Automorphisms and twisted forms of the $N = 1, 2, 3$ Lie conformal superalgebras

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We classify the $N = 1, 2, 3$ superconformal Lie algebras of Schwimmer and Seiberg by means of differential non-abelian cohomology, and describe the general philosophy behind this new technique. The structure of the group (functor) of automorphisms of the corresponding Lie conformal superalgebra is a key ingredient of the proof.

1. Introduction

The purpose of this paper is two-fold.

(1) To bring to the attention of the physics community new ideas, which for lack of a better word we will refer to as differential non-abelian cohomology, that have recently been used to establish some deep results in infinite dimensional Lie theory (see, for example, [2, 4–7, 11, 12]). These powerful methods, based on the seminal work of Demazure and Grothendieck on reductive groups schemes, torsors and descent [3, 14, 15], can be adapted to the study of Lie conformal superalgebras as explained in [10] by what amounts to formally replacing the given base scheme, $\mathrm{Spec}(\mathbb{C}[\mathrm{t}^{\pm 1}])$ in our case, by a differential scheme.

(2) To complete the classification of the $N = 1, 2, 3, 4$ superconformal Lie algebras by providing a *uniform* proof of the cases $N = 1, 2, 3$, exploiting the fact that the relevant Lie conformal superalgebras used as base objects for the twisted loop construction can be described in terms of exterior algebras. We also provide a precise argument that describes the passage from Lie conformal superalgebras to their corresponding Lie superalgebras. (The cases $N = 2, 4$ were done in [10] at the Lie conformal superalgebra level only, and by *ad hoc* methods). These Lie superalgebras are more relevant to physics, where they are commonly referred to as superconformal Lie algebras.

$N = 1, 2, 3, 4$ superconformal Lie algebras are a class of infinite dimensional Lie superalgebras which plays important roles in both mathematics and physics. They are closed related to the twisted loop Lie conformal superalgebra based on a complex Lie conformal superalgebra. They were

first introduced in [9] as the affinization of Lie conformal superalgebras over the complex numbers and then realized as differential Lie conformal algebra in [10]. Based on the point of view taken in [10], a twisted loop Lie conformal superalgebra has both a complex conformal superalgebra structure and an $\mathcal{R}$-Lie conformal superalgebra structure where $\mathcal{R} = (\mathbb{C}[t^{\pm 1}], \frac{d}{dt})$. While the complex structure is of interest in physics, it is the $\mathcal{R}$-structure that allows us to introduce cohomological methods.

The twisted loop Lie conformal superalgebra construction just mentioned is highly reminiscent of the way in which the affine Kac–Moody Lie algebras, which a priori are defined by generators and relations, are explicitly realized in [8]. This classification and construction has recently been established by means of non-abelian étale cohomology [11]. Because of the very special nature of the algebraic fundamental group of the ring $\mathbb{C}[t^{\pm 1}]$, the loop algebras based on a finite-dimensional simple Lie algebra $\mathfrak{g}$ are parameterized by the conjugacy classes of the finite group of symmetries of the corresponding Coxeter–Dynkin diagram. In particular, if the automorphism group of $\mathfrak{g}$ is *connected*, then *all loop algebras based on $\mathfrak{g}$ are trivial*, i.e., isomorphic to $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t^{\pm 1}]$. As the work of Schwimmer and Seiberg had shown, herein lies a story.

A loop inspired procedure as a mean of realizing the N-superconformal Lie algebras is given in the striking[1] [13]. In the $N = 2$ case, the authors show that all the objects in what was until then thought to be an "infinite" family of superconformal algebras were in fact isomorphic, and that only two non-isomorphic classes of superconformal Lie algebras existed.[2] This agrees precisely with the what cohomological point of view for algebras would predict since in the $N = 2$ case the automorphism group $\mathbf{O}_2$ of the corresponding Lie conformal superalgebra has two connected components. By contrast, Schwimmer and Seiberg put forward an infinite family of non-isomorphic $N = 4$ superconformal Lie algebras... even though in this case the automorphism group is connected! How a connected group of automorphism could lead to an infinite family of loop objects, and which cohomology can be used to determine them, is explained in [10]. The crucial idea is that, unlike the case of algebras, the base ring $\mathbb{C}[t^{\pm 1}]$ does not contain enough information to geometrically measure superconformal Lie algebras. It must be replaced by the complex *differential ring* $(\mathbb{C}[t^{\pm 1}], \delta)$ where $\delta = \frac{d}{dt}$. One

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[1]Certainly to mathematicians.

[2]Strictly speaking their work shows that *at most* two classes exist. We entertain no doubts that it was obvious to the authors that the two classes were different.

could in fact take an arbitrary derivation. For example, the case of $\delta = 0$ leads to the classification of "current algebras".

Forms of a given $\mathcal{R}$-Lie conformal superalgebra $(\mathcal{A}, \partial)$ are classified in terms of the non-abelian cohomology pointed set $H^1\big(\mathcal{R}, \mathbf{Aut}(\mathcal{A})\big)$ as explained in [10]. In this paper, we focus on the $N = 1, 2, 3$ Lie conformal superalgebras. The classification of their twisted loop Lie conformal superalgebras will be carried by explicitly computing their automorphism group functors and the corresponding non-abelian cohomology sets.

Concretely, we first review the general theory of differential Lie conformal algebras developed in [10] in Section 2. Then classification will be accomplished along the following lines. In Section 3, we compute the automorphism group functor of the $N = 1, 2, 3$ Lie conformal superalgebras $\mathcal{K}_N$. The construction is quite explicit and we believe of interest to physicists. In Section 4, we complete the classification of forms of $\mathcal{K}_N \otimes_{\mathbb{C}} \mathcal{R}$ up to isomorphism of $\mathcal{R}$-Lie conformal algebras by compute the corresponding non-abelian cohomology set. Centroid considerations are then used to show that no information is lost in the passage from $\mathcal{R}$ to $\mathbb{C}$. Finally in Section 5, we pass from twisted loop Lie conformal superalgebras to their corresponding superconformal Lie algebras, and again show that no collapse occurs to the isomorphism classes. This completes the classification.

Notation: $\mathbb{Z}, \mathbb{N}, \mathbb{Q}$ and $\mathbb{C}$ denote the integers, non-negative integers, rational numbers and complex numbers respectively. $\mathbf{i}$ will denote $\sqrt{-1}$. For a cycle (ijl), ϵ_{ijl} denotes its sign.

$\mathcal{R}, \mathcal{S}_m, \widehat{\mathcal{S}}$ always denote the complex differential algebras $(\mathbb{C}[t^{\pm 1}], \delta_t)$, $(\mathbb{C}[t^{\pm 1}], \delta_t), (\mathbb{C}[t^q, q \in \mathbb{Q}], \delta_t)$, where $\delta_t = \frac{d}{dt}$.

2. Preliminaries

In this section, we will review the general theory of differential Lie conformal superalgebras developed in [10].[3]

Motivated by the affinization of complex Lie conformal superalgebras defined in [9], Lie conformal superalgebras over an arbitrary differential rings were introduced in [10]. In this paper, we only consider the Lie conformal superalgebras over a complex differential ring $\mathcal{D}$. Thus $\mathcal{D} = (D, \delta)$ is a pair

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[3]This concept is not to be confused with that of superconformal Lie algebra as it appears in the physics literature. The two concepts are related, as we will explain in Remark 2.2.

consisting of a commutative associative algebra D over $\mathbb{C}$ and a $\mathbb{C}$-linear derivation $\delta : D \to D$. Complex differential rings form a category, where a morphism $f : (D, \delta) \to (D', \delta')$ is a $\mathbb{C}$-linear ring homomorphism $f : D \to D$ such that $f \circ \delta = \delta' \circ f$.

Definition 2.1 ([10], Definition 1.3). Let $\mathcal{D} = (D, \delta)$ be a complex differential ring, a $\mathcal{D}$-Lie conformal superalgebra is a triple $(\mathcal{A}, \partial_{\mathcal{A}}, (-_{(n)}-)_{n \in \mathbb{N}})$ consisting of

   (i) a $\mathbb{Z}/2\mathbb{Z}$-graded D-module $\mathcal{A} = \mathcal{A}_{\bar{0}} \oplus \mathcal{A}_{\bar{1}}$,
  (ii) $\partial_{\mathcal{A}} \in \mathrm{End}_{\mathbb{C}}(\mathcal{A})$ stabilizing the even and odd parts of $\mathcal{A}$, and
 (iii) a $\mathbb{C}$-bilinear product $(a, b) \mapsto a_{(n)}b, a, b \in \mathcal{A}$ for each $n \in \mathbb{N}$,

satisfying the following axioms for $r \in D, a, b, c \in \mathcal{A}, m, n \in \mathbb{N}$:

  (CS0) $a_{(n)}b = 0$ for $n \gg 0$,
  (CS1) $\partial_{\mathcal{A}}(a)_{(n)}b = -na_{(n-1)}b$ and $a_{(n)}\partial_{\mathcal{A}}(b) = \partial_{\mathcal{A}}(a_{(n)}b) + na_{(n-1)}b$,
  (CS2) $\partial_{\mathcal{A}}(ra) = r\partial_{\mathcal{A}}(a) + \delta(r)a$,
  (CS3) $a_{(n)}(rb) = r(a_{(n)}b)$ and $(ra)_{(n)}b = \sum_{j \in \mathbb{N}} \delta^{(j)}(r)(a_{(n+j)}b)$,
  (CS4) $a_{(n)}b = -p(a, b) \sum_{j \in \mathbb{N}} (-1)^{j+n} \partial_{\mathcal{A}}^{(j)}(b_{(n+j)}a)$, and
  (CS5) $a_{(m)}(b_{(n)}c) = \sum_{j=0}^{m} \binom{m}{j}(a_{(j)}b)_{(m+n-j)}c + p(a, b)b_{(n)}(a_{(m)}c)$,

where $\delta^{(j)} = \frac{1}{j!}\delta^j, \partial_{\mathcal{A}}^{(j)} = \frac{1}{j!}\partial_{\mathcal{A}}^j, j \in \mathbb{N}$ and $p(a, b) = (-1)^{p(a)p(b)}$, $p(a)$ (resp. $p(b)$) is the parity of a (resp. b).

We also use the λ-bracket convention to simplify notation. Recall then that

$$[a_\lambda b] = \sum_{n \in \mathbb{N}} \lambda^{(n)} a_{(n)}b,$$

where λ is a variable, $\lambda^{(n)} = \frac{1}{n!}\lambda^n$ and $a, b \in \mathcal{A}$.

If we consider $\mathbb{C}$ as a differential ring with $\delta = 0$, the $\mathbb{C}$-Lie conformal superalgebras defined above coincide with the usual definition found in [9].

Let $\mathcal{D} = (D, \delta) \to \mathcal{D}' = (D', \delta')$ be an extension of differential rings. A $\mathcal{D}'$-Lie conformal superalgebra can be viewed as a $\mathcal{D}$-Lie conformal superalgebra in the natural way. Conversely, for a $\mathcal{D}$-Lie conformal superalgebra $(\mathcal{A}, \partial_{\mathcal{A}})$, there is a $\mathcal{D}'$-Lie conformal superalgebras structure on $\mathcal{A} \otimes_{\mathcal{D}} \mathcal{D}'$ given by

$$\partial_{\mathcal{A} \otimes_{\mathcal{D}} \mathcal{D}'}(a \otimes r) = \partial_{\mathcal{A}}(a) \otimes r + a \otimes \delta(r),$$

for $a \in \mathcal{A}, r \in D'$, and

$$(a \otimes f)_{(n)}(b \otimes g) = \sum_{j \in \mathbb{N}} (a_{(n+j)}b) \otimes \delta^{(j)}(f)g,$$

for $a, b \in \mathcal{A}, f, g \in D, n \in \mathbb{N}$. With base change defined we make the following definition.

Definition 2.2. Let $\mathcal{D} \to \mathcal{D}'$ be an extension of differential rings and $(\mathcal{A}, \partial)$ a $\mathcal{D}$-Lie conformal superalgebra, a $\mathcal{D}'/\mathcal{D}$-form of $\mathcal{A}$ is a $\mathcal{D}$-Lie conformal superalgebra $\mathcal{L}$ such that

$$\mathcal{L} \otimes_{\mathcal{D}} \mathcal{D}' \cong \mathcal{A} \otimes_{\mathcal{D}} \mathcal{D}'$$

as $\mathcal{D}'$-conformal superalgebras.

