

New structures of knot invariants

KEFENG LIU AND PAN PENG

Based on the proof of the Labastida–Mariño–Ooguri–Vafa conjecture [6], we derive an infinite product formula for Chern–Simons partition functions, the generating function of quantum \mathfrak{sl}_N invariants. Some symmetry properties of the infinite product will also be discussed.

1. Introduction

Chern–Simons theory has been conjectured to be equivalent to a topological string theory by $1/N$ expansion in physics. This duality conjecture builds a fundamental connection in mathematics. On the one hand, Chern–Simons theory leads to the construction of knot invariants; on the other hand, topological string theory gives rise to Gromov–Witten theory in geometry.

Therefore, the Chern–Simons/topological string duality conjecture identifies the generating function of Gromov–Witten invariants as Chern–Simons knot invariants [9]. Based on these thoughts, the existence of a sequence of integer invariants is conjectured [3, 4, 7, 9] in a similar spirit of Gopakumar–Vafa setting [1], which provides an essential evidence of the duality between Chern–Simons theory and topological string theory. This integrality conjecture, called the Labastida–Mariño–Ooguri–Vafa (LMOV) conjecture, was proved in [6].

One important corollary of the LMOV conjecture is to express Chern–Simons partition function as an infinite product derived in this article. The motivation of studying such an infinite product formula is based on a guess on the modularity property of topological string partition functions.

Chern–Simons theory can be approached in mathematics with the help of quantum group theory. Let \mathcal{L} be a link of L components. Quantum \mathfrak{sl}_N invariant of \mathcal{L} , $W_{(A^1, \dots, A^L)}(\mathcal{L}; q, t)$, is defined to be a trace function on the $U_q(\mathfrak{sl}_N)$ modules constructed from the planar diagram of \mathcal{L} and irreducible $U_q(\mathfrak{sl}_N)$ modules V_{A^1}, \dots, V_{A^L} associated to the components of \mathcal{L} . These irreducible $U_q(\mathfrak{sl}_N)$ modules are labeled by Young diagrams A^1, \dots, A^L . However, we will define these invariants more explicitly by the HOMFLY skein model described in Section 2.

The Chern–Simons partition function of \mathcal{L} is the following generating function of quantum \mathfrak{sl}_N invariants:

$$Z_{\text{cs}}(\mathcal{L}; q, t) = 1 + \sum_{A^1, \dots, A^L} W_{(A^1, \dots, A^L)}(\mathcal{L}; q, t) \prod_{\alpha=1}^L s_{A^\alpha}(x^\alpha),$$

where the summation is taken over all the partitions with some of them possibly being empty while not all empty, and $s_{A^\alpha}(x^\alpha)$ is the Schur function of a set of variables $x^\alpha = \{x_i^\alpha\}_{i \geq 1}$. Based on our proof of the LMOV conjecture, we obtain the following infinite product formula (we only show the infinite product formula for a knot. Its generalization to links is similar. Please refer to section 4):

$$Z_{\text{cs}} = \prod_{\mu} \prod_{Q \in \mathbb{Z}/2} \prod_{m=1}^{\infty} \prod_{k=-\infty}^{\infty} \langle 1 - q^{k+m} t^Q x^\mu \rangle^{-m \check{n}_{\mu; k, Q}}.$$

Here, $\langle \cdot \rangle$ is the symmetric product defined by (4.4), and $\check{n}_{\mu; k, Q}$ are invariants related to the integer invariants in the LMOV conjecture. The symmetry property of $\check{n}_{\mu; k, Q}$ is discussed in Section 5. For more details of the infinite product formula for the Chern–Simons partition function of a link \mathcal{L} , please refer to (4.7).

The paper is organized as follows. In Section 2, we define the quantum \mathfrak{sl}_N invariants by the HOMFLY skein model. Chern–Simons theory and the LMOV conjecture are described in Section 3. In Section 4, we will derive the Chern–Simons partition function as an infinite product. In Section 5, we will discuss the symmetry of $q \rightarrow q^{-1}$ and the rank-level duality.

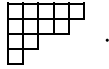
2. Quantum \mathfrak{sl}_N invariants

2.1. Preliminary

We start by reviewing some preliminary on the representations of symmetric groups and symmetric functions.

