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Algebraic K-theory of toric hypersurfaces

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We construct classes in the motivic cohomology of certain 1-parameter families of Calabi–Yau hypersurfaces in toric Fano n-folds, with applications to local mirror symmetry (growth of genus 0 instanton numbers) and inhomogeneous Picard–Fuchs equations. In the case where the family is classically modular the classes are related to Beilinson's Eisenstein symbol; the Abel–Jacobi map (or rational regulator) is computed in this paper for both kinds of cycles. For the "modular toric" families where the cycles essentially coincide, we obtain a motivic (and computationally effective) explanation of a phenomenon observed by Villegas, Stienstra, and Bertin.

0. Introduction

Writing in 1997 on vanishing of constant terms in powers of Laurent polynomials[1]

$$\phi \in \mathbb{C}[\mathbf{T}^n] = \mathbb{C}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}],$$

Duistermaat and van der Kallen [36] proved the following

Completion Theorem. *Given $\phi \in \mathbb{C}[\mathbf{T}^n]$ such that the interior of its Newton polytope contains the origin, there exists a good compactification $\mathcal{X} \supset \mathbf{T}^n$, i.e., the complement $\overline{\mathcal{X}} \backslash \mathbf{T}^n$ is a normal crossings divisor (NCD) in $\overline{\mathcal{X}}$, together with*

(a) *a holomorphic map $\mathcal{X} \to \mathbb{P}^1$ extending ϕ, and*

(b) *a holomorphic form $\Omega \in \Omega^n(\underbrace{\mathcal{X}\backslash\phi^{-1}(\infty)}_{=:\,\mathcal{X}_-})$ extending $\bigwedge^n d\log \underline{x}$.*

For a simple example, take $n = 2$ and

$$\phi = \prod_{i=1}^{2} \left(x_i - \frac{\mu^2+1}{\mu} + \frac{1}{x_i} \right), \qquad \mu \in \mathbb{C}^*.$$

[1] Here $\mathbf{T} = \mathbb{G}_m$.

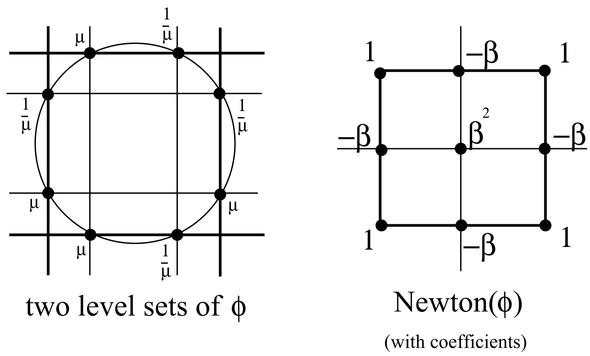


Figure 1: An elliptic pencil.

In the "initial compactification" $\mathbb{P}^1 \times \mathbb{P}^1 (\supset \mathbb{C}^* \times \mathbb{C}^*)$, the level sets $1 - t\phi = 0$ (see figure 1, where $\beta := \frac{\mu^2+1}{\mu}$) complete to a pencil of elliptic curves, with generic member smooth. For ϕ to extend to a well-defined function we must blow $\mathbb{P}^1 \times \mathbb{P}^1$ up at the eight points (marked in the figure) in the base locus; this yields $\mathcal{E} \xrightarrow[1/\phi]{} \mathbb{P}^1_t$ as in the Completion Theorem.

What that result does not address at all is the periods of Ω. Since the Haar form $\frac{1}{(2\pi \mathrm{i})^n} \bigwedge^n d\log \underline{x} := \frac{dx_1}{2\pi \mathrm{i} x_1} \wedge \cdots \wedge \frac{dx_n}{2\pi \mathrm{i} x_n}$ has only rational periods, one might ask under what circumstances this remains true for Ω.

Question 1 (Nori)**.** *Write* $\mathrm{Hg}(-) := \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}(0), -)$*; we have* $\bigwedge^n d\log \underline{x} \in \mathrm{Hg}(H^n(\mathbf{T}^n, \mathbb{Q}(n)))$*. Is* $\Omega \in \mathrm{Hg}(H^n(\mathcal{X}_-, \mathbb{Q}(n)))$*?*

In the above example, the easiest way to compute periods of Ω against topological two-cycles on $\mathcal{E}_-$ is to do a bit of homological algebra. Writing $E_0 := \phi^{-1}(\infty)$, $E_0^{[0]} = \widetilde{E_0} = \amalg^4 \mathbb{P}^1$, $E_0^{[1]} = \mathrm{sing}(E_0)$, we instead can pair two-cocycles in the double-complex of currents

$$\mathcal{D}^{\bullet-4}_{E_0^{[1]}} \xrightarrow[\cdot(2\pi \mathrm{i})]{\text{Gysin}} F^1 \mathcal{D}^{\bullet-2}_{E_0^{[0]}} \xrightarrow[\cdot(2\pi \mathrm{i})]{\text{Gysin}} \underset{\text{(deg. 0)}}{F^2 \mathcal{D}^\bullet_{\mathcal{E}}}$$

against two-cycles in

$$C^{\mathrm{top}}_\bullet(E_0^{[1]}; \mathbb{Q}) \xleftarrow{\text{intersect}} C^{\mathrm{top}}_\bullet(E_0^{[0]}; \mathbb{Q})_\# \xleftarrow{\text{intersect}} C^{\mathrm{top}}_\bullet(\mathcal{E}; \mathbb{Q})_\#$$

(where "#" means chains and their boundaries properly intersect relevant substrata). If $L_1 = \{(x, y) = (\mu, 0)\}$ and $L_2 = \{(x, y) = (\frac{1}{\mu}, 0)\}$ are the sections of $\mathcal{E}$ and $\Gamma = \{\text{path from } (\mu, 0) \text{ to } (\frac{1}{\mu}, 0) \text{ on } \widetilde{E_0}\}$, then we can pair

$$\begin{aligned}&\left\langle \left(\{1,-1,1,-1\}, \left\{ \frac{dx}{x}, -\frac{dy}{y}, -\frac{dx}{x}, \frac{dy}{y} \right\}, \Omega \right), \right. \\ &\left. (\{0,0,0,0\}, \{\Gamma, 0,0,0\}, L_1 - L_2) \right\rangle \\ &= \int_{L_1-L_2} \Omega \,+\, 2\pi\mathrm{i} \int_\Gamma \frac{dx}{x} = -4\pi\mathrm{i} \log \mu.\end{aligned}$$

So the answer is yes precisely when $\mathcal{E}$ has no nontorsion section, or equivalently when

$$\mu \text{ is a root of unity.}$$

This points the way toward some sort of arithmetic restriction on ϕ. (Indeed, the condition on μ, not that on the sections, is the one which generalizes.)

Now assume $K \subset \bar{\mathbb{Q}}$ is a number field, and take $\phi \in K[\mathbf{T}^n]$. If the celebrated Hodge and Bloch–Beilinson conjectures are assumed to hold, an equivalent problem is

Question 2. *Does the "toric symbol" $\{x_1, \ldots, x_n\} \in H^n_{\mathcal{M}}(\mathbf{T}^n, \mathbb{Q}(n))$, or some other symbol with fundamental class $[\bigwedge^n d\log \underline{x}] \in H^n(\mathbf{T}^n, \mathbb{Q}(n))$, extend to $\Xi \in H^n_{\mathcal{M}}(\mathcal{X}_-, \mathbb{Q}(n))$?*

So, in light of the isomorphisms

$$H^n_{\mathcal{M}}(\mathbf{T}^n, \mathbb{Q}(n)) \cong K_n^{\mathrm{alg}}(\mathbf{T}^n)^{(n)}_{\mathbb{Q}} \cong CH^n(\mathbf{T}^n, n)_{(\mathbb{Q})},$$

the question about periods of the "extended Haar form" is replaced by a question about algebraic K-theory. If one does not assume the conjectures then of course this is a stronger criterion than that in Nori's question; but in fact there are very concrete sufficient conditions for an affirmative answer.

To state these conditions we first fix the specific compactifications we will use (for $n \leq 4$). The *Newton polytope* $\Delta := \mathrm{Newton}(\phi)$ is the convex hull in $\mathbb{R}^n$ of the exponent vectors of all nonzero monomials appearing in ϕ. Assume this (hence ϕ) is *reflexive*, i.e., its polar polytope $\Delta^\circ \subset \mathbb{R}^n$ has only integral vertices; and demand that $1 - t\phi(\underline{x})$ be Δ-regular for general t. This last is a mild genericity condition (cf. [3] or Section 2.5 below). (We actually make a weaker, but more technical, assumption in Theorem 3.1 for $n \leq 3$.) Associated to the fan on Δ° is a (compact) toric Fano n-fold $\mathbb{P}_\Delta \supset \mathbf{T}^n$ where the components of the "divisor at ∞" $\mathbb{D} = \mathbb{P}_\Delta \backslash \mathbf{T}^n$ correspond to the facets of Δ. This is usually too singular, and we replace it by $\mathbb{P}_{\tilde{\Delta}}$,[2] the toric variety associated to the fan on a maximal projective triangulation of Δ°. (In the example, $\mathbb{P}^1 \times \mathbb{P}^1 = \mathbb{P}_{\Delta} = \mathbb{P}_{\tilde{\Delta}}$.) Taking Zariski closure of the level sets

$$1 - t\phi(\underline{x}) = 0$$

then leads to a one-parameter family of $\tilde{\mathcal{X}}$ of anticanonical hypersurfaces $\tilde{X}_t \subset \mathbb{P}_{\tilde{\Delta}}$, i.e., Calabi–Yau $(n-1)$-folds. (Again, as in the example, $\tilde{\mathcal{X}}$ is nothing but $\mathbb{P}_{\tilde{\Delta}}$ blown up along [successive proper transforms of] the components of the base locus. Our actual definition of $\tilde{\mathcal{X}}$ in Sections 3 and 4 is slightly different from that used here; note that $\tilde{\mathcal{X}}$ replaces $\mathcal{X}$ in Questions 1 and 2.) If we define $\tilde{\pi} := \frac{1}{\phi} : \tilde{\mathcal{X}} \to \mathbb{P}^1_t$, two more properties all these families have in common is:

- the local system $R^{n-1}\tilde{\pi}_*\mathbb{Q}$ has maximal unipotent monodromy about $t = 0$ (for $n = 4$ an extra assumption is needed for this; cf. Remark 4.1)
- the relative dualizing sheaf $\omega_{\tilde{\mathcal{X}}/\mathbb{P}^1} := K_{\tilde{\mathcal{X}}} \otimes \tilde{\pi}^{-1}\theta^1_{\mathbb{P}^1}$ has

$$\deg \omega_{\tilde{\mathcal{X}}/\mathbb{P}^1} = 1 \quad \text{(cf. Section 10.3).}$$

We write $\mathcal{L} \subset \mathbb{P}^1$ for the discriminant locus of $\tilde{\pi}$, and $\tilde{D} := \tilde{\mathbb{D}} \cap \tilde{X}_t$ for the base locus of the family.

Also writing in 1997, Rodriguez-Villegas [69] introduced the arithmetic condition on ϕ for $n = 2$, that forces the toric symbol $\xi := \{x_1, x_2\}$ in Question 2 to extend. Namely, by decorating the integral points in Δ with the corresponding coefficients (in some field $K \subset \mathbb{C}$) of monomials in ϕ, the coefficients along each edge of Δ yield a one-variable polynomial. If these "edge polynomials" are cyclotomic, then all Tame symbols of ξ are torsion and Villegas says ϕ is *tempered.* In Section 3 of this paper, Villegas's definition is extended to $n \leq 4$ in order to prove Theorem 3.1, which is a stronger version of the following

Theorem 0.1. *Let $\phi \in K[\mathbf{T}^n]$ ($n \leq 4$, K a number field) be reflexive, tempered, and regular. (For $n = 4$ assume also that K is totally real and that the components of the one-skeleton of $\tilde{\mathbb{D}}$ are rational $/K$.) Then Question 2 (and therefore Question 1) has a positive answer.*

For example, for $n = 3$, given a reflexive $\Delta \subset \mathbb{R}^3$ with only triangular facets, ϕ :={characteristic Laurent polynomial of the vertex set of Δ} will

[2] $\tilde{\mathbb{D}}$ will denote the new divisor at infinity (not a desingularization).

satisfy the Theorem. Conversely, we show (cf. Proposition 4.2) that the toric symbol cannot extend if the coefficients of ϕ do not belong to a number field (up to a common constant factor).

The upshot is that we get in each case a family $\Xi_t := \Xi|_{\tilde{X}_t} \in CH^n(\tilde{X}_t, n)$ of Milnor K_2 (resp. K_3, K_4) classes on elliptic curves (resp. $K3$ surfaces, CY three-folds). In Section 4 we show that these classes are always nontorsion by evaluating their image under the Abel–Jacobi map (or "rational regulator map")

$$\begin{array}{cccc} AJ^{n,n}: & H^n_{\mathcal{M}}(\tilde{X}_t, \mathbb{Q}(n)) & \to & H^n_{\mathcal{D}}(\tilde{X}_t, \mathbb{Q}(n)) \\ & \| & & \| \\ & CH^n(\tilde{X}_t, n) & & H^{n-1}(\tilde{X}_t, \mathbb{C}/\mathbb{Q}(n)) \end{array}$$

against a family of topological cycles $\tilde{\varphi}_t$ vanishing at $t = 0$. This yields the formula (Theorem 4.5)

(0.1)

$$\Psi(t) := \langle \tilde{\varphi}_t, AJ(\Xi_t) \rangle \equiv (2\pi \mathrm{i})^{n-1} \left\{ \log(t) + \sum_{m \geq 1} \frac{[\phi^m]_0}{m} t^m \right\} \quad \mathrm{mod}\ \mathbb{Q}(n)$$

(where $[\cdot]_0$ takes the constant term). The treatment of Theorem 0.1 and formula (0.1) (and other material) becomes rather technical in places, partly from the desire to prove results in sufficient generality to accommodate specific key examples. We have included in Sections 1 and 2 a guide to the regulator formulas and aspects of toric geometry that we use.

A fundamental goal of writing this paper has been to broaden the relevancy of (generalized) algebraic cycles and (generalized) normal functions beyond their traditional context of Hodge theory and motives. In particular, we want to persuade the reader that higher cycles are not just to be sought out in the context of the Beilinson conjectures, but instead also are behind things like solutions of inhomogeneous Picard–Fuchs (IPF) equations — even ones arising in string theory. Already in the context of open mirror symmetry in [61], the domainwall tension for D-branes wrapped on the quintic mirror has been interpreted as the Poincaré normal function associated to a family of algebraic one-cycles. This yields not only the solution of an IPF equation, but also data on "counting holomorphic disks" on the real quintic $\subset \mathbb{P}^4$. The higher cycles we consider in this paper are instead related to the local mirror symmetry setting, and their associated "regulator periods" $\Psi(t)$ furnish the mirror map in that context. Hence for $n = 2$, assuming a conjectural "central charge formula" of Hosono [45], we obtain information on the asymptotics of instanton numbers $\{n_d\}$ for $K_{\mathbb{P}_{\Delta^\circ}}$. This story is worked out in Section 5, with explicit computations connecting the exponential growth rate of the $\{n_d\}$ to limits of AJ mappings in Section 6.

The "higher normal functions" $V(t)$ obtained from our generalized cycles, on the other hand, provide solutions to certain IPF equations (cf. Section 4.3). While we do not know if these play any distinguished role in local mirror symmetry, they do play a central part in the Apéry–Beukers irrationality proofs of $\zeta(2)$ and $\zeta(3)$, and provide a missing link for completing the "algebro-geometrization" of these proofs begun by Beukers, Peters, and Stienstra [15, 16, 67, 68]. We will try to convey this link below, but for a complete discussion/proof the reader is referred to [48].

Another number-theoretic phenomenon on which our construction sheds light is the "modularity" of the logarithmic Mahler measure

$$m(t^{-1} - \phi) := \frac{1}{(2\pi \mathrm{i})^n} \int_{|x_1|=\cdots=|x_n|=1} \log |t^{-1} - \phi| \bigwedge^n d\log \underline{x}. \tag{0.2}$$

Specifically, several authors [9, 59, 69, 77] have noted computationally that (for $n = 2, 3$) pullbacks of (0.2) by the inverse of the mirror map frequently yield Eisenstein–Kronecker–Lerch series. In Corollary 4.4, $\Psi(t)$ is related to (0.2), and in Section 10 we use AJ computations (done in Sections 7–9) for Beilinson's Eisenstein symbol to *prove* a general result on pullbacks of Ψ by automorphic functions (Theorem 10.1). This completely explains the observations on Mahler measures.

One more noteworthy application of Theorem 0.1 is to the splitting of the MHS on the cohomology $H^{n-1}(\tilde{X}_0)$ of the "large complex structure" singular fiber. In fact, whenever Question 1 has a positive answer, taking Poincaré residue of $\Omega \in \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}(0), H^n(\tilde{\mathcal{X}}_-, \mathbb{Q}(n)))$ yields

$$\mathrm{Res}(\Omega) \in \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}(0), H_{n-1}(\tilde{X}_0, \mathbb{Q}))$$

hence (dually) a morphism

$$H^{n-1}(\tilde{X}_0, \mathbb{Q}(j)) \to \mathbb{Q}(j) \tag{0.3}$$

of MHS for any j. Now the cycle Ξ produced by the Theorem obviously does *not* extend through $\tilde{X}_0$. Given a second cycle $\mathfrak{Z} \in CH^j(\tilde{\mathcal{X}} \setminus \cup_i X_{t_i}, 2j - n)$ (all $t_i \in \mathcal{L} \setminus \{0\}$) which *does* extend, together with a family $\omega \in \Gamma(\mathbb{P}^1, \tilde{\pi}_* \omega_{\tilde{\mathcal{X}}/\mathbb{P}^1})$ of holomorphic forms, one has the associated (multivalued) normal function

$$\nu(t) = \langle AJ(\mathfrak{Z}|_{\tilde{X}_t}), \omega(t) \rangle$$

over $\mathbb{P}^1\backslash\mathcal{L}$. If $2j = n$ one must also *assume* that $[\iota^*_{\tilde{X}_0}\mathfrak{Z}] = 0 \in H^{2j}(\tilde{X}_0)$. If we normalize ω so that $\widehat{\omega(0)} := \mathrm{im}\{\omega(0)\} \in H_{n-1}(\tilde{X}_0, \mathbb{C})$ is just $[\mathrm{Res}(\Omega)]$, e.g., one could just take $\omega = \nabla_{\delta_t}[AJ_{\tilde{X}_t}(\Xi_t)]$, then the splitting (0.3) gives "meaning" to

$$\lim_{t\to 0}\nu(t) \in \mathbb{C}/\mathbb{Q}(j), \tag{0.4}$$

that is, nontriviality of (0.4) implies nontriviality of $AJ(\mathfrak{Z}|_{\tilde{X}_t})$ as a section of the sheaf of generalized Jacobians $J^{j,\,2j-n}(\tilde{X}_t)$. This "splitting principle" will be elaborated upon in a future work.

In the remainder of this Introduction, we want to convey some of the main ideas behind these applications (including the ones not done in this paper) through three key examples

$$\phi = \frac{(x-1)^2(y-1)^2}{xy}, \quad n = 2, \tag{0.5}$$

$$\phi = \frac{(x-1)(y-1)(z-1)[(x-1)(y-1) - xyz]}{xyz}, \quad n = 3, \tag{0.6}$$

$$\phi = \frac{x^5 + y^5 + z^5 + w^5 + 1}{xyzw}, \quad n = 4, \tag{0.7}$$

all of which satisfy the strengthened version (Theorem 3.1) of Theorem 0.1.

Begin by considering the sequence

$$-4, -4, -12, -48, -240, -1356, -8428, -56000, -392040, -2859120, \ldots$$

of genus zero local instanton numbers $\{n_d\}_{d\geq 1}$ for $K_{\mathbb{P}^1\times\mathbb{P}^1}$ [21]. The related Gromov–Witten invariants $\{N_d\}$ count (roughly speaking) the contribution to the "number of rational curves of degree d" on a CY three-fold made by an embedded $\mathbb{P}^1 \times \mathbb{P}^1$ (when there is one). They have, according to [59], exponential growth rate

$$\lim_{d\to\infty}\left|\frac{n_{d+1}}{n_d}\right| = \lim_{d\to\infty}\left|\frac{N_{d+1}}{N_d}\right| = \mathrm{e}^{\frac{8}{\pi}G}, \tag{0.8}$$

where $G := 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots$ is Catalan's constant. The exponent of (0.8) also appears as a special value of a hypergeometric integral in a formula

(0.9)

$$\frac{8}{\pi}G = \log(16) - \sum_{n\geq 1}\frac{\binom{2n}{n}^2}{16^n n} = -\lim_{\epsilon\to 0}\left\{\int_\epsilon^{\frac{1}{16}} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 4t\right)\frac{dt}{t} - \log(\epsilon)\right\}$$

essentially due to Ramanujan. The surprising fact is that a family of higher cycles, in K_2^{alg} of a family of elliptic curves, is behind (0.8) and (0.9). In order to illustrate how this works, we shall first offer a brief review of the relevant AJ maps.

To begin with, recall Griffiths's AJ map [42] for one-cycles homologous to zero on a smooth projective three-fold $X/\mathbb{C}$. Writing

$$Z = \sum q_i C_i \in Z^2_{\text{hom}}(X), \quad \square := \mathbb{P}^1 \backslash \{1\},$$

we want to know whether Z is *rationally equivalent to zero*:

$$Z \overset{\text{rat}}{\equiv} 0 \quad \Longleftrightarrow \quad \begin{array}{c} \exists W \in Z^2(X \times \square) \text{ (properly intersecting } X \times \{0, \infty\}) \\ \text{with } W \cdot (X \times \{0\}) - W \cdot (X \times \{\infty\}) = Z. \end{array}$$

Here $q_i \in \mathbb{Q}$, and except where otherwise indicated all cycle groups and intermediate Jacobians in this paper are taken $\otimes\mathbb{Q}$. Also note that $Z^p(X)$ denotes complex codimension p algebraic cycles, while $Z^p_{\text{top}}(X)$ (resp. $C^p_{\text{top}}(X)$) means real codimension p (piecewise) smooth topological cycles (resp. chains). The map[3]

$$Z^2_{\text{hom}}(X) \xrightarrow{\widetilde{AJ}} J^2(X) := \frac{H^3(X, \mathbb{C})}{F^2 H^3(X, \mathbb{C}) + H^3(X, \mathbb{Q}(2))} \cong \frac{\{F^2 H^3(X, \mathbb{C})\}^\vee}{\text{im}\{H_3(X, \mathbb{Q}(2))\}}$$
$$\cong \frac{\{\overbrace{\Gamma_{d\text{-closed}}(F^2 A^3_X)}^{\text{test forms}} / d[\Gamma(F^2 A^2_X)]\}^\vee}{\left\{\int_{Z_3^{\text{top}}(X;\mathbb{Q}(2))}(\cdot)\right\}}$$

induced by

$$Z \longmapsto (2\pi \mathrm{i})^2 \int_{\partial^{-1} Z} (\cdot),$$

where $\partial^{-1} Z \in C_3^{\text{top}}(X; \mathbb{Q})$ is any (piecewise smooth) three-chain bounding on Z, descends modulo $\overset{\text{rat}}{\equiv}$ to yield

$$AJ : CH^2_{\text{hom}}(X) \to J^2(X).$$

This is the type of AJ-map which yields the normal functions considered in [61], and detects classes in $K_0(X)^{(2)} \cong CH^2(X)$.

[3] $A^k_X = \oplus_{p+q=k} A^{p,q}_X$ denotes C^∞ k-forms on X.

Now suppose we have an elliptic curve

$$E \subset \mathbb{P}_\Delta = \text{ toric Fano surface,}$$

and would like to detect classes in

$$K_2(E) \underset{\text{“de-loop”}}{\cong} K_0(E \times \underbrace{\check{C}}_{\substack{\text{nodal}\\ \text{affine}\\ \text{curve}}} \times \check{C}) \cong CH^2(\underbrace{E \times \square^2, E \times \partial\square^2}_{X}) \ ,$$

where the right-hand term is a *relative* Chow group and

$$\partial\square^2 := (\{0,\infty\} \times \square) \cup (\square \times \{0,\infty\}) \subset \square^2.$$

The “relative cycles” $Z = \sum q_i C_i \in Z^2(X)$ are just those whose component curves C_i properly intersect[4] $E \times \partial\square^2$ and satisfy $Z \cdot (E \times \partial\square^2) = 0$, and relative rational equivalences are defined similarly, i.e., $W \in Z^2(E \times \square^3)$ must intersect $E \times \partial\square^3$ properly and have $W \cdot (E \times \partial\square^2 \times \square) = 0$. Writing

$$\begin{aligned} \mathbb{I}^2 &:= (\{1\} \times \mathbb{C}^*) \cup (\mathbb{C}^* \times \{1\}) \subset (\mathbb{C}^*)^2, \\ X^\vee &:= (E \times (\mathbb{C}^*)^2,\, E \times \mathbb{I}^2) \end{aligned}$$

for the “Lefschetz dual” variety, the test forms live on $X^\vee$; and

$$\begin{aligned} J^2(X) := \frac{H^3(X,\mathbb{C})}{F^2H^3(X,\mathbb{C}) + H^3(X,\mathbb{Q}(2))} &\cong \frac{\{F^2H^3(X^\vee,\mathbb{C})\}^\vee}{\mathrm{im}\{H_3(X^\vee,\mathbb{Q}(2))\}} \\ \cong \frac{\{H^1(E,\mathbb{C}) \otimes d\log z_1 \wedge d\log z_2\}^\vee}{\mathrm{im}\{H_1(E,\mathbb{Q}) \otimes S^1 \times S^1\}} &\cong \mathrm{Hom}\left(H^1(E,\mathbb{Q}),\, \mathbb{C}/\mathbb{Q}(2)\right). \end{aligned}$$

To produce a map

$$AJ: \ CH^2(X) \to J^2(X),$$

one first notes that $H^i(\square,\partial\square) = \begin{cases} \mathbb{Q}(0), & i=1 \\ 0 & \text{otherwise} \end{cases}$ which implies that

$$Hg^2(H^4(X)) \cong Hg^2(H^2(E) \otimes \mathbb{Q}(0)^{\otimes 2}) = \{0\}.$$

[4] All coskeleta of i.e., components of $E \times \partial\square^2$, and intersections of these components.

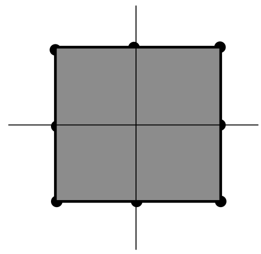


Figure 2: Newton polytope for (0.5).

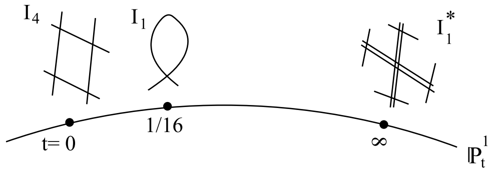


Figure 3: Singular fibers for (0.11).

Thus $CH^2(X) = CH^2_{\text{hom}}(X)$. Hence for any $Z \in Z^2(X)$, we *essentially*[5] have

$$Z = \partial\Gamma \quad \text{in} \quad C^{\text{top}}_{\bullet}(E \times (\mathbb{C}^*)^2,\, E \times \mathbb{I}^2).$$

We can then consider on test forms in $\Gamma_{d\text{-closed}}(A^1_E)$

$$AJ_X(Z) := \int_\Gamma (\cdot) \wedge \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} \in J^2(X), \tag{0.10}$$

which we now turn to computing in one example.

The Laurent polynomial (0.5) has Newton polytope as shown in figure 2, which corresponds to $\mathbb{P}_\Delta = \mathbb{P}^1 \times \mathbb{P}^1$. A projective description of the fibers of $\mathcal{X} \xrightarrow{\pi} \mathbb{P}^1_t$ is then

$$E_t := \{\mathsf{XYZW} = t(\mathsf{X} - \mathsf{W})^2(\mathsf{Y} - \mathsf{W})^2\} \subset \mathbb{P}^1_{\mathsf{X:W}} \times \mathbb{P}^1_{\mathsf{Y:Z}}, \tag{0.11}$$

and after a minimal desingularization at $t = \infty$, π has singular fibers as in figure 3. Now consider the pair of meromorphic functions

$$x := \frac{\mathsf{X}}{\mathsf{W}}, \quad y := \frac{\mathsf{Y}}{\mathsf{Z}} \in \mathbb{C}(E_t)^*$$

[5]For a more precise statement see [50, Section 5.8] and references cited therein.

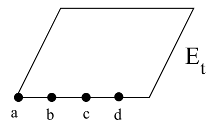


Figure 4: Marked 4-torsion in (0.11).

arising from the toric coordinates; their divisors

$$(x) = 2[b] - 2[d], \quad (y) = 2[a] - 2[c]$$

are supported on marked four-torsion points (see figure 4), and in fact $\mathcal{X}$ is nothing but the modular family over $X_1(4)$.[6] Most importantly, we have the following pair of (chains of) implications

$$x = 0 \text{ or } \infty \quad \Longrightarrow \quad \mathsf{X} \text{ or } \mathsf{W} = 0 \quad \underset{(0.11)}{\overset{\text{use}}{\Longrightarrow}} \quad \mathsf{Y} = \mathsf{Z} \quad \Longrightarrow \quad y = 1,$$

$$y = 0 \text{ or } \infty \quad \Longrightarrow \quad \cdots \quad \Longrightarrow \quad x = 1.$$

Recalling that $1 \notin \square$, if we consider the "graph" (in the sense of calculus, not combinatorics!) of the symbol $\{x, y\}$

$$Z_t := \{(e, x(e), y(e)) \,|\, e \in E_t\} \in Z^2(E \times \square_{z_1} \times \square_{z_2}),$$

then $Z_t \cdot (E \times \partial \square^2) = 0$

$$\Longrightarrow \quad Z_t \in CH^2(X),$$

i.e., Z_t is a relative cycle. Interestingly, this example appears in [22] as the degeneration of a Ceresa cycle on the Jacobian of a nonhyperelliptic genus 3 curve, as that curve acquires two successive nodes.

To construct an explicit three-chain Γ_t bounding on Z_t, we use a procedure similar to that in [13] which was generalized in [47, 50]. First look at the picture of $Z_t \subset E_t \times \square \times \square$ in figure 5. For a first approximation of Γ, "squash" Z_t to $\{1\}$ in the z_1-coordinate and write down the membrane

$$\left\{ (e, \overrightarrow{1.x(e)}, y(e)) \,|\, e \in E \right\} \tag{0.12}$$

which it traces out. Here we recall that for purposes of bounding Z_t, $E_t \times \mathbb{I}^2$ is a sort of "topological trashcan". The path $\overrightarrow{1.x(e)} \subset \mathbb{P}^1 \backslash T_{z_1}$ can be chosen

[6] We use the notation $\overline{Y}_1(4)$ for this in Sections 7–10.

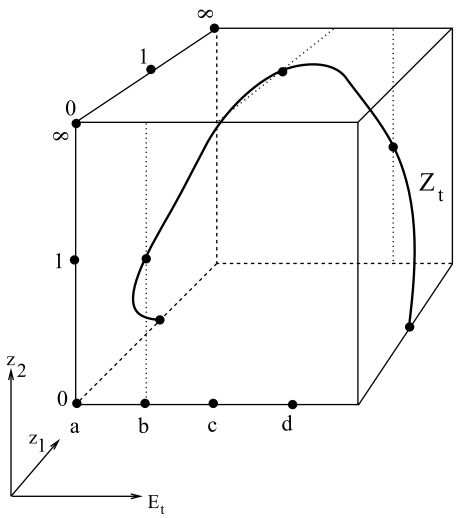


Figure 5: Higher Chow cycle on (0.11).

continuously in $e \in E \backslash T_x$, where $T_x := \{e \in E_t \,|\, x(e) \in \mathbb{R}^{\leq 0} \cup \{\infty\}\}$ is the cut in the branch of $\log(x)$. Along T_x we have a problem, namely that (0.12) has $\{(e, S^1_x, y(e)) \,|\, e \in T_x\}$ as an additional (and unwanted) boundary component. So we squash this component to $\{1\}$ in the z_2-coordinate and continue on, obtaining at last

$$\Gamma_t = \left\{(e, \overrightarrow{1.x(e)}, y(e))\right\}_{e \in E_t} + \left\{(e, S^1_z, \overrightarrow{1.y(e)})\right\}_{e \in T_x} + \{(e, S^1_{z_1}, S^1_{z_2})\}_{e \in \partial^{-1}(T_x \cap T_y)}.$$

Thus (0.10) becomes

$$\begin{aligned} &\int_{\Gamma_t} \omega_E \wedge d\log z_1 \wedge d\log z_2 \\ &= \int_E \omega_E \wedge \log x \, d\log y - 2\pi \mathrm{i} \int_{T_x} \omega_E \log y - 4\pi^2 \int_{\partial^{-1}(T_x \cap T_y)} \omega_E \\ &= \Bigg(\underbrace{\log x \, d\log y - 2\pi \mathrm{i} \log y \delta_{T_x}}_{=: R\{x,y\} \in \mathcal{D}^1(E_t)} - 4\pi^2 \delta_{\partial^{-1}(T_X \cap T_y)} \Bigg) (\omega_E), \end{aligned}$$

where $\mathcal{D}^1$ denotes one-currents; in fact, there is nothing preventing us from taking [Poincaré duals of] topological one-cycles γ as our test forms, and so

$$CH^2(E_t, 2) := CH^2(X_t) \xrightarrow{AJ_{(\mathrm{rel})}} \mathrm{Hom}(H_1(E_t, \mathbb{Q}), \mathbb{C}/\mathbb{Q}(2))$$

is induced (on our cycle) by

$$Z_t \longmapsto \left\{ \gamma \mapsto \int_\gamma R\{x,y\} \right\}. \tag{0.13}$$

Explicit computation on a particular choice of γ_t (using not much more than residue theory; see Section 4.1) yields (0.1), which in this case is

$$\Psi(t) = \int_{\gamma_t} R\{x,y\} \overset{\mathbb{Q}(2)}{\equiv} 2\pi\mathrm{i} \left\{ \log t + \sum_{m\geq 1} \frac{\binom{2m}{m}^2}{m} t^m \right\}. \tag{0.14}$$

Nontriviality of the family of cycles then follows from nonconstancy of the "regulator period" Ψ. Both (0.8) and (0.9) are obtained by computing its value $\Psi(\frac{1}{16})$ at the "conifold point," by pulling back the current $R\{x,y\}$ along a desingularization of the nodal rational curve $E_{\frac{1}{16}}$. (See the "D_5" computation in Section 6.3.) In particular, the relation to the asymptotics of the $\{N_d\}$ (cf. (0.8)) comes from the conjectural *mirror theorem*[7]

$$\frac{1}{(2\pi\mathrm{i})^2} - \sum_{d\geq 1} d^3 N_d Q^d = \frac{\mathcal{Y}(t)}{\left({}_2F_1(\frac{1}{2},\frac{1}{2};1;4t)\right)^3}$$

in which

$$\text{the r.h.s. blows up at } \frac{1}{16}, \text{ and} \tag{0.15}$$

$$\text{the mirror map } Q(t) = \exp\left\{ \frac{\Psi(t)}{2\pi\mathrm{i}} \right\}. \tag{0.16}$$

Equation (0.16) is based on an analysis (Section 5.1) of periods on the (open CY three-fold) mirror manifold of $K_{\mathbb{P}^1\times\mathbb{P}^1}$, which generalizes nicely to higher dimensions (for periods on certain open CY four- and five-folds).

As suggested above, the family of cycles $\{Z_t \in CH^2(X_t,2)\}$ can be canonically constructed on the universal family $\mathcal{E}_1(4) \to Y_1(4) = \Gamma_1(4)\diagdown\mathfrak{H}$ of elliptic curves with a marked four-torsion point. (Similar constructions are possible in any level ≥ 3 and even in higher dimension, by working on *Kuga varieties*, or fiber products of such universal families; this construction is recalled in Section 7.) Using fiberwise double Fourier series for currents on $\mathcal{E}_1(4)$, we obtain a very different expression for the regulator period

[7]The N_d here is actually $N_{2d}^{\langle K_{\mathbb{P}^1\times\mathbb{P}^1}\rangle}$ in Section 5.3.

$\langle \tilde{\varphi}, AJ(Z)\rangle$ as a function of $\tau \in \mathfrak{H}$,

$$\tilde{\Psi}(\tau) \overset{\mathbb{Q}(2)}{\equiv} 2\pi\mathrm{i}\left\{ \frac{2\pi\mathrm{i}}{4}\tau - 4\sum_{\mu\geq 1}\frac{q_0^{\mu}}{\mu}\left(\sum_{r|\mu} r^2\chi_{-4}(r)\right)\right\},$$

where $q_0 = \mathrm{e}^{\frac{2\pi\mathrm{i}}{4}\tau}$. (See Theorem 9.1 and formulas (9.11), (9.16) for the general result.) This must coincide with (0.14) in the sense that

$$\tilde{\Psi}(\tau(t)) \overset{\mathbb{Q}(2)}{\equiv} \Psi(t),$$

where $\tau(t) = \frac{4}{2\pi\mathrm{i}}\log t + t\mathbb{C}[[t]]$ is the period map. The rich interactions between the genus 0 case of the modular/Kuga construction and the toric construction, including a complete classification of the elliptic curve families where the constructions coincide, are explained in Section 10.

Before turning to our next example Laurent polynomial (0.6), we give a brief outline of how the AJ-formulas (0.10), (0.13) for $CH^2(E,2)$ generalize to the setting

$$AJ_X^{p,n} : CH^p(X,n) \to \underbrace{H_{\mathcal{H}}^{2p-n}(X,\mathbb{Q}(p))}_{\text{absolute Hodge cohomology}}.$$

(This will be expanded upon in Section 1; references are [50, Section 5] and [49, Section 8].) Here X is smooth (quasi-projective) and the higher Chow groups satisfy

$$\underbrace{H_{\mathcal{M}}^{2p-n}(X,\mathbb{Q}(p))}_{\text{motivic cohomology}} \cong CH^p(X,n) \cong Gr_{\gamma}^p K_n(X)_{\mathbb{Q}},$$
$$CH^p(X,n) = CH^p(X\times\square^n, X\times\partial\square^n)$$

where $\partial\square^n := \{\underline{z}\in\square^n \,|\, \text{some } z_i = 0 \text{ or } \infty\} \subset \square^n$. When X is singular these isomorphisms fail, but one still has

$$AJ_X^{p,n} : H_{\mathcal{M}}^{2p-n}(X,\mathbb{Q}(p)) \to H_{\mathcal{H}}^{2p-n}(X,\mathbb{Q}(p))$$

which is treated using hyper-resolutions in [49, Section 8].

Recall that the higher Chow groups were defined [14] as the homology of the complex

$$Z^p(X,\bullet) := \frac{\left\{\begin{array}{c}\text{“admissible” cycles in } X\times\square^{\bullet}\text{: components}\\ \text{properly intersect all coskeleta of } X\times\partial\square^{\bullet}\end{array}\right\}}{\{\text{“degenerate” cycles}\}}$$

with differential $\partial_{\mathcal{B}}$ taking the alternating sum of the restrictions to "facets" of $X \times \partial\square^\bullet$. The KLM formula for $AJ^{p,n}$ on X smooth projective (and some quasi-projective cases) is given simply as a map of complexes

$$\begin{aligned} Z^p_{\mathbb{R}}(X,-\bullet) &\to C_{\mathcal{D}}^{2p+\bullet}(X,\mathbb{Q}(p)) \\ &:= C^{2p+\bullet}_{\text{top}}(X;\mathbb{Q}(p)) \oplus F^p\mathcal{D}^{2p+\bullet}(X) \oplus \mathcal{D}^{2p+\bullet-1}(X), \end{aligned} \tag{0.17}$$

where $Z^p_{\mathbb{R}}(X,-\bullet) \subset Z^p(X,-\bullet)$ is a quasi-isomorphic subcomplex. The proper intersection condition is extended to include certain real semi-algebraic subsets of $X \times \square^\bullet$ in order to make the formulas (0.18–20) well-defined (e.g., the intersections of T_{z_i}'s). The (cone) differential on the r.h. complex in (0.17) sends $(a,b,c) \mapsto (-\partial a, -d[b], d[c] - b + \delta_a)$. (0.17) is defined on an irreducible $\mathbb{R}$-admissible cycle $Z \subset X \times \square^n$ by

$$Z \longmapsto (2\pi\mathrm{i})^{p-n}\left((2\pi\mathrm{i})^n T_Z, \Omega_Z, R_Z\right). \tag{0.18}$$

Here T_Z is a C^∞ chain, while Ω_Z and R_Z are currents. Writing

$$\begin{array}{ccc} \square^n_{(z_1,\ldots,z_n)} & \xleftarrow{\pi_\square} & \left\{\begin{matrix}\text{desingularization}\\ \text{of } |Z|\end{matrix}\right\} \\ & & \downarrow{\scriptstyle \pi_X} \\ & & X\,, \end{array}$$

$$\begin{aligned} T_n &:= \bigcap_{i=1}^n T_{z_i} := \bigcap_{i=1}^n \left\{z_i \in (\mathbb{R}^{\le 0} \cup \{\infty\})\right\} \in C^n_{\text{top}}(\square^n) \\ \Omega_n &:= \bigwedge^n d\log z_i := \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n} \in F^n\mathcal{D}^n(\square^n) \\ R_n &:= R\{z_1,\ldots,z_n\} := \sum_{i=1}^n (\pm 2\pi\mathrm{i})^{i-1}\log(z_i)\frac{dz_{i+1}}{z_{i+1}} \wedge \cdots \\ &\qquad \wedge \frac{dz_n}{z_n} \cdot \delta_{T_{z_1}\cap\cdots\cap T_{z_{i-1}}} \in \mathcal{D}^{n-1}(\square^n), \end{aligned} \tag{0.19}$$

the KLM (normal) currents are defined by

$$T_Z := \pi_X\left\{Z \cdot (X \times T_n)\right\}, \quad \left\{\begin{matrix}\Omega_Z \\ R_Z\end{matrix}\right\} := \pi_{X*}\pi_\square{}^*\left\{\begin{matrix}\Omega_n \\ R_n\end{matrix}\right\}. \tag{0.20}$$

Suppose we are given a higher Chow cycle, i.e., a $\partial_{\mathcal{B}}$-closed precycle (= admissible cycle) $Z \in Z^p_{\mathbb{R}}(X,n)$. Then

$$d[R_Z] = \Omega_Z - (2\pi\mathrm{i})^n \delta_{T_Z},$$

or just $-(2\pi\mathrm{i})^n\delta_{T_Z}$ if $\dim X < p$ or $p < n$. So for a symbol $\{\mathbf{f}\} = \{f_1,\dots,f_n\} \in Z^n(U,n)$ (where $f_i \in \mathcal{O}^*(U)$ and U is smooth quasi-projective of dim $< n$), $R_{\{\mathbf{f}\}} = R\{f_1,\dots,f_n\}$ (as in (0.19)) satisfies

$$d[R_{\{\mathbf{f}\}}] = -(2\pi\mathrm{i})^n \delta_{T_{f_1}\cap\cdots\cap T_{f_n}} =: -(2\pi\mathrm{i})^n\delta_{T_{\mathbf{f}}}. \tag{0.21}$$

In Theorem 0.1, $\Xi_t \in Z^n(\tilde{X}_t,n)$ is $\partial_{\mathcal{B}}$-closed and $\dim(\tilde{X}_t) = n-1$; hence

$$R'_{\Xi_t} := R_{\Xi_t} + (2\pi\mathrm{i})^n \delta_{\partial^{-1}T_Z} \in \mathcal{D}^{n-1}(\tilde{X}_t)$$

is d-closed and defines a lift of $AJ(\Xi_t) \in H^{n-1}(\tilde{X}_t, \mathbb{C}/\mathbb{Q}(n))$ to $H^{n-1}(\tilde{X}_t,\mathbb{C})$. This lift is multivalued if t is allowed to vary. We are interested in the higher normal function

$$V(t) := \big\langle [R'_{\Xi_t}], [\omega_t] \big\rangle \tag{0.22}$$

associated to Ξ and a section $\omega \in \Gamma(\mathbb{P}^1, \omega_{\tilde{\mathcal{X}}/\mathbb{P}^1})$ of the dualizing sheaf. If D^{ω}_{PF} is the Picard–Fuchs operator associated to ω (which kills its periods over topological cycles), then nonvanishing of

$$D^{\omega}_{\mathrm{PF}} V(t) =: g_{\Xi,\omega}(t) \in \mathbb{C}(\mathbb{P}^1)$$

implies generic nontriviality of $AJ(\Xi_t)$. This gives a connection to IPF equations, explained in Section 4.3. One way to evaluate (0.22) is to observe that the restriction of Ξ_t to $\tilde{X}^*_t := \tilde{X}_t \cap \mathbf{T}^n$ is $\overset{\mathrm{rat}}{\equiv}$ (by a $\partial_{\mathcal{B}}$-coboundary) to the toric symbol $\{x_1,\dots,x_n\}|_{\tilde{X}^*_t}$, and so

$$[R'_{\Xi_t}|_{\tilde{X}^*_t}] \equiv [R\{x_1|_{\tilde{X}^*_t},\dots,x_n|_{\tilde{X}^*_t}\} + (2\pi\mathrm{i})^n\delta_{\Gamma_t}] \in H^{n-1}(\tilde{X}^*_t,\mathbb{C})$$

for some $\Gamma_t \in C^{\mathrm{top}}_{n-1}(\tilde{X}_t,\tilde{D};\mathbb{Q})$. When we can arrange for Γ_t to vanish (which is true in the calculation below), a careful analytic argument with KLM currents demonstrates that

$$V(t) = \int_{\tilde{X}_t} R\{x_1|_{\tilde{X}_t}, \ldots, x_n|_{\tilde{X}_t}\} \wedge \omega_t. \tag{0.23}$$

What originally got us thinking about higher normal functions was the following integral from a paper [15] of Beukers:

(0.24)
$$\mathcal{R}(\lambda) = \int_0^1 \int_0^1 \int_0^1 \frac{dX\, dY\, dZ}{1-(1-XY)Z-\lambda XYZ(1-X)(1-Y)(1-Z)},$$

with $\mathcal{R}(0) = 2\zeta(3)$. This is the unique linear combination of the generating series of the two sequences $\{a_m\}, \{b_m\}$ used by Apéry to prove irrationality of $\zeta(3)$, with larger radius of convergence than those series. (This leads to Beukers's simpler, geometrically motivated proof.) Substituting $X = \frac{x}{x-1}$, $Y = \frac{y}{y-1}$, $Z = \frac{z}{z-1}$, (0.24) becomes

$$\begin{aligned} &\int\int\int_{T:=T_x\cap T_y\cap T_z} \frac{d\log x \wedge d\log y \wedge d\log z}{\lambda - \frac{(x-1)(y-1)(z-1)(1-x-y+xy-xyz)}{xyz}} \\ &= \int_T \frac{\bigwedge^3 d\log x_i}{\lambda - \phi(\underline{x})} =: \int_T (2\pi \mathrm{i})^3 \hat{\omega}_\lambda, \end{aligned} \tag{0.25}$$

where ϕ is as in (0.6) and (writing $t = \lambda^{-1}$) $\hat{\omega}_\lambda \in \Omega^3(\mathbb{P}_{\tilde{\Delta}})\left\langle \log \tilde{X}_t \right\rangle$ (Δ is shown in figure 6). Differentiating $\hat{\omega}_\lambda$ as a *current* on $\mathbb{P}_{\tilde{\Delta}}$,

$$d[\hat{\omega}_\lambda] = 2\pi\mathrm{i}(\iota_{\tilde{X}_t})_* \mathrm{Res}_{\tilde{X}_t}(\hat{\omega}_\lambda) =: (\iota_{\tilde{X}_t})_* \omega_\lambda \tag{0.26}$$

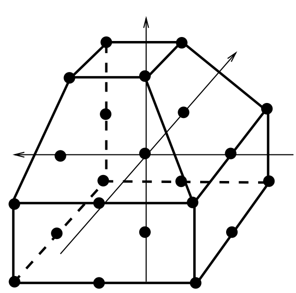


Figure 6: Newton polytope for (0.6).

defines our section $\{\omega_\lambda \in \Gamma(K_{\tilde{X}_t})\}_{t\in\mathbb{P}^1}$ of the dualizing sheaf. Using (0.26) and the generalization

$$d[R_{\{\underline{x}\}}] = \sum\{\text{terms supported on } \tilde{\mathbb{D}}\} + \bigwedge^3 d\log \underline{x} - (2\pi\mathrm{i})^3\delta_T$$

of (0.21) to $\mathbb{P}_{\tilde{\Delta}}$, (0.25) becomes

$$\int_{\mathbb{P}_{\tilde{\Delta}}} (2\pi\mathrm{i})^3\delta_T \wedge \hat{\omega}_\lambda = -\int_{\mathbb{P}_{\tilde{\Delta}}} d[R_{\{\underline{x}\}}] \wedge \hat{\omega}_\lambda$$

$$\overset{\int \text{ by}}{\underset{\text{parts}}{=\!=}} \int_{\mathbb{P}_{\tilde{\Delta}}} R_{\{\underline{x}\}} \wedge \iota_{\tilde{X}_t *}\omega_\lambda = \int_{\tilde{X}_t} R_{\{\underline{x}\}|_{\tilde{X}_t}} \wedge \omega_\lambda,$$

which is (0.23).[8] In fact, $\mathcal{R}(\lambda)$'s interpretation as a higher normal function associated to a family of $K_3(K3)$-classes extending through singular fibers, other than $\lambda = \infty/t = 0$, leads (almost) automatically to the "larger radius of convergence" mentioned above, as well as to its satisfaction of an IPF equation (which then produces a recursion on the $\{b_m\}$).

One knows from [67] that the family of $K3$ surfaces $\tilde{\mathcal{X}}$ associated to (0.6) is the canonical family of Kummer surfaces over $\Gamma_0(6)^{+6}\diagdown\mathfrak{H}^*$. From the toric (Section 4.2) and modular (Section 9.3) computations of the "fundamental regulator period" one gets two rather different expressions

$$\Psi(t) = (2\pi\mathrm{i})^2\left\{\log t + \sum_{m\geq 1}\frac{t^m}{m}\sum_{k=0}^{m}\binom{m}{k}^2\binom{m+k}{k}^2\right\}$$

$$\tilde{\Psi}(\tau) = -12(2\pi\mathrm{i})^3\tau + \frac{(2\pi\mathrm{i})^2}{20}\left\{7\psi_4(q) - 2\psi_4(q^2) + 3\psi_4(q^3) - 42\psi_4(q^6)\right\}$$

(where $q := \mathrm{e}^{2\pi\mathrm{i}\tau}$ and $\psi_4(q) = \sum_{M\geq 1}\frac{q^M}{M}\{\sum_{r|M} r^3\}$) which must coincide modulo $\mathbb{Q}(3)$ under the "period map" $\tau(t) = \frac{\int_{\varphi_1}\omega_t}{\int_{\varphi_0}\omega_t}$ (see Section 10.3).

In general when a toric-hypersurface pencil arising from Theorem 0.1 is modular (in a sense to be made precise in Section 10.3), the limit MHS at $t = 0$ is trivialized by taking $q := \exp(\frac{2\pi\mathrm{i}}{N}\tau(t))$ (for some $N \in \mathbb{Z}$) as the local parameter (or more generally t_0 with $\lim_{t\to 0}\frac{q(t)}{t_0(t)}$ a root of unity). An example of a *non*modular case — with nontrivial LMHS (see Section 10.6) — is

[8] Of course, much of the above needs more thorough justification, as $R\{\underline{x}\}$ is not technically a current on $\mathbb{P}_{\tilde{\Delta}}$, and this will be done in [48].

the mirror quintic family obtained from $\phi = x + y + z + w + \frac{1}{xyzw}$. It follows that the Fermat quintic family $\tilde{\mathcal{X}}$ obtained from (0.7) (of which the mirror quintic is essentially a quotient) also has extensions in $H^3_{\lim}(\tilde{X}_0)$ *not* trivializable by change of parameter. What is still true is that we have the splitting (0.3) of MHS

$$H^3(\tilde{X}_0) \twoheadrightarrow \mathbb{Q}(0)$$

induced by $\langle \cdot , \widehat{\omega(0)} \rangle$, and inducing

$$J^2(\tilde{X}_0) \xrightarrow{\theta} \mathbb{C}/\mathbb{Q}(2).$$

This follows from the existence of Ξ in the theorem, and is *false* if we change the coefficients in (0.7) (e.g., writing instead $\phi = \frac{x^5+2y^5+7z^5+w^5+1}{xyzw}$) without regard for the "generalized temperedness" criterion.

Sticking with the Fermat family, here is why this is important. Let $\mathsf{D}^* := \mathsf{D}\setminus\{0\} \subset \mathbb{P}^1$ be a punctured disk about $t = 0$, and suppose we are given a "local" family of cycles $\{Z_t \in Z^2_{\text{hom}}(\tilde{X}_t)\}_{t\in\mathsf{D}^*}$ satisfying $\mathfrak{Z}^* := \cup_{t\in\mathsf{D}^*} Z_t \overset{\text{hom}}{\equiv} 0$ on $\tilde{\pi}^{-1}(\mathsf{D}^*) \subset \tilde{\mathcal{X}}$. Then by Green *et al.* [40, Section III.B] $\lim_{t\to 0} AJ_{\tilde{X}_t}(Z_t) \in J^2(\tilde{X}_0)$ is well-defined as an invariant of the family of *rational equivalence classes*, and by applying θ so is $\theta(\lim_{t\to 0} AJ_{\tilde{X}_t}(Z_t)) = \lim_{t\to 0} \nu(t) =: \nu(0)$ (cf. (0.4)). In [40, Section IV.C] such a family is constructed, with

$$\Im(\nu(0)) = D_2(\sqrt{-3}),$$

and so the general $Z_t \overset{\text{rat}}{\not\equiv} 0$. Here, D_2 denotes the Bloch–Wigner function.

To conclude, we comment on a few intriguing issues arising in the present work, which might form the basis for later projects. We would like to have a better understanding of the geometry of families of $K3$ surfaces supporting K_3-classes which are *not* Eisenstein symbols. There are scores of Laurent polynomials $\phi \in \mathbb{Q}[\mathbf{T}^3]$ satisfying Theorem 3.1, corresponding to (at least) about a quarter of the 4319 reflexive polytopes in $\mathbb{R}^3$; see Section 3.3. We are only able to show that the generic Picard number $\text{rk}(\text{Pic}(X_\eta)) = 19$ for a handful of these. While there are techniques for obtaining *lower* bounds on this number, we are aware of no methods for (nontrivially) bounding it *above*. Do any of the families have *generic* Picard rank < 19? Are any of them not elliptic fibrations? In fact, on those that admit a *torically defined* elliptic fiber structure, we are able to construct (and partially evaluate the regulator on) families of K_1-classes.

For CY three-folds, it turns out that *none* of the K_4-classes constructed by Theorem 3.1 are Eisenstein symbols, because none of the allowed CY families are classically modular (Proposition 10.3). This would likely be remedied by generalizing the construction to admit singularities on the generic fiber as we have done for $K3$'s; this hard work has yet to be done.

The conjectural mirror theorem of Section 5.4 relates Hodge theory of the (open CY three-fold) B-model family $Y_t := \{1 - t\phi(\underline{x}) + u^2 + v^2 = 0\} \subset (\mathbb{C}^*)^2 \times \mathbb{C}^2$ to enumerative geometry of the (A-model) total space of the canonical bundle $K_{\mathbb{P}_{\Delta^\circ}}$. But the mirror map and the VHS $H^3(Y_t)$ are determined from the data of the underlying elliptic curve family $X_t^* = \{1 - t\phi(\underline{x}) = 0\} \subset (\mathbb{C}^*)^2$ and the toric symbol $\{x_1, x_2\} \in K_2(X_t^*)$ (whose AJ class in $\mathrm{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), H^1(X_t^*, \mathbb{Q}(2)))$ projects to $H^3(Y_t)$, cf. Proposition 5.1ff). The mirror X° of $\{X_t\}$ is the (elliptic curve) zero locus of a section of $K^\vee_{\mathbb{P}_{\Delta^\circ}}$. Is it possible to recast the Gromov–Witten invariants of $K_{\mathbb{P}_{\Delta^\circ}}$ directly in terms of X°, and thus rewrite the mirror theorem in terms of $X_t \longleftrightarrow X^\circ$? A starting point might be to think of $H^{\text{even}}(K_{\mathbb{P}_{\Delta^\circ}})$ as an extension of $H^{\text{even}}(X^\circ)$ by $\mathbb{Q}(0)$ and reduce the quantum product to one on $H^{\text{even}}(X^\circ)$.

Collino [22] has studied the behavior of the Ceresa cycle associated to a nonhyperelliptic genus 3 curve as this curve acquires two successive nodes. Working modulo two-isogenies, with each degeneration a $\mathbb{G}_m$ splits off from the (Jacobian) abelian variety on which the cycle sits. Under this process $CH^2\begin{pmatrix}\text{abelian}\\\text{three-fold}\end{pmatrix} \rightsquigarrow CH^2\begin{pmatrix}\text{abelian}\\\text{surface}\end{pmatrix}, 1\Big) \rightsquigarrow CH^2\Big(\begin{matrix}\text{elliptic}\\\text{curve}\end{matrix}, 2\Big)$, the Ceresa cycle limits to the Eisenstein symbol over $Y_1(4)$, which should be thought of as the intersection of two boundary components in moduli space. Obviously this admits generalization, essentially by considering moduli of genus 3 Jacobians with level N structure. It is of great interest, therefore, to attempt a modular computation of the normal function for such "modular Ceresa cycles," which should limit to an integral of an Eisenstein series. Certain singularities of this normal function in the sense of Griffiths and Green [39] (equivalently, the residues of the corresponding Hodge class [39]), must then be given by the rational residues (in the sense of Section 7.1.5 below) of "$\mathbb{Q}$-Eisenstein series" $E_3^{\mathbb{Q}}(N)$. It is a fundamental property of Eisenstein series that they are determined by their residues.

In fact, there is a beautiful analogy between the picture in Section 4 of [39] and the Eisenstein situation reviewed in Sections 7–8.1. Given a projective variety X^{2p}, a (p,p)-class ζ, and a sufficiently ample line bundle $\mathcal{L} \to X$, the infinitesimal invariant of ζ (pulled back to the incidence variety $\mathcal{X} \subset X \times \mathbb{P}H^0(\mathcal{O}_X(\mathcal{L}))$) maps to certain "residues" over higher-codimension substrata of $X^\vee \subset \mathbb{P}H^0(\mathcal{O}_X(\mathcal{L}))$. An explicit form of Deligne's "Hodge $\Longrightarrow$ Absolute Hodge" conjecture, is that this map should be injective on Hodge classes[9] — that is, that the *rational* $(p,\ p)$ classes are "generalized $\mathbb{Q}$-Eisenstein series." That all such should be motivated by a "generalized Eisenstein symbol" is, of course, the Hodge Conjecture. In the context of Kuga varieties over modular curves (and higher cycles), we have spelled out how Beilinson's work established the relevant (Beilinson-)Hodge Conjecture in Sections 7–8.1 below.

1. Review of the KLM formula

In this expository section, we review a construction of the motivic cohomology groups $H^q_{\mathcal{M}}(X, \mathbb{Q}(p))$ for varieties with "reasonable" singularities, first putting some meat on the bones of the description of higher Chow cycles and formulas for AJ maps from the Introduction. Our choice of material is geared toward what is needed for the reader to follow specific computations in later sections.

1.1. Higher cycle groups and their properties

Let $\square^n := (\mathbb{P}^1 \backslash \{1\})^n$ with coordinates $(z_1, \ldots, z_n)$. For a multi-index $J \subset \{1, \ldots, n\}$ and function $\mathfrak{f} : J \to \{0, \infty\}$, define subsets $\partial^J_{\mathfrak{f}} \square^n := \bigcap_{j \in J} \{z_j = \mathfrak{f}(j)\}$, and put $\partial \square^n := \bigcup_j (\partial^j_0 \square^n \cup \partial^j_\infty \square^n)$. One has obvious inclusion and projection maps

$$\imath_{j,\epsilon} : \square^{n-1} \hookrightarrow \square^n \quad (z_1, \ldots, z_{n-1}) \mapsto (z_1, \ldots, z_{j-1}, \epsilon, z_j, \ldots, z_{n-1})$$

($\epsilon = 0$ or ∞) and

$$\pi_j : \square^n \twoheadrightarrow \square^{n-1} \quad (z_1, \ldots, z_n) \mapsto (z_1, \ldots, \widehat{z_j}, \ldots, z_n).$$

Let X be an algebraic variety defined over an infinite field k, and $Z^p(X \times \square^n)$ the abelian group of codimension-p algebraic cycles defined over k. (It is generated by closed irreducible subvarieties of codimension p.) The *admissible subvarieties* $Z \subset X \times \square^n$ of codimension p, are those for which $Z \cap (X \times \partial^J_{\mathfrak{f}} \square^n)$ has codimension p in $X \times \partial^J_{\mathfrak{f}} \square^n$ $(\forall\ J, \mathfrak{f})$ – i.e. "Z meets

[9]The point is that the map preserves $\mathbb{Q}$-structure and the target $\mathbb{Q}$-structure is "algebraic" (in the sense of being Galois-invariant).

$X \times \partial_{\mathfrak{f}}^{J}\square^{n}$ properly" — and they generate the subgroup

$$c^p(X,n) \subset Z^p(X \times \square^n)$$

of *admissible cycles*. Its quotient by the *degenerate* cycles

$$d^p(X,n) := \sum_j \pi_j^* c^p(X,n-1) \ \subset \ c^p(X,n)$$

defines the *higher Chow precycles*

$$Z^p(X,n) := \frac{c^p(X,n)}{d^p(X,n)}.$$

The pullback/intersection maps

$$\imath_{j,\epsilon}^* : c^p(X,n) \to c^p(X,n-1)$$

are well-defined on admissible cycles. Writing ∂_ϵ^j for the map induced on $Z^p(X,n)$'s, we may define the Bloch differential

$$\begin{gathered}\partial_{\mathcal{B}} : Z^p(X,n) \to Z^p(X,n-1)\,, \\ Z \mapsto \sum_j (-1)^j (\partial_0^j - \partial_\infty^j) Z\,,\end{gathered}$$

which satisfies $\partial_{\mathcal{B}} \circ \partial_{\mathcal{B}} = 0$. A *higher Chow cycle* is a precycle $Z \in \ker(\partial_{\mathcal{B}})$, and the *higher Chow groups* are

$$CH^p(X,n) := H^{-n}\left\{Z^p(X,-\bullet),\partial_{\mathcal{B}}\right\} = \frac{\ker\{\partial_{\mathcal{B}} : Z^p(X,n) \to Z^p(X,n-1)\}}{\operatorname{im}\{\partial_{\mathcal{B}} : Z^p(X,n+1) \to Z^p(X,n)\}},$$

the class of Z in $CH^p(X,n)$ is written $\langle Z \rangle$. There are good reasons for writing this as a cohomology (rather than homology) group; the drawback, of course, is the awkward negative indices. The Bloch–Grothendieck–Riemann–Roch theorem then says that for X smooth

$$K_n^{\mathrm{alg}}(X)_{\mathbb{Q}} \cong \bigoplus_p CH^p(X,n)_{\mathbb{Q}}, \tag{1.1}$$

where the subscript $\mathbb{Q}$ means $\otimes\mathbb{Q}$. More precisely, $CH^p(X,n)_{\mathbb{Q}}$ is the pth Adams graded piece $\mathrm{Gr}_\gamma^p K_n^{\mathrm{alg}}(X)_{\mathbb{Q}}$ of K-theory.

A number of higher Chow groups are familiar: from the geometric side, usual algebraic cycles are

$$CH^p(X) \cong CH^p(X,0),$$

and for X smooth,

$$CH^1(X,1) \cong \Gamma(X,\mathcal{O}_X^*).$$

More generally, since rational equivalences on usual algebraic cycles are given by $\partial_{\mathcal{B}} Z^p(X,1)$, the groups $CH^p(X,1)$ can be thought of as empty rational equivalences.

From the arithmetic side, if we let X be a point $\mathrm{Spec}(k)$, then writing $CH^p(\mathrm{Spec}(k),n) =: CH^p(k,n)$, the Beilinson–Soulé vanishing conjecture (known for $n \leq 3$) says that $CH^p(k,n) = \{0\}$ for $p < \frac{n+1}{2}$. For $n = 2m-1$ odd, if we assume this then one of the extreme terms in (1.1) is $CH^m(k,2m-1)_{\mathbb{Q}}$, which is conjecturally the Bloch group $\mathcal{B}_m(k)_{\mathbb{Q}}$ related to the mth polylogarithm. If k is a number field, then it is known that

$$K_n^{\mathrm{alg}}(k)_{\mathbb{Q}} \cong \begin{cases} 0, & n = 2m, \\ CH^m(k,2m-1)_{\mathbb{Q}}, & n = 2m-1 \end{cases}$$

and that (writing $[k:\mathbb{Q}] = r_1 + 2r_2$)

$$CH^m(k,2m-1)_{\mathbb{Q}} \cong \begin{cases} k^*, & m = 1, \\ \mathbb{Q}^{r_2}, & m \geq 2 \text{ even}, \\ \mathbb{Q}^{r_1+r_2}, & m \geq 3 \text{ odd}. \end{cases}$$

For example, $CH^2(k,3) = \{0\}$ for k totally real ($r_2 = 0$), a fact which we shall use repeatedly. On the other hand, an example of a higher Chow cycle with *non*trivial class, for $k = \mathbb{Q}(\zeta_3)$, is

$$\begin{aligned} Z := & \left\{ \left(1 - \frac{\zeta_3}{t}, 1-t, t\right) \middle| \, t \in \mathbb{P}^1 \right\} \cap \square^3 \\ & + \frac{1}{3} \left\{ \left(1 - \zeta_3, \frac{(t-\zeta_3)^3}{(t-1)^3}, t\right) \middle| \, t \in \mathbb{P}^1 \right\} \cap \square^3 \\ & \in \ker(\partial_{\mathcal{B}}) \quad \subset \quad Z^2(\mathbb{Q}(\zeta_3),3). \end{aligned} \tag{1.2}$$

For more general fields, the other extreme term in (1.1) is the Milnor K-group

$$CH^n(k,n) \cong K_n^M(k) \tag{1.3}$$

(isomorphism due independently to Totaro [79], Nesterenko and Suslin). This is the nth graded piece of the quotient of the exterior algebra $\bigwedge_{\mathbb{Z}}^{\bullet} k^*$ by the ideal generated by terms $\alpha \wedge (1-\alpha)$ (for $\alpha \in k\backslash\{0,1\}$). Alternatively, $K_n^M(k)$ is the free abelian group generated by the symbols $\{\alpha_1, \ldots, \alpha_n\}$, modulo the relations subgroup generated by all elements of the form:

$$\{\alpha_1, \ldots, \alpha_j, \ldots, \alpha_n\} - \{\alpha_1, \ldots, \beta, \ldots, \alpha_n\} - \{\alpha_1, \ldots, \gamma, \ldots, \alpha_n\},$$

where $\alpha_j = \beta\gamma$,

$$\{\alpha_1, \ldots, \alpha_i, \ldots, \alpha_j, \ldots, \alpha_n\} + \{\alpha_1, \ldots, \alpha_j, \ldots, \alpha_i, \ldots, \alpha_n\},$$

and

$$\{\alpha_1, \ldots, \alpha_n\} \quad \text{where } \alpha_i + \alpha_j = 1.$$

Obviously these imply further relations, for example $\{\alpha_1, \ldots, \beta^m, \ldots, \alpha_n\} = m\{\alpha_1, \ldots, \beta, \ldots, \alpha_n\}$ and $\{\alpha_1, \ldots, 1, \ldots, \alpha_n\} = 0$; and if one is working $\otimes\mathbb{Q}$, also $\{\alpha_1, \ldots, \alpha_n\} = 0$ when $\alpha_j = -\alpha_i$, and $\{\alpha_1, \ldots, \zeta, \ldots, \alpha_n\} = 0$ if ζ is a root of 1.