If the extension $\mathcal{D}'/\mathcal{D}$ is a faithfully flat extension, i.e., if the ring extension $D \to D'$ is faithfully flat, the set of isomorphism classes of $\mathcal{D}'/\mathcal{D}$-forms of a $\mathcal{D}$-Lie conformal superalgebra $\mathcal{A}$ is identified with the non-abelian Čech cohomology point set $H^1(\mathcal{D}'/\mathcal{D}, \mathbf{Aut}(\mathcal{A}))$, where $\mathbf{Aut}(\mathcal{A})$ is the group functor from the category of differential extensions of $\mathcal{D}$ to the category of groups which assigns to an extension $\mathcal{D}'$ of $\mathcal{D}$ the group $\mathbf{Aut}(\mathcal{A})(\mathcal{D}')$ of automorphisms of the $\mathcal{D}'$-Lie conformal superalgebra $\mathcal{A} \otimes_{\mathcal{D}} \mathcal{D}'$ (see Theorem 2.16 of [10]).

Remark 2.1. The $\mathcal{D}$-group functor $\mathbf{Aut}(\mathcal{A})$ plays a key role in the classification of $\mathcal{D}'/\mathcal{D}$-forms of a given $\mathcal{D}$-Lie conformal superalgebra $\mathcal{A}$. Let $\mathcal{L}$ be such a form. We make the following definition for future use. We say that a subgroup functor $\mathcal{F}$ of $\mathbf{Aut}(\mathcal{L})$ is *representable* if there exists a scheme $\mathfrak{X}$ over $\mathrm{Spec}(D)$ such that $\mathcal{F}(\mathcal{E}) \simeq \mathrm{Hom}_{D\text{-sch}}(\mathrm{Spec}(E), \mathfrak{X})$ for every differential extension $\mathcal{E} = (E, \delta_{\mathcal{E}})$ of $\mathcal{D}$, where the identifications are "functorial on $\mathcal{E}$". Recall that by Yoneda's correspondence if $\mathfrak{X} = \mathrm{Spec}(A)$ is affine, then

$$\mathrm{Hom}_{D\text{-sch}}(\mathrm{Spec}(E), \mathfrak{X}) \simeq \mathrm{Hom}_{D\text{-alg}}(A, E).$$

In this paper, we are mainly interested in the twisted loop Lie conformal superalgebra $\mathcal{L}(\mathcal{A}, \sigma)$ based on a complex Lie conformal superalgebra $(\mathcal{A}, \partial)$ with respect to an automorphism $\sigma : \mathcal{A} \to \mathcal{A}$ of finite order m. Recall that

this is defined by

$$\mathcal{L}(\mathcal{A}, \sigma) = \bigoplus_{n \in \mathbb{Z}} \mathcal{A}_n \otimes t^{n/m} \subseteq \mathcal{A} \otimes_{\mathbb{C}} \mathcal{S}_m,$$

where $\mathcal{A}_n = \{a \in \mathcal{A} | \sigma(a) = \zeta_m^n a\}, n \in \mathbb{Z}, \zeta_m = e^{\frac{2\pi i}{m}}$ and $\mathcal{S}_m = (\mathbb{C}[t^{\pm \frac{1}{m}}], \delta_t)$.

The twisted loop algebra $\mathcal{L}(\mathcal{A}, \sigma)$ is not only a complex Lie conformal superalgebra but also an $\mathcal{R}$-Lie conformal superalgebra for $\mathcal{R} = (\mathbb{C}[t^{\pm 1}], \delta_t)$. It is *trivialized* by the extension $\mathcal{S}_m / \mathcal{R}$, hence also by $\widehat{\mathcal{S}} = \varinjlim (\mathcal{S}_m, \delta_t)$.[4] This means that after applying the base change $\mathcal{R} \to \widehat{\mathcal{S}}$ our object "splits". To be precise,

$$\mathcal{L}(\mathcal{A}, \sigma) \otimes_{\mathcal{R}} \widehat{\mathcal{S}} \simeq \mathcal{A} \otimes_{\mathbb{C}} \widehat{\mathcal{S}} \simeq (\mathcal{A} \otimes_{\mathbb{C}} \mathcal{R}) \otimes_{\mathcal{R}} \widehat{\mathcal{S}},$$

where all of the above are isomorphisms of $\widehat{\mathcal{S}}$-Lie conformal superalgebras. In other words, $\mathcal{L}(\mathcal{A}, \sigma)$ is *an $\widehat{\mathcal{S}}/\mathcal{R}$-form of $\mathcal{A} \otimes_{\mathbb{C}} \mathcal{R}$*.[5] We will denote in what follows the $\mathcal{R}$-conformal superalgebra $\mathcal{A} \otimes_{\mathbb{C}} \mathcal{R}$ by $\mathcal{A}_{\mathcal{R}}$.

Since $\widehat{\mathcal{S}}/\mathcal{R}$ is faithfully flat, the set of $\mathcal{R}$-conformal isomorphism classes of $\widehat{\mathcal{S}}/\mathcal{R}$-forms of $\mathcal{A} \otimes_{\mathbb{C}} \mathcal{R}$ is identified with the non-abelian Čech cohomology pointed set $H^1(\widehat{\mathcal{S}}/\mathcal{R}, \mathbf{Aut}(\mathcal{A}_{\mathcal{R}}))$.

Furthermore, if $\mathcal{A}_{\mathcal{R}}$ satisfies the following finiteness condition:

(Fin) There exist finitely many elements $a_1, \ldots, a_n \in \mathcal{A}_{\mathcal{R}}$ such that the set $\{\partial_{\mathcal{A}}^{\ell}(ra_i) | r \in R, \ell \geq 0\}$ spans $\mathcal{A}_{\mathcal{R}}$.

(which holds in all of the cases we are interested in), the $H^1(\widehat{\mathcal{S}}/\mathcal{R}, \mathbf{Aut}(\mathcal{A}_{\mathcal{R}}))$ we are after can be identified with the non-abelian continuous cohomology $H^1(\pi_1(R), \mathbf{Aut}(\mathcal{A}_{\mathcal{R}})(\widehat{\mathcal{S}}))$, where $\pi_1(R)$ is the algebraic fundamental group of $\mathrm{Spec}(R)$ at the geometric point $\mathrm{Spec}(\overline{\mathbb{C}(t)})$ (See [10], Proposition 2.29).

In particular for a complex Lie conformal superalgebra $\mathcal{A}$ we can consider $\mathbf{Aut}(\mathcal{A})$, a functor that can be evaluated at any differential ring to yield, in doing so, a group. The group functor $\mathbf{Aut}(\mathcal{A}_{\mathcal{R}})$ considered above is nothing but the restriction of $\mathbf{Aut}(\mathcal{A})$ to the category of differential extensions of $\mathcal{R}$ (such extensions are by definition naturally endowed with a differential ring structure). In particular $\mathbf{Aut}(\mathcal{A})(\widehat{\mathcal{S}}) = \mathbf{Aut}(\mathcal{A}_{\mathcal{R}})(\widehat{\mathcal{S}})$. This

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[4]The introduction of $\widehat{\mathcal{S}}$ is a useful artifice that allows us to compare loop algebras of automorphisms of arbitrary order all at once.

[5]The spirit of this construction is not unlike that of vector bundles, which locally look like a well-understood object, namely the trivial bundle. The difference is in the meaning of "locally", which is to be understood not for the usual topology of $\mathbb{C}^{\times}$, but in a differential version of Grothendieck's *fppf*-topology on $\mathrm{Spec}(R)$. The trivialization takes place by using "only one open set", namely $\mathrm{Spec}(S)$.

makes the cohomology set $H^1(\widehat{\mathcal{S}}/\mathcal{R}, \mathbf{Aut}(\mathcal{A}_{\mathcal{R}}))$ computable, which yields the classification of $\widehat{\mathcal{S}}/\mathcal{R}$-forms of $\mathcal{A} \otimes_{\mathbb{C}} \mathcal{R}$ *up to isomorphisms of $\mathcal{R}$-Lie conformal superalgebras.* Then the centroid trick can be brought in to obtain the passage from the classification of $\mathcal{R}$-Lie conformal superalgebras to that of complex Lie conformal superalgebras.

In this paper, we consider twisted loop conformal superalgebras base on complex Lie conformal superalgebras $\mathcal{K}_N, N = 1, 2, 3$ which are described in [9] as follows:

Let $\Lambda(N)$ be the Grassmann algebra over $\mathbb{C}$ in N variables $\xi_1, \ldots, \xi_N$. $\Lambda(N)$ has a $\mathbb{Z}/2\mathbb{Z}$-grading given by setting each $\xi_i, i = 1, \ldots, N$ to be odd. This gives $\Lambda(N)$ the structure of a complex superalgebra. $\Lambda(N)$ also has a $\mathbb{Z}$-grading in which each $\xi_i, i = 1, \ldots, N$ has degree 1. If f is homogeneous with respect to the $\mathbb{Z}$-grading, we denote by $|f|$ the degree of f. The complex Lie conformal superalgebra $\mathcal{K}_N$ is the naturally $\mathbb{Z}/2\mathbb{Z}$-graded complex vector space

$$\mathcal{K}_N = \mathbb{C}[\partial] \otimes_{\mathbb{C}} \Lambda(N)$$

equipped with the nth product for each $n \in \mathbb{N}$ given by

$$f_{(0)}g = \left(\frac{1}{2}|f| - 1\right)\partial \otimes fg + \frac{1}{2}(-1)^{|f|}\sum_{i=1}^{N}(\partial_i f)(\partial_i g),$$

$$f_{(1)}g = \left(\frac{1}{2}(|f| + |g|) - 2\right)fg,$$

$$f_{(n)}g = 0, \quad n \geqslant 2,$$

where $f, g \in \Lambda(N)$ are homogenous elements with respect to the $\mathbb{Z}$-grading, and ∂_i is the derivative with respect to $\xi_i, i = 1, \ldots, N$.

Remark 2.2. Every twisted loop Lie conformal superalgebra $\mathcal{L}(\mathcal{A}, \sigma)$ based on a complex Lie conformal superalgebra $(\mathcal{A}, \sigma)$ naturally corresponds to a complex Lie superalgebra:

$$\mathrm{Alg}(\mathcal{A}, \sigma) = \mathcal{L}(\mathcal{A}, \sigma)/\left(\partial + \frac{d}{dt}\right)\mathcal{L}(\mathcal{A}, \sigma),$$

where the Lie bracket comes from the zeroth product of $\mathcal{L}(\mathcal{A}, \sigma)$. The classification of loop algebras based on a complex Lie conformal superalgebra $\mathcal{A}$ is thus related, in an essential and meaningful way, to that of their corresponding Lie superalgebras. The Lie superalgebras corresponding to the twisted loop superalgebras base on complex conformal superalgebras $\mathcal{K}_N, N = 1, 2, 3$

are exactly the $N = 1, 2, 3$ *superconformal Lie algebras* described in [13]. We shall study these algebras in Section 5.

3. Automorphism groups of the $N = 1, 2, 3$ Lie conformal superalgebras

The underlying vector space of the complex Lie conformal superalgebras $\mathcal{A}$ that we are interested are all of the form

$$\mathcal{A} = \mathbb{C}[\partial] \otimes_{\mathbb{C}} V$$

for some complex vector superspace V. In particular, $\mathcal{A}$ is a free $\mathbb{C}[\partial]$-supermodule, and has a natural $\mathbb{Z}$-grading (as a superspace) with $\mathcal{A}_n = \mathbb{C}\partial^n \otimes_{\mathbb{C}} V$. Let $\phi \in \mathbf{Aut}(\mathcal{A})$. Since ϕ commutes with the action of ∂ to ask that ϕ preserve the $\mathbb{Z}$-grading is the same than to ask that $\phi(1 \otimes V) \subseteq 1 \otimes V$.[6] This leads us to consider a natural subgroup functor $\mathbf{GrAut}(\mathcal{A})$ of $\mathbf{Aut}(\mathcal{A})$ defined by

$$\mathbf{GrAut}(\mathcal{A})(\mathcal{D}) = \{\phi \in \mathbf{Aut}(\mathcal{A})(\mathcal{D}) | \phi\left(\mathbb{C} \otimes_{\mathbb{C}} V \otimes_{\mathbb{C}} D\right) \subseteq \mathbb{C} \otimes_{\mathbb{C}} V \otimes_{\mathbb{C}} D\},$$

for all complex differential ring $\mathcal{D} = (D, \delta)$. The preservation of degrees is a natural condition to impose, and it carries also a physical meaning.