A partition of n is a tuple of positive integers $\mu = (\mu_1, \mu_2, \dots, \mu_k)$ such that $|\mu| \triangleq \sum_{i=1}^k \mu_i = n$ and $\mu_1 \geq \mu_2 \geq \dots \geq \mu_k > 0$, where $|\mu|$ is called the degree of μ and k is called the length of μ , denoted by $\ell(\mu)$. A partition can be represented by a Young diagram, for example, partition $(5, 4, 2, 1)$ can be

identified as the following Young diagram:



Denote by \mathcal{Y} the set of all Young diagrams. Let χ_A be the character of an irreducible representation of symmetric group, labeled by partition A . Given a partition μ , define $m_j = \text{card}(\mu_k = j; k \geq 1)$. The order of the conjugate class of type μ is given by:

$$z_\mu = \prod_{j \geq 1} j^{m_j} m_j!$$

The orthogonality of the character formula gives

$$\sum_A \frac{\chi_A(C_\mu)\chi_A(C_\nu)}{z_\mu} = \delta_{\mu,\nu} = \begin{cases} 1, & \text{if } \mu = \nu; \\ 0, & \text{otherwise.} \end{cases}$$

The theory of symmetric functions has a close relationship with the representations of symmetric group. The symmetric power functions of a given set of variables $x = \{x_j\}_{j \geq 1}$ are defined as the direct limit of the Newton polynomials:

$$p_n(x) = \sum_{j \geq 1} x_j^n, \quad p_\mu(x) = \prod_{i \geq 1} p_{\mu_i}(x),$$

and we have the following formula which determines the Schur function:

$$s_A(x) = \sum_\mu \frac{\chi_A(C_\mu)}{z_\mu} p_\mu(x).$$

Given $x = \{x_i\}_{i \geq 1}$, $y = \{y_j\}_{j \geq 1}$, define

$$(2.1) \quad x * y = \{x_i \cdot y_j\}_{i \geq 1, j \geq 1}.$$

We also define $x^d = \{x_i^d\}_{i \geq 1}$. The d th Adams operation of a Schur function is given by $s_A(x^d)$.

2.2. The HOMFLY skien models

The quantum group invariants can be defined over any semi-simple Lie algebra \mathfrak{g} . Here, we are particularly interested in $SU(N)$ Chern–Simons

gauge theory, hence we will study the quantum \mathfrak{sl}_N invariants, which can be identified as the colored HOMFLY polynomials.

We will start by introducing the framed HOMFLY polynomial, an invariant of framed oriented links. Define the skein $R_n^n(q, t)$ by linear combinations of oriented n -tangles modulo the following relations:

$$(2.2) \quad \begin{array}{c} \begin{array}{ccc} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} & - & \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} & = & (q^{-\frac{1}{2}} - q^{\frac{1}{2}}) \begin{array}{c} \curvearrowright \quad \curvearrowleft \end{array} , \\ \text{and} & & \begin{array}{c} \curvearrowright \end{array} & = & t^{-\frac{1}{2}} \begin{array}{c} \downarrow \end{array} . \end{array}$$

The HOMFLY skein of an planar annulus, with some designated input and output boundary points, is defined as linear combinations of oriented tangles in the annulus, modulo Reidemeister moves II and III and the above two relations (2.2). The coefficient ring is $\mathbb{Z}[q^{\pm\frac{1}{2}}, t^{\pm\frac{1}{2}}]$ with finitely many products of $(q^{-\frac{k}{2}} - q^{\frac{k}{2}})$ in the denominators. The skein of the annulus is denoted by \mathcal{C} . This is a commutative algebra with a product given by placing one annulus outside another.

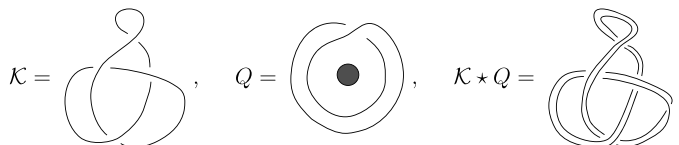
From the HOMFLY skein of annulus, we can obtain the framed HOMFLY polynomial of links, denoted by $\mathcal{H}(\mathcal{L})$. Here we normalize \mathcal{H} as:

$$\mathcal{H}(\text{unknot}) = \frac{t^{-\frac{1}{2}} - t^{\frac{1}{2}}}{q^{-\frac{1}{2}} - q^{\frac{1}{2}}}.$$

These invariants can be recursively computed through the HOMFLY skein.

