y) := (2y^4 - y^2 + 2)x^4 + (y^4 + 24y^2 + 1)x^2 + 2y^4 - y^2 + 2,$$

note that $g_-(x,y) = g_+(ix, iy)$. We define two curves, $C_\pm$, by taking the closure in $\mathbf{P}^1 \times \mathbf{P}^1$ of the loci $g_\pm(x,y) = 0$ in $\mathbb{A}^2$. These curves are isomorphic to the closure of the locus of the lines parametrized by $g_\pm = 0$ in the Grassmannian of lines in $\mathbf{P}^3$. We checked that $C_\pm$ are smooth curves. As they have bidegree $(4,4)$, their genus is $3^2 = 9$.

$$C_+ : P_+(y,v)x^4 - Q_+(y,v)x^2u^2 + P_+(y,u)u^4 = 0 \ (\subset P^1_{(x:u)} \times \mathbf{P}^1_{(y:v)})$$
$$\text{with} \quad P_+(y,v) = 2y^4 + y^2v^2 + 2v^4, \quad Q_+(y,v) = y^4 - 24y^2v^2 + v^4.$$

8.3. Remark: lines in the surface

Some of the lines l_c, parametrized by $c \in C_\pm$, do lie entirely within S. For such a line the coefficients of the degree eight polynomial in $f_c(t)$ must be zero, so:

$$A = B = C = 0.$$

One verifies that there are 32 such lines:

$$l_{(x,y)} \subset S \quad \Longleftrightarrow \quad x^8 + 14x^4 + 1 = 0 \quad \text{and} \quad y^2 = x^2 \text{ or } y^2 = x^{-2}$$

and that the corresponding $c = (x,y)$ are the points of intersection of C_+ and C_-:

$$l_c \subset S \qquad \Longleftrightarrow \quad c \in C_+ \cap C_-.$$

As C_+ and C_- are curves of type $(4,4)$ on $\mathbf{P}^1 \times \mathbf{P}^1$ their intersection number is $4^2 + 4^2 = 32$, in particular, their intersection is transversal. Using the factorization

$$x^8 + 14x^4 + 1 = (x^4 - 2x^3 + 2x^2 + 2x + 1)(x^4 + 2x^3 + 2x^2 - 2x + 1),$$

one easily checks that the line $l_3 \subset S$ from Section 3.2 is one of the lines l_c for $c \in C_+ \cap C_-$.

8.4. The double cover $\widetilde{C}_+$ of C_+

Now we consider the inverse images of the lines l_c with $c \in C_+$ in the CY three-fold X. Locally we have

$$X: \quad w^2 = F(x_0, 1, x_2, x_3), \qquad \pi^{-1}(l_c): \quad w^2 = At^8 + Bt^4 + C,$$

where t runs over $\mathbf{C}$. On C_+ we have $B^2 = 4AC$ and thus, for each $c = (x, y) \in C_+$ with $A(c) \neq 0$ we can rewrite the equation of $\pi^{-1}(l_c)$ as

$$4Aw^2 = 4A^2t^8 + 4ABt^4 + 4AC = 4A^2t^8 + 4ABt^4 + B^2 = (2At^4 + B)^2.$$

The two irreducible components of the $\pi^{-1}(l_c)$ are thus defined by the two factors of

$$(2\sqrt{A}w - (2At^4 + B))(2\sqrt{A}w + (2At^4 + B)).$$

Thus the points of the double cover $\widetilde{C}_+$ of C_+ defined by $\sqrt{A}$, where we view $A = y^8 + 14y^4 + 1$ as a rational function on C_+, parametrize rational curves in X, which map to lines l_c with $c \in C_+$. The double cover $\widetilde{C}_+ \to C_+$ is branched over $4 \cdot 8 = 32$ points, and as C_+ has genus 9, $\widetilde{C}_+$ has genus 33.

8.5. The Abel–Jacobi map

We refer to [30], chapter 12, for this section. We denote by $\mathbb{L}$ the total space of the family of lines in X parametrized by $\widetilde{C}_+$, it is a surface in the product $\widetilde{C}_+ \times X$. In the diagram f, g are the projection maps:

$$\begin{array}{ccccc} \widetilde{C}_+ \times X & \supseteq & \mathbb{L} & \xrightarrow{f} & X \\ & & \downarrow g & & \\ & & \widetilde{C}_+ & & \end{array} \qquad \mathbb{L} := \{(\widetilde{c}, x) \subset \widetilde{C}_+ \times X : \ x \in l_{\widetilde{c}}\}.$$

The subvariety $\mathbb{L}$ has cohomology class, of Hodge type $(2, 2)$,

$$[\mathbb{L}] \in H^4(\widetilde{C}_+ \times X, \mathbf{Z}) = \oplus_{i=0}^2 H^i(\widetilde{C}_+, \mathbf{Z}) \otimes H^{4-i}(X, \mathbf{Z}).$$

By Poincaré duality, $H^1(\widetilde{C}_+, \mathbf{Z})$ is selfdual and thus

$$H^1(\widetilde{C}_+, \mathbf{Z}) \otimes H^3(X, \mathbf{Z}) \cong \mathrm{Hom}_{\mathbf{Z}}(H^1(\widetilde{C}_+, \mathbf{Z}), H^3(X, \mathbf{Z})).$$

As $[\mathbb{L}]$ has type $(2,2)$, it induces a morphism of Hodge structures

$$\phi := [\mathbb{L}]_1 : H^1(\widetilde{C}_+, \mathbf{Z}) \longrightarrow H^3(X, \mathbf{Z}).$$

This map relates the cohomology of $\widetilde{C}_+$ and X. To understand ϕ we use the Abel–Jacobi map.