For $\mathcal{K}_N, N = 1, 2, 3$, the subgroup functor is defined by

$$\mathbf{GrAut}(\mathcal{K}_N)(\mathcal{D})$$
$$= \{\phi \in \mathbf{Aut}(\mathcal{K}_N)(\mathcal{D}) | \phi\left(\mathbb{C} \otimes_{\mathbb{C}} \Lambda(N) \otimes_{\mathbb{C}} D\right) \subseteq \mathbb{C} \otimes_{\mathbb{C}} \Lambda(N) \otimes_{\mathbb{C}} D\},$$

where $\mathcal{D} = (D, \delta)$ is an arbitrary complex differential ring.

The next theorem gives a precise characterization of the group functor $\mathbf{GrAut}(\mathcal{K}_N)$, $N = 1, 2, 3$. We will also show that the functors $\mathbf{GrAut}(\mathcal{K}_N)$ and $\mathbf{Aut}(\mathcal{K}_N)$ are equal when evaluated at complex differential rings that are integral domains, in particular at $\widehat{\mathcal{S}} = (\mathbb{C}[t^q, q \in \mathbb{Q}], \frac{d}{dt})$.

To simplify our notation, we set $\widehat{\partial} = \partial \otimes \mathrm{id} + \mathrm{id} \otimes \delta$, which can be viewed as an operator on $\mathcal{K}_N \otimes_{\mathbb{C}} \mathcal{D}$. And for convenience we identify $f \in \Lambda(N)$ with its image $(1 \otimes f) \otimes 1$ in $\mathcal{K}_N \otimes_{\mathbb{C}} \mathcal{D}$.

Theorem 3.1. *Let $\mathcal{D} = (D, \delta)$ be an arbitrary complex differential ring. For $N = 1, 2, 3$, there is an isomorphism of groups*

$$\iota_{\mathcal{D}} : \mathbf{O}_N(D) \xrightarrow{\sim} \mathbf{GrAut}(\mathcal{K}_N)(\mathcal{D}), \quad A = (a_{ij})_{N \times N} \mapsto \phi_A,$$

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[6]The inclusion is necessarily an equality since ϕ is surjective.

where $\phi_A \in \mathbf{GrAut}(\mathcal{K}_N)(\mathcal{D})$ is given by the following data:

- *For $N = 1$, $\phi_A(1) = 1$ and $\phi_A(\xi_1) = \xi_1 \otimes a_{11}$.*

- *For $N = 2$,*

$$\phi_A(1) = 1 + \xi_1\xi_2 \otimes r, \qquad \phi_A(\xi_1) = \xi_1 \otimes a_{11} + \xi_2 \otimes a_{21},$$
$$\phi_A(\xi_1\xi_2) = \xi_1\xi_2 \otimes \det(A), \qquad \phi_A(\xi_2) = \xi_1 \otimes a_{12} + \xi_2 \otimes a_{22},$$

  where $\begin{pmatrix} 0 & r \\ -r & 0 \end{pmatrix} = 2\delta(A)A^T$.

- *For $N = 3$,*

$$\phi_A(1) = 1 + \sum_{l=1}^{3} \epsilon_{mnl}\xi_m\xi_n \otimes r_l, \quad \phi_A(\xi_j) = \sum_{l=1}^{3} \xi_l \otimes a_{lj} + \xi_1\xi_2\xi_3 \otimes s_j,$$

$$\phi_A(\xi_1\xi_2\xi_3) = \xi_1\xi_2\xi_3 \otimes \det(A), \qquad \phi_A(\xi_i\xi_j) = \epsilon_{ijl} \sum_{l'=1}^{3} \epsilon_{mnl'}\xi_m\xi_n \otimes A_{l'l},$$

  $i, j = 1, 2, 3, i \neq j$, where $A_{l'l}$ is the cofactor of $a_{l'l}$ in A and

$$\begin{pmatrix} 0 & r_3 & -r_2 \\ -r_3 & 0 & r_1 \\ r_2 & -r_1 & 0 \end{pmatrix} = 2\delta(A)A^T,$$
$$\begin{pmatrix} 0 & s_3 & -s_2 \\ -s_3 & 0 & s_1 \\ s_2 & -s_1 & 0 \end{pmatrix} = 2(\det A)A^T\delta(A).$$

This construction is functorial on $\mathcal{D}$. In particular $\mathbf{O}_N \simeq \mathbf{GrAut}(\mathcal{K}_N)$.

Proof. For $A \in \mathbf{O}_N(D)$, the data given in the theorem defines an D-module homomorphism $\phi_A : \Lambda(N) \otimes_\mathbb{C} D \to \Lambda(N) \otimes_\mathbb{C} D$. One establishes that ϕ_A is a module isomorphism by explicitly checking that $\phi_{A^{-1}}$ is its inverse. Then ϕ_A can be extended to a unique D-module isomorphism $\mathcal{K}_N \otimes_\mathbb{C} \mathcal{D} \to \mathcal{K}_N \otimes_\mathbb{C} \mathcal{D}$, which is also denoted by ϕ_A, such that $\widehat{\partial} \circ \phi_A = \phi_A \circ \widehat{\partial}$. A direct computation shows that

$$\phi_A([f \otimes 1_\lambda g \otimes 1]) = [\phi_A(f \otimes 1)_\lambda \phi_A(g \otimes 1)]$$

for all $f, g \in \Lambda(N)$. Since $\mathcal{K}_N = \mathbb{C}[\partial] \otimes_\mathbb{C} \Lambda(N)$ is a free $\mathbb{C}[\partial]$-module, we deduce that $\phi_A \in \mathbf{Aut}(\mathcal{K}_N)(\mathcal{D})$ by [10], Lemma 3.1(ii). Moreover, we observe that $\phi_A \in \mathbf{GrAut}(\mathcal{K}_N)(\mathcal{D})$ by the definition of ϕ_A.

The above yields a function $\iota_{\mathcal{D}} : \mathbf{O}_N(D) \to \mathbf{GrAut}(\mathcal{K}_N)(\mathcal{D})$ such that $A \mapsto \phi_A$. For $\phi_1, \phi_2 \in \mathbf{Aut}(\mathcal{K}_N)(\mathcal{D})$, $\phi_1 = \phi_2$ if and only if $\phi_1(f) = \phi_2(f)$ for all $f \in \Lambda(N)$ by [10], Lemma 3.1(i).[7] In particular, for $A, B \in \mathbf{O}_N(D)$, $\phi_{AB}(f) = (\phi_A \circ \phi_B)(f)$ for all $f \in \Lambda(N)$, thus $\phi_{AB} = \phi_A \circ \phi_B$, i.e., $\iota_{\mathcal{D}}$ is a group homomorphism. By the same reasoning $A = B$ if $\phi_A(f) = \phi_B(f)$ for all $f \in \Lambda(N)$, so that $\iota_{\mathcal{D}}$ is injective.

It remains to show that $\iota_{\mathcal{D}}$ is surjective, namely that given an automorphism $\phi \in \mathbf{GrAut}(\mathcal{K}_N)(\mathcal{D})$ there exists $A \in \mathbf{O}_N(D)$ such that $\phi = \phi_A$.

Case $N = 1$: Since $\phi(\mathbb{C} \otimes \Lambda(1) \otimes D) \subseteq \mathbb{C} \otimes \Lambda(1) \otimes D$ and ϕ preservers the $\mathbb{Z}/2\mathbb{Z}$-grading of $\mathcal{K}_1 \otimes_{\mathbb{C}} \mathcal{D}$, we may assume that $\phi(\xi_1) = \xi_1 \otimes r$ for some $r \in D$.

Since $\phi^{-1} \in \mathbf{GrAut}(\mathcal{K}_N)(\mathcal{D})$, we may also assume that $\phi^{-1}(\xi_1) = \xi_1 \otimes s$ for some $s \in D$. Then $\phi \circ \phi^{-1}(\xi_1) = \xi_1 \otimes rs = \xi_1 \otimes 1$. Thus $rs = 1$. i.e., r is an unit in D.

We deduce from $\phi(\xi_1) = \xi_1 \otimes r$ that $\phi(1) = -2\phi(\xi_1)_{(0)}\phi(\xi_1) = 1 \otimes r^2$. While, $\phi(1)_{(1)}\phi(1) = -2\phi(1)$ implies that $r^4 = r^2$. Since r is an unit in D we obtain $r^2 = 1$. In particular, $r \in \mathbf{O}_1(D)$ and $\phi = \phi_r$.

Case $N = 2$: $\phi(\mathbb{C} \otimes \Lambda(2) \otimes D) \subseteq \mathbb{C} \otimes \Lambda(2) \otimes D$ and ϕ preservers the $\mathbb{Z}/2\mathbb{Z}$-grading of $\mathcal{K}_2 \otimes_{\mathbb{C}} \mathcal{D}$ yield

$$\phi(\xi_1) = \xi_1 \otimes a_{11} + \xi_2 \otimes a_{21}, \text{ and } \phi(\xi_2) = \xi_1 \otimes a_{12} + \xi_2 \otimes a_{22},$$

where $a_{ij} \in D, i, j = 1, 2$.

Let $A = (a_{ij})_{2 \times 2}$. Since ϕ has an inverse in $\mathbf{GrAut}(\mathcal{K}_N)(\mathcal{D})$, the matrix A is necessarily invertible. Now

$$\phi(\xi_1 \xi_2) = -\phi(\xi_1)_{(1)}\phi(\xi_2) = \xi_1\xi_2 \otimes (a_{11}a_{22} - a_{21}a_{12}) = \xi_1\xi_2 \otimes \det(A).$$

Choose $c, r \in D$ such that $\phi(1) = 1 \otimes c + \xi_1\xi_2 \otimes r$. From $\phi(1)_{(1)}\phi(\xi_1\xi_2) = -\phi(\xi_1\xi_2)$ we deduce that $c \cdot \det(A) = \det(A)$. Since A is invertible, $\det(A)$ is an unit in D and therefore $c = 1$.

Since $\phi(\xi_j)_{(0)}\phi(\xi_j) = -\frac{1}{2}\phi(1)$, we have

$$a_{1j}^2 + a_{2j}^2 = 1, \quad \text{and} \quad r = 2\big(\delta(a_{1j})a_{2j} - a_{1j}\delta(a_{2j})\big), \quad j = 1, 2,$$

---

[7] This useful result will be used repeatedly in what follows without further reference.

while $\phi(\xi_1)_{(0)}\phi(\xi_2) = -\frac{1}{2}\widehat{\partial}\phi(\xi_1\xi_2)$ implies that $a_{11}a_{12} + a_{21}a_{22} = 0$. Thus $A = (a_{ij}) \in \mathbf{O}_2(D)$ and

$$
\begin{aligned}
r &= \big(\delta(a_{11})a_{21} - a_{11}\delta(a_{21})\big) + \big(\delta(a_{12})a_{22} - a_{12}\delta(a_{22})\big) \\
&= \big(\delta(a_{11})a_{21} - a_{11}\delta(a_{21})\big) + \big(\delta(a_{12})a_{22} - a_{12}\delta(a_{22})\big) + \delta(a_{11}a_{21} + a_{12}a_{22}) \\
&= 2\big(\delta(a_{11})a_{21} + \delta(a_{12})a_{22}\big),
\end{aligned}
$$

i.e., $\begin{pmatrix} 0 & r \\ -r & 0 \end{pmatrix} = 2\delta(A)A$. It follows $\phi(f) = \phi_A(f)$ for all $f \in \Lambda(2)$. Hence, $\phi = \phi_A$.