2.3. The quantum group invariants

The colored HOMFLY polynomials are defined through *satellite knot*. A satellite of \mathcal{K} is determined by choosing a diagram Q in the annulus. Draw Q on the annular neighborhood of \mathcal{K} determined by the framing to give a satellite knot $\mathcal{K} \star Q$. See the following figure for a satellite of a framed trefoil knot with Q :



We will refer to this construction as *decorating* \mathcal{K} with the pattern Q . The HOMFLY polynomial $\mathcal{H}(\mathcal{K} \star Q)$ of the satellite depends on Q only as an element of the skein \mathcal{C} of the annulus. \mathcal{C} can be regarded as the parameter space for these invariants of \mathcal{K} . We will call it the *HOMFLY satellite invariants of* \mathcal{K} .

There is a known set of idempotent elements, E_λ , one for each partition λ of n . They were originally described in [2, 8]. Take the closure of E_λ , we have $Q_\lambda \in \mathcal{C}$. $\{Q_\lambda\}_{\lambda \in \mathcal{Y}}$ form a basis of \mathcal{C} .

The quantum \mathfrak{sl}_N invariant for the irreducible module V_{A^1, \dots, A^L} , labeled by the corresponding partitions A^1, \dots, A^L , can be identified as the HOMFLY invariants for the link decorated by Q_{A^1}, \dots, Q_{A^L} . Write $\vec{A} = (A^1, \dots, A^L)$. The quantum \mathfrak{sl}_N invariants of the link is given by

$$(2.3) \quad W_{\vec{A}}(\mathcal{L}; q, t) = \mathcal{H}(\mathcal{L} \star \otimes_{\alpha=1}^L Q_{A^\alpha}),$$

and we can define the following invariants:

$$(2.4) \quad Z_{\vec{\mu}}(\mathcal{L}; q, t) = \sum_{\vec{A}=(A^1, \dots, A^L)} \left(\prod_{\alpha=1}^L \chi_{A^\alpha}(C_{\mu^\alpha}) \right) W_{\vec{A}}(\mathcal{L}; q, t).$$

2.4. Notations

Here, we make the following convention for the notations in this article.

- We will consistently denote by \mathcal{L} a link and by L the number of components in \mathcal{L} .
- The irreducible $U_q(\mathfrak{sl}_N)$ module associated to \mathcal{L} will be labeled by their highest weights, thus by Young diagrams. We usually denote it by a vector form $\vec{A} = (A^1, \dots, A^L)$.
- Let $\vec{x} = (x_1, \dots, x_L)$ is L sets of variables, each of which is associated to a component of \mathcal{L} and $\vec{\mu} = (\mu^1, \dots, \mu^L) \in \mathcal{Y}^L$ be a tuple of L partitions. Define:

$$[n] = q^{-\frac{n}{2}} - q^{\frac{n}{2}}, \quad [\vec{\mu}] = \prod_{\alpha=1}^L [\mu^\alpha], \quad \mathfrak{z}_{\vec{\mu}} = \prod_{\alpha=1}^L \mathfrak{z}_{\mu^\alpha},$$

$$\chi_{\vec{A}}(C_{\vec{\mu}}) = \prod_{\alpha=1}^L \chi_{A^\alpha}(C_{\mu^\alpha}), \quad s_{\vec{A}}(\vec{x}) = \prod_{\alpha=1}^L s_{A^\alpha}(x_\alpha), \quad p_{\vec{\mu}}(\vec{x}) = \prod_{\alpha=1}^L p_{\mu^\alpha}(x_\alpha).$$

3. Chern–Simons theory

3.1. Chern–Simons partition function

The *Chern–Simons partition function* of the link \mathcal{L} is the following generating series of quantum group invariants weighted by Schur functions:

$$Z_{\text{cs}}(\mathcal{L}; q, t) = 1 + \sum_{\vec{A}} W_{\vec{A}}(\mathcal{L}; q, t) s_{\vec{A}}(\vec{x}),$$

or equivalently by the reformulated invariants $Z_{\vec{\mu}}$:

$$(3.1) \quad Z_{\text{cs}}(\mathcal{L}; q, t) = 1 + \sum_{\vec{\mu}} \frac{Z_{\vec{\mu}}(\mathcal{L}; q, t)}{\mathfrak{z}_{\vec{\mu}}} p_{\vec{\mu}}(\vec{x}).$$

Free energy is defined by

$$(3.2) \quad F(\mathcal{L}; q, t) = \log Z_{\text{CS}}(\mathcal{L}; q, t) = \sum_{\vec{\mu}} \frac{F_{\vec{\mu}}(\mathcal{L}; q, t)}{\mathfrak{z}_{\vec{\mu}}} p_{\vec{\mu}}(\vec{x}).$$

Here, we expand the free energy and get the definition of $F_{\vec{\mu}}(\mathcal{L}; q, t)$ according to the above formula.