The (Griffiths) intermediate Jacobian

$$\begin{aligned} J(X) &:= (H^{1,2}(X) \oplus H^{0,3}(X))/H^3(X, \mathbf{Z}) \\ &\cong (H^{3,0}(X)^* \oplus H^{2,1}(X)^*)/H^3(X, \mathbf{Z}), \end{aligned}$$

where the * indicates the dual complex vector space, is a 150D complex torus. The Jacobian of $\widetilde{C}_+$ is the abelian variety

$$J(\widetilde{C}_+) := H^{0,1}(\widetilde{C}_+)/H^1(\widetilde{C}_+, \mathbf{Z}) \;\cong\; H^{1,0}(\widetilde{C}_+)^*/H^1(\widetilde{C}_+, \mathbf{Z}).$$

The morphism of Hodge structures ϕ corresponds to the holomorphic map of complex tori

$$\Phi : J(\widetilde{C}_+) \longrightarrow J(X)$$

which is induced, by the Albanese property of $J(\widetilde{C}_+)$, by the (holomorphic) Abel–Jacobi map

$$\Phi_{\widetilde{C}_+} : \widetilde{C}_+ \longrightarrow J(X), \qquad \widetilde{c} \longmapsto \int_\Gamma \in (H^{3,0}(X)^* \oplus H^{2,1}(X)^*)/H^3(X, \mathbf{Z}).$$

Here one fixes a base point $\widetilde{c}_0 \in \widetilde{C}_+$, and $\Gamma \subset X$ is a differentiable 3-chain in X with boundary

$$\partial\Gamma = l_{\widetilde{c}} - l_{\widetilde{c}_0}.$$

We now point out some easy facts on the domain and the codomain of the Abel–Jacobi map Φ_C and thus of ϕ.

The Hodge structure $H^1(\widetilde{C}_+, \mathbf{Q})$ has a decomposition into eigenspaces for the covering involution:

$$H^1(\widetilde{C}_+, \mathbf{Q}) = H^1(\widetilde{C}_+, \mathbf{Q})_+ \oplus H^1(\widetilde{C}_+, \mathbf{Q})_-, \quad H^1(\widetilde{C}_+, \mathbf{Q})_+ \cong H^1(C_+, \mathbf{Q}).$$

The double cover $\widetilde{C}_+ \to C_+$ induces the projection on the first factor in the eigenspace decomposition.

A point $c \in C_+$ corresponds to the cycle $\pi^{-1}(l_c) \subset X$. But as any two lines in $\mathbf{P}^3$ are fibers of a family of lines parametrized by a $\mathbf{P}^1$, the Abel–Jacobi map restricted to the image of $J(C_+) \subset J(\widetilde{C}_+)$ is trivial, since there are no non-constant holomorphic maps from $\mathbf{P}^1$ to a complex torus. Hence Φ factors over the quotient abelian variety $P(\widetilde{C}_+/C_+) := J(\widetilde{C}_+)/J(C_+)$, the Prym variety of the double cover $\widetilde{C}_+ \to C_+$, and ϕ factors over $H^1(\widetilde{C}_+, \mathbf{Q})_-$.

As ϕ is a morphism of Hodge structures, we have $\phi(H^{p,q}(\widetilde{C}_+)) \subset H^{p+1,q+1}(X)$. In Section 6.2 we showed that the G-action splits the rational Hodge structure on $H^3(X, \mathbf{Q})$ into two Hodge substructures:

$$H^3(X, \mathbf{Q}) = H^3(X, \mathbf{Q})_t \oplus H^3(X, \mathbf{Q})_a,$$
$$\begin{cases} H^{3,0}(X)_t \cong \mathbf{C}, & H^{2,1}(X)_t = 0, \\ H^{3,0}(X)_a \cong 0, & H^{2,1}(X)_a \cong \mathbf{C}^{149}, \end{cases}$$

here $H^3(X, \mathbf{Q})_t \cong V_1 \otimes W_1$ and $H^3(X, \mathbf{Q})_a$ is a direct sum of six summands. In particular, $H^{1,2}(X)_a = H^{1,2}(X)$. This implies that the intermediate Jacobian $J(X)$ has a codimension one subtorus

$$J_a(X) := H^{1,2}(X)/H^3(X, \mathbf{Z})_a, \quad H^3(X, \mathbf{Z})_a := H^3(X, \mathbf{Z}) \cap H^3(X, \mathbf{Q})_a,$$

which is actually an abelian variety, polarized by the restriction of the intersection form on $H^3(X, \mathbf{Q})$ to $H^3(X, \mathbf{Q})_a$. Moreover, $J_a(X)$ is isogeneous to a product of 149 elliptic curves (cf. Section 6.2). The image of Φ is thus contained in $J_a(X)$. Thus the map ϕ induces a morphism of Hodge structures:

$$\phi_a : H^1(\widetilde{C}_+, \mathbf{Q})_- \longrightarrow H^3(X, \mathbf{Q})_a.$$

In Section 8.7, we show that Φ, and hence ϕ_a is non-trivial.