The isomorphism (1.3) is induced simply by sending a symbol $\{\alpha_1, \ldots, \alpha_n\}$ to the point $(\alpha_1, \ldots, \alpha_n) \in \square^n\backslash\partial\square^n$ viewed as an admissible zero-cycle (unless some $\alpha_i = 1$, in which case the symbol is sent to 0). When $k = K(\mathcal{X})$ for $\mathcal{X}$ smooth over K, one thinks of $\mathrm{Spec}(k)$ as the generic point $\eta_{\mathcal{X}}$. If $\dim_K \mathcal{X} = \mathsf{d}$, then the zero-cycle (over k) corresponding to a symbol $\{f_1, \ldots, f_n\} \in K_n^M(K(\mathcal{X}))$ should be thought of as the restriction to $\eta_{\mathcal{X}}$ of the d-cycle defined (over K) by the graph of the n meromorphic functions $\{f_j \in K(\mathcal{X})^*\}$. More precisely, if we let $\mathcal{U} = \mathcal{X}\backslash\{\cup_{j=1}^n |(f_j)|\}$, then this graph

$$\{(x\,;\, f_1(x), \ldots, f_n(x)) \,|\, x \in \mathcal{U}\} \subset \mathcal{U} \times \square^n$$

is a $\partial_{\mathcal{B}}$-closed admissible precycle; we write $\{\mathbf{f}\} = \{f_1, \ldots, f_n\} \in Z^n(\mathcal{U}, n)$ (still called a "symbol") and $\langle\{\mathbf{f}\}\rangle \in CH^n(\mathcal{U}, n)$. It restricts to the "synonymous" Milnor K-theory element in $CH^n(\eta_{\mathcal{X}}, n) \cong K_n^M(K(\mathcal{X}))$. In the constructions we study below, $\langle\{\mathbf{f}\}\rangle$ will frequently extend to a class in $CH^n(\mathcal{X}, n)$, even as the closure of $\{\mathbf{f}\}$ in $\mathcal{X} \times \square^n$ fails to be admissible. The mechanisms for dealing with this are the Bloch moving lemma, residue maps and the localization sequence, which we now explain from a general perspective.

Let $F: Y \to X$ be a proper morphism of varieties over k, of relative dimension r; push-forward of cycles induces a homomorphism

$$CH^p(Y,n) \xrightarrow{F_*} CH^{p-r}(X,n).$$

On the other hand, if F is *any* morphism of *smooth* varieties, then there is a pullback homomorphism

$$CH^p(X,n) \xrightarrow{F^*} CH^p(Y,n),$$

though it is not in general well-defined on cycles $Z \in Z^p(X,n)$ (e.g., Z may not intersect $\mathrm{im}(F)$ properly). We will say how to deal with this in Section 1.3.

Here we only need the case of $F = \jmath : Y \hookrightarrow X$ an open embedding, where (for restriction of cycles $Z \mapsto \jmath^* Z$) no issues arise. Write $D = X \backslash Y$ for the complement, which we assume is of pure codimension 1 in X. (While X is smooth, D can be singular.)

The Bloch moving lemma [11] says that

$$\frac{Z^p(X,\bullet)}{Z^{p-1}(D,\bullet)} \xrightarrow{\jmath^*} Z^p(Y,\bullet)$$

is a quasi-isomorphism. Intuitively, this means that we can modify (or "move") a $\partial_{\mathcal{B}}$-closed precycle on Y by adding a $\partial_{\mathcal{B}}$-exact cycle, so that it extends to an admissible precycle on X. Since $\partial_{\mathcal{B}}$ of this extended precycle is supported on D, we get a residue map

$$\mathrm{Res}: \ CH^p(Y,n) \to CH^{p-1}(D,n-1).$$

This fits in the long-exact *localization sequence*

$$\to CH^p(X,n) \overset{\jmath^*}{\to} CH^p(Y,n) \overset{\mathrm{Res}}{\to} CH^{p-1}(D,n-1) \overset{\imath_*}{\to} CH^p(X,n-1) \overset{\jmath^*}{\to},$$

which says that for extending a higher cycle-*class* $\langle Z \rangle$ from Y to X, we must only check vanishing of $\mathrm{Res}(\langle Z \rangle)$. Nothing like this happens for *ordinary* algebraic cycles, which always extend.

The difficulty with this is that D may be singular, in which case it is not necessarily practical to directly check vanishing of something in its higher Chow groups. It is better to break it into smooth substrata and check vanishing of classes on these, an idea which leads to the *local-global spectral* *sequence.* Writing $\mathsf{d} = \dim(X)$, let

$$\emptyset \subseteq D^{\mathsf{d}} \subseteq D^{\mathsf{d}-1} \subseteq \cdots \subseteq D^2 \subseteq D^1 = D$$

be a filtration of D by subvarieties D^j of pure dimension $\mathsf{d} - j$, with each $D^{j,*} := D^j \backslash D^{j+1}$ smooth. Putting

$$E(p)_1^{a,b} := \begin{cases} CH^{p-a}(D^{a,*}, -a-b), & a \geq 1, \\ CH^p(Y, -b), & a = 0, \\ 0 & a < 0. \end{cases}$$
$$d_1 = \mathrm{Res}: \ E(p)_1^{a,b} \to E(p)_1^{a+1,b}$$

leads to a fourth-quadrant spectral sequence converging to $CH^p(X, -a-b)$. In particular,

$$\begin{aligned} &\mathrm{im}\,\{CH^p(X,n) \to CH^p(Y,n)\} \\ &\quad \cong E(p)_\infty^{0,-n} = \{(\cap_{j\geq 1} \ker(d_j)) \subset CH^p(Y,n)\}\,, \end{aligned}$$

where the target of each d_j is a subquotient of $CH^{p-j}(D^{j,*}, n-1)$. How to compute the d_j for $j \geq 2$ is described in [47]; also see [49, Section 3.4].

1.2. Abel–Jacobi maps for higher cycles

For most of this paper we shall work rationally, that is, all cycle groups are implicitly $\otimes\mathbb{Q}$ (and we omit the subscript $\mathbb{Q}$); one exception is Section 5 where the AJ computation on $CH^2(X^*_{\underline{a}}, 2)$ is done integrally. Henceforth we adopt this convention, and assume that the field of definition k for X is a subfield of $\mathbb{C}$. In this subsection we also take X to be smooth, and let $X^{an}_{\mathbb{C}}$ denote the complex analytic space associated to $X \otimes_k \mathbb{C}$. Note that $\mathbb{Q}(p) = (2\pi\sqrt{-1})^p\mathbb{Q}$ has, by convention, Hodge type $(-p, -p)$.

The coarsest invariant attached to a higher Chow cycle is its fundamental class

$$\begin{aligned} \mathrm{cl}^{p,n}_X : CH^p(X,n) \to Hg^{p,n}(X^{\mathrm{an}}_{\mathbb{C}}) &:= \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}(0), H^{2p-n}(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Q}(p))) \\ &\cong F^p H^{2p-n}(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{C}) \cap H^{2p-n}(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Q}). \end{aligned}$$

(We will take the $(\cdot)^{\mathrm{an}}_{\mathbb{C}}$ to be "understood" when required from here on.) This is followed by a secondary invariant, the Abel–Jacobi map

$$AJ_X^{p,n} : \underbrace{\ker(\mathrm{cl}^{p,n})}_{=:CH^p(X,n)_{\mathrm{hom}}} \longrightarrow J^{p,n}(X) := \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), H^{2p-n-1}(X,\mathbb{Q}(p)))$$
$$\cong \frac{W_{2p}H^{2p-n-1}(X,\mathbb{C})}{F^pW_{2p}H^{2p-n-1}(X,\mathbb{C}) + W_{2p}H^{2p-n-1}(X,\mathbb{Q})}.$$

One has the short-exact sequence

$$0 \to J^{p,n}(X) \to H^{2p-n}_{\mathcal{H}}(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Q}(p)) \to Hg^{p,n}(X) \to 0,$$

so that absolute Hodge cohomology (resp. Deligne cohomology if X is projective) and the cycle-class map

$$c_{\mathcal{H}}(\text{resp. } c_{\mathcal{D}}) : CH^p(X,n) \to H^{2p-n}_{\mathcal{H}}(X,\mathbb{Q}(p))$$

collects both pieces of information together. This is how the results of [49,50] are formulated.

The situation can simplify vastly: $Hg^{p,n}(X)$ vanishes if $n > p$, or $p > \mathsf{d}(= \dim X)$, or X is projective and $n \geq 1$; in those cases $CH^p(X,n) = CH^p(X,n)_{\mathrm{hom}}$. When $n \geq p$ or $p \geq \mathsf{d}$, $F^pH^{2p-n-1}(X,\mathbb{C}) = \{0\}$ and $W_{2p}H^{2p-n-1}(X) = H^{2p-n-1}(X)$, so that

$$J^{p,n}(X) \cong H^{2p-n-1}(X, \mathbb{C}/\mathbb{Q}(p))$$
$$\cong \mathrm{Hom}(H_{2p-n-1}(X,\mathbb{Q}), \mathbb{C}/\mathbb{Q}(p)).$$

If X is a point, then $J^{p,n}(X) = 0$ unless $n = 2p-1$, in which case it is $\mathbb{C}/\mathbb{Q}(p)$.

These invariants are functorial with respect to pullback, pushforward, and residue maps. Here is a special case which gets substantial use in Sections 4 and 5: let $Y \subset X$ be a Zariski open subset with complement $D = \cup D_i$, where the D_i are irreducible hypersurfaces and $D_i^* := D_i \backslash \{\cup_{j\neq i}(D_j \cap D_i)\}$ are smooth. Given $\Xi \in CH^{n+1}(Y, n+1)$ where $\mathsf{d} = n$, let $\xi_i \in CH^n(D_i^*, n)$ be the residues of Ξ on the D_i^*. Consider topological cycles $\gamma_i \in Z^{\mathrm{top}}_{n-1}(D_i^*)$ of real dimension $n-1$ and let $\Gamma \in C^{\mathrm{top}}_{n+1}(X\backslash\{\cup_{i<j} D_i \cap D_j\})$ be such that $\Gamma \cap D_i^* = \gamma_i$ for each i; and put $\gamma = \partial\Gamma \in Z^{\mathrm{top}}_n(Y)$. Then noting that $J^{n,n}(Y) \cong$

$\mathrm{Hom}(H_{n-1}(X,\mathbb{Q}),\mathbb{C}/\mathbb{Q}(n))$ etc., we have that (mod $\mathbb{Q}(n+1)$)

$$AJ_Y^{n+1,n+1}(\Xi)(\gamma) = 2\pi\sqrt{-1}\sum_i AJ^{n,n}_{D_i^*}(\xi_i)(\gamma_i). \tag{1.4}$$

One writes $\gamma = \mathrm{Tube}(\{\gamma_i\})$, $\{\xi_i\} = \mathrm{Res}(\Xi)$, and says that Res and Tube are adjoint.

The $\{AJ^{p,n}\}$ are frequently called *regulator maps* due to their close relationship with the Beilinson regulator. Assume X is projective and defined over a number field F; then by composing structure morphisms $X \to \mathrm{Spec}(F) \to \mathrm{Spec}(\mathbb{Q})$ we may actually view X as a variety over $\mathbb{Q}(=: k)$. Only then, $X_{\mathbb{C}}^{\mathrm{an}}$ looks like the disjoint union of all Galois conjugates of the original X/k. Applying $c_{\mathcal{D}}^{p,n} = AJ^{p,n}$ for this X to those cycle classes which lift to an integral model $\mathbb{X} \to \mathrm{Spec}(\mathcal{O}_F)$, and composing with the projection to real Deligne cohomology yields[10]

$$r_{\mathrm{Be}}^{p,n} : CH^p(\mathbb{X},n) \to \left(H_{\mathcal{D}}^{2p-n}(X_{\mathbb{C}}^{\mathrm{an}},\mathbb{R}(p))\right)^{\mathrm{DR}}.$$

Now suppose $n \geq 2$ (or $n = 1$, but with additional fiddling). The right-hand side has a natural rational substructure which allows one to measure the covolume of the image up to a multiplicative rational constant, and the *Beilinson conjectures* assert that this is $\underset{\mathbb{Q}^*}{\sim} L(H^{2p-n-1}(X),p)$. (When $X = \mathrm{Spec}(F)$ this is essentially Borel's *theorem*.) This relation to the cohomological L-function is the source of the arithmetic interest of the AJ maps.

Continuing to assume X smooth projective, but defined over any subfield of $\mathbb{C}$, we define a map of complexes of the form (0.17) inducing $AJ_X^{p,n}$. In order that the currents which we shall associate to precycles be well-defined, we must further restrict what it means for these precycles to be admissible. First, for any meromorphic function $f \in \mathbb{C}(\mathcal{X})$ on a smooth quasi-projective variety, let T_f be the real-codimension-1 chain $\overline{f^{-1}(\mathbb{R}^-)}$ oriented so that $\partial T_f = (f)$. For $j \notin I \subset \{1,\ldots,n\}$, $\mathfrak{f} : I \to \{0,\infty\}$, write $\partial_{\mathfrak{f},\mathbb{R}}^{I,j}\square^n = \partial_{\mathfrak{f}}^I\square^n \cap \left\{\cap_{\ell\notin I,\ell\leq j} T_{z_\ell}\right\}$ (and $\partial_{\mathfrak{f}}^I\square^n$ for $j=0$). Then the subcomplex of $\mathbb{R}$-*admissible cycles*

$$Z_{\mathbb{R}}^p(X,-\bullet) \subset Z^p(X,-\bullet)$$

[10] "DR" is an involution on real Deligne cohomology; cf. [46] or [71] for more details on this paragraph.

is defined by demanding that cycles meet properly (as real analytic varieties) all $X \times \partial^{I,j}_{\mathsf{f},\mathbb{R}}\square^n$. In [49, Section 8.2], it is shown that this inclusion is a quasi-isomorphism, so that the $Z^p_{\mathbb{R}}(X,-\bullet)$ still compute $CH^p(X,n)$; and that the restricted cycles satisfy a Bloch moving lemma.

We now describe the terms of the Deligne cohomology complex $C^{2p+\bullet}_{\mathcal{D}}(X, \mathbb{Q}(p))$ from (0.17), which computes $H^{2p+*}_{\mathcal{D}}(X,\mathbb{Q}(p))$. Again for smooth quasi-projective (d-dimensional) $\mathcal{X}$, a-currents $D^a(\mathcal{X})$ are simply functionals on compactly supported C^∞ forms of degree $2\mathsf{d}-a$, with $F^bD^a(\mathcal{X})$ killing $\Gamma_c(F^{\mathsf{d}-b+1}\Omega^{2\mathsf{d}-a}_{\mathcal{X}^\infty})$. Elementary examples include

- the current of integration against a real-codimension-a C^∞-Borel-Moore[11] chain Γ on $\mathcal{X}$, denoted δ_Γ;
- differential a-forms with log poles along subvarieties of $\mathcal{X}$ (and any behavior "at infinity");
- the 0-current $\log f$ (for $f \in \mathbb{C}(\mathcal{X})^*$), which denotes the branch with imaginary part in $(-\pi,\pi)$ and a discontinuity along T_f.

Exterior derivative is defined as the adjoint of that for C^∞ forms, so that e.g.,

$$d[\log f] = \frac{df}{f} - 2\pi\sqrt{-1}\delta_{T_f},$$

and the resulting complex of currents computes de Rham cohomology of $\mathcal{X}$. Now let $T \in C^{2p-n}_{\mathrm{top}}(X;\mathbb{Q}(p))$ be a chain, and $\Omega \in F^pD^{2p-n}(X)$ and $R \in D^{2p-n-1}(X)$ currents, so that $(T,\Omega,R) \in C^{2p-n}_{\mathcal{D}}(X,\mathbb{Q}(p))$; then the (cone) differential is defined by

$$D(T,\Omega,R) := (-\partial T, -d[\Omega], d[R] - \Omega + \delta_T).$$

The KLM formula, which has been given as a map of complexes in the Introduction, simply says that for $Z \in Z^p_{\mathbb{R}}(X,n)$ $\partial_{\mathcal{B}}$-closed, $AJ^{p,n}_X(Z)$ is represented by

$$\left((2\pi\sqrt{-1})^pT_Z,\ (2\pi\sqrt{-1})^{p-n}\Omega_Z,\ (2\pi\sqrt{-1})^{p-n}R_Z\right) \tag{1.5}$$

[11]This means (roughly) that Γ can extend to the "boundary" of $\mathcal{X}$, i.e., should be considered as a relative chain on $(\bar{\mathcal{X}}, \bar{\mathcal{X}}\backslash\mathcal{X})$. More precisely, one works with so called "integral currents," but this level of precision will not concern us below.

in $H^{2p-n}_{\mathcal{D}}(X,\mathbb{Q}(p))$. The meaning of (for example) $R_Z := (\pi_X)_*(\pi_\square)^* R_n \in D^{2p-n-1}(X)$ in (0.20), is that for a C^∞ form $\omega \in \Gamma(\Omega^{2\mathsf{d}-2p+n+1}_{X^\infty})$ on X,

$$\int_X R_Z \wedge \omega \ = \ \int_Z \pi_\square^* R_n \wedge \pi_X^* \omega$$

for Z irreducible; and then $R_{\sum m_i Z_i} := \sum m_i R_{Z_i}$. The classes of T_Z and Ω_Z represent $\mathrm{cl}^{p,n}(Z)$; assuming this is 0 (automatic if $n>0$), there exist $\Gamma \in C^{2p-n-1}_{\mathrm{top}}(X;\mathbb{Q})$ and $\tilde{\Omega} \in F^p D^{2p-n-1}(X)$ with $\partial\Gamma = T_Z$, $d[\tilde{\Omega}] = \Omega_Z$. Adding $D\left((2\pi\sqrt{-1})^p\Gamma, (2\pi\sqrt{-1})^{p-n}\tilde{\Omega}, 0\right)$ to (1.5) gives $(0,0,(2\pi\sqrt{-1})^{p-n}R'_Z)$ where the closed $(2p-n-1)$-current

$$R'_Z := R_Z - \tilde{\Omega} + (2\pi\sqrt{-1})^n \delta_\Gamma$$

now represents a lift of $AJ^{p,n}(Z)$ to $H^{2p-n-1}(X,\mathbb{C})$. For $n=0$, this recovers the Griffiths AJ map.

If $n \geq p$ or $p \geq \mathsf{d}$, $F^p D^{2p-n}(X) = \{0\}$ and

$$R'_Z = R_Z + (2\pi\sqrt{-1})^n \delta_\Gamma.$$

In this range, we are merely after a $\mathbb{C}/\mathbb{Q}(p)$-valued functional on topological $(2p-n-1)$-cycles, and this is just given by

$$\begin{array}{rcl} Z^p(X,n) & \stackrel{AJ^{p,n}}{\longrightarrow} & \mathrm{Hom}(H_{2p-n-1}(X,\mathbb{Q}),\ \mathbb{C}/\mathbb{Q}(p)) \\ Z & \longmapsto & \left\{ [\gamma] \mapsto (2\pi\sqrt{-1})^{p-n} \displaystyle\int_\gamma R_Z \right\} \end{array}$$

since $\int_\gamma (2\pi\sqrt{-1})^p \delta_\Gamma \in \mathbb{Q}(p)$. In fact, this formula works for *quasi*-projective X (cf. [50, Section 5.9]).

To ease their use for the reader, we survey some properties and examples of the R-currents. We have

$$\begin{aligned} R_1 &= \log z_{(1)}, \\ R_2 &= \log z_1 d\log z_2 - (2\pi\sqrt{-1}) \log z_2 \delta_{T_{z_1}}, \\ R_3 &= \log z_1 d\log z_2 \wedge d\log z_3 + (2\pi\sqrt{-1}) \log z_2 d\log z_3 \delta_{T_{z_1}} \\ &\quad + (2\pi\sqrt{-1})^2 \log z_3 \delta_{T_{z_1} \cap T_{z_2}}, \end{aligned}$$

and in general $R_n = R_{n-1} \wedge d\log z_n + (2\pi\sqrt{-1})^{n-1} \log z_n \delta_{T_{n-1}}$. That the KLM formula gives a morphism of complexes is one consequence of the residue formula

$$d[R_n] - \Omega_n + (2\pi\sqrt{-1})\delta_{T_n} = 2\pi\sqrt{-1}\sum_{i=1}^{n}(-1)^i R(z_1,\ldots,\widehat{z_i},\ldots,z_n)\delta_{(z_i)}.$$

Here is another: if in (1.4), we take $\Xi = \{f_1,\ldots,f_{n+1}\}$ ($f_i \in \mathbb{C}(X)^*$, X quasi-projective) and $D_i = |(f_i)|$ (that the $|(f_i)|$ do not share components is a big assumption), then the formula is

$$\int_\gamma \underbrace{R(f_1,\ldots,f_{n+1})}_{R_\Xi} \underset{\mathbb{Q}(n+1)}{\equiv} 2\pi\sqrt{-1}\sum_{i=1}^{n+1}(-1)^i \int_{\gamma_i} \underbrace{R(f_1,\ldots,\widehat{f_i},\ldots,f_{n+1})}_{R_{\xi_i}}.$$

Finally, we look at the AJ map over a point,

$$Z^m(k,2m-1) \longrightarrow \mathbb{C}/\mathbb{Q}(m)$$

sending

$$Z \longmapsto \frac{R_Z}{(2\pi\sqrt{-1})^{m-1}},$$

where $R_Z = \int_Z R_{2m-1}$. If $m=1$ this just sends $\alpha \in k^*$ to $\log\alpha$, a map related (essentially via the r_{Be} discussion above) to the Dirichlet regulator. The remaining maps are tied to the Borel regulator; we shall compute $AJ^{2,3}$ on (1.2) to demonstrate the process. Only the first term $(1-\frac{\zeta_3}{t}, 1-t, t)_{t\in\mathbb{P}^1} =:$ Z_0 will contribute, and $\int_{Z_0} R_3$ is computed by pulling back to $\mathbb{P}^1$. So

$$\begin{aligned}
&AJ(Z)\\
&\quad= \frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{P}^1} R\left(1-\frac{\zeta_3}{t}, 1-t, t\right)\\
&\quad= \frac{1}{2\pi\sqrt{-1}}\int_{\mathbb{P}^1}\Bigg\{\log\left(1-\frac{\zeta_3}{t}\right)\underbrace{d\log(1-t)\wedge d\log t}_{0}\\
&\qquad\quad + 2\pi\sqrt{-1}\log(1-t)d\log(t)\delta_{T_{1-\frac{\zeta_3}{t}}} + (2\pi\sqrt{-1})^2(\log t)\delta_{\underbrace{T_{1-t}\cap T_{1-\frac{\zeta_3}{t}}}_{\emptyset}}\Bigg\}\\
&\quad= -\int_{T_{1-\frac{\zeta_3}{t}}}\log(1-t)d\log t = -\int_0^{\zeta_3}\log(1-t)\frac{dt}{t} = Li_2(\zeta_3).
\end{aligned}$$

In fact, denoting by $\bar{Z}$ the complex conjugate cycle, we obtain

$$AJ(Z-\bar{Z}) = Li_2(\zeta_3) - Li_2(\bar{\zeta}_3) = \sqrt{-3}L(\chi_{-3},2).$$

1.3. Higher cycles on singular varieties

Let X be smooth projective over $k \subset \mathbb{C}$, and $V \overset{\imath}{\subset} X$ a nonsingular closed subvariety. Define $Z^p(X,n)_V \subset Z^p(X,n)$ to consist of those admissible precycles which meet all $V \times \partial_{\mathfrak{f}}^I \square^n$ properly. Cycle-theoretic intersection $Z \mapsto Z \cdot (V \times \square^n)$ then defines a morphism of complexes

$$\imath^* : Z^p(X,-\bullet)_V \to Z^p(V,-\bullet).$$

Levine's moving lemma says that $Z^p(X,-\bullet)_V \hookrightarrow Z^p(X,-\bullet)$ is a quasi-isomorphism, so that $\imath^*$ induces pullback maps

$$CH^p(X,n) \to CH^p(V,n).$$

Replacing V by a finite collection $\mathcal{S} = \{S_\alpha\}$ of (possibly singular) closed subvarieties, we define $Z^p(X,-\bullet)_{\mathcal{S}}$ by imposing the proper intersection condition with respect to each $S_i \times \partial_{\mathfrak{f}}^I \square^n$. This still yields a (quasi-isomorphic) subcomplex computing $CH^p(X,n)$. (There is a version of this whole story for $Z^p_{\mathbb{R}}$'s too, cf. [49].)

If V is singular, then the best possible pullback maps are not to higher Chow groups, since these play the role of motivic "Borel–Moore homology" in general and pullback is most natural for *cohomology* groups. To construct the *motivic cohomology groups* $H^{2p-n}_{\mathcal{M}}(V,\mathbb{Q}(p))$ ($\cong CH^p(V,n)$ for smooth), one first replaces V by a diagram of smooth quasi-projective varieties called a *hyper-resolution*. Taking $Z^p(\cdot,-\bullet)$ of this diagram, the associated simple complex then computes $H_{\mathcal{M}}$. In what follows we explain how to do this in the cases required below, in ad hoc fashion. The general procedure is described for example in [53].

First, suppose $V = \cup_{i=1}^N V_i$ is a "smooth normal crossing divisor", in particular that all $V_I = \cap_{i\in I} V_i$ are smooth of dimension $\mathsf{d} - |I|$. Denote by V^I the collection of all V_J with $J \supsetneq I$, and put

$$Z_V^{a,b}(p) := \underset{|I|=a+1}{\oplus} Z^p(V_I,-b)_{V^I} \tag{1.6}$$

with differentials $\partial_{\mathcal{B}} : Z^{a,b} \to Z^{a,b+1}$ and

$$\sum_{|I|=a+1} \sum_{i \notin I} (-1)^{\langle i \rangle_I} (\imath_{V_{I\cup\{i\}} \subset V_I})^* = \mathfrak{I} : \; Z^{a,b} \to Z^{a+1,b},$$

where $\langle i \rangle_I :=$ the position of i in $\{1, \ldots, N\} \backslash I$. Then (1.6) is a double complex; and its associated simple complex $Z_V^\bullet(p) := \oplus_{a+b=\bullet} Z_V^{a,b}(p)$ (differential $\mathbb{D} = \partial_{\mathcal{B}} + (-1)^b \mathfrak{I}$) has $H^{-n}(Z_V^\bullet(p)) \cong H_{\mathcal{M}}^{2p-n}(V, \mathbb{Q}(p))$. The pullback map from $CH^p(X,n)$ to this is given by sending $Z \in Z^p(X,n)_{\{V_I\}_{I \subset \{1,\ldots,N\}}}$ to the element of $Z_V^{-n}(p)$ consisting of $\{Z \cdot (V_i \times \square^n)\}_{i=1}^N \in Z_V^{0,-n}(p)$ and 0 in each $Z_V^{a,-a-n}(p)$ $(a \geq 1)$. We shall need the AJ map for $H_{\mathcal{M}}$ of a NCD in Section 6 and it is introduced there.

Second, suppose V is irreducible but singular, with subvariety $S \, (\overset{\iota}{\hookrightarrow} V)$ the support of its singularities. Let $\beta : \tilde{V} \to V$ be a resolution of singularities, and assume that both $E := \beta^{-1}(S) \, (\overset{\tilde{\iota}}{\hookrightarrow} \tilde{V})$ and S are smooth NCDs. Motivated by the commutative square

$$\begin{array}{ccc} E & \overset{\tilde{\iota}}{\to} & \tilde{V} \\ \beta|_E \downarrow & & \downarrow \beta \\ S & \overset{\iota}{\to} & V \end{array}$$

we consider the simple (cone) complex associated to the double complex

$$Z^p(\tilde{V}, -\bullet)_{\{E_I\}} \oplus Z_S^\bullet(p) \underset{\tilde{\iota}^* - (\beta|_E)^*}{\longrightarrow} Z_E^\bullet(p).$$

(Here the $(\beta|_E)^*$ has to be done componentwise.) So a class in $H_{\mathcal{M}}^{2p-n}(V, \mathbb{Q}(p))$ is represented by a "triple" $(Z, \mathfrak{Z}, \Xi) \in Z^p(\tilde{V}, n)_{\{E_I\}} \oplus Z_S^{-n}(p) \oplus Z_E^{-n-1}(p)$ with $\partial_{\mathcal{B}} Z = 0$, $\mathbb{D}\mathfrak{Z} = 0$, and $\mathbb{D}\Xi = \tilde{\iota}^* Z - (\beta|_E)^* \mathfrak{Z}$. Moreover, we obtain a long-exact sequence

$$\to H_{\mathcal{M}}^{2p-n-1}(E, \mathbb{Q}(p)) \to H_{\mathcal{M}}^{2p-n}(V, \mathbb{Q}(p)) \overset{\beta^* \oplus \iota^*}{\longrightarrow} CH^p(\tilde{V}, n) \oplus H_{\mathcal{M}}^{2p-n}(S, \mathbb{Q}(p))$$
$$\overset{\tilde{\iota}^* - (\beta|_E)^*}{\longrightarrow} H_{\mathcal{M}}^{2p-n}(E, \mathbb{Q}(p)) \to .$$

This is used in the constructions of Section 3.

2. Preliminaries on toric varieties

A complex toric n-fold X is a normal, irreducible algebraic variety containing the algebraic torus $\mathbb{G}_m^n(\mathbb{C}) \cong (\mathbb{C}^*)^n$ as a Zariski-open subset and extending its obvious action on itself. the key references for this and the next subsection are [5, 23, Sections 3–4, 38, 66], and especially [3]. We start by summarizing the two standard constructions of toric varieties, from fans and from polytopes, focusing on the local affine coordinate systems in which we shall compute.

2.1. Cones and flags: the affine case

The core definition, from this point of view, is the affine toric variety $U_{\mathbb{C}}$ associated to a (rational convex polyhedral) cone

$$\mathfrak{c} := \mathbb{R}_{\geq 0}\langle \underline{\mathsf{v}}_1, \ldots, \underline{\mathsf{v}}_\ell \rangle \subset \mathbb{R}^n,$$

with integral generators $\underline{\mathsf{v}}_i \in \mathbb{Z}^n$. Under the standard inner product $\langle \cdot, \cdot \rangle$, the dual cone

$$\mathfrak{c}^\circ := \{\underline{\mathsf{w}} \in \mathbb{R}^n \mid \langle \underline{\mathsf{w}}, \underline{\mathsf{v}} \rangle \geq 0 \ \forall \underline{\mathsf{v}} \in \mathfrak{c}\}$$

gives rise to an abelian subgroup

$$S_{\mathfrak{c}} := \mathfrak{c}^\circ \cap \mathbb{Z}^n,$$

which has a finite generating set $\{\underline{\mathsf{w}}_1, \ldots, \underline{\mathsf{w}}_k\}$ by Gordan's lemma. The subalgebra of Laurent polynomials

$$A_{\mathfrak{c}} := \mathbb{C}[\underline{x}^{\underline{\mathsf{w}}_1}, \ldots, \underline{x}^{\underline{\mathsf{w}}_k}] \subset \mathbb{C}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$$

then produces

$$U_{\mathfrak{c}} := \mathrm{Spec} A_{\mathfrak{c}} \supset \mathrm{Spec}\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] = (\mathbb{G}_m)^n$$

as a *scheme*. If we consider the map $\mathbb{C}[\underline{\mathsf{w}}_1, \ldots, \underline{\mathsf{w}}_k] \to \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ given by $\underline{\mathsf{w}}_i \mapsto \underline{x}^{\underline{\mathsf{w}}_i}$ with kernel $I_{\mathfrak{c}}$, then $A_{\mathfrak{c}} \cong \frac{\mathbb{C}[\underline{\mathsf{w}}_1, \ldots, \underline{\mathsf{w}}_k]}{I_{\mathfrak{c}}}$ and as a *variety*,

$$U_{\mathfrak{c}} \cong V(I_{\mathfrak{c}}) := \left\{ \underline{W} \in \mathbb{C}^k \mid f(\underline{W}) = 0 \ \forall f \in I_{\mathfrak{c}} \right\} \subseteq \mathbb{C}^k.$$

Thinking of the x_i as *toric coordinates* on $(\mathbb{C}^*)^n$, the $W_i (= \underline{x}^{\underline{\mathsf{w}}_i}$ in $A_{\mathfrak{c}})$ generate precisely those monomials[12] in them which extend to regular functions on $U_{\mathfrak{c}}$. That is, $A_{\mathfrak{c}}$ is the coordinate ring $\mathbb{C}[U_{\mathfrak{c}}]$.

---

[12] A monomial (resp. Laurent monomial) in k variables W_i is a product $\prod W_i^{\xi_i}$, $\xi_i \in \mathbb{Z}_{\geq 0}$ (resp. $\mathbb{Z}$).

Now for the basic combinatorial considerations. First, by the *dimension* of a cone $\mathfrak{c}$ we just mean that of $\mathbb{R}_{\mathfrak{c}} := \mathbb{R}\langle \underline{\mathsf{v}}_1, \ldots, \underline{v}_\ell \rangle$. Natural subcones are the *faces*, i.e., intersections $\mathfrak{c} \cap \{L = 0\}$ for $L \in (\mathbb{R}^n)^\vee$ satisfying $L \geq 0$ on $\mathfrak{c}$, those of codimension (resp. dimension) one being called *facets* (resp. *edges*). One says that $\mathfrak{c}$ is *simplicial* ($\iff$ $U_{\mathfrak{c}}$ orbifold) if the $\{\underline{\mathsf{v}}_i\}_{i=1}^{\ell}$ are a basis of $\mathbb{R}_{\mathfrak{c}}$ and *smooth* ($\iff$ $U_{\mathfrak{c}}$ smooth) if moreover $\mathbb{Z}^n \cap \mathbb{R}_{\mathfrak{c}} = \mathbb{Z}\langle \underline{\mathsf{v}}_1, \ldots, \underline{\mathsf{v}}_\ell \rangle$. In the latter situation we have simply $U_{\mathfrak{c}} \cong \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}$.

Of particular importance is the setting where $\mathfrak{c}$ is simplicial of dimension n, in which case $\mathfrak{c}^\circ$ is as well. Let $\varepsilon_1, \ldots, \varepsilon_n$ denote the edges of $\mathfrak{c}^\circ$, and $\underline{\mathsf{w}}_1, \ldots, \underline{\mathsf{w}}_n$ the unique integral generators of each $\varepsilon_i \cap \mathbb{Z}^n$. In general, $\underline{\mathsf{w}}_1$ or $\underline{\mathsf{w}}_2$ will *not* suffice to generate $\mathbb{R}_{\geq 0}\langle \varepsilon_1, \varepsilon_2 \rangle \cap \mathbb{Z}^n$; let $\underline{\tilde{\mathsf{w}}}_1, \ldots, \underline{\tilde{\mathsf{w}}}_{k_2}$ be the required additional generators. Likewise, $\underline{\mathsf{w}}_1, \underline{\mathsf{w}}_2, \underline{\mathsf{w}}_3$ and $\underline{\tilde{\mathsf{w}}}_1, \ldots, \underline{\tilde{\mathsf{w}}}_{k_2}$ will not generate $\mathbb{R}_{\geq 0}\langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle \cap \mathbb{Z}^n$; and we must introduce $\underline{\tilde{\mathsf{w}}}_{k_2+1}, \ldots, \underline{\tilde{\mathsf{w}}}_{k_3}$. Continuing in this fashion up to $\underline{\tilde{\mathsf{w}}}_{k_n}$, our affine coordinates on $U_{\mathfrak{c}}$ are then the $\{\underline{x}^{\underline{\mathsf{w}}_j}\}_{j=1}^{n}$ and $\{\underline{x}^{\underline{\tilde{\mathsf{w}}}_j}\}_{j=1}^{k_n}$. Instead of W_i we shall write

$$z_i = \underline{x}^{\underline{\mathsf{w}}_i}, \qquad u_j = \underline{x}^{\underline{\tilde{\mathsf{w}}}_j} \tag{2.1}$$

and

$$(u_1, \ldots, u_{k_2}) =: \underline{u}_2,\ (u_{k_2+1}, \ldots, u_{k_3}) =: \underline{u}_3, \ldots, (u_{k_{n-1}+1}, \ldots, u_{k_n}) =: \underline{u}_n,$$

organized so that powers of the $\underline{u}_{k_m}$ are expressible in $z_1, \ldots, z_m$ but not $z_1, \ldots, z_{m-1}$. If $\mathfrak{c}$ is smooth then (we can take) $k_n = 0$, so that there are no u_j's.

If $\mathfrak{c}'$ is nonsimplicial (of dimension n) the procedure still works, with the difference that one gets more than n $\{z_i'\}$, hence relations amongst their powers as well. A *wedge* in $\mathfrak{c}'$ is a simplicial n-dimensional subcone

$$\mathfrak{c} = \mathbb{R}_{\geq 0}\langle \underline{\mathsf{v}}_1, \ldots, \underline{\mathsf{v}}_n \rangle \subseteq \mathfrak{c}'$$

such that $\mathbb{R}\langle \underline{\mathsf{v}}_1, \ldots, \underline{\mathsf{v}}_k \rangle \cap \mathfrak{c}'$ is a face of $\mathfrak{c}'$ for each $k = 1, \ldots, n$. In this case there are orderings of the edges of $(\mathfrak{c}')^\circ \subseteq \mathfrak{c}^\circ$ so that $\mathbb{R}_{\geq 0}\langle \varepsilon_1', \ldots, \varepsilon_k' \rangle \subseteq \mathbb{R}_{\geq 0}\langle \varepsilon_1, \ldots, \varepsilon_k \rangle$ $(k = 1, \ldots, n)$; hence z_k' and the $\underline{u}_k'$ can be written as monomials in the $z_1, \ldots, z_\ell$ and $\underline{u}_1, \ldots, \underline{u}_\ell$ exactly when $\ell \geq k$ (or $\ell = n$, if $k \geq n$). One consequence of this, to be used in Section 2.5, is that on $U_{\mathfrak{c}}$ $z_k = 0$ (which implies $\underline{u}_k = 0$) we have $z_k' = 0$ (or $z_n' = z_{n+1}' = \cdots = 0$, if $k = n$). It also gives us a rational morphism $U_{\mathfrak{c}} \to U_{\mathfrak{c}'}$ compatible with the inclusion of the torus.

2.2. Fans and polytopes: complete varieties

This is, of course, a special case of the general covariance of the assignment $\mathfrak{c} \mapsto U_{\mathfrak{c}}$ under inclusions of cones. When the inclusion is that of a *face*, the induced rational map is actually an embedding, which leads to the standard gluing construction. If cones $\mathfrak{c}_1$ and $\mathfrak{c}_2$ share $\mathfrak{c}_1 \cap \mathfrak{c}_2$ as a face, then the embedding $U_{\mathfrak{c}_1\cap\mathfrak{c}_2} \hookrightarrow U_{\mathfrak{c}_1} \sqcup U_{\mathfrak{c}_2}$ is closed, hence the quotient (by the induced equivalence relation) Hausdorff. Iterating this process, we get a toric n-fold X_Σ associated to any *fan* Σ in $\mathbb{R}^n$: that is, a finite collection (closed under taking faces) of *strongly* convex cones ($\mathfrak{c} \cap (-\mathfrak{c}) = \{0\}$) whose intersections are faces of each. If the support $|\Sigma| := \cup_{\mathfrak{c}\in\Sigma}\mathfrak{c}$ is all of $\mathbb{R}^n$, then we say Σ is complete. In any case the $\{U_{\mathfrak{c}}\}_{\mathfrak{c}\in\Sigma}$ give a Zariski-open cover of X_Σ.

The i-dimensional cones $\mathfrak{c} \in \Sigma$ are in one-to-one correspondence with the codimension-i torus orbits in X_Σ (as $U_{\mathfrak{c}}$ contains a unique $(n-i)$-dimensional orbit). We get a morphism $X_{\Sigma'} \xrightarrow{\mu} X_\Sigma$ whenever each cone of Σ' is contained in a cone of Σ; if moreover $|\Sigma'| = |\Sigma|$ then we say Σ' refines Σ, and μ is surjective. In this case it may be described as a sequence of blow-ups at (closures of) torus orbits corresponding to the cones of Σ which get broken up in Σ'.

Now the toric variety of a complete fan is complete but not necessarily projective. To remedy this, consider an n-dimensional polytope $\Delta \subset \mathbb{R}^n$ with integer vertices and $\underline{0}$ in its *interior*. Denote by

$$\Delta^\circ := \{\underline{\mathsf{v}} \in \mathbb{R}^n \mid \langle \underline{\mathsf{v}}, \underline{\mathsf{w}} \rangle \geq -1\}$$

its dual (convex) polytope, which may not have integer vertices. The *faces* of Δ are the intersections $\Delta \cap \{L = 0\}$ for affine functions L (on $\mathbb{R}^n$) satisfying $L|_\Delta \geq 0$. The dimension of a face σ of Δ is $\dim(\mathbb{R}_\sigma)$, for $\mathbb{R}_\sigma$ the smallest affine subspace of $\mathbb{R}^n$ containing σ; write $\Delta(i)$ for the set of codimension-i faces. Combinatorial duality produces a one-to-one correspondence ($\sigma \longleftrightarrow \sigma^\circ$) between $\Delta(i)$ and $\Delta^\circ(n-i+1)$, e.g., vertices $\Delta(n)$ and facets $\Delta^\circ(1)$. Let $\Sigma(\Delta^\circ)$ be the complete fan consisting of cones on all the faces of Δ°; then the toric n-fold

$$\mathbb{P}_\Delta := X_{\Sigma(\Delta^\circ)}$$

is projective. One can see this scheme-theoretically, by checking that

$$\text{(2.2)} \qquad \begin{aligned} \mathbb{P}_\Delta &:= \operatorname{Proj}\left(\mathbb{C}\left[\left\{x_0^\ell \underline{x}^{\underline{m}} \,|\, \underline{m} \in \ell\Delta \cap \mathbb{Z}^n,\ \ell \in \mathbb{Z}_{\geq 0}\right\}\right]\right) \\ &\hookleftarrow \operatorname{Proj}\left(\mathbb{C}\left[x_0, x_1^{\pm 1}, \ldots, x_n^{\pm 1}\right]\right) = (\mathbb{G}_m)^n. \end{aligned}$$

Remark 2.1. In fact, $\Sigma(\Delta^\circ)$ is just the normal fan of Δ, making this substitution in the definition of $\mathbb{P}_\Delta$ extends (2.2) to the case when $\underline{0} \notin \mathrm{int}(\Delta)$.

A more concrete perspective will, however, be valuable: this involves the construction of an ample invertible sheaf. First, note that the toric coordinates $x_1, \ldots, x_n$ give rational functions on $\mathbb{P}_\Delta$. For each $\sigma \in \Delta(i)$ $(0 < i \leq n)$, pick an "origin" $\underline{o}_\sigma \in \mathbb{R}_\sigma \cap \mathbb{Z}^n$, and take a basis $\underline{w}_1^\sigma, \ldots, \underline{w}_{n-i}^\sigma$ for $(\mathbb{R}_\sigma \cap \mathbb{Z}^n) - \underline{o}_\sigma$. We may complete this to a basis $\underline{w}_1^\sigma, \ldots, \underline{w}_n^\sigma$ for $\mathbb{Z}^n$, in such a way that

$$\mathbb{R}_{\geq 0} \left\langle \underline{w}_1^\sigma, -\underline{w}_1^\sigma, \ldots, \underline{w}_{n-i}^\sigma, -\underline{w}_{n-i}^\sigma; \underline{w}_{n-i+1}^\sigma, \ldots, \underline{w}_n^\sigma \right\rangle \supset \Delta - \underline{o}_\sigma.$$

This yields an invertible change of toric coordinates, to $x_j^\sigma := \underline{x}^{\underline{w}_j^\sigma}$ $(j = 1, \ldots, n)$, and then

$$\mathbb{D}_\sigma^* := \left\{ x_1^\sigma, \ldots, x_{n-i}^\sigma \in \mathbb{C}^* \right\} \cap \left\{ x_{n-i+1}^\sigma = \cdots = x_n^\sigma = 0 \right\} \subseteq \mathbb{P}_\Delta$$

is precisely the torus orbit ($\cong (\mathbb{C}^*)^{n-i}$) corresponding to σ°. Writing $\mathbb{D}_\sigma := \overline{\mathbb{D}_\sigma^*}$ for the Zariski closure,

$$\mathbb{D} := \bigcup_{\sigma \in \Delta(1)} \mathbb{D}_\sigma = \coprod_{i=1}^{n} \left(\coprod_{\sigma \in \Delta(i)} \mathbb{D}_\sigma^* \right)$$

is the complement of $(\mathbb{C}^*)^n$ in $\mathbb{P}_\Delta$. The face structure of Δ exactly describes (combinatorially speaking) the intersection behavior of $\mathbb{D}$. Furthermore, if one considers σ as a polytope in $\mathbb{R}_\sigma$ relative to the integer $\mathbb{Z}^n \cap \mathbb{R}_\sigma$, then (by Remark 2.1) $\mathbb{P}_\sigma$ is defined; and in fact $\mathbb{D}_\sigma \cong \mathbb{P}_\sigma$.

Also denoting by $\mathbb{D}$ the divisor $\sum_{\sigma \in \Delta(1)} [\mathbb{D}_\sigma]$, a standard result is that $\mathcal{O}_\Delta(1) := \mathcal{O}(\mathbb{D})$ is ample. Its sections are given by Laurent polynomials with exponent vectors supported on Δ:

$$\begin{aligned} H^0(\mathbb{P}_\Delta, \mathcal{O}_\Delta(1)) &\cong \{ f \in \mathbb{C}(\mathbb{P}_\Delta)^* \mid (f) + \mathbb{D} \geq 0 \} \cup \{0\} \\ &= \left\{ \sum_{\underline{m} \in \Delta \cap \mathbb{Z}^n} \alpha_{\underline{m}} \underline{x}^{\underline{m}} \mid \alpha_{\underline{m}} \in \mathbb{C} \right\}. \end{aligned} \tag{2.3}$$

It is sections of $\mathcal{O}(\ell\mathbb{D})$ (for ℓ sufficiently large) that yield the projective embedding.

An integral convex polytope Δ is called *reflexive* if Δ° has integer vertices too. (In view of $(\Delta^\circ)^\circ = \Delta$, the dual of a reflexive polytope is also reflexive.) An equivalent condition — that $\underline{0}$ be the *unique* integer interior point of Δ — leads easily to

$$\left(\frac{dx_1}{x_1}\wedge\cdots\wedge\frac{dx_n}{x_n}\right) = -\mathbb{D}\quad(\text{on } \mathbb{P}_\Delta),$$

so that $\mathbb{D}$ is an anticanonical divisor. Consequently the anticanonical sheaf $-K_{\mathbb{P}_\Delta}$ is $\mathcal{O}_\Delta(1)$ and is therefore simple, making $\mathbb{P}_\Delta$ *Fano*. Henceforth we assume Δ is reflexive; up to unimodular transformation of $\mathbb{Z}^n$, it is known that there are 16 (resp. 4319, 473800776) possibilities when $n = 2$ (resp. $3, 4$).

2.3. Toric smoothing constructions

Partial desingularizations of $\mathbb{P}_\Delta$ can be produced by subdividing faces of Δ° and replacing $\Sigma(\Delta^\circ)$ by the refinement obtained from te fan on the subdivision. In particular, a *maximal triangulation* of $\partial\Delta^\circ$ is finite collection $\underline{\theta} = \{\theta_\alpha\}$ if simplices, closed under taking faces, such that:

- $\cup_\alpha \theta_\alpha = \partial\Delta^\circ$,
- the union of vertices of the $\{\theta_\alpha\}$ is $\partial\Delta^\circ \cap \mathbb{Z}^n$,
- $\theta_\alpha \cap \theta_\beta$ (if nonempty) is a common face of θ_α and θ_β ($\forall\ \alpha, \beta$).

Associated to each such $\underline{\theta}$ is a refinement $\Sigma(\underline{\theta})$ of $\Sigma(\Delta^\circ)$ consisting of the cones $\tilde{\mathfrak{c}}_\alpha := \mathbb{R}_{\geq 0}\langle\theta_\alpha\rangle$. A *projective support* for $\underline{\theta}$ is a continuous function $h : \mathbb{R}^n \to \mathbb{R}$ which is convex $(h(\underline{x}+\underline{y}) \leq h(\underline{x}) + h(\underline{y})\ \forall x, y)$ and restricts to *distinct* $\mathbb{Q}$-linear functions on distinct n-dimensional cones $\tilde{\mathfrak{c}}_\alpha$. When $\underline{\theta}$ has a projective support, it is called a *maximal projective triangulation* (these always exist), and a theorem of Batyrev [3] asserts that $X_{\Sigma(\underline{\theta})}$ is projective, with (at worst) singularities in codimension ≥ 4 (of $\mathbb{Q}$-factorial terminal type). Moreover, the morphism $X_{\Sigma(\underline{\theta})} \xrightarrow{\mu} \mathbb{P}_\Delta$ is crepant, i.e., $\mu^* K_{\mathbb{P}_\Delta} = K_{X_{\Sigma(\underline{\theta})}}$; Batyrev [3] calls μ a maximal projective crepant partial (MPCP) desingularization of $\mathbb{P}_\Delta$.

There is a convenient way to visualize this process in terms of *real* (non-integral) polytopes, which is not in the literature and will be immensely helpful in the sections ahead. For $\epsilon > 0$, define a function on the vertices of $\underline{\theta}$

$$H_\epsilon : \mathbb{Z}^n \cap \partial\Delta^\circ \longrightarrow \mathbb{R}^n$$

by

$$\underline{v} \longmapsto (1 - h(\underline{v})\epsilon)\underline{v}.$$

Lemma 2.1. *For $\epsilon > 0$ sufficiently small, the set of vertices of* conv $(\mathrm{im}(H_\epsilon))$ *is exactly* $\mathrm{im}(H_\epsilon)$.

Proof (Sketch). Suppose otherwise; then (taking $\epsilon_0 > 0$ sufficiently small) there are distinct $\underline{v}_i \in \mathbb{Z}^n \cap \partial\Delta^\circ$ $(i = 1, \ldots, \delta)$ and continuous $t_i : [0, \epsilon_0) \to [0, 1]$ $(i = 1, \ldots, \delta)$ satisfying $0 \leq \sum_{i=1}^{\delta} t_i(\epsilon) \leq 1$, such that

$$H_\epsilon(\underline{v}_0) = \sum_{i=1}^{\delta} t_i(\epsilon) H_\epsilon(\underline{v}_i). \tag{2.4}$$

Let σ° denote the smallest form of Δ° containing $\underline{v}_0$. Evaluating at $t = 0$ gives $\underline{v}_0 = \sum_{i=1}^{\delta} t_i(0)\underline{v}_i$, and so the $\underline{v}_i$ belong to σ° and $\sum_{i=1}^{\delta} t_i(0) = 1$; by convexity, $h(\underline{v}_0) \leq \sum_{i=1}^{\delta} t_i(0) h(\underline{v}_i)$.

If the $\{\underline{v}_i\}_{i=0}^{\delta}$ are all in one simplex θ_α, then they are linearly independent and (by linearity of $h|_{\mathfrak{c}_\alpha}$) so are the $\{H_\epsilon(\underline{v}_i)\}_{i=0}^{\delta}$, contradicting (2.4).

So the $\{\underline{v}_i\}_{i=0}^{\delta}$ are *not* all in one simplex, and then convexity of h becomes strict: $h(\underline{v}_0) < \sum_{i=1}^{\delta} t_i(0) h(\underline{v}_i)$, implying that for $\epsilon > 0$

$$1 - h(\underline{v}_0)\epsilon \; > \sum_{i=0}^{\delta} t_i(0)(1 - h(\underline{v}_i)\epsilon).$$

By continuity

$$1 - h(\underline{v}_0)\epsilon \; > \sum_{i=0}^{\delta} t_i(\epsilon)(1 - h(\underline{v}_i)\epsilon)$$

for $\epsilon \in (0, \epsilon_0)$, so that (2.4) becomes

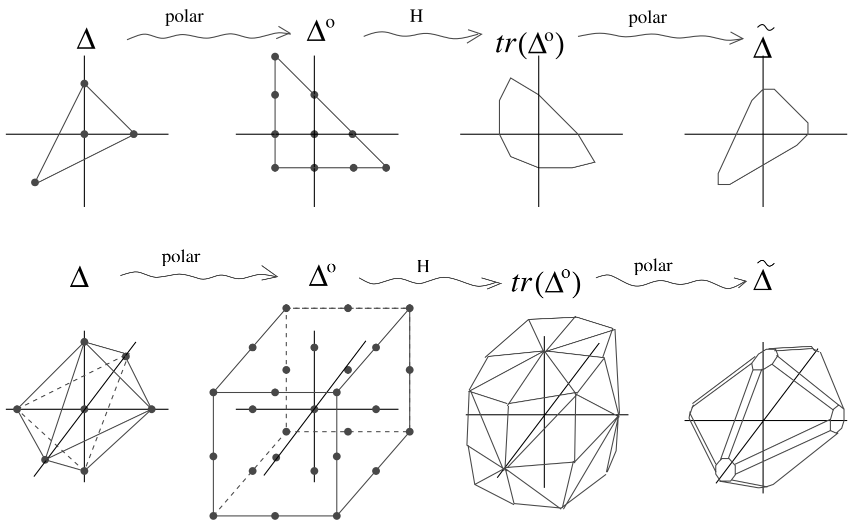
$$\underline{v}_0 = \sum_{i=1}^{\delta} \left(t_i(\epsilon) \frac{1 - h(\underline{v}_i)\epsilon}{1 - h(\underline{v}_0)\epsilon} \right) \underline{v}_i \;=:\; \sum_{i=1}^{\delta} \tau_i(\epsilon)\underline{v}_i$$

with $\sum_{i=1}^{\delta} \tau_i(\epsilon) < 1$. Since all $\underline{v}_i \in \sigma^\circ$ $(i = 0, \ldots, \delta)$, this is impossible. □

Thinking of $\epsilon \in \mathbb{R}_{>0}$ as fixed, we define polytopes in $\mathbb{R}^n$ by

$$\mathrm{tr}(\Delta^\circ) := \mathrm{conv}(\mathrm{im}(H_\epsilon))$$
$$\tilde{\Delta} := \mathrm{tr}(\Delta^\circ)^\circ.$$

Note that $h \geq 0$, $tr(\Delta^\circ) \subseteq \Delta^\circ$, and $\tilde{\Delta} \supset \Delta$; here are some pictures:



As ϵ tends to 0, $\tilde{\Delta}$ (resp. $\mathrm{tr}(\Delta^\circ)$) tends to Δ (resp. Δ°). Given a face of $\tilde{\Delta}$ (resp. $\mathrm{tr}(\Delta^\circ)$), we can consider the smallest face of Δ (resp. Δ°) it limits into (resp. onto). (For $\mathrm{tr}(\Delta^\circ)$ only, one can also use the *map* to Δ° produced by radial projection.) This defines maps (for each k)

$$\underset{j \leq k}{\cup} \mathrm{tr}(\Delta^\circ)(j) \longrightarrow \underset{j \leq k}{\cup} \Delta^\circ(j)$$
$$\underset{i \geq k}{\cup} \tilde{\Delta}(i) \longrightarrow \underset{i \geq k}{\cup} \Delta(i),$$

and the "preimage faces" of a face of Δ° (resp. Δ) are said to lie over it. For faces $\tilde{\sigma}^\circ$ of $\mathrm{tr}(\Delta^\circ)$ lying over a face σ° of Δ°, the projected image gives a simplex $\theta(\tilde{\sigma}^\circ) \subseteq \sigma^\circ$ from the triangulation. To faces $\tilde{\sigma}$ of $\tilde{\Delta}$ we shall associate an affine subspace containing $\tilde{\sigma}$ and then letting ϵ tend to 0. If $\tilde{\sigma}$ lies over σ, then $\mathbb{R}_\sigma \subset \mathbb{R}_{\tilde{\sigma}}$.

Now the point of all this is that by Lemma 2.1, $\sum(\mathrm{tr}(\Delta^\circ)) = \sum(\underline{\theta})$ and so putting

$$\mathbb{P}_{\tilde{\Delta}} := X_{\Sigma(\underline{\theta})}$$

recovers all the one-to-one correspondences previously encountered (for $\mathbb{P}_\Delta$). Let $\tilde{\sigma} \in \tilde{\Delta}(n-i)$; then starting from $\mathbb{R}_{\tilde{\sigma}}$, the same procedure as above yields coordinates $\{x_j^{\tilde{\sigma}}\}_{j=1}^n$ and $\mathbb{D}^*_{\tilde{\sigma}} \subset \mathbb{P}_{\tilde{\Delta}}$, the i-dimensional orbit associated to

$\tilde{\sigma}^\circ \in \mathrm{tr}(\Delta^\circ)(i+1)$. Moreover, $\tilde{\Delta}$ describes the "divisor at ∞"

$$\tilde{\mathbb{D}} := \mathbb{P}_{\tilde{\Delta}} \backslash (\mathbb{C}^*)^n = \bigcup_{\tilde{\sigma} \in \tilde{\Delta}(1)} \mathbb{D}_{\tilde{\sigma}} = \coprod_{j=1}^{n} \left(\coprod_{\tilde{\sigma} \in \tilde{\Delta}(j)} \mathbb{D}^*_{\tilde{\sigma}} \right)$$

in $\mathbb{P}_{\tilde{\Delta}}$. For example, since each facet of $\mathrm{tr}(\Delta^\circ)$ is a simplex, each j-face contains $j+1$ vertices, and so each i-face of $\tilde{\Delta}$ abuts $i+1$ facets, making $\tilde{\mathbb{D}}$ a NCD on the smooth part of $\mathbb{P}_{\tilde{\Delta}}$. Since μ is crepant,

$$H^0\left(\mathbb{P}_{\tilde{\Delta}}, -K_{\mathbb{P}_{\tilde{\Delta}}}\right) \cong \left\{ \sum_{\underline{m} \in \Delta \cap \mathbb{Z}^n} \alpha_{\underline{m}} x^{\underline{m}} \,\middle|\, \alpha_{\underline{m}} \in \mathbb{C} \right\} \tag{2.5}$$

and $\tilde{\mathbb{D}} = \sum_{\tilde{\sigma} \in \tilde{\Delta}(1)} [\mathbb{D}_{\tilde{\sigma}}]$ is additionally an anticanonical divisor, though $\mathbb{P}_{\tilde{\Delta}}$ may not be Fano.

2.4. Local coordinates

Summarizing the story so far, affine charts for $\mathbb{P}_{\tilde{\Delta}}$ are obtained from monomial generators for the integral points of the cones dual to the cones on $\mathrm{tr}(\Delta^\circ)$. The cones on Δ° likewise provide affine charts for $\mathbb{P}_\Delta$; and in both cases the relations between the monomials produce local equations for the toric variety. The two sets of affine charts are related by blow-up along coordinate subspaces, and locally μ is just the proper transform. On the level of torus orbits we can easily describe μ as follows: If $\tilde{\sigma} \in \tilde{\Delta}(i-k)$ lies over $\sigma \in \Delta(i)$ then $\mu(\mathbb{D}^*_{\tilde{\sigma}}) = \mathbb{D}^*_\sigma$, and the toric coordinates on $\mathbb{D}^*_{\tilde{\sigma}} \cong \mathbb{D}^*_\sigma \times (\mathbb{C}^*)^k$ can be written as $\{x_1^{\tilde{\sigma}}, \ldots, x_{n+k-i}^{\tilde{\sigma}}\} = \{x_1^\sigma, \ldots, x_{n-i}^\sigma; y_1^{\tilde{\sigma}}, \ldots, y_k^{\tilde{\sigma}}\}$ where the $y_j^{\tilde{\sigma}}$ are blow-up coordinates.

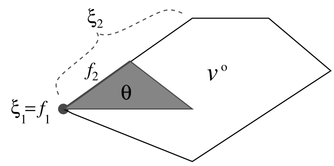
We elaborate on the affine charts for $\mathbb{P}_{\tilde{\Delta}}$. These are in one-to-one correspondence with the facets $\mathrm{tr}(\Delta^\circ)(1)$, or (dually) with the vertices $\tilde{\Delta}(n)$. we need the following general statement:

Lemma 2.2. *Let $\tilde{\sigma}^\circ \in \mathrm{tr}(\Delta^\circ)(i+1)$, with dual face $\tilde{\sigma} \in \tilde{\Delta}(n-i)$; let $\underline{p} \in \tilde{\sigma} \backslash \partial\tilde{\sigma}$ be any interior point. The dual of the $(n-i)$-cone $\mathbb{R}_{\geq 0} \langle \tilde{\sigma}^\circ \rangle = \mathbb{R}_{\geq 0} \langle \theta(\tilde{\sigma}^\circ) \rangle$ is then the n-cone $\mathbb{R}_{\geq 0} \left\langle \tilde{\Delta} - \underline{p} \right\rangle$.*

Now, given a vertex $\tilde{v} \in \tilde{\Delta}(n)$, let $\kappa(\tilde{v})$ denote the dual of $\mathfrak{c}(\tilde{v}^\circ) := \mathbb{R}_{\geq 0} \langle \tilde{v}^\circ \rangle$, and $U_{\tilde{v}} := U_{\mathfrak{c}(\tilde{v}^\circ)} \subset \mathbb{P}_{\tilde{\Delta}}$. According to the Lemma, *$\kappa(\tilde{v})$ is the cone through $\tilde{v}$ or $\tilde{\Delta}$, with $\tilde{v}$ translated to $\underline{0}$*. So the coordinate rings $A_{\tilde{v}} :=$

$\mathbb{C}[\underline{x}^{\kappa(\tilde{v})\cap\mathbb{Z}^n}]$ of the affine neighborhoods $U_{\tilde{v}}$ can be read off *directly from the geometry of* $\tilde{\Delta}$.

Dropping tildes, the same story goes through for Δ. Let $v \in \Delta(n)$ with dual facet $v^\circ \in \Delta(1)$. In *any* triangulation of v°, there exists a simplex θ and sequences of faces (with subscript denoting $1+$ dimension) $f_1 \subsetneq f_2 \subsetneq \cdots \subsetneq f_{n-1}$ of θ and $\xi_1 \subsetneq \xi_2 \subsetneq \cdots \subsetneq \xi_{n-1}$ of v° such that $f_i \subseteq \xi$ $(\forall i)$, e.g.,



Since $\theta = \tilde{v}^\circ$ for some $\tilde{v} \in \tilde{\Delta}(n)$ lying over v, this shows we can choose $\tilde{v}$ so that $\mathfrak{c}(\tilde{v}^\circ)$ is a wedge in $\mathfrak{c}(v^\circ)$. The map $U_{\tilde{v}} \to U_v$ induced by μ can then be described exactly as at the end of Section 2.1.

We conclude with a brief description of singularities of $\mathbb{P}_{\tilde{\Delta}}$. Consider a simplex $\theta \subset \mathbb{R}^{n-1}$: if its vertices lie in $\mathbb{Z}^{n-1}$, then $\mathrm{vol}(\theta) = \frac{q}{(n-1)!}$ for some $q \in \mathbb{Z}_{>0}$. If $\theta \cap \mathbb{Z}^{n-1}$ is nothing but these vertices, θ is *elementary*; if $q = 1$, θ is *regular*. For $n \leq 3$, elementary implies regular; for $n = 4$ the simplices with vertices $\underline{0}$, $(1,0,0)$, $(0,1,0)$, $(1,p,q)$, where $0 < p < q$ and $(p,q) = 1$, are elementary but irregular. Now let $\tilde{v} \in \tilde{\Delta}(n)$ lie over $v \in \Delta(n)$. By maximality of $\underline{\theta}$, the $(n-1)$-simplex $\theta(\tilde{v}^\circ) \subset v^\circ \subset \mathbb{R}_{v^\circ}$ $(\cong \mathbb{R}^{n-1})$ is elementary, relative to the integer lattice $\mathbb{R}_{v^\circ} \cap \mathbb{Z}^n$ $(\cong \mathbb{Z}^{n-1})$. Our observations in Section 2.1 essentially amount to the statement that the point $\mathbb{D}_{\tilde{v}}$ (in $\mathbb{P}_{\tilde{\Delta}}$) is smooth if and only if the integral generators of edges of $\kappa(\tilde{v})$ generate $\kappa(\tilde{v}) \cap \mathbb{Z}^n$. One easily shows that this is equivalent to regularity of $\theta(\tilde{v}^\circ)$, which shows $\mathbb{P}_{\tilde{\Delta}}$ is smooth for $n \leq 3$ and has isolated ($\mathbb{Q}$-factorial terminal) singularities for $n = 4$ (cf. [3, 2.2.8]).

2.5. Anticanonical hypersurfaces

Let $\Delta \subset \mathbb{R}^n$ be a reflexive polytope with $(2 \leq)n \leq 4$, and

$$F = \sum_{\underline{m} \in \Delta \cap \mathbb{Z}^n} \alpha_{\underline{m}} \underline{x}^{\underline{m}} \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$$

a nonzero Laurent polynomial with *support* (i.e., monomial exponent set) $\mathfrak{M}_F := \{\underline{m} \in \mathbb{Z}^n \mid \alpha_{\underline{m}} \neq 0\}$ contained in Δ. Let $X_F \subset \mathbb{P}_\Delta$ be the zero-locus of the section of $-K_{\mathbb{P}_\Delta}$ given by F (cf. (2.3)). If F is constant, $X_F = \mathbb{D}$; if $\mathrm{conv}(\mathfrak{M}_F) = \Delta$ then it is the Zariski closure of $X_F^* := \{\underline{x} \mid F(\underline{x}) = 0\} \subset$

$(\mathbb{C}^*)^n$. We do not treat, and will not need, the "in between" cases where X_F contains some but not all components of $\mathbb{D}$. Recall that while $\mathbb{P}_\Delta$ may have singularities (in codimension ≥ 2), the torus orbits $\mathbb{D}^*_\sigma$ are smooth. We say that F is Δ-*regular* ([3, 3.1.1]) when the intersections

$$D^*_{F,\sigma} := X_F \cap \mathbb{D}^*_\sigma \subset \mathbb{D}^*_\sigma$$

(taken over all faces of Δ) as well as $X^*_F \subset (\mathbb{C}^*)^n$, are reduced (irreducible components have multiplicity 1) and smooth of codimension one. Put $D_F := \cup_{\sigma\in\Delta(1)} D_{F,\sigma} = X_F \backslash X^*_F$, where $D_{F,\sigma} := \overline{D^*_{F,\sigma}}$.

Fixing a maximal projective triangulation of Δ°, F also yields (cf. (2.5)) an element of $H^0(\mathbb{P}_{\tilde{\Delta}}, -K_{\mathbb{P}_{\tilde{\Delta}}})$ whose vanishing locus $\tilde{X}_F$ is $\tilde{\mathbb{D}}$ for F constant and the closure of $\tilde{X}^*_F$ $(:= X^*_F)$ in $\mathbb{P}_{\tilde{\Delta}}$ if $\mathrm{conv}(\mathfrak{M}_F) = \Delta$. If F is Δ-regular, $\tilde{X}_F$ is (a) the preimage of X_F under $\mu : \mathbb{P}_{\tilde{\Delta}} \to \mathbb{P}_\Delta$ and (b) smooth, hence (using the adjunction formula to obtain $K_{\tilde{X}_F} \cong \mathcal{O}_{\tilde{X}_F}$) (c) a Calabi–Yau $(n-1)$-fold.

To get a handle on the $D^{(*)}_{F,\sigma}$ and $D^{(*)}_{F,\tilde{\sigma}} := \tilde{X}_F \cap \mathbb{D}^{(*)}_{\tilde{\sigma}}$, we need the *face polynomials* of F attached to each $\sigma \in \Delta(i)$. In the notation of Section 2.2, these are obtained by rewriting $\underline{x}^{-\underline{o}_\sigma} F(\underline{x})$ in the $\{x^\sigma_j\}_{j=1}^n$ and setting $x^\sigma_{n-i+1} = \cdots = x^\sigma_n = 0$ to get a Laurent polynomial $(=: F_\sigma)$ in $x^\sigma_1, \ldots, x^\sigma_{n-i}$. The support $\mathfrak{M}_{F_\sigma}$ of F_σ lies in $\sigma - \underline{o}_\sigma$, and its vanishing locus is $D^*_{F,\sigma}$ (under the isomorphism $(\mathbb{C}^*)^{n-i} \cong \mathbb{D}^*_\sigma$). So for example, necessary criteria for Δ-regularity of F are that its vertex polynomials be *nonzero* constants and its edge (one-variable) polynomials have no multiple roots. This condition on vertices (i.e., that $\underline{v} \in \Delta(n) \implies \alpha_{\underline{v}} \neq 0$) implies, in turn, that $\Delta = \mathrm{conv}(\mathfrak{M}_F)$.