3.2. Transformation function

Define the following *transformation function* for two partitions, A and B , of n :

$$(3.3) \quad \mathcal{T}_{AB}(x) = \sum_{\mu} \frac{\chi_A(C_{\mu})\chi_B(C_{\mu})}{\mathfrak{z}_{\mu}} p_{\mu}(x).$$

By a simple algebra, its inverse is given by

$$(3.4) \quad \mathcal{T}_{AB}^{-1}(x) = \sum_{\mu} \frac{\chi_A(C_{\mu})\chi_B(C_{\mu})}{\mathfrak{z}_{\mu}} \frac{1}{p_{\mu}(x)}.$$

One immediately sees the following:

$$\begin{aligned}
 (3.5) \quad \sum_A \mathcal{T}_{AB}(x) s_A(y) &= \sum_A \sum_{\mu} \frac{\chi_A(C_{\mu}) \chi_B(C_{\mu})}{\mathfrak{z}_{\mu}} p_{\mu}(x) s_A(y) \\
 &= \sum_{\mu} \frac{\chi_B(C_{\mu})}{\mathfrak{z}_{\mu}} p_{\mu}(y) \sum_A \chi_A(C_{\mu}) s_A(x) \\
 &= \sum_{\mu} \frac{\chi_B(C_{\mu})}{\mathfrak{z}_{\mu}} p_{\mu}(y) p_{\mu}(x) \\
 &= s_B(x * y).
 \end{aligned}$$

Here in the last step, $x * y$ is defined in (2.1). Similarly, we have

$$(3.6) \quad \sum_A \mathcal{T}_{AB}^{-1}(x) s_A(x * y) = s_B(y)$$

Define variables $q^{\ell} = \{q^{j-\frac{1}{2}}\}_{j \geq 1}$ and $q^{d\ell} = \{q^{d(j-\frac{1}{2})}\}_{j \geq 1}$. We have

$$(3.7) \quad \frac{1}{[n]} = p_n(q^{\ell}).$$

3.3. LMOV conjecture

Based on the large N duality between Chern–Simons gauge theory and topological string theory, Labastida, Mariño, Ooguri and Vafa made a conjecture on the existence of an infinite sequence of integer invariants in Chern–Simons theory [3, 4, 9].

Free energy has the following expansion:

$$F = \sum_{\vec{A}} \sum_{d=1}^{\infty} \frac{1}{d} f_{\vec{A}}(q^d, t^d) s_{\vec{A}}((\vec{x})^d),$$

where

$$s_{\vec{A}}((\vec{x})^d) = \prod_{\alpha=1}^L s_{A^{\alpha}}(\{(x_j^{\alpha})^d\}_{j \geq 1}).$$

The LMOV conjecture proved in [6] can be stated as the following theorem:

Theorem 3.1 Liu–Peng. *Notations are as above. Define*

$$(3.8) \quad P_{\vec{B}}(q, t) = \sum_{\vec{A}} f_{\vec{A}}(q, t) \prod_{\alpha=1}^L \mathcal{T}_{A^\alpha B^\alpha}(q^\ell).$$

Then

$$(3.9) \quad P_{\vec{B}}(q, t) \in [1]^{-2} \cdot \mathbb{Z}[[1]^2, t^{\pm\frac{1}{2}}].$$

4. Infinite product formula

To derive an infinite product formula, we will state the result for a knot at first, since the notations in the computation for a knot are relatively simpler.

4.1. The case of a knot

Set $y = x * q^\ell$, then $p_n(x * q^\ell) = p_n(x) \cdot p_n(q^\ell)$. We have

$$p_\mu(y) = p_\mu(x)p_\mu(q^\ell).$$

Consider free energy weighted by the Schur function of y , LMOV conjecture implies the following reformulation of free energy:

$$\begin{aligned} F(q, t; y) &= \sum_{d=1}^{\infty} \sum_A \frac{1}{d} f_A(q^d, t^d) s_A(y^d) \\ &= \sum_{d=1}^{\infty} \sum_A \frac{1}{d} \sum_B \mathcal{T}_{AB}^{-1}(q^{d\ell}) P_B(q^d, t^d) s_A(y^d), \end{aligned}$$