Case $N = 3$: By reasoning as above we may assume that

$$
(3.1) \qquad \phi(\xi_j) = \xi_1 \otimes a_{1j} + \xi_2 \otimes a_{2j} + \xi_3 \otimes a_{3j} + \xi_1\xi_2\xi_3 \otimes s_j,
$$

where $a_{ij}, s_j \in D, i, j = 1, 2, 3$, and that the matrix $A = (a_{ij})$ is invertible. For $i \neq j$, we have

$$
(3.2) \qquad
\begin{aligned}
\phi(\xi_i\xi_j) &= -\phi(\xi_i)_{(1)}\phi(\xi_j) \\
&= \xi_1\xi_2 \otimes (a_{1i}a_{2j} - a_{2i}a_{1j}) + \xi_2\xi_3 \otimes (a_{2i}a_{3j} - a_{3i}a_{2j}) \\
&\quad + \xi_3\xi_2 \otimes (a_{3i}a_{2j} - a_{2i}a_{3j}) \\
&= \epsilon_{ijk}(\xi_1\xi_2 \otimes A_{3k} + \xi_2\xi_3 \otimes A_{1k} + \xi_3\xi_1 \otimes A_{2k}),
\end{aligned}
$$

where A_{ij} is the cofactor of a_{ij} in A. While,

$$
(3.3) \qquad
\begin{aligned}
\phi(\xi_1\xi_2\xi_3) &= -2\phi(\xi_1\xi_2)_{(1)}\phi(\xi_3) \\
&= \xi_1\xi_2\xi_3 \otimes (A_{13}a_{13} + A_{23}a_{23} + A_{33}a_{33}) \\
&= \xi_1\xi_2\xi_3 \otimes \det(A).
\end{aligned}
$$

Write $\phi(1) = 1 \otimes c + \xi_1\xi_2 \otimes r_3 + \xi_2\xi_3 \otimes r_2 + \xi_3\xi_1 \otimes r_1$, $c, r_j \in D, j = 1, 2, 3$. Then $\phi(1)_{(1)}\phi(\xi_1\xi_2\xi_3) = -\frac{1}{2}\phi(\xi_1\xi_2\xi_3)$ yields $c \cdot \det(A) = \det(A)$. Thus $c = 1$. We will show that $A \in \mathbf{O}_3(D)$. First, $\phi(\xi_j)_{(0)}\phi(\xi_j) = -\frac{1}{2}\phi(1), j = 1, 2, 3$ yields

$$
a_{1j}^2 + a_{2j}^2 + a_{3j}^2 = 1, \quad j = 1, 2, 3,
$$

and for $i \neq j$, $\phi(\xi_i)_{(0)}\phi(\xi_j) = -\frac{1}{2}\widehat{\partial}\phi(\xi_i\xi_j)$ implies

$$
a_{1i}a_{1j} + a_{2i}a_{2j} + a_{3i}a_{3j} = 0.
$$

Thus, $A = (a_{ij}) \in \mathbf{O}_3(D)$.

Finally, we consider $r_j, s_j, j = 1, 2, 3$. Since $\phi(1)_{(1)}\phi(\xi_j) = -\frac{3}{2}\phi(\xi_j)$, $j = 1, 2, 3$,

$$\frac{1}{2}\epsilon_{lmn}(a_{mj}r_n - a_{nj}r_m) = \delta(a_{lj}), \quad \text{and} \quad r_1 a_{1j} + r_2 a_{2j} + r_3 a_{3j} = s_j,$$

for $l, j = 1, 2, 3$. Writing the first equation in matrix form, we obtain

(3.4)
$$\frac{1}{2}\begin{pmatrix} 0 & r_3 & -r_2 \\ -r_3 & 0 & r_1 \\ r_2 & -r_1 & 0 \end{pmatrix} A = \delta(A)$$

$$\Longrightarrow \begin{pmatrix} 0 & r_3 & -r_2 \\ -r_3 & 0 & r_1 \\ r_2 & -r_1 & 0 \end{pmatrix} = 2\delta(A)A^{\mathrm{T}},$$

because $A \in \mathbf{O}_3(D)$. A direct computation shows that

$$2A^{\mathrm{T}}\delta(A) = A^{\mathrm{T}} \begin{pmatrix} 0 & r_3 & -r_2 \\ -r_3 & 0 & r_1 \\ r_2 & -r_1 & 0 \end{pmatrix} A$$

$$= \begin{pmatrix} 0 & \sum_{l=1}^{3} r_l A_{l3} & -\sum_{l=1}^{3} r_l A_{l2} \\ -\sum_{l=1}^{3} r_l A_{l3} & 0 & \sum_{l=1}^{3} r_l A_{l1} \\ \sum_{l=1}^{3} r_l A_{l2} & -\sum_{l=1}^{3} r_l A_{l1} & 0 \end{pmatrix}.$$

Since $A \in \mathbf{O}_3(D)$, $A_{ij} = \det(A)a_{ij}, i, j = 1, 2, 3$. So

$$\sum_{l=1}^{3} r_l A_{lj} = \det(A) \sum_{l=1}^{3} r_l a_{lj} = \det(A)s_j.$$

Hence,

(3.5)
$$\begin{pmatrix} 0 & s_3 & -s_2 \\ -s_3 & 0 & s_1 \\ s_2 & -s_1 & 0 \end{pmatrix} = 2\det(A)A^{\mathrm{T}}\delta(A).$$

Summarizing (3.1)–(3.5), we obtain $\phi(f) = \phi_A(f)$, for all $f \in \Lambda(3)$. Hence, $\phi = \phi_A$. $\qquad\square$

Corollary 3.1. $\mathbf{GrAut}(\mathcal{K}_N), N = 1, 2, 3$, *is representable by a smooth affine $\mathbb{C}$-scheme of finite type.* $\square$

Recall that for the classification of twisted loop Lie conformal superalgebras based on $\mathcal{K}_N$ the crucial group to compute is $\mathbf{Aut}(\mathcal{K}_N)(\widehat{\mathcal{S}})$. The next theorem shows that $\mathbf{Aut}(\mathcal{K}_N)$ and $\mathbf{O}_N$ coincide when evaluated at complex differential rings that are integral domains. In particular for $\widehat{\mathcal{S}}$.

Theorem 3.2. *Let $\mathcal{D} = (D, \delta)$ be a complex differential ring such that D is an integral domain. Then for $N = 1, 2, 3$ the natural inclusion*

$$\mathbf{GrAut}(\mathcal{K}_N)(\mathcal{D}) \subseteq \mathbf{Aut}(\mathcal{K}_N)(\mathcal{D})$$

is an equality.

Proof. We have to show that every $\phi \in \mathbf{Aut}(\mathcal{K}_N)(\mathcal{D})$ satisfies

$$\phi\left(\mathbb{C} \otimes_{\mathbb{C}} \Lambda(N) \otimes_{\mathbb{C}} D\right) \subseteq \mathbb{C} \otimes_{\mathbb{C}} \Lambda(N) \otimes_{\mathbb{C}} D.$$

Case $N = 1$: Since $(\mathcal{K}_1 \otimes_{\mathbb{C}} \mathcal{D})_{\bar{1}} = \mathbb{C}[\partial] \otimes_{\mathbb{C}} \xi_1 \otimes_{\mathbb{C}} D$ and ϕ preserve the $\mathbb{Z}/2\mathbb{Z}$-grading of $\mathcal{K}_1 \otimes_{\mathbb{C}} \mathcal{D}$, we may assume that

$$\phi(\xi_1) = \sum_{n=0}^{M} \widehat{\partial}^n(\xi_1 \otimes s_n),$$

where $s_n \in D, n = 0, \ldots, M$ and $s_M \neq 0$. Then

$$[\phi(\xi_1)_\lambda \phi(\xi_1)] = -\frac{1}{2} \sum_{m=0}^{M} \sum_{n=0}^{M} (-\lambda)^m (\widehat{\partial} + \lambda)^n (1 \otimes s_m s_n) \text{ while}$$

$$\phi([\xi_{1\lambda} \xi_1]) = \phi\left(\left(-\frac{1}{2}\right) 1\right) = -\frac{1}{2}\phi(1).$$

The leading term (i.e., the term with highest degree with respect to λ) in the right-hand sides of above two equations are $\frac{1}{2}(-1)^{M+1}\lambda^{2M}(1 \otimes s_M^2)$ and $-\frac{1}{2}\phi(1)$, respectively. Since D is an integral domain, $s_M \neq 0$ implies $s_M^2 \neq 0$. Thus $[\phi(\xi_1)_\lambda \phi(\xi_1)] = \phi([\xi_{1\lambda} \xi_1])$ yields $M = 0$, i.e.,

$$\phi(\xi_1) = \xi_1 \otimes s_0, \quad 0 \neq s_0 \in D.$$

and $\phi(1) = -2[\phi(\xi_1)_\lambda \phi(\xi_1)] = 1 \otimes s_0^2$. Hence,

$$\phi\left(\mathbb{C} \otimes \Lambda(1) \otimes D\right) \subseteq \mathbb{C} \otimes \Lambda(1) \otimes D.$$

Case $N = 2$: Write $\phi(\xi_1\xi_2) = \sum_{m=0}^{M} \widehat{\partial}^m(1 \otimes s_m) + x$ with $x \in \mathbb{C}[\partial] \otimes \xi_1\xi_2 \otimes D$, $s_m \in D, m = 0, \ldots, M, s_M \neq 0$. Then

$$
0 = [\phi(\xi_1\xi_2)_\lambda \phi(\xi_1\xi_2)]
$$
$$
= \sum_{m,n=0}^{M} (-\lambda)^m (\widehat{\partial} + \lambda)^n (-\partial 1 \otimes s_m s_n - 2 \otimes \delta(s_m)s_n - \lambda 2 \otimes s_m s_n)
$$
$$
+ \left[\sum_{m=0}^{M} \widehat{\partial}^m (1 \otimes s_m)_\lambda x \right] + \left[x_\lambda \sum_{m=0}^{M} \widehat{\partial}^m (1 \otimes s_m) \right] + [x_\lambda x].
$$

Observing that all terms in the last row of the above equation are contained in $\mathbb{C}[\partial] \otimes \xi_1\xi_2 \otimes D$ and $(\mathcal{K}_2 \otimes_\mathbb{C} D)_{\bar{0}} = (\mathbb{C}[\partial] \otimes 1 \otimes D) \oplus (\mathbb{C}[\partial] \otimes \xi_1\xi_2 \otimes D)$, we obtain

$$
0 = \sum_{m,n=0}^{M} (-\lambda)^m (\widehat{\partial} + \lambda)^n (\partial 1 \otimes s_m s_n + 2 \otimes \delta(s_m)s_n + \lambda 2 \otimes s_m s_n).
$$

By comparing the coefficients of λ it follows that $s_M^2 = 0$, hence that $s_M = 0$ since D is an integral domain. This contradicts our assumption that $s_M \neq 0$. Thus

$$
\phi(\xi_1\xi_2) = x = \sum_{m=0}^{M'} \widehat{\partial}^m (\xi_1\xi_2 \otimes c_m),
$$

where $c_m \in D, m = 1, \ldots, M', c_{M'} \neq 0$.

Similarly, we may assume that

$$
\phi(1) = \sum_{m=0}^{\widetilde{M'}} \widehat{\partial}^m (1 \otimes r'_m) + \sum_{m=0}^{\widetilde{M}} \widehat{\partial}^m (\xi_1\xi_2 \otimes r_m),
$$

where $r'_{m'}, r_m \in \widehat{\mathcal{S}}, m' = 0, \ldots, \widetilde{M'}, m = 0, \ldots, \widetilde{M}$. Then $[\phi(1)_\lambda \phi(\xi_1\xi_2)] = -(\widehat{\partial} + \lambda)\phi(\xi_1\xi_2)$ yields $r'_{\widetilde{M'}} c_{M'} = 0$ if $\widetilde{M'} + M' > 0$. Since $c_{M'} \neq 0$ and D is an integral domain, $r'_{\widetilde{M'}} = 0$ if $\widetilde{M'} + M' > 0$. Thus $\widetilde{M'} = M' = 0$, i.e.,

$$
\phi(\xi_1\xi_2) = \xi_1\xi_2 \otimes c,
$$
$$
\phi(1) = 1 \otimes r' + \sum_{m=0}^{\widetilde{M}} \widehat{\partial}^m (\xi_1\xi_2 \otimes r_m),
$$

where $0 \neq c \in D, 0 \neq r' \in D$ and $r_m \in D, m = 0, \ldots, \widetilde{M}$.