If $\tilde{\sigma} \in \tilde{\Delta}(i-k)$ lies over $\sigma \in \Delta(i)$ then (in the notation of Section 2.4) setting $F_{\tilde{\sigma}}(x^\sigma_1, \ldots, x^\sigma_{n-i}; y^{\tilde{\sigma}}_1, \ldots, y^{\tilde{\sigma}}_k) := F_\sigma(x^\sigma_1, \ldots, x^\sigma_{n-i})$, $F_{\tilde{\sigma}} = 0$ cuts $D^*_{F,\tilde{\sigma}} \cong D^*_{F,\sigma} \times (\mathbb{C}^*)^k$ out of $\mathbb{D}^*_{\tilde{\sigma}} := \mathbb{D}^*_\sigma \times (\mathbb{C}^*)^k$. So Δ-regularity of F guarantees that $D^*_{F,\tilde{\sigma}}$ is empty if $\tilde{\sigma}$ lies over a vertex (or is one) and is otherwise smooth and reduced. From this and from the fact that (off singularities $\tilde{X}_F$ avoids) $\tilde{\mathbb{D}}$ is a NCD, one may deduce that $\tilde{D}_F := \tilde{X}_F \cap \tilde{\mathbb{D}} = \tilde{X}_F \backslash \tilde{X}^*_F$ is one too.

We can describe the local affine equation of X_F in any neighborhood $U_{\underline{v}} \subset \mathbb{P}_\Delta$ (for $\underline{v} \in \Delta(n)$) as follows. Set $\mathfrak{c}' := \mathfrak{c}(v^\circ)$ and $\kappa(\underline{v}) := (\mathfrak{c}')^\circ$ as in Section 2.4, so that

$$\Phi_{\underline{v}} := \underline{x}^{-\underline{v}} F(\underline{x})$$

has support in $\kappa(\underline{v})$. Writing $\{\underline{w}'_i, \underline{\tilde{w}}_j\}$ for generators of $\kappa(\underline{v}) \cap \mathbb{Z}^n$ (à la Section 2.1, with $\underline{w}'_i \longleftrightarrow$ edges of $\kappa(\underline{v})$), the monomial terms of $\Phi_{\underline{v}}$ can be expressed in terms of $\mathbb{Z}_{\geq 0}$-powers of the $\{z'_k = \underline{x}^{\underline{w}'_k};\, u'_\ell := \underline{x}^{\underline{\tilde{w}}'_\ell}\}$. Since

$\mathrm{conv}(\mathfrak{M}_F) = \Delta$ and the edges of $\kappa(\underline{v})$ lead to other vertices of $\Delta - \underline{v}$, $\Phi_{\underline{v}}$ has a nonzero constant term c_0 and nonzero terms of the form $c_i(z_i')^{k_i}$ ($k_i \in \mathbb{Z}_{>0}$) for each i. Clearly its vanishing locus is exactly $U_{\underline{v}} \cap X_F$.

Referring to Section 2.4, we can choose $\tilde{\underline{v}} \in \tilde{\Delta}(n)$ lying over $\underline{v}$ so that $\mathfrak{c} := \mathfrak{c}(\tilde{v}^\circ)$ is a wedge in $\mathfrak{c}'$. $\Phi_{\underline{v}}$ pulls back to a regular function on $U_{\tilde{\underline{v}}}$ ($\subset \mathbb{P}_{\tilde{\Delta}}$) cutting out $\tilde{X}_F \cap U_{\tilde{\underline{v}}}$. Let $\mathfrak{f}_i \in tr(\Delta^\circ)(n-i+1)$ denote the distinguished flag of faces of $\tilde{v}^\circ$ ($\theta(\mathfrak{f}_i) = f_i = \theta(\tilde{v}^\circ)$, $i = 1, \ldots, n-1$; and $\mathfrak{f}_n := \tilde{v}^\circ$), and $\tilde{\sigma}_i \in \tilde{\Delta}(i)$ their duals (incl. $\tilde{\sigma}_n = \tilde{\underline{v}}$). The algorithm from Section 2.1 produces[13] $\{z_j, \underline{u}_j\}_{j=1}^n$ satisfying $\mathbb{D}_{\tilde{\sigma}_i} \cap U_{\underline{v}} = \{z_{n-i+1} = \cdots = z_n = 0\}$, and we can decompose

(2.6)
$$c_0^{-1}\Phi_{\underline{v}} \;=\; 1 + \Phi_{\underline{v},1}(z_1) + \Phi_{\underline{v},2}(z_1, z_2; \underline{u}_2) + \cdots + \Phi_{\underline{v},n}(z_1, \ldots, z_n; \underline{u}_2, \ldots, \underline{u}_n)$$

so that $\Phi_{\underline{v},i}$ consists of those monomial terms in $z_1, \ldots, z_i$ and $\underline{u}_2, \ldots, \underline{u}_i$ vanishing when $z_i = 0$. Since $(z_i')^{k_i}$ is such a monomial, none of the $\Phi_{\underline{v},i}$ are identically zero. (In fact, $\Phi_{\underline{v}}|_{\mathbb{D}_{\tilde{\sigma}_i}} = 1 + \Phi_{\underline{v}_1} + \cdots + \Phi_{\underline{v},n-i}$ is essentially the edge polynomial associated to the face of Δ that $\tilde{\sigma}_i$ lies over.)

Finally, it will be important in Sections 4.1 and 4.2 that the monomial term $c\underline{x}^{-\underline{v}}$ (in $\Phi_{\underline{v}}$) which comes from the interior point of Δ, lies in $\Phi_{\underline{v},n}$. This is simply because $-\underline{v}$ lies in the interior of $\kappa(\underline{v})$. Moreover, since the anticanonical hypersurface in $\mathbb{P}_{\tilde{\Delta}}$ associated to the Laurent polynomial 1 is $\tilde{\mathbb{D}}$, the variety cut out by $\underline{x}^{-\underline{v}}$ is $\tilde{\mathbb{D}} \cap U_{\tilde{v}}$. This is the (reduced) union of the $\mathbb{D}_{\tilde{\sigma}} \cap U_{\tilde{v}}$, over facets $\tilde{\sigma} \in \Delta(1)$ containing $\tilde{v}$. While these are the hypersurfaces where the z_i vanish, this vanishing map not be to first order; and thus as a monomial in the $\{z_i, \underline{u}_i\}$, $\underline{x}^{-\underline{v}}$ may involve some u's. On the other hand, if $\theta(\tilde{v}^\circ)$ is a regular simplex (always true for $n = 2$ or 3), $U_{\tilde{v}}$ is smooth and isomorphic to $\mathbb{C}^n$ with coordinates $\{z_i\}$, and we have

$$\underline{x}^{-\underline{v}} = z_1 \cdot \cdots \cdot z_n.$$

This is used in several places below.

3. Constructing motivic cohomology classes on families of CY-varieties

The goal of this section is a combinatorial machine for producing one-parameter families of Calabi–Yau $(n-1)$-folds[14] $\tilde{X}_t$ that carry nontrivial

[13] There are only $\{\underline{u}_j\}$ for $n = 4$.

[14] The small tilde does not denote a desingularization; $\tilde{X}_t$ can be singular.

elements $\Xi_t \in H^n_{\mathcal{M}}(\tilde{X}_t, \mathbb{Q}(n))\ \forall t \in \mathbb{P}^1 \backslash \{0\}$, for $n = 2, 3, 4$. For $n = 2$, our construction is a slight extension of work [69] of Villegas. The $\tilde{X}_t$ are considered as fibers of a total space $\tilde{\mathcal{X}}_-$, which can itself be singular and on which we will actually construct a global class Ξ pulling back to the Ξ_t.

We remind the reader that for $\tilde{X}_t$ smooth, working $\otimes\mathbb{Q}$ (as is our convention in this paper)

$$H^n_{\mathcal{M}}(\tilde{X}_t, \mathbb{Q}(n)) \xrightarrow{\cong} CH^n(\tilde{X}_t, n) \xleftarrow{\cong} \mathrm{Gr}^n_\gamma K_n(\tilde{X}_t).$$

Our construction still yields something in $H^n_{\mathcal{M}}$ for singular members of the family, though in that case $CH^n(\tilde{X}_t, n) \cong \mathrm{Gr}^n_\gamma G_n(\tilde{X}_t)$ and both isomorphisms above fail. However, by taking hyper-resolutions as in Section 1.3, $H^n_{\mathcal{M}}$ can still be represented by higher Chow precycles, which allows for explicit computation [49] of the Abel–Jacobi map

$$AJ^{n,n} : H^n_{\mathcal{M}}(\tilde{X}_t, \mathbb{Q}(n)) \to H^{n-1}(\tilde{X}_t, \mathbb{C}/\mathbb{Q}(n))$$

in terms of currents and C^∞ chains. We will partially compute AJ in Section 4, and deal with the degenerate fibers (in some cases) in Section 6.

3.1. Toric data

Our $\tilde{X}_t$'s will be hypersurfaces in partial desingularizations $\mathbb{P}_{\tilde{\Delta}}$ of toric Fano n-folds. To start the construction, let

$$\sum_{\underline{m} \in \mathbb{Z}^n} \alpha_{\underline{m}} \underline{x}^{\underline{m}} = \phi \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$$

be a Laurent polynomial with coefficients in a number field $K \subset \mathbb{C}$, and set

$$\Delta := \mathrm{conv}(\mathfrak{M}_\phi).$$

Definition 3.1. (i) ϕ is *reflexive* if Δ is a reflexive polytope.
(ii) ϕ is *regular* if $\lambda - \phi$ is Δ-regular for general $\lambda \in \mathbb{C}$.

We henceforth assume ϕ reflexive, and consider the one-parameter family of anticanonical hypersurfaces

$$\mathbb{P}^1 \times \mathbb{P}_\Delta \supset \mathcal{X} \xrightarrow{\pi} \mathbb{P}^1$$

given by taking the Zariski closure of

$$\{1 - t\phi = 0\} \subset \mathbb{C} \times (\mathbb{C}^*)^n.$$

Alternately, writing $\lambda := t^{-1}$ we can think of $\mathcal{X}$ as the closure of $\lambda - \phi = 0$. The reader may wonder why we restrict so early on to a variable internal coefficient (i.e. λ) and algebraic values of the other (external) coefficients. That we lose no generality in doing so will be established later, in Proposition 4.2.

Denote the fibres of our family by $X^\lambda = X_t := \pi^{-1}(t)$. Its base locus is the intersection of X^λ with

$$X_0 = \mathbb{D} \subset \mathbb{P}_\Delta$$

for any $\lambda \in \mathbb{C}$. Since the face polynomials of $\lambda - \phi$ (cf. Section 2.5) are just multiples of the ϕ_σ, this is

$$D := X^\lambda \cap \mathbb{D} = \bigcup_{\sigma \in \Delta(i)} D_\sigma = \coprod_{i=1}^{n-1} \left(\coprod_{\text{sigma} \in \Delta(i)} D_\sigma^* \right) \tag{3.1}$$

where $D_\sigma^{(*)} := D_{\phi,\sigma}^{(*)}$. Since $\operatorname{conv}(\mathfrak{M}_\phi) = \Delta$, these are always of codimension 1 in $\mathbb{D}_\sigma^{(*)}$. Regularity of ϕ is therefore equivalent to the D_σ^* being nonsingular and reduced for all $\sigma \in \Delta(i)$, $i = 1, \dots, n-1$.

Choose a (maximal, projective) triangulation of the dual Δ°, and let $\mathbb{P}_{\tilde{\Delta}} \xrightarrow{\mu} \mathbb{P}_\Delta$ be the corresponding MPCP-desingularization. By taking the closure of $1 - t\phi = 0$ in $\mathbb{P}^1 \times \mathbb{P}_{\tilde{\Delta}}$, we get the family $\tilde{\mathcal{X}} \xrightarrow{\tilde{\pi}} \mathbb{P}^1$ with fibers $\tilde{X}_t (= \tilde{X}^\lambda)$ and base locus $\tilde{D} := \tilde{X}^\lambda \cap \tilde{\mathbb{D}} = \cup_{\tilde{\sigma} \in \tilde{\Delta}(1)} D_{\tilde{\sigma}}$. If $\tilde{v} \in \tilde{\Delta}(n)$ is dual to a regular simplex $\theta(\tilde{v}^\circ)$, the local equation of $\tilde{X}^\lambda$ in $U_{\tilde{v}}$ is of the form $P(z_1, \dots, z_n) - \lambda z_1 \cdots z_n = 0$ (with P a polynomial determined from ϕ and $\tilde{v}$ as in Section 2.5). Assuming ϕ regular (which we shall not always do), $\tilde{\mathcal{X}}$ is the μ-preimage of $\mathcal{X}$, $\tilde{D}$ is a NCD, and the $\tilde{X}_t$ are smooth CY $(n-1)$-folds for $t \in \mathbb{P}^1$ outside a finite set $\mathcal{L}$ (the *discriminant locus*).

We recall some notation from Section 1: given nonvanishing holomorphic functions $f_1, \dots, f_\ell \in \Gamma(Y, \mathcal{O}_Y^*)$ on a quasi-projective variety Y, the symbol $\{f_1, \dots, f_\ell\} \in Z^\ell(Y, \ell)$ denotes the higher Chow cycle given by their graph in $Y \times (\mathbb{P}^1 \backslash \{1\})^\ell$. Its class $\langle \{f_1, \dots, f_\ell\} \rangle \in CH^\ell(Y, \ell)$ maps to an element in Milnor K-theory $K_\ell^M(\mathbb{C}(Y)) \cong CH^\ell(\eta_Y, \ell)$ which is also denoted $\{f_1, \dots, f_\ell\}$.

Definition 3.2. ϕ is *tempered* if the toric-coordinate symbols $\{x_1^\sigma, \ldots, x_{n-i}^\sigma\}$ give trivial[15] classes in $CH^{n-i}(D_\sigma^*, n-i)$ for all $i \geq 1$ and $\sigma \in \Delta(i)$.

Remark 3.1. (a) Here we are thinking of the D_σ^* as being cut out by the face polynomials $\phi_\sigma(x_1^\sigma, \ldots, x_{n-i}^\sigma)$. For faces $\tilde{\sigma} \in \tilde{\Delta}(i-k)$ over σ, since

$$\phi_{\tilde{\sigma}}(x_1^\sigma, \ldots, x_{n-i}^\sigma; y_1^{\tilde{\sigma}}, \ldots, y_k^{\tilde{\sigma}}) = \phi_\sigma(x_1^\sigma, \ldots, x_{n-i}^\sigma),$$

the natural symbols $\{x_1^\sigma, \ldots x_{n-i}^\sigma; y_1^{\tilde{\sigma}}, \ldots, y_k^{\tilde{\sigma}}\} \in CH^{n-i+k}(D_{\tilde{\sigma}}^*, n-i+k)$ are also trivial if ϕ is tempered.

(b) Though we have been working over $\mathbb{C}$, the above constructions and definitions descend to K. Provided one is willing to work over a suitable algebraic extension of K (or $\bar{\mathbb{Q}}$), we can discuss irreducible components of the D_σ^*. For $n-i=1$, the D_σ^* components are points and must have root-of-unity coordinates x_1^σ if ϕ is tempered. (Hence we recover Villegas's prescription for $n=2$, that the ϕ_σ be cyclotomic $\forall\, \sigma \in \Delta(1)$.) For $n-i=2$, the tempered condition is equivalent to $\{x_1^\sigma, x_2^\sigma\}$ giving torsion classes in K_2^M of the $\bar{\mathbb{Q}}$-function fields of the irreducible component curves C of D_σ^*, since $ker\{CH^2(C,2) \to CH^2(\eta_C, 2)\} = \oplus_{p \in C(\bar{\mathbb{Q}})} CH^1(p,2) = 0$.

Now assume that ϕ is regular and $n \leq 4$. For $\tilde{\sigma}_i \in \tilde{\Delta}(i)$ we may define iterated residue maps

$$CH^n(\mathbb{P}_{\tilde{\Delta}} \backslash \tilde{\mathbb{D}}, n) \to CH^{n-1}(\mathbb{D}_{\tilde{\sigma}_1}^*, n-1) \to \cdots \to CH^{n-i}(\mathbb{D}_{\tilde{\sigma}_i}^*, n-i),$$

given a choice of flag $(\tilde{\sigma}_i \subsetneq)\, \tilde{\sigma}_{i-1} \subsetneq \cdots \subsetneq \tilde{\sigma}_1,\ \tilde{\sigma}_j \in \tilde{\Delta}(j)$. The composition is independent of the choice, and is denoted $\mathrm{Res}^i_{\tilde{\sigma}_i}$; a similar construction yields $\mathrm{Res}^i_{\tilde{\sigma}} : CH^n(\tilde{X}_t \backslash \tilde{D}, n) \to CH^{n-i}(D_{\tilde{\sigma}}^*, n-i)$ for $t \notin \mathcal{L}$. If we remove tildes, the Res^i_σ still make sense; note in particular that all singularities (on $\mathbb{P}_\Delta$, X_t, $\mathbb{D}_\sigma$, D_σ for any σ) are in codimension ≥ 2. For example, if $\sigma' \subsetneq \sigma$ ($\sigma' \in \Delta(i+1)$, $\sigma \in \Delta(i)$) with toric coordinates $x_1^\sigma = x_1^{\sigma'}$, $\ldots$, $x_{n-i-1}^\sigma = x_{n-i-1}^{\sigma'}$, x_{n-i}^σ on $\mathbb{D}_{\sigma'}^*$, one has a smooth affine neighborhood $\mathbb{D}_{\sigma'}^* \times \mathbb{A}^1_{x_{n-i}^\sigma} \subset \mathbb{D}_\sigma$. This allows for easy computation of the iterated residues.

Let $\xi := \langle \{x_1, \ldots, x_n\} \rangle \in CH^n\left((\mathbb{C}^*)^n = \mathbb{P}_{\tilde{\Delta}} \backslash \tilde{\mathbb{D}} = \mathbb{P}_\Delta \backslash \mathbb{D},\, n\right)$ denote the class of the coordinate symbol. For $t \notin \mathcal{L}$ this restricts to $\xi_t \in CH^n(X_t^* = \tilde{X}_t^*,\, n)$, either by pulling back the $\{x_i\}$ directly or by invoking contravariant functoriality of higher Chow groups ($\otimes\mathbb{Q}$) for arbitrary morphisms between smooth varieties [54].

[15] We are working $\otimes\mathbb{Q}$; so this means what would usually be meant by "torsion."

Lemma 3.1. *The diagram*

$$\begin{array}{ccc} CH^n(\mathbb{P}_{\tilde{\Delta}}\backslash\tilde{\mathbb{D}}, n) & \xrightarrow{\mathrm{Res}^i_{\tilde{\sigma}}} & CH^{n-i}(\mathbb{D}^*_{\tilde{\sigma}}, n-i) \\ \downarrow{\scriptstyle I^*_t} & & \downarrow{\scriptstyle I^*_{\tilde{\sigma}}} \\ CH^n(\tilde{X}^*_t, n) & \xrightarrow{\mathrm{Res}^i_{\tilde{\sigma}}} & CH^{n-i}(D^*_{\tilde{\sigma}}, n-i) \end{array}$$

commutes for any $\tilde{\sigma} \in \Delta(i)$, as does a similar diagram with all tildes removed.

Proof. With or without tildes, this is based on iterated application ($\ell = 0, 1, \ldots, i-1$) of a quasi-isomorphism which may be proved using the moving lemmas of [11, 54]. Writing

$$\mathbb{D}^{[i]} := \bigcup_{\sigma\in\Delta(i)} \mathbb{D}_\sigma,\ \ \mathbb{D}^{[0]} := \mathbb{P}_\Delta,\ \ D^{[i]} := X_t \cap \mathbb{D}^{[i]},$$

this is

$$\frac{Z^{n-\ell}(\mathbb{D}^{[\ell]}\backslash\mathbb{D}^{[\ell+2]}, \bullet)_{D^{[\ell]}\backslash D^{[\ell+2]}}}{\iota_*\left(Z^{n-\ell-1}(\mathbb{D}^{[\ell+1]}\backslash\mathbb{D}^{[\ell+2]}, \bullet)_{D^{[\ell+1]}\backslash D^{[\ell+2]}}\right)} \xrightarrow{\simeq} Z^{n-\ell}\left(\mathbb{D}^{[\ell]}\backslash\mathbb{D}^{[\ell+1]}, \bullet\right)_{D^{[\ell]}\backslash D^{[\ell+1]}}.$$

A $\partial_{\mathcal{B}}$-closed element on the r.h.s. can therefore be moved into good position, extended to $\mathbb{D}^{[\ell]}\backslash\mathbb{D}^{[\ell+2]}$, and differentiated (to yield a cycle supported on $\mathbb{D}^{[\ell+1]}\backslash\mathbb{D}^{[\ell+2]}$), compatibly with pullbacks to X_t. □

The point is to use the lemma to compute the $Res^i_{\tilde{\sigma}\ \mathrm{or}\,\sigma}$ (bottom row) on ξ_t. For one thing, it is clear that the result is constant in t and descends to $CH^{n-i}\left((D^*_{\tilde{\sigma}\ \mathrm{or}\,\sigma})_K, n-i\right)$. The next result follows easily from the lemma combined with the foregoing discussion.

Proposition 3.1. *For $t \notin \mathcal{L}$, $\sigma \in \Delta(i)$, and $\tilde{\sigma} \in \tilde{\Delta}(i-k)$ lying over σ in the above sense,*

$$\begin{aligned} \mathrm{Res}^i_\sigma \xi_{(t)} &= (I^*_\sigma)\left\langle \pm\{x^\sigma_1, \ldots, x^\sigma_{n-i}\}\right\rangle, \\ \mathrm{Res}^{i-k}_{\tilde{\sigma}} \xi_{(t)} &= (I^*_{\tilde{\sigma}})\left\langle \pm\{x^\sigma_1, \ldots, x^\sigma_{n-i}, y^{\tilde{\sigma}}_1, \ldots, y^{\tilde{\sigma}}_k\}\right\rangle, \end{aligned}$$

where the parenthetical expressions are optional.

It follows that if all $\mathrm{Res}^i_\sigma \xi_t$ are trivial (hence, if ϕ is tempered), then so are all $\mathrm{Res}^i_{\tilde{\sigma}} \xi_t$ — in particular, all $\mathrm{Res}^1_{\tilde{\sigma}}$'s.

Remark 3.2. (i) The regularity assumption on ϕ is not strictly necessary for these results. For $n=2$, we need only ask that the general $\tilde{X}_t$ (equivalently, X_t) be nonsingular; whereas for $n=3$ A-D-E (rational) singularities are allowed (on $\tilde{X}_t$) provided they occur in $\tilde{D}^{[2]} := \cup_{\tilde{\sigma}\in\tilde{\Delta}(2)} D_{\tilde{\sigma}}$. Note however that in Proposition 3.1 the formulas for $\mathrm{Res}^i_{\sigma \text{ or } \tilde{\sigma}} \xi_t$ (not ξ) are multiplied by the multiplicity of (components of) $D_{\sigma \text{ or } \tilde{\sigma}}$ in case these are nonreduced.

(ii) The Res^i_σ, $\mathrm{Res}^{i-k}_{\tilde{\sigma}}$ are trivially 0 on $CH^n(\tilde{X}^*_t, n)$ (hence on ξ_t) for $i=n$ (in particular, for $\tilde{\sigma}$ lying over a point), since $D_\sigma, D_{\tilde{\sigma}} = \emptyset$ in that case.

3.2. Completing the coordinate symbol

Turning our attention to the family, we define ($\lambda = t^{-1}$)

$$\tilde{\mathcal{X}} := \{(\lambda, x)\,|\, x \in \tilde{X}^\lambda\} \subseteq \mathbb{P}^1_\lambda \times \mathbb{P}_{\tilde{\Delta}}.$$

Recalling that $\tilde{X}_0 = \tilde{X}^\infty = \tilde{\mathbb{D}}$, set

$$\tilde{\mathcal{X}}_- := \tilde{\mathcal{X}} \backslash (\{\infty\} \times \tilde{X}^\infty) \subset \mathbb{A}^1_\lambda \times \mathbb{P}_{\tilde{\Delta}},$$

and noting that $\tilde{\mathcal{X}}_- \cap \mathbb{A}^1 \times \tilde{\mathbb{D}} \cong \mathbb{A}^1 \times \tilde{D}$,

$$\tilde{\mathcal{X}}^*_- := \tilde{\mathcal{X}}_- \backslash \mathbb{A}^1 \times \tilde{D} = \{(\lambda, x)\,|\, x \in (\tilde{X}^\lambda)^*\} \subset \mathbb{A}^1 \times (\mathbb{C}^*)^n.$$

Definition 3.3. We say ξ *($\in H^n_{\mathcal{M}}((\mathbb{C}^*)^n, \mathbb{Q}(n))$) completes to a family of motivic cohomology classes*, if $\exists\, \Xi \in H^n_{\mathcal{M}}(\tilde{\mathcal{X}}_-, \mathbb{Q}(n))$ such that the pullbacks of ξ, Ξ to $H^n_{\mathcal{M}}((\tilde{X}^\lambda)^*, \mathbb{Q}(n))$ agree $\forall\, \lambda \in \mathbb{A}^1$. That is, in the diagram

$$\begin{array}{ccccc} \Xi \in & H^n_{\mathcal{M}}(\tilde{\mathcal{X}}_-, \mathbb{Q}(n)) & & H^n_{\mathcal{M}}((\mathbb{C}^*)^n, \mathbb{Q}(n)) & \ni \xi \\ \downarrow & \downarrow{\scriptstyle (\iota^\lambda)^*} & & \downarrow{\scriptstyle (I^\lambda)^*} & \downarrow \\ \Xi^\lambda \in & H^n_{\mathcal{M}}(\tilde{X}^\lambda, \mathbb{Q}(n)) & \xrightarrow[r^\lambda]{} & H^n_{\mathcal{M}}((\tilde{X}^\lambda)^*, \mathbb{Q}(n)) & \ni \xi^\lambda \end{array} \tag{3.2}$$

we must have for each λ, $r^\lambda(\Xi^\lambda) = \xi^\lambda$. (Here $\tilde{\mathcal{X}}_-$, $\tilde{X}^\lambda$, and even $(\tilde{X}^\lambda)^*$ may all be singular.)

To state general conditions under which we can produce such a Ξ, we introduce some more notation (mainly for subsets of $\tilde{D}$). When ϕ is not regular, it has a nonempty irregularity locus

$$\mathcal{I} := \text{ union over all } \tilde{\sigma} \text{ of singularities or nonreduced components of } D^*_{\tilde{\sigma}}$$

(which is just where $\phi_{\tilde{\sigma}}$ vanishes together with all its partials). Writing $\mathbb{I}^n := \cup_i \{x_i = 1\} \subset \mathbb{P}_{\tilde{\Delta}}$ (where $\{x_i\}_{i=1}^n \subset K(\mathbb{P}_{\tilde{\Delta}})^*$ extend the $(\mathbb{C}^*)^n$-coordinates), set

$$\mathcal{J} := \text{ union of all } D_{\tilde{\sigma}},\ \tilde{\sigma} \in \tilde{\Delta}(1), \text{ which are not contained in } \mathbb{I}^n \cap \tilde{\mathbb{D}}.$$

For $n = 3$ specifically, where we will allow A_1-singularities (ordinary double points) on the general $\tilde{X}^\lambda$ (but only at $\tilde{D}^{[2]}$), write $\mathcal{A}$ ($\subseteq \mathcal{I}$) for the collection of these,

$$\begin{aligned} \{\alpha_1, \ldots, \alpha_k\} &:= \mathcal{A} \cap \mathcal{J}, \quad \text{and} \\ \{\mathcal{D}_1, \ldots, \mathcal{D}_\ell\} &:= \text{irreducible curves in } \tilde{D} \\ &\qquad \text{avoiding the set } (\mathcal{A} \backslash \mathcal{A} \cap \mathcal{J}) \cup (\mathcal{I} \backslash \mathcal{A}). \end{aligned}$$

There is a linear map of vector spaces

$$\mathcal{E} : \mathbb{Q}\langle \mathcal{D}_1, \ldots, \mathcal{D}_\ell \rangle \to \mathbb{Q}\langle \alpha_1, \ldots, \alpha_k \rangle$$

obtained by sending generators $[\mathcal{D}_i] \mapsto \sum_{\alpha_j \in \mathcal{D}_i} [\alpha_j]$.

Theorem 3.1. *Let ϕ be reflexive and tempered, $n \leq 4$. Also assume in case*

$\underline{n = 2}$*: the general X^λ is nonsingular.*

$\underline{n = 3}$*:* (a) *the general $\tilde{X}^\lambda$ is nonsingular apart from A_1-singularities at points $\mathcal{A} \subseteq \mathbb{I}^3 \cap \tilde{\mathbb{D}}^{[2]}$;*

(b) *$\mathcal{I} \subseteq \mathbb{I}^3 (\cap \tilde{\mathbb{D}})$, $\mathcal{I} \cap \mathcal{J} \subseteq \mathcal{A}$; and*

(c) *either*

(i) *$\mathcal{E}$ is surjective, or*

(ii) *K is totally real and the irreducible component curves of $\tilde{D}$ are nonsingular and defined over K.*

$\underline{n = 4}$*:* (a) *ϕ is regular,*

(b) *K is totally real, and*

(c) *each irreducible component of each D_σ, $\sigma \in \Delta(2)$ resp. $\Delta(3)$, admits a dominant morphism defined over K from $\mathbb{A}^1$ resp. $\mathbb{A}^0$.*

Then ξ completes to a family of motivic cohomology classes (see Definition 3.3).

Remark 3.3. (i) For ease of application we have stated the additional requirements for $n = 2, 4$ in terms of X^{λ}, D; whereas for $n = 3$ they are phrased in terms of $\tilde{X}^{\lambda}, \tilde{D}$. (We are *not* saying all singularities must be A_1's on X^{λ}; just that A_1's are all that remains after passing to $\tilde{X}^{\lambda}$.)

(ii) The additional requirements for $n = 3$ may be significantly relaxed if all we want to do is complete ξ to a class in $H^n_{\mathcal{M}}(\tilde{X}^{\lambda}, \mathbb{Q}(n))$ for some fixed λ. Obviously, taking λ to be very general and spreading out would then also yield a class in $H^n_{\mathcal{M}}(\tilde{\mathcal{X}}_- \times_{\rho, \mathbb{A}^1} U, \mathbb{Q}(n))$ for some étale neighborhood $U \xrightarrow{\rho} \mathbb{A}^1$ — i.e. not on the family $\tilde{\mathcal{X}}_-$ but on a finite pullback. Here are two possibilities:

(1) Drop "general" in (a), drop requirement (b), assume (c)(i) (but only make $\{\mathcal{D}_i\}$ avoid $\mathcal{A} \backslash \mathcal{A} \cap \mathcal{J}$ in the definition of $\mathcal{E}$). If $\tilde{X}^{\lambda}$ is smooth, (c)(i) is empty.

(2) Allow A–D–E singularities (call the set of these $\mathcal{A}'$): more precisely, $\tilde{X}^{\lambda}$ nonsingular except at $\mathcal{A}' \subseteq \mathbb{I}^3 \cap \mathbb{D}^{[2]}$; and each irreducible component of $\mathcal{J}$ contains at most one point of $\mathcal{A}'$. (We should also note that $\tilde{X}^{\lambda}$ is still a [singular] $K3$ surface in this case, and its minimal desingularization is a smooth $K3$.)

(iii) With the caveat that the following simplification comes at the expense of important examples, all three additional requirements (for $n = 3$) may be done away with if we assume ϕ regular: in fact, (a), (b), and (c)(i) collapse.

(iv) We make no claim that this result is exhaustive for $n = 3$ or 4. Indeed, if (for $n = 3$) the general $\tilde{X}^{\lambda}$ is nonsingular and $\mathcal{I} \subset (\cup D^*_{\tilde{\sigma}}) \cap \mathbb{I}^3$ consists of K-rational points (K totally real), then (although we may not have $\mathcal{I} \cap \mathcal{J} = \emptyset$) the conclusion still holds.

Proof. Noting that $\tilde{\mathcal{X}}^*_- \cong (\mathbb{C}^*)^n$ and that the resulting map

$$H^n_{\mathcal{M}}(\tilde{\mathcal{X}}_-, \mathbb{Q}(n)) \xrightarrow{r} H^n_{\mathcal{M}}((\mathbb{C}^*)^n, \mathbb{Q}(n))$$

completes Equation (3.2) to a commutative diagram, it suffices to construct $\Xi \in r^{-1}(\xi)$.

Before doing so, we briefly sketch how the map (ι^{δ}) to $H^n_{\mathcal{M}}(\tilde{X}^{\delta}, \mathbb{Q}(n))$ can be computed explicitly in terms of higher Chow cycles, when $\delta \in \mathcal{L}$ ($\implies \tilde{X}^{\delta}$ is singular with desingularization $\widetilde{\tilde{X}^{\delta}}$). For simplicity, assume $sing(\tilde{X}^{\delta}) =: \mathcal{S}$, $\widetilde{\tilde{X}^{\delta}} \times_{\tilde{X}^{\delta}} \mathcal{S} =: \mathcal{S}'$, and $\tilde{\mathcal{X}}_-$ are smooth: then H^{-n} of

$$\hat{Z}^n(\tilde{X}^{\delta}, -\bullet) := \text{Cone}\left\{ Z^n(\widetilde{\tilde{X}^{\delta}}, -\bullet)_{\mathcal{S}'} \oplus Z^n(\mathcal{S}, -\bullet) \xrightarrow[\text{pullbacks}]{\text{diff. of}} Z^n(\mathcal{S}', -\bullet) \right\} [-1]$$

computes $H^n_{\mathcal{M}}(\tilde{X}^\delta, \mathbb{Q}(n))$. (In general, Z^n of $\mathcal{S}, \mathcal{S}'$ must each be replaced by a Cone complex, also denoted $\hat{Z}^n$.) Assuming Ξ has been produced, and representing it by a cycle in $\underbrace{Z^n(\tilde{\mathcal{X}}_-, n)}_{\mathcal{S}\cup\tilde{X}^\delta}$, a representative of $(\iota^\delta)^*\Xi$ is obtained by pulling back to $\widetilde{\tilde{X}^\delta}$ and $\mathcal{S}$ (which gives a triple of the form $(*, *, 0)$).

Now, we will first explain the construction of Ξ in case the total space $\tilde{\mathcal{X}}_-$ (and fixed general $\tilde{X}^\lambda$) is nonsingular, as is the case when ϕ is regular. (However, we don't assume that $\tilde{D}$ is a NCD or even that its components are smooth.) In the (commutative) diagram

$$\begin{array}{ccccc}
\xi \in & CH^n((\tilde{\mathcal{X}}^*_-)_K, n) & \xrightarrow{\mathrm{Res}^1_{\tilde{\sigma}}} & CH^{n-1}((D^*_{\tilde{\sigma}} \times \mathbb{A}^1)_K, n-1) & \xleftarrow{\cong} CH^{n-1}((D^*_{\tilde{\sigma}})_K, n-1) \\
\downarrow & \downarrow{\scriptstyle (\iota^\lambda)^*} & & \downarrow & \swarrow \\
\xi^\lambda \in & CH^n((\tilde{X}^\lambda)^*_{\mathbb{C}}, n) & \xrightarrow{\mathrm{Res}^1_{\tilde{\sigma}}} & CH^{n-1}((D^*_{\tilde{\sigma}})_{\mathbb{C}}, n-1), &
\end{array}$$

our hypothesis that ϕ is tempered (together with Proposition 3.1) implies $\mathrm{Res}^1_{\tilde{\sigma}}\xi^\lambda = 0$, hence that $\mathrm{Res}^1_{\tilde{\sigma}}\xi = 0 \ \forall \ \tilde{\sigma} \in \tilde{\Delta}(1)$. The local-global spectral sequence

$$E_1^{i,-j}(n) := \begin{cases} CH^n(\tilde{\mathcal{X}}^*_-, j)\,[\cong H^{2n-j}_{\mathcal{M}}((\mathbb{C}^*)^n, \mathbb{Q}(n))] & , \ i = 0, \\ \oplus_{\tilde{\sigma}\in\tilde{\Delta}(i)} CH^{n-i}(D^*_{\tilde{\sigma}} \times \mathbb{A}^1, j-i) & , \ i > 0, \\ 0 & , \ i < 0 \end{cases}$$

with $d_1 : E_1^{0,-n}(n) \to E_1^{1,-n}(n)$ given by $\oplus_{\tilde{\sigma}\in\tilde{\Delta}(1)}\mathrm{Res}^1_{\tilde{\sigma}}$, has

$$\begin{aligned} E_\infty^{0,-n}(n) &\cong \mathrm{im}\left\{CH^n(\tilde{\mathcal{X}}_-, n) \to CH^n(\tilde{\mathcal{X}}^*_-, n)\right\} \\ &\cong \bigcap \ker\left\{d_i : E_i^{0,-n}(n) \to E_i^{i,-n-i+1}(n)\right\} \\ &\cong \begin{cases} \ker(d_1) & , \ \text{for } n = 2, 3, \\ \ker(d_1) \cap \ker(d_2) & , \ \text{for } n = 4 \end{cases} . \end{aligned}$$

The intersection has meaning since $E_{i+1}^{0,-n} = \ker(d_i) \subset E_i^{0,-n}$. (Warning: the d_i are not the above Res^i for $i > 1$; see [47] for a description.) So for $n = 2, 3$ we automatically get the desired class $\Xi \in CH^n(\tilde{\mathcal{X}}_-, n) \cong H^n_{\mathcal{M}}(\tilde{\mathcal{X}}_-, \mathbb{Q}(n))$.

For $n = 4$, the stated conditions imply that the $\{D^*_{\tilde{\sigma}}\}_{\tilde{\sigma}\in\tilde{\Delta}(2)}$ are Zariski-open subsets $U \subseteq \mathbb{A}^1_K$ (obtained by omitting points with coordinates $\in K$). Since $CH^1(\mathrm{pt.}, 3)$ is zero, $CH^2(U, 3) \cong CH^2(\mathbb{A}^1_K, 3) \cong CH^2(\mathrm{Spec}(K), 3) \cong$

$K_3^{\mathrm{ind}}(K) = 0$ for K totally real (# field); since $E_2^{2,-5}(4)$ is a subquotient of $\oplus_{\tilde{\sigma}\in\tilde{\Delta}(2)} CH^2((D^*_{\tilde{\sigma}} \times \mathbb{A}^1)_K, 3)$ we are done.

So we have reduced to examining additional complications arising from the case of $\tilde{\mathcal{X}}_-$ singular insofar as this is allowed by the conditions of the theorem. If $n = 2$, the singularities occur in $\tilde{D} \times \mathcal{L}$ and are always rational (surface) singularities of type A_1, A_2, or A_3 (see [2] for definition). The last observation is verified using the table of 16 two-dimensional reflexive polytopes in [18]. Briefly, a singularity $Q \in \mathrm{sing}(\tilde{\mathcal{X}}_-)$ occurs due to a multiple root r_Q of $\phi_\sigma(x_1^\sigma)$ for some $\sigma \in \Delta(1)$. In a neighborhood of $\{(x_1^\sigma - r_Q,\, x_2,\, \lambda - \delta) = (0,0,0)\} = Q$ the equation of $\tilde{\mathcal{X}}_-$ is of the form

$$0 = (x_1^\sigma - r_Q)^k \Psi_1(x_1^\sigma - r_Q) \,+\, (x_2^\sigma)^{\ell(>0)} \Psi_2(x_1^\sigma - r_Q, x_2) \,-\, (\lambda - \delta) \\ \times\, (x_1^\sigma - r_Q) x_2^\sigma - (\lambda - \delta) x_2^\sigma,$$

where Ψ_1, Ψ_2 are holomorphic ($\neq 0$ at Q) and $2 \leq k \leq 4$. (Note $(\lambda - \delta)x_2^\sigma$ is quadratic and nonzero, and is not canceled out.) At any rate, the canonical desingularization [2] produces $\widetilde{\tilde{\mathcal{X}}_-} \xrightarrow{b} \tilde{\mathcal{X}}_-$ with $b^{-1}(Q) =$ a chain $\mathbb{R}_Q$ of (1, 2, or 3) rational curves for each $Q \in \mathrm{sing}(\tilde{\mathcal{X}}_-)$. Writing $\widetilde{\tilde{\mathcal{X}}_-}^{\,*} := b^{-1}(\tilde{\mathcal{X}}_-^*) \cong (\mathbb{C}^*)^2$, there are some extra Res^1's of $\xi \in CH^2(\widetilde{\tilde{\mathcal{X}}_-}^{\,*}, 2)$ to deal with, in $CH^1(\mathbb{U}_Q, 1)$ for $\mathbb{U}_Q \subseteq \mathbb{R}_Q$ Zariski open. But this is clearly just (for $Q = \{(r_Q, \delta)\} \in D_{\tilde{\sigma}} \times \mathcal{L}$ as above) $\{r_Q\}$, which is necessarily a root of unity (due to the tempered requirement), hence trivial. So ξ comes from $\Xi \in CH^2(\widetilde{\tilde{\mathcal{X}}_-}, 2)$. In view of the long-exact sequence [with $\sqcup = \sqcup_{Q\in\mathrm{sing}(\tilde{\mathcal{X}}_-)}$]

$$\to H^2_{\mathcal{M}}(\tilde{\mathcal{X}}_-, \mathbb{Q}(2)) \to CH^2(\widetilde{\tilde{\mathcal{X}}_-}, 2) \oplus CH^2(\sqcup Q, 2) \to H^2_{\mathcal{M}}(\sqcup \mathbb{R}_Q, 2) \to$$

and the identification of $CH^2(Q,2)$ and $H^2_{\mathcal{M}}(\mathbb{R}_Q, \mathbb{Q}(2))$ (working over $\bar{K} = \bar{\mathbb{Q}}$) with $K_2^M(\bar{\mathbb{Q}}) = 0$, Ξ descends to $H^2_{\mathcal{M}}(\tilde{\mathcal{X}}_-, \mathbb{Q}(2))$.

If $n = 3$, then we admit fiberwise A_1-singularities α; since these live in $\tilde{D}^{[2]}$, their location in $\mathbb{P}_{\tilde{\Delta}}$ is fixed as λ varies. So for each $\alpha \in \mathcal{A}$, $\{\alpha\} \times \mathbb{A}^1 \subseteq \mathrm{sing}(\tilde{\mathcal{X}}_-)$. Since these are ordinary double points, a minimal resolution for the generic fiber is effected merely by blowing up $\mathbb{P}_{\tilde{\Delta}}$ at each α. (The proper transform $\hat{\mathcal{X}}_- \subset Bl_{\mathcal{A}}(\tilde{\mathcal{X}}_-)$ of $\tilde{\mathcal{X}}_-$ is still possibly singular over a discriminant set $=: \mathcal{L} \subset \mathbb{A}^1$.) We write $\hat{\mathcal{X}}_- \xrightarrow{B} \tilde{\mathcal{X}}_-$ for the resulting morphism, which has its own "exceptional divisors" $B^{-1}(\alpha \times \mathbb{A}^1)$ and proper transforms $\hat{D}\,(\times\mathbb{A}^1)$ of $\tilde{D}\,(\times\mathbb{A}^1)$.

Let $\mathbb{P}^2_\alpha$ denote the exceptional divisor in $Bl_{\mathcal{A}}(\mathbb{P}_{\tilde{\Delta}})$ over $\alpha \in D_{\tilde{\sigma}}$, $\tilde{\sigma} \in \tilde{\Delta}(2)$; and let X, Y, Z be homogeneous coordinates with $X = 0$, $Y = 0$ the equations of $\mathbb{P}^2_\alpha \cap \hat{\mathbb{D}}_{\tilde{\sigma}_1}$, $\mathbb{P}^2_\alpha \cap \hat{\mathbb{D}}_{\tilde{\sigma}_2}$ (where $\tilde{\sigma}_1, \tilde{\sigma}_2$ are the facets of $\tilde{\Delta}$ meeting $\tilde{\sigma}$). The equation for $B^{-1}(\alpha \times \mathbb{A}^1) \subseteq \mathbb{P}^2_\alpha \times \mathbb{A}^1_{(\lambda)}$ must be of the form

$$f(X, Y, Z) + \lambda XY = 0 \tag{3.3}$$

with $f \not\equiv 0$ of homogeneous degree 2.

Let $\{p_i\}_{i=1}^4$ denote the (not necessarily distinct) points of intersection of $f = 0$ and $XY = 0$. Stereographic projection, say, through p_1 to the $Z = 0$ line "uniformizes" the conic (uniformly in λ), so that $B^{-1}(\alpha \times \mathbb{A}^1) \cong \mathbb{P}^1 \times \mathbb{A}^1 =: \mathbb{P}^1_\alpha \times \mathbb{A}^1$. (If (c)(ii) holds, then this can be done over K.) Clearly the $\{p_i\}$ are the points where $\hat{D}_{\tilde{\sigma}_1}, \hat{D}_{\tilde{\sigma}_2}$ meet the conic (3.3). Since they and their images $q_i \in \mathbb{P}^1_\alpha$ under projection are constant in λ, we see that

$$B^{-1}(\alpha \times \mathbb{A}^1) \cap (\hat{D}_{\tilde{\sigma}_j} \times \mathbb{A}^1) = \left\{ \begin{array}{ll} q_1 \cup q_2 & \text{if } j = 1 \\ & \\ q_3 \cup q_4 & \text{if } j = 2 \end{array} \right\} \times \mathbb{A}^1 \subseteq \mathbb{P}^1_\alpha \times \mathbb{A}^1,$$

for $j = 1, 2$.

Suppose a component $D_{\mathcal{I}}$ of (say) $D_{\tilde{\sigma}_1}$ passing through α belongs to $\mathcal{I}$. Since $\mathcal{I} \subseteq \mathbb{I}^3 \cap \tilde{\mathbb{D}}$, some $x_i \equiv 1$, and another $x_j \equiv 0$ or ∞ on $D_{\mathcal{I}}$. Hence $D_{\mathcal{I}}$ is a double line (double in the sense of the multiplicity of $\tilde{X}^\lambda \cdot D_{\tilde{\sigma}_1}$ there); this means that $p_1 = p_2$ and no other components of $D_{\tilde{\sigma}_1}$ pass through α. It follows that any component of $\mathcal{J}$ passing through α belongs to $D_{\tilde{\sigma}_2}$ and has tangent line (at α) distinct from $T_\alpha D_{\mathcal{I}}$ (i.e., $\{p_1, p_2\}$ and $\{p_3, p_4\}$ are disjoint). Since $\mathcal{I} \cap \mathcal{J} \subseteq \mathcal{A}$, this argument makes it clear that the proper B-transforms of $\mathcal{I}\,(\times \mathbb{A}^1)$ and $\mathcal{J}\,(\times \mathbb{A}^1)$ do not meet.

Now $\hat{\mathcal{S}} := \operatorname{sing}(\hat{\mathcal{X}}_-) \subseteq \hat{\mathcal{I}} \times \mathcal{L}$, hence does not intersect $\hat{\mathcal{J}} \times \mathbb{A}^1$ (the proper transform of $\mathcal{J} \times \mathbb{A}^1$). Let $\widetilde{\hat{\mathcal{X}}_-} \xrightarrow{\beta} \hat{\mathcal{X}}_-$ be a desingularization (which is an $\cong$ off $\operatorname{sing}(\hat{\mathcal{X}}_-)$), and write $\mathcal{Q}_\alpha := \beta^{-1}(\mathbb{P}^1_\alpha \times \mathbb{A}^1_{(\lambda)})$, $\cup_{\alpha \in \mathcal{A}} \mathcal{Q}_\alpha =: \mathcal{Q}$. Obviously $\beta^{-1}(\hat{\mathcal{J}} \times \mathbb{A}^1) \cong \hat{\mathcal{J}} \times \mathbb{A}^1$, so we may write $\mathcal{Q}^- := \mathcal{Q} \backslash (\hat{\mathcal{J}} \times \mathbb{A}^1) \cap \mathcal{Q}$; the $\mathcal{Q}_\alpha$ are rational surfaces, and the $\mathcal{Q}^-_\alpha$ have rational curves missing. Finally, put $\mathcal{S} := \operatorname{sing}(\tilde{\mathcal{X}}_-) = B(\hat{\mathcal{S}}) \cup (\mathcal{A} \times \mathbb{A}^1)$ and $b := B \circ \beta : \widetilde{\hat{\mathcal{X}}_-} \to \tilde{\mathcal{X}}_-$, and note that $b^{-1}(\mathcal{S}) = \beta^{-1}(\hat{\mathcal{S}}) \cup \mathcal{Q}$. As above, we want to use the l.e.s.

$$\begin{aligned} &\to H^3_{\mathcal{M}}(\tilde{\mathcal{X}}_-, \mathbb{Q}(3)) \to CH^3(\widetilde{\hat{\mathcal{X}}_-}, 3) \oplus H^3_{\mathcal{M}}(\mathcal{S}, \mathbb{Q}(3)) \\ &\xrightarrow{i^* - b^*} H^3_{\mathcal{M}}(b^{-1}(\mathcal{S}), \mathbb{Q}(3)) \to \end{aligned}$$

to obtain a class Ξ in the first term from a pair $(\Xi_0, 0)$ in the middle, with $i^*\Xi_0 = 0$.

To construct Ξ_0, begin with the coordinate symbol $\xi \in Z^3((\widetilde{\hat{\mathcal{X}}_-} \backslash b^{-1}(\tilde{D})) \cong (\mathbb{C}^*)^3, 3)$, which (as $\mathcal{I} \subseteq \mathbb{I}^3$) obviously extends to $\xi \in Z^3_{\partial_{\mathcal{B}}-cl}(\widetilde{\hat{\mathcal{X}}_-} \backslash \hat{\mathcal{J}} \times \mathbb{A}^1, 3)_{\beta^{-1}(\hat{\mathcal{S}} \cup \mathcal{Q}^-)}$. (It actually pulls back to 0 on $\beta^{-1}(\hat{\mathcal{S}})$ and $\mathcal{Q}^-$.) Clearly the Res^1's are all 0. Combining this with the moving lemmas of Levine and Bloch, there exist $\Gamma \in Z^3(\widetilde{\hat{\mathcal{X}}_-} \backslash \hat{\mathcal{J}} \times \mathbb{A}^1, 4)_{\beta^{-1}(\hat{\mathcal{S}}) \cup \mathcal{Q}^-}$ and $\Xi_0 \in Z_{\partial_{\mathcal{B}}-cl}(\widetilde{\hat{\mathcal{X}}_-}, 3)_{\beta^{-1}(\hat{\mathcal{S}}) \cup \mathcal{Q}\,[=b^{-1}(\mathcal{S})]}$ such that $\xi + \partial_{\mathcal{B}}\Gamma$ is the restriction of Ξ_0. The pull-back of Ξ_0 to $b^{-1}(\mathcal{S})$ gives a cocycle in the complex computing $H_{\mathcal{M}}$, $\hat{Z}^3(b^{-1}(\mathcal{S}), -\bullet) :=$

$$\mathit{Cone}\left\{\hat{Z}^3(\beta^{-1}(\hat{\mathcal{S}}), -\bullet)_{\beta^{-1}(\hat{\mathcal{S}}) \cap \mathcal{Q}} \oplus Z^3(\mathcal{Q}, -\bullet)_{\beta^{-1}(\hat{\mathcal{S}}) \cap \mathcal{Q}} \to \hat{Z}^3(\beta^{-1}(\hat{\mathcal{S}}) \cap \mathcal{Q}, -\bullet)\right\}[-1].$$

This can be "moved" by a coboundary (in the cone complex) to essentially an element of $Z^3_{\partial_{\mathcal{B}}-\mathrm{cl}}(\mathcal{Q}, 3)_{\beta^{-1}(\hat{\mathcal{S}}) \cap \mathcal{Q}}$ supported on $\mathcal{Q} \cap \hat{\mathcal{J}} \times \mathbb{A}^1$. Moreover, the components of $\mathcal{Q}_\alpha \cap \hat{\mathcal{J}} \times \mathbb{A}^1$ ($\alpha \in \mathcal{A}$) are pairwise disjoint $\mathbb{A}^1$'s which are $\overset{\mathrm{rat}}{\equiv}$ (as divisors) on $\mathcal{Q}_\alpha$ by functions $\hat{f}_\alpha \in \bar{\mathbb{Q}}(\mathcal{Q}_\alpha)$ restricting to 1 on $\mathcal{Q}_\alpha \cap \beta^{-1}(\hat{\mathcal{S}})$. (Pull back to $\mathcal{Q}_\alpha$ $f \in \bar{\mathbb{Q}}(\mathbb{P}^1_\alpha)^*$ which has $(f) = q_3 - q_4$ and $f(q_1 = q_2) = 1$, in the only nontrivial situation.) Since $CH^2(\mathbb{A}^1, 3) \cong CH^2(\mathrm{pt.}, 3)$ one can move the elements of $Z^3(\mathcal{Q}_\alpha, 3)$ so as to make them constant along each of the supporting $\mathbb{A}^1$'s, and then "collect" all these constant cycles along only one such $\mathbb{A}^1$, by using [$\partial_{\mathcal{B}}$-coboundaries of] cycles (of the form $\mathfrak{A} \otimes \hat{f}_\alpha \in Z^3(\mathcal{Q}_\alpha, 4)$) restricting to 0 at $\mathcal{Q}_\alpha \cap \beta^{-1}(\hat{\mathcal{S}})$. The constant $\mathbb{A}^1$-supported cycles are then killed by adding constant cycles on the $b^{-1}(\mathcal{D}_j \times \mathbb{A}^1) \cong \mathcal{D}_j \times \mathbb{A}^1$ to Ξ_0, via $Z^2(\mathcal{D}_j \times \mathbb{A}^1, 3) \hookrightarrow Z^3(\widetilde{\hat{\mathcal{X}}_-}, 3)$. That we have "enough" $\mathcal{D}_j$'s to kill all constant cycles on the $\mathcal{Q}_\alpha$'s is guaranteed (if (c)(i) holds) by surjectivity of $\mathcal{E}$. Alternatively, if (c)(ii) holds then all of the above is valid over K (as opposed to $\bar{K}$), and K totally real $\implies$ the $CH^2(\mathbb{A}^1_K, 3)$-classes embedded in the $\mathcal{Q}_\alpha$'s self-annihilate. □

3.3. Examples of ϕ satisfying the Theorem

Here are specific ways to realize the conditions of the Theorem (in particular, the tempered condition); ϕ is defined over a number field K as usual.

Corollary 3.1. *Let ϕ be reflexive with cyclotomic edge polynomials and root-of-unity vertex coefficients. Furthermore for*

$\underline{n=2}$: *Assume the general X_t is nonsingular.*

$\underline{n=3}$: *Assume the facets of Δ have no interior points, and that ϕ is regular.*

$\underline{n=4}$: *Assume the facets of Δ are elementary three-simplices (all points of Δ other than $\{\underline{0}\}$ are vertices), with coefficients ± 1 only (except at $\{\underline{0}\}$).*[16]

Then ξ completes.

Example 3.1. Take ϕ to be an arbitrary constant plus the characteristic (Laurent) polynomial of the *vertex set* of any reflexive polytope Δ satisfying the relevant assumption in boldface. This will be regular in case $n=2,4$, and also for $n=3$ provided none of the facets are of the form (c) (see proof below) with $\frac{a}{2^m}$, $\frac{b}{2^m}$ both odd for the same $m \in \mathbb{Z}^{\geq 0}$. Out of the 899 reflexive three-polytopes with interior-point-free facets, this leaves us with 239 [65].

Remark 3.4. For $n=3$, we can also allow triangular facets σ with interior points, provided the only monomials appearing (with nonzero coefficients) in ϕ_σ correspond to the vertices of σ. This gets us up to 1071 resp. 358 three-polytopes, depending on whether the special type (c) facets are admitted [65].

Proof of Corollary. For $n=2$ it suffices to show ϕ tempered, and this is obvious.

For $n=3$, one can easily classify (up to shift and unimodular transformation) facets σ with no interior points. Viewed in a two-plane $\mathbb{R}_\sigma$, they are all convex hulls of three or four points: (a) $\{(0,0), (2,0), (0,2)\}$, (b) $\{(0,0), (0,1), (a,0)\}$, or (c) $\{(0,0), (0,1), (a,0), (b,1)\}$ (with $a,b \in \mathbb{N}$). In each case $\phi_\sigma(x_1^\sigma, x_2^\sigma) = 0$ can only yield ($D_\sigma^* =$) a Zariski open subset of a rational curve. (Since ϕ is regular, D_σ is also nonsingular.) For $\sigma' \in \Delta(2)$, $\phi_{\sigma'}$ cyclotomic implies that $\{x_1^{\sigma'}\}$ gives 0 in $CH^1(D_{\sigma'}^*, 1)$. Hence (for $\sigma \in \Delta(1)$) $\{x_1^\sigma, x_2^\sigma\} \in \left\{\ker(\text{Tame}) \subseteq CH^2(D_\sigma^*, 2)\right\} = \text{im}\left\{\text{CH}^2(D_\sigma, 2) \to CH^2(D_\sigma^*, 2)\right\}$. But $CH^2(\mathbb{P}^1_K, 2) \cong K_2^M(K) = 0$ (in fact, $K_2^M(\bar{\mathbb{Q}}) = 0$), and so ϕ is tempered. The remaining conditions follow from regularity by Remark 3.3(iii).

For $n=4$, the tempered condition is again clear for edges $\sigma'' \in \Delta(3)$, so fix $\sigma' \subset \sigma$, $\sigma \in \Delta(1)$ and $\sigma' \in \Delta(2)$; σ is a triangle and σ' a tetrahedron. Any two edges of σ' (viewed as integral vectors) generate $\mathbb{R}_{\sigma'} \cap \mathbb{Z}^4$, and so one may choose the monomials $x_1^{\sigma'}, x_2^{\sigma'}$ so that $\phi_{\sigma'} = 1 + x_1^{\sigma'} + x_2^{\sigma'}$ (ignoring the ± 1 issue). This makes plain the $\mathbb{A}^1_{\bar{\mathbb{Q}}}$-uniformizability of $D_{\sigma'}$ (condition

[16] There are 151 such reflexive four-polytopes, with a maximum of 12 vertices. [65]

(c) of Theorem 3.1), since $\phi_{\sigma'} = 0$ is the equation of D^*_σ (in local toric coordinates); it is also clear that $\{x_1^{\sigma'}, x_2^{\sigma'}\} \in CH^2(D^*_{\sigma'}, 2)$ vanishes. Next, one can choose monomials $x_1^\sigma (:= x_1^{\sigma'})$, $x_2^\sigma (:= x_2^{\sigma'})$, x_3^σ generating $\mathbb{R}_\sigma \cap \mathbb{Z}^4$ such that $\phi_\sigma = 1 + x_1^\sigma + x_2^\sigma + (x_1^\sigma)^a (x_2^\sigma)^b (x_3^\sigma)^c$ $(a, b \in \mathbb{Z}^{\geq 0}$, $c \in \mathbb{N})$. We must show that $\{x_1^\sigma, x_2^\sigma, x_3^\sigma\}$ vanishes in $CH^3(D^*_\sigma, 3)$, where $D^*_\sigma \cong \{(x_1^\sigma, x_2^\sigma, x_3^\sigma) \in (\mathbb{C}^*)^3 \,|\, \phi_\sigma(\underline{x}^\sigma) = 0\}$. This requires a short calculation for which we rewrite $x_i^\sigma =: y_i$ and write elements of $CH^3(D^*_\sigma, 3)$ as symbols — as if they were in $K_3^M(\bar{\mathbb{Q}}(D_\sigma))$. However, we have explicitly checked that the following relations actually hold over D^*_σ (for the relevant graph cycles) and not just $\eta_{D^*_\sigma}$:

$$\{y_1, y_2, y_3\} = \frac{1}{c}\left\{y_1, y_2, y_1^a y_2^b y_3^c\right\} = \frac{1}{c}\left\{-\frac{y_1}{y_2}, -y_2, -y_1^a y_2^b y_3^c\right\}$$
$$= \frac{1}{c}\left\{-\frac{y_1}{y_2}, -\left(1 + \frac{y_1}{y_2}\right) y_2, -y_1^a y_2^b y_3^c\right\} = \frac{1}{c}\left\{-\frac{y_1}{y_2}, -(y_1 + y_2), -y_1^a y_2^b y_3^c\right\}.$$

Using $1 + y_1 + y_2 + y_1^a y_2^b y_3^c = 0$ yields

$$\frac{1}{c}\left\{-\frac{y_1}{y_2}, -(y_1 + y_2), 1 + (y_1 + y_2)\right\},$$

which is zero (again over all of D^*_σ). Hence ϕ is tempered. Regularity of ϕ (i.e., Δ-regularity of $\phi - \lambda$ for general λ) along the faces is obvious from the explicit equations for $\phi_\sigma, \phi_{\sigma'}, \phi_{\sigma''}$ (and irregularities in the torus $(\mathbb{C}^*)^4$ for generic λ are impossible by a simple calculus argument). □

Example 3.2. For $n = 4$, there are examples (where ξ completes) that do not fall under the aegis of Theorem 3.1 — e.g., $\phi = x_1^{-1} x_2^{-1} x_3^{-1} x_4^{-1} (1 + \sum_{i=1}^4 x_i^5)$, which gives the Fermat quintic family in $\mathbb{P}^4$. One must verify directly that $\langle\{\underline{x}\}\rangle \in CH^4(\tilde{\mathcal{X}}^*_-, 4)$ lies in $\ker(d_1) \cap \ker(d_2)$, in the local-global spectral sequence described in the Theorem's proof. This means checking that the residues of (a representative of) $\langle\{\underline{x}\}\rangle$ in $\oplus_{\tilde{\sigma} \in \tilde{\Delta}(1)} Z^3(D^*_{\tilde{\sigma}} \times \mathbb{A}^1, 3)$ are killed by relations (in $Z^3(D^*_{\tilde{\sigma}} \times \mathbb{A}^1, 4)$), then that differences of residues of these relations in $\oplus_{\tilde{\sigma} \in \tilde{\Delta}(2)} Z^2(D^*_{\tilde{\sigma}} \times \mathbb{A}^1, 3)$ are trivialized as well. This is left to the reader.

Remark 3.5. For $n = 2$, one can sometimes avoid going modulo torsion and complete ξ to a class $\tilde{\Xi} \in H^2_{\mathcal{M}}(\widetilde{\tilde{\mathcal{X}}_-}, \mathbb{Z}(2))$ ($\cong CH^2(\widetilde{\tilde{\mathcal{X}}_-}, 2)$ but without our implicit $\otimes\mathbb{Q}$ convention). Namely, for each edge σ, let $x^\sigma_{(1)} = x_1^{a_\sigma} x_2^{b_\sigma}$ (where $(a_\sigma, b_\sigma) = 1$) generate $\mathbb{R}_\sigma \cap \mathbb{Z}^2$. Then it suffices to require (besides smoothness of the general X^λ) the edge polynomial ϕ_σ to have only (-1)

as root if a_σ and b_σ are both odd, and only $(+1)$ as root otherwise. This follows simply from (integral) computation of the $Tame$ symbol of $\{x_1, x_2\}$.

We conclude this section with a discussion of what can be done for an arbitrary reflexive three-polytope Δ if we are only after getting a Ξ^λ for general λ (as in Remark 3.3(ii)). An arbitrary facet $\sigma \in \Delta(1)$ inherits the integral structure $\mathbb{Z}^3 \cap \mathbb{R}_\sigma$ (and is obviously not in general itself reflexive).

Fact 3.1. [65] *Up to shift and unimodular transformation, there are* 344 *possibilities for* σ*, and they all satisfy* $\ell(\sigma) > 2\ell^*(\sigma)$*.*

Fix an isomorphism $\mathbb{Z}^2 \overset{\cong}{\to} \mathbb{Z}^3 \cap \mathbb{R}_\sigma$, and denote the corresponding toric coordinates on $\mathbb{D}^*_{\tilde{\sigma}}$ by x_1^σ, x_2^σ. Writing $\ell'(\sigma) := \ell(\sigma) - \ell^*(\sigma) - 1$, let $\mathfrak{M}_\sigma = \mathfrak{M}^*_\sigma \cup (\mathfrak{M}_\sigma \setminus \mathfrak{M}^*_\sigma) = \{\underline{m}_i^*\}_{i=1}^{\ell^*(\sigma)} \cup \{\underline{m}_j'\}_{j=0}^{\ell'(\sigma)}$ be the decomposition of $\sigma \cap \mathbb{Z}^2$ into interior and edge points. The ample linear system $|\mathcal{O}_{\mathbb{D}_{\tilde{\sigma}}}(1)| \cong \mathbb{P}^{\ell(\sigma)-1}$ is parametrized by Laurent polynomials

$$\phi_{\sigma;[\underline{\alpha}:\underline{\beta}]}(\underline{x}^\sigma) := \sum_{i=1}^{\ell^*(\sigma)} \alpha_i \cdot (\underline{x}^\sigma)^{\underline{m}_i^*} + \sum_{j=0}^{\ell'(\sigma)} \beta_j \cdot (\underline{x}^\sigma)^{\underline{m}_j'} = A_{\underline{\alpha}}(\underline{x}^\sigma) + B_{\underline{\beta}}(\underline{x}^\sigma),$$

and consists (generically) of genus-$\ell^*(\sigma)$ curves. Let $\mathcal{V}_\sigma^{irr} \subset \mathbb{P}^{\ell(\sigma)-1}$ be the locus of (ϕ_σ cutting out) $\ell^*(\sigma)$-nodal irreducible rational curves C_{ϕ_σ} in this system. It seems entirely reasonable to hope that

$$\mathcal{V}_\sigma^{irr} \text{ is nonempty for all } \sigma \in \Delta(1) \tag{3.4}$$

is satisfied for all reflexive $\Delta \subset \mathbb{R}^3$; this may be decidable by applying the tropical methods of [57]. In fact one has

Fact 3.2. [57,80] *If* $\mathcal{V}_\sigma^{irr} \neq \emptyset$*, its Zariski closure* $\overline{\mathcal{V}_\sigma^{irr}}$ *(the so-called Severi variety) is a codimension-*$\ell^*(\sigma)$ *irreducible subvariety of* $\mathbb{P}^{\ell(\sigma)-1}$*.*

Here, then, is our "most general" example for $n = 3$:

Proposition 3.2. *For a reflexive three-polytope* Δ *satisfying* (3.4)*, there exists a tempered Laurent polynomial* ϕ *(with Newton polytope* Δ*) defining a family of (generically smooth)* $K3$ *surfaces* $\{\tilde{X}_t\}$ *such that (for general* t*) the toric symbol completes to a* $CH^3(\tilde{X}_t, 3)$*-class* Ξ_t*.*

Proof. Let $\mathcal{U} \subset \mathbb{P}^{\ell(\sigma)-1}$ be the complement of the $\mathbb{P}^{\ell^*(\sigma)-1}$ defined by $\underline{\beta} = \underline{0}$. Since $\dim(\overline{\mathcal{V}_\sigma^{irr}}) = \ell(\sigma) - \ell^*(\sigma) - 1 > \ell^*(\sigma) - 1$ by Facts 3.1 and 3.2,

$\overline{\mathcal{V}_\sigma^{irr}} \cap \mathcal{U} \neq \emptyset$. Consider the projection $\mathcal{U} \overset{\rho}{\to} \mathbb{P}^{\ell'(\sigma)}$ induced by $[\underline{\alpha} : \underline{\beta}] \mapsto [\underline{\beta}]$; we contend that its restriction to $\overline{\mathcal{V}_\sigma^{irr}} \cap \mathcal{U}$ is generically an immersion.