where

$$(4.1) \quad [1]^2 \cdot P_B(q, t) = \sum_{Q \in \mathbb{Z}/2} \sum_{g=0}^{\infty} N_{B;g,Q} (q^{-\frac{1}{2}} - q^{\frac{1}{2}})^{2g} t^Q$$

is a polynomial of $[1]^2$ and $t^{\pm\frac{1}{2}}$ with integer coefficients, $N_{B;g,Q}$. By (3.6), we have

$$F(q, t; y) = \sum_{d=1}^{\infty} \sum_B \frac{1}{d} P_B(q^d, t^d) s_B(x^d).$$

There exist integers $n_{B;g,Q}$ such that

$$(4.2) \quad \sum_{g=0}^{\infty} N_{B;g,Q} (q^{-\frac{1}{2}} - q^{\frac{1}{2}})^{2g} = \sum_{g=0}^{\infty} n_{B;g,Q} \sum_{k=0}^g q^{g-2k}.$$

This is due to the equivalence of two integral bases of $U_q(\mathfrak{sl}_2)$ modules: one is obtained by all the irreducible modules of $U_q(\mathfrak{sl}_2)$, V_n , for each $n \geq 0$; the other one is obtained by $V_1^{\otimes n}$ for $n \geq 0$ since $V_1 \otimes V_n = V_{n+1} \oplus V_{n-1}$.

By (3.9) $N_{B;g,Q}$ vanish for sufficiently large g and $|Q|$, thus $n_{B;g,Q}$ vanish for sufficiently large g and $|Q|$. We have

$$\begin{aligned} P_B(q, t) &= \sum_{Q \in \mathbb{Z}/2} \sum_{g=0}^{\infty} N_{B;g,Q} (q^{-\frac{1}{2}} - q^{\frac{1}{2}})^{2g-2} t^Q \\ &= \sum_Q \frac{t^Q}{(q^{-\frac{1}{2}} - q^{\frac{1}{2}})^2} \sum_{g=0}^{\infty} n_{B;g,Q} \sum_{k=0}^g q^{g-2k} \\ &= \sum_Q t^Q \sum_{m=1}^{\infty} m q^m \sum_{g=0}^{\infty} n_{B;g,Q} \sum_{k=0}^g q^{g-2k} \\ &= \sum_{Q \in \mathbb{Z}/2} \sum_{m=1}^{\infty} \sum_{g=0}^{\infty} \sum_{k=0}^g m n_{B;g,Q} q^{g-2k+m} t^Q, \end{aligned}$$

which leads to

$$(4.3) \quad \begin{aligned} F &= \sum_B \sum_{d=1}^{\infty} \sum_{Q \in \mathbb{Z}/2} \sum_{m=1}^{\infty} \sum_{g=0}^{\infty} \sum_{k=0}^g \frac{1}{d} m n_{B;g,Q} t^{dQ} q^{d(g-2k+m)} s_B(x^d) \\ &= \sum_{Q,m,g} \sum_{k=0}^g \sum_{B,\mu} m n_{B;g,Q} \frac{\chi_B(C_\mu)}{\mathfrak{z}_\mu} \sum_{d=1}^{\infty} \frac{1}{d} q^{d(g-2k+m)} p_\mu(x^d) t^{dQ}. \end{aligned}$$

Now focus on the following computation:

$$\begin{aligned} \sum_{d \geq 1} \frac{1}{d} \omega^d p_\mu(x^d) &= \sum_{d \geq 1} \frac{1}{d} \omega^d \prod_{j=1}^{\ell(\mu)} \sum_{i \geq 1} x_k^{d\mu_j} \\ &= \sum_{d \geq 1} \frac{1}{d} \omega^d \sum_{i_1, \dots, i_\ell} (x_{i_1}^{\mu_1} \dots x_{i_\ell}^{\mu_\ell})^d \end{aligned}$$

$$\begin{aligned}
 &= \sum_{i_1, \dots, i_\ell} \sum_{d \geq 1} \frac{1}{d} (\omega x_{i_1}^{\mu_1} \cdots x_{i_\ell}^{\mu_\ell})^d \\
 &= - \sum_{i_1, \dots, i_\ell} \log(1 - \omega x_{i_1}^{\mu_1} \cdots x_{i_\ell}^{\mu_\ell}).
 \end{aligned}$$