8.6. Remark

The surface S contains two G-orbits of lines, cf. Section 3.2. In Remark 8.3, we observed that the lines in the orbit of l_3 deform to four tangent lines of S. Thus these lines, viewed as cycles with multiplicity two, deform in X. On the other hand, we checked that the normal bundle of the line l_5 in X is $\mathcal{O}(-1)^{\oplus 2}$, and thus these lines are rigid in X. The 1-cycle $l_3 - l_5$ might therefore give a non-trivial element in the Griffiths group of X. To check this, one could try to determine the image of this cycle in the intermediate Jacobian $J(X)$, in particular the component in the isogeny

factor $J_t(X) := H^{0,3}(X)/H^3(X,\mathbf{Z})_t$. The conjectures of Bloch and Beilinson imply that the rank of the Griffiths group of X over a number field L is the order of vanishing of the L-series of the G_L-representation on $W_{1,\ell}$. As this is the restriction of the $G_{\mathbf{Q}}$-representation associated to the weight four newform $f120$ (cf. Section 7.2), one might be able to compute the order of zero of the L-series in case $L = \mathbf{Q}(\zeta_{60})$, the field of definition of the 1-cycle. We refer to [3] for an interesting example involving Schoen's rigid CY three-fold.

In case one has a family of CY three-folds and a relative family of 1-cycles, there is an associated inhomogeneous Picard–Fuchs equation. A nice example, involving the Dwork family of quintic three-folds, appears in [22].

8.7. The Abel–Jacobi map is non-trivial

To show that the Abel–Jacobi map Φ is non-trivial, we consider the differential $\mathrm{d}\Phi_{\widetilde{C}_+,\widetilde{c}}$ of the map $\Phi_{\widetilde{C}_+}$ which induces Φ at a point $\widetilde{c} \in \widetilde{C}_+$. This differential is called the infinitesimal Abel–Jacobi map. There is a natural map from the deformation space of the pair (l, X), where l is a smooth curve in the CY three-fold X, to the deformation space of X. If this map is not surjective, then the infinitesimal Abel–Jacobi map is non-zero ([31], Lemma 5.1).

The deformations X_ϵ of X are given by deformations S_ϵ of Maschke's octic S, thus they are defined by $F + \epsilon G$, where G is homogeneous of degree 8 in $X_0, \ldots, X_3$ and $\epsilon^2 = 0$. Considering a rational curve $l_{\widetilde{c}}$ in X, which maps onto the line l_c, this curve deforms to X_ϵ iff the line l_c deforms to a four tangent line $l_{c,\epsilon}$ of S_ϵ. As l_c is parametrized by $(x : 1 : ty : t)$, with $c = (x, y) \in C_+$, any deformation of l_c can be parametrized as

$$l_{c,\epsilon} : t \longmapsto (x + \epsilon(a + ct), 1, ty + \epsilon(b + dt), t),$$

for some $a, b, c, d \in \mathbf{C}$. Substituting this parametrization in $F + \epsilon G$ one obtains $h(t) + \epsilon k(t)$, for certain polynomials $h, k \in \mathbf{C}[t]$. As $c \in C_+$, we have $h(t) = f_c(t)^2$ and $l_{c,\epsilon}$ is four tangent to S_ϵ iff $f_c(t)^2 + \epsilon k(t) = (f_c(t) + \epsilon m(t))^2$ for some polynomial m. Equivalently, $k(t) = 2f_c(t)m(t)$ for some polynomial m. Taking a point $c = (x, y) \in C_+$ with $y = 2$ and taking $G = X_0^2 X_1^4 X_2^2$, we found that the coefficient of t^2 in $k(t)$ mod $f_c(t)$ is a non-zero constant (i.e., independent of a, b, c, d), hence there is no polynomial $m(t)$ such that $k(t) = 2f_c(t)m(t)$ and thus the infinitesimal Abel–Jacobi map in $\widetilde{c}$ is non-zero.

9. The geometry and arithmetic of $\widetilde{C}_+$

9.1. Outline

The map ϕ from Section 8.5 is defined by the algebraic cycle $[\mathbb{L}]$ on $\tilde{C}_+ \times X$ which is defined over $\mathbf{Q}$. Hence, after suitable restrictions, we also get maps

$$\phi_{a,\ell} : H^1_{\text{ét}}(\widetilde{C}_+, \mathbf{Q}_\ell)_- \longrightarrow H^3_{\text{ét}}(X, \mathbf{Q}_\ell)_a,$$

which are, up to a Tate twist, compatible with the action of the Galois group $G_{\mathbf{Q}}$. Thus if $H^1_{\text{ét}}(\widetilde{C}_+, \mathbf{Q}_\ell)_-$ has a $G_{\mathbf{Q}}$-subrepresentation which does not occur in $H^3_{\text{ét}}(X, \mathbf{Q}_\ell)_a$, then that subrepresentation is mapped to zero by $\phi_{a,\ell}$ and hence also ϕ_a has a non-trivial kernel.

Our results indicate that the 66D $G_{\mathbf{Q}}$-representation on $H^1_{\text{ét}}(\widetilde{C}_+, \mathbf{Q}_\ell)$ might be the direct sum of 33 2D $G_{\mathbf{Q}}$-representations. If this is the case, then the Jacobian of $\widetilde{C}_+$ is isogeneous to the product of 33 elliptic curves.

Up to isomorphism, only the Galois representations associated to six newforms (and their twists by $\sigma_{1,0,0}$) of weight two appear in $H^1_{\text{ét}}(\widetilde{C}_+, \mathbf{Q}_\ell)$. Up to twist by $\sigma_{1,0,0}$, three of the six $G_{\mathbf{Q}}$-representations in $H^1_{\text{ét}}(\widetilde{C}_+, \mathbf{Q}_\ell)$ occur only in $H^1_{\text{ét}}(\widetilde{C}_+, \mathbf{Q}_\ell)_+$ and none of these representations or their twist appear, even after a Tate twist, in $H^3_{\text{ét}}(X, \mathbf{Q}_\ell)_a$. The remaining $G_{\mathbf{Q}}$-representations appear only in $H^1(\tilde{C}_+, \mathbf{Q}_\ell)_-$. All of these do appear, after Tate twist, in $H^3_{\text{ét}}(X, \mathbf{Q}_\ell)_a$.