Indeed, otherwise a generic $C_{\phi_\sigma} \in \mathcal{V}_\sigma^{\text{irr}}$ deforms while keeping its intersection with the boundary $\underbrace{\mathbb{D}_{\tilde{\sigma}} \setminus (\mathcal{C}^*)^2}$ =: D fixed. The normal bundle of the composition $f : \mathbb{P}^1 \cong \widetilde{C_{\phi_\sigma}} \twoheadrightarrow C_{\phi_\sigma} \hookrightarrow \mathbb{D}_{\tilde{\sigma}}$ is $N_f := f^*(\theta^1_{\mathbb{D}_{\tilde{\sigma}}})/\theta^1_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(-2 + f^*(\mathsf{D}))$. A deformation of this form would yield a nonzero section of $N_f(-f^*(\mathsf{D})) \cong \mathcal{O}_{\mathbb{P}^1}(-2)$, which is impossible.

Since $\dim(\overline{\mathcal{V}_\sigma^{\text{irr}}}) = \ell'(\sigma)$ we conclude that $\rho(\mathcal{V}_\sigma^{\text{irr}} \cap \mathcal{U}) \subset \mathbb{P}^{\ell'(\sigma)}$ is open, and therefore contains a Zariski-dense subset corresponding to cyclotomic edge polynomials (with distinct roots on each edge). So we get countably many $\phi_{\sigma;[\underline{\alpha}:\underline{\beta}]}$ defining irreducible nodal rational curves C_{ϕ_σ} with regular, cyclotomic edge polynomials; and $\underline{\alpha}, \underline{\beta}$ can be taken to lie in $\bar{\mathbb{Q}}$.

Globalizing this to the three-polytope, there is a choice of $\phi(x_1, x_2, x_3)$, all of whose facet polynomials ϕ_σ are of this form. Clearly, ϕ is tempered if the classes $\{x_1^\sigma, x_2^\sigma\} \in K_2(\bar{\mathbb{Q}}(\widetilde{C_{\phi_\sigma}})) \cong K_2(\bar{\mathbb{Q}}(\mathbb{P}^1))$ vanish. But since the edges of ϕ_σ are cyclotomic, $\{x_1^\sigma, x_2^\sigma\} \in \ker(\text{Tame}) = K_2(\bar{\mathbb{Q}}) = \{0\}$. □

4. The fundamental regulator period

The one-parameter families $\{\tilde{X}_t\}$ of CY toric hypersurfaces produced by Theorem 3.1 have in a neighborhood of $t = 0$ a canonical family of cycles $\tilde{\varphi}_t$ vanishing (in $H_{n-1}(\tilde{X}_0)$) at $t = 0$. What we aim to do in this section, is to pair $\tilde{\varphi}_t$ against the regulator image

$$AJ(\Xi_t) \in H^{n-1}(\tilde{X}_t, \mathbb{C}/\mathbb{Q}(n)) \cong \text{Hom}_{\mathbb{Q}}\left(H_{n-1}(\tilde{X}_t, \mathbb{Q}), \mathbb{C}/\mathbb{Q}(n)\right)$$

over a punctured disk $\bar{D}^*_{|t_0|}(0)$ extending to the singular fiber (at $t_0 \in \mathcal{L}$) nearest the one at $t = 0$. The resulting (multivalued) function is called the "fundamental regulator period;" the "fundamental period" is just the period of a canonical holomorphic form $\tilde{\omega}_t \in \Omega^{n-1}(\tilde{X}_t)$ over $\tilde{\varphi}_t$. The regulator computation has some surprisingly beautiful and easy corollaries related to differential equations, number theory, and local mirror symmetry.

For the next two subsections, it will suffice to assume

(a) ϕ is reflexive with root-of-unity vertex coefficients (denoted ζ);

(b) the generic $\tilde{X}_t$ has at worst Gorenstein orbifold singularities — in this case $\mathcal{L} \subset \mathbb{P}^1$ records only the "more" singular fibers where the local system $R^{n-1}\tilde{\pi}_*\mathbb{Q}$ has monodromy — and these lie in $\tilde{D}$; and

(c) ξ completes to $\Xi_t \in H^n_{\mathcal{M}}(\tilde{X}_t, \mathbb{Q}(n))$ as in Definition 3.3.
So in principle n could be > 4. The importance of (a) is that it amounts to a choice of the parameter t normalizing (in fact, for $n = 2$ trivializing) the rational limit mixed Hodge structure at 0.

Remark 4.1. By Lian *et al.* [55], one knows that $R^{n-1}\tilde{\pi}_*\mathbb{Q}$ of the family $\{\tilde{X}_t\}$ has maximal unipotent monodromy about $t = 0$, provided [for $n = 4$] $\mathbb{P}_{\tilde{\Delta}}$ is smooth. Alternately, there is the following simpler argument using the Clemens–Schmid sequence: SSR replaces $\tilde{X}_0$ by a NCD $'\tilde{X}_0$, and

$$H_{n-1}('\tilde{X}_0)(-n+1) \to H^{n-1}('\tilde{X}_0) \to H^{n-1}_{\lim}(\tilde{X}_t) \xrightarrow{N} H^{n-1}_{\lim}(\tilde{X}_t)$$

is exact (with $\mathbb{Q}$-coefficients), where $N = \log(T)$ and weights of $H^{n-1}('\tilde{X}_0)$ [resp. $H_{n-1}('\tilde{X}_0)(-n+1)$] lie in $[0, n-1]$ [resp. $[n-1, 2n-2]$]. So maximal unipotent monodromy of $T \iff N^{n-1} \neq 0 \iff \mathrm{Hom}_{\text{MHS}}(\mathbb{Q}(0), \ker(N)) \neq \{0\} \iff \mathrm{Hom}_{\text{MHS}}(\mathbb{Q}(0), H^{n-1}('\tilde{X}_0)) \neq \{0\} \iff H^0('\tilde{X}_0^{[n-2]}) \to H^0('\tilde{X}_0^{[n-1]})$ is not surjective (where $'\tilde{X}_0^{[i]} :=$ desingularization of ith coskeleton of $'\tilde{X}_0$). The last criterion follows from the fact that the dual graph of $'\tilde{X}_0$ is $\partial\{tr(\Delta^\circ)\}$, which is topologically a triangulation of S^{n-1}.

4.1. The vanishing cycle and fundamental period

Pick a vertex $\underline{v} \in \Delta(n)$ and $\tilde{v} \in \tilde{\Delta}(n)$ lying over it as in the end of Section 2.5. The local affine equation for $\tilde{X}^\lambda$ in $U_{\tilde{v}}$ is obtained by dividing out the $\zeta \underline{x}^{\underline{v}}$ term from $\lambda - \phi(\underline{x})$ and writing the result in the $\{z_i, \underline{u}_i\}_{i=1}^n$. Organizing terms as in (2.6), we have $0 = \Phi_{\underline{v}}(\underline{z}, \underline{u}) =$

$$1 + \phi_1(z_1) + \phi_2(z_1, z_2; \underline{u}_2) + \cdots + \{\phi_n(z_1, \ldots, z_n; \underline{u}) - \lambda \underline{z}^{\underline{\mu}_1} \underline{u}^{\underline{\mu}_2}\},$$

and

$$\Phi_{\underline{v}, \tilde{\sigma}_i}(z_1, \ldots, z_{n-i}; \underline{u}) := \Phi_{\underline{v}}|_{\mathbb{D}_{\tilde{\sigma}_i}} = 1 + \sum_{k \leq n-i} \phi_k$$

for $i = 1, \ldots, n$. Here the $\mathbb{D}_{\tilde{\sigma}_i}$ are (as in Section 2.5) where $z_{n-i+1} = \cdots = z_n = 0$, with $\mathbb{D}_{\tilde{\sigma}_1}$ given by $z_n = 0$ in particular.

Define on $\mathbb{P}_\Delta$, $\Omega_t \in \Gamma(\hat{\Omega}^n_{\mathbb{P}_\Delta}(\log X_t))$ by

$$\Omega_t := \frac{d\log x_1 \wedge \cdots \wedge d\log x_n}{1 - t\phi(\underline{x})} = \lambda \frac{\bigwedge^n d\log \underline{x}}{\lambda - \phi(\underline{x})},$$

and let

$$\omega_t := \mathrm{Res}_{X_t}(\Omega_{X_t}) \in \hat{\Omega}^{n-1}(X_t);$$

these have μ^*-pullbacks $\tilde{\Omega}_t$, $\tilde{\omega}_t (\in \Omega^{n-1}(\tilde{X}_t))$. Let $\epsilon > 0$ and define the real n-torus

$$\hat{\mathbb{T}}^n_{\underline{v},\epsilon} := \{|z_1| = \cdots = |z_n| = \epsilon\} \cap \mathbb{P}_{\tilde{\Delta}} \in Z^{\text{top}}_n(\mathbb{P}_{\tilde{\Delta}} \backslash \tilde{X}_t \cup \tilde{\mathbb{D}}).$$

For fixed $\epsilon > 0$ it is clear (using $\Phi_{\underline{v}}$ above) that for $|\lambda| >$ some fractional power of $\frac{1}{\epsilon}$, i.e., for $|t| < \delta(\epsilon)$ sufficiently small, $\hat{\mathbb{T}}^n_{\underline{v},\epsilon}$ avoids $\tilde{X}_t$. One has the "membrane"

$$\Gamma_{\underline{v},\epsilon} := \{|z_1| = \cdots = |z_{n-1}| = \epsilon,\ |z_n| \leq \epsilon\} \in C^{\text{top}}_{n+1}(\mathbb{P}_{\tilde{\Delta}} \backslash \tilde{\mathbb{D}}^-)$$

where $\tilde{\mathbb{D}}^- := \bigcup_{\tilde{\sigma} \neq \tilde{\sigma}_1} \mathbb{D}_{\tilde{\sigma}}$; this bounds on the real n-torus:

$$\partial \Gamma_{\underline{v},\epsilon} = (-1)^{n-1} \hat{\mathbb{T}}^n_{\underline{v},\epsilon}.$$

We specify our family of vanishing cycles by demanding that for $|t| < \delta(\epsilon)$

$$-\tilde{\varphi}_t \overset{\text{hom}}{\equiv} \tilde{X}_t \cap \Gamma_{\underline{v},\epsilon} \in Z^{\text{top}}_{n-1}(\tilde{X}_t).$$

Now the exponent vectors $\underline{m}_i$ relating $\{z_i\} \longleftrightarrow \{x_j\}$ ($z_i = \underline{x}^{\underline{m}_i}$) form a rationally invertible matrix. Hence, $\hat{\mathbb{T}}^n_{\underline{v},\epsilon} = \{|x_i| = \epsilon^{q_i}\ (\forall i)\} \subset (\mathbb{C}^*)^n \subset \mathbb{P}_{\tilde{\Delta}}$ for some rational numbers q_i. Note that (only for $n = 4$) the $\{z_i\}$ need not parametrize $\hat{\mathbb{T}}^n_{\underline{v},\epsilon}$ on their own, while the $\{x_i\}$ do. (The $|z_1| = \cdots = |z_n| = \epsilon$ definition conceals the role played by the $\{u_i\}$.) For the fundamental period we have therefore

$$\begin{aligned} (4.1) \qquad A(t) &:= \int_{\tilde{\varphi}_t} \tilde{\omega}_t = \int_{\tilde{\varphi}_t} \text{Res}_{\tilde{X}_t}(\tilde{\Omega}_t) = \frac{1}{2\pi \text{i}} \int_{\text{Tube}(\tilde{\varphi}_t) = \hat{\mathbb{T}}^n_{\underline{v},\epsilon}} \tilde{\Omega}_t \\ &= \frac{1}{2\pi\text{i}} \int_{\cap_{i=1}^n \{|x_i| = \epsilon^{q_i}\}} \left(\sum_{m=0}^{\infty} t^m \phi(\underline{x})^m \right) \bigwedge^n d\log \underline{x} \\ &= (2\pi\text{i})^{n-1} \sum_{m=0}^{\infty} \frac{t^m}{(2\pi\text{i})^n} \oint \phi(\underline{x})^m \bigwedge^n d\log \underline{x} \\ &= (2\pi\text{i})^{n-1} \sum_{m=0}^{\infty} [\phi(\underline{x})^m]_0 t^m, \end{aligned}$$

where $[\cdot]_0$ takes the constant term of a Laurent polynomial. While we proved this for $|t| < \delta(\epsilon)$ (which implies $|t\phi(\underline{x})| < 1$ on $\hat{\mathbb{T}}^n_{\underline{v},\epsilon}$), the period and the power series extend to $D^*_{|t_0|}(0)$ and agree there since both functions are analytic.

4.2. The period of the Milnor regulator current

Given a symbol $\langle\{f_1,\ldots,f_n\}\rangle \in CH^n(Y,n)$ as in Section 3.1 (but with Y *smooth* quasi-projective of dim $< n$), recall from Section 1.2 that $AJ\left\langle\{\underline{f}\}\right\rangle \in H^{n-1}(Y,\mathbb{C}/\mathbb{Q}(n))$ is represented by the regulator current

$$R_n\{\underline{f}\} = \log f_1 d\log f_2 \wedge\cdots\wedge d\log f_n \;-\; (2\pi\mathrm{i})\delta_{T_{f_1}} \wedge R_{n-1}\{f_2,\ldots,f_n\} \;\in\; \mathcal{D}^{n-1}(Y), \tag{4.2}$$

where

$$T_f := f^{-1}\{\mathbb{R}^{\leq 0}\cup\{\infty\},\ \text{oriented from } \infty \text{ to } 0\}$$

is the "cut" in $\arg(f)\in(-\pi,\pi)$. ($R_1\{f\}$ is just the 0-current $\log f$.) Note that in (4.2) we have omitted the $\mathbb{Q}(n)$-valued δ-current; modulo this, R_n is d-closed.

Remark 4.2. (i) Though we will not check this explicitly, the real-admissibility requirements described in Section 1.2 are satisfied in the calculations below.

(ii) If the integral cohomology of Y is torsion-free, as in the case of an open elliptic curve, we can replace $\mathbb{Q}(n)$ by $\mathbb{Z}(n)$.

The vanishing cycle $\tilde{\varphi}_t$ extends to a multivalued section of $\mathbb{R}^{n-1}\tilde{\pi}_*\mathbb{Z}$ over $\mathbb{P}^1\backslash\mathcal{L}$, and

$$\Psi(t) := AJ(\Xi_t)(\tilde{\varphi}_t) \tag{4.3}$$

yields a multivalued holomorphic function. (See the discussion preceding Corollary 4.3; it remains multivalued after going modulo $\mathbb{Q}(n)$, due to monodromy of $\tilde{\varphi}_t$.) We want to compute $\Psi(t)$ for $t\in U_\epsilon := \{|t|<\delta(\epsilon)\ \text{and}\ \arg(t) \in (-\frac{\pi}{4},\frac{\pi}{4})\}$. Consider the diagrams

$$\begin{array}{ccc}
\xi_t := \langle\{x_1,\ldots,x_n\}\rangle & \longleftarrow & \Xi_t \\
CH^n(\tilde{X}_t\backslash\tilde{D},n) & \xleftarrow{\;r_t\;} & H^n_{\mathcal{M}}(\tilde{X}_t,\mathbb{Q}(n)) \\
\downarrow{\scriptstyle AJ} & & \downarrow{\scriptstyle AJ} \\
H^{n-1}(\tilde{X}_t\backslash\tilde{D},\mathbb{C}/\mathbb{Q}(n)) & \xleftarrow{\;\jmath^*\;} & H^{n-1}(\tilde{X}_t,\mathbb{C}/\mathbb{Q}(n)),
\end{array}$$

$$\begin{array}{ccc} \hat{\xi}_t := \langle\{\lambda-\phi(\underline{x}), x_1,\ldots,x_n\}\rangle & \longmapsto & \left(\xi_t, \mathrm{Res}^1_{\tilde{\sigma}}\hat{\xi}_t\right) \\ CH^{n+1}(\mathbb{P}_{\tilde{\Delta}}\backslash\tilde{\mathbb{D}}\cup\tilde{X}_t, n+1) & \xrightarrow{\mathrm{Res}} & CH^n(\tilde{X}_t\backslash\tilde{D}, n)\oplus CH^n(\mathbb{D}^*_{\tilde{\sigma}_1}\backslash D^*_{\tilde{\sigma}_1}, n) \\ \downarrow{\scriptstyle AJ} & & \downarrow{\scriptstyle AJ} \\ H^n(\mathbb{P}_{\tilde{\Delta}}\backslash\tilde{\mathbb{D}}\cup\tilde{X}_t, \mathbb{C}/\mathbb{Q}(n+1)) & \xrightarrow{\mathrm{Res}} & H^{n-1}(\tilde{X}_t\backslash\tilde{D}, \mathbb{C}/\mathbb{Q}(n))\oplus H^{n-1}(\mathbb{D}^*_{\tilde{\sigma}_1}\backslash D^*_{\tilde{\sigma}_1}, \mathbb{C}/\mathbb{Q}(n)), \end{array}$$

$$\begin{array}{ccccc} [\hat{\mathbb{T}}^n_{\underline{v},\epsilon}] & \longleftarrow & \left([\Gamma_{\underline{v},\epsilon}\cap\tilde{X}_t], [\Gamma_{\underline{v},\epsilon}\cap\mathbb{D}_{\tilde{\sigma}_1}]\right) & \longmapsto & [\tilde{\varphi}_t] \\ H_n(\mathbb{P}_{\tilde{\Delta}}\backslash\tilde{\mathbb{D}}\cap\tilde{X}_t, \mathbb{Q}) & \xleftarrow{\mathrm{Tube}} & H_{n-1}(\tilde{X}_t\backslash\tilde{D}, \mathbb{Q})\oplus H_{n-1}(\mathbb{D}^*_{\tilde{\sigma}_1}\backslash D^*_{\tilde{\sigma}_1}, \mathbb{Q}) & \xrightarrow{(\jmath_*,0)} & H_{n-1}(\tilde{X}_t, \mathbb{Q}). \end{array}$$

These suggest that

$$\begin{aligned} \Psi(t) &= AJ(\xi_t)(\Gamma_{\underline{v},\epsilon}\cap\tilde{X}_t) \\ &= -\frac{1}{2\pi\mathrm{i}}AJ(\hat{\xi}_t)(\hat{\mathbb{T}}^n_{\underline{v},\epsilon}) + (-1)^n AJ(\mathrm{Res}^1_{\tilde{\sigma}_1}\hat{\xi}_t)(\Gamma_{\underline{v},\epsilon}\cap\mathbb{D}_{\tilde{\sigma}_1}), \end{aligned}$$

the first term of which we can compute directly using the regulator formula (4.2); we will show the second zero by an induction argument.

Working on $\mathbb{P}_{\tilde{\Delta}}\backslash\tilde{X}_t\cup\tilde{\mathbb{D}}$, we have

$$\begin{aligned} \text{(4.4)}\qquad & \frac{1}{2\pi\mathrm{i}}AJ(\hat{\xi}_t)(\hat{\mathbb{T}}^n_{\underline{v},\epsilon}) \\ &= \frac{1}{2\pi\mathrm{i}}\int_{\hat{\mathbb{T}}^n_{\underline{v},\epsilon}} R\{\lambda-\phi(\underline{x}), x_1,\ldots,x_n\} \\ &= \frac{1}{2\pi\mathrm{i}}\oint_{\cap_{i=1}^n\{|x_i|=\epsilon^{q_i}\}} \log(\lambda-\phi)\bigwedge^n d\log\underline{x}, \end{aligned}$$

since $t\in U_\epsilon$ and $\underline{x}\in\hat{\mathbb{T}}^n_{\underline{v},\epsilon} \implies |\phi(\underline{x})|\le\frac{1}{\delta(\epsilon)}<|\lambda|$ and $\arg(\lambda)\in(-\frac{\pi}{4},\frac{\pi}{4}) \implies \underline{x}\notin T_{\lambda-\phi(\underline{x})}$. Using $\lambda-\phi=t^{-1}(1-t\phi)$ and $|t\phi|<1$, we see the latter

$$= -(2\pi\mathrm{i})^{n-1}\left\{\log t + \sum_{m\ge1}\frac{[\phi(\underline{x})^m]_0 t^m}{m}\right\}.$$

On the other hand, we can manipulate the regulator current in (4.4) by only {coboundary on $\mathbb{P}_{\tilde{\Delta}}\backslash\tilde{X}_t\cup\tilde{\mathbb{D}}$}+{$\mathbb{Q}(n)$-currents} to obtain a rational multiple of $R\{\Phi_{\underline{v}}, z_1,\ldots,z_n\}$. This is done by using multilinearity and anti-commutativity relations for symbols valid in $CH^n(\mathbb{P}_{\tilde{\Delta}}\backslash\tilde{X}_t\cup\tilde{\mathbb{D}})$ and the map of complexes in [50]. The relations are used first to multiply $\lambda-\phi$ by $\underline{x}^{-\underline{v}}$

(which just gives $\Phi_{\underline{v}}(\underline{z};\underline{u})$, and then to turn $\{x_1,\ldots,x_n\}$ into $q\cdot\{z_1,\ldots,z_n\}$. (Here, $q\in\mathbb{Q}^*$ is the inverse of the determinant of the matrix of exponent vectors mentioned above.) Hence (4.4) $=$

$$\frac{q}{2\pi\mathrm{i}}\int_{\hat{\mathbb{T}}^n_{\underline{v},\epsilon}} R\{\Phi_{\underline{v}},z_1,\ldots,z_n\},$$

and enlarging the domain to $\mathbb{P}_{\tilde{\Delta}}\backslash\tilde{\mathbb{D}}^-$ and using $(-1)^{n-1}\hat{\mathbb{T}}^n_{\underline{v},\epsilon}=\partial\Gamma_{\underline{v},\epsilon}$ gives

$$\begin{aligned}\frac{-q}{2\pi\mathrm{i}}&\int_{\Gamma_{\underline{v},\epsilon}} d[R\{\Phi_{\underline{v}};\underline{z}\}]\\ &=q\left(\int_{\Gamma_{\underline{v},\epsilon}\cap\tilde{X}_t} R\{z_1,\ldots,z_n\}\pm\int_{\Gamma_{\underline{v},\epsilon}\cap\mathbb{D}_{\tilde{\sigma}_1}} R\{\Phi_{\underline{v},\tilde{\sigma}_1},z_1,\ldots,z_{n-1}\}\right)\\ &=-\int_{\tilde{\varphi}_t} R\{x_1,\ldots,x_n\}\pm q\int_{\partial\Gamma^{(1)}_{\underline{v},\epsilon}} R\{\Phi_{\underline{v},\tilde{\sigma}_1},z_1,\ldots,z_{n-1}\},\end{aligned}$$

where the switch from $R\{\underline{z}\}$ back to $R\{\underline{x}\}$ (in the first term) is valid on $\tilde{X}_t^*$ and

$$\begin{aligned}\Gamma^{(i)}_{\underline{v},\epsilon}:=\{|z_1|=\cdots=|z_{n-i-1}|=\epsilon,\ |z_{n-i}|\le\epsilon,\ |z_{n-i+1}|=\cdots=|z_n|=0\}\\ \in C^{\mathrm{top}}_{n-i+1}(\mathbb{D}_{\tilde{\sigma}_i}).\end{aligned}$$

Of course $\int_{\tilde{\varphi}_t} R\{\underline{x}\}\equiv\Psi(t)\mod\mathbb{Q}(n)$.

Now we may argue inductively: working on $\mathbb{D}_{\tilde{\sigma}_i}$, if $\mathfrak{o}\in\mathbb{N}$ is the order of vanishing of z_{n-i} along $\mathbb{D}_{\tilde{\sigma}_{i+1}}$,

$$\int_{\partial\Gamma^{(i)}_{\underline{v},\epsilon}} R\{\Phi_{\underline{v},\tilde{\sigma}_i},z_1,\ldots,z_{n-i}\}=\pm\int_{\Gamma^{(i)}_{\underline{v},\epsilon}} d[R]$$

$$2\pi\mathrm{i}\left(\pm\mathfrak{o}\int_{\Gamma^{(i)}_{\underline{v},\epsilon}\cap\mathbb{D}_{\tilde{\sigma}_{i+1}}} R\{\Phi_{\underline{v},\tilde{\sigma}_{i+1}},z_1,\ldots,z_{n-i-1}\}\pm\int_{\Gamma^{(i)}_{\underline{v},\epsilon}\cap D_{\tilde{\sigma}_i}} R\{z_1,\ldots,z_{n-i}\}\right).$$

Since $D_{\tilde{\sigma}_i}$ is defined by vanishing of $\Phi_{\underline{v},\tilde{\sigma}_i}=1+\phi_1+\cdots+\phi_{n-i}$, which is ≈ 1 on $\Gamma^{(i)}_{\underline{v},\epsilon}$, $\Gamma^{(i)}_{\underline{v},\epsilon}\cap D_{\tilde{\sigma}_i}=\emptyset$ and this becomes

$$\pm 2\pi\mathrm{i}\int_{\partial\Gamma^{(i+1)}_{\underline{v},\epsilon}} R\{\Phi_{\underline{v},\tilde{\sigma}_{i+1}},z_1,\ldots,z_{n-i-1}\}$$

for $i<n-1$. When $i=n-1$, $\Gamma^{(n-1)}_{\underline{v},\epsilon}\cap\mathbb{D}_{\tilde{\sigma}_n(=\underline{v})}$ is just the origin, $\Phi_{\underline{v},\tilde{\sigma}_n}$ is 1, and

$$\int_{\Gamma^{(n-1)}_{\underline{v},\epsilon}} R\{\Phi_{\underline{v}},\tilde{\sigma}_n\}=\log 1=0.$$

We have proved

Theorem 4.1. *Assuming hypotheses (a)–(c) at the beginning of the section, the fundamental regulator period for Ξ_t is*

$$\Psi(t) \equiv (2\pi\mathrm{i})^{n-1}\{\log t + \sum_{m\geq 1} \frac{[\phi^m]_0}{m} t^m\} \quad \bmod \mathbb{Q}(n), \tag{4.5}$$

for all $t \in U_\epsilon$.

Remark 4.3. (a) For $\tilde{X}_t$ smooth, $AJ(\Xi_t)$ is represented (by Kerr *et al.* [50]) by the class of a closed $(n-1)$-current $R'_{\Xi_t} := R_{\Xi_t} + (2\pi\mathrm{i})^n \delta_{\partial^{-1}T_{\Xi_t}}$ (modulo cycles modifying the membrane $\partial^{-1}T_{\Xi_t}$) in $H^{n-1}(\tilde{X}_t, \mathbb{C})/\mathrm{im}\{H_{n-1}(\tilde{X}_t, \mathbb{Q}(n))\}$, and $\Psi(t) \equiv \int_{\tilde{\varphi}_t} [R'_{\Xi_t}]$. For brevity, we denote $R'_{\Xi_t} =: R'_t$. We think of $[R'_t]$ as a multivalued section of $\mathcal{H}^{n-1}_{\tilde{\mathcal{X}}/\mathbb{P}^1} := R^{n-1}\tilde{\pi}_*\mathbb{C} \otimes \mathcal{O}_{\mathbb{P}^1}$ over $\mathbb{P}^1 \backslash \mathcal{L}$.

(b) Theorem 4.1 is valid $\bmod\, \mathbb{Z}(2)$ if $n = 2$, Remark 3.5 applies, and vertex coefficients of ϕ are all 1.

(c) The apparent similarity (of the $\sum_{m\geq 1}$ in the theorem) to the formal group law in [17] is somewhat deceptive, as their $\ell(t)$ would correspond to $\sum_{m\geq 0} \frac{[\phi^m]_0}{m+1} t^{m+1}$ in the present notation.

Now assume henceforth that the general $\tilde{X}_t$ is nonsingular (or is a surface with A_1 singularities). The Gauss–Manin connection ∇ kills periods hence $H^{n-1}(\tilde{X}_t, \mathbb{Q}(n))$-ambiguities in $[R'_t]$, and $\nabla[R'_t] \in \Gamma(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}\langle \log \mathcal{L}\rangle \otimes \mathcal{F}^{n-1}\mathcal{H}^{n-1}_{\tilde{\mathcal{X}}/\mathbb{P}^1})$ (see [47]). Writing $\delta_t := t\partial_t := t\frac{d}{dt}$, this implies that

$$\nabla_{\delta_t}[R'_t] = f(t)[\tilde{\omega}_t]$$

for $f \in \bar{K}(\mathbb{P}^1)^*$. To find f, we take periods of both sides:

$$\frac{1}{(2\pi\mathrm{i})^{n-1}} t\frac{d}{dt} \int_{\tilde{\varphi}_t} [R'_t] = \frac{f(t)}{(2\pi\mathrm{i})^{n-1}} \int_{\tilde{\varphi}_t} \tilde{\omega}_t$$

and for $t \in U_\epsilon$ this becomes

$$t\frac{d}{dt}\left\{\log t + \sum_{m\geq 1} \frac{[\phi^m]_0}{m} t^m\right\} = f(t) \sum_{m\geq 0} [\phi^m]_0 t^m.$$

So $f(t) \equiv 1$ on U_ϵ, hence on $\mathbb{P}^1$. There exists a Picard–Fuchs operator $D_{\mathrm{PF}} = \delta_t^r + \sum_{k=0}^{r-1} g_k(t)\delta_t^k$ $(g_k \in \bar{K}(\mathbb{P}^1)^*,\ r \leq rk(R^{n-1}\tilde{\pi}_*\mathbb{C}))$ satisfying $D_{PF}A(t) = 0$, and $\nabla_{PF}[\tilde{\omega}_t] = 0$.

Corollary 4.1. *On $\mathbb{P}^1\backslash\mathcal{L}$, $\nabla_{\delta_t}[R'_t] = [\tilde{\omega}_t]$, and the periods of R'_t (e.g., $\Psi(t)$) satisfy the homogeneous equation $(D_{\mathrm{PF}} \circ \delta_t)(\cdot) = 0$.*

Corollary 4.2. *The classes $\Xi_t \in H^n_{\mathcal{M}}(\tilde{X}_t, \mathbb{Q}(n))$ and $\xi_t \in CH^n(\tilde{X}^*_t, n)$ are (AJ-)nontrivial for general $t \in \mathbb{P}^1$.*

Proof. There are several simple ways to see this; the first is that Theorem 4.1 $\implies$ $\Psi(t) \to \infty$ as $t \to 0$, which obviously shows

$$0 \neq AJ(\xi_t) \in \mathrm{Hom}_{\mathbb{Q}}(H_{n-1}(\tilde{X}^*_t, \mathbb{Q}), \mathbb{C}/\mathbb{Q}(n)).$$

One can also use nonvanishing of the infinitesimal invariant $\nabla[R'_t]$, and there is an abstract way to do this which bypasses Corollary 4.1 (and the theorem). Recall $\tilde{\mathcal{X}}^*_- \cong (\mathbb{C}^*)^n$, and consider the diagram

$$\begin{array}{ccccc} CH^n(\tilde{\mathcal{X}}^*_-, n) & \xleftarrow{\jmath^*} & H^n_{\mathcal{M}}(\tilde{\mathcal{X}}_-, n) & \xrightarrow{\{AJ_t\}_{t\in\mathbb{P}^1\backslash\mathcal{L}}} & H^0\left(\mathbb{P}^1\backslash\mathcal{L}\,,\; \mathcal{H}^{n-1}_{\tilde{\mathcal{X}}/\mathbb{P}^1}/R^{n-1}\tilde{\pi}_*\mathbb{Q}(n)\right) \\ \downarrow{\scriptstyle cl} & & \downarrow{\scriptstyle cl} & & \downarrow{\scriptstyle \nabla} \\ F^nH^n(\tilde{\mathcal{X}}^*_-, \mathbb{C}) & \xleftarrow{\jmath^*} & F^nH^n(\tilde{\mathcal{X}}_-, \mathbb{C}) & \hookrightarrow & H^0\left(\mathbb{P}^1\backslash\mathcal{L}\,,\; \Omega^1_{\mathbb{P}^1} \otimes \mathcal{F}^{n-1}\mathcal{H}^{n-1}_{\tilde{\mathcal{X}}/\mathbb{P}^1}\right) \end{array}$$

in which

$$\jmath^*(\Omega_\Xi) = \jmath^*(\mathrm{cl}(\Xi)) = \mathrm{cl}\,\langle\{\underline{x}\}\rangle = [\bigwedge^n d\log\underline{x}] \neq 0.$$

(Note that this implies that $\bigwedge^n d\log\underline{x}$ extends to a holomorphic form on $\tilde{\mathcal{X}}_-$, namely Ω_Ξ.) One could also base a proof on Corollary 4.5 below, when its hypothesis $(r = n)$ holds. □

To put the last result in context, we recall the vanishing theorem of [47] as it applies to the case of CY's. For $X/\mathbb{C}$ smooth projective of dimension $n-1$, let

$$K^M_n(X) := \mathrm{im}\{CH^n(X, n) \to K^M_n(\mathbb{C}(X))\},$$

and

$$\begin{aligned} &\underline{H}^{n-1}(\eta_X, \mathbb{C}/\mathbb{Q}(n)) \\ &\quad := \mathrm{im}\left\{ H^{n-1}(X, \mathbb{C}/\mathbb{Q}(n)) \quad\to \varinjlim_{\substack{D\subset X \\ \mathrm{codim.}\,1}} H^{n-1}(X\backslash D, \mathbb{C}/\mathbb{Q}(n)) \right\} \\ &\quad \cong \mathrm{Gr}^0_N H^{n-1}(X, \mathbb{C}/\mathbb{Q}(n)), \end{aligned}$$

where $N^\bullet$ is the coniveau filtration. (This is nonzero for a CY since $[\omega] \notin N^1$; for a surface it is H^2_{tr}.) Then the AJ map

$$K_n^M(X) \to \underline{H}^{n-1}(\eta_X, \mathbb{C}/\mathbb{Q}(n))$$

is zero for X a CY arising as a very general complete intersection in $\mathbb{P}^{n+r}$ of multidegree $(D_0, \ldots, D_r)$, $\sum D_j = n + r + 1$, and $n \geq 3$ ($X \neq$curve). (Probably a similar result holds with $\mathbb{P}^{n+r}$ replaced by another toric Fano variety.) In contrast, a general member of a one-parameter family arising from Theorem 3.1 is still rather special, ϕ having coefficients in a number field which are further restricted by the tempered requirement. In fact, since $0 \neq [\tilde{\omega}_t] = \nabla_{\delta_t}[R'_t] \in \frac{N^0}{N^1} H^{n-1}(\tilde{X}_t, \mathbb{C}/\mathbb{Q}(n))$ for general t and $\nabla_{\delta_t}\mathcal{N}^1\mathcal{H}^{n-1} \subseteq \mathcal{N}^1\mathcal{H}^{n-1}$, we see that generically $0 \neq [R'_t] \in Gr^0_N$ and hence that $\{\underline{x}\} \in K_n^M(\tilde{X}_t)$ is (AJ-)nontrivial.

So far, little to nothing has been said regarding the behavior of $\Psi(t)$ globally or near $t_1 \in \mathcal{L}\backslash\{0\} =: \mathcal{L}^*$. Fix a base point $0' \in U_\epsilon$, let $\mathfrak{P}$ denote the space of C^∞ paths $P : [0,1] \to \mathbb{P}^1\backslash\{0\}$ satisfying $P(0) = 0'$, $P([0,1)) \subset \mathbb{P}^1\backslash\mathcal{L}$, and write $P([0,1]) =: |P|$. Define a projection $\rho : \mathfrak{P} \to \mathbb{P}^1\backslash\{0\}$ by $\rho(P) := P(1)$, and let $\Phi_P = \cup_{t\in|P|}\tilde{\varphi}_t$ (with $[\tilde{\varphi}_{\rho(P)}] \in H_{n-1}(\tilde{X}_{\rho(P)}, \mathbb{Z})$) be a "topological continuation" of the vanishing cycle. There is an obvious equivalence relation on $\mathfrak{P}^\circ := \rho^{-1}(\mathbb{P}^1\backslash\mathcal{L})$ — namely, $P_1, P_2 \in \rho^{-1}(t)$ are equivalent iff the restriction of $R^{n-1}\tilde{\pi}_*\mathbb{Z}$ to $|P_1| \cup |P_2|$ is trivial. Extend this to $t \in \mathcal{L}^*$ by requiring only that the union of $(|P_1| \cup |P_2|)\backslash\{t\}$ with some subset of $D^*_\varepsilon(t)$ have trivial monodromy. Denote the quotient spaces by $\check{\mathfrak{P}}^\circ \subset \check{\mathfrak{P}}$, topologizing the latter in analogy with the extended upper half-plane. Note that $\mathcal{L}^*$ splits into finite and (unipotent and nonunipotent) infinite monodromy fibers; ρ^{-1} of the former should be thought of as points interior to $\check{\mathfrak{P}}$, ρ^{-1} of the latter as cusps.

We want to clarify the following

Assertion: $\Psi(t)$ lifts to a well-defined, continuous function on $\check{\mathfrak{P}}$ with holomorphic restriction to $\check{\mathfrak{P}}^\circ$.

To do this, we must finish defining $\Psi(t)$ by observing that (4.3) makes sense (in $\mathbb{C}/\mathbb{Q}(n)$) even for $t \in \mathcal{L}^*$ once the homology class $\tilde{\varphi}_t \in H_{n-1}(\tilde{X}_t, \mathbb{Z})$ is fixed. Since the MHS $H^n(\tilde{X}_t)$ has weights $\leq n$, $\mathrm{Hom}_{\text{MHS}}(\mathbb{Q}(0), H^n(\tilde{X}_t, \mathbb{Q}(n))) = \{0\}$ and $H^n_{\mathcal{H}}(\tilde{X}_t, \mathbb{Q}(n)) \cong \mathrm{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), H^{n-1}(\tilde{X}_t, \mathbb{Q}(n))) \cong H^{n-1}(\tilde{X}_t, \mathbb{C}/\mathbb{Q}(n))$. So $AJ(\Xi_t)$ is at least defined in the last group (though we will not say how to compute it until Section 6), and (4.3) simply pairs homology and cohomology.

Fix $t \in \mathbb{P}^1\backslash\{0\}$, $P \in \rho^{-1}(t)$ and Φ_P (hence $\tilde{\varphi}_t$). By functoriality of KLM currents (moving Ξ if necessary to lie in $Z^n(\tilde{\mathcal{X}}_-, n)_{\tilde{X}_t}$), $\int_{\tilde{\varphi}_t} R_{\Xi_t} = \int_{\tilde{\varphi}_t} R_\Xi$ for any $t \in \mathbb{P}^1\backslash\{0\}$. If we accept (in anticipation of Section 6.1) that $AJ(\Xi_t)(\tilde{\varphi}_t) \equiv \int_{\tilde{\varphi}_t} R_{\Xi_t}$ even for $t \in \mathcal{L}^*$, then (4.3) gives

$$\Psi(t) = \int_{\tilde{\varphi}_t} R_\Xi = \int_{\Phi_P} d[R_\Xi] + \int_{\tilde{\varphi}_{0'}} R_\Xi \overset{\mathbb{Q}(n)}{\equiv} \int_{\Phi_P} \Omega_\Xi + \Psi(0')$$

for the continuation of Ψ corresponding to P. The Assertion follows, using $\Omega_\Xi \in \Omega^n(\tilde{\mathcal{X}}_-)$ and Morera's theorem for the holomorphicity (which we already know in any case), and "smoothing out" any $\mathbb{Q}(n)$-discrepancies.

As for the local behavior of (the multivalued function) $\Psi(t)$ at $t_1 \in \mathcal{L}^*$ on $\mathbb{P}^1$, this must be consistent with the continuity on $\check{\mathfrak{P}}$. In $q := t - t_1$ we have in general $\Psi =$ holomorphic plus terms of the form $q^\beta(\log^k q)H(q)$ where $\beta \in \mathbb{Q}^+$, $k \in \{1, \ldots, n-1\}$, and H is holomorphic. For example, in the unipotent case suppose we have monodromy $T\tilde{\varphi}_t = \tilde{\varphi}_t + \eta_t$; then $\eta_t \in \mathrm{im}(T - I)$ implies (by Clemens–Schmid) that η_{t_1} is zero in $H_{n-1}(X_{t_1}, \mathbb{Z})$, hence pairs to 0 (mod $\mathbb{Q}(n)$) with $AJ(\Xi_{t_1})$. Moreover, if $\eta_t \in \ker(T - I)$ then we simply have $\Psi = \Psi_0(q) + q(\log q)\Psi_1(q)$ where Ψ_0, Ψ_1 are holomorphic (and single-valued).

Now let t_0 be the smallest nonzero element of $\mathcal{L}$; i.e. (at least if ϕ is regular) $\frac{1}{t_0}$ is the critical value of ϕ of largest finite modulus. Of course, there might be more than one element of smallest ($\neq 0$) modulus; in this event just choose one. Putting the above discussion together with Corollary 4.1 yields

Corollary 4.3. *The* $\Psi(t)$ *computation in Theorem* 4.1 *holds* $\forall t \in \bar{D}^*_{|t_0|}$.

Proof. The convergence and continuity of $\sum \frac{[\phi^m]_0}{m} t^m$ at the boundary follows from a bit of Tauberian theory, combined with the fact that $A(t) = \delta_t \Psi(t)$ has at worst a $\log^{n-1}(t - t_0)$ pole at t_0. Then one invokes continuity of $\Psi(t)$ itself. □

We conclude with a number-theoretic application. Various authors [9, 29, 69] have noticed a relation between the logarithmic Mahler measure $\mathbf{m}$ of a Laurent polynomial $Q(x_1, \ldots, x_n)$ and real regulator periods (or special values of L-functions) associated to the variety $Q = 0$. Writing

$$\hat{\mathbb{T}}^n := \{|x_1| = \cdots = |x_n| = 1\} \subset (\mathbb{C}^*)^n,$$

this is

$$\mathbf{m}(Q) := \frac{1}{(2\pi \mathrm{i})^n} \int_{\hat{\mathbb{T}}^n} \log |Q| \bigwedge^n d\log \underline{x},$$

the real regulator is just the composition

$$H^n_{\mathcal{M}}(\tilde{X}_t, \mathbb{Q}(n)) \xrightarrow{AJ} H^{n-1}(\tilde{X}_t, \mathbb{C}/\mathbb{Q}(n)) \xrightarrow{\pi_{\mathbb{R}}} H^{n-1}(\tilde{X}_t, \mathbb{R}(n-1)),$$

where (on the level of currents) $\pi_{\mathbb{R}}$ takes R'_{Ξ_t} to its "$(2\pi\mathrm{i})^{n-1}$·real"-part $r_{\Xi_t} \in \mathcal{D}^{n-1}_{\mathbb{R}(n-1)}(\tilde{X}_t)$. (The latter is $(2\pi\mathrm{i})^n$·Goncharov's current [43], up to coboundary.) In the present context the two are related as follows.

Corollary 4.4. *Under the conditions of Theorem* 4.1*,*

$$-Re\left(\frac{1}{(2\pi\mathrm{i})^{n-1}}\Psi(t)\right) = \frac{-1}{(2\pi\mathrm{i})^{n-1}} \int_{\tilde{\varphi}_t} [r_t] = \mathbf{m}(t^{-1} - \phi)$$

for all t in

$$\mathcal{S} := \overline{\{\text{connected component of } (\mathbb{P}^1 \backslash \{\tfrac{1}{\phi(\hat{\mathbb{T}}^n)}\}) \text{ containing } \{0\}\}} \backslash \{0\} \subseteq \mathbb{P}^1,$$

where the bar denotes analytic closure.

Proof. Consider the equation

$$\begin{aligned}\frac{1}{(2\pi\mathrm{i})^{n-1}} \int_{\tilde{\varphi}_t} [R'_t] &= \log t + \sum_{m \geq 1} \frac{[\phi^m]_0}{m} t^m \\ &= \frac{-1}{(2\pi\mathrm{i})^n} \int_{\hat{\mathbb{T}}^n} \log(t^{-1} - \phi) \bigwedge^n d\log \underline{x},\end{aligned}$$

where the first equality holds by Theorem 4.1 for (say) $t \in U_\epsilon$, and the second for $t (\neq 0)$ such that $|t| < |\phi(\underline{x})|^{-1}$ $\forall \underline{x} \in \hat{\mathbb{T}}^n$. (Note that $|\phi|$ is bounded above on $\hat{\mathbb{T}}^n$.) Now the l.h.s. is analytic multivalued on $\mathbb{P}^1 \backslash \mathcal{L}$, while the r.h.s. is analytic multivalued as long as $(0 \neq) t$ does not pass through $\{\frac{1}{\phi(\hat{\mathbb{T}}^n)}\}$ (so that log retains a continuous single-valued branch on the image $t^{-1} - \phi(\hat{\mathbb{T}}^n)$). Since they agree on an analytic open set, they continue to agree on (the covering space of) the obvious connected component of $\mathbb{P}^1 \backslash \mathcal{L} \cup \{\frac{1}{\phi(\hat{\mathbb{T}}^n)}\}$. Taking real parts of both sides kills multivaluedness. To see this on the r.h.s., replace $\frac{\bigwedge^n d\log \underline{x}}{(2\pi\mathrm{i})^n}$ by $\bigwedge^n d\arg \underline{x}$; for the l.h.s., one easily sees that $\tilde{\varphi}_t$ has no monodromy on $\mathcal{S}$ (though $[R'_t]$ may, which is harmless). The equality thus extends to the analytic closure by continuity, erasing $\mathcal{L}\backslash\{0\}$ (where $\int_{\tilde{\varphi}_t}[r_t]$ is finite). $\square$

4.3. The higher normal function

For this subsection, take the family $\tilde{\mathcal{X}} \xrightarrow{\tilde{\pi}} \mathbb{P}^1$ to be as in (the assumptions of) Theorem 3.1. Given any (possibly singular) fiber $\tilde{X}_{t\neq 0}$, we have $AJ(\Xi_t) \in H^{n-1}(\tilde{X}_t, \mathbb{C}/\mathbb{Q}(n))$. If $\mathcal{R}_t \in H^{n-1}(\tilde{X}_t, \mathbb{C})$ is any lift of this class, then since $\tilde{\omega}_t = \frac{1}{2\pi \mathrm{i}}\mathrm{Res}_{\tilde{X}_t}\tilde{\Omega}_t \in H^{n+1}_{\tilde{X}_t}(\mathbb{P}_{\tilde{\Delta}}, \mathbb{C}) \cong H_{n-1}(\tilde{X}_t, \mathbb{C})$, the pairing $\langle \mathcal{R}_t, [\tilde{\omega}_t]\rangle \in \mathbb{C}$ makes sense. For $\tilde{X}_t$ smooth and $\mathcal{R}_t = [R'_t]$ as in Remark 4.3(a), this is just $\int_{\tilde{X}_t} R'_t \wedge \tilde{\omega}_t$.

Definition 4.1. The *higher normal function associated to* Ξ is the multivalued function

$$\nu(t) := \langle \mathcal{R}_t, [\tilde{\omega}_t]\rangle$$

on $\mathbb{P}^1\backslash\mathcal{L}$, where $\mathcal{R}_t$ is a (multivalued) continuous family of lifts of $AJ_{\tilde{X}_t}(\Xi_t)$.

This is a highly transcendental function, but applying D_{PF} kills the ambiguities (which are periods of $\tilde{\omega}$) and produces $g(t) := D_{\mathrm{PF}}\nu(t) \in \bar{K}(\mathbb{P}^1)$ (see [25]). Viewed as an element of $\bar{K}(\mathbb{P}^1)/D_{\mathrm{PF}}\bar{K}(\mathbb{P}^1)$, g is the class of a certain extension of $\mathcal{D}$-modules attached to Ξ. Alternatively, it is the inhomogeneous term of the Picard–Fuchs equation

$$D_{\mathrm{PF}}(\cdot) = g$$

satisfied by ν, and its nonvanishing would give another proof of nontriviality of Ξ_t: $g \neq 0$ implies that $\nu \neq$ a period of $\tilde{\omega}$ which means $\mathcal{R}_t \notin H^{n-1}(\tilde{X}_t, \mathbb{Q}(n))$ [general t] and hence that general $AJ(\Xi_t) \not\equiv 0$. Note that conversely, if the $\mathbb{C}$-span of the $\{\nabla^i_{\delta_t}[\tilde{\omega}_t]\}_{i=0}^{r-1}$ is a (complexified) Hodge structure for general t, then it is possible to show (using $\nabla_{\delta_t}\mathcal{R}_t = [\tilde{\omega}_t]$ from Corollary 4.1) $g \neq 0$.

The study of inhomogeneous PF equations for higher normal functions was initiated by del Angel and Müller-Stach [24–26]. Their work focused on families of higher cycles $\eta_t \in CH^p(X_t, 2p-n)$ $(p < n = \dim X + 1)$, in which case $\int_{X_t} R'_{\eta_t} \wedge \omega_t$ reduces to integration of ω_t over a real membrane. Here we want to demonstrate that the case $p = n$ is also accessible and interesting.

The *Yukawa coupling* is the function $\mathcal{Y} \in K(\mathbb{P}^1)$ defined by

$$\mathcal{Y}(t) := \left\langle [\tilde{\omega}_t], \nabla_{\delta_t}^{n-1}[\tilde{\omega}_t] \right\rangle$$

for $t \notin \mathcal{L}$. (A_1-singularities for such t are harmless here, as $[\tilde{\omega}]$ lifts to $H^{n-1}(\widetilde{\tilde{X}_t})$.) The next result implies this is the inhomogeneous term in many cases including that of elliptic curves ($n = 2$) and $K3$ surfaces ($n = 3$) with generic Picard rank 19.

Corollary 4.5. *If the order of D_{PF} is $(r =)\, n$, i.e., if the $\mathcal{D}$-module generated by $[\tilde{\omega}_t]$ has rank n, then $g = \mathcal{Y}$.*

Proof. Compute first

$$\delta_t \left\langle \mathcal{R}_t, [\tilde{\omega}_t] \right\rangle = \left\langle [\tilde{\omega}_t], [\tilde{\omega}_t] \right\rangle + \left\langle \mathcal{R}_t, \nabla_{\delta_t}[\tilde{\omega}_t] \right\rangle = \left\langle \mathcal{R}_t, \nabla_{\delta_t}[\tilde{\omega}_t] \right\rangle,$$

then inductively

$$\delta_t^{j<n} \left\langle \mathcal{R}_t, [\tilde{\omega}_t] \right\rangle = \delta_t \left\langle \mathcal{R}_t, \nabla_{\delta_t}^{j-1}[\tilde{\omega}_t] \right\rangle = \left\langle [\tilde{\omega}_t], \nabla_{\delta_t}^{j-1}[\tilde{\omega}_t] \right\rangle + \left\langle \mathcal{R}_t, \nabla_{\delta_t}^{j}[\tilde{\omega}_t] \right\rangle.$$

By Hodge type and Griffiths transversality, this

$$= \left\langle \mathcal{R}_t, \nabla_{\delta_t}^{j}[\tilde{\omega}_t] \right\rangle.$$

Hence, with $D_{PF} = \delta_t^n + \sum_{k=0}^{n-1} g_k(t)\delta_t^k$,

$$D_{\mathrm{PF}}\nu(t) = \mathcal{Y}(t) + \left\langle \mathcal{R}_t, \nabla_{\mathrm{PF}}[\tilde{\omega}_t] = 0 \right\rangle = \mathcal{Y}(t).$$

□

Remark 4.4. For $r = n = 2, 3, 4$ $\mathcal{Y}(t)$ is computed by an obvious differential equation. To state it, recall that by Lian *et al.* we have maximal unipotent monodromy at $t = 0$. Hence $g_j(t) = tf_j(t)$ for f_j holomorphic at $t = 0$, and with $q_2 = 1$, $q_3 = \frac{2}{3}$, $q_4 = \frac{1}{2}$ we get $\delta_t\mathcal{Y}(t) = -q_n t f_{n-1}(t)\mathcal{Y}(t) \implies \mathcal{Y}(t) = \kappa \exp\{-q_n \int f_{n-1}(t)dt\}$. From above, $\mathcal{Y} = g$ must be a rational function, and $f_{n-1}(t) = -\frac{M}{q_n} \cdot \frac{\mathcal{Y}'(t)}{\mathcal{Y}(t)}$ (for $M \in \mathbb{Z}$). (If one has maximal unipotent monodromy also at $t = \infty$, then M can be determined also.) The value of κ requires more precise (e.g., modular) information about the family. Note that for $n = 2$, $n = 3$ and $rk(\mathrm{Pic}) = 19$, or $n = 4$ and $h^3 = 4$, Corollary 4.1 implies that $g \neq 0$ and hence that $\kappa \neq 0$.

We prove next an interesting result on the monodromy of (a choice of branch of) ν. Recall from Section 3.3 the definitions (for all n) of $\mathcal{J}, \mathcal{I} \subseteq \tilde{D}$ and for $n = 3$ set $\mathcal{D}$:=normalization of $\mathcal{J}$ at $\mathcal{J} \cap \mathcal{A}$. From the proof of Theorem 3.1, $\hat{\mathcal{X}} \xrightarrow{B} \tilde{\mathcal{X}}$ is the simultaneous resolution of the A_1-singularities $\mathcal{A}(\times\mathbb{P}^1)$, and $\mathcal{D}$ is just the proper transform of $\mathcal{J}$ (along $\hat{X}_t \to \tilde{X}_t$). Let $\mathcal{J}^-$ be the union of the $D_{\tilde{\sigma}}$'s that are not in $\mathbb{I}$ *and* not of the form $\{x_{i_1} + x_{i_2} = 1, x_{i_3}^{\pm 1} = 0\}$. For all n, let $\overset{\circ}{\mathbb{T}}{}^n := \mathbb{R}^-_{x_1} \times \cdots \times \mathbb{R}^-_{x_n} \subset (\mathbb{C}^*)^n$ with analytic closure $\mathbb{T}^n \subset \mathbb{P}_{\tilde{\Delta}}$; note that its class in $H_n(\mathbb{P}_{\tilde{\Delta}}, \tilde{\mathbb{D}})$ is Lefschetz dual to that of the n torus $\hat{\mathbb{T}}^n$ in $H_n((\mathbb{C}^*)^n)$. Let $\mathcal{K}$ denote the analytic closure of $\phi(\overset{\circ}{\mathbb{T}}{}^n)$ in $\mathbb{P}^1_\lambda$, with (open) complement $U := \mathbb{P}^1 \backslash \mathcal{K} \subseteq \mathbb{A}^1_\lambda$, and set $\tilde{\mathcal{X}}_U := \tilde{\pi}^{-1}(U) \subseteq \tilde{\mathcal{X}}_-$, $\tilde{\mathcal{X}}_{\mathcal{K}} := \tilde{\pi}^{-1}(\mathcal{K}) \subseteq \tilde{\mathcal{X}}$. (If U is not connected, replace it by a single connected component, and augment $\mathcal{K}$ by the other connected components.) Finally, let $X := \tilde{X}^{\lambda_0}$ be a very general fiber (with $\lambda_0 \in U$).

Proposition 4.1. (a) *Let $\tilde{\mathcal{X}}_-$ be one of the families from Theorem* 3.1 *with nonsingular general fiber and assume* $\ker\{H_{n-2}(\mathcal{J}) \to H_{n-2}(X)\} = 0$*. Then there exists a single-valued family of cohomology classes $\mathcal{R}^\lambda \in H^{n-1}(\tilde{X}^\lambda, \mathbb{C})$ lifting $AJ(\Xi^\lambda)$ for $\lambda \in U$. (This includes singular fibers $[= U \cap \mathcal{L}]$ unless $n = 2$ and $\mathcal{J} \cap \mathcal{I}$ is nonempty.)*

(b) *For $n = 3$ and $\mathcal{A}$ nonempty (the case excepted above), $H_1(\mathcal{J}^- \backslash \mathcal{J}^- \cap \mathcal{A}) = 0$ so the conclusion of* (a) *holds as stated. If we assume instead $H_1(\mathcal{D}) = 0$, then the conclusion only holds with $\tilde{X}^\lambda$ replaced by $\hat{X}^\lambda$ (and $\mathcal{R}^\lambda$ lifts $AJ(\Xi_0^\lambda) \in H^{n-1}(\hat{X}^\lambda, \mathbb{C}/\mathbb{Q}(n))$).*

Remark 4.5. (i) For $n = 2$, the assumption of (a) says $\mathcal{J}$ is one point; for $n = 3$ it says $H_1(\mathcal{J}) = 0$: $\mathcal{J}$ is a configuration of rational curves whose associated graph has no loop.

(ii) The continuation of $\mathcal{R}^\lambda$ around a loop not in U may no longer be single-valued over U.

(iii) A relaxation of the hypotheses (e.g., allowing singularities in the general fiber, ϕ not regular) may be necessary to produce examples for $n = 4$.

Proof. We do this under the assumption that the total space $\tilde{\mathcal{X}}$ is nonsingular. (While such examples come out of Theorem 3.1, we do not know if any of these survive the extra requirements for this proposition; nevertheless, the main ideas are contained in our "artificial" proof, and the more general situation is treated with cone complexes as in Theorem 3.1's proof.) Write $\overline{Z}^p(\cdot, n)$ for $\partial_{\mathcal{B}}$-closed higher Chow precycles.

In the proof of Theorem 3.1 we started by "completing" $\xi = \{\underline{x}\} \in \overline{Z}^n(\tilde{\mathcal{X}}_- \backslash \mathcal{J} \times \mathbb{A}^1, n)$ to $\Xi \in \overline{Z}^n(\tilde{\mathcal{X}}_-, n)$ restricting to $\xi + \partial_{\mathcal{B}}\gamma$ (on $\tilde{\mathcal{X}}_- \backslash \mathcal{J} \times$

$\mathbb{A}^1$); since $\xi \in \overline{Z}^n_{\mathbb{R}}(\tilde{\mathcal{X}}_- \backslash \mathcal{J} \times \mathbb{A}^1, n)_{X\backslash\mathcal{J}(\times\{x_0\})}$, we may arrange to have

$$\Xi \in \overline{Z}^n_{\mathbb{R}}(\tilde{\mathcal{X}}_-, n)_X\,, \quad \gamma \in Z^n_{\mathbb{R}}(\tilde{\mathcal{X}}_- \backslash \mathcal{J} \times \mathbb{A}^1, n+1)_{X\backslash\mathcal{J}}\,,$$

the first pulling back to $\Xi^{\lambda_0} \in \overline{Z}^n_{\mathbb{R}}(X,n)$. We take the analytic closure of the ∂-closed Borel–Moore C^∞ chain T_ξ on $\tilde{\mathcal{X}}_- \backslash \mathcal{J} \times \mathbb{A}^1$ to get $\overline{T_\xi} \in Z^{\text{top}}_n(\tilde{\mathcal{X}}, \tilde{X}_0 \cup \mathcal{J} \times \mathbb{P}^1)$. Since $(\tilde{\mathcal{X}}_U \backslash \mathcal{J} \times U) \cap \mathbb{T}^n = \emptyset$ by construction, we see that $\overline{T_\xi}$ maps to 0 in $Z^{\text{top}}_n(\tilde{\mathcal{X}}, \tilde{\mathcal{X}}_{\mathcal{K}} \cup \mathcal{J} \times \mathbb{P}^1)$. Clearly $\overline{T_\Xi} \in Z^{\text{top}}_n(\tilde{\mathcal{X}}, \tilde{X}_0)$ maps to $\overline{T_\xi} + \partial\overline{T_\gamma}$ in $Z^{\text{top}}_n(\tilde{\mathcal{X}}, \tilde{X}_0 \cup \mathcal{J} \times \mathbb{P}^1)$, hence to $\partial\overline{T_\gamma}$ in $Z^{\text{top}}_n(\tilde{\mathcal{X}}, \tilde{\mathcal{X}}_{\mathcal{K}} \cup \mathcal{J} \times \mathbb{P}^1)$; and so in $Z^{\text{top}}_n(\tilde{\mathcal{X}}, \tilde{\mathcal{X}}_{\mathcal{K}})$, $\overline{T_\Xi}$ is homologous to a cycle $\tau \in Z^{\text{top}}_n(\mathcal{J} \times (\mathbb{P}^1, \mathcal{K})) \cong Z^{\text{top}}_n(\mathcal{J} \times (\overline{U}, \partial\overline{U}))$ (where $\partial\overline{U} := \overline{U} \backslash U$). (The latter may be put in good position with respect to X, since T_Ξ is.)

Now $0 = F^n H^n(X,\mathbb{C}) \cap H^n(X, \mathbb{Q}(n))$ implies that $0 \overset{\text{hom}}{\equiv} T_{\Xi^{\lambda_0}} = T_\Xi \cap X$ (on X) which tells us that $\tau \cap X \overset{\text{hom}}{\equiv} 0$ (on X). Moreover, $H_n(\mathcal{J} \times (\overline{U}, \partial\overline{U})) = H_{n-2}(\mathcal{J}) \otimes H_2(\overline{U}, \partial\overline{U}) \cong H_{n-2}(\mathcal{J})$ since U connected implies $H_2(\overline{U}, \partial\overline{U}) = \mathbb{Q}$, $\mathcal{K}$ connected yields U simply connected which says $H_1(\overline{U}, \partial\overline{U}) = 0$, and obviously $H_0(\overline{U}, \partial\overline{U}) = 0$. Hence, $\ker\{H_{n-2}(\mathcal{J}) \to H_{n-2}(X)\} = 0$ so $\tau \overset{\text{hom}}{\equiv} 0$ and $\exists\, \Gamma \in Z^{\text{top}}_{n+1}(\tilde{\mathcal{X}}, \tilde{\mathcal{X}}_{\mathcal{K}})$ with $\partial\Gamma = T_\Xi \pmod{\tilde{\mathcal{X}}_{\mathcal{K}}}$, and we define $R'_\Xi := R_\Xi + (2\pi\mathrm{i})^n \delta_\Gamma \in \mathcal{D}^{n-1}(\tilde{\mathcal{X}}_U)$. One has $d[R'_\Xi] = \Omega_\Xi \in F^n\mathcal{D}^n(\tilde{\mathcal{X}}_U)$.

This Ω_Ξ, being a d-closed $(n,0)$-current, is in fact C^∞ (i.e., holomorphic) by standard regularity results. On $\tilde{\mathcal{X}}_U$ it is cohomologous to 0, hence $d\eta$ there for some C^∞ $(n-1)$-form η. Hence $R'_\Xi - \eta$ is closed and $\exists$ $(n-2)$-current κ such that $R'_\Xi - \eta + d[\kappa]$ is C^∞ (in the same class); obviously $R'_\Xi + d[\kappa]$ is also C^∞ (but not closed), and so pulls back to every fiber to give a continuous family of (closed C^∞ forms and hence) classes in $\{H^{n-1}(\tilde{X}^\lambda, \mathbb{C})\}_{\lambda \in U}$ (including singular fibers).

Next pick any $\lambda_1 \in U$, put $X_1 := \tilde{X}^{\lambda_1}$; we must show $[\iota^*_{X_1}(R'_\Xi + d[\kappa])]$ lifts $AJ(\iota^*_{X_1}\Xi_1) \in H^{n-1}(X, \mathbb{C}/\mathbb{Q}(n))$ for some "move" Ξ_1 of Ξ. Namely, use $\mathcal{M} \in Z^n_{\mathbb{R}}(\tilde{\mathcal{X}}_-, n+1)$ to get $\Xi_1 := \Xi + \partial_{\mathcal{B}}\mathcal{M} \in \overline{Z}^n_{\mathbb{R}}(\tilde{\mathcal{X}}_-, n)_{X_1}$, and $\mu \in C^{\text{top}}_{n+2}(\tilde{\mathcal{X}}, \tilde{\mathcal{X}}_{\mathcal{K}})$ to move Γ to $\Gamma_1 := \Gamma - T_{\mathcal{M}} + \partial\mu \in C^{\text{top}}_{n+1}(\tilde{\mathcal{X}}, \tilde{\mathcal{X}}_{\mathcal{K}})_{X_1}$. Note that $\partial\Gamma_1 = \partial\Gamma - \partial T_{\mathcal{M}} = T_\Xi - \partial T_{\mathcal{M}} = T_{\Xi_1}$, so that $R'_{\Xi_1} := R_{\Xi_1} + (2\pi\mathrm{i})^n\delta_{\Gamma_1}$ has $d[R'_{\Xi_1}] = \Omega_{\Xi_1} = \Omega_\Xi$. Moreover, the d-closed pullback $\iota^*_{X_1} R'_{\Xi_1} = R_{\iota^*_{X_1}\Xi_1} + (2\pi\mathrm{i})^n \delta_{\partial^{-1}(\iota^*_{X_1} T_{\Xi_1})}$ so its class lifts $AJ(\iota^*_{X_1}\Xi_1)$. Now we compare the two things pulled back, $\iota^*_{X_1}$ of $\mathcal{R}'_{\Xi_1}$ and $\mathcal{R}'_\Xi + d[\kappa]$:

$$R'_{\Xi_1} = R_\Xi + d\left[\frac{R_{\mathcal{M}}}{2\pi\mathrm{i}}\right] + (2\pi\mathrm{i})^n \delta_{T_{\mathcal{M}} + \Gamma_1}$$

$$= R_\Xi + d\left[\frac{R_{\mathcal{M}}}{2\pi \mathrm{i}} + (2\pi\mathrm{i})^n \delta_\mu\right] + (2\pi\mathrm{i})^n \delta_\Gamma$$
$$= R'_\Xi + d[=: S],$$

hence $R'_{\Xi_1} - R'_\Xi - d[\kappa] = d[S - \kappa]$. If $S - \kappa$ does not pull back to X_1, it is replaceable by something that does (since the l.h.s. does). □

Stiller [78] studied monodromy of solutions to inhomogeneous equations, in the case where the corresponding homogeneous equation $D_{\mathrm{PF}}(\cdot) = 0$ is solved by the period functions associated to an elliptic modular surface. It would be interesting to compare his formula ([78], Theorem 10) with the following for $n = 2$.

Corollary 4.6. *In the situation of Proposition* 4.1*(*(a) *or* (b)*), the inhomogeneous equation $D_{\mathrm{PF}}(\cdot) = g$ admits a solution single-valued in U (i.e., also finite at $U \cap \mathcal{L}$, except possibly when $n = 2$ and $\mathcal{J} \cap \mathcal{I} \neq \emptyset$).*

Of course, this is most interesting in case $\mathrm{ord}(D_{\mathrm{PF}}) = n$ and Corollary 4.5 also applies.

As an application of higher normal functions and Corollary 4.1, we consider the problem of producing *linearly independent* families of higher Chow cycles over $\mathcal{P} := \mathbb{P}^1_t \backslash \mathcal{T}$, where $\mathcal{T} \ni \{0\}$ is a collection of points. Since the idea will be to produce independent topological invariants $[\Omega] \in F^n H^n(\tilde{\mathcal{X}}_{\mathcal{P}}, \mathbb{C}) \cap H^n(\tilde{\mathcal{X}}_{\mathcal{P}}, \mathbb{Q}(n))$ ($\tilde{\mathcal{X}}_{\mathcal{P}} := \tilde{\pi}^{-1}(\mathcal{P})$), larger $\mathcal{T}$ is better. In fact, $\mathcal{T} = \{(t =)\, 0\}$ will not do, as $F^n H^n(\tilde{\mathcal{X}}_-, \mathbb{C}) \cong F^n H^n((\mathbb{C}^*)^n, \mathbb{C}) \cong \mathbb{C}\langle \Omega_\Xi = \bigwedge^n d\log \underline{x}\rangle$ has rank 1.

Suppose we have a rational map (defined $/\bar{\mathbb{Q}}$) of families satisfying the conditions of Theorem 3.1:

$$\begin{array}{ccc} \tilde{\mathcal{X}}_{\mathcal{P}} & \overset{\mathfrak{A}}{\dashrightarrow} & {}'\tilde{\mathcal{X}}_- \\ {\scriptstyle \tilde{\pi}}\downarrow & & \downarrow{\scriptstyle '\tilde{\pi}} \\ \mathcal{P} & \xrightarrow{\alpha} & \mathbb{P}^1\backslash\{0\}. \end{array}$$

That is, we have Zariski open $\mathcal{V}_{\mathcal{P}} \subseteq \tilde{\mathcal{X}}_{\mathcal{P}}$, hence some blow-up $\mathfrak{Y}_{\mathcal{P}} \xrightarrow{\mathfrak{B}} \tilde{\mathcal{X}}_{\mathcal{P}}$, mapping to ${}'\tilde{\mathcal{X}}_-$ over α. Write $\mathfrak{A}_t : \tilde{X}_t - - > {}'\tilde{X}_{\alpha(t)}$, $u_i := \mathfrak{A}^*({}'x_i) \in \bar{\mathbb{Q}}(\tilde{\mathcal{X}}_{\mathcal{P}})^*$. If $\mathfrak{A}$ is the restriction of a rational map $\mathbb{P}_{\tilde{\Delta}} \times \mathbb{P}^1 - - > \mathbb{P}_{'\tilde{\Delta}} \times \mathbb{P}^1$ given by $(x_1, \ldots, x_n; t) \mapsto (f_1(\underline{x}; t), \ldots, f_n(\underline{x}; t); \alpha(t)) = ({}'x_1, \ldots, {}'x_n; {}'t)$, then $u_i = f_i(\underline{x}; t)$.