Let $\omega = t^Q q^{g-2k+m}$ and apply the above computation in (4.3),

$$F = \sum_{Q, m, k} \sum_{B, \mu} (-m n_{B; g, Q}) \frac{\chi_B(C_\mu)}{\mathfrak{z}^\mu} \sum_{i_1, \dots, i_{\ell(\mu)}} \log \left(1 - q^{g-2k+m} t^Q x_{i_1}^{\mu_1} \cdots x_{i_{\ell(\mu)}}^{\mu_{\ell(\mu)}} \right).$$

Define the symmetric product as shown in the following formula:

$$(4.4) \quad \langle 1 - \omega x^\mu \rangle = \prod_{x_{i_1}, \dots, x_{i_{\ell(\mu)}}} \left(1 - \omega x_{i_1}^{\mu_1} \cdots x_{i_{\ell(\mu)}}^{\mu_{\ell(\mu)}} \right),$$

and

$$(4.5) \quad \tilde{n}_{\mu; g, Q} = \sum_B \frac{\chi_B(C_\mu)}{\mathfrak{z}^\mu} n_{B; g, Q}.$$

Therefore, for $Z_{CS} = \exp F$, we obtain the following infinite product formula:

$$(4.6) \quad Z_{CS}(\mathcal{K}; q, t; y) = \prod_{\mu \in \mathcal{Y}} \prod_{Q \in \mathbb{Z}/2} \prod_{m=1}^{\infty} \prod_{g=0}^{\infty} \prod_{k=0}^g \langle 1 - q^{g-2k+m} t^Q x^\mu \rangle^{-m \tilde{n}_{\mu; g, Q}}.$$

Remark 4.1. In the above infinite product formula, since for a given μ , $\tilde{n}_{\mu; g, Q}$ vanish for sufficiently large g and $|Q|$ due to the vanishing property of $n_{B; g, Q}$, the products involved with Q and g are in fact finite products for a fixed partition, μ .

4.2. The case of a link

After going over subsection 4.1, we find that the computation can be carried over to the case of a link. Given a link \mathcal{L} of L components, let $\vec{y} = (y_1, \dots, y_L)$ and $\vec{x} = (x_1, \dots, x_L)$ satisfying $y_i = q^\ell * x_i$, for $i = 1, \dots, L$.

We define $n_{\vec{B};g,Q}$ and $\check{n}_{\vec{\mu};g,Q}$ as the following:

$$\sum_{g=0}^{\infty} N_{\vec{B};g,Q} = \sum_{g=0}^{\infty} n_{\vec{B};g,Q} \sum_{k=0}^g q^{g-2k}, \quad \check{n}_{\vec{\mu};g,Q} = \sum_{\vec{B}} \frac{\chi_A(C_{\vec{\mu}})}{\mathfrak{z}_{\vec{\mu}}} n_{\vec{B};g,Q}.$$

Again, the vanishing result of $N_{\vec{B};g,Q}$ implies that $n_{\vec{B};g,Q}$ and $\check{n}_{\vec{\mu};g,Q}$ vanish for sufficiently large g and $|Q|$.

Let $\vec{\mu} = (\mu^1, \dots, \mu^L)$ and $\vec{x} = (x_1, \dots, x_L)$. Denote by ℓ_i the length of μ^i . Generalize the symmetric product in (4.4) to $\vec{\mu}$ and \vec{x} as:

$$\langle 1 - \omega x_1^{\mu^1} \cdots x_L^{\mu^L} \rangle = \prod_{\alpha=1}^L \prod_{i_{\alpha,1}, \dots, i_{\alpha, \ell_{\alpha}}} \left(1 - \omega \prod_{\alpha=1}^L ((x_{\alpha})_{i_{\alpha,1}}^{\mu_{\alpha,1}^{\alpha}} \cdots (x_{\alpha})_{i_{\alpha, \ell_{\alpha}}}^{\mu_{\alpha, \ell_{\alpha}}^{\alpha}}) \right).$$

Follow the similar computation, we have the infinite product formula for the Chern–Simons partition function of \mathcal{L} :

(4.7)

$$Z_{CS}(\mathcal{L}; q, t; \vec{y}) = \prod_{\vec{\mu} \in \mathcal{Y}^L} \prod_{Q \in \mathbb{Z}/2} \prod_{m=1}^{\infty} \prod_{g=0}^{\infty} \prod_{k=0}^g \langle 1 - q^{g-2k+m} t^Q x_1^{\mu^1} \cdots x_L^{\mu^L} \rangle^{-m \check{n}_{\vec{\mu};g,Q}}.$$

Similar as Remark 4.1, the products related to Q and g are finite for a fixed $\vec{\mu}$.