In conclusion, we do not find an obvious obstruction to the surjectivity of $\phi_{a,\ell}$. We will leave a more detailed study of ϕ_a and $\phi_{a,\ell}$ to another occasion.

9.2. The Galois representation on $H^1_{\text{ét}}(\widetilde{C}_+, \mathbf{Q}_\ell)$

The genus 33 curve $\widetilde{C}_+$ is a double cover, defined by $t^2 = A$, of the genus 9 curve $C_+ \subset \mathbf{P}^1 \times \mathbf{P}^1$ defined by $g_+(x, y) = Px^4 - Qx^2 + P = 0$ where

$$A := y^8 + 14y^4 + 1, \quad P := 2y^4 + y^2 + 2, \quad Q := y^4 - 24y^2 + 1.$$

Hence there is a commutative diagram (see Section 8.4)

$$\begin{array}{ccc} C_+ & \longleftarrow & \widetilde{C}_+ \\ \downarrow & & \downarrow \\ \mathbf{P}^1_y & \longleftarrow & C_3, \quad C_3 : \ t^2 = A, \end{array}$$

where C_3 is a hyperelliptic genus three curve.

The covering involution induces the decomposition of $G_{\mathbf{Q}}$-representations, similar to the decomposition of the Hodge structures in Section 8.5:

$$H^1_{\text{ét}}(\widetilde{C}_+, \mathbf{Q}_\ell) = H^1_{\text{ét}}(\widetilde{C}_+, \mathbf{Q}_\ell)_+ \oplus H^1_{\text{ét}}(\widetilde{C}_+, \mathbf{Q}_\ell)_-,$$
$$H^1_{\text{ét}}(\widetilde{C}_+, \mathbf{Q}_\ell)_+ \cong H^1_{\text{ét}}(C_+, \mathbf{Q}_\ell).$$

In particular, $H^1_{\text{ét}}(C_3, \mathbf{Q}_\ell) \subset H^1_{\text{ét}}(\widetilde{C}_+, \mathbf{Q})_-$. As $\sharp C_+(\mathbf{F}_p) = 1 - \operatorname{tr}(F_p | H^1_{\text{ét}}(\widetilde{C}_+, \mathbf{Q}_\ell)_+) + p$, we get:

$$\sharp\widetilde{C}_+(\mathbf{F}_p) = 1 - \operatorname{tr}(F_p | H^1_{\text{ét}}(\widetilde{C}_+, \mathbf{Q}_\ell)) + p = \sharp C_+(\mathbf{F}_p) - \operatorname{tr}(F_p | H^1_{\text{ét}}(\widetilde{C}_+, \mathbf{Q}_\ell)_-).$$

We computed the cardinality of $\widetilde{C}_+(\mathbf{F}_p)$ and $C_+(\mathbf{F}_p)$ for all primes p with $7 \le p \le 1000$. The results are consistent with the following formula:

$$\operatorname{tr}(F_p | H^1_{\text{ét}}(\widetilde{C}_+, \mathbf{Q}_\ell)_-) \stackrel{?}{=} (9 + 3\sigma_{1,0,0}(F_p))b_p + (5 + \sigma_{1,0,0}(F_p))c_p + (4 + 2\sigma_{1,0,0}(F_p))d_p$$

with b_p, c_p, d_p the Fourier coefficients of the newforms $f24B$, $f120E$ and $f15C$, cf. Section 7.2. If this equality holds for all primes $p > 5$, or at least for a large set of primes (but we do not know a good bound for this set), then we would have an isomorphism of $G_{\mathbf{Q}}$-representations

$$H^1_{\text{ét}}(\widetilde{C}_+, \mathbf{Q}_\ell)_- \stackrel{?}{\cong} {V_{f24B,\ell}}^{\oplus 9} \oplus {V'_{f24B,\ell}}^{\oplus 3} \oplus {V_{f120E,\ell}}^{\oplus 5} \oplus V'_{f120E,\ell} \oplus {V_{f15C,\ell}}^{\oplus 4} \oplus {V'_{f15C,\ell}}^{\oplus 2},$$

where the twist by $\sigma_{1,0,0}$ of a $G_{\mathbf{Q}}$-representation is denoted by

$$V'_* := V_* \otimes \sigma_{1,0,0}.$$

Note that the three representations and their twists in this decomposition appear, after a Tate twist, in $H^3(X, \mathbf{Q}_\ell)_a$. See Remark 9.3 for a possible geometric approach to the conjectured decomposition, with the observations made there one can actually prove that

$$H^1_{\text{ét}}(C_3, \mathbf{Q}_\ell) \cong {V_{f24B,\ell}}^{\oplus 2} \oplus V'_{f24B,\ell}.$$

Similarly, we conjecture (and we checked equality of the traces of F_p for primes p with $5 < p < 1000$):

$$H^1_{\text{ét}}(C_+, \mathbf{Q}_\ell) \stackrel{?}{\cong} V_{f210,\ell}{}^{\oplus 3} \oplus V_{f840,\ell}{}^{\oplus 2} \oplus V'_{f840,\ell} \oplus V_{f1680,\ell} \oplus V'_{f1680,\ell}{}^{\oplus 2},$$

where the f_N are newforms of weight 2 on $\Gamma_0(N)$ characterized by the Fourier coefficients in the table below:

	p	11	13	17	19	23	29	31	37	...	79	83	89	97
$f_{210}, k=2$	b'_p	4	−2	2	−4	−8	6	−8	−2	...	0	12	2	10
$f_{840}, k=2$	c'_p	−4	−2	2	−4	0	−10	0	6	...	8	−4	10	10
$f_{1680}, k=2$	d'_p	−4	−2	2	−4	0	−2	−8	−2	...	8	−4	2	−14

Note that none of these Galois representations occurs (even after Tate twist) in $H^3_{\text{ét}}(X, \mathbf{Q}_\ell)$.