Now we consider the odd part $(\mathcal{K}_2 \otimes_{\mathbb{C}} \mathcal{D})_{\bar{1}} = \mathbb{C}[\partial] \otimes (\mathbb{C}\xi_1 \oplus \mathbb{C}\xi_2) \otimes D$. Write

$$\phi(\xi_j) = \sum_{m=0}^{M_{1j}} \widehat{\partial}^m (\xi_1 \otimes a_{1j,m}) + \sum_{n=0}^{M_{2j}} \widehat{\partial}^n (\xi_2 \otimes a_{2j,n}),$$

where $a_{ij,m} \in D, i, j = 1, 2,$ and $m = 0, \ldots, M_{ij}$. Then it follows from $[\phi(\xi_1)_\lambda \phi(\xi_1 \xi_2)] = -\frac{1}{2}\phi(\xi_2)$ and $[\phi(\xi_2)_\lambda \phi(\xi_1 \xi_2)] = \frac{1}{2}\phi(\xi_1)$ that

$$\phi(\xi_1) = \xi_1 \otimes a_{11} + \xi_2 \otimes a_{21}, \text{ and } \phi(\xi_2) = \xi_1 \otimes a_{12} + \xi_2 \otimes a_{22},$$

where $a_{ij} \in D, i, j = 1, 2$.

Next, we consider $\phi(1)$. We deduce from $[\phi(1)_\lambda \phi(\xi_i)] = -(\widehat{\partial} + \frac{3}{2}\lambda)\phi(\xi_i)$, $i = 1, 2$ that $\phi(1) = 1 \otimes r' + \xi_1 \xi_2 \otimes r_0$, where $r', r_0 \in D$. It follows that:

$$\phi\left(\mathbb{C} \otimes \Lambda(2) \otimes D\right) \subseteq \mathbb{C} \otimes \Lambda(2) \otimes D.$$

Case $N = 3$**:** Let $\mathcal{B} = \mathbb{C}[\partial] \otimes_{\mathbb{C}} (\mathbb{C}\xi_1\xi_2 \oplus \mathbb{C}\xi_2\xi_3 \oplus \mathbb{C}\xi_3\xi_1)$, then

$$(\mathcal{K}_3 \otimes_{\mathbb{C}} \mathcal{D})_{\bar{0}} = (\mathbb{C}[\partial] \otimes 1 \otimes D) \oplus (\mathcal{B} \otimes_{\mathbb{C}} \mathcal{D}).$$

We may assume $\phi(\xi_i \xi_j) = \sum_{m=0}^{M} \widehat{\partial}^m (1 \otimes s_m) + x_{ij}, i \neq j$, where $s_m \in D$ and $x_{ij} \in \mathcal{B} \otimes_{\mathbb{C}} \mathcal{D}$. Since $\mathcal{B} \otimes_{\mathbb{C}} \mathcal{D}$ is an ideal of $(\mathcal{K}_3 \otimes_{\mathbb{C}} \mathcal{D})_{\bar{0}}$ and $[\phi(\xi_i \xi_j)_\lambda \phi(\xi_i \xi_j)] = 0$, we deduce that $s_M^2 = 0$. Then $s_M = 0$ because D is an integral domain. It follows that $\phi(\xi_i \xi_j) = x_{ij} \in \mathcal{B} \otimes_{\mathbb{C}} \mathcal{D}$. Hence, $\phi|_{\mathcal{B} \otimes_{\mathbb{C}} \mathcal{D}}$ is an automorphism of the $\mathcal{D}$-Lie conformal superalgebra $\mathcal{B} \otimes_{\mathbb{C}} \mathcal{D}$. Since $\mathcal{B} \simeq \mathrm{Curr}(\mathfrak{so}_3(\mathbb{C}))$ and $\mathfrak{so}_3(\mathbb{C})$ is a finite-dimensional simple complex Lie algebra, by [10] Corollary 3.17, we obtain

$$(3.6) \qquad \phi(\xi_i \xi_j) = \epsilon_{ijl}(\xi_1\xi_2 \otimes b_{3l} + \xi_2\xi_3 \otimes b_{1l} + \xi_3\xi_1 \otimes b_{2l}), \quad i \neq j,$$

where $(b_{l'l})_{3 \times 3} \in \mathbf{GL}_3(D)$. Next we consider $\phi(1)$.

Claim:

$$(3.7) \qquad \phi(1) = 1 + \xi_1\xi_2 \otimes r_3 + \xi_2\xi_3 \otimes r_1 + \xi_3\xi_1 \otimes r_2, \qquad r_1, r_2, r_3 \in D.$$

Indeed, we can write $\phi(1) = \sum_{m=0}^{M} \widehat{\partial}^m(1 \otimes s_m) + x$ with $s_i \in D, i = 0, \ldots,$ $M, x \in \mathcal{B} \otimes_{\mathbb{C}} \mathcal{D}$. We may assume $s_M \neq 0$ because ϕ is an isomorphism. Then

$$
\begin{aligned}
[\phi(1)_\lambda \phi(1)] = {} & \sum_{m,n=0}^{M} (-\lambda)^m (\widehat{\partial} + \lambda)^n (-\partial 1 \otimes s_m s_n - 2 \otimes \delta(s_m)s_n - \lambda 2 \otimes s_m s_n) \\
& + \left[\sum_{m=0}^{M} \widehat{\partial}^m (1 \otimes s_m)_\lambda x \right] + \left[x_\lambda \sum_{n=0}^{M} \widehat{\partial}^n (1 \otimes s_n) \right] + [x_\lambda x].
\end{aligned}
$$

Note that all terms in the second row of the above equation are contained in $\mathbb{C}[\lambda] \otimes_{\mathbb{C}} \mathcal{B} \otimes_{\mathbb{C}} \mathcal{D}$. If $M > 0$, we deduce that $s_M^2 = 0$ by comparing the coefficients of λ^{2M+1} in $[\phi(1)_\lambda \phi(1)] = -(\widehat{\partial} + 2\lambda)\phi(1)$. Since D is an integral domain, $s_M = 0$. This contradicts $s_M \neq 0$. Hence, $M = 0$, i.e., $\phi(1) = 1 \otimes s_0 + x$ with $x \in \mathcal{B} \otimes_{\mathbb{C}} \mathcal{D}$.

$$
\begin{aligned}
[\phi(1)_\lambda \phi(1)] = {} & -\partial 1 \otimes s_0^2 - 2 \otimes \delta(s_0)s_0 - \lambda 2 \otimes s_0^2 \\
& + [(1 \otimes s_0)_\lambda x] + [x_\lambda (1 \otimes s_0)] + [x_\lambda x], \\
-(\widehat{\partial} + 2\lambda)\phi(1) = {} & -\partial 1 \otimes s_0 - 1 \otimes \delta(s_0) - \lambda 2 \otimes s_0 - (\widehat{\partial} + 2\lambda)x.
\end{aligned}
$$

It follows that $s_0^2 = s_0$. Since $s_0 \neq 0$, $s_0 = 1$ because D is an integral domain, i.e., $\phi(1) = 1 + x$ with $x \in \mathcal{B} \otimes_{\mathbb{C}} \mathcal{D}$.

We further write $x = \sum_{l=0}^{N} \widehat{\partial}^l (\xi_1 \xi_2 \otimes r_{3l} + \xi_2 \xi_3 \otimes r_{1l} + \xi_3 \xi_1 \otimes r_{2l}), r_{2l} \in D$. Observing that $\phi(\xi_i \xi_j) = \epsilon_{ijk}(\xi_1 \xi_2 \otimes b_{3k} + \xi_2 \xi_3 \otimes b_{1k} + \xi_3 \xi_1 \otimes b_{2k})$, we obtain

$$
\begin{aligned}
[\phi(1)_\lambda \phi(\xi_i \xi_j)] = {} & -\epsilon_{ijk}(\partial \xi_1 \xi_2 \otimes b_{3k} + \partial \xi_2 \xi_3 \otimes b_{1k} + \partial \xi_3 \xi_1 \otimes b_{2k}) \\
& - \epsilon_{ijk} \lambda (\xi_1 \xi_2 \otimes b_{3k} + \xi_2 \xi_3 \otimes b_{1k} + \xi_3 \xi_1 \otimes b_{2k}) \\
& + \epsilon_{ijk} \sum_{l=0}^{N} (-\lambda)^l (\xi_1 \xi_2 \otimes (r_{1l} b_{2k} - r_{2l} b_{1k}) \\
& + \xi_2 \xi_3 \otimes (r_{2l} b_{3k} - r_{3l} b_{2k})) \\
& + \epsilon_{ijk} \sum_{l=0}^{N} (-\lambda)^l \xi_3 \xi_1 \otimes (r_{3l} b_{1k} - r_{1l} b_{3k}).
\end{aligned}
$$

Note that

$$-(\widehat{\partial} + \lambda)\phi(\xi_i\xi_j) = -\epsilon_{ijk}(\partial\xi_1\xi_2 \otimes b_{3k} + \partial\xi_2\xi_3 \otimes b_{1k} + \partial\xi_3\xi_1 \otimes b_{2k})$$
$$- \epsilon_{ijk}(\xi_1\xi_2 \otimes \delta(b_{3k}) + \xi_2\xi_3 \otimes \delta(b_{1k}) + \xi_3\xi_1 \otimes \delta(b_{2k}))$$
$$- \epsilon_{ijk}\lambda(\xi_1\xi_2 \otimes b_{3k} + \xi_2\xi_3 \otimes b_{1k} + \xi_3\xi_1 \otimes b_{2k}).$$

Then $[\phi(1)_\lambda\phi(\xi_i\xi_j)] = -(\widehat{\partial} + \lambda)\phi(\xi_i\xi_j), i, j = 1, 2, 3, i \neq j$ imply that

$$r_{il}b_{jk} - r_{jl}b_{ik} = 0,$$

for all $i, j, k = 1, 2, 3, i \neq j, l \geqslant 1$. In matrix form, these are equivalent to

$$\begin{pmatrix} 0 & -r_{3l} & r_{2l} \\ r_{3l} & 0 & -r_{1l} \\ -r_{2l} & r_{1l} & 0 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = 0, \quad \forall l \geqslant 1.$$

Hence, $r_{1l} = r_{2l} = r_{3l} = 0$ for all $l \geqslant 1$ because $(b_{ij})_{3\times3} \in \mathbf{GL}_3(D)$, i.e.,

$$\phi(1) = 1 + \xi_1\xi_2 \otimes r_{30} + \xi_2\xi_3 \otimes r_{10} + \xi_3\xi_1 \otimes r_{20},$$

where $r_{10}, r_{20}, r_{30} \in D$. This completes the proof of the claim.