4.3. The case of the unknot

The Chern–Simons partition function of the unknot is given by

(4.8)

$$Z_{CS}(\text{unknot}; q, t) = 1 + \sum_{A \in \mathcal{Y}} \dim_q V_A \cdot s_A(x).$$

Here, $\dim_q V_A$ is the quantum dimension of the irreducible $U_q(\mathfrak{sl}_N)$ module V_A . The formula of quantum dimension is given by (the formula of the quantum dimension can be found in many literatures, for example, [6]):

$$\dim_q V_A = \sum_{\mu} \frac{\chi_A(C_{\mu})}{\mathfrak{z}_{\mu}} \prod_{j=1}^{\ell(\mu)} \frac{t^{-\frac{\mu_j}{2}} - t^{\frac{\mu_j}{2}}}{q^{-\frac{\mu_j}{2}} - q^{\frac{\mu_j}{2}}}.$$

A similar computation as shown in the previous section leads to the following infinite product formula:

$$(4.9) \quad Z_{\text{CS}}(\text{unknot}; q, t; y) = \prod_{m=1}^{\infty} \prod_i \frac{(1 - q^m t^{\frac{1}{2}} x_i)^m}{(1 - q^m t^{-\frac{1}{2}} x_i)^m}.$$

By a comparison with (4.6), we have

$$\check{n}_{\mu; g, Q} = \begin{cases} -\text{sgn}(Q), & \text{if } Q = \pm \frac{1}{2}, \mu = (1) \text{ and } g = 0; \\ 0, & \text{otherwise.} \end{cases}$$

5. Symmetry property

We will discuss some symmetry properties of the infinite product formula given in Section 4. Here, we will express it for a knot. The case of links is exactly similar, otherwise we will make a remark of their difference.

5.1. The symmetry of $q \rightarrow q^{-1}$

In the derivation of the infinite product formula, we assume $|q| < 1$ for Taylor expansion of $\frac{1}{[1]^2}$. In the case of $|q| > 1$, the Taylor expansion is given by

$$\frac{1}{[1]^2} = \sum_{m=1}^{\infty} m q^{-m}.$$

Therefore, the infinite product formula will be:

$$\begin{aligned} Z_{\text{CS}}(\mathcal{K}; q, t; y) &= \prod_{\mu \in \mathcal{Y}} \prod_{Q \in \mathbb{Z}/2} \prod_{m=1}^{\infty} \prod_{g=0}^{\infty} \prod_{k=0}^g \langle 1 - q^{g-2k} q^{-m} t^Q x^\mu \rangle^{-m \check{n}_{\mu; g, Q}} \\ &= \prod_{\mu \in \mathcal{Y}} \prod_{Q \in \mathbb{Z}/2} \prod_{m=1}^{\infty} \prod_{g=0}^{\infty} \prod_{k=0}^g \langle 1 - q^{-g+2(g-k)} q^{-m} t^Q x^\mu \rangle^{-m \check{n}_{\mu; g, Q}} \\ &= \prod_{\mu \in \mathcal{Y}} \prod_{Q \in \mathbb{Z}/2} \prod_{m=1}^{\infty} \prod_{g=0}^{\infty} \prod_{k=0}^g \langle 1 - (q^{-1})^{g-2k+m} t^Q x^\mu \rangle^{-m \check{n}_{\mu; g, Q}}. \end{aligned}$$

The substitution of k to $g - k$ is used in the last equation. This is the symmetry of $q \rightarrow q^{-1}$ for the infinite product formula.

5.2. Rank-level duality

Rank-level duality is essentially a symmetry of quantum group invariants relating a labeling color to its transpose, which can be expressed as the following identity:

$$(5.1) \quad W_{A^t}(s^{-1}, -v) = W_A(s, v),$$

where $s = q^{\frac{1}{2}}$, $v = t^{\frac{1}{2}}$. From [6], we can actually obtain the following stronger version:

$$(5.2) \quad W_{A^t}(s^{-1}, v) = (-1)^{|A|} W_A(s, v),$$

$$(5.3) \quad W_A(s, -v) = (-1)^{|A|} W_A(s, v).$$

Remark 5.1. The above two equations of the strong version rank-level duality can be interpreted by the $1/N$ expansion for the Chern–Simons partition function and the monodromy of $t \rightarrow e^{2\pi i t}$.