9.3. Quotients of $\widetilde{C}_+$

The curve $\tilde{C}_+$ has a big automorphism group, which one can use to decompose the cohomology and to find quotient curves, of lower genus, of $\tilde{C}_+$. One might hope these automorphisms, and further automorphisms of the quotient curves not induced automorphisms of $\tilde{C}_+$, might suffice to prove the conjectural decomposition of the étale cohomology. However, we did not succeed in carrying this out, but the partial results we obtained were quite helpful in finding the conjectured decomposition.

The group $(\mathbf{Z}/2\mathbf{Z})^2$, generated by the involutions

$$\iota_1 : (x, y, t) \longmapsto (-x, y, t), \quad \iota_2 : (x, y, t) \longmapsto (x, y, -t).$$

on $\widetilde{C}_+$. From this one finds the following quotient curves.

The curve $\overline{C}_+ = \widetilde{C}_+/(\mathbf{Z}/2\mathbf{Z})^2$ is a double cover $\overline{C}_+ \to \mathbf{P}^1_y$ defined by $Px^2 - Qx + P = 0$. In particular, it is a hyperelliptic curve of genus 3, defined by $s^2 = Q^2 - 4P^2$.

The quotient of $\widetilde{C}_+$ by the product $\iota_1\iota_2$ of these involutions is a curve C_{17} of genus 17, which is a double cover of $\overline{C}_3$.

The curve $C_{13} := \widetilde{C}_+/\iota_2$ has genus 13, it is a double cover of both $\overline{C}_+$ and of C_3, each of these covers is branched in $2 \cdot 8 = 16$ points. This curve is again a $(\mathbf{Z}/2\mathbf{Z})^2$-cover $\mathbf{P}^1_y$, two quotients by involutions are $\overline{C}_+$ and C_3, the third quotient by an involution is a 2:1 cover of $\mathbf{P}^1_y$ branched over the

$8+8$ ramification points of the other two double covers, so it is a curve C_7 of genus 7 with equation $u^2 = A(Q^2 - 4P^2) = (y^8 + 14y^4 + 1)(y^4 - 24y^2 + 1)^2 - 4(2y^4 + y^2 + 2))$. Thus C_7 has an involution $(u, y) \mapsto (u, -y)$, with quotient curve of genus three. Another involution on C_7, which is fixed point free, is $(u, y) \mapsto (-u, -y)$ and it has a genus four quotient, etc. Using similar involutions, one can find genus one quotients of C_3 which lead to the decomposition of $H^1_{\text{ét}}(C_3, \mathbf{Q}_\ell)$ given in the previous section.

10. Trace formulas

10.1. Outline

In this section, we prove the formulas in Proposition 10.1. The first two, which give the Euler characteristic and the dimension $h^n(X)_{\text{pr}}$ of the primitive cohomology group of a hypersurface in projective space, are well known. The second is a formula of Chênevert [8], which determines the trace of an automorphism of a hypersurface on the primitive cohomology. We derive it directly from the Lefschetz trace formula. The last is an easy generalization of Chênevert's formula to cyclic ramified covers of projective space, like Maschke's CY. These trace formulas are remarkable and easy to use, since they involve only the degree of X and the dimensions of certain eigenspaces of σ, but not the specific geometry of $X \subset \mathbf{P}^{n+1}$.

Proposition 10.1. *Let X be a smooth hypersurface of degree d and dimension n in $\boldsymbol{P}^{n+1}$ defined by an equation $F = 0$. Let $\sigma : \boldsymbol{C}^{n+2} \to \boldsymbol{C}^{n+2}$ be a linear map such that $F(\sigma(x)) = F(x)$. Let r be a divisor of d and let $Y \to \boldsymbol{P}^{n+1}$ be the cyclic r:1 cover of $\boldsymbol{P}^{n+1}$ branched along X and let $\widetilde{\sigma} \in Aut(Y)$ be the automorphism induced by σ. Then we have:*

1. *The Euler characteristic of X is given by:*

$$\chi(X) = n + 2 + \frac{1}{d}((1-d)^{n+2} - 1),$$
$$h^n_{\text{pr}}(X) := \dim H^n(X, \boldsymbol{Q})_{\text{pr}} = \frac{(-1)^n}{d}((1-d)^{n+2} + d - 1).$$

2. *(Chênevert's formula [8]) For $\alpha \in \boldsymbol{C}$ such that $\alpha^d = 1$, let m_α be the multiplicity of the eigenvalue α of σ, where we put $m_\alpha = 0$ if α is not an eigenvalue of σ. Then the trace of automorphism of X induced by σ on the primitive cohomology of X is the following sum over all the*

d-th roots of unity:

$$\mathrm{tr}(\sigma^*|H^n(X, \boldsymbol{Q})_{\mathrm{pr}}) = \frac{(-1)^n}{d} \sum_{\alpha^d=1} (1-d)^{m_\alpha}.$$

3. *With the notations as above, we have:*

$$\mathrm{tr}(\widetilde{\sigma}^*|H^{n+1}(Y, \boldsymbol{Q})_{\mathrm{pr}}) = \frac{(-1)^{n+1}}{d} \left(\sum_{\alpha^d=1} (1-d)^{m_\alpha} - r \sum_{\gamma^{d/r}=1} (1-d)^{m_\gamma} \right).$$