Next, we consider the odd part $(K_3 \otimes_\mathbb{C} \mathcal{D})_{\bar{1}}$. First $[\phi(\xi_i\xi_j)_\lambda\phi(\xi_1\xi_2\xi_3)] = 0$ for all $i \neq j$ yield that $\phi(\xi_1\xi_2\xi_3) = \sum_{m=0}^M \widehat{\partial}^m(\xi_1\xi_2\xi_3 \otimes c_m), c_m \in D$. Considering $[\phi(1)_\lambda\phi(\xi_1\xi_2\xi_3)] = -(\widehat{\partial} + \frac{1}{2}\lambda)\phi(\xi_1\xi_2\xi_3)$, we deduce that

(3.8) $$\phi(\xi_1\xi_2\xi_3) = \xi_1\xi_2\xi_3 \otimes c, \quad 0 \neq c \in D.$$

A similar argument applies to $\phi(\xi_j)$. From $[\phi(\xi_j)_\lambda\phi(\xi_1\xi_2\xi_3)] = \epsilon_{jmn}\phi(\xi_m\xi_n)$ and $[\phi(1)_\lambda\phi(\xi_j)] = -(\widehat{\partial} + \frac{3}{2}\lambda)\phi(\xi_j)$, we obtain

(3.9) $$\phi(\xi_j) = \xi_1 \otimes a_{1j} + \xi_2 \otimes a_{2j} + \xi_3 \otimes a_{3j} + \xi_1\xi_2\xi_3 \otimes s_j,$$

where $a_{ij}, s_j \in D$. Summarizing (3.6) to (3.9), we obtain

$$\phi\left(\mathbb{C} \otimes \Lambda(3) \otimes D\right) \subseteq \left(\mathbb{C} \otimes \Lambda(3) \otimes D\right).$$

This completes the proof. $\qquad\square$

Remark 3.1. The integral assumption on D is not superflous. Consider the complex differential ring $\mathcal{D} = (D, \delta)$, where $D = \mathbb{C} \oplus \mathbb{C}\tau, \tau^2 = 0$ (the algebra

of dual numbers) and $\delta = 0$. For the $\mathcal{D}$-Lie conformal superalgebra $\mathcal{K}_1 \otimes_{\mathbb{C}}$
$\mathcal{D} = \mathbb{C}[\partial] \otimes_{\mathbb{C}} (\mathbb{C} \oplus \mathbb{C}\xi_1) \otimes_{\mathbb{C}} D$ it is easy to check that

$$\phi(\partial^{\ell} \otimes 1 \otimes s) = \partial^{\ell} \otimes 1 \otimes s + \partial^{\ell+1} \otimes 1 \otimes \tau s, \ell \geqslant 0, s \in D,$$
$$\phi(\partial^{\ell} \otimes \xi_1 \otimes s) = \partial^{\ell} \otimes \xi_1 \otimes s + \partial^{\ell+1} \otimes \xi_1 \otimes \tau s, \ell \geqslant 0, s \in D,$$

define an automorphism of the $\mathcal{D}$-Lie conformal superalgebra of $\mathcal{K}_1 \otimes_{\mathbb{C}} \mathcal{D}$.
But $\phi \notin \mathbf{GrAut}(\mathcal{K}_1)(\mathcal{D})$.

4. Forms of the $N = 1, 2, 3$ Lie conformal superalgebras

In this section, we classify the twisted loop Lie conformal superalgebras
based on the complex Lie conformal superalgebra $\mathcal{K}_N, N = 1, 2, 3$. As pre-
viously mentioned, the classification can be completed in two steps. The
first step is to classify the $\widehat{\mathcal{S}}/\mathcal{R}$-forms of $\mathcal{K}_N \otimes_{\mathbb{C}} \mathcal{R}$, and then look at the
passage from isomorphic classes of the $\mathcal{R}$-Lie conformal superalgebras to
isomorphic classes of the complex Lie conformal superalgebras. Recall that
$\widehat{S} = \mathbb{C}[t^q, q \in \mathbb{Q}]$ is the algebraic simply connected cover of $R = \mathbb{C}[t^{\pm 1}]$, and
that the algebraic fundamental group $\pi_1(R)$ of $\mathrm{Spec}(R)$ at the geometric
point $\mathrm{Spec}(\overline{\mathbb{C}(t)})$ can be identified with $\widehat{\mathbb{Z}} = \varprojlim \mathbb{Z}/m\mathbb{Z}$ via our canonical
choice of compatible roots of unity $\zeta_m = e^{\frac{\mathrm{i}2\pi}{m}}$. The explicit continuous action
of the profinite group $\widehat{\mathbb{Z}}$ on $\widehat{S}$ is given by $^{\overline{1}}t^{p/m} = \zeta_m^p t^{p/m}$, where $\overline{1}$ is the image
of 1 in $\widehat{\mathbb{Z}}$ under the canonical homomorphism $\mathbb{Z} \to \widehat{\mathbb{Z}}$.[8] We thus have natural
continuous actions of $\pi_1(R)$ on $\mathbf{Aut}(\mathcal{K}_N)(\widehat{\mathcal{S}})$ and $\mathbf{O}_N(\widehat{S}), N = 1, 2, 3$. Note
that the isomorphism $\iota : \mathbf{O}_N(\widehat{S}) \to \mathbf{Aut}(\mathcal{K}_N)(\widehat{\mathcal{S}})$ is $\pi_1(R)$-equivariant.

Theorem 4.1. *Let $N = 1, 2, 3$. There are exactly two $\widehat{\mathcal{S}}/\mathcal{R}$-forms (up to
isomorphism of $\mathcal{R}$-Lie conformal superalgebras) of $\mathcal{K}_N \otimes_{\mathbb{C}} \mathcal{R}$. These are
$\mathcal{L}(\mathcal{K}_N, \mathrm{id})$ and $\mathcal{L}(\mathcal{K}_N, \omega_N)$, where $\omega_N : \mathcal{K}_N \to \mathcal{K}_N$ is the automorphism of
the complex Lie conformal superalgebra $\mathcal{K}_N$ given by*

$$\begin{array}{llll} \omega_1: & 1 \mapsto 1, & \xi_1 \mapsto -\xi_1, \\ \omega_2: & 1 \mapsto 1, & \xi_1 \mapsto -\xi_1, \\ & \xi_2 \mapsto \xi_2, & \xi_1\xi_2 \mapsto -\xi_1\xi_2, \\ \omega_3: & 1 \mapsto 1, & \xi_j \mapsto -\xi_j, j = 1, 2, 3, \\ & \xi_i\xi_j \mapsto \xi_i\xi_j, i \neq j, & \xi_1\xi_2\xi_3 \mapsto -\xi_1\xi_2\xi_3. \end{array}$$

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[8]The continuous action of $\widehat{\mathbb{Z}}$ is determined by the action of $\overline{1}$ because $\mathbb{Z}$ is dense
in $\widehat{\mathbb{Z}}$.

Proof. By [10] Theorem 2.16, the $\widehat{\mathcal{S}}/\mathcal{R}$-forms of $\mathcal{K}_N \otimes_{\mathbb{C}} \mathcal{R}$ are parametrized by the non-abelian Čech cohomology set $H^1\big(\widehat{\mathcal{S}}/\mathcal{R}, \mathbf{Aut}(\mathcal{K}_N)\big)$.

Since $\mathcal{K}_N \otimes_{\mathbb{C}} \mathcal{R}$ is spanned by

$$\{\widehat{\partial}^\ell (rf) | r \in R, \ell \geqslant 0, f = \xi_{i_1} \ldots \xi_{i_t}, 1 \leqslant i_1 < \cdots < i_t \leqslant N\},$$

we see that $\mathcal{R}$-Lie conformal superalgebra $\mathcal{K}_N \otimes_{\mathbb{C}} \mathcal{R}$ satisfies the finiteness condition of Proposition 2.29 of [10], and this allows us to identify the cohomology set $H^1\big(\widehat{\mathcal{S}}/\mathcal{R}, \mathbf{Aut}(\mathcal{K}_N)\big)$ with the "usual" non-abelian (continuous) cohomology set $H^1\big(\pi_1(R), \mathbf{Aut}(\mathcal{K}_N)(\widehat{\mathcal{S}})\big)$. Our problem is thus reduced to computing $H^1\big(\pi_1(R), \mathbf{Aut}(\mathcal{K}_N)(\widehat{\mathcal{S}})\big)$.

The loop algebras $\mathcal{L}(\mathcal{K}_N, \mathrm{id})$ and $\mathcal{L}(\mathcal{K}_N, \omega_N)$ correspond to the classes $[\alpha]$ and $[\beta]$ in $H^1\big(\pi_1(R), \mathbf{Aut}(\mathcal{K}_N)(\widehat{\mathcal{S}})\big)$ given by the (constant) cocycles $\alpha, \beta : \pi_1(R) \to \mathbf{Aut}(\mathcal{K}_N)(\widehat{\mathcal{S}})$ defined by $1 \mapsto \mathrm{id}$ and $1 \mapsto \omega_N$, respectively.

Since the isomorphism $\iota : \mathbf{O}_N(\widehat{S}) \to \mathbf{Aut}(\mathcal{K}_N)(\widehat{\mathcal{S}})$ of Theorem 3.1 is $\pi_1(R)$-equivariant we are reduced to determining $H^1\big(\widehat{\mathbb{Z}}, \mathbf{O}_N(\widehat{S})\big)$. The classes in $H^1\big(\widehat{\mathbb{Z}}, \mathbf{O}_N(\widehat{S})\big)$ corresponding to $[\alpha]$ and $[\beta]$ will be still denoted in the same way.

Consider the split exact sequence of $\widehat{\mathbb{Z}}$-groups

$$(4.1) \qquad 1 \longrightarrow \mathbf{SO}_N(\widehat{S}) \longrightarrow \mathbf{O}_N(\widehat{S}) \xrightarrow{\det} \mathbb{Z}/2\mathbb{Z} \to 1,$$

where $\widehat{\mathbb{Z}}$ acts on $\mathbb{Z}/2\mathbb{Z}$ trivially and "det" is the determinant map. It yields the exact sequence of pointed sets

$$(4.2) \qquad H^1\big(\widehat{\mathbb{Z}}, \mathbf{SO}_N(\widehat{S})\big) \longrightarrow H^1\big(\widehat{\mathbb{Z}}, \mathbf{O}_N(\widehat{S})\big) \xrightarrow{\psi} H^1(\widehat{\mathbb{Z}}, \mathbb{Z}/2\mathbb{Z}).$$

Since $\widehat{\mathbb{Z}}$ acts on $\mathbb{Z}/2\mathbb{Z}$ trivially, we have $H^1(\widehat{\mathbb{Z}}, \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$. Since the short exact sequence (4.1) is split, ψ admits a section hence is surjective by general considerations. This is also explicitly clear in our situation since ψ visibly maps $[\alpha]$ and $[\beta]$ to the two distinct classes in $H^1(\widehat{\mathbb{Z}}, \mathbb{Z}/2\mathbb{Z})$. It remains to show that ψ is injective.

The fiber of ψ over the trivial class of $H^1(\widehat{\mathbb{Z}}, \mathbb{Z}/2\mathbb{Z})$ is measured by $H^1\big(\widehat{\mathbb{Z}}, \mathbf{SO}_N(\widehat{S})\big)$, while the fiber over the non-trivial class is measured by $H^1\big(\widehat{\mathbb{Z}}, {}_\beta\mathbf{SO}_N(\widehat{S})\big)$ where ${}_\beta\mathbf{SO}_N$ is the group scheme over R obtained from $\mathbf{SO}_N$ by twisting by β.[9] Since every finite connected étale cover of $\mathrm{Spec}(\mathbb{C}[t^{\pm 1}])$ is of the form $\mathrm{Spec}(\mathbb{C}[t^{\pm \frac{1}{m}}])$ with m a positive integer, all such

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[9]Strictly speaking the twist is by the pullback $\psi_*([\beta])$. The abuse of terminology and notation is standard.

covers have trivial Picard group. By [11], Theorem 3.1 (i) $H^1_{\text{ét}}(R, \mathfrak{G})$ vanishes for every reductive group scheme $\mathfrak{G}$ over R, in particular for $\mathbf{SO}_N$ and $_\beta\mathbf{SO}_N$. On the other hand by the isotriviality theorem of [4, 5] we have $H^1_{\text{ét}}(R, \mathfrak{G}) \simeq H^1\big(\pi_1(R), \mathfrak{G}(\widehat{S})\big) = H^1\big(\widehat{\mathbb{Z}}, \mathfrak{G}(\widehat{S})\big)$. This finishes the proof of injectivity. $\qquad\square$

Now we consider the relation between isomorphism of twisted loop algebras based on $\mathcal{K}_N$ as $\mathcal{R}$-Lie conformal superalgebras and isomorphism of these objects as complex Lie conformal superalgebras. Recall that for any $\mathcal{R}$-Lie conformal superalgebra $\mathcal{A}$, the centroid[10] $\text{Ctd}_\mathcal{R}(\mathcal{A})$ is defined to be

$$\text{Ctd}_\mathcal{R}(\mathcal{A}) = \{\chi \in \text{End}_{R\text{-smod}}(\mathcal{A}) | \chi(a_{(n)}b) = a_{(n)}\chi(b) \text{ for all } a, b \in \mathcal{A}, n \in \mathbb{N}\},$$

where $\text{End}_{R\text{-smod}}(\mathcal{A})$ is the set of homogeneous R-supermodule endomorphisms $\mathcal{A} \to \mathcal{A}$ of degree $\bar{0}$.