By pulling ${}'\Xi$ back to $\mathfrak{Y}_{\mathcal{P}}$ and pushing forward along $\mathfrak{B}$ we obtain

$$\Theta := \mathfrak{A}^*({}'\Xi) = \text{completion of } \{\underline{u}\} \in CH^n(\tilde{\mathcal{X}}_{\mathcal{P}}, n).$$

Clearly $\Omega_\Theta = \mathfrak{A}^*(\Omega_{'\Xi})$, and this is a holomorphic form; since the fibers of $\tilde{\pi}$ are CY, $[\Omega_\Theta] = [(\tilde{\pi}^* G)\Omega_\Xi]$ for some $G \in \bar{\mathbb{Q}}(\mathbb{P}^1)^*$. On the fibers we have $\mathfrak{A}_t^*[{}'\tilde{\omega}_{\alpha(t)}] = G(t)[\tilde{\omega}_t]$, and $\mathfrak{A}_t^*({}'\mathcal{R}_{\alpha(t)}) =: \mathcal{S}_t$ lifting $AJ(\Theta_t)$. Corollary 4.1 for ${}'\Xi$ says $\nabla_{\delta_{\alpha(t)}} {}'\mathcal{R}_{\alpha(t)} = [{}'\tilde{\omega}_{\alpha(t)}]$, and applying $\mathfrak{A}^*$ gives $\nabla_{\delta_{\alpha(t)}} \mathcal{S}_t = G(t)[\tilde{\omega}_t]$, or

$$\nabla_{\delta_t} \mathcal{S}_t = \frac{t\alpha'(t)}{\alpha(t)} G(t)[\tilde{\omega}_t].$$

Comparing this with $\nabla_{\delta_t} \mathcal{R}_t = [\tilde{\omega}_t]$ (and noting that ∇_{δ_t} removes the ambiguities in the lifts of AJ of Θ_t, Ξ_t), we obtain:

Corollary 4.7. *If $\frac{t\alpha'}{\alpha} G$ is not a rational constant, then the families of classes Θ_t, $\Xi_t \in CH^n(\tilde{X}_t, n)$ are (AJ-)independent.*

There are examples where $\alpha(t) = \pm\frac{1}{t}$ and $G(t) = t$ for $n = 2$ and 3, see [48].

We can also compare the higher normal functions $\nu(t) := \langle \mathcal{R}_t, [\tilde{\omega}_t] \rangle$, $\epsilon(t) := \langle \mathcal{S}_t, [\tilde{\omega}_t] \rangle$. If $0 \neq g := D_{PF}\nu$, and $\frac{t\alpha'}{\alpha} G$ is not a rational constant, then from

$$D_{\mathrm{PF}}\epsilon = \frac{t\alpha'}{\alpha} G g$$

one may deduce independence of the families of Milnor K-theory classes $\{\underline{x}\}, \{\underline{u}\} \in K_n^M(\mathbb{C}(\tilde{X}_t))$ for $n = 2, 3$.

In the event that α is of infinite order (rather than e.g., an involution like $t \mapsto \pm\frac{1}{t}$), iteratively applying the above construction (for α, $\alpha \circ \alpha$, $\alpha \circ \alpha \circ \alpha$, etc. which of course requires shrinking $\mathcal{P}$ at each stage) would give explicit countable generation for $CH^n(\text{generic fiber}, n)$. However it seems likely (already for $n = 2$, by comparing with the proof of infinite generation in [22, Section 7] that this is not possible without allowing α to be *algebraic* and replacing the Zariski neighborhood $\mathcal{P}$ with an étale one; the relevant (geometric) generic fiber is then defined over $\overline{\mathbb{Q}(\mathbb{P}^1)}$ (rather than $\bar{\mathbb{Q}}(\mathbb{P}^1)$).

4.4. Appendix

Before turning to mirror symmetry and examples, we wish to answer an interesting question of the third referee. Up to this point we have dealt with *sufficient* conditions under which the coordinate symbol completes; the Proposition below gives a *necessary* condition.

Let $\Delta \subset \mathbb{R}^n$ $(n = 2, 3, 4)$ be a reflexive polytope and $F = \sum_{\underline{m} \in \Delta \cap \mathbb{Z}^n} \alpha_{\underline{m}} \underline{x}^{\underline{m}} \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ a fixed Δ-regular Laurent polynomial. Assume, for some $\underline{\nu} \in \Delta(n)$, that we have normalized $\alpha_{\underline{\nu}} = 1$. We write $X^* := \{\underline{x} \in (\mathbb{C}^*)^n \mid F(\underline{x}) = 0\}$ and $\tilde{X} \subset \mathbb{P}_{\tilde{\Delta}}$ for its (smooth) Zariski closure, and consider the coordinate symbol $\xi := \langle\{x_1|_{X^*}, \ldots, x_n|_{X^*}\}\rangle \in CH^n(X^*, n)$.

Proposition 4.2. *If ξ is the restriction of a class $\Xi \in CH^n(\tilde{X}, n)$, then for every $\underline{m} \in \Delta \cap \mathbb{Z} \backslash \{\underline{0}\}$ we have $\alpha_{\underline{m}} \in \bar{\mathbb{Q}}$.*

This justifies our restrictions in Section 3, to the effect that only $\alpha_{\underline{0}}$ is allowed to vary, and moreover that ϕ be defined over a number field. The proof has been postponed to this section because it rests on a variant of Corollary 4.1:

Lemma 4.1. *Let $\Delta' \subset \mathbb{R}^\ell$ ($\ell = 2, 3$) be a polytope, not necessarily reflexive, with integer interior points $\{\underline{\mu}_j\}_{j=1}^{g(>0)}$, and set $U = \{s \in \mathbb{C} \mid |s| < \epsilon\}$. Consider a one-parameter family*

$$\mathcal{Y}^* = \left\{ (\underline{y}, s) \in (\mathbb{C}^*)^\ell \times U \;\middle|\; G_s(\underline{y}) = 0 \right\}$$

of smooth hypersurfaces with smooth compactification $\mathcal{Y} \subset \mathbb{P}_{\tilde{\Delta}'} \times U$, where

$$G_s(\underline{y}) := \sum_{j=1}^{g} \beta_j(s) \underline{y}^{\underline{\mu}_j} + \sum_{\underline{\mu}' \in \partial\Delta' \cap \mathbb{Z}^\ell} \gamma_{\underline{\mu}'} \underline{y}^{\underline{\mu}'}.$$

Finally let $\xi'_s := \left\langle \{y_1, \ldots, y_\ell\}|_{Y_s^*} \right\rangle \in CH^\ell(Y_s^*, \ell)$ be the family of coordinate symbols on fibers of $\mathcal{Y}^* \xrightarrow[\pi_{\mathcal{Y}^*}]{} U$. Then

- (a) *the forms $\omega_j(s) := \mathrm{Res}_{Y_s}\left(\frac{\underline{y}^{\underline{\mu}_j}\, d\log y_1 \wedge \cdots \wedge d\log y_\ell}{G_s(\underline{y})}\right)$ give a basis for $\Omega^{\ell-1}(Y_s)$ ($\forall s \in U$), hence for its isomorphic image under $H^{\ell-1,0}(Y_s) \hookrightarrow H^{\ell-1}(Y_s^*)$; and*
- (b) *under the Gauss–Manin connection on the relative $(\ell-1)$th cohomology of $\pi_{\mathcal{Y}^*}$, $\nabla_{\partial_s}[AJ_{Y_s^*}(\xi'_s)] = \sum_{j=1}^{g} \beta'_j(s)[\omega_j(s)]|_{Y_s^*}$.*

Proof. (a) Is due to [5] (see the top of p. 386).

For (b), look at the analytic higher Chow cycle $\xi' := \langle\{y_1, \ldots, y_\ell\}\rangle \in CH^\ell(\mathcal{Y}^*, \ell)$. Although $\Omega_{\xi'}$ is nonzero, its pullback to fibers is zero by type, and $H^{\ell-1}(\mathcal{Y}^*) \cong H^{\ell-1}(Y_s^*)$. So $0 = \mathrm{cl}_{\mathcal{Y}^*}(\xi') = [\Omega_{\xi'}] = [T_{\xi'}]$, and there exists an $(\ell+1)$-chain Γ on $\mathcal{Y}$ with $|\partial\Gamma - T_{\xi'}| \subset \mathcal{Y} \backslash \mathcal{Y}^*$, meeting fibers properly. The restriction of $\tilde{R}_{\xi'} := R_{\xi'} + (2\pi\sqrt{-1})^\ell T_\Gamma \in D^{\ell-1}(\mathcal{Y}^*)$ to each Y_s^* is closed, with class in $H^{\ell-1}(Y_s^*,\mathbb{C})$ a lift of $AJ_{Y_s^*}(\xi_s')$. Writing $\Omega_\ell := d\log y_1 \wedge \cdots \wedge d\log y_\ell$ and $\mathcal{G}(\underline{y},s) := G_s(\underline{y})$, we compute

$$d[\tilde{R}_{\xi'}] = \Omega_{\xi'} = \Omega_\ell = \mathrm{Res}_{\mathcal{Y}^*}\left(\Omega_\ell \wedge d\log\mathcal{G}\right)$$
$$= \mathrm{Res}_{\mathcal{Y}^*}\left(\frac{\Omega_\ell \wedge \frac{\partial\mathcal{G}}{\partial s}ds}{\mathcal{G}}\right) = \sum_{j=1}^{g} \beta_j'(s)\mathrm{Res}_{\mathcal{Y}^*}\left(\frac{\underline{y}^{\underline{\mu}_j}\Omega_\ell}{\mathcal{G}}\right) \wedge ds.$$

Since $\nabla_{\partial_s}[AJ_{Y_s^*}(\xi_s')]$ is represented by the interior product of $d[\tilde{R}_{\xi'}]$ with a lift of $\partial/\partial s$, this gives the result. $\square$

Proof of Proposition 4.2. We use the notation from Sections 2.5, 3.1 and take $n = 4$ for concreteness (the other two cases are treated in the same way). If ξ "completes" to Ξ, it must be in the kernel of

$$\mathrm{Res}^j_{\tilde{\sigma}} : CH^4(X^*,4) \to CH^{4-j}(D^*_{\tilde{\sigma}}, 4-j)$$

for each $j = 1,2,3$ and $\tilde{\sigma} \in \tilde{\Delta}(j)$. By Proposition 3.1, it follows that for each $\sigma \in \Delta(i)$ $(i = 1,2,3)$, $\left\langle\{x_1^\sigma,\ldots,x_{4-i}^\sigma\}\right\rangle \in CH^{4-i}(D^*_\sigma, 4-i)$ must be trivial.

For an edge $\sigma \in \Delta(3)$, $\dim(D^*_\sigma) = 0$, and triviality of $\langle\{x_1^\sigma\}\rangle$ means that F_σ is cyclotomic. This implies that $\alpha_{\underline{m}} \in \bar{\mathbb{Q}}$ for $\underline{m} \in \sigma \cap \mathbb{Z}^n$ $(\forall\sigma \in \Delta(3))$. Moreover, since the one-skeleton of Δ is connected, we see that $\alpha_{\underline{\nu}} = 1$ for *every* vertex $\underline{\nu} \in \Delta(4)$.

Now let $\sigma \in \Delta(2)$ be a two-face, and *assume σ has at least one integer interior point $\underline{m}_0$ for which $\alpha_{\underline{m}_0} \notin \bar{\mathbb{Q}}$.* Write $\ell = 2$, $\Delta' := \mathrm{conv}(\mathfrak{M}_{F_\sigma})$, $G_0 := F_\sigma$, $(y_1,y_2) := (x_1^\sigma, x_2^\sigma)$, $Y_0^* = D_\sigma^*$, and $\xi_0' := \left\langle\{y_1,y_2\}|_{Y_0^*}\right\rangle$. Taking $\bar{\mathbb{Q}}$-spreads of Y_0^* and ξ_0' yields a family of curves $\mathcal{Y}^*_{\mathcal{S}} \to \mathcal{S}$ (defined over $\bar{\mathbb{Q}}$) over a quasi-projective variety, with a family of *trivial* higher Chow cycles on the fibers. Pulling back along a holomorphic map $U \to \mathcal{S}$, we are exactly in the situation of Lemma 4.1, with at least one $\beta_j'(s) \neq 0$ (from spreading $\alpha_{\underline{m}_0}$). Together (a) and (b) obviously contradict the triviality ξ_s' (hence $[AJ_{Y_s^*}(\xi_s')]$) inherits from ξ_0'. We conclude that $\alpha_{\underline{m}} \in \bar{\mathbb{Q}}$ for all $\sigma \in \Delta(2)$ and $\underline{m} \in \sigma \cap \mathbb{Z}^n$.

It remains to consider facets $\sigma \in \Delta(1)$, where the same assumption leads via spreading out to the setting of Lemma 4.1 (with $\ell = 3$) and a contradiction. Hence $\alpha_{\underline{m}} \in \bar{\mathbb{Q}}$ for any $\underline{m} \in \partial\Delta \cap \mathbb{Z}^n$, and since Δ is reflexive we are done. $\square$

5. An application to local mirror symmetry

For any reflexive polytope $\Delta \subset \mathbb{R}^n$ $(n = 2,3,4)$, the total space of $\mathsf{K}_{\mathbb{P}_{\Delta^\circ}}$ may be viewed as a noncompact CY $(n+1)$-fold. If we let $F \in \mathbb{C}[x_1^{\pm1},\ldots,x_n^{\pm1}]$

range over Laurent polynomials with $\mathrm{Conv}(\mathfrak{M}_F) = \Delta$, then the family

$$Y_F := \{F(\underline{x}) + u^2 + v^2 = 0\} \subset (\mathbb{C}^*)^n \times \mathbb{C}^2$$

of $(n+1)$-folds is the mirror dual of $\mathsf{K}_{\mathbb{P}_{\Delta^\circ}}$. These are CY, since the holomorphic form

$$\eta_F := 2\mathrm{i} \cdot \mathrm{Res}_{Y_F}\left(\frac{\bigwedge^n d\log \underline{x} \wedge du \wedge dv}{F + u^2 + v^2}\right) \in \Omega^{n+1}(Y_F)$$

yields a nonvanishing global section of the canonical bundle (i.e., K_{Y_F}). Its periods may be interpreted in terms of regulator periods on the $X_F^* := \{F(\underline{x}) = 0\} \subset (\mathbb{C}^*)^n$. We work out this story in Section 5.1 and use it to compute the mirror map for $n = 2$ in Section 5.3. Only in Section 5.4 (and the end of Section 5.1) do we once again require F to be tempered, in order to link up with Section 3, 4, 6 and study asymptotic growth of local Gromov–Witten numbers for $\mathsf{K}_{\mathbb{P}_{\Delta^\circ}}$.

5.1. Periods of an open CY three-fold

Let $X_F \subset \mathbb{P}_\Delta$ be the Zariski closure of X_F^*, with crepant resolution $\tilde{X}_F \subset \mathbb{P}_{\tilde{\Delta}}$; denote the inclusion $J : X_F^* \hookrightarrow \tilde{X}_F$. We assume F is Δ-regular, so that $\tilde{X}_F$ is smooth and the $D_{\tilde{\sigma}}$ reduced ($\forall i \geq 1$, $\tilde{\sigma} \in \tilde{\Delta}(i)$). Write $\{\underline{x}\} := \{x_1, \ldots, x_n\} \in CH^n((\mathbb{C}^*)^n, n)$ and $\xi_F := I^*\{\underline{x}\} \in CH^n(X_F^*, n)$ for its restriction to $X_F^* \xhookrightarrow{I} (\mathbb{C}^*)^n$. We use a somewhat nonstandard definition

$$H_{n-1}^{\mathrm{tr}}(\tilde{X}_F) := \mathrm{im}\{H_{n-1}(X_F^*, \mathbb{Q}) \xrightarrow{J_*} H_{n-1}(\tilde{X}_F, \mathbb{Q})\}$$

for the "transcendental part" of homology; clearly this is everything for $n = 2$ and contains the orthogonal complement of $\mathrm{Pic}(\tilde{X}_F)$ for $n = 3$. Also define

$$\mathcal{K}_{n-1}(X_F^*) := \ker\{H_{n-1}(X_F^*, \mathbb{Q}) \xrightarrow{I_*} H_{n-1}((\mathbb{C}^*)^n, \mathbb{Q})\}.$$

Lemma 5.1. *$\mathcal{K}_{n-1}(X_F^*)$ surjects onto $H_{n-1}^{\mathrm{tr}}(\tilde{X}_F)$; that is, every class Γ in $H_{n-1}^{\mathrm{tr}}(\tilde{X}_F, \mathbb{Q})$ has a representative $\gamma \in Z_{n-1}^{\mathrm{top}}(X_F^*; \mathbb{Q})$ that bounds in $(\mathbb{C}^*)^n$.*

Proof. Choose an edge $\sigma_1 \in \Delta(n-1)$ and vertex $\underline{\nu} \in \Delta(n)$ on σ_1. (More precisely, we take $\tilde{\sigma}_1 \in \tilde{\Delta}(n-1)$ and $\tilde{\underline{\nu}} \in \tilde{\Delta}(n)$ sitting "over" these.) Repeat the construction of Section 4.1 so that $\Phi_{\underline{\nu}} = 0$ locally describes $\tilde{X}_F$ and

$1+\phi_1(z_1)$ gives (up to a constant) the edge polynomial of σ_1. Fix a root $r(\in \mathbb{C}^*)$ of this, define in $Z^{\text{top}}_{n-1}(X^*_F;\mathbb{Z})$

$$\delta_{\sigma_1} := \{\Phi_{\underline{\nu}}=0\}\cap\{|z_2|=\cdots=|z_n|=\epsilon\}\cap\{|z_1-r| \text{ “small”}\}$$

and notice $\delta_{\sigma_1} \overset{\text{hom}}{\equiv} 0$ on $\tilde{X}_F$. Write $z_1(=x_1^{\sigma_1}) =: \underline{x}^{\underline{m}(\sigma_1)}$.

Define projections and inclusions

$$\begin{array}{ccc} (\mathbb{C}^*)^n \xrightarrow{\pi_i} & \{\underline{x}\in(\mathbb{C}^*)^n \,|\, x_i=1\} \cong (\mathbb{C}^*)^{n-1} & \xhookrightarrow{\iota_i} (\mathbb{C}^*)^n \\ & \uparrow & \\ & \{x_i=1,\, |x_j|=1\,\forall j\neq i\} =: \hat{\mathbb{T}}_i^{n-1}. & \end{array}$$

We can orient everything so that ${\pi_i}_*(I(\delta_{\sigma_1})) \overset{\text{hom}}{\equiv} m_i(\sigma_1)\hat{\mathbb{T}}_i^{n-1}$; hence $I(\delta_{\sigma_1}) \equiv \sum_{i=1}^n m_i(\sigma_1){\iota_i}_*(\hat{\mathbb{T}}_i^{n-1})$. Now the $\{\underline{m}(\sigma_1)\}$ (taken over all such edges) generate $\mathbb{Q}^n$; hence the $\{I(\delta_{\sigma_1})\}$ generate $H_{n-1}((\mathbb{C}^*)^{n-1},\mathbb{Q})$.

Given $\Gamma\in H^{\text{tr}}_{n-1}(\tilde{X}_F)$, let γ^0 be a representative in $Z^{\text{top}}_{n-1}(X^*_F)$. We may choose an appropriate sum δ of δ_{σ_1}'s with $I(\gamma^0) \overset{\text{hom}}{\equiv} I(\delta)$; clearly $\delta \overset{\text{hom}}{\equiv} 0$ on $\tilde{X}_F$, and so taking $\gamma := \gamma^0-\delta$ we are done. □

Remark 5.1. When $|\gamma^0|\subseteq X^*_F\cap\{\mathbb{R}^n \text{ or } (\mathrm{i}\mathbb{R})^n\}$, $I(\gamma^0)$ bounds on $(\mathbb{C}^*)^n$ without modification by a δ. [*Proof*: For any cycle $\mathfrak{Z}$ on $(\mathbb{C}^*)^n$, $\text{Box}^n(\mathfrak{Z}) := \mathfrak{Z} + \sum_{k=1}^n(-1)^k\sum_{|I|=k}(\iota_I\circ\pi_I)_*\mathfrak{Z} \overset{\text{hom}}{\equiv} 0$; since $H_{n-1}((\mathbb{C}^*)^{j<n-1})=0$, it follows that $I(\gamma^0)-\sum_{i=1}^n(\iota_i\circ\pi_i)_* I(\gamma^0)$ bounds (in $(\mathbb{C}^*)^n$). But if γ^0 has real support then each $(\pi_i)_* I(\gamma^0)$ “cancels itself out”, being of the same real dimension as the *real* part of the target (=disjoint union of copies of $(\mathbb{R}^+)^{n-1}$).] This is essentially used for the real, nonvanishing cycle L_0 (for real t near 0) in Appendix A of [45]. However, the procedure (employed there) of “bounding” the vanishing cycles $\{K_j\}$ with *noncompact* membranes is unnecessary in view of Lemma 5.1, and also incorrect in homology.

Lemma 5.2. *If $\gamma\in Z^{\text{top}}_{n-1}(X^*_F;\mathbb{Z})$ has $I(\gamma)=\partial\mu$, for $\mu\in C^{\text{top}}_n((\mathbb{C}^*)^n;\mathbb{Z})$, then*

$$\int_\gamma R(\xi_F) \equiv \int_\mu \wedge^n d\log\underline{x} \qquad \text{mod } \mathbb{Z}(n).$$

Proof. On $(\mathbb{C}^*)^n$, $\bigwedge^n d\log\underline{x} = d[R\{\underline{x}\}] \pm (2\pi\mathrm{i})^n\delta_{T_{\underline{x}}}$, and so

$$\int_\mu \wedge^n d\log\underline{x} \equiv \int_\mu d[R\{\underline{x}\}] = \int_{\partial\mu} R\{\underline{x}\} = \int_\gamma I^*R\{\underline{x}\}.$$

□

We want to construct cycles in $Z^{\mathrm{top}}_{n+1}(Y_F)$ over which to integrate η_F. Considering Y_F as a fiber bundle over $(\mathbb{C}^*)^n$, we have (for $n = 2$) the picture displayed in figure 7. In a topological sense, we may view Y as the disjoint union of an S^1-bundle over $(\mathbb{C}^*)^n$ with a copy of X_F^*. More precisely, if $P : Y_F \twoheadrightarrow (\mathbb{C}^*)^n$ sends $(\underline{x}, u, v) \mapsto \underline{x}$, then

$$\begin{aligned} \underline{x} \in (\mathbb{C}^*)^n \backslash X_F^* \quad &\Longrightarrow \quad P^{-1}(\underline{x}) \cong \mathbb{C}^* \ \text{(homotopic to } S^1\text{)}, \\ \underline{x} \in X_F^* \quad &\Longrightarrow \quad P^{-1}(\underline{x}) \cong \{u^2 + v^2 = 0\} =: W = W_1 \cup W_2, \end{aligned}$$

where $W_i \cong \mathbb{A}^1_{\mathbb{C}}$. In fact, $Y_F \supset X_F^* \times W$ and we can write $W = W_1 \amalg W_2^*$ ($W_2^* := W_2 \backslash \{(0,0)\}$); the complement $Y_F \backslash (X_F^* \times W_1)$ is then homotopic to $(\mathbb{C}^*)^n \times S^1$.

Consider the long-exact sequence

$$(5.1) \qquad \begin{array}{ccc} \uparrow & & \\ H_n(Y_F \backslash X_F^* \times W_1) & \xrightarrow[\cong]{} & H_{n-1}((\mathbb{C}^*)^n) \oplus H_n((\mathbb{C}^*)^n) \\ {\scriptstyle tube}\uparrow & & \uparrow{\scriptstyle (I_*,0)} \\ H_{n-1}(X_F^* \times W_1) & \xrightarrow[\cong]{} & H_{n-1}(X_F^*) \\ {\scriptstyle \cap}\uparrow & & \\ H_{n+1}(Y_F) & & \\ \uparrow & & \\ H_{n+1}(Y_F \backslash X_F^* \times W_1) & \xrightarrow[\cong]{\text{"forget } S^1\text{"}} & H_n((\mathbb{C}^*)^n) \\ {\scriptstyle tube}\uparrow & & \uparrow{\scriptstyle I_*=0} \\ H_n(X_F^* \times W_1) & \xrightarrow[\cong]{} & H_n(X_F^*). \\ \uparrow & & \end{array}$$

The bottom I_* is 0 because the dual map $[F^n]H^n((\mathbb{C}^*)^n) \to H^n(X_F^*)$ must be, as $\dim(X_F^*) = n - 1$ implies that $F^n H^n(X_F^*) = \{0\}$. The $H_{n-1}(X_F^*) \to H_n((\mathbb{C}^*)^n)$ is essentially the composition of $\mathrm{Tube} : H_{n-1}(X_F^*) \to H_n((\mathbb{C}^*)^n \backslash X_F^*)$ with $H_n((\mathbb{C}^*)^n \backslash X_F^*) \to H_n((\mathbb{C}^*)^n)$; it is 0 for a similar reason.

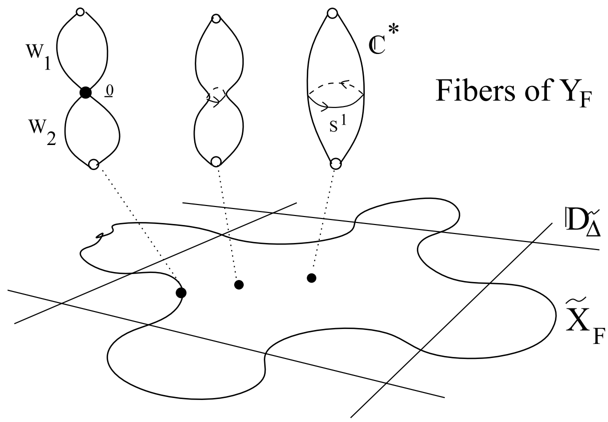


Figure 7: The open CY $(n+1)$-fold Y.

Using any $\hat{\mathbb{T}}^n_{\underline{\nu},\epsilon} \in Z^{\text{top}}_n((\mathbb{C}^*)^n \backslash X^*_F)$ (see Section 4.1) and the topological "S^1-bundle" structure of $Y_F \backslash (X^*_F \times W_1)$, gives a cycle $\hat{\mathbb{T}}^{n+1}_Y \in Z^{\text{top}}_{n+1}(Y_F)$. Now (5.1) becomes the short-exact sequence

$$\mathbb{Q}\left\langle \hat{\mathbb{T}}^{n+1}_Y \right\rangle \to H_{n+1}(Y_F) \to \mathcal{K}_{n-1}(X^*_F).$$

To construct explicitly an isomorphism

$$M : \mathcal{K}_{n-1}(X^*_F) \to H_{n+1}(Y_F) \Big/ \mathbb{Q}\left\langle \hat{\mathbb{T}}^{n+1}_Y \right\rangle ,$$

let γ, μ be as in Lemma 5.2 ($\mathbb{Q}$-coefficients). The cycle (representing) $M(\gamma)$ will have support in $P^{-1}(|\mu|)$, with S^1-fibers over $\text{Int}|\mu|$ and point fibers over $|\partial\mu| = |\gamma|$. More precisely, $M(\gamma) \cap P^{-1}(\underline{x})$ (for $\underline{x} \in |\mu|$) is given by

$$V \in [-\sqrt{|F(\underline{x})|}, \sqrt{|F(\underline{x})|}] , \quad v = \mathrm{e}^{\frac{\mathrm{i}}{2}\arg(-F(\underline{x}))} V , \quad u = \pm\sqrt{-(v^2 + F(\underline{x}))}.$$

Note that $\mathbb{Q}\left\langle \hat{\mathbb{T}}^{n+1}_Y \right\rangle$ absorbs the ambiguity arising from the choice of μ.

Lemma 5.3. *For γ, μ as in Lemma 5.2,*

$$\int_{M(\gamma)} \eta_F = 2\pi\mathrm{i} \int_\mu \wedge^n d\log\underline{x}\,.$$

Moreover, $\int_{\hat{\mathbb{T}}^{n+1}_Y} \eta_F = (2\pi\mathrm{i})^{n+1}$.

Proof. Writing $u' := u + \mathrm{i}v$, $v' := u - \mathrm{i}v$, we have (away from $v' = 0$) $\eta_F = \mathrm{Res}_{Y_F}\left(\frac{\bigwedge^n d\log \underline{x} \wedge du' \wedge dv'}{F(\underline{x}) + u'v'}\right) = \bigwedge^n d\log \underline{x} \wedge d\log u'$. The result is now immediate (by integrating "first" over the S^1 fibers of $M(\gamma)$). □

Lemmas 5.2 and 5.3 imply the following

Proposition 5.1. *The periods of η_F are precisely the $\mathbb{C} \to \mathbb{C}/\mathbb{Q}(n+1)$ lifts of the $2\pi\mathrm{i}\int_\gamma R(\xi_F)$ for $\gamma \in \mathcal{K}_{n-1}(X_F^*)$, including the lifts $(2\pi\mathrm{i})^{n+1}\mathbb{Q}$ of 0.*

If we now assume $F = \hat{F}$ is tempered, plus additional assumptions for $n = 4$ (cf. Theorem 3.1), then $\xi_{\hat{F}}$ comes from some $\Xi_{\hat{F}} \in CH^n(\tilde{X}_{\hat{F}}, n)$, and so $R(\xi_{\hat{F}})$ has no residues to separate periods over $\gamma_1, \gamma_2 (\in \mathcal{K}_{n-1})$ with $J_*\gamma_1 = J_*\gamma_2$. Therefore (using Lemma 5.1), we get

Corollary 5.1. *The periods of $\eta_{\hat{F}}$ may be expressed in terms of the regulator periods of "transcendental cycles": $\int_{(\cdot)} \eta_{\hat{F}}$ is the composition*

$$\frac{H_{n+1}(Y_{\hat{F}})}{M(\ker(J_*) \cap \mathcal{K}_{n-1}) + \mathbb{Q}\left\langle \hat{\mathbb{T}}_Y^{n+1} \right\rangle} \xrightarrow[\cong]{M^{-1}} \frac{\mathcal{K}_{n-1}(X_{\hat{F}}^*)}{\ker(J_*) \cap \mathcal{K}_{n-1}} \xrightarrow[\cong]{\text{Lemma 5.1}}$$
$$H_{n-1}^{\mathrm{tr}}(\tilde{X}_{\hat{F}}) \xrightarrow{2\pi\mathrm{i}\int_{(\cdot)} R(\Xi_{\hat{F}})} \mathbb{C}/\mathbb{Q}(n+1).$$

In particular, if we put ourselves in a one-parameter family setting $\hat{F} = 1 - t\phi(\underline{x})$ for ϕ as in Section 2, then Corollaries 4.1 and 5.1 get

Corollary 5.2. *The $\mathcal{D}$-submodule of $\mathcal{H}^{n-1}_{\tilde{X}_t}$ generated by $[\tilde{\omega}_t]$ is a quotient of the submodule of $\mathcal{H}^{n+1}_{Y_t}$ generated by $[\eta_t]$, via*

$$\nabla_{\mathrm{PF}}^{(Y,\eta)} = \nabla_{\mathrm{PF}}^{(\tilde{X},\tilde{\omega})} \circ \nabla_{\delta_t}.$$

Remark 5.2. If $\tilde{\varphi}_0$ is a vanishing cycle (as in Section 4), with $\mathcal{K}_{n-1} \ni \varphi_0 \overset{J_*}{\longmapsto} \tilde{\varphi}_0$, then by Theorem 4.1 and Corollary 5.1

$$\frac{\int_{M(-\varphi_0)} \eta_t}{\int_{\hat{\mathbb{T}}_Y^{n+1}} \eta_t} = \frac{-2\pi\mathrm{i}\int_{\tilde{\varphi}_0} R(\Xi_t)}{(2\pi\mathrm{i})^{n+1}} = \frac{\Psi(t)}{-(2\pi\mathrm{i})^n} \sim \frac{\log t}{2\pi\mathrm{i}}$$

as $t \to 0$. So this period ratio is custom-made for defining a mirror map.

5.2. The canonical bundle as a CY toric variety

We specialize to the case $n = 2$ for the remainder of the section. Let $\Delta \subset \mathbb{R}^2$ be a reflexive polytope with vertices $\underline{\nu}^{(1)}, \ldots, \underline{\nu}^{(r+2)}$ numbered counter-clockwise. Together with $\underline{\nu}^{(0)} = \{\underline{0}\}$, these are the "relevant integral points" of Δ (any interior points of edges are excluded). We have a (partial) triangulation $\mathrm{tr}(\Delta)$ using the segments $\mathfrak{s}^{(k)} = [\underline{\nu}^{(0)}, \underline{\nu}^{(k)}]$, and write $\underline{\nu}^{(i,j)} := \underline{\nu}^{(j)} - \underline{\nu}^{(i)}$.

A fan Σ_Δ is obtained by taking cones on $\mathrm{tr}(\Delta) \times \{1\} \subset \mathbb{R}^3$. The generators of $\Sigma_\Delta(1)$ are $\{\hat{\underline{\nu}}^{(0)}, \ldots, \hat{\underline{\nu}}^{(r+2)}\}$ where $\hat{\underline{\nu}}^{(k)} = (\underline{\nu}^{(k)}, 1)$. The associated toric variety Y° is the total space of $\mathsf{K}_{\mathbb{P}_{\Delta^\circ}} \xrightarrow{\rho} \mathbb{P}_{\Delta^\circ}$. The line bundle K_{Y° is trivialized by a [global nonvanishing] "tautological section", making Y° an open CY three-fold. If edges of Δ have interior integral points $\underline{u}^{(\ell)}$ then Y° is singular (but normal). When we refer to the "singular case" resp. "smooth case" below, this is what is meant.

The curves $C_i^\circ \subset Y^\circ$ dual to subfans $\Sigma_{\mathfrak{s}^{(i)}}$ are in 1–1 correspondence with edges of Δ°, and are supported on the "0-section" $D_0^\circ \cong \mathbb{P}_{\Delta^\circ} \subset Y^\circ$. The $[C_i^\circ]$ generate $H_2(Y^\circ, \mathbb{Z})$, and the *Mori cone* (of effective curves) in $H_2(Y^\circ, \mathbb{R})$ is just obtained by taking $\mathbb{R}^{\geq 0}$-linear combinations of them. We assume henceforth that the Mori cone with *this* integral structure is smooth (cf. [23, p. 32]; this implies simplicial). A simple example where both Y° and Mori are smooth is shown in figure 8.

The divisors D_i° dual to subfans $\Sigma_{\underline{\nu}^{(i)}}$, $i = 0, \ldots, r+2$, generate $H^2(Y^\circ, \mathbb{Q})$. If $\mathbb{P}_{\Delta^\circ}$ (and Y°) are smooth then the $D_i^\circ = \rho^{-1}(C_i^\circ)$. Otherwise, using the $\underline{u}^{(\ell)}$ to refine Σ_Δ yields the crepant resolution $Y^\circ \xleftarrow{\hat{p}} \tilde{Y}^\circ$ over $\mathbb{P}_{\Delta^\circ} \xleftarrow{p} \mathbb{P}_{\tilde{\Delta}^\circ}$. Denote the exceptional divisors E_ℓ° (for p) and $\hat{E}_\ell^\circ := \tilde{\rho}^{-1}(E_\ell^\circ)$ (for $\hat{p}$); we have $H^2(Y^\circ, \mathbb{Q}) \cong \ker\{H^2(\tilde{Y}^\circ) \to H^2(\cup \hat{E}_\ell^\circ)\}$. Writing $\tilde{C}_i^\circ = p^* C_i^\circ$

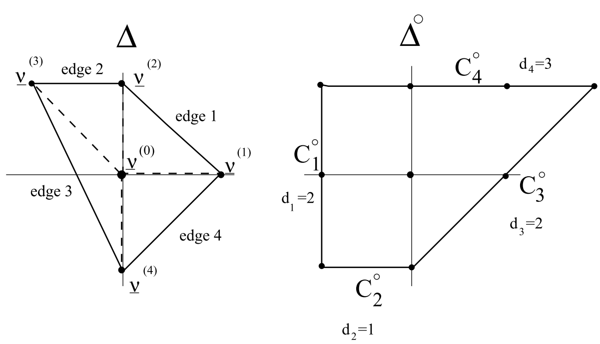


Figure 8: Local mirror CY 3-fold data.

for the proper transforms, the D_i are then represented by cycles on $\tilde{Y}^\circ$ of the form $\tilde{D}_j^\circ := \tilde{\rho}^{-1}(\tilde{C}_i^\circ) + \sum_\ell \beta_\ell^i \hat{E}_\ell^\circ$ for $\beta_\ell^i \in \mathbb{Q}$ satisfying $(\tilde{C}_i^\circ + \sum \beta_\ell^i E_\ell^\circ) \cdot E_k^\circ = 0\ \forall i, k$.

Intersections $M_{ij} := \left\langle C_i^\circ, D_j^\circ \right\rangle$ under the pairing $H^2(Y^\circ) \times H_2(Y^\circ) \to \mathbb{Q}$ are then computed by $\tilde{C}_i^\circ \cdot \tilde{D}_j^\circ$. These need not be integers (see [37]) but the matrix $[M_{ij}]_{i,j\geq 1}$ is symmetric. The *Kähler cone* is the dual of Mori in $H^2(Y^\circ, \mathbb{R})$ under this pairing; it is represented by divisors $\{D = \sum \alpha_j D_j \mid \langle C_i, D\rangle \geq 0\,(\forall i)\}$.

In general, we have in $H^2(Y^\circ)$

$$D_0^\circ \equiv -\sum_{i\geq 1} D_i^\circ \equiv \rho^{-1}(\mathsf{K}_{\mathbb{P}_{\Delta^\circ}}) \equiv -\rho^{-1}(X^\circ),$$

where X° is any anticanonical (elliptic curve) hypersurface in good position with respect to $\mathbb{D}_{\Delta^\circ}$. Writing $d_i - 1 :=$ number of interior points of the edge of Δ° dual to $\underline{\nu}^{(i)}$, we have $(i \geq 1)$

$$-\left\langle C_i^\circ, D_0^\circ \right\rangle_{Y^\circ} = \left\langle C_i^\circ, X^\circ \right\rangle_{\mathbb{P}_{\Delta^\circ}} = d_i.$$

Put $e_i - 1 :=$ number of interior points on the edge "next" (in the counter-clockwise direction) to $\underline{\nu}^{(i)}$. We are in the singular case iff some $e_i > 1$.

We are interested in a very explicit (and standard) presentation of the Mori cone: first, we write down generators for the integral relations on the $\underline{\hat{\nu}}^{(i)}$ as follows. For any $k \in \{1, \ldots, r+2\}$, let $\ell_{k-1}^{(k)} \underline{\hat{\nu}}^{(k-1)} + \ell_{k+1}^{(k)} \underline{\hat{\nu}}^{(k+1)}$ be the minimal $\mathbb{Z}^+$-linear combination lying in the line containing $\mathfrak{s}^{(k)}$, and then choose $\ell_k^{(k)} \in \mathbb{Z}$, $\ell_0^{(k)} \in \mathbb{Z}^{\leq 0}$ such that

$$\ell_0^{(k)} \underline{\hat{\nu}}^{(0)} + \ell_{k-1}^{(k)} \underline{\hat{\nu}}^{(k-1)} + \ell_k^{(k)} \underline{\hat{\nu}}^{(k)} + \ell_{k+1}^{(k)} \underline{\hat{\nu}}^{(k+1)} = 0. \tag{5.2}$$

Note that $\ell_{k-1}^{(k)}$ is replaced by $\ell_{r+2}^{(k)}$ for $k = 1$, and $\ell_{k+1}^{(k)}$ by $\ell_1^{(k)}$ for $k = r+2$.

Remark 5.3. One can show that these take the form

$$\ell_0^{(k)} = \frac{-e_k e_{k-1} d_k}{e_{(k,k-1)}}, \quad \ell_{k-1}^{(k)} = \frac{e_k}{e_{(k,k-1)}}, \quad \ell_k^{(k)} = \frac{e_k e_{k-1} d_k - e_k - e_{k-1}}{e_{(k,k-1)}},$$
$$\ell_{k+1}^{(k)} = \frac{e_{k-1}}{e_{(k,k-1)}},$$

where $e_{(k,k-1)} := \gcd(e_k, e_{k-1})$.

This procedure determines a vector $\underline{\ell}^{(k)} \in \mathbb{Z}^{r+3}$ with

$$d_k \ell_j^{(k)} = -\ell_0^{(k)} M_{kj} = \left\langle -\ell_0^{(k)} C_k^\circ, D_j^\circ \right\rangle .$$

(In the smooth case, $d_k = -\ell_0^{(k)}$.) That is, the relations vectors $\underline{\ell}^{(i)}$ are essentially the rows of M with denominators cleared; write L for the new matrix.

The Mori cone can be represented by the $\mathbb{R}^{\geq 0}$-span $\mathcal{M} \subset \mathbb{R}^{r+3}$ of rows of L; by our above assumption (on Mori), $\mathcal{M}$ is simplicial. However, the integral structures may not be the same in the "singular case," so $\mathcal{M}$ may not be smooth. More concretely, write $\mathbb{M} := \{\mathbb{R}\text{-span of } \underline{\ell}^{(i)}\} \subset \mathbb{R}^{r+3}$, with integral lattice $\mathbb{M}_\mathbb{Z} = \mathbb{M} \cap \mathbb{Z}^{r+3}$, and $\mathcal{M}_\mathbb{Z} = \mathcal{M} \cap \mathbb{M}_\mathbb{Z}$. Then the affine toric variety

$$U_\Delta := \operatorname{Spec}\left\{\mathbb{C}[\underline{a}^{\underline{m}} \,|\, \underline{m} \in \mathcal{M}_\mathbb{Z}]\right\}$$

is just $\mathbb{A}^r$ in the smooth case but can be singular in the singular case.

Using the fact that $\mathcal{M}$ is simplicial, take the $\{\underline{\ell}^{(i_k)}\}_{k=1}^r$ which cannot be written as $\mathbb{R}^{\geq 0}$-linear combinations of the other $\{\underline{\ell}^{(j)}\}$. (In the singular case, if any $\underline{\ell}^{(i)}$ are the same, we choose the one for which the "dual" d_i is minimized.) Note that $\mathcal{M}$ is smooth iff $\mathbb{Z}^{\geq 0}\left\langle \{\underline{\ell}^{(i_k)}\} \right\rangle$ is all of $\mathcal{M}_\mathbb{Z}$. Next, let $\alpha_k^i \in \mathbb{Q}$ be such that $J_m^\circ := \sum_{j=1}^{r+2} \alpha_m^j D_j^\circ$ satisfy

$$\langle C_{i_k}^\circ, J_m^\circ \rangle_{Y^\circ} \left(= \sum_{j=1}^{r+2} \alpha_m^j \left| \frac{d_{i_k}}{\ell_0^{(i_k)}} \right| \ell_j^{(i_k)} \right) = \delta_{km}.$$

(That is, if we omit a couple of rows from L, the $\{\alpha_m^j\}$ give linear combinations of the columns that yield $\hat{e}_m \in \mathbb{R}^r$.) These $\{J_m^\circ\}$ then generate the Kähler cone. We have $\sum d_{i_k} J_k^\circ \equiv -D_0^\circ$ since $\sum_k d_{i_k} \left\langle C_{i_j}^\circ, J_k^\circ \right\rangle = d_{i_j} = -\left\langle C_{i_j}^\circ, D_0^\circ \right\rangle$.

Remark 5.4. The $\{\alpha_m^j\}$ are nonnegative, since the Kähler cone lies in the effective divisor cone, see [23]. It follows that $\mathcal{M}_\mathbb{Z} \supseteq \mathbb{M} \cap (\mathbb{Z}^{\geq 0})^{r+3}$.

Now we use this construction to identify the complex structure moduli we will use, for the anticanonical hypersurface $X_{\underline{a}}$ given by the Zariski closure of

$$F_{\underline{a}}(\underline{x}) := \sum_{i=0}^{r+2} a_i \underline{x}^{\underline{\nu}^{(i)}} = 0$$

in $\mathbb{P}_\Delta$. The coordinate patch in simplified polynomial moduli space $\overline{\mathcal{M}}_{\mathrm{simp}}$ (cf. [23]) on which it is natural to work is just U_Δ, with coordinates

$$t_k := \underline{a}^{\underline{\ell}^{(i_k)}}, \quad k = 1, \ldots, r.$$

In the singular case, to parametrize U_Δ one really needs all $r+2$ of the $\underline{a}^{\underline{\ell}^{(i)}} =: s_i$ together with their relations, but the functions we consider will be defined in terms of the $\{t_k\}$. Moreover, the inclusion of $\mathcal{M}_{\mathbb{Z}}$ into the true Mori integral lattice (generated by the $\{C^\circ_{i_k}\}$) defines a smooth finite cover $\mathbb{A}^r \cong \tilde{U}_\Delta \to U_\Delta$ with coordinates $\{\tilde{t}_k\}$ satisfying $(\tilde{t}_k)^{\mu_k} = t_k$, for $\mu_k := \frac{|\ell_0^{(i_k)}|}{d_{i_k}} = \frac{e_{i_k} e_{i_k - 1}}{e_{(i_k, i_k - 1)}}$. This is where we really want to work.

5.3. Construction of the mirror map via regulator periods

The family $Y_{\underline{a}} := \{u^2 + v^2 + F_{\underline{a}}(\underline{x}) = 0\} \subset (\mathbb{C}^*)^2 \times \mathbb{C}^2$ treated (in greater generality) above, with holomorphic form $\eta_{\underline{a}}$, is considered to be the mirror of $\mathsf{K}_{\mathbb{P}_{\Delta^\circ}}$. This is in part because its periods satisfy the relevant GKZ equations $\mathcal{D}_k(\cdot) = 0$.[17] The $\mathcal{D}_k$ are essentially the push-forwards, under the map $(\mathbb{C}^*)^{r+3} \to (\mathbb{C}^*)^r$ given by $\underline{a} \mapsto \underline{t}$, of

$$\tilde{\mathcal{D}}_k = \prod_{\{j \,|\, \ell_j^{(i_k)} > 0\}} \partial_{a_j}^{|\ell_j^{(i_k)}|} - \prod_{\{j \,|\, \ell_j^{(i_k)} < 0\}} \partial_{a_j}^{|\ell_j^{(i_k)}|}.$$

In view of Proposition 5.1, we will work instead with regulator periods on $X^*_{\underline{a}}$ to construct the (inverse of the) mirror map. This will be a map from complex structure parameters $\underline{\tilde{t}}$ to complexified Kähler parameters

$$\tilde{U}_\Delta \supset \tilde{\mathcal{P}} \rightsquigarrow \left\{\mathbb{Z}\left\langle\{J_k^\circ\}_{k=1}^r\right\rangle \subset H^{1,1}(Y^\circ, \mathbb{Q})\right\} \otimes_{\mathbb{Z}} (\mathbb{C}/\mathbb{Z}), \tag{5.3}$$

where $\tilde{\mathcal{P}} \to \mathcal{P} \to D^*_\varepsilon(0)^{\times r}$ are small punctured polycylinders centered at $\underline{0}$ in $\tilde{U}_\Delta \to U_\Delta \to \mathbb{A}^r$.

We will follow the method of Sections 4.1 and 4.2 for computing these periods, taking $\underline{\nu} := \underline{\nu}^{(j)}$ and $z_1 := \underline{x}^{e_j^{-1}\underline{\nu}^{(j,j+1)}}$ (see beginning of Section 4.1).

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[17] For a more thorough conceptual treatment of local mirror symmetry, the reader is encouraged to consult [21, 27, 45].

The local affine equation of $\tilde{X}_{\underline{a}}$ is then given by

$$(f_{\underline{a}}(\underline{z})+a_0)z_1z_2 = a_j + a_{j+1}z_1^{e_j} + \phi_2(z_1,z_2) + a_0z_1z_2 = 0,$$

where $\phi_2(z_1,0)=0$. Assuming $0<|a_i|\ll|a_0|$ $(\forall i)$ [hence $0<|t_k|\ll 1$ $(\forall k)$], consider the family of cycles

$$\hat{\varphi}_0^{(j)} := \{|z_1|=\epsilon\,,\ |z_2|\leq\epsilon\}\cap\tilde{X}_{\underline{a}}\subset X^*_{\underline{a}}.$$

This may be thought of as a vanishing cycle being pinched to the "point at vertex $\underline{\nu}^{(j)}$" as $a_j\to 0$.

As in Section 4.2 we set (working integrally)

$$\xi_{\underline{a}} := \{x_1,x_2\}\equiv\{(-1)^{\sigma_j}z_1,(-1)^{\sigma_{j-1}}z_2\}\in CH^2(X^*_{\underline{a}},2),$$

where $\sigma_j := \left|\frac{\nu_1^{(j,j+1)}\nu_2^{(j,j+1)}}{e_j^2}\right|$ gives essentially the sign from Remark 3.5.

In $CH^3((\mathbb{C}^*)^2\backslash X^*_{\underline{a}},3)$ we define

$$\begin{aligned}\hat{\xi}_{\underline{a}} &:= \left\{a_0+f_{\underline{a}}(\underline{z}),\,(-1)^{\sigma_j}z_1,\,(-1)^{\sigma_{j-1}}z_2\right\}\\ &\equiv\left\{(-1)^{\sigma_j+\sigma_{j-1}}(a_j+a_{j+1}z_1^{e_j}+\mathcal{O}(z_2)),\,(-1)^{\sigma_j}z_1,\,(-1)^{\sigma_{j-1}}z_2\right\}.\end{aligned}$$

This has residue $\xi_{\underline{a}}$ along $\tilde{X}_{\underline{a}}$, so that

$$\begin{aligned}(5.4)\quad &\frac{1}{2\pi\mathrm{i}}AJ(\xi_{\underline{a}})(\hat{\varphi}_0^{(j)})\\ &=\frac{1}{(2\pi\mathrm{i})^2}AJ(\hat{\xi}_{\underline{a}})(|z_1|=|z_2|=\epsilon)-\frac{1}{2\pi\mathrm{i}}AJ(\mathrm{Res}^1_{\{z_2=0\}}\hat{\xi}_{\underline{a}})(|z_1|=\epsilon)\\ &=\int_{|z_1|=|z_2|=\epsilon}\log(a_0+f_{\underline{a}}(\underline{z}))\frac{d\log(z_1)}{2\pi\mathrm{i}}\wedge\frac{d\log(z_2)}{2\pi\mathrm{i}}\\ &\quad-\int_{|z_1|=\epsilon}\log((-1)^{\sigma_j+\sigma_{j-1}}(a_j+a_{j+1}z_1^{e_j}))\frac{d\log(z_1)}{2\pi\mathrm{i}}\\ &=\log(a_0)-\sum_{k\geq 1}\frac{1}{k}\left[\left(-\frac{1}{a_0}f_{\underline{a}}(\underline{z})\right)^k\right]_0-\log((-1)^{\sigma_j+\sigma_{j-1}}a_j)\\ &=-\log\left((-1)^{\sigma_j+\sigma_{j-1}}\frac{a_j}{a_0}\right)-H(\underline{a}).\end{aligned}$$

Here $[\cdot]_0$ takes the terms constant in z_1, z_2. Now in the smooth case (essentially following pp. 160–161 [23])

$$\begin{aligned} H(\underline{a}) &= \sum_{m\geq 1} \frac{1}{m} \sum_{\ell_1,\ldots,\ell_{r+2}} \frac{(\sum \ell_j)!}{\prod(\ell_j!)} \cdot \frac{\prod a_i^{\ell_i}}{(-a_i)^{\sum \ell_i}} \\ &= \sum_{m\geq 1} \frac{1}{m} \sum_{n_1,\ldots,n_r} \frac{(\sum n_k |\ell_0^{(i_k)}|)!}{\prod_j (\sum n_k \ell_j^{(i_k)})!} \cdot \prod_k ((-1)^{\ell_0^{(i_k)}} t_k)^{n_k}. \end{aligned}$$

The first big $\sum$ is over nonnegative integers $\{\ell_j\}$ satisfying $\sum \ell_j = m$, $\sum \ell_j \underline{\nu}^{(j)} = 0$; the second is over integers $\{n_k\}$ with $\sum n_k \underline{\ell}^{(i_k)} \in \mathbb{Z} \times (\mathbb{Z}^{\geq 0})^{r+2}$ and $\sum n_k |\ell_0^{(i_k)}| = m$. By Remark 5.4 we can take these $n_k \geq 0$, and so H is holomorphic (and well-defined) in a neighborhood of $\underline{0}$ in U_Δ. In the singular case, we replace $\sum_{n_1,\ldots,n_r}$ by a sum over $\mathbb{M} \cap (\mathbb{Z}^{\geq 0})^{r+3}$ (which involves nonredundant choices of $\{n_i\}_{i=1}^{r+2}$) and use all the $\ell^{(i)}$ and s_i (not just the $\ell^{(i_k)}$ and t_k). The resulting H is defined on U_Δ and pulls back to a holomorphic function on $\tilde{U}_\Delta$. Henceforth it will be written $H(\underline{s})$.

Clearly the "log"-term of (5.4) makes no sense on U_Δ or even $\tilde{U}_\Delta$; this reflects the fact that $\xi_{\underline{a}}$ is not invariant under the action of the torus $(\mathbb{C}^*)^2$. But the periods of $R\{x_1, x_2\}$ over cycles in

$$\mathcal{K}(X^*_{\underline{a}}) := \ker\{H_1(X^*_{\underline{a}}, \mathbb{Z}) \to H_1((\mathbb{C}^*)^2, \mathbb{Z})\}$$

are torus-invariant, and r *distinguished vanishing cycles* in $\mathcal{K}(X^*_{\underline{a}})$ are given by

$$\varphi_0^{[k]} := -\sum_{j=1}^{r+2} \ell_j^{(i_k)} \hat{\varphi}_0^{(j)} \quad k = 1, \ldots, r\,.$$

The map $H_1(X^*_{\underline{a}}) \twoheadrightarrow H_1(\tilde{X}_{\underline{a}})$ induced by inclusion sends $\varphi_0^{[k]}$ to $\ell_0^{(i_k)}$ times a primitive vanishing cycle $\tilde{\varphi}_0$. If $\varphi_1 \in \mathcal{K}(X^*_{\underline{a}})$ is a lift of a complimentary generator $-\tilde{\varphi}_1$, then $AJ(\xi_{\underline{a}})(\varphi_1)$ and the $AJ(\xi_{\underline{a}})(\varphi_0^{[k]})$ form a $\mathbb{Q}$-basis for the periods (modulo $\mathbb{Q}(2)$) of $AJ(\xi_{\underline{a}}) = [R\{x, y\}]$ over cycles in $\mathcal{K}(X^*_{\underline{a}})$. One should view the $\varphi_0^{[k]}$ as differing by loops around points of $D \subset \tilde{X}_{\underline{a}}$, hence the $AJ(\xi_{\underline{a}})(\varphi_0^{[k]})$ as differing by residues.

Now we slightly change our notation to bring it in line with [45]. Write (multivalued) functions of $\underline{t}$

$$\tilde{w}^{(0)} := (2\pi\mathrm{i})^3 = \int_{\hat{\mathbb{T}}^3_Y} \eta_{\underline{a}}\,,$$
$$\tilde{w}^{(1)}_k := 2\pi\mathrm{i}\,AJ(\xi_{\underline{a}})(\varphi_0^{[k]}) = \int_{M(\varphi_0^{[k]})} \eta_{\underline{a}}\,,$$
$$\tilde{w}^{(2)} := 2\pi\mathrm{i}\,AJ(\xi_{\underline{a}})(\varphi_1) = \int_{M(\varphi_1)} \eta_{\underline{a}},$$

and normalize these by setting $w^{(\cdot)}_{\cdot} := \tilde{w}^{(\cdot)}_{\cdot}/\tilde{w}^{(0)}$.

Theorem 5.1. *The $w^{(1)}_k$ are well-defined $\mathbb{C}/\mathbb{Z}$-valued functions on $\mathcal{P}$, given by $\frac{1}{2\pi\mathrm{i}}$ times*

$$\log\left((-1)^{\epsilon_k} t_k\right) + |\ell_0^{(i_k)}| H(\underline{s}),$$

where $\epsilon_k := \sum_{j=1}^{r+2} (\sigma_j + \sigma_{j-1})\ell_j^{(i_k)}$.

Definition 5.1. The (inverse) mirror map (5.3) is given by

$$(\tilde{t}_1, \ldots, \tilde{t}_r) \longmapsto \sum_{k=1}^{r} J_k^\circ \otimes W_k^{(1)}(\underline{\tilde{t}}),$$

where $W_k^{(1)}(\underline{\tilde{t}}) := \frac{1}{\mu_k} w_k^{(1)}(\underline{s}(\underline{\tilde{t}}))$.

Remark 5.5. (i) Hosono [45] considers the (conjectural!) map

$$\mathrm{mir}:\ K^c(Y^\circ) \to H_3(Y, \mathbb{Z})$$

arising from Kontsevich's homological mirror symmetry conjecture, and proposes that one should have $\hat{\mathbb{T}}^3_Y = \mathrm{mir}(\mathcal{O}_{\mathrm{pt}})$, $\frac{1}{\mu_k} M(\varphi_0^{[k]}) = \mathrm{mir}(\mathcal{O}_{C^\circ_{i_k}}(-J_k^\circ))$, $M(\varphi_1) = \mathrm{mir}(\mathcal{O}_{D_0})$.

(ii) Set $\delta_T := \sum_{j=1}^r |\ell_0^{(i_j)}| \delta_{t_j}$. The $W_k^{(1)}$ are logarithmic integrals of periods of $\omega_{\underline{a}} := \mathrm{Res}\left(\frac{\frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2}}{F_{\underline{a}}(x_1, x_2)}\right)$ in the (limited) sense that

$$\delta_T W_k^{(1)} = \frac{d_{i_k}}{(2\pi\mathrm{i})^2} \int_{\tilde{\varphi}_0} \omega_{\underline{a}}$$

for each k. We also write (after [21]) $\partial_S := \sum_{k=1}^r d_{i_k} \partial_{W_k^{(1)}}$.

5.4. Growth of local Gromov–Witten invariants

Define (on $\tilde{\mathcal{P}}$) the Gromov–Witten prepotential

$$\mathcal{F}_{\mathrm{loc}}(\underline{W}^{(1)}) := \frac{1}{2}\sum_{j,\ell}\langle J_j^\circ|_{\mathbb{P}_{\Delta^\circ}}, J_\ell^\circ\rangle W_j^{(1)}W_\ell^{(1)} + \left\{\begin{matrix}\text{lower-order}\\ \text{terms}\end{matrix}\right\}(\underline{W}^{(1)})$$
$$- \sum_{k_1,\ldots,k_r}\left(\sum_{j=1}^{r} d_{i_j}k_j\right) N_{k_1,\ldots,k_r}Q_1^{k_1}\cdots Q_r^{k_r},$$

where $Q_j := \exp(2\pi\mathrm{i}\,W_j^{(1)})$ and $N_{\underline{k}}$ is the genus zero (local) G–W invariant "counting rational curves in [the total space of] $\mathsf{K}_{\mathbb{P}_{\Delta^\circ}}$" of homology class $\sum k_j[C_{i_j}^\circ] \in H_2(Y^\circ, \mathbb{Z})$. (See [56, Section 6.1] for a precise definition.) Chiang *et al.* [21] originally obtained (essentially) this expression by writing a compact CY three-fold $\mathfrak{X}$ (with prepotential $\mathcal{F}$) as a torically described elliptic fibration over $\mathbb{P}_{\Delta^\circ}$, and taking the limit of [a suitable partial of] $\mathcal{F}$ under degeneration of the fiber. Morally, the resulting (local) $N_{\underline{k}}$ were supposed to measure the contribution of the zero-section $\mathbb{P}_{\Delta^\circ}$ to G–W invariants of $\mathfrak{X}$.

Here then is the fundamental local mirror symmetry prediction:

Conjecture 5.1 [21, 45]. *For a suitable choice of* φ_1*,*

$$\mathcal{F}_{\mathrm{loc}}(\underline{W}^{(1)}) = w^{(2)}(\underline{\tilde{t}}) \tag{5.5}$$

under the mirror map.

To summarize: the first regulator period yields the mirror map; the second gives the prepotential.

We will now pull (5.5) back to a "diagonal slice" of $\tilde{\mathcal{P}}$ where residual differences between the $w_k^{(1)}$ vanish. Write

$$\phi := \sum_{j=1}^{r+2}\alpha_j \underline{x}^{\underline{\nu}^{(j)}}\ , \quad F_{\phi,t}(\underline{x}) := 1 - t\phi(\underline{x}); \tag{5.6}$$

this gives $a_0 = 1$, $a_j = t\alpha_j$,

$$t_k = (-1)^{\ell_0^{(i_k)}}\left(\prod_{j=1}^{r+2}\alpha_j^{\ell_j^{(i_k)}}\right)t^{|\ell_0^{(i_k)}|}.$$

If we further set

$$\alpha_j := (-1)^{\sigma_j+\sigma_{j-1}+1}, \tag{5.7}$$

then $t_k(t) = (-1)^{\epsilon_k} t^{|\ell_0^{(i_k)}|}$, and the "slice" is given by $\tilde{t}_k(t) := \zeta_k t^{d_{i_k}}$ ($\zeta_k =$ some root of unity with μ_kth power $(-1)^{\epsilon_k}$; the choice will not affect calculations). The pullback of $W_k^{(1)}(\underline{\tilde{t}})$ under $t \mapsto \underline{\tilde{t}}(t)$ is then simply

$$W_k^{(1)}(t) = \frac{d_{i_k}}{2\pi \mathrm{i}} \{\log t \,+\, H(t)\} =: d_{i_k} w^{(1)}(t),$$

where $H(t)$ $(:= H(\underline{s}(t)))$ can frequently be easier to determine than $H(\underline{s})$.

So the map of families $\{F_{\phi,t}(\underline{x}) = 0\} \to \{F_{\underline{a}}(\underline{x}) = 0\}/(\mathbb{C}^*)^3$ induces a "diagonal" embedding $\mathfrak{D} : w^{(1)} \mapsto (d_{i_1} w^{(1)}, \ldots, d_{i_r} w^{(1)})$ of Kähler moduli. Clearly $\mathfrak{D}^* \circ \partial_S = \partial_{w^{(1)}} \circ \mathfrak{D}^*$, and by (5.5)

$$\mathfrak{D}^* \mathcal{F}_{\mathrm{loc}}(\underline{W}^{(1)}) = w^{(2)}(t(w^{(1)})),$$

it follows that

$$\mathfrak{D}^* \partial_S^2 \mathcal{F}_{\mathrm{loc}}(\underline{W}^{(1)}) = \left(\frac{d}{dw^{(1)}}\right)^2 w^{(2)}(t(w^{(1)})). \tag{5.8}$$

For the l.h.s. of (5.8),

$$\begin{aligned} \partial_S^2 \mathcal{F}_{\mathrm{loc}} &= \sum_{j,\ell} d_{i_j} d_{i_\ell} \left\langle J_j^\circ|_{\mathbb{P}_{\Delta^\circ}}, J_\ell^\circ \right\rangle_{Y^\circ} \\ &\quad - (2\pi \mathrm{i})^2 \sum_{k_1,\ldots,k_r} \left(\sum_{j=1}^r d_{i_j} k_j\right)^3 N_{k_1,\ldots,k_r} Q_1^{k_1} \cdots Q_r^{k_r} \\ &= \langle -\mathsf{K}_{\mathbb{P}_{\Delta^\circ}}, -\mathsf{K}_{\mathbb{P}_{\Delta^\circ}} \rangle_{\mathbb{P}_{\Delta^\circ}} - (2\pi \mathrm{i})^2 \sum_{D \geq 1} D^3 \sum_{\{\underline{k} \,|\, \sum d_{i_j} k_j = D\}} N_{\underline{k}} \underline{Q}^{\underline{k}}. \end{aligned}$$

Thinking of $\underline{k}$ as the homology class $\sum k_j [C_{i_j}^\circ] \in H_2(Y^\circ) = H_2(\mathbb{P}_{\Delta^\circ})$, we have $\langle \underline{k}, X^\circ \rangle_{\mathbb{P}_{\Delta^\circ}} = \sum k_j d_{i_j}$; hence applying $\mathfrak{D}^*$ yields

$$\sum_{i=1}^{r+2} d_i \;-\; (2\pi \mathrm{i})^2 \sum_{D \geq 1} D^3 \left(\sum_{\{\underline{k} \,|\, \langle \underline{k}, X^\circ \rangle = D\}} N_{\underline{k}} \right) Q^D,$$

where $Q = \exp(2\pi \mathrm{i}\, w^{(1)})$. Note that the constant term just records the number of components $\mathcal{N}_0$ of the singular fiber of the diagonal family at $t = 0$

(after a minimal desingularization of the total space). We also rechristen the sum in parentheses $N_D^{\langle X^\circ\rangle}$. It would be very interesting to have an interpretation of these numbers in terms of X° alone, since the mirror map is defined only in terms of X (not Y). To venture out on a limb, can one suitably define a class in K_2 of (the nerve of) the Fukaya category (of X°), which completes X° to a datum "mirror" to the family $\{X_t\}$ together with $\{\xi_t \in K_2(X_t)\}$? Is there then a "regulator" of this class which pairs with $\mathcal{O}_{D_0}|_{X^\circ}$ (recall $M(\varphi_1)$'s conjectural mirror is $\mathcal{O}_{D_0}$) to yield the prepotential $\mathcal{F}_{\mathrm{loc}}$?

For the r.h.s. of (5.8), write $\pi^{(1)}$ and $\pi^{(2)}$ for the $\frac{\text{periods}}{(2\pi\mathrm{i})^2}$ of $\omega_t := \mathrm{Res}_{X_{\phi,t}}\left(\frac{\bigwedge d\log \underline{x}}{F_{\phi,t}}\right)$; then $\delta_t w^{(\ell)}(t) = \pi^{(\ell)}(t)$ $(\ell = 1, 2)$. So we have

$$\frac{d}{dw^{(1)}} w^{(2)} = \frac{\delta_t w^{(2)}}{\delta_t w^{(1)}} = \frac{\pi^{(2)}}{\pi^{(1)}},$$

and applying one more $\frac{d}{dw^{(1)}}$ yields

$$\frac{\delta_t\left(\frac{\pi^{(2)}}{\pi^{(1)}}\right)}{\delta_t w^{(1)}} = \frac{\pi^{(1)}\delta_t\pi^{(2)} - \pi^{(2)}\delta_t\pi^{(1)}}{(\pi^{(1)})^3}.$$

Writing this in terms of functions from Sections 4.1, 4.3 for the diagonal family $\tilde{X}_{\phi,t}$ (and dividing l.h.s. and r.h.s. by $(2\pi\mathrm{i})^2$), we have the following equality of a G–W generating function and Yukawa coupling:

$$\frac{\mathcal{N}_0}{(2\pi\mathrm{i})^2} - \sum_{D\geq 1} D^3 N_D^{\langle X^\circ\rangle} Q^D = \frac{\mathcal{Y}(t)}{(A(t))^3} \tag{5.9}$$

under the mirror map. The latter is just the local analytic isomorphism $t \mapsto Q(t)$ $[Q(0) = 0]$, extending at least to $\overline{D_{|t_0|}}$. (Recall $\tilde{X}_{\phi,t_0}$ is the singular fiber nearest $t = 0$ in the punctured diagonal family.) The r.h.s. of (5.9) blows up at t_0 since $\mathcal{Y}(t) \sim \frac{1}{t-t_0}$ and $A(t) \sim \log(t - t_0)$ (up to constants) for $t \to t_0$. Hence the l.h.s. series has radius of convergence $|Q(t_0)| = \exp\{\Re(2\pi\mathrm{i}\, w^{(1)}(t_0))\} = \exp\{\frac{1}{2\pi}\Im(\Psi(t_0))\}$ where $\Psi(t) = (2\pi\mathrm{i})^2 w^{(1)}(t)$. If there is more than one t_0 of minimal modulus, one should of course pick the one that minimizes $|Q(t_0)|$; but in every case we have tested, symmetry ensures that this is independent of the choice.