Proof. 1. The first formula can be obtained from the Gauss–Bonnet formula $\chi(X) = c_n(T_X)$. The normal bundle sequence implies that

$$c(T_X) = \frac{c(T_{\mathbf{P}^{n+1}|X})}{c(\mathcal{O}(d))} = \frac{(1+h)^{n+2}}{1+dh} = \left(\sum_{i=0}^{n+2} \binom{n+2}{i} h^i \right) \left(\sum_j (-d)^j h^j \right)$$

where h is the hyperplane class on X, so $h^n = d$ and $h^i = 0$ for $i > n$. In degree n we get:

$$c_n(T_X) = d \sum_{k=0}^{n} \binom{n+2}{k} (-d)^{n-k}.$$

Comparing this with

$$(1-d)^{n+2} = d^2 \left(\sum_{k=0}^{n} \binom{n+2}{k} (-d)^{n-k} \right) - d(n+2) + 1$$

gives the formula for $\chi(X)$.

By Lefschetz's hyperplane theorem, one has $H^{2i}(X, \mathbf{Q}) \cong H^{2i}(\mathbf{P}^{n+1}, \mathbf{Q}) \cong \mathbf{Q}$ if $2i \neq n$ and $H^{2i+1}(X, \mathbf{Q}) = 0$ if $2i+1 \neq n$. Thus the non-primitive cohomology of X contributes $d+1$ to the Euler characteristic, the primitive cohomology is concentrated in $H^n(X, \mathbf{Q})$, and its dimension is $h^n_{\mathrm{pr}}(X) = \chi(X) - (n+1)$. Hence

$$h^n_{\mathrm{pr}}(X) = (-1)^n \left(1 + \frac{1}{d}((1-d)^{n+2} - 1) \right) = \frac{(-1)^n}{d}((1-d)^{n+2} + d - 1).$$

2. An eigenvalue of σ is denoted by β and its multiplicity by m_β, so that $\sum m_\beta = n+2$. Let $\mathbf{P}_\beta \subset \mathbf{P}^{n+1}$ be the projectivization of the eigenspace of σ with eigenvalue β, it has dimension $m_\beta - 1$. Let $X_\beta :=$

$X \cap \mathbf{P}_\beta$, then $X_\beta = \mathbf{P}_\beta \subset X$ if $\beta^d \neq 1$ and else X_β is a smooth subvariety of $\mathbf{P}_\beta$ of dimension $m_\beta - 2$ and degree d (cf. [8], Lemma 2.3).

The Lefschetz fixed point formula for an automorphism σ of X is:

$$\sum (-1)^i \mathrm{tr}(\sigma^* | H^i(X, \mathbf{Q})) = \sum_j \chi(X_j),$$

where the right-hand sum is over the components of the fixed point set of σ in X (cf. [25]). As we observed above, the right-hand side can be written as a sum over the eigenvalues β of σ:

$$\sum_j \chi(X_j) = \sum_\beta \chi(X_\beta) = \sum_{\beta^d \neq 1} \chi(\mathbf{P}_\beta) + \sum_{\beta^d = 1} \chi(X_\beta),$$

where the X_β are now all smooth hypersurfaces of degree d in $\mathbf{P}_\beta \cong \mathbf{P}^{m_\beta - 1}$. Thus, we get:

$$\sum_j \chi(X_j) = \sum_{\beta^d \neq 1} m_\beta \; + \sum_{\beta^d = 1} m_\beta + \tfrac{1}{d}((1-d)^{m_\beta} - 1),$$

which, using $\sum m_\beta = n + 2$, simplifies to

$$\sum_j \chi(X_j) = n + 2 + \tfrac{1}{d} \sum_{\beta^d = 1} (1-d)^{m_\beta} - 1).$$

If $\alpha^d = 1$ and α is not an eigenvalue of σ, then $m_\alpha = 0$ and thus $(1 - d)^{m_\alpha} - 1 = 0$. There are d complex numbers with $\alpha^d = 1$ and so:

$$\sum_j \chi(X_j) = n + 2 + \sum_{\alpha^d = 1} \tfrac{1}{d}((1-d)^{m_\alpha} - 1) = n + 1 + \tfrac{1}{d} \sum_{\alpha^d = 1} (1-d)^{m_\alpha}.$$

As σ^* is trivial on the non-primitive cohomology, the left-hand side is

$$\sum (-1)^i \mathrm{tr}(\sigma^* | H^i(X, \mathbf{Q})) = n + 1 + (-1)^n \mathrm{tr}(\sigma^* | H^n((X, \mathbf{Q})_{\mathrm{pr}}),$$

hence Chênevert's formula follows.

3. Let $r \in \mathbf{Z}_{\geq 0}$ divide d and let Y be the r:1 cover of $\mathbf{P}^{n+1}$ branched along X. The variety Y has a natural embedding in the total space of the line bundle $\mathcal{O}(d/r)$ over $\mathbf{P}^{n+1}$, where it is defined by $t^r = F$. The automorphism σ of $\mathbf{P}^{n+1}$ lifts to an automorphism $\widetilde{\sigma}$ of the tautological line bundle $\mathcal{O}(-1) \subset \mathbf{P}^{n+1} \times \mathbf{C}^{n+2}$, its action is induced by (σ, σ) on $\mathbf{C}^{n+2} \times \mathbf{C}^{n+2}$. This induces an action, again denoted by $\widetilde{\sigma}$, on $\mathcal{O}(d/r)$ such that if $x \in \mathbf{C}^{n+2}$ is an eigenvector of σ with eigenvalue β, then $\widetilde{\sigma}$ acts as scalar multiplication by $\beta^{-d/r}$ on the fiber of $\mathcal{O}(d/r)$ over $\langle x \rangle \in \mathbf{P}^{n+1}$.