Lemma 4.1. *Let* $\mathcal{A} = \mathcal{L}(\mathcal{K}_N, \sigma)$ *where* σ *is an automorphism of* $\mathcal{K}_N$ *of order* m, *where* $N = 1, 2, 3$. *Then the canonical map* $R \to \text{Ctd}_\mathbb{C}(\mathcal{A})$ *is a* $\mathbb{C}$-*algebra isomorphism.*

Proof. Recall that for $r \in R$, there is an endomorphism $r_\mathcal{A} : \mathcal{A} \to \mathcal{A}, a \mapsto ra$, which is an element of $\text{Ctd}_\mathbb{C}(\mathcal{A})$. Then the canonical map $R \to \text{Ctd}_\mathbb{C}(\mathcal{A})$ is defined by $r \mapsto r_\mathcal{A}$. Since $\sigma \circ \partial = \partial \circ \sigma$, we obtain,

(4.3)
$$\begin{aligned}
(\mathcal{K}_N)_i &= \{a \in \mathcal{K}_N | \sigma(a) = \zeta_m^i a\} \\
&= \mathbb{C}[\partial] \otimes_\mathbb{C} \text{span}_\mathbb{C}\{f \in \Lambda(N) | \sigma(f) = \zeta_m^i f \text{ and } f \text{ homogeneous}\}.
\end{aligned}$$

Recall that

$$\mathcal{A} = \sum_{i=0}^{m-1} (\mathcal{K}_N)_i \otimes t^{\frac{i}{m}} k[t^{\pm 1}] \subseteq \mathcal{K}_N \otimes_\mathbb{C} \widehat{S},$$

and

$$\mathcal{K}_N \otimes_\mathbb{C} \widehat{S} = \bigoplus_{\substack{l \geqslant 0 \\ k=0,\ldots,N}} (\mathcal{K}_N \otimes_\mathbb{C} \widehat{S})_{l,k},$$

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[10]The centroid of a $\mathcal{R}$-Lie conformal superalgebra is originally defined in [10] Section 2.4, which is analogous to the centroid of root graded Lie algebras studied in [1].

where $(\mathcal{K}_N \otimes_{\mathbb{C}} \widehat{\mathcal{S}})_{l,k} = \{(\partial^l f) \otimes s | f \in \Lambda(N) \text{ homogeneous of degree } k, s \in \widehat{S}\}$. Hence,

$$(4.4) \quad \mathcal{A} = \bigoplus_{\substack{l \geqslant 0 \\ k = 0, \ldots, N}} \left(\mathcal{A} \cap (\mathcal{K}_N \otimes_{\mathbb{C}} \widehat{\mathcal{S}})_{l,k} \right)$$

$$= \operatorname{span}_{\mathbb{C}} \left\{ (\partial^l f) \otimes s \Big|_{\substack{f \in \Lambda(N) \text{ homogeneous,} \\ s \in \widehat{S} \text{ and } \ell \geqslant 0 \text{ such that } \partial^l f \otimes s \in \mathcal{A}}} \right\}.$$

Let $\chi \in \operatorname{Ctd}_{\mathbb{C}}(\mathcal{A})$. Let $L = -1 \otimes 1 \in \mathcal{A}$. For $f \in \Lambda(N)$ homogeneous of degree $|f|$,

$$L_{(1)}((\partial^\ell f) \otimes s) = \left(2 + \ell - \frac{1}{2}|f| \right)(\partial^\ell f) \otimes s, \qquad \forall s \in \widehat{S}, \ell \geqslant 0.$$

We therefore see that $\mathcal{K}_N \otimes_{\mathbb{C}} \widehat{\mathcal{S}}$ can be decomposed into the direct sum of eigenspaces of $L_{(1)}$ and

$$(\mathcal{K}_N \otimes_{\mathbb{C}} \widehat{\mathcal{S}})_2 = \{a \in \mathcal{K}_N \otimes_{\mathbb{C}} \widehat{\mathcal{S}} | L_{(1)}a = 2a\}$$
$$= \operatorname{span}_{\mathbb{C}} \{1 \otimes s, (\partial \xi_i \xi_j) \otimes s' | i < j, s, s' \in \widehat{S}\}.$$

Applying χ on $L_{(1)}L = 2L$ we obtain $L_{(1)}\chi(L) = 2\chi(L)$, thus we may assume that

$$\chi(L) = L \otimes r + \sum_{i<j} (\partial \xi_i \xi_j) \otimes s_{ij},$$

where $r, s_{ij} \in \widehat{S}$. Then

$$0 = \chi(L_{(2)}L) = L_{(2)}(L \otimes s) + \sum_{i<j} L_{(2)}\big((\partial \xi_i \xi_j) \otimes s_{ij}\big) = 2 \sum_{i<j} \xi_i \xi_j \otimes s_{ij}.$$

It follows $s_{ij} = 0, i < j$. So $\chi(L) = L \otimes r$ and $r \in R$ because $\chi(L) = L \otimes r \in \mathcal{A}$.

For $f \in \Lambda(N)$ homogeneous of degree $|f|$ and $s \in \widehat{S}$ such that $f \otimes s \in \mathcal{A}$,

$$\left(2 - \frac{1}{2}|f| \right) \chi(f \otimes s) = \chi((f \otimes s)_{(1)}L) = (f \otimes s)_{(1)}\chi(L)$$

$$= (f \otimes s)_{(1)}L \otimes r = \left(2 - \frac{1}{2}|f| \right) f \otimes sr.$$

Since $|f| \leqslant 3$, $2 - \frac{1}{2}|f| \neq 0$, thus $\chi(f \otimes s) = f \otimes sr$.

Let $f \in \Lambda(N)$ homogeneous of degree $|f|$, $s \in \widehat{S}$ and $\ell \geqslant 0$ such that $(\partial^{l+1})f \otimes s \in \mathcal{A}$. By (4.3), we have $(\partial^l f) \otimes s \in \mathcal{A}$. Then

$$\chi\big((\partial^{l+1}f) \otimes s\big) = \chi\big(L_{(0)}((\partial^l f) \otimes s)\big) = L_{(0)}\chi\big((\partial^l f) \otimes s\big).$$

By induction, $\chi\big((\partial^\ell f) \otimes s\big) = (\partial^\ell f) \otimes sr$ for all $\ell \geq 0, s \in \widehat{S}$ such that $(\partial^\ell f) \otimes s \in \mathcal{A}$.

Therefore, (4.4) implies $\chi(a) = ar$ for all $a \in \mathcal{A}$. Thus the canonical map $R \to \mathrm{Ctd}_{\mathbb{C}}(\mathcal{A})$ is surjective. Injectivity is clear. $\qquad\square$

Theorem 4.2. *There are exactly two twisted loop Lie conformal superalgebra (up to isomorphism of complex Lie conformal superalgebras) based on each $\mathcal{K}_N, N = 1, 2, 3$. These are $\mathcal{L}(\mathcal{K}_N, \mathrm{id})$ and $\mathcal{L}(\mathcal{K}_N, \omega_N)$.*

Proof. Each twisted loop Lie conformal superalgebra based on $\mathcal{K}_N$ is an $\widehat{\mathcal{S}}/\mathcal{R}$-form of $\mathcal{K}_N \otimes_{\mathbb{C}} \mathcal{R}$. It follows from Theorem 4.1 that there exists exactly two of them up to isomorphism of $\mathcal{R}$-Lie conformal superalgebras, namely $\mathcal{L}(\mathcal{K}_N, \mathrm{id})$ and $\mathcal{L}(\mathcal{K}_N, \omega_N)$. By Lemma 4.1 and Corollary 2.36 of [10] we conclude that $\mathcal{L}(\mathcal{K}_N, \mathrm{id})$ and $\mathcal{L}(\mathcal{K}_N, \omega_N)$ remain non-isomorphic when viewed as complex Lie conformal superalgebras.[11] $\qquad\square$

5. Passage from Lie conformal superalgebras to superconformal Lie algebras

In the previous section, we have shown that there are only two twisted loop Lie conformal superalgebras based on $\mathcal{K}_N$ up to isomorphism of complex Lie conformal superalgebras, namely $\mathcal{L}(\mathcal{K}_N, \mathrm{id})$ and $\mathcal{L}(\mathcal{K}_N, \omega_N)$. By factoring the image of $\partial + \delta_t$, we obtain two Lie superalgebras $\mathrm{Alg}(\mathcal{K}_N, \mathrm{id})$ and $\mathrm{Alg}(\mathcal{K}_N, \omega_N)$. The crucial point is that these two Lie superalgebras are non-isomorphic. They can be distinguished by the eigenvalues of the Virasoro operator.[12] In this section, we will give a rigorous proof of this non-isomorphism statement when $N = 3$, which is the most involved of all three cases.

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[11]The situation is quite different for loop algebras based on finite-dimensional simple algebras. Here non-isomorphic objects over R may become isomorphic when viewed as complex algebras. Remarkably enough, this does not happen in the case of Lie algebras. The reason is that in the outer group of automorphisms (symmetries of the Dynkin diagram), every element is conjugate to its inverse. See [11] for details.

[12]For $N = 4$ this argument is mentioned, without proof, in [10].

Let $\mathrm{Alg}(\mathcal{K}_3, \mathrm{id}) = \mathcal{L}(\mathcal{K}_3, \mathrm{id})/(\partial + \delta_t)\mathcal{L}(\mathcal{K}_3, \mathrm{id})$, where the Lie bracket on $\mathrm{Alg}(\mathcal{K}_3, \mathrm{id})$ is given by the zeroth product of $\mathcal{L}(\mathcal{K}_3, \mathrm{id})$. For $a \in \mathcal{L}(\mathcal{K}_3, \mathrm{id})$, let $\bar{a}$ be its image in $\mathrm{Alg}(\mathcal{K}_3, \mathrm{id})$. Set

$$L_m = -\overline{1 \otimes t^{m+1}}, \qquad\qquad G_\alpha^i = 2\overline{\xi_i \otimes t^{\alpha + \frac{1}{2}}},$$
$$T_m^i = 2\mathbf{i}\epsilon_{ijl}\overline{\xi_j \xi_l \otimes t^m}, \qquad \Psi_\alpha = -2\mathbf{i}\overline{\xi_1 \xi_2 \xi_3 \otimes t^{\alpha - \frac{1}{2}}}.$$

for $i = 1, 2, 3, m \in \mathbb{Z}, \alpha \in \frac{1}{2} + \mathbb{Z}$. Then $\{L_m, T_m^i, G_\alpha^i, \Psi_\alpha | i = 1, 2, 3, m \in \mathbb{Z}, \alpha \in \frac{1}{2} + \mathbb{Z}\}$ is a basis of $\mathrm{Alg}(K_3, \mathrm{id})$. For $m, n \in \mathbb{Z}, \alpha, \beta \in \frac{1}{2} + \mathbb{Z}, i, j = 1, 2, 3$, the following relations hold:

$$[L_m, L_n] = (m - n)L_{m+n}, \qquad [L_m, T_n^i] = -nT_{m+n}^i, \quad [T_m^i, T_n^j] = \mathbf{i}\epsilon_{ijl}T_{m+n}^l,$$
$$[L_m, \Psi_\alpha] = -\left(\frac{1}{2}m + \alpha\right)\Psi_{m+\alpha}, \; [T_m^i, \Psi_\alpha] = 0, \qquad\quad [\Psi_\alpha, \Psi_\beta] = 0,$$
$$[L_m, G_\alpha^i] = \left(\frac{1}{2}m - \alpha\right)G_{m+\alpha}^i, \quad [T_m^i, G_\alpha^j] = \mathbf{i}\epsilon_{ijl}G_{m+\alpha}^l + \delta_{ij}m\Psi_{m+\alpha},$$
$$[G_\alpha^i, \Psi_\beta] = T_{\alpha+\beta}^i, \qquad\qquad\qquad [G_\alpha^i, G_\beta^j] = 2\delta_{ij}L_{\alpha+\beta} + \mathbf{i}\epsilon_{ijl}(\alpha - \beta)T_{\alpha+\beta}^l.$$