Theorem 5.2. *Let Δ be a reflexive polytope $\subseteq \mathbb{R}^2$ such that the Mori cone of $Y^\circ := \mathsf{K}_{\mathbb{P}_{\Delta^\circ}}$ is smooth, determine $\phi(\underline{x})$ by (5.6), (5.7), and let $\Psi(t)$ and*

$|t_0|$ be as in Corollary 4.3. Assume Conjecture 5.1. Then the local Gromov–Witten invariants of Y° have exponential growth-rate

$$\limsup_{D\to\infty} |N_D^{\langle X^\circ\rangle}|^{\frac{1}{D}} = \mathrm{e}^{\frac{-1}{2\pi}\Im(\Psi(t_0))}. \tag{5.10}$$

Remark 5.6. (i) In Section 6 we will describe a procedure for computing the "regulator period" $\Psi(t_0)$ on a singular elliptic fiber of Kodaira type I_n. This identifies with the image of an indecomposable K_3 class under the composition

$$K_3^{\mathrm{ind}}(\bar{\mathbb{Q}}) \cong H^2_{\mathcal{M},\mathrm{hom}}(\tilde{X}_{t_0}/\bar{\mathbb{Q}}\,,\, \mathbb{Q}(2)) \xrightarrow{AJ^{2,2}} H^1(\tilde{X}_{t_0}, \mathbb{C}/\mathbb{Q}(2)) \cong \mathbb{C}/\mathbb{Q}(2),$$

which (after taking the imaginary part) coincides (up to a factor of 2) with the Borel regulator. This explains the occurrence of Dirichlet L-functions in results of [59] related to (5.10). (We will be more precise about the field of definition in Section 6.)

(ii) Equation (5.9) gives, for $t = 0$, the correct value $\mathcal{Y}(0) = 2\pi\mathrm{i}\,\mathcal{N}_0$.

Finally, we want to explain how "reasonable" assumptions on the $\{N_D^{\langle X^\circ\rangle}\}$ lead to a more precise characterization of their growth. (The argument is similar to that in [20] but more rigorous.) Let $d := \gcd\{d_i \,|\, i = 1, \ldots, r+2\}$, put $\tilde{\Psi}(t) = d \cdot \{\Psi(t) - \Re(\Psi(t_0))\}$, and define "normalized" quantities

$$\tilde{N}_D := -d^3 N_{d\cdot D}^{\langle X^\circ\rangle} \mathrm{e}^{-\mathrm{i}\frac{d\cdot D}{2\pi}\Re(\Psi(t_0))}, \quad \tilde{Q} := \exp\left\{\frac{-\mathrm{i}}{2\pi}\tilde{\Psi}(t)\right\}.$$

Reindexing, (5.9) becomes

$$-\frac{\mathcal{N}_0}{4\pi^2} + \sum_{D\geq 1} D^3 \tilde{N}_D \tilde{Q}^D = \frac{\mathcal{Y}(t)}{A^3(t)},$$

and we assume

(a) the $\tilde{N}_D$ are uniformly positive (or negative) for sufficiently large D. Next define n_D (> 0) by

$$\tilde{N}_D = \pm \mathrm{e}^{\frac{-D}{2\pi\mathrm{i}}\tilde{\Psi}(t_0)} D^{-3} n_D,$$

and assume that

(b) $\lim_{D\to\infty} n_D \log^2 D$ exists (in the extended reals $\mathbb{R}^{\geq 0} \cup \{\infty\}$), i.e., that the $\tilde{N}_D$ "do not oscillate too much" in the limit.

Now asymptotically as $t \to t_0$ (keeping $t - t_0 \in \mathbb{R}$ and $|t| < |t_0|$), $\pi^{(1)} \sim -m\pi^{(2)}(t_0)\frac{\log|t-t_0|}{2\pi \mathrm{i}}$ (where $m \in \mathbb{Z}^+$ is essentially the number of components of X_{t_0}); logarithmically integrating this, we have $x := \frac{1}{2\pi \mathrm{i}}(\tilde{\Psi}(t_0) - \tilde{\Psi}(t)) = \frac{d}{2\pi \mathrm{i}}(\Psi(t_0) - \Psi(t)) \sim d \cdot m \cdot \pi^{(2)}(t_0)\left(\frac{t}{t_0} - 1\right)$ $\log|t - t_0|$. This implies the r.h.s. of

(c) $\pm\sum_{D\geq 1} n_D \mathrm{e}^{-Dx} = \frac{\mathcal{Y}(t)}{A^3(t)} + \frac{\mathcal{N}_0}{4\pi^2}$ is asymptotic to $\frac{d}{mx\log^2(t-t_0)} \sim \frac{d}{mx\log^2 x}$, where we can replace $t \to t_0$ by $x \to 0^+$.

We need a result from Laplace Tauberian theory.

Lemma 5.4. *Given a sequence $\{n_k\}$ of real numbers satisfying*

(a′) *n_k positive (or at least $n_k \geq -\frac{C}{\log^2 k}$ for some $C > 0$),*

(b′) *$\lim_{n\to\infty} n_k \log^2 k$ exists (finite or infinite),*

(c′) *$\sum_{k=0}^{\infty} n_k \mathrm{e}^{-kx} \sim \frac{1}{x\log^2 x}$ as $x \to 0^+$.*

(Here (a′) is the "Tauberian" hypothesis.) Then $n_k \sim \frac{1}{\log^2 k}$ as $k \to \infty$. That is, $n_k \log^2 k \to 1$.

Proof. For $m_k := \begin{cases} 1, & k = 0 \\ 0, & k = 1 \\ \frac{1}{\log^2 k}, & k \geq 2 \end{cases}$, it is an exercise in elementary analysis to prove $\sum_{k=0}^{\infty} m_k \mathrm{e}^{-kx} \sim \frac{1}{x\log^2 x}$ $(x \to 0^+)$, e.g., in the form $\lim_{y\to\infty} \sum_{k=2}^{\infty} \frac{1}{y}\left(\frac{\log^2 y}{\log^2 k} - 1\right)\mathrm{e}^{-\frac{k}{y}} = 0$. Now let $N(k)$, $M(k)$ be the respective kth partial sums of n_k, m_k, viewed as functions on $\mathbb{R}^{\geq 0}$. Hypothesis (c′) obviously implies $\int_0^{\infty} \mathrm{e}^{-kx} dN(k) \sim \int_0^{\infty} \mathrm{e}^{-kx} dM(k)$ (for $x \to 0^+$) and then (using (a′)) [31] gives $N(k) \sim M(k)$ for $k \to \infty$. Hypothesis (b′) says $\lim_{k\to\infty} \frac{n_k}{m_k}$ exists (finite or $+\infty$), in which case it must equal $\lim_{k\to\infty} \frac{N(k)}{M(k)}$, which is 1. □

In our situation this yields $n_D \sim \frac{d}{m\log^2 D}$, hence the following result:

Corollary 5.3. *Under assumptions (a) and (b) above (and the conditions of Theorem 5.2), the "normalized" G–W invariants have asymptotic behavior*

$$\tilde{N}_D \sim \pm\frac{d}{m}\frac{\exp\left\{-\frac{D\cdot\tilde{\Psi}(t_0)}{2\pi \mathrm{i}}\right\}}{D^3\log^2 D}$$

for $D \to \infty$.

Remark. It seems likely that one could use a Fourier Tauberian argument to eliminate the assumptions.

6. First examples: limits of regulator periods

A well-traveled road in dealing with computations for one-parameter families of varieties is to attempt to recognize "modularity" in some suitable sense. For example, this approach was employed in [32,33] to describe mirror maps and Picard–Fuchs equations for families of CYs. Here (in Section 10) we use it, for the families (and higher cycles) produced by Theorem 3.1, to compute the cycle class, higher normal function, and regulator periods — especially their limiting values at cusps. The central purpose of this section, in contrast, is to illustrate a procedure inspired by Bloch [12] for computing these "special values" of $\Psi(t)$ (at singular fibers), that does not rely on modularity. This leads to a formula (Proposition 6.3) for essentially the $\tilde{\Psi}(t_0)$ of Theorem 5.2/Corollary 5.3, which we apply to some key examples in Section 6.3. Throughout this section $\tilde{\mathcal{X}}_{-}$ is as in Theorem 3.1 (so that Ξ and Ψ have the established meaning).

6.1. AJ map for singular fibers

Fixing $\alpha \in \mathcal{L}^*$, write $\tilde{X}_\alpha =: Y = \cup Y_i$ with Y_i irreducible, $\tilde{\varphi}_\alpha = \sum \varphi_i$ for $\varphi_i \in C^{\text{top}}_{n-1}(Y_i)$; we do not require that $\tilde{\pi}^{-1}(\alpha) = \sum m_i Y_i$ to be reduced, here or in the $Y = \text{NCD}$ case. Assume further that $\Xi \in Z^n_{\partial_{\mathcal{B}}-\text{cl.}}(\tilde{\mathcal{X}}_{-}, n)_Y$ so that the $\Xi_i := \Xi \cdot Y$ are defined. Our first goal is to verify the claim from Section 4.2 (cf. the discussion leading up to Corollary 4.3) that

$$AJ(\Xi_\alpha)(\tilde{\varphi}_\alpha) = \int_{\tilde{\varphi}_\alpha} R_\Xi = \sum_i \int_{\varphi_i} R_{\Xi_i}, \tag{6.1}$$

to this end we review briefly the computation of $AJ(\Xi_\alpha)$ from Section 8 of [49]. The (somewhat technical) general conditions under which it (hence (6.1)) is valid are described in [49] following Proposition 8.17, and allow for all singular curves, as well as any local-normal-crossing or nodal singularities.

Here we shall focus on the case $Y = \text{NCD}$, writing $Y_I := \cap_{i \in I} Y_i$, $Y^{[j]} := \amalg_{|I|=j+1} Y_I$, and Y^I for the collection $\{Y_J \cap Y_I\}_{J \cap I = \emptyset}$ of subsets of Y_I. This "hyper-resolution" of Y gives rise to fourth quadrant double complexes

$$\begin{array}{rl|c} Z_Y^{\ell,m}(n) := Z^n(Y^{[\ell]},-m)_{\#} & & C^Y_{\ell,m}(n) := C^{\text{top}}_{2n+m-1}(Y^{[\ell]};\mathbb{Q}), \\ := \oplus_{|I|=\ell+1} Z^n_{\mathbb{R}}(Y_I,-m)_{Y^I}, & & \text{(piecewise } C^\infty \text{ chains)} \\ \partial_{\mathcal{B}} : Z_Y^{\ell,m}(n) \to Z_Y^{\ell,m+1}(n), & & \partial_{\text{top}} : C^Y_{\ell,m}(n) \to C^Y_{\ell,m-1}(n), \\ \mathfrak{I} : Z_Y^{\ell,m}(n) \to Z_Y^{\ell+1,m}(n), & & Gy : C^Y_{\ell,m}(n) \to C^Y_{\ell-1,m}(n), \end{array}$$

where $\mathfrak{I}$ (resp. Gy) is the alternating sum (cf. [49] for signs) of pullbacks (resp. pushforwards). These have associated simple complexes/total differentials/(co)homology

$$\begin{array}{c|c} Z_Y^{\bullet}(n) := \mathbf{s}^{\bullet} Z_Y^{\bullet,\bullet}(n), & C^Y_{\bullet}(n) := \mathbf{s}_{\bullet} C^Y_{\bullet,\bullet}(n), \\ \underline{\underline{\partial_{\mathcal{B}}}} := \partial_{\mathcal{B}} \pm \mathfrak{I}, & \underline{\underline{\partial_{\text{top}}}} := \partial_{\text{top}} \pm Gy, \\ H^*(\overline{Z_Y^{\bullet}(n)}) \cong H^{2n+*}_{\mathcal{M}}(Y,\mathbb{Q}(n)), & H_*(\overline{C^Y_{\bullet}(n)}) \cong H_{2n+*-1}(Y). \end{array}$$

The KLM currents ($\mathfrak{Z} \mapsto T_{\mathfrak{Z}}, \Omega_{\mathfrak{Z}}, R_{\mathfrak{Z}}$) give a map of complexes (described in full in [49]) inducing an Abel–Jacobi map from $H^{2n+*}_{\mathcal{M}}(Y,\mathbb{Q}(n))$ to

$$\begin{aligned} H^{2n+*}_{\mathcal{D}}(Y,\mathbb{Q}(n)) &\overset{*<0}{\cong} \operatorname{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), H^{2n+*-1}(Y,\mathbb{Q}(n))) \\ &\overset{*\leq -n}{\cong} H^{2n+*-1}(Y,\mathbb{C}/\mathbb{Q}(n)). \end{aligned}$$

For $* = -n$ in particular, this is

$$AJ_Y^{n,n} : H^n_{\mathcal{M}}(Y,\mathbb{Q}(n)) \to \operatorname{Hom}(H_{n-1}(Y,\mathbb{Q}),\mathbb{C}/\mathbb{Q}(n)). \tag{6.2}$$

To compute this for $\dim(Y) = n-1$, let

$$\begin{aligned} \mathfrak{Z} &= \sum_{\ell} \{\mathfrak{Z}^{[\ell]} \in Z^n_{\mathbb{R}}(Y^{[\ell]}, n+\ell)\} \in \{\ker(\underline{\underline{\partial_{\mathcal{B}}}}) \subset Z_Y^{-n}(n)\}, \\ \gamma &= \sum_{\ell} \{\gamma^{[\ell]} \in C^{\text{top}}_{n-\ell-1}(Y^{[\ell]};\mathbb{Q})\} \in \{\ker(\underline{\underline{\partial_{\text{top}}}}) \subset C^Y_{-n}(n)\}, \end{aligned} \tag{6.3}$$

with each $\gamma^{[\ell]}$ (resp. $\mathfrak{Z}^{[\ell]}$) decomposing into $\{\gamma_I\}_{|I|=\ell+1}$ (resp. $\{\mathfrak{Z}_I\}_{|I|=\ell+1}$). Then

$$AJ_Y^{n,n}(\mathfrak{Z})(\gamma) \equiv \sum_{\ell \geq 0} \int_{\gamma^{[\ell]}} R_{\mathfrak{Z}^{[\ell]}} = \sum_{\ell \geq 0} \sum_{|I|=\ell+1} \int_{\gamma_I} R_{\mathfrak{Z}_I} \tag{6.4}$$

gives a well-defined pairing $H^{-n}(Z_Y^{\bullet}(n)) \times H_{-n}(C^Y_{\bullet}(n)) \to \mathbb{C}/\mathbb{Q}(n)$. Now consider the map

$$I_Y^* : Z^n_{\mathbb{R},\partial_{\mathcal{B}}-\text{cl.}}(\tilde{\mathcal{X}}_-, n)_Y \to \{\ker(\underline{\underline{\partial_{\mathcal{B}}}}) \subset Z_Y^{-n}(n)\}$$

given by restricting to the irreducible components of Y. That is, if $\mathfrak{Z} = I_Y^* \Xi$ then $\mathfrak{Z}^{[0]}$ is the collection $\{\iota_{Y_i}^* \Xi\}$ while $\mathfrak{Z}^{[\ell]} = 0$ for $\ell > 0$. Let γ be the $\underline{\underline{\partial_{\text{top}}}}$-cycle corresponding to $\tilde{\varphi}_\alpha$: i.e., $\gamma^{[0]} = \{\varphi_i\}$, while the $\gamma^{[\ell]}(\neq 0)$ comprise iterated boundaries of the φ_i. Then

$$AJ(\Xi_\alpha)(\tilde{\varphi}_\alpha) = AJ(\mathfrak{Z})(\gamma) \overset{(6.4)}{=} \sum_i \int_{\varphi_i} R_{\iota_{Y_i}^* \Xi = \Xi_i}$$

confirms (6.1).

Continuing to assume Y a (connected) NCD of dimension $n-1$, we want to say something about the *value* of (6.1) in $\mathbb{C}/\mathbb{Q}(n)$. Place the "weight" filtration

$$W_\beta H_{\mathcal{M}}^{2n+*}(Y, \mathbb{Q}(n)) := \operatorname{im}\{H^*(\mathbf{s}^\bullet Z_Y^{(\bullet \geq -n-\beta),\bullet}(n)) \to H^*(Z_Y^\bullet(n))\}$$

on motivic cohomology, and note that $W_{-2n+1} H_{\mathcal{M}}^n(Y, \mathbb{Q}(n))$ consists of those classes representable by $\underline{\underline{\partial_{\mathcal{B}}}}$-cocycles supported on points $p_I := Y_I$, $|I| = n$. (For simplicity we assume these are each one point.) This is compatible with the weight filtration on the generalized Jacobians in the sense that $AJ_Y^{n,r}$ is "filtered" by maps

$$W_\bullet H_{\mathcal{M}}^{2n-r}(Y, \mathbb{Q}(n)) \xrightarrow{W_\bullet AJ_Y^{n,r}} \operatorname{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), W_{\bullet-1} H^{2n-r-1}(Y, \mathbb{Q}(n))).$$

In particular the target of $W_{-2n+1} AJ_Y^{n,n}$ is $\operatorname{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), \mathbb{Q}(n)^{\oplus b_Y}) \cong (\mathbb{C}/\mathbb{Q}(n))^{\oplus b_Y}$, where $b_Y := \operatorname{rk}\{\operatorname{coker}(H^0(Y^{[n-2]}) \to H^0(Y^{[n-1]}))\}$. For $Y = \tilde{X}_\alpha$ a degenerate CY, $b_Y = 0$ or 1: $b_Y = 1$ implies maximal quasi-unipotent monodromy about α; and in the unipotent case, maximal monodromy $\implies$ $b_Y = 1$.

We need to be more precise about the field of definition: recall that $\tilde{\mathcal{X}}_-$ is defined over a number field K; it may be that $\alpha \notin K$, and that to "separate components" of Y requires an algebraic field extension larger than $K(\alpha)$.

Definition 6.1. $\mathsf{L}/K(\alpha)$ is a *splitting field* for the NCD Y iff all the components Y_I of the hyper-resolution are defined over L. Furthermore, Y is *simple* iff all Y_I are rational.

With such a choice of L, and assuming $b_Y = 1$, we have

$$(6.5) \quad W_{-2n+1} H_{\mathcal{M}}^n(Y/\mathsf{L}, \mathbb{Q}(n)) \cong CH^n(\operatorname{Spec}(\mathsf{L}), 2n-1) \cong K_{2n-1}^{\text{alg}}(\mathsf{L}) \otimes \mathbb{Q}.$$

Let $\gamma, \mathfrak{Z}$ be as in (6.3) with $\gamma^{[n-1]} = \{q_I[p_I]\}_{|I|=n}$ ($q_I \in \mathbb{Q}$) and $[\mathfrak{Z}] \in (6.5)$. Then $\mathfrak{Z} \equiv \{\mathfrak{W}_I\}_{|I|=n}$ modulo $\underline{\underline{\partial_{\mathcal{B}}}}$-coboundary, and

$$AJ_Y^{n,n}(\mathfrak{Z})(\gamma) = AJ^{2n-1,n}_{\mathrm{Spec}(\mathsf{L})}\left(\sum \pm q_I \mathfrak{W}_I\right) \in \mathbb{C}/\mathbb{Q}(n), \tag{6.6}$$

where in light of (6.5) $AJ^{2n-1,n}_{\mathrm{Spec}(\mathsf{L})}$ should be thought of essentially as the Borel regulator. The key result, which the computations below will reflect (but not use), is

Proposition 6.1. *Let $n = 2$ or 3, $Y = \tilde{X}_\alpha$ be a simple NCD with abelian splitting field extension $\mathsf{L}/\mathbb{Q}$, and if $n = 3$ assume L totally real. Then $H^n_{\mathcal{M}}(Y/\mathsf{L}, \mathbb{Q}(n)) = W_{-2n+1}H^n_{\mathcal{M}}(Y/\mathsf{L}, \mathbb{Q}(n))$, and $\Psi(\alpha)$ is a sum of Dirichlet L-series $L(\chi, n)$ with algebraic coefficients.*

Remark. For L nonabelian one might hope to relate the collection of values of Ψ at (some) points of $\mathcal{L}^*$ to Artin L-series corresponding to a representation of $\mathrm{Gal}(\mathsf{L}/\mathbb{Q})$.

Proof. In order to "move" an arbitrary $\underline{\underline{\partial_{\mathcal{B}}}}$-cocycle (in $Z_Y^{-n}(n)$) into $Z_Y^{n-1,-2n+1}(n)$, we need only know that (for $n = 2$) $CH^2(Y_i, 2) = \{0\}$ ($\forall i$) and (for $n = 3$) $CH^3(Y_i, 3)$ and $CH^3(Y_{ij}, 4)$ are 0 ($\forall i, j$). This follows from vanishing of $CH^p(\mathbb{P}^1_{\mathsf{L}}, n) \cong_{n.c.} CH^p(\mathsf{L}, n) \oplus CH^{p-1}(\mathsf{L}, n)$ and (for $S := Bl_{\{p_1,\ldots,p_N\}}(\mathbb{P}^2)$)

$$CH^p(S_{\mathsf{L}}, n) \cong CH^p(\mathsf{L}, n) \oplus CH^{p-1}(\mathsf{L}, n)^{\oplus(N+1)} \oplus CH^{p-2}(\mathsf{L}, n).$$

Now since Ξ is (like $\mathcal{X}$) defined over K, its pullback to (the components of) Y is defined over L. The last statement (of the proposition) then follows from Beilinson's fundamental result [7, 62] on higher regulators of a cyclotomic field ($\supset \mathsf{L}$), together with (6.5) and (6.6). □

For actually computing (6.1) we shall take a different approach, for which one may drop the assumption that Y is a NCD. Using the fact that Ξ and ξ differ by a $\partial_{\mathcal{B}}$-coboundary on $\tilde{\mathcal{X}}^*_-$, $\int_{\tilde{\varphi}_t} R_\Xi \equiv \int_{\tilde{\varphi}_t} R_\xi$ (mod $\mathbb{Q}(n)$) provided $\tilde{\varphi}_t$ does not meet $\tilde{D}$. For $t = \alpha$ this yields

$$\Psi(\alpha) \overset{\mathbb{Q}(n)}{\equiv} \sum_i \int_{\varphi_i} R\{x_1|_{Y_i}, \ldots, x_n|_{Y_i}\}. \tag{6.7}$$

In the event that $(t=)\alpha=t_0$ (at the boundary of convergence of (4.5)), using Corollary 4.3 gives

$$\log(t_0)+\sum_{k\geq 1}\frac{[\phi^k]_0}{k}t^k \overset{\mathbb{Q}(1)}{\equiv} \frac{1}{(2\pi\mathrm{i})^{n-1}}\sum_i\int_{\varphi_i}R\{\underline{x}\}|_{Y_i}, \tag{6.8}$$

in particular, if $t_0\in\mathbb{R}^+$ and $K\subset\mathbb{R}$ then the l.h.s. $=\Re$(r.h.s.).

These formulas are of greatest practical use — i.e., the r.h.s. of (6.7) and (6.8) is directly computable — when the $\{Y_I\}$ are rational (and explicitly parametrized). This is automatic for $n=2$, but unfortunately (at least for (6.8)) doesn't tend to occur at t_0 for $n=3$ — in all the examples we have analyzed (see e.g., Sections 6.4 and 10.5), the $K3$ acquires a node there.

We conclude with a general result which best captures the sense in which "singular" $AJ_{\tilde{X}_\alpha}(\Xi_\alpha)$ is a *limit* of "smooth" $\{AJ_{\tilde{X}_t}(\Xi_t)\}$. Let $\mathcal{X}\overset{\pi}{\to}\mathcal{S}$ be a proper, dominant morphism of smooth varieties with $\dim(\mathcal{S})=1$ and unique singular fiber X_0; since $\mathcal{S}$ is not required to be complete, this can be arranged by omitting other singular fibers. Assume X_0 is a *reduced* NCD so that the local degeneration (over a disk with coordinate s)

$$\begin{array}{ccccccc}
\mathcal{X}^*_\Delta & \hookrightarrow & \mathcal{X}_\Delta & \overset{\iota_{X_0}}{\hookleftarrow} & X_0 & = & \cup Y_i \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \bar{f}} & & \downarrow & & \\
\Delta^* & \overset{\jmath}{\hookrightarrow} & \Delta & \hookleftarrow & \{0\} & &
\end{array}$$

is semistable; and let $\Xi^*\in CH^p(\mathcal{X}\backslash X_0,r)$. Define the local system $\mathbb{H}_{\mathbb{Q}}:=R^{2p-r-1}f_*\mathbb{Q}(p)$, cohomology sheaves $\mathcal{H}:=R^{2p-r-1}f_*\mathbb{C}\otimes\mathcal{O}_{\Delta^*}$ with holomorphic Hodge subsheaves $\mathcal{F}^m$, and Jacobian sheaf (via the s.e.s.)

$$\mathbb{H}_{\mathbb{Q}}\hookrightarrow\frac{\mathcal{H}}{\mathcal{F}^p}\twoheadrightarrow\mathcal{J}^{p,r}. \tag{6.9}$$

Then Ξ^* gives rise to the higher normal function

$$\nu_{\Xi^*}(s):=AJ_{X_s}(\Xi_s)\in\Gamma(\Delta^*,\mathcal{J}^{p,r}),$$

where $\Xi_s:=\iota^*_{X_s}(\Xi^*)$. Writing $T\in\mathrm{Aut}(\mathbb{H}_{\mathbb{Q}})$ for the (unipotent) monodromy operator (with $N:=\log T$), consider the Clemens–Schmid exact sequence of MHS

$$\cdots\to H^{2p-r-1}(X_0)\overset{\rho}{\to}H^{2p-r-1}_{\lim}(X_s)\overset{N}{\to}H^{2p-r-1}_{\lim}(X_s)(-1)\to\cdots$$

and the canonically extended sheaves $\mathcal{H}_e$, $\mathcal{F}_e^p$, and

$$\jmath_* \mathbb{H}_{\mathbb{Q}} \hookrightarrow \frac{\mathcal{H}_e}{\mathcal{F}_e^p} \twoheadrightarrow \mathcal{J}_e^{p,r} \tag{6.10}$$

over Δ. Set

$$J^{p,r}_{\lim}(X_s) := \frac{\mathcal{H}_{e,0}}{(\jmath_*\mathbb{H}_{\mathbb{Q}})_0 + \mathcal{F}^p_{e,0}} \cong \mathrm{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), H^{2p-r-1}_{\lim}(X_s, \mathbb{Q}(p)))$$

and $J^{p,r}(X_0) := \mathrm{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), H^{2p-r-1}(X_0, \mathbb{Q}(p)))$, where $(\jmath_*\mathbb{H}_{\mathbb{Q}})_0$ is the stalk of the local system at 0 (i.e., invariant cycles), while $\mathcal{H}_{e,0}$ and $\mathcal{F}^p_{e,0}$ are the fibers (over 0) of the corresponding holomorphic vector bundles. Then ρ induces

$$J(\rho) : \; J^{p,r}(X_0) \to J^{p,r}_{\lim}(X_s).$$

Note that any section $\nu \in \Gamma(\Delta, \mathcal{J}_e^{p,r})$ has a well-defined "value" $\nu(0) \in J^{p,r}_{\lim}(X_s)$.

Proposition 6.2. *Suppose* $\mathrm{Res}_{X_0}(\Xi^*) \in CH^{p-1}(X_0, r-1)\,(\cong H^{2p-r+1}_{\mathcal{M},X_0}(\mathcal{X}, \mathbb{Q}(p)))$ *is zero. Then* ν_{Ξ^*} *lifts uniquely to a section* $\nu \in \Gamma(\Delta, \mathcal{J}_e^{p,r})$*, and we define* $\lim_{s\to 0} \nu_{\Xi^*}(s) := \nu(0) \in J^{p,r}_{\lim}(X_s)$*. Furthermore, if* $\Xi \in CH^p(\mathcal{X}, r)$ *restricts to* Ξ^* *then*

$$\lim_{s\to 0} \nu_{\Xi^*}(s) = J(\rho)(AJ_{X_0}(\iota^*_{X_0}\Xi)).$$

Proof (Sketch) The existence of Ξ follows from Bloch's moving lemma [11], and we can put it into good position relative to X_0. Since

$$\iota^*_{X_0}(\mathrm{cl}(\Xi)) \in \mathrm{Hom}_{\text{MHS}}(\mathbb{Q}(0), H^{2p-r}(X_0, \mathbb{Q}(p))) = \{0\},$$

and X_0 is a deformation retract of $\mathcal{X}_\Delta$, the restriction of $\mathrm{cl}(\Xi) = [\Omega_\Xi] = (2\pi\mathrm{i})^p[T_\Xi]$ to $\mathcal{X}_\Delta$ (hence to $\mathcal{X}^*_\Delta$) is trivial.[18] So the image of ν_{Ξ^*} in $H^1(\Delta^*, \mathbb{H}_{\mathbb{Q}})$ vanishes, and its lift to $\Gamma(\Delta^*, \frac{\mathcal{H}}{\mathcal{F}^p})$ is actually computed by fiberwise integration of the completed regulator current $R''_{(\Xi|_{\mathcal{X}_\Delta})} := R_\Xi|_{\mathcal{X}_\Delta} - d^{-1}(\Omega_\Xi|_{\mathcal{X}_\Delta}) + (2\pi\mathrm{i})^p \delta_{\partial^{-1}(T_\Xi|_{\mathcal{X}_\Delta})}$ against sections of $\bar{f}_* F^{n-p} A^{2(n-p)+r-1}_{\mathcal{X}/\mathcal{S}}(\log X_0) (n = \dim \mathcal{X})$. As $s \to 0$ these integrals do not blow up, so the lift extends to $\tilde{\nu} \in \gamma(\Delta, \frac{\mathcal{H}_e}{\mathcal{F}_e^p})$; this has image $\nu \in \Gamma(\Delta, \mathcal{J}_e^{p,r})$. (In fact, at $s = 0$ they compute $AJ_{X_0}(\iota^*_{X_0}\Xi)$ by generalizing the argument used to prove (6.1) above.) The uniqueness of ν is a simple argument using the long-exact cohomology sequences of (6.9), (6.10). □

[18] After this step, remaining details are similar to those in [40, Section 3].

6.2. Formula for AJ on a Néron N-gon

Returning to the setting of Theorem 3.1, we will now compute the r.h.s. of (6.7) for Kodaira type I_N degenerations of elliptic curves. Specialize to the case $n=2$, $\tilde{X}_\alpha = Y = \cup_{i=1}^N Y_i$ with each $Y_i \cong \mathbb{P}^1$, $Y_{i_0 i_1}$ nonempty iff $i_0 - i_1 \equiv \pm 1 \bmod N$, and $Y^{[2]} \cap \tilde{D} = \emptyset$. Let $z_i : Y_i \overset{\cong}{\to} \mathbb{P}^1$ be such that $z_i(Y_{i,i-1}) = \infty$, $z_i(Y_{i,i+1}) = 0$, and $\tilde{\varphi}_\alpha = \varepsilon_\alpha \cdot \sum_{i=1}^N T_{z_i}$ (for some $\varepsilon \in \mathbb{Z}$). Then restrictions of toric coordinates $x_1|_{Y_i}, x_2|_{Y_i}$ will be written

$$f_i(z_i) = A_i \prod_j \left(1 - \frac{\alpha_{ij}}{z_i}\right)^{d_{ij}}, \quad g_i(z_i) = B_i \prod_k \left(1 - \frac{z_i}{\beta_{ik}}\right)^{e_{ik}}$$

(with no α_{ij} or β_{ik} 0 or ∞); note that $\sum_j d_{ij} = \sum_k e_{ik} = 0$ $(\forall i)$ and

$$(f_i(0), g_i(0)) = \left(A_i \prod_i \alpha_{ij}^{d_{ij}},\, B_i\right), \quad (f_i(\infty), g_i(\infty)) = \left(A_i,\, B_i \prod_k \beta_{ik}^{-e_{ik}}\right).$$

Since Y is a singular fiber in a family of elliptic curves produced via a *tempered* Laurent polynomial, $\mathrm{Tame}_\xi\{f_i, g_i\}$ is torsion for every $\xi \in |(f_i)| \cup |(g_i)|$. We do *not* require that $|(f_i)| \cap |(g_i)| = \emptyset$, so for sums over both j and k the notation $\sum'_{j,k}$ means to omit terms for which $\alpha_{ij} = \beta_{ik}$. In particular, we set

$$\mathcal{N}_{f_i,g_i} := {\sum_{j,k}}' d_{ij} e_{ik} \left[\frac{\alpha_{ij}}{\beta_{ik}}\right] \in \mathbb{Z}[\mathbb{P}^1 \backslash \{0,\infty\}]$$

and $\mathcal{N}_\alpha := \sum_i \mathcal{N}_{f_i,g_i}$. Another important notational point is that $\log z$ is regarded as a 0-current with branch cut along T_z, so that $\left(\text{with } d\log z := \frac{dz}{z}\right)$ $\delta_{T_z} = \frac{1}{2\pi \mathrm{i}}(d\log z - d[\log z])$; also $d\left[\frac{d\log z}{2\pi\mathrm{i}}\right] = \delta_{\{0\}} - \delta_{\{\infty\}}$. While this approach "keeps track of branches of log," a nasty side effect is that $\log a - \log b \neq \log \frac{a}{b}$; although the discrepancy lies in $\mathbb{Z}(1)$ this becomes significant when multiplied by another function.

Now recalling that

$$R\{f,g\} := \log f\, d\log g - 2\pi\mathrm{i}(\log g)\delta_{T_f},$$

one easily checks that (in $\mathcal{D}^1(Y_i \backslash |(f_i)| \cup |(g_i)|)$)

$$R\{f_i, g_i\} \equiv {\sum_{j,k}}' d_{ij} e_{ik} R\left\{1 - \frac{\alpha_{ij}}{z_i}, 1 - \frac{z_i}{\beta_{ik}}\right\} + R\{f_i, B_i\} + R\{A_i, g_i\},$$

where the equivalence is generated by $d\{$0-currents which are 0 at $z=0,\infty\}$ and $\delta_{\{\mathbb{Z}(2)[\frac{1}{2}]-\text{chains}\}}$. This gives the r.h.s. of (6.7) (for now omitting ε_α)

$$\sum_{i,j,k}{}' d_{ij}e_{ik}\int_{T_{z_i}} R\left\{1-\frac{\alpha_{ij}}{z_i},1-\frac{z_i}{\beta_{ik}}\right\} - 2\pi\mathrm{i}\sum_i \log B_i \int_{T_{z_i}} \delta_{T_{f_i}}$$
$$+\sum_i \log A_i \int_{T_{z_i}} d\log g_i.$$

Rewriting $\int_{T_{z_i}}(\cdot)$ as $\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{P}^1}\left(\frac{dz_i}{z_i}-d[\log z_i]\right)\wedge(\cdot)=\frac{-1}{2\pi\mathrm{i}}\int_{\mathbb{P}^1}(\cdot)\wedge\frac{dz_i}{z_i}+\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{P}^1}(\log z_i)d(\cdot)$ yields

(6.11)
$$\sum_{i,j,k}{}' d_{ij}e_{ik}\left(\int_{T_{1-\frac{\alpha_{ij}}{z_i}}}\log\left(1-\frac{z_i}{\beta_{ik}}\right)\frac{dz_i}{z_i}+\int_{\mathbb{P}^1}\frac{\log z_i}{2\pi\mathrm{i}}d\left[R\left\{1-\frac{\alpha_{ij}}{z_i},1-\frac{z_i}{\beta_{ik}}\right\}\right]\right)$$
$$+\frac{1}{2\pi\mathrm{i}}\sum_i \log B_i\int_{\mathbb{P}^1}\left\{(\log f_i)d\left[\frac{dz_i}{z_i}\right]-(\log z_i)d\left[\frac{df_i}{f_i}\right]\right\}$$
$$+\frac{1}{2\pi\mathrm{i}}\sum_i \log A_i\int_{\mathbb{P}^1}(\log z_i)d\left[\frac{dg_i}{g_i}\right].$$

The directed line segments (for distinct $a,b\in\mathbb{C}^*$)

$$T_{1-\frac{a}{z}}=\mathrm{e}^{\mathrm{i}\arg a}[0,|a|],\quad T_{1-\frac{z}{b}}=\mathrm{e}^{\mathrm{i}\arg b}[-\infty,|b|]$$

in $\mathbb{P}^1$ do not intersect unless $\arg a\equiv\arg b \pmod{2\pi\mathbb{Z}}$ and $|b|<|a|$, in which case a global perturbation as in Section 9 of [47] may be deployed to kill the intersection. Since in general

$$d[R\{f,g\}]=2\pi\mathrm{i}(\log f|_{(g)}-\log g|_{(f)})-(2\pi\mathrm{i})^2\delta_{T_f\cdot T_g},$$

(6.11) becomes ($\Psi(\alpha)\overset{\mathbb{Q}(2)}{\equiv}$)

$$(6.12)\qquad -\sum_{i,j,k}{}' d_{ij}e_{ik}\left\{Li_2\left(\frac{\alpha_{ij}}{\beta_{ik}}\right)+(\log\alpha_{ij}-\log\beta_{ik})\log\left(1-\frac{\alpha_{ij}}{\beta_{ik}}\right)\right\}$$
$$+\sum_i \log g_i(0)(\log f_i(0)-\log f_i(\infty))$$
$$-\sum_i \log B_i\sum_j d_j\log\alpha_{ij}+\sum_i\log A_i\sum_k e_{ik}\log\beta_{ik}.$$

This is the best we can do without further information.

Next, suppose that we know $\Psi(\alpha)$ is pure imaginary (up to $\mathbb{Q}(2)$), or just want its imaginary part. Taking $\Im\{(6.12)\}$ gives

(6.13)

$$\begin{aligned}
&-\sum_{i,j,k}' d_{ij}e_{ik}\left\{\Im Li_2\left(\frac{\alpha_{ij}}{\beta_{ik}}\right)+\log\left|\frac{\alpha_{ij}}{\beta_{ik}}\right|\arg\left(1-\frac{\alpha_{ij}}{\beta_{ik}}\right)\right\}\\
&\quad+\sum_i \log|g_i(0)|(\arg f_i(0)-\arg f_i(\infty))+\sum_i \arg(g_i(0))\log\left|\frac{f_i(0)}{f_i(\infty)}\right|\\
&\quad-\sum_i \arg(g_i(0))\log\left|\frac{f_i(0)}{f_i(\infty)}\right|+\sum_i \arg(f_i(\infty))\log\left|\frac{g_i(0)}{g_i(\infty)}\right|\\
&\quad-\sum_i \log|B_i|\sum_j d_{ij}\arg\alpha_{ij}+\sum_i \log|A_i|\sum_k e_{ik}\arg\beta_{ik}\\
&\quad-\sum_i\sum_j d_{ij}\arg\alpha_{ij}\log\left|\prod_k{}'\left(1-\frac{\alpha_{ij}}{\beta_{ik}}\right)^{e_{ik}}\right|\\
&\quad+\sum_i\sum_k e_{ik}\arg\beta_{ik}\log\left|\prod_j{}'\left(1-\frac{\alpha_{ij}}{\beta_{ik}}\right)^{d_{ij}}\right|,
\end{aligned}$$

where the $\prod_k'$, $\prod_j'$ mean to omit terms which are 0. The last four terms of (6.13) may be rearranged to give

$$\begin{aligned}
&\sum_i\sum_{\xi\in\mathbb{C}^*}\arg(\xi)\log\left|\frac{\left\{A_i\prod_j'\left(1-\frac{\alpha_{ij}}{\xi}\right)^{d_{ij}}\right\}^{\nu_\xi(g_i)}}{\left\{B_i\prod_k'\left(1-\frac{\xi}{\beta_{ik}}\right)^{e_{ik}}\right\}^{\nu_\xi(f_i)}}\right|\\
&\quad=\sum_i\sum_{\xi\in\mathbb{C}^*}\arg(\xi)\log|\mathrm{Tame}_\xi\{f_i,g_i\}|=0.
\end{aligned}$$

The second and third rows of (6.13), after obvious cancelations, yield the collapsing sum

$$\sum_i\{\log|g_i(0)|\arg f_i(0)-\log|g_i(\infty)|\arg f_i(\infty)\}=0.$$

This leaves us with the first row, which is just

$$-\sum_{i,j,k}{}' d_{ij}e_{ik}D_2\left(\frac{\alpha_{ij}}{\beta_{ik}}\right) =: -D_2(\mathcal{N}_\alpha),$$

where $D_2(z) := \Im(Li_2(z)) + \log|z|\arg(1-z)$ is the (real, single-valued) Bloch–Wigner function. Summarizing this discussion and combining with (6.8) gives immediately

Proposition 6.3. *For a family of elliptic curves as in Theorem 3.1 ($n = 2$), with $\tilde{X}_\alpha$ a Néron N-gon (including cases $N = 1, 2$), $\Psi(\alpha) \overset{\mathbb{Q}(2)}{\equiv} \varepsilon_\alpha \cdot$ (6.12) with $\Im(\Psi(\alpha)) = -\varepsilon_\alpha D_2(\mathcal{N}_\alpha)$. In particular if $\alpha = t_0$, and $K(t_0) \subset \mathbb{R}$, we have*

$$\log\left|\frac{1}{t_0}\right| - \sum_{k\geq 1}\frac{[\phi^k]_0}{k}t_0^k = \frac{\varepsilon_\alpha}{2\pi}D_2(\mathcal{N}_{t_0}), \tag{6.14}$$

plus or minus πi if $t_0 < 0$.

If the family $\tilde{\mathcal{X}}_-$ or a $t \mapsto t^\kappa$ quotient thereof has just three singular fibers, then the l.h.s. of (6.12) is a special value of a "hypergeometric integral" or Meijer G-function, and such identities seem to go back essentially to Ramanujan. In addition, the Meijer G-functions studied in [59] for the E_6, E_7, E_8 cases below are nothing but $\frac{1}{2\pi\mathrm{i}}$ times the regulator period $\Psi(t^\kappa)$.

We should emphasize that (6.14) (as derived above) is a *motivic* identity which directly reflects the limit AJ result Proposition 6.2.

6.3. Examples $\mathbf{D_5, E_6, E_7, E_8}$

We turn now to four "mirror pairs" of elliptic curve families with common fundamental periods. The Laurent polynomials ϕ_{I}, ϕ_{II} in the first column of the table below have dual Newton polytopes and are of the type considered in Example 3.1. The corresponding $\tilde{\mathcal{X}}_{\mathrm{I}}, \tilde{\mathcal{X}}_{\mathrm{II}}$ are smooth and the second column lists their Kodaira fiber types over $t = 0$, $t \in \mathcal{L} \cap \mathbb{C}*$, and $t = \infty$ (in that order). These two families share a common degree-κ quotient (over simply $t \mapsto t^\kappa$ for each $\tilde{\mathcal{X}}_{II}$), whose singular fibers (after a minimal desingularization of the total space) are listed next. This is followed by the Dynkin diagram type of the dual graph of the singular fiber over $t^\kappa = \infty$ (in the quotient), which we use to "identify" each example. The vanishing-cycle periods about $t = 0$ (being pullbacks from the quotient families) take the form $A_{\mathrm{I}}(t) = A_{\mathrm{II}}(t) = \sum_{m\geq 0} a_m t^{\kappa m}$, and so $\Psi_{\mathrm{I}}(t) = \Psi_{\mathrm{II}}(t) = 2\pi\mathrm{i}\left(\log t + \sum_{m\geq 1}\frac{a_m}{\kappa m}t^{\kappa m}\right)$.

Finally, if we take $\phi = \phi_{\mathrm{II}}$ in Section 5, then the $\{N_D^{\langle X^\circ \rangle}\}$ are local Gromov–Witten invariants of the Y°_{II} indicated and these will have exponential growth rate $\exp\left(-\Re\left(\frac{\Psi_{\mathrm{II}}(t_0)}{2\pi \mathrm{i}}\right)\right)$ by (5.10).

ϕ_{I} ϕ_{II}	Fibers of $\tilde{\mathcal{X}}$	κ	Fibers of $\widetilde{\tilde{X}/\mathbb{Z}_\kappa}$	Type at ∞	a_m	t_0	Y°_{II}
$\left(x+\frac{1}{x}\right)\left(y+\frac{1}{y}\right)$ $x+\frac{1}{x}+y+\frac{1}{y}$	$I_4, 2I_2, I_4$ $I_8, 2I_1, I_2$	2	I_4, I_1, I_1^*	D_5	$\binom{2m}{m}^2$	$\frac{1}{4}$	$K_{\mathbb{P}^1\times\mathbb{P}^1}$
$\frac{x^2}{y}+\frac{y^2}{x}+\frac{1}{xy}$ $x+y+\frac{1}{xy}$	$I_3, 3I_3, I_0$ $I_9, 3I_1, I_0$	3	I_3, I_1, IV^*	E_6	$\binom{3m}{m,m,m}$	$\frac{1}{3}$	$K_{\mathbb{P}^2}$
$\frac{x}{y}+\frac{y^3}{x}+\frac{1}{xy}$ $x+y+\frac{1}{x^2y}$	$I_4, 4I_2, I_0$ $I_8, 4I_1, I_0$	4	I_2, I_1, III^*	E_7	$\binom{4m}{2m,m,m}$	$\frac{1}{2\sqrt{2}}$	$K_{\mathbb{P}(1,1,2)}$
$\frac{x}{y}+\frac{y^2}{x}+\frac{1}{xy}$ $x+y+\frac{1}{x^3y^2}$	$I_6, 6I_1, I_0$ $I_6, 6I_1, I_0$	6	I_1, I_1, II^*	E_8	$\binom{6m}{3m,2m,m}$	$\frac{1}{4^{\frac{1}{3}}\sqrt{3}}$	$K_{\mathbb{P}(1,2,3)}$

Obviously, we may use *either* $\tilde{\mathcal{X}}_{\mathrm{I}}$ or $\tilde{\mathcal{X}}_{\mathrm{II}}$ to compute $\Psi_{\mathrm{II}}(t_0)(=\Psi_{\mathrm{I}}(t_0))$, and for E_6, E_7, E_8 we will use $\tilde{\mathcal{X}}_{\mathrm{I}}$. For D_5, we use instead the family $\tilde{\mathcal{X}}$ produced by $\phi := \frac{(x-1)^2(y-1)^2}{xy}$, with $t_0 = \frac{1}{16}$ and $A(t) = \sum_{m\geq 0}\binom{2m}{m}^2 t^m$ (hence $\Psi_{\mathrm{II}}(t) = \frac{1}{2}\Psi(t^2)$); in fact, its minimal desingularization *is* the quotient family.

What we now do in each case is find an explicit parametrization of (each component of) $\tilde{X}_{t_0}$ via $\{f_i, g_i\}$, then compute $\mathcal{N} := \mathcal{N}_{t_0}$ and $D_2(\mathcal{N})$. First, to record some notation: we shall consider L-functions $L(\chi, s) := \sum_{k\geq 1}\frac{\chi(k)}{k^s}$ of primitive Dirichlet characters

$$\begin{aligned}
\chi_{-3}(\cdot) &= 0, 1, -1, \ldots \quad (\mathrm{mod}\ 3),\\
\chi_{-4}(\cdot) &= 0, 1, 0, -1, \ldots \quad (\mathrm{mod}\ 4),\\
\chi_{+i,5}(\cdot) &= 0, 1, \mathrm{i}, -\mathrm{i}, -1, \ldots \quad (\mathrm{mod}\ 5),\\
\chi_{-i,5}(\cdot) &= 0, 1, -\mathrm{i}, \mathrm{i}, -1, \ldots \quad (\mathrm{mod}\ 5),\\
\chi_{-8}(\cdot) &= 0, 1, 0, 1, 0, -1, 0, -1, \ldots \quad (\mathrm{mod}\ 8)
\end{aligned}$$

at $s = 2$. An easy way to get such values is by taking Bloch–Wigner of roots of unity: e.g., for $\zeta_a = \mathrm{e}^{\frac{2\pi\mathrm{i}}{a}}$,

$$D_2(\zeta_a) = \Im(Li_2(\zeta_a)) + 0 = \sum_{k\geq 1}\frac{\Im(\zeta_a^k)}{k^2}.$$

To simplify $D_2(\mathcal{N})$ to terms of this form, we manipulate $\mathcal{N}$ in a quotient of the pre-Bloch group $\mathcal{B}_2(\mathbb{C})$. Namely, work in $\mathbb{Z}[\mathbb{P}^1_{\mathbb{C}}\backslash\{0,1,\infty\}]$ modulo (the subgroup generated by) relations: $[\xi] + [\frac{1}{\xi}]$; $[1-\xi] + [\xi]$; $[\xi] + [\bar{\xi}]$; and $\sum_{i=1}^5 [\xi_i]$ where (with subscripts mod 5) $\xi_i = 1 - \xi_{i+1}\xi_{i-1}$ ($\forall i$), pictured as in figure 9. (These are all well-known relations on D_2, see [12].)

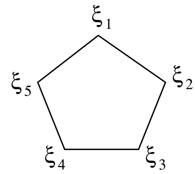


Figure 9: Mnemonic for 5-term relations.

D$_5$: In $\mathbb{P}^1 \times \mathbb{P}^1$, $1 - \frac{1}{16}\frac{(x-1)^2(y-1)^2}{xy} = 0$ is an I_1 normalized by

$$f(z) = -\frac{\left(1+\frac{1}{z}\right)^2}{\left(1-\frac{1}{z}\right)^2}, \quad g(z) = -\frac{\left(1+\frac{z}{i}\right)^2}{\left(1-\frac{z}{i}\right)^2}.$$

Hence $\mathcal{N} = 8[-\mathrm{i}] - 8[\mathrm{i}] \equiv -16[\mathrm{i}]$, and

$$D_2(\mathcal{N}) = -16D_2(\mathrm{i}) = -16L(\chi_{-4}, 2).$$

(So in fact the correct $D_2(\mathcal{N}_{t_0})$ to use for ϕ_{II} is $-8L(\chi_{-4}, 2)$.)

E$_6$: In $\mathbb{P}^2$, $0 = 1 - \frac{1}{3}\frac{x^3+y^3+1}{xy} = \frac{-1}{3xy}(1+x+y)(1+\zeta_3 x+\bar{\zeta}_3 y)(1+\bar{\zeta}_3 x+\zeta_3 y)$ is normalized by

$$f_1(z_1) = \bar{\zeta}_3\frac{\left(1-\frac{\zeta_3}{z_1}\right)}{\left(1-\frac{1}{z_1}\right)}, \quad g_1(z_1) = \frac{\left(1-\frac{z_1}{\bar{\zeta}_3}\right)}{(1-z_1)},$$
$$f_2(z_2) = \frac{\left(1-\frac{\zeta_3}{z_2}\right)}{\left(1-\frac{1}{z_2}\right)}, \quad g_2(z_2) = \bar{\zeta}_3\frac{\left(1-\frac{z_2}{\bar{\zeta}_3}\right)}{(1-z_2)},$$
$$f_3(z_3) = \zeta_3\frac{\left(1-\frac{\zeta_3}{z_3}\right)}{\left(1-\frac{1}{z_3}\right)}, \quad g_3(z_3) = \zeta_3\frac{\left(1-\frac{z_3}{\bar{\zeta}_3}\right)}{(1-z_3)},$$

so that $\mathcal{N} = 3[\bar{\zeta}_3] - 6[\zeta_3] \equiv -9[\zeta_3]$ and

$$D_2(\mathcal{N}) = -9D_2(\zeta_3) = -\frac{9\sqrt{3}}{2}L(\chi_{-3}, 2).$$

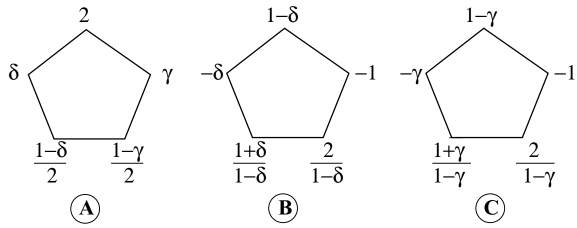


Figure 10: 5-term relations for E7.

$\underline{\mathrm{E}_7}$: In $\mathbb{P}(1,1,2)$, $0 = 1 - \frac{1}{2\sqrt{2}}\frac{x^2+y^4+1}{xy} = \frac{-1}{2\sqrt{2}xy}(x + \mathrm{i}y^2 - \sqrt{2}y - \mathrm{i})(x - \mathrm{i}y^2 - \sqrt{2}y + \mathrm{i})$ is normalized by

$$f_1(z_1) = -\sqrt{2}\frac{\left(1-\frac{\gamma}{z_1}\right)\left(1-\frac{\delta}{z_1}\right)}{\left(1+\frac{1}{z_1}\right)^2}, \quad g_1(z_1) = \frac{1-z_1}{1+z_1},$$
$$f_2(z_2) = \sqrt{2}\frac{\left(1-\frac{\gamma}{z_2}\right)\left(1-\frac{\delta}{z_2}\right)}{\left(1+\frac{1}{z_2}\right)^2}, \quad g_2(z_2) = \frac{1-z_2}{1+z_2},$$

where $\gamma := \mathrm{i}(\sqrt{2}-1)$, $\delta := \mathrm{i}(\sqrt{2}+1)$ (and $\gamma\delta = -1$). We read off

$$\mathcal{N} = 2[\gamma] + 2[\delta] - 2[-\gamma] - 2[-\delta] - 2[-1] = 4[\gamma] + 4[\delta]$$

using $\bar{\gamma} = -\gamma$, $\bar{\delta} = -\delta$. Now using the three five-term relations pictured in figure 10, together with $\frac{1+\gamma}{1-\gamma} = \zeta_8$, $\frac{1+\delta}{1-\delta} = \zeta_8^3$, we have

$$\begin{aligned}[\gamma] + [\delta] &\overset{A}{\equiv} 2([\gamma] + [\delta]) + \left[\frac{1-\delta}{2}\right] + \left[\frac{1-\gamma}{2}\right] \\ &\equiv -[-\gamma] - [1-\gamma] - [-\delta] - [1-\delta] - \left[\frac{2}{1-\gamma}\right] - \left[\frac{2}{1-\delta}\right] \\ &\overset{B,C}{\equiv} [\zeta_8] + [\zeta_8^3].\end{aligned}$$

Hence

$$\begin{aligned}D_2(\mathcal{N}) &= 4D_2(\zeta_8) + 4D_2(\zeta_8^3) = -2\mathrm{i}\sum_{k\geq 1} k^{-2}\{\zeta_8^k + \zeta_8^{3k} - \zeta_8^{5k} - \zeta_8^{7k}\} \\ &= 4\sqrt{2}L(\chi_{-8}, 2).\end{aligned}$$

$\underline{\mathrm{E}_8}$: In $\mathbb{P}(1,2,3)$, $1-\frac{x^2+y^3+1}{4^{\frac{1}{3}}3^{\frac{1}{2}}xy}=0$ is an I_1 whose normalization takes the form

$$f(z)=\sqrt{3}\frac{\prod_{j=1}^{3}\left(1-\frac{\alpha_j}{z}\right)}{\left(1-\frac{1}{z}\right)^3},\quad g(z)=\sqrt[3]{2}\frac{\prod_{k=1}^{2}\left(1-\frac{z}{\beta_k}\right)}{(1-z)^2},$$

where $\prod\alpha_j=\prod\beta_k=1$, $g(\alpha_j)=-\zeta_3^j$ and $f(\beta_k)=(-1)^k\mathrm{i}$.

Conjecture. $\sum_{i,j}\left[\frac{\alpha_j}{\beta_k}\right]-3\sum_k\left[\frac{1}{\beta_k}\right]-2\sum_j[\alpha_j]\equiv\frac{20}{3}[\mathrm{i}]$.

If this is true then $D_2(\mathcal{N})=\frac{20}{3}L(\chi_{-4},2)$.

In each of these four cases, $\varepsilon_{t_0}=-1$ and multiplying (6.14) by κ yields

$$\log\left|\frac{1}{t_0^\kappa}\right|-\sum_{m\geq1}\frac{a_m}{m}(t_0^\kappa)^m=\frac{-\kappa}{2\pi}D_2(\mathcal{N}_{t_0});\tag{6.15}$$

or on an individual basis $\Big($writing $G:=\sum_{k\geq0}\frac{(-1)^k}{(2k+1)^2}$ for Catalan's constant$\Big)$

$$D_5:\log 16-\sum_{m\geq1}\frac{\binom{2m}{m}^2}{m(16)^m}=\frac{8}{\pi}G,$$

$$E_6:\log 27-\sum_{m\geq1}\frac{(3m)!}{m(m!)^3(27)^m}=\frac{27\sqrt{3}}{4\pi}L(\chi_{-3},2),$$

$$E_7:\log 64-\sum_{m\geq1}\frac{(4m)!}{m(2m)!(m!)^2(64)^m}=\frac{8\sqrt{2}}{\pi}L(\chi_{-8},2),$$

$$E_8:\log 432-\sum_{m\geq1}\frac{(6m)!}{m(3m)!(2m)!m!(432)^m}\overset{?}{=}\frac{20}{\pi}G.$$

Of these identities, D_5 and E_6 were known to [69], while E_7 and E_8 were conjectured on the basis of numerical experiment in [19, 59]. The latter two examples (modulo the E_8 Conjecture) make the strongest case for the method of Proposition 6.3; they are not amenable to the approach in Section 10.4 since $\tilde{\mathcal{X}}_{\mathrm{I}}$, $\tilde{\mathcal{X}}_{\mathrm{II}}$, $\tilde{\mathcal{X}}_{\mathrm{II}}/\mathbb{Z}_\kappa$ all fail to be modular in the sense required there.

The four cases in this section correspond to fundamental examples in the local mirror symmetry literature. The instanton numbers that appear in [21, Table 7; 59, Table 1; 77, Ex. 1–4] ("rational") have the same exponential growth rates as our $\{N_{\kappa D}^{\langle X^\circ\rangle}\}$, namely $\exp\{$r.h.s. of (6.15)$\}$. The "κD"

(instead of D) appears due to a discrepancy in indexing of cohomology classes.

6.4. Other examples

We begin with an elliptic curve family for which $\Psi(t_0)$ involves more than one Dirichlet character: the universal curve with a marked five-torsion point, or "A_5" family. This arises via minimal desingularization of the $\tilde{\mathcal{X}}$ obtained from

$$\phi = \frac{(1-x)(1-y)(1-x-y)}{xy},$$

and is birational to the family considered by Beukers [15] in relation to irrationality of $\zeta(2)$. This has

$$A(t) = \sum_{m\geq 0}\left(\sum_{\ell=0}^{m}\binom{m}{\ell}^2\binom{m+\ell}{\ell}\right)t^m, \quad t_0 = \frac{-11\pm 5\sqrt{5}}{2},$$

with singular fibers I_5, I_1, I_1, I_5; $X_{t_0} = \overline{\{1-t\phi=0\}}$ is normalized by

$$f(z) = \gamma\frac{\left(1-\frac{1}{z}\right)^2}{\left(1-\frac{\zeta_5^2}{z}\right)\left(1-\frac{\zeta_5^3}{z}\right)}, \quad g(z) = \gamma\frac{\left(1-\frac{z}{\zeta_5}\right)^2}{\left(1-\frac{z}{\zeta_5^4}\right)\left(1-\frac{z}{\zeta_5^3}\right)},$$

where $\gamma = -\frac{1+\sqrt{5}}{2} = 2\Re(\zeta_5^2) = \bar{\zeta}_5^{\,2}(\bar{\zeta}_5+1) = \zeta_5^2(\zeta_5+1)$. This gives $\mathcal{N} = -4[\zeta_5] - 4[\zeta_5^2] + [\zeta_5^3] + 6[\zeta_5^4] \equiv -10[\zeta_5] - 5[\zeta_5^2]$. Writing $\delta_\pm := \sqrt{\frac{5\pm\sqrt{5}}{8}}$ ($\delta_+ = \Im(\zeta_5)$, $\delta_- = \Im(\zeta_5^2)$) and $\lambda_0 = \frac{11+5\sqrt{5}}{2}$, we compute

$$\begin{aligned} D_2(\mathcal{N}) = &-5\left\{\left(1+\frac{\mathrm{i}}{2}\right)\delta_+ + \left(\frac{1}{2}-\mathrm{i}\right)\delta_-\right\}L(\chi_{+\mathrm{i},5},2) \\ &-5\left\{\left(1-\frac{\mathrm{i}}{2}\right)\delta_+ + \left(\frac{1}{2}+\mathrm{i}\right)\delta_-\right\}L(\chi_{-\mathrm{i},5},2) \end{aligned}$$

and

$$\log\lambda_0 - \sum_{m\geq 1}\frac{\sum_{\ell=0}^{m}\binom{m}{\ell}^2\binom{m+\ell}{\ell}}{m\lambda_0^m} = -\frac{D_2(\mathcal{N})}{2\pi}\ (\in\mathbb{R}^+).$$

Turning to $n=3$, consider the irregular (but reflexive and tempered) Laurent polynomial

$$\phi = \left(1-\frac{1}{x}\right)\left(1-\frac{1}{y}\right)\left(1-\frac{1}{z}\right)(1-x-y+xy-xyz).$$

This gives rise to the ("Apéry") family $\tilde{\mathcal{X}}$ of singular $K3$'s related to irrationality of $\zeta(3)$ from the Introduction. The general fiber has seven A_1 (node) singularities and Theorem 3.1 applies (with $K = \mathbb{Q}$), producing $\Xi \in H^3_{\mathcal{M}}(\tilde{\mathcal{X}}_-, \mathbb{Q}(3))$. The degenerations occur over $\mathcal{L} = \left\{0, t_0, \frac{1}{t_0}, \infty\right\}$ where $t_0 = (\sqrt{2}-1)^4$; X_{t_0} and $X_{\frac{1}{t_0}}$ just have extra nodes ($\implies$ order 2 monodromy), while X_0 and X_∞ are unions of rational surfaces (and the corresponding monodromies maximally unipotent). One can therefore use (6.7) (but with a different choice φ'_∞ of topological two-cycle) to directly compute $AJ(\Xi_\infty)(\varphi'_\infty) = -2\zeta(3)$. This is done in [49] (Example 10.21) and is behind the assertion about $V(0)$ in the Introduction.

Now for the $n = 2$ families A_5, D_5, E_6, we can take advantage of their modularity to obtain an alternate computation of $\lim_{t\to t_0}\Psi(t)$; this is carried out for D_5 in Example 10.1. Similarly, by identifying the Apéry $K3$ family as modular (and Ξ essentially as an Eisenstein symbol), one can compute that (one continuation of) $\Psi(\infty) = -48\zeta(3)$, see Examples 10.2 and 10.5. More interestingly, we can even use (9.17) and (9.18) to compute $\Psi(t_0)$, which is not amenable to (6.8) (due to the nodal degeneration). Since the fixed point $\tau_0 = \frac{\mathrm{i}}{\sqrt{6}} \in \mathbb{H}$ of $\begin{pmatrix} 0 & \frac{-1}{\sqrt{6}} \\ \sqrt{6} & 0 \end{pmatrix}$ corresponds to t_0, we have (with $'\widehat{\varphi_{\mathbf{f},+6}}$ as in (10.5))

$$\Psi((\sqrt{2}-1)^4) \overset{\mathbb{Q}(3)}{\equiv} (2\pi\mathrm{i})^3 \frac{\mathrm{i}}{\sqrt{6}} \mathsf{H}^{[2]}_{[\mathrm{i}\infty]}({}'\varphi_{\mathbf{f},+6}) + \frac{1}{2\pi\mathrm{i}} {\sum_n}' \lim_{M\to\infty} {\sum_{m=-M}^{M}}' \frac{'\widehat{\varphi_{\mathbf{f},+6}}(m,n)}{m(m\frac{\mathrm{i}}{\sqrt{6}}+n)^3},$$

or dividing by $-4\pi^2$,

$$4\log(\sqrt{2}-1) + \sum_{k\geq 1} \frac{(\sqrt{2}-1)^{4k}}{k} \left\{ \sum_{j=0}^{k} \binom{k}{j}^2 \binom{k+j}{j}^2 \right\} = 4\sqrt{6}\pi - \frac{\sqrt{6}}{8\pi^3} \sum_{n\in\mathbb{Z}\setminus\{0\}} \sum_{m\geq 1} {}'\widehat{\varphi_{\mathbf{f},+6}}(m,n) \frac{\left(\frac{m^2}{18} - n^2\right)}{\left(\frac{m^2}{6}+n^2\right)^3}.$$

Presumably something more can be said about the r.h.s. but we have not attempted this.

7. The classically modular analogue: Beilinson's Eisenstein symbol

The next three sections run parallel to what was done for the toric symbols in Sections 3 and 4: here we will construct the basic higher cycles, and in Sections 8 and 9 compute the cycle class and evaluate the fiberwise AJ map on them (and consider some variations on the basic cycles). Starting from an $(\ell+1)$-tuple of functions on an elliptic curve with divisors supported on N-torsion (or the $(\ell+1)$ divisors themselves, or even just their Pontryagin product), the goal is essentially to construct a family of $CH^{\ell+1}(\cdot,\ell+1)$-cycles on the ℓth fiber product of the universal elliptic curve with marked N-torsion over $\Gamma(N)\diagdown\mathfrak{H}$. The idea comes from work of Bloch for $\ell=2$ [12,13], and first appeared in the generality considered here (but for infinite level) in [8]. Interesting aspects of the story include the relationship between the "vertical" choice of divisors and the "horizontal" values of the resulting global cycle's residues over the cusps; and the role played by modular forms and especially Eisenstein series. Much of the material in this section (and Section 8.1) is expository, but is set up to better enable the AJ computations (and for potentially easier reading) than the presentations in the existing literature, amongst which we have found [8,28,30,72] to be especially helpful.

7.1. Motivation via the Beilinson–Hodge Conjecture

For a quasi-projective variety V defined over $\bar{\mathbb{Q}}$, this conjecture predicts that the cycle-class map

$$\mathrm{cl}_V^{p,r} : CH^p(V,r) \to \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}(0), H^{2p-r}(V_{\mathbb{C}}^{\mathrm{an}}, \mathbb{Q}(p)))$$

should surject, i.e., that there "exist enough cycles." In the context below (with $p=r=\ell+1$), it translates to the statement that every Eisenstein series is, in a precise sense, the fundamental class of an "Eisenstein cycle" (or "symbol"). This case will be proved in Section 8.1 when we compute the classes of the symbols constructed in Section 7.3. In a sense our motivation is backwards since the Eisenstein material was originally a major piece of evidence leading to the conjecture.

7.1.1. Construction of Kuga modular varieties. $\mathbb{Z}^{2\ell}$ acts on $\mathfrak{H}\times\mathbb{C}^\ell$ ($\mathfrak{H}$ = upper half-plane) by

$$((m_1,n_1),\ldots,(m_\ell,n_\ell))\cdot(\tau;z_1,\ldots,z_\ell) := (\tau;\ z_1+m_1\tau+n_1,\ldots,z_\ell + m_\ell\tau+n_\ell)$$

and we quotient

$$\mathbb{Z}^{2\ell}\diagdown\mathfrak{H}\times\mathbb{C}^{\ell} \;=:\; \mathcal{E}^{[\ell]} \xrightarrow{\pi} \mathfrak{H}.$$

Recall $\Gamma(N) := \ker\{SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N\mathbb{Z})\} = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \middle| \begin{smallmatrix} ad-bc=1 \\ a\equiv 1\equiv d(N) \\ b\equiv 0\equiv c(N) \end{smallmatrix} \right\}$ and take $\Gamma \subset SL_2(\mathbb{Z})$ s.t. $\{-\mathrm{id}\} \notin \Gamma$ and $\Gamma \supset \Gamma(N)$ for some $N \geq 3$ (such a Γ is a *congruence subgroup* of $SL_2(\mathbb{Z})$).