We now give a proof of this strong version of the rank-level duality. (5.2) is equivalent to Proposition 5.2 in [6]:

$$W_{A^t}(q, t) = (-1)^{|A|} W_A(q^{-1}, t).$$

To prove (5.3), we will investigate the HOMFLY polynomial invariants of knots first. By the HOMFLY skein relation:

$$(5.4) \quad t^{-\frac{1}{2}} \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} - t^{\frac{1}{2}} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} = (q^{-\frac{1}{2}} - q^{\frac{1}{2}}) \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array},$$

in the HOMFLY polynomial, the degrees of t are either all integers or half integers. This can be done through a resolution of the crossings. The unknot obviously satisfies this property. Suppose that for any planar diagram representing a link with number of crossings less or equal to n , the degrees of t in the HOMFLY polynomial of the link are either integers or half integers, we immediately see that switching positive crossing to negative crossing will keep this property by (5.4). This will reduce the verification to the case of disjoint unknots. To generalize this result to the quantum \mathfrak{sl}_N invariants, we will apply (5.31) in [6] to directly obtain that $Z_\mu(q, t)$ defined in (3.1)

satisfies that

$$t^{-|\mu|/2} Z_\mu \in \mathbb{Q}[(q^{-\frac{1}{2}} - q^{\frac{1}{2}})^{\pm 1}, t^{\pm 1}],$$

by the observation that in $v^d - v^{-d}$, changing the sign of v will result in a multiplication of $(-1)^d$. This completes the proof of (5.3).

Equation (5.2) leads to the symmetry of the following integer invariants by expansion:

$$(5.5) \quad N_{A^t; g, Q} = (-1)^{|A|} N_{A; g, -Q}.$$

Applying it to (4.5) and combining the fact that

$$\chi_{A^t}(C_\mu) = (-1)^{|\mu| - \ell(\mu)} \chi_A(C_\mu),$$

we have the following symmetry about μ and Q :

$$(5.6) \quad \check{n}_{\mu; g, -Q} = (-1)^{\ell(\mu)} \check{n}_{\mu; g, Q}.$$

This is (4.44) in [7], the rank-level duality of the $SU(N)_k$ and $SU(k)_N$ Chern–Simons gauge theories.

6. Concluding remarks

The infinite product formula derived in Section 4 is interestingly related to a conjecture on the modularity property of the Chern–Simons partition function, which is hardly seen from the knot theory point of view. We hope this formula will shed a new light on the study of knot invariants. A complete understanding in this aspect will lead to a deep insight of the Chern–Simons/topological string duality.

References

- [1] R. Gopakumar and C. Vafa, *On the gauged theory/geometry correspondence*, Adv. Theor. Math. Phys. **3** (1999), 1415.
- [2] A. Gyoja, *A q -analogue of Young symmetrisers*, Osaka J. Math. **23** (1986), 841–852.
- [3] J. M. F. Labastida and M. Mariño, *A new point of view in the theory of knot and link invariants*, J. Knot Theory Ramif. **11** (2002), 173.

- [4] J. M. F. Labastida, M. Mariño and C. Vafa, *Knots, links and branes at large N* , High Energy Phys. **11** (2000), 7.
- [5] K. Liu and P. Peng, *On a proof of the Labastida-Mariño-Ooguri-Vafa conjecture*, Math. Res. Lett. **17**(3) (2010), 493–506.
- [6] K. Liu and P. Peng, *Proof of the Labastida-Mariño-Ooguri-Vafa conjecture*, J. Differential Geom. **85**(3) (2010), 479–525.
- [7] M. Mariño and C. Vafa, *Framed knots at large N* , in ‘Orbifolds in mathematics and physics (Madison, WI, 2001)’, Contemp. Math., **310**, American Mathematical Society, Providence, RI, 2002, 183–204.
- [8] H. R. Morton and A. K. Aiston, *Idempotents of Hecke algebra of type A* , J. Knot Theory Ramif. **7** (1998), 463–487.
- [9] H. Ooguri and C. Vafa, *Knot invariants and topological strings*, Nucl. Phys. B, **577** (2000), 419.

CENTER OF MATHEMATICAL SCIENCES
ZHEJIANG UNIVERSITY, BOX 310027
HANGZHOU, PEOPLE’S REPUBLIC OF CHINA

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA AT LOS ANGELES, BOX 951555
LOS ANGELES, CA 90095-1555, USA
E-mail address: liu@math.ucla.edu

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ARIZONA
617 N. SANTA RITA AVE.
TUCSON, AZ 85721, USA
E-mail address: ppeng@math.arizona.edu

RECEIVED FEBRUARY 24, 2011