Next we consider $\mathrm{Alg}(\mathcal{K}_3, \omega_3) = \mathcal{L}(\mathcal{K}_3, \omega_3)/(\partial + \delta_t)\mathcal{L}(\mathcal{K}_3, \omega_3)$, where the Lie bracket is given by the corresponding zeroth product as before. Since ω_3 acts on the even part $(\mathcal{K}_3)_{\bar{0}}$ as the identity and on the odd part $(\mathcal{K}_3)_{\bar{1}}$ as $-\mathrm{id}$, we have by definition

$$\mathcal{L}(\mathcal{K}_3, \omega_3) = ((\mathcal{K}_3)_{\bar{0}} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]) \oplus ((\mathcal{K}_3)_{\bar{1}} \otimes_{\mathbb{C}} t^{\frac{1}{2}}\mathbb{C}[t, t^{-1}]).$$

As before for $a \in \mathcal{L}(\mathcal{K}_3, \omega_3)$ we let $\bar{a}$ denote its image in $\mathrm{Alg}(\mathcal{K}_3, \omega_3)$. Let

$$L_m = -\overline{1 \otimes t^{m+1}}, \qquad\qquad G_m^i = 2\overline{\xi_i \otimes t^{m + \frac{1}{2}}},$$
$$T_m^i = 2\mathbf{i}\epsilon_{ijl}\overline{\xi_j \xi_l \otimes t^m}, \qquad \Psi_m = -2\mathbf{i}\overline{\xi_1 \xi_2 \xi_3 \otimes t^{m - \frac{1}{2}}},$$

for $i = 1, 2, 3, m \in \mathbb{Z}$. Then $\{L_m, T_m^i, G_m^i, \Psi_m | i = 1, 2, 3, m \in \mathbb{Z}\}$ is a basis of $\mathrm{Alg}(\mathcal{K}_3, \omega_3)$ and these elements satisfy the following relations for $m, n \in \mathbb{Z}$ and $i, j = 1, 2, 3$:

$$[L_m, L_n] = (m - n)L_{m+n}, \qquad [L_m, T_n^i] = -nT_{m+n}^i, \quad [T_m^i, T_n^j] = \mathbf{i}\epsilon_{ijl}T_{m+n}^l,$$
$$[L_m, \Psi_n] = -\left(\frac{1}{2}m + n\right)\Psi_{m+n}, \; [T_m^i, \Psi_n] = 0, \qquad\quad [\Psi_m, \Psi_n] = 0,$$
$$[L_m, G_n^i] = \left(\frac{1}{2}m - n\right)G_{m+n}^i, \quad [T_m^i, G_n^j] = \mathbf{i}\epsilon_{ijl}G_{m+n}^l + \delta_{ij}m\Psi_{m+n},$$
$$[G_m^i, \Psi_n] = T_{m+n}^i, \qquad\qquad\qquad [G_m^i, G_n^j] = 2\delta_{ij}L_{m+n} + \mathbf{i}\epsilon_{ijl}(m - n)T_{m+n}^l.$$

To prove that these two Lie superalgebras are not isomorphic, we first establish an auxiliary useful Lemma. Consider the subspace of $\mathrm{Alg}(\mathcal{K}_3, \mathrm{id})$ and $\mathrm{Alg}(\mathcal{K}_3, \omega_3)$ with bases composed of the respective $\{L_m, T_m^i | i = 1, 2, 3, m \in \mathbb{Z}\}$. The above relations show that these are in fact sub Lie superalgebras which are isomorphic. We will denote them by $\mathfrak{g}$.

Lemma 5.1 [Rigidity of the Virasoro element]. *If ϕ is an automorphism of $\mathfrak{g}$, then $\phi(L_0) = \pm L_0$.*

Proof. $\mathfrak{g}$ can be decomposed into the direct sum of weight spaces with respect to L_0, i.e.,

$$\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n,$$

where $\mathfrak{g}_n = \{x \in \mathfrak{g} | [L_0, x] = nx\} = \mathrm{span}_{\mathbb{C}}\{L_{-n}, T_{-n}^i, i = 1, 2, 3\}$, $n \in \mathbb{Z}$. If $x \in \mathfrak{g}_n$ we say that x is homogeneous of weight n and write $\mathrm{wt}(x) = n$. Note that $[\mathfrak{g}_m, \mathfrak{g}_n] \subseteq \mathfrak{g}_{m+n}$.

Recall that an element x of a Lie superalgebra $\mathfrak{g}$ over $\mathbb{C}$ is called *locally finite* if for every fixed $y \in \mathfrak{g}$, $\{(\mathrm{ad}x)^n y, n \in \mathbb{N}\}$ spans a finite-dimensional subspace, or equivalently, every element of $\mathfrak{g}$ lies inside a finite-dimensional $\mathrm{ad}x$-stable subspace. We first show that the locally finite elements of $\mathfrak{g}$ are precisely the elements of $\mathfrak{g}_0$. For $n > 0$, let x be an arbitrary non-zero element in $\mathfrak{g}_n$. We claim that there exists a homogeneous element $y \in \mathfrak{g}$ such that $(\mathrm{ad}x)^m(y) \neq 0$ and $\mathrm{wt}((\mathrm{ad}x)^m(y)) < \mathrm{wt}((\mathrm{ad}x)^{m+1}(y))$ for all $m \in \mathbb{N}$.

We may assume $x = aL_{-n} + b_1 T_{-n}^1 + b_2 T_{-n}^2 + b_3 T_{-n}^3$. If $a \neq 0$, let $y = L_{-2n}$, then

$$(\mathrm{ad}x)^m(y) = m! a^m n^m L_{-(m+2)n} + c_{1m} T_{-(m+2)n}^1 + c_{2m} T_{-(m+2)n}^2$$
$$+ c_{3m} T_{-(m+2)n}^3$$
$$\neq 0$$

and $\mathrm{wt}((\mathrm{ad}x)^m y) = (m+2)n, m \in \mathbb{N}$. If $a = 0$, $x = b_1 T_{-n}^1 + b_2 T_{-n}^2 + b_3 T_{-n}^3$ with $(b_1, b_2, b_3) \neq (0, 0, 0)$. Take $y = L_{-1}$, then

$$(\mathrm{ad}x)^m(y) = n(n+1) \cdots (n+m-1)(b_1 T_{-n-m}^1 + b_2 T_{-n-m}^2 + b_3 T_{-n-m}^3) \neq 0$$

and $\mathrm{wt}((\mathrm{ad}x)^m y) = m + n$.

Now, let $x \in \mathfrak{g}$ be locally finite, we can write, $x = x_{-M'} + \cdots + x_0 + \cdots + x_M$ with $x_l \in \mathfrak{g}_l, l = -M', \ldots, M$ and $M', M \geqslant 0$. If $M > 0$ and $x_M \neq 0$, then there is a homogeneous element $y \in \mathfrak{g}$ such that

$\mathrm{wt}\big((\mathrm{ad}x_M)^m y\big) < \mathrm{wt}\big((\mathrm{ad}x_M)^{m+1}y\big)$. Then

$$(\mathrm{ad}x)^m(y) = (\mathrm{ad}x_M)^m(y) + \sum_{\mathrm{wt}z < \mathrm{wt}((\mathrm{ad}x_M)^m(y))} z.$$

Thus $\{(\mathrm{ad}x)^m(y)|m \in \mathbb{N}\}$ is linear independent. This contradicts the locally finiteness of x. So $M = 0$. Similarly, we can prove $M' = 0$. Hence, $x \in \mathfrak{g}_0$.

Conversely, it is easy to check every elements in $\mathfrak{g}_0$ is locally finite. Moreover, since $[L_0, L_0] = 0, [L_0, T_0^i] = 0, [T_0^i, T_0^j] = \mathbf{i}\epsilon_{ijl}T_0^l$, $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{g}$, isomorphic to $\mathfrak{gl}_2(\mathbb{C})$.

If ϕ is an automorphism of $\mathfrak{g}$, then it preserves locally finite elements. So the restriction of ϕ to $\mathfrak{g}_0$ induces an automorphism of $\mathfrak{g}_0$. Observe that the center of $\mathfrak{g}_0$ is $\mathbb{C}L_0$. So $\phi(L_0) = aL_0$ for some $0 \neq a \in \mathbb{C}$.

Since $a[L_0, \phi(L_{-n})] = [\phi(L_0), \phi(L_{-n})] = n\phi(L_{-n})$, $\phi(L_{-n})$ is a weight vector of L_0 with weight $\frac{n}{a}$ and all weights of L_0 are integers, it follows $\frac{n}{a}$ is an integer for all n. Hence $a = \pm 1$. $\square$

Remark 5.1. Similar results hold for $N = 1, 2$.

Proposition 5.1. *For $N = 1, 2, 3$ the Lie superalgebras $\mathrm{Alg}(\mathcal{K}_N, \mathrm{id})$ and $\mathrm{Alg}(\mathcal{K}_N, \omega_N)$ are not isomorphic.*

Proof. As mentioned above we only give an explicit prove in the case $N = 3$. Suppose that $\phi : \mathrm{Alg}(\mathcal{K}_3, \mathrm{id}) \to \mathrm{Alg}(\mathcal{K}_3, \omega_3)$ is an isomorphism of Lie superalgebras. Then it induces an isomorphism on the even parts which by rigidity satisfies $\phi(L_0) = \pm L_0$. But the eigenvalues of L_0 in $\mathrm{Alg}(\mathcal{K}_3, \mathrm{id})_{\bar{1}}$ are contained in $\frac{1}{2} + \mathbb{Z}$, while the eigenvalues of $\phi(L_0) = \pm L_0$ in $\mathrm{Alg}(\mathcal{K}_3, \omega_3)_{\bar{1}}$ are all integers; a contradiction. $\square$

6. A conjecture on representability

Let $\mathcal{A}$ be one of the complex N-Lie conformal superalgebras, where $N = 1, 2, 3, 4$. (In particular, $\mathcal{A} = \mathcal{K}_N$ if $N \neq 4$. If $N = 4$ the description of $\mathcal{A}$ is given in detail in both [9, 10]). Let $\mathcal{L}$ be an $\widehat{\mathcal{S}}/\mathcal{R}$-form of $\mathcal{A} \otimes_{\mathbb{C}} \mathcal{R}$, Consider the corresponding $\mathcal{R}$-group functor of automorphisms $\mathbf{Aut}(\mathcal{L})$ (see Remark 2.1). The above construction for $N \neq 4$ and that of [10] in the case $N = 4$ show that there is a suitable degree 0 subspace $\mathcal{V}$ of $\mathcal{L}$ such that $\mathcal{L} = \bigoplus_{n \in \mathbb{N}} \widehat{\partial}^n \mathcal{V}$ since the descent data defining $\mathcal{L}$ preserves the suitable degree 0 space $\mathbb{C} \otimes V \otimes \widehat{\mathcal{S}}$ of $\mathcal{A} \otimes \widehat{\mathcal{S}} = \mathbb{C}[\partial] \otimes_{\mathbb{C}} V \otimes_{\mathbb{C}} \widehat{\mathcal{S}}$. One can define in a natural way a subgroup functor $\mathbf{GrAut}(\mathcal{L})$ of $\mathbf{Aut}(\mathcal{L})$.

Conjecture 6.1. *If $N \neq 4$ the $\mathcal{R}$-group functor $\mathbf{GrAut}(\mathcal{L})$ is representable by an a affine group scheme of finite type whose connected component of the identity is simple (in the sense of [15]).*

After a "differential" version of faithfully flat descent we may assume that $\mathcal{L}$ is split, that is $\mathcal{L} = \mathcal{A} \otimes_{\mathbb{C}} \mathcal{R}$. For $N \neq 4$ we have $\mathbf{GrAut}(\mathcal{L}) = \mathbf{O}_{N,R}$, where this last denotes the orthogonal group $\mathbf{O}_N$ over $\mathrm{Spec}(R)$.

In view of [10], for $N = 4$ the natural conjecture to make is that in the split case $\mathbf{GrAut}(\mathcal{L}) = \left(\mathbf{SL}_2 \times \mathrm{SL}_2(\mathbb{C})_R\right)/\boldsymbol{\mu}_{2,R}$ where $\mathrm{SL}_2(\mathbb{C})_R$ is the constant R-group scheme with underlying (abstract) group $\mathrm{SL}_2(\mathbb{C})$, and $\boldsymbol{\mu}_{2,R}$ embeds diagonally. We refrain, however, from making such a conjecture.

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