Now $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ acts on $\mathfrak{H}^* := \mathfrak{H} \cup \mathbb{P}^1(\mathbb{Q})$ by $\gamma(\tau) = \frac{a\tau+b}{c\tau+d}$, and we define *modular curves*

$$\overline{Y}_\Gamma := \Gamma\diagdown\mathfrak{H}^* \supset \Gamma\diagdown\mathfrak{H} =: Y_\Gamma$$

with the *cusps* as complement:

$$\begin{aligned} \kappa_\Gamma := \overline{Y}_\Gamma \backslash Y_\Gamma &= \frac{\left\{ \frac{r}{s} \in \mathbb{P}^1(\mathbb{Q}) \;\middle|\; \begin{matrix} \exists p, q \in \mathbb{Z}/N\mathbb{Z} \text{ s.t.} \\ pr + qs \equiv 1 \mod N \end{matrix} \right\}}{\Gamma} \\ &= \frac{\left\{ (-s,r) \in (\mathbb{Z}/N\mathbb{Z})^2 \,\middle|\, |\langle(-s,r)\rangle| = N \right\}}{\left\langle \begin{matrix} (-s,r) \sim \gamma.(-s,r) = (-cr-ds, ar+bs) \\ (-s,r) \sim (s,-r) \end{matrix} \right\rangle}. \end{aligned}$$

One has also the *elliptic points*

$$\varepsilon_\Gamma := \left(\underbrace{\{\tau \in \mathfrak{H} \mid \exists \gamma \in \Gamma \text{ s.t. } \gamma(\tau) = \tau\}}_{=:\, \widetilde{\varepsilon}_\Gamma} \Big/ \Gamma \right) \subset Y_\Gamma.$$

Now let Γ act on $\mathcal{E}^{[\ell]} \backslash \pi^{-1}(\widetilde{\varepsilon}_\Gamma)$ by

$$\gamma.(\tau; [z_1, \ldots, z_\ell]_\tau) := \left(\gamma(\tau); \left[\frac{z_1}{c\tau+d}, \ldots, \frac{z_\ell}{c\tau+d} \right]_{\gamma(\tau)} \right);$$

the quotient is denoted $\mathcal{E}^{[\ell]}_\gamma \xrightarrow{\pi_\Gamma} Y_\Gamma \backslash \varepsilon_\Gamma$ and Shokurov's smooth compatification [75] is $\overline{\mathcal{E}}^{[\ell]}_\Gamma \xrightarrow{\overline{\pi}_\Gamma} \overline{Y}_\Gamma$ (we just need its existence).

7.1.2. Monodromy on $\mathcal{E}^{[\ell]}_\Gamma$. To understand monodromy about $\varepsilon_\Gamma \cup \kappa_\Gamma$, first take $\ell = 1$ and let α resp. β be the families of one-cycles $[0,1]$ resp. $[0,\tau]$ on fibers E_τ of $\mathcal{E}^{[1]} \to \mathfrak{H}$. Each $\gamma \in \Gamma$ should be thought of as a composition of monodromy transformations with action

$$\alpha \mapsto a\alpha + c\beta, \quad \beta \mapsto b\alpha + d\beta.$$

If γ *fixes* $\frac{r}{s} \in \mathbb{P}^1(\mathbb{Q})$ (resp. $\tau_0 \in \mathfrak{H}$) then it corresponds to going around (some number of times) $[\frac{r}{s}] \in \kappa_\Gamma$ (resp. $[\tau_0] \in \varepsilon_\Gamma$). The ε_Γ are just the finite monodromy points, with order$=3$ and monodromy locally of the form $\left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$ in an appropriate basis (Kodaira type IV^*). If we had not required $-\mathrm{id} \notin \Gamma$ then they could have order 2 or 4). If $\Gamma = \Gamma(N)$ then $\varepsilon_\Gamma = \emptyset$.

To put all the cusps on an equal footing with regard to monodromy matrices, given $\frac{r}{s} \in \mathbb{P}^1(\mathbb{Q})$ pick $p, q \in \mathbb{Z}$ such that $pr + qs = 1$ and define a "local monodromy group"

$$M_\Gamma\left(\left[\frac{r}{s}\right]\right) := \begin{pmatrix} p & q \\ -s & r \end{pmatrix} \mathrm{Stab}_\Gamma\left(\frac{r}{s}\right) \begin{pmatrix} r & -q \\ s & p \end{pmatrix},$$

which is generated by $\left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right)$ $\left(\text{or } \left(\begin{smallmatrix} -1 & -m \\ 0 & -1 \end{smallmatrix}\right)\right)$ for some $m|N$ $\left(\text{resp. } m|\frac{N}{2}\right)$. For $\overline{\mathcal{E}}^{[1]}$, this yields a fiber of type I_m (resp. I_m^*) in Kodaira's classification; we subdivide $\kappa_\Gamma =: \kappa_\Gamma^I \cup \kappa_\Gamma^{I^*}$.

For $\ell \geq 1$, one has an isomorphism of VHS

$$\mathcal{H}^\ell_{\mathcal{E}^{[\ell]}/Y} \cong \bigoplus_{0 \leq a \leq \lfloor \frac{\ell}{2} \rfloor} \left(\mathcal{H}^1_{\mathcal{E}^{[1]}/Y}(-a)^{\otimes(\ell-2a)}\right)^{\oplus \binom{\ell}{\ell-2a,a,a}}$$

so that monodromy about type I cusps is (maximally) unipotent for all ℓ, while that about type I^* cusps is only unipotent for ℓ even (by considering ℓth symmetric powers of $\left(\begin{smallmatrix} -1 & -m \\ 0 & -1 \end{smallmatrix}\right)$).

7.1.3. MHS on the singular fibers of $\overline{\boldsymbol{\mathcal{E}}}^{[\boldsymbol{\ell}]}_{\boldsymbol{\Gamma}}$. We will use the notation $E^{[\ell]}_{\Gamma,y} (\cong E^{[\ell]}_\tau$ for some $\tau \in \mathfrak{H})$ for smooth fibers and $\hat{E}^{[\ell]}_{\Gamma,y_0}$ for singular fibers, which are NCDs in the Shokurov compactification. (Note: $\hat{E}^{[\ell]}_{\Gamma,y_0}$ does not count multiple fiber-components with multiplicity.)

(A) Elliptic points. ($y_0 \in \varepsilon_\Gamma$) Take a degree-3 cover $\overline{\tilde{\tilde{Y}}}_\Gamma \xrightarrow{\mu} \overline{Y}_\Gamma$ with ramification index 3 at $\tilde{y_0} \mapsto y_0$, and let $\overline{\tilde{\tilde{\mathcal{E}}}}^{[\ell]}_\Gamma$ be a smooth resolution of $\overline{\mathcal{E}}^{[\ell]}_\Gamma \times_\mu \tilde{Y}_\Gamma$. This maps to $'\overline{\tilde{\tilde{\mathcal{E}}}}^{[\ell]}_\Gamma$ where

(a) $'\overline{\tilde{\tilde{\mathcal{E}}}}^{[\ell]}_\Gamma \backslash {}'\tilde{E}^{[\ell]}_{\Gamma,\tilde{y_0}} = \overline{\tilde{\tilde{\mathcal{E}}}}^{[\ell]}_\Gamma \backslash \tilde{E}^{[\ell]}_{\Gamma,\tilde{y_0}}$ (here $\tilde{E}^{[\ell]}_{\Gamma,\tilde{y_0}}$ is possibly singular)

(b) ${}'\tilde{E}^{[\ell]}_{\Gamma,\tilde{y_0}}$ is the ℓth self-product of a *smooth* elliptic curve ($\tau = e^{\frac{2\pi\mathrm{i}}{3}}$ or $\mathrm{e}^{\frac{2\pi\mathrm{i}}{6}}$), yielding a diagram

$$\begin{array}{ccccc} {}'\tilde{\tilde{\overline{\mathcal{E}}}}{}^{[\ell]}_{\Gamma} & \xleftarrow{\;p\;} & \tilde{\tilde{\overline{\mathcal{E}}}}{}^{[\ell]}_{\Gamma} & \xrightarrow{\;\mathcal{M}\;} & \overline{\mathcal{E}}^{[\ell]}_{\Gamma} \\ & {\scriptstyle {}'\tilde{\pi}}\searrow & \downarrow{\scriptstyle \tilde{\pi}} & & \downarrow{\scriptstyle \bar{\pi}} \\ & & \tilde{\tilde{\overline{Y}}}_{\Gamma} & \xrightarrow{\;\mu\;} & \overline{Y}_{\Gamma} \end{array}$$

Now $(a)+(b) \implies H^{\ell+1}(\tilde{\tilde{\overline{\mathcal{E}}}}{}^{[\ell]}_{\Gamma}\backslash \tilde{E}^{[\ell]}_{\Gamma,y_0}) = W_{\ell+2}H^{\ell+1}(\tilde{\tilde{\overline{\mathcal{E}}}}{}^{[\ell]}_{\Gamma}\backslash \tilde{E}^{[\ell]}_{\Gamma,\tilde{y_0}})$, while $\frac{1}{3}\mathcal{M}_*\mathcal{M}^*$ is the identity on $H^{\ell+1}(\overline{\mathcal{E}}^{[\ell]}_{\Gamma}\backslash E^{[\ell]}_{\Gamma,y_0})$. By the localization sequence

$$\to H^{\ell+1}(\overline{\mathcal{E}}^{[\ell]}_{\Gamma}\backslash \hat{E}^{[\ell]}_{\Gamma,y_0}) \to H_\ell(\hat{E}^{[\ell]}_{\Gamma,y_0})(-(\ell+1)) \to H^{\ell+2}(\overline{\mathcal{E}}^{[\ell]}_{\Gamma}) \to,$$

$H_\ell(\hat{E}^{[\ell]}_{\Gamma,y_0})$ is a pure HS of weight $-\ell$.

(B) Nonunipotent cusps. ($y_0 \in \kappa^{I^*}_{\Gamma}$, ℓ odd) Even in the quasi-unipotent/ non-semistable degeneration setting, if the total space is smooth (with NCD central fiber) the Wang sequence, relative homology sequence, and deformation retract business goes through, yielding a long-exact sequence

(7.1)
$$\to H_{\ell+2}(\hat{E}^{[\ell]}_{\Gamma,y_0}(-(\ell+1)) \xrightarrow{\;\xi\;} H^{\ell}(\hat{E}^{[\ell]}_{\Gamma,y_0} \to H^{\ell}(E^{[\ell]}_{\Gamma,y}) \xrightarrow{\;T-I\;} H^{\ell}(E^{[\ell]}_{\Gamma,y}) \to;$$

here ξ is a morphism of MHS (as $\iota^*_{y_0}\circ(\iota_{y_0})_*$, it is motivic). For the monodromy matrix, taking ℓth symmetric power of $\left(\begin{smallmatrix}-1 & -m\\ 0 & -1\end{smallmatrix}\right)$ for $\ell \geq 1$ odd gives $T = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1\end{pmatrix}$; hence $T-I$ has maximal rank and ξ is surjective. Since $H_{\ell+2}(\hat{E})(-(\ell+1))$ has weights $\geq \ell$ and $H^{\ell}(\hat{E})$ weights $\leq \ell$, we find again that $H^{\ell}(\hat{E})$ (hence $H_\ell(\hat{E}^{[\ell]}_{\Gamma,y_0})$) is a *pure* HS.

(C) Unipotent cusps. ($y_0 \in \kappa^{I^*}_{\Gamma}$ and ℓ even; $y_0 \in \kappa^{I}_{\Gamma}$) Start with $\ell = 1$: taking $y = [\mathrm{i}\infty]$ as our prototypical such cusp and assuming an I_m degeneration there, the choice of local parameter $q^{\frac{1}{m}} =: \tilde{q} := \exp\left(\frac{2\pi\mathrm{i}}{m}\tau\right) = \exp\left(\frac{2\pi\mathrm{i}}{m}\frac{\int_\beta dz}{\int_\alpha dz}\right)$ splits the LMHS:

$$H^1_{\lim_{\tilde{q}\to 0}}(E_{\Gamma,\tilde{q}}) \cong \mathbb{Q}(0)\oplus\mathbb{Q}(-1).$$

Similarly, $H^{\ell}_{\lim}(E^{[\ell]}_{\Gamma,\tilde{q}})$ is a $\oplus$ of copies of $\mathbb{Q}(0)$ thru $\mathbb{Q}(-\ell)$ — in particular *one* copy of $\mathbb{Q}(0)$. (Think of this as a consequence of the fact that the periods are all powers of $m\log\tilde{q}$; the $\mathbb{Q}(0)$ corresponds to $\alpha^{\times\ell}$ with period 1.) Equation (7.1) becomes the Clemens–Schmid sequence

$$\to H_{\ell+2}(\hat{E}^{[\ell]}_{\Gamma,y_0})(-(\ell+1)) \xrightarrow{\xi} H^{\ell}(\hat{E}^{[\ell]}_{\Gamma,y_0}) \to H^{\ell}_{\lim}(E^{[\ell]}_{\Gamma,y}) \xrightarrow{N} H^{\ell}_{\lim}(E^{[\ell]}_{\Gamma,y}) \to$$

(where $N = \log(T)$ now makes sense); since N is of type $(-1,-1)$ it kills $\mathbb{Q}(0)$. By the same reasoning as above, $\mathrm{im}(\xi)$ has pure weight ℓ; so $H^{\ell}(\hat{E}^{[\ell]}_{\Gamma,y_0})$ is completely split into $\mathbb{Q}(-j)$'s (independent of the choice of parameter), in particular $H^{\ell}(\hat{E}^{[\ell]}_{\Gamma,y_0}) \cong \mathbb{Q}(0)\oplus\mathcal{H}$ where $W_0\mathcal{H}=\{0\}$.

Conclusion:

$\mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}(0), H_{\ell}(\hat{E}^{[\ell]}_{\Gamma,y_0}))$ is $\{0\}$ in cases (A) and (B) (or for a smooth fiber), and one copy of $\mathbb{Q}(0)$ for case (C).

7.1.4. Residues and Beilinson–Hodge Let $\mathfrak{p}\subset Y_{\Gamma}\backslash\varepsilon_{\Gamma}$ be a finite point set, and consider open subsets of $\begin{matrix}\mathcal{E}^{[\ell]}_{\Gamma}\\ \bigcap \\ \overline{\mathcal{E}}^{[\ell]}_{\Gamma}\end{matrix}$

$$\begin{array}{ccccl} (\mathcal{E}^{[\ell]}_{\Gamma})^{\circ} & \xrightarrow{\pi^{\circ}_{\Gamma}} & Y^{\circ}_{\Gamma} & := Y_{\Gamma}\backslash\varepsilon_{\Gamma}\cup\mathfrak{p} & = \overline{Y}_{\Gamma}\backslash\mathfrak{P}, \\ \bigcap & & \bigcap & & \\ (\overline{\mathcal{E}}^{[\ell]}_{\Gamma})^{\circ} & \xrightarrow{\overline{\pi}^{\circ}_{\Gamma}} & \overline{Y}^{\circ}_{\Gamma} & := \overline{Y}_{\Gamma}\backslash\kappa^{[\ell]}_{\Gamma}, & \end{array}$$

where $\mathfrak{P} := \kappa^{I}_{\Gamma}\cup\kappa^{I^*}_{\Gamma}\cup\varepsilon_{\Gamma}\cup\mathfrak{p}$, and $\kappa^{[\ell]}_{\Gamma} := \begin{cases}\kappa_{\Gamma}, & \ell \text{ odd}\\ \kappa^{I}_{\Gamma}, & \ell \text{ even}\end{cases}$ consists of the unipotent cusps. Applying $\mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}(0), \text{—}\otimes\mathbb{Q}(\ell+1))$ to the "localization sequence"

$$\begin{aligned} 0 \to \operatorname{coker}&\left\{\jmath^*_{\ell+1} : H^{\ell+1}(\overline{\mathcal{E}}^{[\ell]}_{\gamma}) \to H^{\ell+1}((\mathcal{E}^{[\ell]}_{\Gamma})^{\circ})\right\} \\ &\xrightarrow{\oplus\frac{\mathrm{Res}_{y_0}}{(2\pi\mathrm{i})^{\ell}}} \oplus_{y_0\in\mathfrak{P}} H_{\ell}(\overset{(\wedge)}{E}{}^{[\ell]}_{\Gamma,y_0})(-(\ell+1)) \\ &\xrightarrow{(2\pi\mathrm{i})^{\ell+1}\left(\oplus(\imath_{y_0})_*\right)} \ker\left\{\jmath^*_{\ell+2} :\ H^{\ell+2}(\overline{\mathcal{E}}^{[\ell]}_{\Gamma}) \to H^{\ell+2}((\mathcal{E}^{[\ell]}_{\Gamma})^{\circ})\right\} \to 0 \end{aligned}$$

gives

$$\begin{aligned} \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}(0), \operatorname{coker}(\jmath^*_{\ell+1})\otimes\mathbb{Q}(\ell+1)) &\cong \bigoplus_{y_0\in\mathfrak{P}} \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}(0), H_{\ell}(\overset{(\wedge)}{E}{}^{[\ell]}_{\Gamma,y_0})) \\ &\overset{\text{by §7.1.3}}{\cong} \oplus_{y_0\in\kappa^{[\ell]}_{\Gamma}}\mathbb{Q}(0), \end{aligned}$$

since $\ker(\jmath^*_{\ell+2})$ has pure weight $\ell+2$ (and $\ell \geq 1$). Using

$$0 \to \mathrm{im}(\jmath^*_{\ell+1}) \to H^{\ell+1}((\mathcal{E}^{[\ell]}_\Gamma)^\circ) \to \mathrm{coker}(\jmath^*_{\ell+1}) \to 0,$$

we then clearly have $\mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}(0), H^{\ell+1}((\overline{E}^{[\ell]}_\gamma)^\circ, \mathbb{Q}(\ell+1))) \subset$

$$\mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}(0), H^{\ell+1}((\mathcal{E}^{[\ell]}_\Gamma)^\circ, \mathbb{Q}(\ell+1))) \xhookrightarrow{\oplus \frac{\mathrm{Res}}{(2\pi \mathrm{i})^\ell}} \oplus_{\left[\frac{r}{s}\right]\in\kappa^{[\ell]}_\Gamma} \mathbb{Q}.$$

Claim 7.1. The composition

$$\begin{array}{ccc} CH^{\ell+1}\left((\overline{E}^{[\ell]}_\Gamma)^\circ, \ell+1\right) & & \\ \downarrow [\cdot] & \searrow & \\ \mathrm{Hom}_{\mathrm{MHS}}\left(\mathbb{Q}(0), H^{\ell+1}\left((\overline{E}^{[\ell]}_\Gamma)^\circ, \mathbb{Q}(\ell+1)\right)\right) & \xrightarrow[\oplus \frac{\mathrm{Res}}{(2\pi \mathrm{i})^\ell}]{} & \oplus_{\left[\frac{r}{s}\right]\in\kappa^{[\ell]}_\Gamma}\mathbb{Q} \end{array}$$

is surjective.

If this is true, then we have clearly proved that for any $\mathfrak{P}$ as just described

$$CH^{\ell+1}((\mathcal{E}^{[\ell]}_\Gamma)^\circ, \ell+1) \twoheadrightarrow \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}(0), H^{\ell+1}((\mathcal{E}^{[\ell]}_\Gamma)^\circ, \mathbb{Q}(\ell+1))),$$

which is the relevant special case of the Beilinson–Hodge conjecture.

7.1.5. Holomorphic forms of top degree. Clearly on $\mathcal{E}^{[\ell]}(\to \mathfrak{H})$ these are of the form

$$\Omega^{\ell+1}_F := F(\tau) dz_1 \wedge \cdots \wedge dz_\ell \wedge d\tau,$$

for F holomorphic ($F \in \mathcal{O}(\mathfrak{H})$). For this to descend to $\mathcal{E}^{[\ell]}_\Gamma$ (recalling from Section 7.1.1 the action of $\gamma \in \Gamma \subset SL_2(\mathbb{Z})$ on $\mathcal{E}^{[\ell]} \backslash \pi^{-1}(\widetilde{\varepsilon}_\Gamma)$), we must have

$$\Omega^{\ell+1}_F = \gamma^* \Omega^{\ell+1}_F = F(\gamma(\tau)) \frac{dz_1}{c\tau+d} \wedge \cdots \wedge \frac{dz_\ell}{c\tau+d} \wedge \frac{\overbrace{(ad-bc)}^{=1} d\tau}{(c\tau+d)^2},$$

which is equivalent to

$$F(\tau) = \frac{F(\gamma(\tau))}{(c\tau+d)^{\ell+2}} =: F|^{\ell+2}_\gamma(\tau)(\forall \gamma \in \Gamma). \tag{7.2}$$

Definition 7.1. (i) $F \in \mathcal{O}(\mathfrak{H})$ and (7.2) holds if and only if $F(\tau)$ an automorphic form of weight $\ell+2$ with respect to Γ.

(ii) $\lim_{\tau\to i\infty} F(\tau) =: \mathfrak{R}_{[i\infty]}(F) < \infty$ if and only if $F(\tau)$ bounded at $i\infty$.

(iii) $\mathfrak{R}_{[i\infty]}(F) = 0$ if and only if $F(\tau)$ cusp at $i\infty$.

Now assuming F automorphic of weight $\ell+2$ (w.r.t. some Γ):

(iv) F is *cusp* (*resp. bounded*) at $\left[\frac{r}{s}\right]$ if and only if $F|^{\ell+2}_{\left(\begin{smallmatrix} r & -q \\ s & p\end{smallmatrix}\right)}$ cusp (resp. bounded) at $i\infty$, where p, q are chosen so that the matrix $\in SL_2(\mathbb{Z})$; and

(v) F *cusp (resp. modular) form* of weight $\ell+2$ (w.r.t. Γ) if and only if F cusp (resp. bounded) at every cusp($\in \kappa_\Gamma$).

Remark. Unconventionally, a *meromorphic modular form* will mean the same thing as *modular form* except that poles at cusps κ_Γ and elliptic points $\widetilde{\varepsilon}_\Gamma$ are permitted. (For each cusp $\left[\frac{r}{s}\right]$, this means $q^{-K}F|^{\ell+2}_{\left(\begin{smallmatrix} r & -q \\ s & p\end{smallmatrix}\right)}$ is bounded at $i\infty$ for some $K \in \mathbb{Z}^+$.) We write $A_{\ell+2}(\Gamma)$ (resp. $S_{\ell+2}(\Gamma)$, $M_{\ell+2}(\Gamma)$, $\check{M}_{\ell+2}(\Gamma)$) for automorphic (resp. cusp, modular, mero. modular) forms.

Example 7.1. Let $F \in A_{\ell+2}(\Gamma)$. If the cusp $[i\infty] \in \kappa_\Gamma$ is type I_m then $\left(\begin{smallmatrix} 1 & m \\ 0 & 1\end{smallmatrix}\right) \in \Gamma$, so that $F(\tau+m) = F(\tau)$; if type I_m^* then $\left(\begin{smallmatrix} -1 & -m \\ 0 & -1\end{smallmatrix}\right) \in \Gamma$, ensuring $F(\tau+m) = (-1)^{\ell+2}F(\tau)$. Either way, $\tilde{q} := q^{\frac{1}{m}}$ (see Section 7.1.3(C)) gives a local coordinate on $\overline{Y}_\Gamma$ at $[i\infty]$. In the unipotent case, we conclude that F has a Laurent expansion $F(\tau) = \sum_{k\in\mathbb{Z}} a_k \tilde{q}^k$; in the nonunipotent ($I_m^*$ and ℓ odd) case we get instead $F(\tau) = \sum_{k\in\mathbb{Z}\text{ odd}} a_k \tilde{q}^{\frac{k}{2}}$ (Ω_F still gives a well-defined holomorphic $form$ on the quotient $\mathcal{E}_\Gamma$). Evidently, the "bounded" condition says in both cases that $a_k = 0$ for $k < 0$ (and "cusp" forms have no constant term); so in the nonunipotent case, bounded implies cusp.

Shokurov [75] proved the following:

Proposition 7.1. (i) $\Omega^{\ell+1}(\overline{\mathcal{E}}_\Gamma^{[\ell]} \backslash \pi^{-1}(\kappa_\Gamma)) = \{\Omega_F \,|\, F \in A_{\ell+2}(\Gamma)\}$, *i.e., such Ω_F extend holomorphically across the singular fibers over elliptic points;*

(ii) $\Omega^{\ell+1}(\overline{\mathcal{E}}_\Gamma^{[\ell]})\langle \log(\overline{\pi}_\Gamma^{-1}(\kappa_\Gamma))\rangle = \Omega^{\ell+1}(\overline{\mathcal{E}}_\Gamma^{[\ell]})\langle \log(\overline{\pi}_\Gamma^{-1}(\kappa_\Gamma^{[\ell]})\rangle = \{\Omega_F \,|\, F \in M_{\ell+2}(\Gamma)\}$*; and*

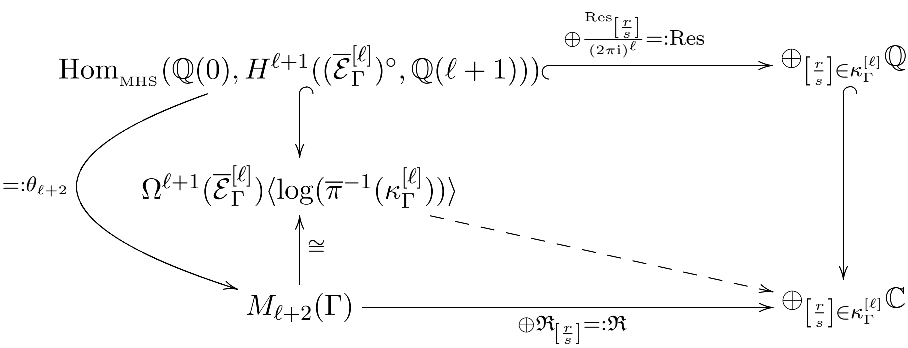
(iii) $\Omega^{\ell+1}(\overline{\mathcal{E}}_\Gamma^{[\ell]}) = \{\Omega_F \,|\, F \in S_{\ell+2}(\Gamma)\}$.

This gives the dictionary between automorphic forms and holomorphic forms that we will need. To start relating modular forms to Beilinson–Hodge, make the following.

Definition 7.2. Given $F \in M_{\ell+2}(\Gamma)$ and $\left[\frac{r}{s}\right] \in \kappa_\Gamma^{[\ell]}$, take any $\left(\begin{smallmatrix} r & -q \\ s & p \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ and set

$$\mathfrak{R}_{\left[\frac{r}{s}\right]}(F) := \lim_{\tau \to i\infty} F|^{\ell+2}_{\left(\begin{smallmatrix} r & -q \\ s & p \end{smallmatrix}\right)}(\tau) = \lim_{\tau \to i\infty} \frac{F\left(\frac{r\tau - q}{s\tau + p}\right)}{(s\tau + p)^{\ell+2}} \in \mathbb{C}.$$

This gives an interpretation of residues, in the sense that the following diagram commutes:



where the vertical isomorphism sends $F \mapsto (2\pi \mathrm{i})^{\ell+1}\Omega_F$.

Definition 7.3. $M^{\mathbb{Q}}_{\ell+2}(\Gamma) := \mathrm{im}(\Theta_{\ell+2}) =$ modular forms corresponding to holomorphic forms with log poles (at cuspidal fibers) *and rational periods.*

By pure thought we have

Proposition 7.2. (i) $\mathfrak{R}$ *is surjective;*

(ii) $\mathfrak{R}|_{M^{\mathbb{Q}}_{\ell+2} \otimes \mathbb{C}}$ *is injective; and*

(iii) $(M^{\mathbb{Q}}_{\ell+2} \otimes \mathbb{C}) \oplus S_{\ell+2} \hookrightarrow M_{\ell+2}(\Gamma)$.

Proof. Since $\ker(\mathfrak{R}) = S_{\ell+2}(\Gamma)$, the kernel of the dotted arrow is actually $\Omega^{\ell+1}(\overline{E}_\Gamma^{[\ell]})$. This arrow must *surject*, since the $\oplus\mathbb{Q}$'s (hance $\oplus\mathbb{C}$'s) correspond to weight $2\ell + 2 > \ell + 2$ in $\oplus H_\ell(\hat{E}^{[\ell]}_{\Gamma, \left[\frac{r}{s}\right]})(-(\ell+1))$ (hence cannot be absorbed by the next term in the localization sequence); (i) follows. Injectivity of Res implies (ii), which in turn implies (iii). $\square$

Now if Claim 7.1 holds, we have also $M^{\mathbb{Q}}_{\ell+2}(\Gamma) \twoheadrightarrow \oplus\mathbb{Q}$, hence $M^{\mathbb{Q}} \otimes \mathbb{C} \twoheadrightarrow \oplus\mathbb{C}$ (hence $\cong$), which would imply

$$M_{\ell+2}(\Gamma) = (M^{\mathbb{Q}}_{\ell+2}(\Gamma) \otimes \mathbb{C}) \oplus S_{\ell+2}(\Gamma). \tag{7.3}$$

7.1.6. Reduction to $(\Gamma =)\Gamma(N)$. Assume $SL_2(\mathbb{Z}) \supset \Gamma \supset \Gamma(N)$. Since $\Gamma(N) \trianglelefteq SL_2(\mathbb{Z})$, $\Gamma(N) \trianglelefteq \Gamma$ and the coset representatives $\{\gamma_i\}_{i=1}^{[\Gamma:\Gamma(N)]}$ act on the sheets of the branched cover $\overline{Y}_{\Gamma(N)} \xrightarrow{\overline{\rho}} \overline{Y}_{\Gamma}$, and also on

$$\begin{array}{ccc} \mathcal{E}^{[\ell]}_{\Gamma(N)} \backslash \pi^{-1}_{\Gamma(N)}(\overline{\rho}^{-1}(\varepsilon_\Gamma)) & \xrightarrow{\mathcal{P}^{[\ell]}_{\Gamma(N)/\Gamma}} & \mathcal{E}^{[\ell]}_{\Gamma} \\ \downarrow{\scriptstyle \pi_{\Gamma(N)}} & & \downarrow{\scriptstyle \pi_\Gamma} \\ Y_{\Gamma(N)} \backslash \overline{\rho}^{-1}(\varepsilon_\Gamma) & \xrightarrow{\rho_{\Gamma(N)/\Gamma}} & Y_\Gamma \backslash \varepsilon_\Gamma. \end{array}$$

We can interpret the action of this $\mathcal{P}$ on holomorphic forms (and eventually, algebraic cycles) in terms of modular forms and residues:

$$\begin{array}{ccc} \Omega^{\ell+1}\left(\overline{\mathcal{E}}^{[\ell]}_{\Gamma(N)}\right)\left\langle \log \overline{\pi}^{-1}_{\Gamma(N)}(\overline{\rho}^{-1}(\varepsilon_\Gamma) \cup \kappa_{\Gamma(N)})\right\rangle & \underset{\mathcal{P}^*}{\overset{\mathcal{P}_*}{\rightleftarrows}} & \Omega^{\ell+1}\left(\overline{\mathcal{E}}^{[\ell]}_{\Gamma}\right)\left\langle \log \overline{\pi}^{-1}_{\Gamma}(\kappa_\Gamma)\right\rangle \\ \uparrow{\scriptstyle \cong} & & \uparrow{\scriptstyle \cong} \\ M_{\ell+2}(\Gamma(N)) & \underset{F(\tau) \leftarrow F(\tau)}{\overset{F(\tau) \mapsto \sum_i F|^{\ell+2}_{\gamma_i}(\tau)}{\rightleftarrows}} & M_{\ell+2}(\Gamma) \\ \downarrow{\scriptstyle \mathfrak{R}} & & \downarrow{\scriptstyle \mathfrak{R}} \\ \Upsilon_2(N) := \oplus_{\kappa^{[\ell]}_{\Gamma(N)}} \mathbb{C} & \underset{\text{pull-back } \mathsf{P}^{[\ell]}_{\Gamma(N)/\Gamma}}{\overset{\text{trace } \mathsf{T}^{[\ell]}_{\Gamma(N)/\Gamma}}{\rightleftarrows}} \; \text{(of } \mathbb{C}\text{-valued functions on cusps)} & \oplus_{\kappa^{[\ell]}_{\Gamma}} \mathbb{C} =: \Upsilon_2(\Gamma) \end{array}$$

More precisely (for the "trace"): given $[\frac{r_0}{s_0}] \in \kappa^{[\ell]}_{\Gamma}$, the image of an element $\{\beta : \kappa^{[\ell]}_{\Gamma(N)} \to \mathbb{C}\} \in \Upsilon_2(N)$ takes value $(\mathsf{T}_*\beta)([\frac{r_0}{s_0}]) = \sum_{[\frac{r}{s}] \in \overline{\rho}^{-1}([\frac{r_0}{s_0}])} \mathrm{ord}_{[\frac{r}{s}]}(\overline{\rho}) \cdot \beta([\frac{r}{s}])$. This map is surjective since unipotent cusps cover unipotent cusps; though when ℓ is odd, unipotent $([\frac{r}{s}] \in \kappa^{[\ell]}_{\Gamma(N)})$ can map to nonunipotent $([\frac{r}{s}] \in \kappa^{I^*}_{\Gamma})$, in which case the value is lost.

The main point is that

Claim 7.1 (hence Beilinson–Hodge) for $\Gamma(N)$ implies Claim 7.1 for Γ,

since the trace surjects and one can use $\mathcal{P}_*$ on higher Chow cycles, to push them from $\mathcal{E}^{[\ell]}_{\Gamma(N)}$ to $\mathcal{E}^{[\ell]}_{\Gamma}$. We write $\overline{Y}_{\Gamma(N)} =: \overline{Y}(N)$, $\kappa_{\Gamma(N)} =: \kappa(N)$, etc. for simplicity.

Why do we want to do make this reduction? $Y(N)$ is the moduli space of elliptic curves with "completely marked N-torsion" (in particular, two marked generators), so $\mathcal{E}(N)(:= \mathcal{E}_{\Gamma(N)})$ has N^2 N-torsion sections — ideal for building relative higher Chow cycles (from functions with divisors supported on that N-torsion). Also, all cusps are (unipotent) of type I_N. One reason why we exclude $N = 2$ is that this is $false$ — there are two cusps of type I_2 and one of type I_2^*. The downside is that $\overline{Y}(N)$ has genus zero only for $N = (2), 3, 4, 5$.

For the cusps, writing $\mathfrak{G}(N)$ for the set of subgroups of $(\mathbb{Z}/N\mathbb{Z})^2$ isomorphic to $\mathbb{Z}/N\mathbb{Z}$, we have $\kappa^{[\ell]}(N) =$

$$\kappa(N) = \frac{\left\{ (-s,r) \in (\mathbb{Z}/N\mathbb{Z})^2 \,\middle|\, |\langle(-s,r)\rangle| = N \right\}}{\langle (-s,r) \sim (s,-r) \rangle} = \bigcup_{G \in \mathfrak{G}(N)} G^* / \langle \pm 1 \rangle$$

$$\cong PSL_2(\mathbb{Z}/N\mathbb{Z}) \Big/ \left\langle \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\rangle ;$$

since each $G \in \mathfrak{G}(N)$ has $|G^*| = \phi_{\mathrm{euler}}(N)$,

$$|\kappa(N)| = \frac{\phi_{\mathrm{euler}}(N)}{2} \cdot |\mathfrak{G}(N)| = \frac{N}{2} \prod_{p|N} \left(1 - \frac{1}{p}\right) \cdot N \prod_{p|N} \left(1 + \frac{1}{p}\right)$$

$$= \frac{N^2}{2} \prod_{p|N} \left(1 - \frac{1}{p^2}\right).$$

Now given a field $K \subseteq \mathbb{C}$ set[19]

$$\begin{aligned} \Phi_m^K(N) &:= \{K\text{-valued functions on } (\mathbb{Z}/N\mathbb{Z})^m\}, \\ \Phi_m^K(N)_\circ &:= \{\varphi \in \Phi_m^K(N) \,|\, \varphi(\bar{0}, \ldots, \bar{0}) = 0)\}, \\ \Phi_m^K(N)^\circ &:= \ker\{\text{augmentation map: } \Phi_m^K(N) \to K\}. \end{aligned}$$

Ultimately, $\Phi_2^K(N)^\circ$ will be divisors $(\otimes\mathbb{Q})$ of degree 0 on N-torsion.

Choose once and for all a representative $(-s, r)$ for each cusp $\sigma \in \kappa(N)$ (s.t. $\sigma = [\frac{r}{s}]$) and a matrix $\begin{pmatrix} p & q \\ -s & r \end{pmatrix} \in SL_2(\mathbb{Z})$. Writing

$$\begin{aligned} \pi_{[\frac{r}{s}]} : (\mathbb{Z}/N\mathbb{Z})^2 &\twoheadrightarrow \mathbb{Z}/N\mathbb{Z}, & \iota_{[\frac{r}{s}]} : \mathbb{Z}/N\mathbb{Z} &\hookrightarrow (\mathbb{Z}/N\mathbb{Z})^2, \\ (m,n) = a(p,q) + b(-s,r) &\mapsto a, & a &\mapsto a(-s,r), \end{aligned}$$

[19]Notationally, we drop $m = 1$ or $K = \mathbb{C}$.

one has

$$(\pi_{[\frac{r}{s}]})_* : \Phi_2(N)^{(\circ)} \xrightarrow{\text{trace}} \Phi(N)^{(\circ)}, \quad (\iota_{[\frac{r}{s}]})^* : \Phi_2(N)_{(\circ)} \xrightarrow{\text{pullback}} \Phi(N)_{(\circ)},$$

etc.

7.2. Divisors with N-torsion support

Here we collect together related material on finite Fourier transforms, L-functions, and meromorphic functions on $\mathcal{E}(N)$ with divisors supported on the N-torsion sections. The technical "(p,q)-vertical" subsection will be used in Section 9 to compute the AJ map.

7.2.1. Some Fourier theory. We define Fourier transforms

$$\widehat{\ }: \Phi(N)^{(\circ)} \xrightarrow{\cong} \Phi(N)_{(\circ)},$$
$$\varphi(a) \mapsto \widehat{\varphi}(k) := \sum_{a\in \mathbb{Z}/N\mathbb{Z}} \varphi(a)\mathrm{e}^{-\frac{2\pi \mathrm{i}}{N}ka},$$
$$\widehat{\ }: \Phi_2(N)^{(\circ)} \xrightarrow{\cong} \Phi_2(N)_{(\circ)},$$
$$\varphi(m,n) \mapsto \widehat{\varphi}(\mu,\eta) := \sum_{(m,n)\in(\mathbb{Z}/N\mathbb{Z})^2} \varphi(m,n)\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}(\mu n-\eta m)}.$$

One can show (easily) that for $\varphi_0 \in \Phi(N)$, $\varphi \in \Phi_2(N)$

$$\frac{1}{N}\cdot \widehat{\pi^*_{[\frac{r}{s}]}\varphi_0} = (\iota_{[\frac{r}{s}]})_*\widehat{\varphi_0}, \tag{7.4}$$

$$\widehat{(\pi_{[\frac{r}{s}]})_*\varphi} = \iota^*_{[\frac{r}{s}]}\widehat{\varphi}, \tag{7.5}$$

and also $(\pi^*_{[\frac{r}{s}]}\widehat{\varphi_0})(\cdot) = (\widehat{\iota_{[\frac{r}{s}]})_*\varphi_0}(-\cdot)$. Note that for N prime, one has (dividing by $\frac{\phi_{\text{euler}}(N)}{2}$ = the number of cusps "in" each $\mathbb{Z}/N\mathbb{Z}$ subgroup) $\widehat{\varphi} = \frac{2}{\phi_{\text{euler}}(N)}\sum_{\sigma\in\kappa(N)}(\iota_{[\frac{r}{s}]})_*(\iota_{[\frac{r}{s}]})^*\widehat{\varphi}$ implies that $\varphi = \frac{2}{N\cdot\phi_{\text{euler}}(N)}\sum_\sigma (\pi_{[\frac{r}{s}]})^*(\pi_{[\frac{r}{s}]})_*\varphi$ for $\varphi\in\Phi_2(N)^\circ$ but this does not hold for N *not* prime. Finally, if $\mu_a : \mathbb{Z}/N\mathbb{Z} \xrightarrow{\cong} \mathbb{Z}/N\mathbb{Z}$ is multiplication (mod N) by $a\in(\mathbb{Z}/N\mathbb{Z})^*$, one has

$$\widehat{\mu_a^*\varphi_0} = \mu^*_{a^{-1}}\widehat{\varphi_0}. \tag{7.6}$$

One wonders why undergraduates do not learn these discrete Fourier transforms in linear algebra (or at least before the continuous/L^2/L^1 theory), considering that future mathematicians might use them in number theory and engineers in MATLAB. Moreover, together with Bernoulli numbers and polynomials, they have a very attractive application to computing series yielding rational multiples of powers of π. Recall that the Bernoulli numbers

$$B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = \frac{-1}{30}, B_5 = 0, \text{ etc.}$$

satisfy $\sum_{k=0}^{\infty} B_k \frac{t^k}{k!} = \frac{t\mathrm{e}^t}{\mathrm{e}^t - 1}$. If we define Bernoulli polynomials

$$B_k(x) := \sum_{j=0}^{k} \binom{k}{j} B_j x^{k-j}$$

(e.g., $B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x,\ B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}$) then they consequently satisfy $\sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!} = \frac{t\mathrm{e}^{t(1+x)}}{\mathrm{e}^t - 1}$. One also has (for $k \geq 2$) $B_k = \begin{cases} \frac{-k!}{(2\pi\mathrm{i})^k} 2\zeta(k), & k \text{ even} \\ 0, & k \text{ odd} \end{cases}$ and correspondingly $B_k(x) = \frac{(-1)^{k-1}k!}{(2\pi\mathrm{i})^k} \sum'_{m\in\mathbb{Z}} \frac{\mathrm{e}^{-2\pi\mathrm{i}mx}}{m^k}$.

For us the key calculation is: given $\varphi \in \Phi(N)$ (and $\ell \geq 1$),

$$\begin{aligned} \sum_{a=0}^{N-1} \varphi(a) B_{\ell+2}\left(\frac{a}{N}\right) &= \frac{(-1)^{\ell+1}(\ell+2)!}{(2\pi\mathrm{i})^{\ell+2}} \sum_{a=0}^{N-1} \varphi(a) \sum_{m\in\mathbb{Z}}{}' \frac{\mathrm{e}^{-2\pi\mathrm{i}m\frac{a}{N}}}{m^{\ell+2}} \\ &= \frac{(-1)^{\ell+1}(\ell+2)!}{(2\pi\mathrm{i})^{\ell+2}} \sum_{m\in\mathbb{Z}}{}' \frac{1}{m^{\ell+2}} \underbrace{\sum_{a=0}^{N-1} \varphi(a) \mathrm{e}^{-\frac{2\pi\mathrm{i}}{N}ma}}_{\widehat{\varphi}(m)} \\ &= \frac{(-1)^{\ell+1}(\ell+2)!}{(2\pi\mathrm{i})^{\ell+2}} \tilde{L}(\widehat{\varphi}, \ell+2), \end{aligned}$$

where $\tilde{L}(\widehat{\varphi}, \ell+2) := \sum'_{m\in\mathbb{Z}} \frac{\widehat{\varphi}(m)}{m^{\ell+2}}$ (thinking of $\widehat{\varphi}$ as an N-periodic function on $\mathbb{Z}$). Note that, by this calculation, if $\varphi \in \Phi^{\mathbb{Q}}(N)$ then *regardless* of rationality of $\widehat{\varphi}$, $\tilde{L}(\widehat{\varphi}, \ell+2)$ is always in $\mathbb{Q}(\ell+2)$.

Example 7.2 (For the undergraduates)**.** $N = 4$, $\varphi = 0, 1, 0, -1; \ldots \overset{\mathrm{FT}}{\mapsto} \widehat{\varphi} = 0, 2\mathrm{i}, 0, -2\mathrm{i}; \ldots$. Say we want to compute $1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \sum_{M\geq 0} \frac{(-1)^M}{(2M+1)^3}$. This is $\frac{1}{2} \cdot \frac{1}{(-2\mathrm{i})} \cdot \sum_{m\in\mathbb{Z}}{}' \frac{\widehat{\varphi}(m)}{m^3} = \frac{-1}{4\mathrm{i}} \cdot \frac{(2\pi\mathrm{i})^3}{(-1)^2 3!} \sum_{a=0}^{3} \varphi(a) B_3\left(\frac{a}{4}\right) = \frac{-8\pi^3\mathrm{i}}{-4\mathrm{i}\cdot 6} \left(B_3(\frac{1}{4}) - B_3\left(\frac{3}{4}\right)\right) = \frac{\pi^3}{3}\left(\frac{3}{64} - \left(\frac{-3}{64}\right)\right) = \frac{\pi^3}{32}$. Much more complicated rational numbers (than $\frac{1}{32}$) usually arise.

7.2.2. The horospherical map. Now we establish the central number-theoretic Lemma 7.1 which will ultimately translate to "surjectivity of residues of higher Chow cycle classes onto the cusps," hence Beilinson–Hodge. Define for $\sigma \in \kappa(N)$, $\mathbb{Q} \subseteq K \subseteq \mathbb{C}$

$$\begin{aligned} \mathsf{H}_\sigma^{[\ell]} : \Phi_2^K(N)^\circ &\to K \\ \varphi &\mapsto \frac{(-1)^\ell(\ell+1)}{(\ell+2)!} \sum_{a=0}^{N-1} \left((\pi_\sigma)_*\varphi\right)(a) \cdot B_{\ell+2}\left(\frac{a}{N}\right). \end{aligned}$$

If the following is true for $K = \mathbb{C}$ then it holds for any K:

Lemma 7.1. $\left(\oplus_{\sigma\in\kappa(N)}\mathsf{H}_\sigma^{[\ell]}\right) : \Phi_2^K(N)^\circ \to \Upsilon_2^K(N)$ *is surjective.*

Proof. Let $(\Phi(N)^\circ \supset)$

$$\begin{aligned} \Upsilon^{[\ell]}(N) &:= \left\{\text{functions on } (\mathbb{Z}/N\mathbb{Z})^* \text{ satisfying } f(-y) = (-1)^\ell f(y)\right\} \\ &\cong \left\{\text{functions on those cusps } (-s,r) \text{ ``contained'' in any one } G \in \mathfrak{G}(N)\right\}. \end{aligned}$$

Writing

$$\begin{aligned} \mathsf{L}^{[\ell]} : \Phi(N)_\circ &\to \mathbb{C} \\ \xi &\mapsto -\frac{\ell+1}{(2\pi \mathrm{i})^{\ell+2}} \tilde{L}(\xi, \ell+2), \end{aligned}$$

by results of Section 7.2.1 we have

$$\bigoplus_\sigma \mathsf{H}_\sigma^{[\ell]} = \bigoplus_\sigma \mathsf{L}^{[\ell]} \circ \widehat{\ } \circ (\pi_\sigma)_* = \bigoplus_\sigma \mathsf{L}^{[\ell]} \circ \iota_\sigma^* \circ \widehat{\ } = \bigoplus_{G\in\mathfrak{G}(N)} \bigoplus_{a\in(\mathbb{Z}/N\mathbb{Z})^*} \mathsf{L}^{[\ell]} \circ \mu_a^* \circ \iota_{\sigma_G}^* \circ \widehat{\ },$$

for σ_G some choice of generator $(-s,r)$ for each $G \subset (\mathbb{Z}/N\mathbb{Z})^2$. Obviously

$$\left(\bigoplus_{G\in\mathfrak{G}(N)} \Upsilon^{[\ell]}(N)\right) \subseteq \text{image}\left\{\bigoplus_{G\in\mathfrak{G}(N)} \iota_{\sigma_G}^* : \Phi_2(N)_\circ \to \bigoplus_{\mathfrak{G}(N)} \Phi(N)_\circ\right\},$$

and $\widehat{\ } : \Phi_2(N)^\circ \to \Phi_2(N)_\circ$ is also obviously surjective; so it will suffice to check the following

Sublemma: $\left(\oplus_{a\in(\mathbb{Z}/N\mathbb{Z})^*} \mathsf{L}^{[\ell]} \circ \mu_a^*\right)\Big|_{\Upsilon(N)} : \Upsilon^{[\ell]}(N)\,(\subseteq \Phi(N)_\circ) \xrightarrow{\cong} \Upsilon^{[\ell]}(N)$.

Proof. Working over $\mathbb{C}$, $\Upsilon^{[\ell]}(N)$ is spanned (depending on ℓ) by even or odd Dirichlet characters (mod N) $\{\chi_i\}_{i=1}^{\frac{1}{2}\phi_{\text{euler}}(N)}$. These satisfy (by definition) $(\mu_a^*\chi)(b) = \chi(a)\cdot\chi(b)$. So $\left(\mathsf{L}^{[\ell]} \circ \mu_a^*\right)(\chi_i) = \chi_i(a)\cdot \mathsf{L}^{[\ell]}(\chi_i)$, and by Neukirch [62, Section VII.2] $\tilde{L}(\chi_i, \ell+2) \neq 0$. We may therefore *divide* $\frac{\chi_i(\cdot)}{\mathsf{L}^{[\ell]}(\chi_i)} =: \tilde{\chi}_i(\cdot)$, so that $\left(\mathsf{L}^{[\ell]} \circ \mu_a^*\right)(\tilde{\chi}_i) = \chi_i(a)$. Thus each χ_i appears in the image (in $\Upsilon^{[\ell]}(N)$) of this map, and since they span $\Upsilon^{[\ell]}(N)$ we are done. □

We can be more explicit and produce a "rational basis for the surjection" of Lemma 7.1 (onto $\Upsilon^{[\ell]}(N)$).

Proposition 7.3. *There exists a unique "fundamental vector" $\varphi_N^{[\ell]} \in \Phi^{\mathbb{Q}}(N)^\circ$ satisfying $\mathsf{H}_{\sigma'}^{[\ell]}\left(\frac{1}{N}\pi_\sigma^*(\varphi_N^{[\ell]})\right) = \delta_{\sigma\sigma'}$ $(\forall \sigma, \sigma' \in \kappa(N))$.*

Proof. The proof of the sublemma implies the existence of $\varphi \in \Phi(N)^\circ$ with (i) $\mathsf{L}^{[\ell]}(\widehat{\varphi}) = 1$, (ii) $\mathsf{L}^{[\ell]}(\widehat{\mu_a^*\varphi}) = 0$ $\forall a \in (\mathbb{Z}/N\mathbb{Z})^* \backslash \{\pm 1\}$, and (iii) $\widehat{\varphi}(n) = 0$ $\forall n$ not relatively prime to N. If we ask that (iv) $\varphi(-a) = (-1)^\ell \varphi(a)$ $(\forall a)$, then φ is uniquely determined. Conditions (i)–(iii) translate to (somewhat redundantly expressed) $\mathbb{Q}$-linear conditions on φ:

(i′) $1 = \frac{(-1)^\ell(\ell+1)}{(\ell+2)!} \sum_{c=0}^{N-1} \varphi(c) B_{\ell+2}\left(\frac{c}{N}\right)$.

(ii′) $0 = \sum_{c=0}^{N-1} \varphi(ac) B_{\ell+2}\left(\frac{c}{N}\right)$ $(\forall a \not\equiv \pm 1 \ (N)$ with $\gcd(a, N) = 1)$.

(iii′) $0 = \sum_{b=0}^{r-1} \varphi(a + b\frac{N}{r})$ $(\forall a = 0, \ldots, \frac{N}{r} - 1)$ for each $r (\neq 1, N)$ dividing N.

Then $\mathsf{H}_{\sigma'}^{[\ell]}(\frac{1}{N}\pi_\sigma^*\varphi) = \mathsf{L}^{[\ell]}(\frac{1}{N}\widehat{\pi_{\sigma'_*}\pi_\sigma^*\varphi}) = \mathsf{L}^{[\ell]}(\iota_{\sigma'}^*\iota_{\sigma_*}\widehat{\varphi})$, which is 0 if σ' "belongs to a different subgroup" than σ (using condition (iii) if N is not prime); otherwise it becomes $\mathsf{L}^{[\ell]}(\mu_{a^{-1}}^*\widehat{\varphi})$ $(= 0$ if $\sigma' \not\equiv \sigma [\leftrightarrow a \not\equiv \pm 1]$, by (ii); or $= 1$ by (i)). □

Example 7.3. Here are a few of the fundamental vectors for $\ell = 1, 2$ (where we list the values $\varphi(0), \ldots, \varphi(N-1)$)

$$\varphi_3^{[1]} = 0, -\frac{81}{2}, \frac{81}{2}; \quad \varphi_4^{[1]} = 0, -32, 0, 32; \quad \varphi_5^{[1]} = 0, -25, -\frac{25}{2}, \frac{25}{2}, 25;$$
$$\varphi_3^{[2]} = -162, 81, 81; \quad \varphi_6^{[2]} = -\frac{432}{5}, -\frac{216}{5}, \frac{216}{5}, \frac{432}{5}, \frac{216}{5}, -\frac{216}{5}.$$

7.2.3. Pontryagin products. Consider the map

$$\left(\Phi_2^{\mathbb{Q}}(N)^\circ\right)^{\otimes \ell+1} \xrightarrow{*^{\ell+1}} \Phi_2^{\mathbb{Q}}(N)^\circ$$

$$\varphi_1 \otimes \cdots \otimes \varphi_{\ell+1} \mapsto (\varphi_1 * \cdots * \varphi_{\ell+1})(m, n) := \sum_{\substack{\{m_i, n_i\} \in (\mathbb{Z}/N\mathbb{Z})^{2\ell+2} \\ \sum (m_i, n_i) \overset{(N)}{\equiv} (m,n)}} \prod_{i=1}^{\ell+1} \varphi_i(m_i, n_i)$$

which becomes Pontryagin product when $\Phi_2(N)^\circ$ is interpreted as divisors on N-torsion.

Lemma 7.2. (i) $\overset{\ell+1}{*}$ *is surjective;*

(ii) $\widehat{\varphi_1 * \cdots * \varphi_{\ell+1}} = \prod_{i=1}^{\ell+1} \widehat{\varphi}_i$.

Proof. Condition (ii) is a trivial computation.

For (i) write $\beta_N(m,n) := \begin{cases} \frac{N^2-1}{N^2} & (m,n) \equiv (0,0) \\ \frac{1}{N^2} & \text{otherwise} \end{cases}$, and let $\varphi \in \Phi_2^{\mathbb{Q}}(N)^\circ$. Then $\varphi * \underbrace{\beta_N * \cdots * \beta_N}_{\ell \text{ times}} = \varphi$. □

7.2.4. Decomposition into (p,q)-verticals. For $(p,q) \in (\mathbb{Z}/N\mathbb{Z})^2$ such that $\langle (p,q) \rangle \cong \mathbb{Z}/N\mathbb{Z}$, define in $\Phi_2^{\mathbb{Q}}(N)^\circ$ a subgroup of "(p,q)-*vertical*-degree-0" functions

$$\Phi_2^{\mathbb{Q}}(N)^\circ_{(p,q)} := \left\{ \varphi \in \Phi_2^{\mathbb{Q}}(N) \mid \sum_{a \in \mathbb{Z}/N\mathbb{Z}} \varphi(a(p,q) + (m,n)) = 0 \quad \forall (m,n) \in (\mathbb{Z}/N\mathbb{Z})^2 \right\}.$$

Inside this we have the *set*

$$\mathfrak{S}(N)_{(p,q)} := \left\{ \text{translates of the function } \varphi_{(p,q)}(m,n) := \begin{cases} -2, & (m,n) \overset{(N)}{\equiv} (0,0) \\ 1, & (m,n) \overset{(N)}{\equiv} \pm(p,q) \\ 0 & \text{otherwise} \end{cases} \right\}.$$

The next result says that $\varphi \in \Phi_2^{\mathbb{Q}}(N)^\circ$ can be written as a sum of Pontryagin products where each term contains only functions from $\mathfrak{S}(N)_{(p,q)}$ for some (p,q).

Decomposition Lemma. (i) *The map*

$$\mathbb{Q}[\mathfrak{S}(N)_{(p,q)}^{\times(\ell+1)}] \to \Phi_2^{\mathbb{Q}}(N)^\circ_{(p,q)}$$
$$\sum a_j[(\varphi_1^{(j)}, \ldots, \varphi_{\ell+1}^{(j)})] \mapsto \sum a_j \varphi_1^{(j)} * \cdots * \varphi_{\ell+1}^{(j)}$$

is surjective $(\ell \geq 0)$;

(ii) *If* $\sigma \in \kappa(N)$ *corresponds to* $\begin{pmatrix} p & q \\ -s & r \end{pmatrix} \in SL_2(\mathbb{Z})$ *(see the end of* §7.1.6*), then*

$$\Phi_2^{\mathbb{Q}}(N)^\circ_{(p,q)} \supset \pi_\sigma^* \Phi^{\mathbb{Q}}(N)^\circ;$$

(iii) $\oplus_{G \in \mathfrak{G}(N)} \pi_{\sigma_G}^* \Phi^{\mathbb{Q}}(N)^\circ \twoheadrightarrow \Phi_2^{\mathbb{Q}}(N)^\circ$ *(σ_G as in the proof of Lemma* 7.1*).*

Proof. (i) First note $\otimes^{\ell+1}\Phi_2^{\mathbb{Q}}(N)^{\circ}_{(p,q)} \xrightarrow{*^{\ell+1}} \Phi_2^{\mathbb{Q}}(N)^{\circ}_{(p,q)}$ using

$$\beta_N^{(p,q)}(m,n) := \begin{cases} \frac{N-1}{N}, (m,n) \overset{(N)}{\equiv} (0,0), \\ \frac{1}{N}, \quad (m,n) \in \langle (p,q)\rangle \setminus \{(0,0)\}, \\ 0 \quad \text{otherwise} \end{cases}$$

in place of β_N above; so it suffices to prove case $\ell = 0$. Put $\varphi^{\{k\}}_{(p,q)}(m,n) := \varphi_{(p,q)}((m,n) - k(p,q))$ and $\Delta_{(p,q)}(m,n) := \begin{cases} 1, \quad (m,n) \overset{(N)}{\equiv} (p,q) \\ -1, \quad (m,n) \overset{(N)}{\equiv} (0,0) \\ 0 \quad \text{otherwise} \end{cases}$. Translates of $\Delta_{(p,q)}$ clearly generate $\Phi_2^{\mathbb{Q}}(N)^{\circ}_{(p,q)}$, and $\sum_{k=1}^{N} \frac{k}{N}\varphi^{\{k\}}_{(p,q)} = \Delta_{(p,q)}$.

(ii) Obvious.

(iii) Follows from

$$\widehat{\Phi_2(N)^{\circ}} = \Phi_2(N)_{\circ} = \sum_{G\in\mathfrak{G}(N)} (\iota_{\sigma_G})_*(\Phi(N)_{\circ}) = \sum_{G\in\mathfrak{G}(N)} \widehat{(\pi_{\sigma_G})^*(\Phi(N)}^{\circ}).$$

□

7.2.5. Functions with divisors supported on N-torsion. Writing $\mathcal{E}(N)$, $\mathcal{E}$ for $\mathcal{E}^{[1]}(N)$, $\mathcal{E}^{[1]}$ we have

$$\begin{array}{ccccc} \mathcal{E} & \xrightarrow[\mathcal{P}_N]{} & \mathcal{E}(N) & \hookrightarrow & \overline{\mathcal{E}}(N) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi(N)} & & \downarrow{\scriptstyle \overline{\pi}(N)} \\ \mathfrak{H} & \xrightarrow[\rho_N]{} & Y(N) & \hookrightarrow & \overline{Y}(N). \end{array}$$

Let $U(N) \overset{\jmath(N)}{\subset} \mathcal{E}(N)$ be the complement of the N^2 N-torsion sections; there is a "relative divisor" map[20]

$$\mathcal{O}^*(U(N)) \xrightarrow{\div} \Phi_2(N)^{\circ}$$
$$f \mapsto \varphi_f$$

(which ignores divisor components supported on the singular fibers over cusps $\{\hat{E}_{y_0}(N) \,|\, y_0 \in \kappa(N)\}$). Now assume p, q have been chosen as in the

[20]Note that $\mathcal{O}^*(U(N)) \subset \mathbb{C}(\overline{\mathcal{E}}(N))^*$.

beginning of Section 7.2.4. Taking any r, s such that $\gamma := \left(\begin{smallmatrix} p & q \\ -s & r \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, define $\mathfrak{F}(N)_\gamma :=$

$$\left\{ f \in \mathcal{O}^*(U(N)) \;\middle|\; \begin{array}{c} \mathcal{P}_N^* f \text{ has ``}(p,q)\text{-vertical'' } T_{\mathcal{P}_N^* f} \text{ over the hyperbolic} \\ \text{geodesic } (\tau \in)\, \mathcal{A}_\gamma := \{ \frac{ibr-q}{ibs+p} \,|\, b \in \mathbb{R}^+ \} \subset \mathfrak{H} \\ \text{connecting } [\frac{r}{s}] \text{ and } [\frac{-q}{p}], \text{in the sense} \\ \text{that its support in } E_\tau \text{ lies in one connected} \\ \text{component of } W_\tau^{(p,q)}(N) := \\ \{ \xi(p\tau+q) + \frac{m\tau+n}{N} \,|\, m,n \in {}^{\mathbb{Z}}\!/\!{}_{N\mathbb{Z}},\ \xi \in {}^{\mathbb{C}}\!/\!{}_{\mathbb{Z}} \} \end{array} \right\}.$$

Lemma 7.3. (i) $\div$ *is surjective.*

(ii) $\div(\mathfrak{F}(N)_\gamma) \supset \mathfrak{S}(N)_{(p,q)}$.

Remark 7.1. (a) Together with the Decomposition Lemma, (ii) ensures that we can actually compute with the KLM formula (because we are able to work with functions with known T_f on π^{-1} of the arc $\mathcal{A}_\gamma$).

(b) It is obvious that the definition of $\mathfrak{F}(N)_\gamma$ only depends on the coset of γ in $SL_2(\mathbb{Z})/\Gamma(N)$, but we will not need this.

Proof. (i) Working on $\mathcal{E}$, we will construct a meromorphic function $f \in \operatorname{im}(\mathcal{P}_N^*)$ with divisor $\sum_{(m,n)\in({}^{\mathbb{Z}}/{}_{N\mathbb{Z}})^2} a_{m,n} \left[\frac{m\tau+n}{N}\right]$ for any given $\{a_{m,n}\}_{(m,n)\in({}^{\mathbb{Z}}/{}_{N\mathbb{Z}})^2}$ satisfying $\sum a_{m,n} m \overset{(N)}{\equiv} 0 \overset{(N)}{\equiv} \sum a_{m,n} n$ and $\sum a_{m,n} = 0$. In fact, we can choose $\{\tilde{a}_{m,n}\}_{(m,n)\in\mathbb{Z}^2}$ (all but finitely many zero) "lifting" $\{a_{m,n}\}$ such that $\sum \tilde{a}_{m,n} m = 0 = \sum \tilde{a}_{m,n} n$ exactly; this leads (following [13, p. 8.8]) to the construction of a function f_0 on $\mathfrak{H} \times \mathbb{C}$ descending to $\mathcal{E}$:

$$f_0(\tau, z) := \prod_{k\in\mathbb{Z}} \prod_{(m,n)\in\mathbb{Z}^2} \left(1 - e^{2\pi i(k\tau + z - \frac{m\tau+n}{N})}\right)^{a_{m,n}}. \tag{7.7}$$

Factoring f_0 if necessary, we may assume that some $(m_0, n_0) \in (\mathbb{Z}/N\mathbb{Z})^2$ has $a_{m_0,n_0} = 0$; then

$$f(\tau, z) := \frac{f_0(\tau, z)}{f_0\left(\tau, \frac{m_0\tau+n_0}{N}\right)} \text{ descends to } \mathcal{E}(N). \tag{7.8}$$

(ii) We will use the proof of (i) to construct $f \in \mathfrak{F}(N)_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)}$ with

$$\varphi_f(m,n) = \begin{cases} -1, & (m,n) \overset{(N)}{\equiv} (\pm 1, 0) \\ 2, & (m,n) \overset{(N)}{\equiv} (0,0) \\ 0, & \text{otherwise} \end{cases};$$

then the idea is simply to translate and pull back (using the action of $\left(\begin{smallmatrix} p & q \\ -s & r \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ on $\mathcal{E}(N)$ induced from that on $\mathcal{E}$) this f.

Taking $\tilde{a}_{0,0} = 2$, $\tilde{a}_{1,0} = \tilde{a}_{-1,0} = -1$ (all other $\tilde{a}_{m,n} = 0$) in (7.7), one easily computes that (with $\tau = \mathrm{i}y \in \mathrm{i}\mathbb{R}^+$) $f_0(\mathrm{i}y, \mathrm{i}Y) \in \mathbb{R}^{\leq 0}$ for $Y \in (\frac{-y}{N}, \frac{y}{N})$. So on each $E_{\tau=\mathrm{i}y}$, $|T_{f_0}| \supset \{z = \mathrm{i}Y \mid Y \in [\frac{-y}{N}, \frac{y}{N}]\}$, while f_0 is of *degree* 2; it follows that T_{f_0} is just the sum of two directed line segments, from $\pm\frac{\tau}{N} (= \pm\frac{\mathrm{i}y}{N})$ to 0. In (7.8), we take $(m_0, n_0) = (0, 1)$, and check that $T_f = T_{f_0}$ over $\tau = \mathrm{i}y$ ($y \in \mathbb{R}^+$), or equivalently that $f(\mathrm{i}y, \frac{1}{N}) \in \mathcal{R}^+$. To do this, observe that $\overline{f_0(\mathrm{i}y, \overline{z})}$ is (a) holomorphic and has (b) the same divisor as $f_0(\mathrm{i}y, z)$ and (c) the same leading coefficient of power series expansion at $z = 0$ ($f_0 = Cz^2 + \cdots$, where $[0 \neq] C \in \mathbb{R}^+$ since T_{f_0} is vertical). Thus $\overline{f_0(\overline{z})} = f_0(z)$, which implies $\overline{f_0(\frac{1}{N})} = f_0(\frac{1}{N})$ $(\in \mathbb{R})$. Since $\frac{1}{N} \notin T_{f_0}$, $f_0(\frac{1}{N}) \in \mathbb{R}^+$. $\square$

Now we can obtain meromorphic functions on $\mathcal{E}^{[\ell]}(N)$ by noting that $\mathcal{E}^{[\ell]}(N) = \times^{\ell}_{Y(N)} \mathcal{E}(N)$, $\mathcal{E}^{[\ell]} = \times^{\ell}_{\mathfrak{H}} \mathcal{E}$, and (by abuse of notation) writing the projections to these factors

$$\begin{array}{ccc} \mathcal{E}^{[\ell]}(N) & \xrightarrow[z_i]{\hspace{2em}} & \mathcal{E}(N) \\ & \searrow \quad \swarrow & \\ & Y(N) & \end{array}$$

so that $f(z_i)$ denotes $z_i^* f$, etc.

7.3. Construction of the Eisenstein symbols

7.3.1. Eisenstein series. Since the cycle-class computation (Section 8.1) will show that these series actually yield modular forms, we will not bother proving this directly. Note that for the double sums $\sum'_{m,n}$ means to omit $(m, n) = (0, 0)$.

For $N \geq 3$ and $\ell \in \mathbb{Z}^+$ define

$$\mathbb{E}_{\ell+2}(\Gamma(N)) := \left\{ F \in \mathcal{O}(\mathfrak{H}) \;\middle|\; F \text{ of form } \sum_{(m,n)\in\mathbb{Z}^2}{}' \frac{\psi(m,n)}{(m\tau+n)^{\ell+2}} \text{ for } \psi \in \Phi_2(N) \right\}.$$

(The series is necessarily convergent.)

Lemma 7.4. *The map*

$$\Phi_2(N)^\circ \xrightarrow{\mathsf{E}^{[\ell]}} \mathbb{E}_{\ell+2}(\Gamma(N))$$

defined by

$$\varphi \mapsto E_\varphi^{[\ell]}(\tau) := \frac{-(\ell+1)}{(2\pi \mathrm{i})^{\ell+2}} \sum_{(m,n)\in\mathbb{Z}^2}{}' \frac{\widehat{\varphi}(m,n)}{(m\tau+n)^{\ell+2}}$$

is surjective.