   Let $y \in Y$ be a fixed point of $\widetilde{\sigma}$, then its image $x \in \mathbf{P}^{n+1}$ is a fixed point for σ, hence $x \in \mathbf{P}_\beta$ for some eigenvalue β of σ. In case $\beta^d \neq 1$, $\mathbf{P}_\beta \subset X$, hence $F(x) = 0$ and $x = y$ in the total space of $\mathcal{O}(d/r)$. Conversely, any point $x \in \mathbf{P}_\beta \subset Y$ is a fixed point of $\widetilde{\sigma}$. In case $\beta^d = 1$, there are two possibilities: if $x \in X_\beta = X \cap \mathbf{P}_\beta$, then $x = y$ and conversely any point in $X_\beta \subset Y$ is a fixed point of $\widetilde{\sigma}$. If $x \notin X_\beta$, then we must have $\beta^{-d/r} = 1$, in that case all points in the preimage Y_β of $\mathbf{P}_\beta$ are fixed points of $\widetilde{\sigma}$. The variety Y_β is a smooth r:1 cover of $\mathbf{P}_\beta$ branched along X_β. Thus the fixed point set of $\widetilde{\sigma}$ in Y is:

$$Y^{\widetilde{\sigma}} = \left(\coprod_{\beta^{d/r} \neq 1} X_\beta \right) \coprod \left(\coprod_{\beta^{d/r}=1} Y_\beta \right).$$

   The Euler characteristic of the fixed point set $Y^{\widetilde{\sigma}}$ is thus given by

$$\chi(Y^{\widetilde{\sigma}}) = \sum_{\beta^d \neq 1} \chi(\mathbf{P}_\beta) + \sum_{\beta^d = 1, \beta^{d/r} \neq 1} \chi(X_\beta) + \sum_{\beta^{d/r}=1} \chi(Y_\beta),$$

   where the X_β which appear are smooth hypersurfaces of degree d in $\mathbf{P}_\beta \cong \mathbf{P}^{m_\beta - 1}$. The Hurwitz formula for the branched r:1 cover $Y_\beta \to \mathbf{P}_\beta$ gives:

$$\chi(Y_\beta) = r\chi(\mathbf{P}_\beta) - (r-1)\chi(X_\beta) = rm_\beta - (r-1)(m_\beta + \frac{1}{d}((1-d)^{m_\beta} - 1)).$$

   Therefore we get:

$$\begin{aligned} \chi(Y^{\widetilde{\sigma}}) = &\sum_{\beta^d \neq 1} m_\beta + \sum_{\beta^d = 1, \beta^{d/r} \neq 1} m_\beta + \frac{1}{d}((1-d)^{m_\beta} - 1) \\ &+ \sum_{\beta^{d/r}=1} m_\beta - \frac{r-1}{d}((1-d)^{m_\beta} - 1), \end{aligned}$$

where we sum over all eigenvalues β of σ. As $\sum m_\beta = n+2$ we obtain:

$$\chi(Y^{\widetilde{\sigma}}) = n+2+\frac{1}{d}\sum_{\beta^d=1,\beta^{d/r}\neq 1}((1-d)^{m_\beta}-1) \; - \frac{r-1}{d}\sum_{\beta^{d/r}=1}((1-d)^{m_\beta}-1).$$

Since $(1-d)^{m_\alpha}-1=0$ if $m_\alpha=0$, we can rewrite the formula with a sum over all d-th roots of unity α and a sum over all d/r-th roots of unity γ:

$$\chi(Y^{\widetilde{\sigma}}) = n+2+\frac{1}{d}\sum_{\alpha^d=1}((1-d)^{m_\alpha}-1) - \frac{r}{d}\sum_{\gamma^{d/r}=1}((1-d)^{m_\gamma}-1),$$

which simplifies to

$$\chi(Y^{\widetilde{\sigma}}) = n+2+\frac{1}{d}\sum_{\alpha^d=1}(1-d)^{m_\alpha} - \frac{r}{d}\sum_{\gamma^{d/r}=1}(1-d)^{m_\gamma}.$$

As any automorphism $\widetilde{\sigma}$ is the identity on the non-primitive cohomology of Y, we find that

$$\operatorname{tr}(\widetilde{\sigma}|H^{n+1}(Y)_{\mathrm{pr}}) = \frac{(-1)^{n+1}}{d}\left(\sum_{\alpha^d=1}(1-d)^{m_\alpha} - r\sum_{\gamma^{d/r}=1}(1-d)^{m_\gamma}\right).$$

This concludes the proof of the proposition. □

10.2. Remark

In case $r=d$, Y is a hypersurface in the total space of $\mathcal{O}(1)$, which can be identified with $\mathbf{P}^{n+2}-\{(0:\ldots:0:1)\}$. The bundle projection to $\mathbf{P}^{n+1}$ is given by $(x_0:\ldots:x_{n+1}:x_{n+2})\mapsto(x_0:\ldots:x_{n+1})$ and Y is defined by $F(x_0,\ldots,x_{n+1})=x_{n+2}^d$. The action of $\widetilde{\sigma}$ is as σ on $x_0,\ldots,x_{n+1}$ and is trivial on x_{n+2}. Hence the eigenspaces $\mathbf{P}_\beta$ of σ and $\widetilde{\sigma}$ are the same if $\beta\neq 1$ and the dimension of $\mathbf{P}_1$ increases by one. One easily verifies that our generalization of Chênevert's formula in this case gives the original formula.

References

[1] F. Bardelli, *On Grothendieck's generalized Hodge conjecture for a family of threefolds with trivial canonical bundle*, J. Reine Angew. Math. **422** (1991), 165–200.

[2] V. V. Batyrev, *Birational Calabi–Yau n-folds have equal Betti numbers*, in: New trends in algebraic geometry (Warwick, 1996), 1–11, LMS Lecture Note Ser., 264, Cambridge University Press, Cambridge, 1999.

[3] S. Bloch, *Algebraic cycles and values of L-functions. II*, Duke Math. J. **52** (1985), 379–397.

[4] S. Boissière and A. Sarti, *Counting lines on surfaces*, Ann. Sci. Norm. Super. Pisa Cl. Sci. **6** (2007), 39–52.

[5] P. Candelas, X. de la Ossa and F. Rodriguez-Villegas, *Calabi–Yau manifolds over finite fields. II.*, in: Calabi–Yau varieties and mirror symmetry (Toronto, ON, 2001), 121–157, Fields Inst. Commun., 38, AMS 2003.

[6] P. Candelas, private communication.

[7] B. L. Cerchiai and B. van Geemen, *From qubits to E7*, J. Math. Phys. **51** (2010), 122203.

[8] G. Chenevert, *Representations on the cohomology of smooth projective hypersurfaces with symmetries*, `arXiv:0908.1748`.

[9] D. A. Cox and S. Katz, Mirror Symmetry and Algebraic Geometry. AMS 1999.

[10] W. Chen and Y. Ruan, *A new cohomology theory of orbifold*, Comm. Math. Phys. **248** (2004), 1–31.

[11] L. Dieulefait, *On the modularity of rigid Calabi–Yau threefolds: epilogue*. Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **377** (2010); Issledovaniya po Teorii Chisel. **10**, 44–49, 241.

[12] A. Garbagnati and M. Schütt, *Enriques Surfaces — Brauer groups and Kummer structures*, `arXiv:1006.4952`, to appear in Mich. Math. J.

[13] F. Q. Gouvêa and N. Yui, *Rigid Calabi–Yau threefolds over $\boldsymbol{Q}$ are modular*, Expo. Math. **29** (2011), 142–149.

[14] R. Goodman and N. R. Wallach, *Representations and invariants of the classical groups*, Cambridge University Press, Cambridge, 1998.

[15] K. Hulek and H. Verrill, *On the modularity of Calabi–Yau threefolds containing elliptic ruled surfaces*, with an appendix by Luis V. Dieulefait, in: AMS/IP Stud. Adv. Math., **38**, Mirror symmetry. V, 1934, AMS, 2006.

[16] M. Kisin, *Modularity of 2-adic Barsotti–Tate representations*, Invent. Math. **178** (2009), 587–634.

[17] R. Lazarsfeld, *A Barth-type theorem for branched coverings of projective space*, Math. Ann. **249** (1980), 153-162.

[18] R. Lee and S. H. Weintraub, *The Siegel modular variety of degree two and level four*, Mem. Amer. Math. Soc. **133** (1998), 1–58.

[19] W. Bosma, J. Cannon and C. Playoust, *The Magma algebra system. I. The user language*, J. Symbol. Comput., **24** (1997), 235–265.

[20] H. Maschke, *Ueber die quaternäre, endliche, lineare Substitutionsgruppe der Borchardt'schen Moduln*, Math. Ann. **30** (1887), 496–515.

[21] D. R. Morrison, *Picard–Fuchs equations and mirror maps for hypersurfaces*, in: Essays on mirror manifolds, 241–264, Int. Press, Hong Kong, 1992.

[22] D. R. Morrison, J. Walcher, *D-branes and normal functions*, Adv. Theor. Math. Phys. **13** (2009), 553–598.

[23] B. J. Birch and W. Kuyk (ed), *Modular functions of one variable IV*, Springer LNM **476**, 1975.

[24] A. Mustata, *Degree 1 curves in the Dwork pencil and the Mirror Quintic*, to appear in Math. Ann. available at http://www.springerlink.com/content/1432-1807/.

[25] C. A. M. Peters, *Some applications of the Lefschez fixed point theorems in (complex) algebraic geometry*, in: The Lefschetz centennial Conf., Part I (Mexico City, 1984), 213–221, Contemp. Math., **58**, Amer. Math. Soc., Providence, RI, 1986.

[26] J. Quer, *Q-curves and abelian varieties of* GL_2*-type*, Proc. London Math. Soc. **81** (2000), 285–317.

[27] S. Rams and M. Schütt, *The Barth quintic surface has Picard number* 41, `arXiv:1107.2210`.

[28] M. Schütt, *CM newforms with rational coefficients*, Ramanujan J. **19** (2009), 187–205.

[29] M. Schütt, *Modularity of Maschke's octic and Calabi–Yau threefold*, to appear in Commun. Number Theory Phys. **5**(4) (2011), `arXiv:1112.0405`.

[30] C. Voisin, *Hodge theory and complex algebraic geometry I.* Cambridge University Press, 2002.

[31] R. F. Westhoff, *Computing the infinitesimal invariants associated to deformations of subvarieties*, Pacific J. Math. **183** (1998), 375–397.

Dipartimento di Matematica
Università di Milano
Via Saldini 50
I-20133 Milano
Italia
E-mail address: gilberto.bini@unimi.it
E-mail address: lambertus.vangeemen@unimi.it

Received October 8, 2011