Proof. Let $\psi_0 := \begin{cases} N^{\ell+2}-1, & (m,n) \overset{(N)}{\equiv} (0,0) \\ -1 & \text{otherwise} \end{cases}$; then $\sum' \frac{\psi_0(m,n)}{(m\tau+n)^{\ell+2}}$ is obviously 0. This implies that we may assume $\psi \in \Phi_2(N)_\circ$ ($\implies \psi = \widehat{\varphi}$, $\varphi \in \Phi_2(N)^\circ$) in the definition. □

Put $\mathbb{E}^{\mathbb{Q}}_{\ell+2}(\Gamma(N)) := \mathsf{E}^{[\ell]}\left(\Phi_2^{\mathbb{Q}}(N)^\circ\right)$. (Clearly $\mathbb{E}_{\ell+2} = \mathbb{E}^{\mathbb{Q}}_{\ell+2} \otimes \mathbb{C}$.)

Lemma 7.5. *For $E_\varphi^{[\ell]} \in \mathbb{E}^{\mathbb{Q}}_{\ell+2}(\Gamma(N))$, $\lim_{\tau\to \mathrm{i}\infty} E_\varphi^{[\ell]}(\tau) = \mathsf{H}^{[\ell]}_{[\mathrm{i}\infty]}(\varphi)$ $(\in \mathbb{Q})$.*

Proof. $\lim_{\tau\to i\infty} \sum'_{m,n} \frac{\widehat{\varphi}(m,n)}{(m\tau+n)^{\ell+2}} = \sum'_n \frac{\widehat{\varphi}(0,n)}{n^{\ell+2}} = \sum'_n \frac{(\iota^*_{[\mathrm{i}\infty]}\widehat{\varphi})(n)}{n^{\ell+2}} = \sum'_n \frac{\widehat{\pi_{[\mathrm{i}\infty]_*}\varphi}(n)}{n^{\ell+2}}$

$$= \tilde{L}(\widehat{\pi_{[\mathrm{i}\infty]_*}\varphi}, \ell+2) = \frac{-(2\pi\mathrm{i})^{\ell+2}}{\ell+1}\mathsf{H}^{[\ell]}_{[\mathrm{i}\infty]}(\varphi),$$

by Sections 7.2.1 and 7.2.2. □

7.3.2. Group actions. Writing $\mathfrak{S}_\ell$ for the symmetric group, let $\mathcal{G} := \mathfrak{S}_\ell \ltimes (\mathbb{Z}/2\mathbb{Z})^\ell$ act on $\mathfrak{H} \times \mathbb{C}^\ell$ by

$$(c, \underline{\epsilon})(\tau;\, z_1, \ldots, z_\ell) := (\tau;\, (-1)^{\epsilon_1} z_{c(1)}, \ldots, (-1)^{\epsilon_\ell} z_{c(\ell)});$$

this descends to $\mathcal{E}^{[\ell]}$ and $\mathcal{E}^{[\ell]}(N)$. Fixing N, let $\Lambda^\ell := (\mathbb{Z}/N\mathbb{Z})^{2\ell}$ act on $\mathcal{E}^{[\ell]}$ via translations

$$\mathrm{tr}_{\underline{\lambda}}(\tau;\, z_1, \ldots, z_\ell) := \left(\tau;\, z_1 + \frac{\lambda_1\tau+\lambda_2}{N}, \ldots, z_\ell + \frac{\lambda_{2\ell-1}\tau+\lambda_{2\ell}}{N}\right);$$

this descends to $\mathcal{E}^{[\ell]}(N)$.

7.3.3. Inclusions and open subsets of

$$\begin{array}{ccccc} \mathcal{E}^{[\ell]}(N) & \supset & \bar{U}^{[\ell]}(N) & \supset & \tilde{U}^{[\ell]}(N) \\ & & \cup & & \cup \\ & & U^{[\ell]}(N) & \supset & \hat{U}^{[\ell]}(N) \end{array}$$

(to be defined). Writing "FP" for fixed points, set

$$\hat{W}_N^{[\ell]} := \bigcup_{\underline{\lambda}\in\Lambda^\ell} \mathrm{tr}_{\underline{\lambda}} \left\{ \cup_{(c,\underline{\epsilon})\in\mathcal{G}} \mathrm{FP}((c,\underline{\epsilon})) \right\} \subset \mathcal{E}^{[\ell]} , \qquad \hat{W}^{[\ell]}(N) := \mathcal{P}_N(W_N^{[\ell]}),$$
$$\hat{U}^{[\ell]}(N) := \mathcal{E}^{[\ell]}(N)\backslash \hat{W}^{[\ell]}(N).$$

Next, generalize $U(N)$ in two different ways:

$$U^{[\ell]}(N) := \times^{\ell}_{Y(N)} U(N), \quad \bar{U}^{[\ell]}(N) := \mathcal{E}^{[\ell]}(N)\backslash\{N^{2\ell}\ N\text{-torsion sections}\}.$$

The inclusion $\mathfrak{H}\times\mathbb{C}^\ell \hookrightarrow \mathfrak{H}\times\mathbb{C}^{\ell+1}$ given by

$$(z_1,\ldots,z_\ell)\mapsto(-z_1, z_1-z_2,\ldots,z_{\ell-1}-z_\ell, z_\ell) =: (u_1,\ldots,u_{\ell+1})$$

descends to define maps $\iota:\ \mathcal{E}^{[\ell]}\hookrightarrow\mathcal{E}^{[\ell+1]}$ and

$$\iota(N):\ \mathcal{E}^{[\ell]}(N)\hookrightarrow\mathcal{E}^{[\ell+1]}(N).$$

Finally, put

$$\tilde{U}^{[\ell]}(N) := \iota(N)^{-1}\left(U^{[\ell+1]}(N)\right).$$

To summarize,

$$\left.\begin{array}{c} \bar{U}^{[\ell]}(N) \\ \hat{U}^{[\ell]}(N) \\ \tilde{U}^{[\ell]}(N) \\ U^{[\ell]}(N) \end{array}\right\} \begin{array}{c}\text{means the}\\ \text{"complement of}\\ \text{translates of"}\end{array} \left\{\begin{array}{c} z_1=\cdots=z_\ell=0, \\ \text{all } z_i=\pm z_j,\ z_i=0, \\ z_1=0,\ z_1=z_2,\ldots,\ z_{\ell-1}=z_\ell,\ z_\ell=0, \\ z_1=0,\ z_2=0,\ldots,\ z_\ell=0 \end{array}\right.$$

and makes sense in $\mathcal{E}^{[\ell]}$ or $\mathcal{E}^{[\ell]}(N)$ (where in $\mathcal{E}^{[\ell]}$ these open sets are denoted instead $\bar{U}_N^{[\ell]}$, $\hat{U}_N^{[\ell]}$, etc.). Denote the U-complements (i.e., the translates of the sets on the r.h.s.) by $\bar{W}$, $\hat{W}$, etc.

7.3.4. Completion of symbols. Write $\mathbb{Q}[\mathcal{O}^*(U(N))]$ for the $\otimes\mathbb{Q}$ free-abelian group on the set of elements of $\mathcal{O}^*(U(N))$, and recall $\square := \mathbb{P}^1\backslash\{1\}$.

To each monomial $\mathbf{f} := f_1 \otimes \cdots \otimes f_{\ell+1} \in \otimes^{\ell+1}\mathbb{Q}[\mathcal{O}^*(U(N))]$ we associate the graph cycle $\{\mathbf{f}\} :=$

$$\{f_1(u_1), \ldots, f_{\ell+1}(u_{\ell+1})\} := \Big\{(\tau;\, \underline{u};\, f_1(\tau, u_1), \ldots, f_{\ell+1}(\tau, u_{\ell+1}))\,\Big|\, (\tau, \underline{u}) \in U^{[\ell+1]}(N)\Big\} \subset U^{[\ell+1]}(N) \times \square^{\ell+1}.$$

Its pullback by $\iota(N)$ should be thought of as the symbol

$$\iota^*\{\mathbf{f}\} := \{f_1(-z_1), f_2(z_1 - z_2), \ldots, f_\ell(z_{\ell-1} - z_\ell), f_{\ell+1}(z_\ell)\}, \tag{7.9}$$

which is evidently in good position (i.e. yields a higher Chow precycle) over all of $\bar{U}^{[\ell]}(N)$. To kill $\partial_{\mathcal{B}}$ of this symbol in $\hat{W}^{[\ell]}(N)$, we flip it about components of $\hat{W}^{[\ell]}(N)$ and subtract the result: writing $\tilde{\mathcal{G}} := \mathcal{G} \ltimes \Lambda^\ell$, define

$$\tilde{\mathcal{G}}^* := \frac{1}{\ell! 2^\ell N^{2\ell}} \left\{ \sum_{(c,\underline{\epsilon},\underline{\lambda}) \in \tilde{\mathcal{G}}} (-1)^{\mathrm{sgn}(\sigma) + \sum \epsilon_i} (c, \underline{\epsilon})^* (\mathrm{tr}_{\underline{\lambda}})^* \right\},$$

and $\tilde{\mathcal{G}}_0^*$ if signs are *removed*. (There is also $\mathcal{G}^*$, defined by forgetting the $\frac{1}{N^{2\ell}} \sum_{\underline{\lambda}} (\mathrm{tr}_{\underline{\lambda}})^*$ part.)

Now consider the diagram

$$\begin{array}{ccc}
\otimes^{\ell+1}\mathbb{Q}\left[\mathcal{O}^*(U(N))\right] & & \\
\downarrow \{\cdot\} & & \\
Z^{\ell+1}_{\partial_{\mathcal{B}}\text{-cl.}}\left(U^{[\ell+1]}(N), \ell+1\right)_{\{\sum u_i = 0\}} & & \tilde{\mathcal{G}}^* \circ \iota(N)^* \circ \{\cdot\}, \text{ followed by Zariski closure} \\
\downarrow \iota(N)^* & & \\
Z^{\ell+1}_{\partial_{\mathcal{B}}\text{-cl.}}\left(\tilde{U}^{[\ell]}(N), \ell+1\right) & & \\
\downarrow \tilde{\mathcal{G}}^* & & \downarrow \\
\left[Z^{\ell+1}_{\partial_{\mathcal{B}}\text{-cl.}}\left(\hat{U}^{[\ell]}(N), \ell+1\right)\right]^{\tilde{\mathcal{G}}} & \xleftarrow{\text{restriction}} & \left[Z^{\ell+1}_{\partial_{\mathcal{B}}\text{-cl.}}\left(\bar{U}^{[\ell]}(N), \ell+1\right)\right]^{\tilde{\mathcal{G}}}
\end{array}$$

in which we denote the images of $\mathbf{f}$ as follows:

$$\begin{array}{ccc}
\mathbf{f} & \longmapsto & \\
\downarrow & & \\
\{\mathbf{f}\} & & \\
\downarrow & & \\
\iota^*_{(N)}\{\mathbf{f}\} & & \\
\downarrow & & \downarrow \\
Z_{\mathbf{f}} & \longleftarrow & \overline{Z_{\mathbf{f}}}.
\end{array}$$

Unless $\alpha_1 + \cdots + \alpha_{\ell+1} \neq 0 \quad \forall\{\alpha_1, \ldots, \alpha_{\ell+1}\} \in |(f_1)| \times \cdots \times |(f_{\ell+1})|$, extending the cycle over the N-torsion sections to $\mathcal{E}^{[\ell]}(N)$ requires a "move" (by adding a $\partial_{\mathcal{B}}$-coboundary). This condition just says that $0 \notin |(f_1)| * \cdots * |(f_{\ell+1})|$. Such a move always exists, as

$$\left[CH^{\ell+1}(\mathcal{E}^{[\ell]}(N), \ell+1)\right]^{\tilde{\mathcal{G}}} \xrightarrow[\text{restriction}]{\cong} \left[CH^{\ell+1}(\hat{U}^{[\ell]}(N), \ell+1)\right]^{\tilde{\mathcal{G}}}.$$

Of course this eliminates well-definedness on the level of *precycles* (but not cycle-class) for the resulting

$$\mathfrak{Z}_{\mathbf{f}} \in Z^{\ell+1}_{\partial_{\mathcal{B}}\text{-cl.}}(\mathcal{E}^{[\ell]}(N), \ell+1).$$

Proposition 7.4. *We have a well-defined map of precycles*

$$\begin{aligned} \otimes^{\ell+1}\mathbb{Q}[\mathcal{O}^*(U(N))] &\longrightarrow [Z^{\ell+1}_{\partial_{\mathcal{B}}\text{-}cl.}(\bar{U}^{[\ell]}(N), \ell+1)]^{\tilde{\mathcal{G}}} \\ \boldsymbol{f} &\longmapsto \overline{Z_{\boldsymbol{f}}}. \end{aligned}$$

Going modulo relations, this induces a well-defined map

$$\begin{array}{ccc}
\mathcal{O}^*(U(N))^{\otimes \ell+1} & \longrightarrow & \left[CH^{\ell+1}(\bar{U}^{[\ell]}(N), \ell+1)\right]^{\tilde{\mathcal{G}}} \\
& \searrow & \uparrow \cong \\
\boldsymbol{f} & & \left[CH^{\ell+1}(\mathcal{E}^{[\ell]}(N), \ell+1)\right]^{\tilde{\mathcal{G}}} \\
& \searrow & \\
& & \langle \mathfrak{Z}_{\boldsymbol{f}} \rangle .
\end{array}$$

8. Fundamental class computations

8.1. Cycle class of the Eisenstein symbol

8.1.1. More Fourier theory. Now we introduce "fiberwise Fourier series" for

$$\begin{array}{c} \mathcal{E} \\ \pi \downarrow \; \big) \, \mathfrak{e} := \text{zero section.} \\ \mathfrak{H} \end{array}$$

Writing coordinates $(\tau, u = x + y\tau)$ on $\mathcal{E}$, and $\nu := \bar{\tau} - \tau$, we note that du is only well-defined in $\Omega^1(\mathcal{E}/\mathfrak{H})$, whereas

$$\widetilde{du} := du - \frac{\bar{u} - u}{\nu} d\tau \in A^{1,0}(\mathcal{E}) \quad [\text{resp. } \widetilde{d\bar{u}} := \overline{\widetilde{du}} \in A^{0,1}(\mathcal{E})]$$

make sense on $\mathcal{E}$.

Let $\Gamma := \Gamma(\mathfrak{H}, R^1\pi_*\mathbb{Z}) \cong \mathbb{Z}\langle[\alpha], [\beta]\rangle$, so that $\gamma = m[\beta] + n[\alpha] =$"$(m, n)$"$\in \Gamma$ has period $\omega(\gamma) := \pi_*(du \cdot \delta_\gamma) = m\tau + n$ against du; and write

$$\overline{\chi_\gamma}(u) := \exp(2\pi\mathrm{i}(mx - ny)), \quad d\overline{\chi_\gamma}(u) = \frac{2\pi\mathrm{i}}{\nu}\{\overline{\omega(\gamma)}du - \omega(\gamma)d\bar{u}\}\overline{\chi_\gamma}.$$

Associate to a current $\mathcal{K} \in \mathcal{D}^M(\mathcal{E})$ "Fourier coefficients"

$$\hat{\mathcal{K}}(\gamma) := \begin{cases} \pi_*(\mathcal{K} \cdot \overline{\chi_\gamma}) \in \mathcal{D}^{M-2}(\mathfrak{H}), & M \geq 2, \\ \nu^{-1}\pi_*(\mathcal{K} \cdot \overline{\chi_\gamma}\widetilde{du} \wedge \widetilde{d\bar{u}}) \in \mathcal{D}^M(\mathfrak{H}), & M < 2 \end{cases}$$

for each $\gamma \in \Gamma$. (Note: $\nu^{-1} du \wedge d\bar{u} = dx \wedge dy$.)

Lemma 8.1. (i) *If $\mathcal{K} \in A^M(\mathcal{E})$ $(M < 2)$ then*

$$\mathfrak{e}^*\mathcal{K} = \sum_{\gamma \in \Gamma} \hat{\mathcal{K}}(\gamma),$$

and the r.h.s. is absolutely convergent.

(ii) *Recalling the notation of Section* 7.2.5,[21] *if* $\mathcal{K} \in \mathcal{D}^0(E_\tau)$ *is a smooth function on the complement of* $W_\tau^{(p,q)}(N)\backslash\{$*connected component of* $u = 0\}$, *then*

$$\mathcal{K}(0) = \mathfrak{e}^*\mathcal{K} = \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}}^{P.V.} \hat{\mathcal{K}}(jp - ks, jq + kr)$$

where $\sum_{j\in\mathbb{Z}}^{P.V.} := \lim_{J\to\infty} \sum_{j=-J}^{J}$ *(or alternatively, add* $\pm j$ *terms then sum* $j \geq 0$*; obviously the singularities are* L^1*-integrable since* $\mathcal{K}$ *is a current).*

Proof. Condition (i) is just the statement "$\mathcal{K}(0) =\{$inverse FT evaluated at $0\}= \sum\{$Fourier coefficients$\}$" for smooth functions.

(ii) Say $(p, q) = (1, 0)$, $M = 0$. Then (working on some E_τ) put $G_k(x) := \int_0^1 \mathcal{K}(x, y)\mathrm{e}^{-2\pi i n y} dy \in \mathcal{D}^0(\mathbb{C}/\mathbb{Z})$; this restricts to a *smooth function* on the complement of $\{\frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N}\}$. By Wilcox and Myers [84, Corollary 41.4] $G_k(0) = \sum_{j\in\mathbb{Z}}^{\text{P.V.}} \widehat{G_k}(j) = \sum_{j\in\mathbb{Z}}^{\text{P.V.}} \int_0^1 G_k(x)\mathrm{e}^{2\pi \mathrm{i} jx} dx \overset{\text{Fubini}}{=\!=\!=} \sum_{j\in\mathbb{Z}}^{\text{P.V.}} \int\int_{E_\tau} \mathcal{K}(x, y) \overline{\chi_{(j,k)}} dx \wedge dy = \sum_{j\in\mathbb{Z}}^{\text{P.V.}} \hat{\mathcal{K}}(j, k)$. But the $\{G_k(0)\}$ are the Fourier coefficients of the smooth function $\mathcal{K}(0, y) \implies \mathcal{K}(0, 0) = \sum G_k(0)$. $\square$

Lemma 8.2. *If* $F \in \mathcal{D}^0(\mathcal{E})$, $\frac{\partial F}{\partial \bar{u}} \in \mathcal{D}^0(\mathcal{E})$ *is defined and* $\widehat{\frac{\partial F}{\partial \bar{u}}}(\gamma) = \frac{2\pi \mathrm{i}\omega(\gamma)}{\nu} \hat{F}(\gamma)$.

Lemma 8.3. *Let* $f \in \mathcal{O}^*(U_N)$, *and write* $\widehat{\varphi_f}(\gamma) := \widehat{\varphi_f}(m, n)$.

(i) $\widehat{\delta_{(f)}}(\gamma) = \widehat{\varphi_f}(\gamma)$*;*

(ii) $\widehat{\log f}(\gamma) = \frac{\int_{T_f} \overline{\chi_\gamma} du}{\omega(\gamma)}$ *for* $\gamma \neq (0, 0)$, *while* $\widehat{\log f}(0) = 0$ *if* $f \in \mathfrak{F}(N)_{\left(\begin{smallmatrix} p & q \\ -s & r \end{smallmatrix}\right)}$*;*

(iii) $d \log f = \alpha_f \widetilde{du} + \beta_f d\tau \implies \widehat{\alpha_f}(0) = \widehat{\beta_f}(0) = 0$ *while for* $\gamma \neq (0, 0)$,

$$\widehat{\alpha_f}(\gamma) = \frac{-\widehat{\varphi_f}(\gamma)}{\omega(\gamma)} \quad \textit{and} \quad \widehat{\beta_f}(\gamma) = \frac{\widehat{\phi_f}(\gamma)}{2\pi \mathrm{i}(\omega(\gamma))^2}.$$

[21] Warning: in this section we are no longer using γ to denote $\left(\begin{smallmatrix} p & q \\ -s & r \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$.

Proof. Lemmas 8.2 and 8.3(i), (iii) (which uses 8.2) are essentially done in [8]. For (ii) (and to get a feel for how the others go),

$$\begin{aligned}\widehat{\log f}(\gamma) &= \nu^{-1}\pi_*(\log f\,\overline{\chi_\gamma}du\wedge d\bar{u})\\ &= (2\pi\mathrm{i})^{-1}\omega(\gamma)^{-1}\pi_*\left(\log f\,\frac{2\pi\mathrm{i}}{\nu}\{\overline{\omega(\gamma)}du-\omega(\gamma)d\bar{u}\}\overline{\chi_\gamma}\wedge du\right)\\ &= (2\pi\mathrm{i})^{-1}\omega(\gamma)^{-1}\pi_*(\log f\,d\overline{\chi_\gamma}\wedge du)\\ &= (2\pi\mathrm{i})^{-1}\omega(\gamma)^{-1}\left\{-\pi_*(\overline{\chi_\gamma}d[\log f]\wedge du)+\underbrace{\pi_*(\overline{\chi_\gamma}\tfrac{df}{f}\wedge du)}_{0}\right\}\\ &= \omega(\gamma)^{-1}\pi_*(\overline{\chi_\gamma}\delta_{T_f}\wedge du)=\frac{\int_{T_f}\overline{\chi_\gamma}du}{\omega(\gamma)},\end{aligned}$$

where at the end we have used $d[\log f]=\frac{df}{f}-2\pi\mathrm{i}\delta_{T_f}$. As for $\widehat{\log f}(0)$, we have $\widehat{\log|f|}(0)=\nu^{-1}\pi_*(\log|f|\,du\wedge d\bar{u})=0$ since $\log|f|du\wedge d\bar{u}=d\log|f|\wedge d\bar{u}=d[\log|f|d\bar{u}]$. Now, using our prototype (from the proof of Lemma 7.3(ii)) for $f\in\mathfrak{F}(N)_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)}$ with $\overline{f(\bar{z})}=f(z)$, one finds ($\tau\in i\mathbb{R}^+$) that $\pi_*(\arg f\,du\wedge d\bar{u})=\pi_*(\arg\bar{f}\,du\wedge d\bar{u})=\pi_*(-\arg f\,du\wedge d\bar{u})$. (A similar argument works in general.) $\square$

Lemma 8.4. *Let* $f\in\mathfrak{F}(N)_{\left(\begin{smallmatrix}p&q\\-s&r\end{smallmatrix}\right)}$, $\gamma=(m,n)$. *Then over* $(\tau\in)\,\mathcal{A}_{\left(\begin{smallmatrix}p&q\\-s&r\end{smallmatrix}\right)}\subset\mathfrak{H}$,

$$\int_{T_f}\overline{\chi_\gamma}d\begin{Bmatrix}u\\ \bar{u}\end{Bmatrix}=\frac{p\begin{Bmatrix}\tau\\ \bar{\tau}\end{Bmatrix}+q}{2\pi\mathrm{i}(mq-np)}\widehat{\varphi_f}(\gamma)$$

if $mq-np\neq0$; *otherwise the l.h.s. is* 0.

Proof. Represent T_f as a sum of straight paths of the following type, assuming $(f)=\sum_{K=0}^{N-1}a_K\left[K\frac{p\tau+q}{N}+L\frac{-s\tau+r}{N}\right]$ ($L\in\{0,\ldots,N-1\}$ fixed). For the paths, write

$$\begin{aligned}\mathsf{P}:\ [0,1]&\hookrightarrow E_\tau\\ t&\mapsto L\frac{-s\tau+r}{N}+t(p\tau+q);\end{aligned}$$

then

$$T_f = \sum_K a_K \left\{ \frac{N-K}{N} \cdot \mathsf{P}\left([0, {}^{K}\!/\!{}_{N}]\right) - \frac{K}{N} \cdot \mathsf{P}\left([{}^{K}\!/\!{}_{N}, 1]\right) \right\} + b \cdot \mathsf{P}([0,1])$$
$$=: \tilde{T}_f + S_f,$$

where $b \in \mathbb{Q}$. We have

$$\begin{aligned}\mathsf{P}^*(\overline{\chi_\gamma} du) &= \mathrm{e}^{2\pi\mathrm{i}\left\{m\left(\frac{Lr}{N}+qt\right)-n\left(\frac{-Ls}{N}+pt\right)\right\}}(p\tau+q)dt \\ &= \mathrm{e}^{2\pi\mathrm{i}t(mq-np)}\mathrm{e}^{\frac{2\pi\mathrm{i}L}{N}(mr+ns)}(p\tau+q)dt.\end{aligned}$$

Now $\frac{1}{p\tau+q}\int_{S_f}\overline{\chi_\gamma}du = b\cdot \mathrm{e}^{\frac{2\pi\mathrm{i}L}{N}(mr+ns)}\int_0^1 \mathrm{e}^{2\pi\mathrm{i}t(mq-np)}dt$ is obviously 0 if $mq-np\neq 0$; but if $mq-np=0$ then

$$\begin{aligned}\frac{\mathrm{e}^{\frac{-2\pi\mathrm{i}L}{N}(mr+ns)}}{p\tau+q}\int_{T_f}\overline{\chi_\gamma}du &= \int_{T_f} du = \frac{1}{2\pi\mathrm{i}}\int\left(\frac{df}{f} - d[\log f]\right)\wedge du \\ &= \frac{1}{2\pi\mathrm{i}}\int(\log f)d[du] = 0.\end{aligned}$$

For $mq-np\neq 0$ we have

$$\begin{aligned}\frac{1}{p\tau+q}\int_{\tilde{T}_f}\overline{\chi_\gamma}du &= \mathrm{e}^{\frac{2\pi\mathrm{i}L}{N}(mr+ns)}\sum_K a_K\left\{\frac{N-K}{N}\right. \\ &\quad\times\left.\int_0^{\frac{K}{N}}\mathrm{e}^{2\pi\mathrm{i}t(mq-np)}dt - \frac{K}{N}\int_{\frac{K}{N}}^1 \mathrm{e}^{2\pi\mathrm{i}t(mq-np)}dt\right\} \\ &= \frac{\mathrm{e}^{2\pi\mathrm{i}\frac{L}{N}(mr+ns)}}{2\pi\mathrm{i}(mq-np)}\left(\sum_K a_K\mathrm{e}^{2\pi\mathrm{i}\frac{K}{N}(mq-np)} - \sum_K a_K\right) \\ &= \frac{1}{2\pi\mathrm{i}(mq-np)}\sum_K a_K\overline{\chi_\gamma}\left(K\frac{p\tau+q}{N} + L\frac{-s\tau+r}{N}\right) \\ &= \frac{\widehat{\varphi_f}(\gamma)}{2\pi\mathrm{i}(mq-np)}\end{aligned}$$

(where we have used that $\sum a_K = 0$). □

Remark 8.1. Lemma 8.3(iii) can be read $\int_{E_\tau}\overline{\chi_\gamma}d\log f\wedge d\bar{u} = \frac{-\nu\widehat{\varphi_f}(\gamma)}{\omega(\gamma)}$.

8.1.2. Main computation; proof of Beilinson-Hodge. We now use the Fourier "technology" to compute

$$CH^{\ell+1}(\mathcal{E}^{[\ell]}(N),\ell+1) \xrightarrow{[\cdot]} \mathrm{Hom}_{\text{MHS}}\Big(\mathbb{Q}(0), H^{\ell+1}\Big(\mathcal{E}^{[\ell]}(N),\mathbb{Q}(\ell+1)\Big)\Big)$$

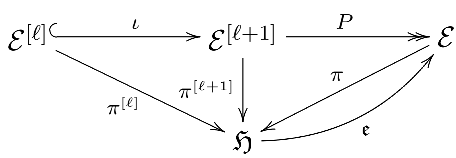
for

$$\mathfrak{Z}_{\mathbf{f}} \longmapsto \Omega_{\mathfrak{Z}_{\mathbf{f}}} \in \Omega^{\ell+1}(\overline{\mathcal{E}}^{[\ell]}(N))\left\langle \log \pi^{-1}(\kappa(N))\right\rangle.$$

By Section 7.1.5, $\mathcal{P}_N^*\Omega_{\mathfrak{Z}_{\mathbf{f}}} = (2\pi\mathrm{i})^{\ell+1}\Omega_{F_{\mathbf{f}}} = (2\pi\mathrm{i})^{\ell+1}F_{\mathbf{f}}(\tau)dz_1\wedge\cdots\wedge dz_\ell\wedge d\tau$ for some $F_{\mathbf{f}}(\tau)\in M^{\mathbb{Q}}_{\ell+2}(\Gamma(N))$, and it is *this modular form* we must identify. Consider $\Omega_{\iota(N)^*\{\mathbf{f}\}} \in \Omega^{\ell+1}(\overline{\mathcal{E}}^{[\ell]}(N))\left\langle \log\left(\overline{\tilde{W}^{[\ell]}(N)}\cup\pi^{-1}(\kappa(N))\right)\right\rangle$, which pulls back by $\tilde{\mathcal{G}}^*$ to $\Omega_{\overline{Z_{\mathbf{f}}}}$. The latter is *not* affected by moving $\overline{Z_{\mathbf{f}}}$ into good position over $\bar{W}^{[\ell]}(N)$ and completing it to $\mathfrak{Z}_{\mathbf{f}}$; so $\Omega_{\mathfrak{Z}_{\mathbf{f}}} = \tilde{\mathcal{G}}^*\Omega_{\iota(N)^*\{\mathbf{f}\}} = \tilde{\mathcal{G}}^*\iota(N)^*d\log f_1(u_1)\wedge\cdots\wedge d\log f_{\ell+1}(u_{\ell+1})$.

Write $\mathfrak{A}_{\{\mathbf{f}\}} := (-1)^\ell\Omega_{\mathcal{P}_N^*\{\mathbf{f}\}}\wedge\widetilde{d\bar{u}_1}\wedge\cdots\wedge\widetilde{d\bar{u}_\ell}\in A^{\ell+1,\ell}(\mathcal{E}^{[\ell+1]})\left\langle\log W_N^{[\ell+1]}\right\rangle$ and $\iota^*\mathfrak{A}_{\{\mathbf{f}\}} = \mathcal{P}_N^*\Omega_{\iota(N)^*\{\mathbf{f}\}}\wedge\widetilde{d\bar{z}_1}\wedge\cdots\wedge\widetilde{d\bar{z}_\ell}\in A^{\ell+1,\ell}(\mathcal{E}^{[\ell]})\left\langle\log\tilde{W}_N^{[\ell]}\right\rangle\subset\mathcal{D}^{\ell+1,\ell}$ $(\mathcal{E}^{[\ell]})$. Using the diagram

(8.1)



where $P(\tau;[u_1,\ldots,u_{\ell+1}]_\tau) := (\tau;[u_1+\cdots+u_{\ell+1}]_\tau)$, we compute $\pi_*^{[\ell]}(\iota^*\mathfrak{A}_{\{\mathbf{f}\}})$ in two different ways.

For the first,

$$\begin{aligned}\pi_*^{[\ell]}(\iota^*\mathfrak{A}_{\{\mathbf{f}\}}) &= \pi_*^{[\ell]}(\tilde{\mathcal{G}}_0\iota^*\mathfrak{A}_{\{\mathbf{f}\}}) = \pi_*^{[\ell]}\{\tilde{\mathcal{G}}^*(\mathcal{P}_N^*\Omega_{\iota(N)^*\{\mathbf{f}\}})\wedge\widetilde{d\bar{z}_1}\wedge\cdots\wedge\widetilde{d\bar{z}_\ell}\}\\ &= \pi_*^{[\ell]}\left\{(2\pi\mathrm{i})^{\ell+1}F_{\mathbf{f}}(\tau)dz_1\wedge\cdots\wedge dz_\ell\wedge d\tau\wedge\widetilde{d\bar{z}_1}\wedge\cdots\wedge\widetilde{d\bar{z}_\ell}\right\}\\ &= (-1)^{\binom{\ell+1}{2}}(2\pi\mathrm{i})^{\ell+1}\nu^\ell F_{\mathbf{f}}(\tau)d\tau\in A^{1,0}(\mathfrak{H}).\end{aligned}$$

For the second,

$$\pi_*^{[\ell]}(\iota^*\mathfrak{A}_{\{\mathbf{f}\}}) = \mathfrak{e}^*P_*\mathfrak{A}_{\{\mathbf{f}\}} \overset{\text{Lemma 8.1}(i)}{=\!=\!=\!=\!=} \sum_{\gamma\in\Gamma}\widehat{P_*\mathfrak{A}_{\{\mathbf{f}\}}}(\gamma)$$

$$= \nu^{-1} \sum_{\gamma\in\Gamma} \pi_*(\overline{\chi_\gamma} P_* \mathfrak{A}_{\{\mathbf{f}\}} \wedge \widetilde{du} \wedge \widetilde{d\bar{u}})$$
$$= \nu^{-1} \sum_{\gamma\in\Gamma} \pi_*^{[\ell+1]} \left((P^* \overline{\chi_\gamma}) \mathfrak{A}_{\{\mathbf{f}\}} \wedge (\widetilde{du_1} + \cdots + \widetilde{du_{\ell+1}}) \wedge P^* \widetilde{d\bar{u}} \right).$$

Writing $d\log(\mathcal{P}_N^* f_i(u_i)) = \alpha_i \widetilde{du_i} + \beta_i d\tau$, this

$$= (-1)^{\binom{\ell+2}{2}} \nu^{-1} \sum_{\gamma\in\Gamma} \sum_{i=1}^{\ell+1} \pi_*^{[\ell+1]}$$
$$\times \left\{ \left(\prod_{k=1}^{\ell+1} \overline{\chi_\gamma}(u_k) \right) \beta_i \prod_{j\neq i} \alpha_j \, \widetilde{du_1} \wedge \widetilde{d\bar{u}_1} \wedge \cdots \wedge \widetilde{du_{\ell+1}} \wedge \widetilde{d\bar{u}_{\ell+1}} \wedge d\tau \right\}$$
$$= (-1)^{\binom{\ell+2}{2}} \nu^{\ell} \sum_{\gamma\in\Gamma} \sum_{i=1}^{\ell+1} \widehat{\beta}_i(\gamma) \prod_{j\neq i} \widehat{\alpha}_i(\gamma) d\tau$$
$$= \frac{(-1)^\ell(\ell+1)}{2\pi\mathrm{i}} (-1)^{\binom{\ell+2}{2}} \nu^{\ell} \sum_{\gamma\in\Gamma}{}' \frac{\prod_{i=1}^{\ell+1} \widehat{\varphi_{f_i}}(\gamma)}{(\omega(\gamma))^{\ell+2}}.$$

So defining $\varphi_{\mathbf{f}} := \varphi_{f_1} * \cdots * \varphi_{f_{\ell+1}} \in \Phi_2^{\mathbb{Q}}(N)^\circ$ (and linearly extending this to sums of "monomials" $f_1 \otimes \cdots \otimes f_{\ell+1}$), we have proved

Theorem 8.1. $F_{\boldsymbol{f}}(\tau) = \frac{-(\ell+1)}{(2\pi\mathrm{i})^{\ell+2}} \sum_{m,n\in\mathbb{Z}^2} \frac{\widehat{\varphi}_{\boldsymbol{f}}(m,n)}{(m\tau+n)^{\ell+2}} = E^{[\ell]}_{\varphi_{\boldsymbol{f}}}(\tau)$.

Together with Lemma 7.2(i), the Decomposition Lemma (i), and Lemma 7.4, this immediately yields

Corollary 8.1. $\mathbb{E}_{\ell+2}(\Gamma(N)) \subset M_{\ell+2}(\Gamma(N))$, $\mathbb{E}^{\mathbb{Q}}_{\ell+2}(\Gamma(N)) \subset M^{\mathbb{Q}}_{\ell+2}(\Gamma(N))$.

(In particular, the map $\mathcal{O}^*(U(N))^{\otimes \ell+1} \to \mathbb{E}^{\mathbb{Q}}_{\ell+2}(\Gamma(N))$ defined by $f_1 \otimes \cdots \otimes f_{\ell+1} \mapsto E^{[\ell]}_{\varphi_{\mathbf{f}}}(\tau)$ is surjective.)

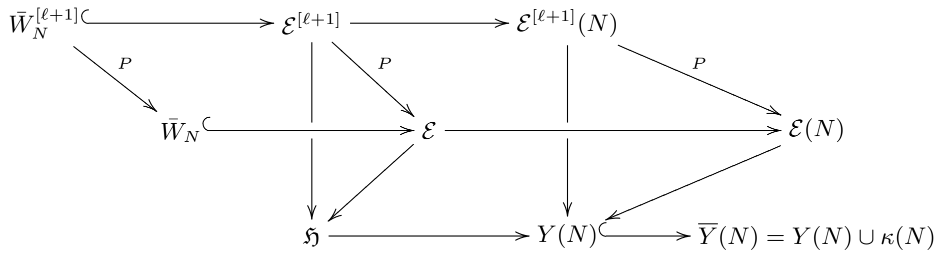
What is striking here is how simple cycles (once they are constructed) make it to prove statements about related objects: in this case, that *Eisenstein series are modular forms*; in the same spirit we can *identify their "values" at cusps*, and show that *they yield all holomorphic forms with log poles and $\mathbb{Q}$-periods.*

Corollary 8.2. *For $\sigma \in \kappa(N)$,*

$$\frac{1}{(2\pi\mathrm{i})^\ell} Res_\sigma(\Omega_{\mathfrak{Z}_f}) = \mathfrak{R}_\sigma(F_{\boldsymbol{f}}) = \mathsf{H}^{[\ell]}_\sigma(\varphi_{\boldsymbol{f}}) = \frac{-(\ell+1)}{(2\pi\mathrm{i})^{\ell+2}} \tilde{L}(\widehat{(\pi_\sigma)_*\varphi_{\boldsymbol{f}}}, \ell+2).$$

Proof. The outer equalities are just Sections 7.1.5 and 7.2.2, respectively ($\forall\sigma$). For $\sigma = [\mathrm{i}\infty]$, $\mathfrak{R}_{[\mathrm{i}\infty]}(F_{\mathbf{f}}) := \lim_{\tau\to\mathrm{i}\infty} F_{\mathbf{f}}(\tau) = \lim_{\tau\to\mathrm{i}\infty} E^{[\ell]}_{\varphi_{\mathbf{f}}}(\tau) = \mathsf{H}^{[\ell]}_{[\mathrm{i}\infty]}(\varphi_{\mathbf{f}})$ by Section 7.3.1.

Now $SL_2(\mathbb{Z})$ acts compatibly on the diagram



since $\Gamma(N) \trianglelefteq SL_2(\mathbb{Z})$. In particular, the action on connected components of $\bar{W}_N$ (the union of N-torsion sections) induces an action (by pullback) on $\Phi_2^{\mathbb{Q}}(N)^\circ$ compatible with Pontryagin $*$ and pullbacks of functions $\in \mathcal{O}^*(U_N)$, etc. Explicitly, $M_\sigma := \begin{pmatrix} r & -q \\ s & p\end{pmatrix}$ sends: (in $\kappa(N)$) $[\mathrm{i}\infty] \mapsto [\frac{r}{s}] =: \sigma$, (in $\mathfrak{H}$) $\tau \mapsto \frac{r\tau - q}{s\tau + p} =: \tau_0$, (in $\bar{W}_N$) $m\frac{\tau}{N} + n\frac{1}{N} \mapsto \frac{1}{N}\frac{m\tau+n}{s\tau+p} = (mp-ns)\frac{\tau_0}{N} + (mq+nr)\frac{1}{N} =: \mu\frac{\tau_0}{N} + \eta\frac{1}{N}$, and (in $\Phi_2^{\mathbb{Q}}(N)^\circ$, by pullback) $\varphi_{\mathbf{f}}(\mu,\eta) \mapsto \left(\begin{pmatrix} r & -q \\ s & p\end{pmatrix}^* \varphi_{\mathbf{f}}\right)(m,n) := \varphi_{\mathbf{f}}(mp-ns, mq+nr)$. So

$$\begin{aligned}\left(\pi_{[\mathrm{i}\infty]_*}\begin{pmatrix} r & -q \\ s & p\end{pmatrix}^* \varphi_{\mathbf{f}}\right)(m) &= \sum_{n\in\mathbb{Z}/N\mathbb{Z}} \varphi_{\mathbf{f}}(mp-ns, mq+nr) \\ &= \sum_n \varphi_{\mathbf{f}}(m(p,q) + n(-s,r)) \\ &= \left(\pi_{[\frac{r}{s}]_*}\varphi_{\mathbf{f}}\right)(m),\end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{Res}_\sigma}{(2\pi\mathrm{i})^\ell}(\Omega_{\mathfrak{Z}_{\mathbf{f}}}) &= \frac{\mathrm{Res}_{[\mathrm{i}\infty]}}{(2\pi\mathrm{i})^\ell}(M_\sigma^*\Omega_{\mathfrak{Z}_{\mathbf{f}}}) \\ &= \frac{\mathrm{Res}_{[\mathrm{i}\infty]}}{(2\pi\mathrm{i})^\ell}\left(\Omega_{\mathfrak{Z}_{M_\sigma^*\mathbf{f}}}\right) = \frac{-(\ell+1)}{(2\pi\mathrm{i})^{\ell+2}}\tilde{L}\left(\widehat{\pi_{[\mathrm{i}\infty]_*}\varphi_{M_\sigma^*\mathbf{f}}},\, \ell+2\right) \\ &= \frac{-(\ell+1)}{(2\pi\mathrm{i})^{\ell+2}}\tilde{L}\left(\widehat{\pi_{\sigma_*}\varphi_{\mathbf{f}}},\, \ell+2\right).\end{aligned}$$

□

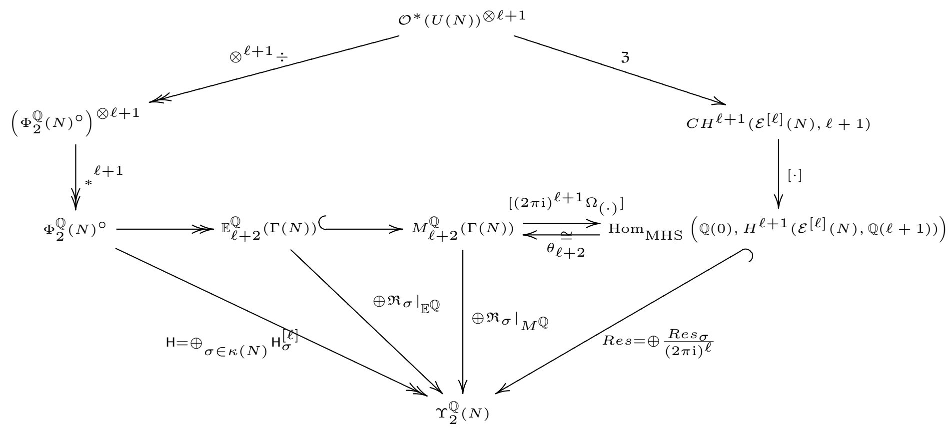
Corollary 8.3. (i) *Claim 7.1 holds for* $\Gamma(N)$ *(this implies Beilinson-Hodge for* $\mathcal{E}^{[\ell]}(N)$*).*

(ii) $\mathbb{E}^{\mathbb{Q}}_{\ell+2}(\Gamma(N)) = M^{\mathbb{Q}}_{\ell+2}(\Gamma(N)) \cong \mathrm{Hom}_{\text{MHS}}\big(\mathbb{Q}(0), H^{\ell+1}(\mathcal{E}^{[\ell]}(N), \mathbb{Q}(\ell+1)\big),$ *with dimension* $|\kappa(N)|$.

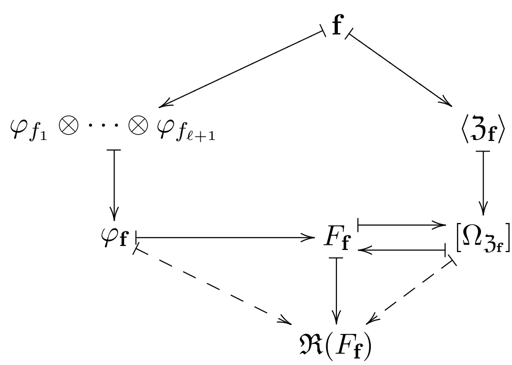
(iii) $M_{\ell+2}(\Gamma(N)) = \mathbb{E}_{\ell+2}(\Gamma(N)) \oplus S_{\ell+2}(\Gamma(N))$.

Remark. Note that $\dim_{\mathbb{C}} \mathbb{E} = \dim_{\mathbb{Q}} \mathbb{E}^{\mathbb{Q}} = \dim_{\mathbb{Q}} M^{\mathbb{Q}} \leq \dim_{\mathbb{C}} M$ in general.

Proof. It is basically contained in the diagram



(The arrows around the outer left surject by Sections 7.2.5, 7.2.3, 7.2.2 (resp.), as does the map to $\mathbb{E}^{\mathbb{Q}}_{\ell+2}$ by Section 7.3.1; the map *from* $\mathbb{E}^{\mathbb{Q}}_{\ell+2}$ injects by Corollary 8.1 and Res by Section 7.1.4. The upper pentagon commutes by Theorem 8.1, and the lower triangles by Corollary 8.2.) We can track $\mathbf{f} := f_1 \otimes \cdots \otimes f_{\ell+1}$ though the diagram:



To see (i), note the composition $\mathsf{H} \circ *^{\ell+1} \circ \otimes^{\ell+1} \div$ surjective $\implies$ Res $\circ [\cdot] \circ \mathfrak{Z}$ surjective $\implies$ Res $\circ [\cdot]$ surjective (=Claim 7.1) (which implies $[\cdot]$ is surjective (=Beilinson–Hodge)).

For (ii), Res $\circ [\cdot] \circ \mathfrak{Z}$ surjective implies that $[\cdot] \circ \mathfrak{Z}$ is surjective (and Res $\cong$) which shows $\theta_{\ell+2} \circ [\cdot] \circ \mathfrak{Z}$ is surjective and hence that $\mathbb{E}^{\mathbb{Q}} \subseteq M^{\mathbb{Q}}$ is equality. Finally, $\dim \Upsilon_2^{\mathbb{Q}}(N) = |\kappa(N)|$.

Now (iii) follows from Equation (7.3). □

Remark 8.2. Corollary 8.3 holds for arbitrary congruence subgroups Γ (between $\Gamma(N)$ and $SL_2(\mathbb{Z})$), given an appropriate definition of Eisenstein series for Γ. This is (referring to Section 7.1.6)

$$\mathbb{E}^{\mathbb{Q}}_{\ell+2}(\Gamma) := \mathfrak{R}^{-1}\left(\mathsf{P}^{[\ell]}_{\Gamma(N)/\Gamma}(\Upsilon_2^{\mathbb{Q}}(\Gamma))\right) \cap \mathbb{E}^{\mathbb{Q}}_{\ell+2}(\Gamma(N)),$$

the important point being that these are generated by $\varphi \in \Phi_2(N)^\circ$ satisfying $\mathsf{H}^{[\ell]}_{[\frac{r}{s}]}(\varphi) = \mathsf{H}^{[\ell]}_{[\frac{r'}{s'}]}(\varphi)$ *whenever* $[\frac{r}{s}]$, $[\frac{r'}{s'}] \in \kappa(N)$ *map to the same cusp in* $\kappa(\Gamma)$. We will look at this condition further below (in Section 8.2.1).

Also, a version of the above construction can be made to work for $\mathbb{P}\Gamma(2)$ (by choosing an $\cong$ subgroup of $SL_2(\mathbb{Z})$) if ℓ is even, but we have omitted this.

8.1.3. Additional calculations for the cycle class The results of Section 8.1.2 lead naturally to a basis for $\mathbb{E}^{\mathbb{Q}}_{\ell+2}(\Gamma(N))$ whose elements correspond to holomorphic $(\ell+1)$-forms with $\mathbb{Q}(\ell+1)$ periods and *log poles along the fiber over exactly one cusp* σ. (In some sense this is the most explicit confirmation of Beilinson–Hodge.)

Writing

$$\Gamma(N)_{\mathrm{i}\infty} := \mathrm{Stab}(\mathrm{i}\infty \in \mathfrak{H}^*) = \left\{ \begin{pmatrix} 1 & aN \\ 0 & 1 \end{pmatrix} \right\} \subset \Gamma(N),$$
$$PSL_2(\mathbb{Z})_{\mathrm{i}\infty} := \mathrm{Stab}(\mathrm{i}\infty \in \mathfrak{H}^*) = \left\{ \pm \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \right\} \subset PSL_2(\mathbb{Z}),$$

we have a short-exact sequence

$$\Gamma(N)_{\mathrm{i}\infty} \backslash \Gamma(N) \longrightarrow PSL_2(\mathbb{Z})_{\mathrm{i}\infty} \backslash PSL_2(\mathbb{Z}) \longrightarrow \underbrace{\left\langle \left(\begin{smallmatrix} \bar{1} & \bar{a} \\ \bar{0} & \bar{1} \end{smallmatrix}\right) \right\rangle \backslash PSL_2(\mathbb{Z}/N\mathbb{Z})}_{\cong \kappa(N)} .$$

Hence

$$\begin{aligned}
E_\varphi^{[\ell]}(\tau) &= \frac{-(\ell+1)}{(2\pi \mathrm{i})^{\ell+2}} \sum_{(m,n)\in\mathbb{Z}^2}{}' \frac{\widehat{\varphi}(m,n)}{(m\tau+n)^{\ell+2}} \\
&= \frac{-(\ell+1)}{(2\pi \mathrm{i})^{\ell+2}} \sum_{\substack{\pm(m_0,n_0)\in\mathbb{Z}^2/\pm \\ \text{rel. prime} \\ \updownarrow\, \gamma = \begin{pmatrix} * & * \\ m_0 & n_0\end{pmatrix} \\ \sum_{\gamma\in\frac{PSL_2(\mathbb{Z})}{PSL_2(\mathbb{Z})_{i\infty}}}}} \sum_{\mathfrak{z}\in\mathbb{Z}}{}' \frac{\widehat{\varphi}(\mathfrak{z}m_0,\mathfrak{z}n_0)}{(\mathfrak{z}m_0\tau+\mathfrak{z}n_0)^{\ell+2}} \\
&= \frac{-(\ell+1)}{(2\pi \mathrm{i})^{\ell+2}} \sum_{\substack{\sigma\in\kappa(N) \\ \| \\ [\frac{r}{s}]}} \sum_{\substack{\gamma'\in\frac{\Gamma(N)}{\Gamma(N)_{i\infty}}\cdot\begin{pmatrix} p & q \\ -s & r\end{pmatrix} \\ \updownarrow\, \gamma' = \begin{pmatrix} * & * \\ m_0 & n_0\end{pmatrix} \\ \sum_{\substack{(m_0,n_0)\text{ rel. prime,} \\ \overset{(N)}{\equiv}(-s,r)}}}} \sum_{\mathfrak{z}\in\mathbb{Z}}{}' \frac{\widehat{\varphi}(\mathfrak{z}m_0,\mathfrak{z}n_0)}{\mathfrak{z}^{\ell+2}(m_0\tau+n_0)^{\ell+2}}.
\end{aligned}$$

Now since (in the sum) $(m_0,n_0) \overset{(N)}{\equiv} (-s,r)$, $\widehat{\varphi}(\mathfrak{z}m_0,\mathfrak{z}n_0) = \widehat{\varphi}(-\mathfrak{z}s,\mathfrak{z}r) = (\iota^*_{[\frac{r}{s}]}\widehat{\varphi})(\mathfrak{z}) = \widehat{\pi_{[\frac{r}{s}]_*}\varphi}(\mathfrak{z})$ and the above

$$\begin{aligned}
&= \sum_{\sigma\in\kappa(N)} \left[\frac{-(\ell+1)}{(2\pi \mathrm{i})^{\ell+2}} \sum_{\mathfrak{z}\in\mathbb{Z}}{}' \frac{\widehat{\pi_{[\frac{r}{s}]_*}\varphi}(\mathfrak{z})}{\mathfrak{z}^{\ell+2}} \right] \sum_{\substack{(m_0,n_0)\overset{(N)}{\equiv}(-s,r) \\ \gcd(m_0,n_0)=1}} \frac{1}{(m_0\tau+n_0)^{\ell+2}} \\
&=: \sum_{\sigma\in\kappa(N)} \mathsf{H}_\sigma^{[\ell]}(\varphi)\tilde{E}_\sigma^{[\ell]}(\tau),
\end{aligned}$$

where the $\sum_{\mathfrak{z}} = \tilde{L}(\widehat{\pi_{[\frac{r}{s}]_*}\varphi}, \ell+2)$ and $\mathsf{H}_\sigma^{[\ell]}(\varphi)$ ($\sigma = [\frac{r}{s}]$) is the entire bracketed quantity.

Proposition 8.1. (i) *We have, for* $\sigma = [\frac{r}{s}]$,

$$\begin{aligned}
\tilde{E}_\sigma^{[\ell]}(\tau) &= \sum_{\substack{(m_0,n_0)\in\mathbb{Z}^2 \\ \textit{rel. prime,} \\ \overset{(N)}{\equiv}(-s,r)}} \frac{1}{(m_0\tau+n_0)^{\ell+2}} \\
&= \sum_{\substack{(\alpha',\beta')\in\mathbb{Z}^2 \\ \gcd(r+N\alpha',s+N\beta')=1}} \frac{1}{(r+N\alpha'-(s+N\beta')\tau)^{\ell+2}}
\end{aligned}$$

$$= \sum_{\substack{(\alpha,\beta)\in\mathbb{Z}^2\\ \gcd(1+N\alpha,N\beta)=1}} \frac{1}{[(1+\alpha N)(r-s\tau)+\beta N(q+p\tau)]^{\ell+2}}.$$

In particular,

$$\tilde{E}^{[\ell]}_{[\mathrm{i}\infty]}(\tau) = \sum_{\substack{(\alpha,\beta)\in\mathbb{Z}^2\\ \gcd(1+N\alpha,N\beta)=1}} \frac{1}{(1+N\alpha-N\beta\tau)^{\ell+2}}.$$

(ii) *The* $\{\tilde{E}^{[\ell]}_{\sigma}(\tau)\}_{\sigma\in\kappa(N)}$ *give a basis for the* $\mathbb{E}^{\mathbb{Q}}_{\ell+2}(\Gamma(N))$*, satisfying* $\mathfrak{R}_{\sigma'}(\tilde{E}^{[\ell]}_{\sigma}) = \delta_{\sigma\sigma'}$.

(iii) *Given* $\boldsymbol{f} \in \mathcal{O}^*(U(N))^{\otimes \ell+1}$,

$$F_{\boldsymbol{f}}(\tau) = \sum_{\sigma\in\kappa(N)} \mathsf{H}^{[\ell]}_{\sigma}(\varphi_{\boldsymbol{f}})\tilde{E}^{[\ell]}_{\sigma}(\tau).$$

Proof. For (ii), pick for each σ a $\varphi \in \Phi^{\mathbb{Q}}_2(N)^{\circ}$ so that $\mathsf{H}^{[\ell]}_{\sigma'}(\varphi) = \delta_{\sigma\sigma'}$, and plug into the computation above. The remainder is clear. □

Next, we have a q-series expansion at $[\mathrm{i}\infty]$ for the usual Eisenstein series associated to a "divisor on N-torsion" $\varphi \in \Phi^{\mathbb{Q}}_2(N)^{\circ}$: write $q_0 := \mathrm{e}^{\frac{2\pi\mathrm{i}\tau}{N}} =$"$q^{\frac{1}{N}}$", $\xi_N(a) := \mathrm{e}^{\frac{2\pi\mathrm{i}a}{N}}$, ${}^{\ell}\widehat{\varphi}(m,n) := \widehat{\varphi}(m,n) + (-1)^{\ell}\widehat{\varphi}(-m,-n)$.

Proposition 8.2.

$$E^{[\ell]}_{\varphi}(\tau) = \mathsf{H}^{[\ell]}_{[\mathrm{i}\infty]}(\varphi) + \frac{(-1)^{\ell+1}}{N^{\ell+2}\ell!}\sum_{M\geq 1} q_0^M \left\{ \sum_{r|M} r^{\ell+1}\left(\sum_{n_0\in\mathbb{Z}/N\mathbb{Z}} \xi_N(n_0 r)\cdot {}^{\ell}\widehat{\varphi}\left(\frac{M}{r}, n_0\right)\right)\right\}.$$

Proof. Essentially in [44] for ℓ even (also see [57]), but can be derived from scratch using ideas in [76] (will be done below for q-series of regulator periods). □

Since q_0 is the local coordinate at $[\mathrm{i}\infty] \in \overline{Y}(N)$, this yields a *power-series expansion* for $F_{\mathbf{f}}$ there. We have not tried to directly compute q-expansions for the $\tilde{E}^{[\ell]}_{\sigma}$, but one can plug $\varphi := \frac{1}{N}\pi^*_{\sigma}\varphi^{[\ell]}_N$ into $E^{[\ell]}_{\varphi}$ to have the same effect (see Proposition 7.3). We are particularly interested in the case $\sigma = [\mathrm{i}\infty]$. First, a simplification of Proposition 8.2:

Corollary 8.4. *For $\varphi_0 \in \Phi^{\mathbb{Q}}(N)^\circ$, $\varphi := \frac{1}{N}\pi^*_{[\mathrm{i}\infty]}\varphi_0$, we have*

$$E^{[\ell]}_\varphi(\tau) = \frac{(-1)^\ell}{\ell!(\ell+2)}\sum_{a=0}^{N}\varphi_0(a)B_{\ell+2}\left(\frac{a}{N}\right) + \frac{(-1)^{\ell+1}}{N^{\ell+1}\ell!}\sum_{\mu\geq 1} q_0^{N\mu}\left\{\sum_{r|\mu} r^{\ell+1}\cdot {}^\ell\varphi_0(r)\right\},$$

where ${}^\ell\varphi_0(a) = \varphi_0(a) + (-1)^\ell\varphi_0(-a)$.

Proof. ${}^\ell\widehat{\varphi} = {}^\ell(\widehat{\tfrac{1}{N}\pi^*_{[\mathrm{i}\infty]}\varphi_0}) = \iota_{[\mathrm{i}\infty]_*}{}^\ell\widehat{\varphi_0}$ implies that $\sum_{n_0}\xi_N(n_0 r)\cdot {}^\ell\widehat{\varphi}(\frac{M}{r}, n_0) = 0$ if $N \nmid \frac{M}{r}$; otherwise $= \sum_{n_0}\xi_N(n_0 r)\cdot {}^\ell\widehat{\varphi_0(n_0)} = N\cdot {}^\ell\varphi_0(r)$. Put $M = \mu N$. □

Now take φ_0 to be the "fundamental vector" $\varphi^{[\ell]}_N$; then

$$E^{[\ell]}_\varphi(\tau) = 1 + \frac{2(-1)^{\ell+1}}{N^{\ell+1}\ell!}\sum_{\mu\geq 1} q_0^{N\mu}\left\{\sum_{r|\mu} r^{\ell+1}\varphi^{[\ell]}_N(r)\right\}$$

has $\mathfrak{R}_\sigma(E^{[\ell]}_\varphi) = \delta_{\sigma,[\mathrm{i}\infty]}$.

Example 8.1. If $\ell = 1$ and $N = 3$, from Example 7.3 we get

$$1 - 9\sum_{\mu\geq 1} q_0^{3\mu}\left\{\sum_{r|\mu} r^2\chi_{-3}(r)\right\}.$$

8.2. Push-forwards of the construction

8.2.1. Eisenstein symbols for other congruence subgroups Γ. Recall that this means $\Gamma(N) \subseteq \Gamma \subseteq SL_2(\mathbb{Z})$ ($N \geq 3$), $\{-\mathrm{id}\} \notin \Gamma$; that automatically $\Gamma(N) \trianglelefteq \Gamma$; and that there are corresponding quotients $(\mathcal{E}^{[\ell]}(N)\backslash\text{fibers}) \xrightarrow{\mathcal{P}^{[\ell]}_{\Gamma(N)/\Gamma}} \mathcal{E}^{[\ell]}_\Gamma$, $(Y(N)\backslash\text{pts.}) \xrightarrow{\rho_{\Gamma(N)/\Gamma}} Y_\Gamma\backslash\varepsilon_\Gamma$. Our main examples will be

$$\Gamma_1(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) \,\middle|\, a \overset{(N)}{\equiv} 1 \overset{(N)}{\equiv} d,\, c \overset{(N)}{\equiv} 0 \right\} = \left\langle \Gamma(N), \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \right\rangle,$$
$$\Gamma_1'(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) \,\middle|\, a \overset{(N)}{\equiv} 1 \overset{(N)}{\equiv} d,\, b \overset{(N)}{\equiv} 0 \right\} = \left\langle \Gamma(N), \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) \right\rangle$$
$$= \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \Gamma_1(N) \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right).$$

Already for $\Gamma_1^{(')}(N)$, N not prime, one has type I_m^* cusps — e.g., $\overline{Y}_1'(4)$ has cusps $[i\infty]$ (I_4), $[0]$ (I_1), $[2]$ (I_1^*). (Also, $Y_1^{(')}(3)$ has an elliptic point, but for simplicity our notation will ignore this fact.)

However, we will consider also "traditional" congruence subgroups that do not fit our convention e.g.,

$$\Gamma_0(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) \,\middle|\, c \overset{(N)}{\equiv} 0 \right\} \ (\ni \{-\mathbf{id}\}),$$

for which one has $\overline{Y}_\Gamma$ but no canonically defined $\mathcal{E}_\Gamma^{[\ell]}$ (though when $N = 3, 4, 6$ one can get around this problem by observing that $SL_2(\mathbb{Z}) \twoheadrightarrow PSL_2(\mathbb{Z})$ sends $\Gamma_1(N) \xrightarrow{\cong} \mathbb{P}\Gamma_0(N)$). We will also consider (in Section 8.2.2)

$$\Gamma^{+N} := \left\langle \Gamma, \iota_N := \begin{pmatrix} 0 & -1/\sqrt{N} \\ \sqrt{N} & 0 \end{pmatrix} \right\rangle \ (\not\subseteq SL_2(\mathbb{Z}))$$

for $\Gamma = \Gamma_0(N), \Gamma_1(N)$.

We will now (e.g., using $\mathcal{P}^{[\ell]}_{\Gamma(N)/\Gamma^*}$) push the $\{\mathfrak{Z}_\mathbf{f}\}$ constructed in Section 7.3.4 forward to cycles (on families) over these new Y_Γ. The aim in doing this is to produce more Eisenstein symbols (on families of abelian varieties or CY's) that live over genus 0 curves, in order to link up with those cases of the construction of Sections 3 and 4 which are classically modular. We note that, while $g(\overline{Y}(N)) = 0$ only for $N =$ (2,) 3, 4, 5, on the other hand $Y_1^{(')}(2-10, 12)$ and $Y_0(2-10, 12, 13, 16, 18, 25)$ are all rational.

To get a feel for the behavior of cusps under the various $\overline{\rho}_{\Gamma'/\Gamma}$, consider the maps $\overline{Y}(N) \to \overline{Y}_1(N) \to \overline{Y}_0(N) \to \overline{Y}_0(N)^{+N}$ for N prime, with (resp.) $\frac{N^2-1}{N}$ (all I_N), $N-1$ (half each of I_N, I_1), 2 (I_N, I_1), and 1 cusp(s). Since N is prime, one has a correspondence $\kappa(N) \cong \frac{(\mathbb{Z}/N\mathbb{Z})^2 \setminus \{(0,0)\}}{\langle \pm \mathrm{id} \rangle}$, and one can picture how these get equated (e.g., for $N = 5$) as in figure 11, where circles are chosen representatives of equivalence classes. Flipping about the diagonal gives the picture for $\kappa(5) \to \kappa_1'(5)$.

For $\Gamma' \subset \Gamma$ if index r, $\overline{\rho}_{\Gamma'/\Gamma} : \overline{Y}_{\Gamma'} \to \overline{Y}_\Gamma$ is of degree r; if $\Gamma' \trianglelefteq \Gamma$ then $\overline{\rho}_{\Gamma'/\Gamma}$ (omitting cusps/elliptic points and their preimages) is a Galois covering, so

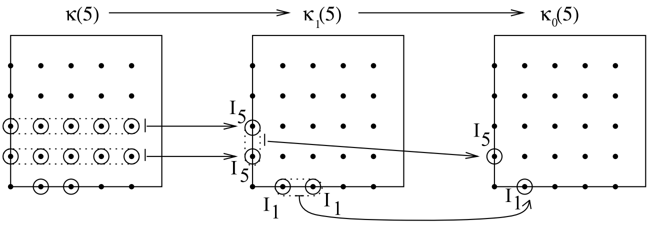
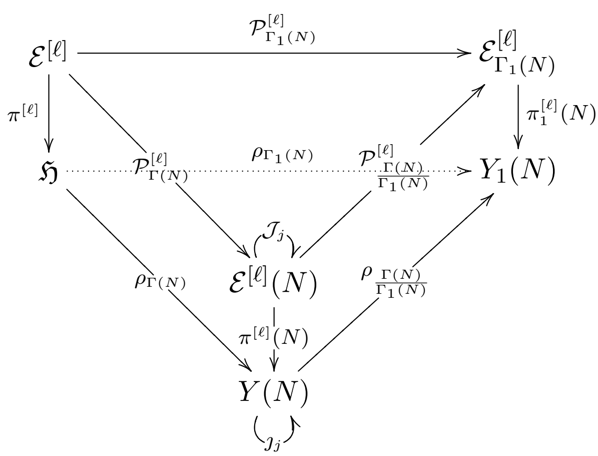


Figure 11: Behavior of cusps under brached coverings.

that one has deck transformations $\{\jmath_j\}_{j=1}^{r}$ satisfying $\sum \jmath_j^* = \rho^*\rho_*$ (on forms, cycles, etc.), and corresponding transformations on the Kuga varieties. For example, one has a diagram $(j = 1, \ldots, N)$



(and a similar diagram for $\Gamma_1^{'}(N)$) where ${}^{(')}\mathcal{J}_j$ and ${}^{(')}\jmath_j$ are induced by the action of coset representatives $\gamma_j^{(')} = \left(\begin{smallmatrix} 1 & j \\ 0 & 1 \end{smallmatrix}\right)$ [resp. $\left(\begin{smallmatrix} 1 & 0 \\ j & 1 \end{smallmatrix}\right)$]$\in SL_2(\mathbb{Z})$ for $\Gamma_1^{(')}(N)/\Gamma(N)$, on $\mathcal{E}^{[\ell]}$ and $\mathfrak{H}$. Now define

$$\mathfrak{Z}_{\mathbf{f},1^{(')}} := \frac{1}{N}\left(\mathcal{P}^{[\ell]}_{\Gamma(N)/\Gamma_1^{(')}(N)}\right)_* \mathfrak{Z}_{\mathbf{f}} \in CH^{\ell+1}(\mathcal{E}^{[\ell]}_{\Gamma_1^{(')}(N)}, \ell+1);$$

then we have

$$F_{\mathbf{f},1^{(')}} := \theta_{\ell+2}(\Omega_{\mathfrak{Z}_{\mathbf{f},1^{(')}}}) = \theta_{\ell+2}\left(\left(\mathcal{P}^{[\ell]}_{\Gamma(N)/\Gamma_1^{(')}(N)}\right)^* \mathfrak{Z}_{\mathbf{f},1^{(')}}\right)$$

$$= \frac{1}{N}\theta_{\ell+2}\left(\sum_{j=1}^{N}{}^{(\prime)}\mathcal{J}_j^*\Omega_{\mathfrak{Z}_{\mathbf{f}}}\right)$$
$$= \frac{1}{N}\sum_{j=1}^{N}\theta_{\ell+2}(\Omega_{\mathfrak{Z}_{\mathbf{f}}})|^{\ell+2}_{\gamma_j^{(\prime)}} = \frac{1}{N}\sum_{j=1}^{N}F_{\mathbf{f}}|^{\ell+2}_{\gamma_j^{(\prime)}},$$

i.e.,

$$F_{\mathbf{f},1}(\tau) = \frac{1}{N}\sum_{j=0}^{N-1}F_{\mathbf{f}}(\tau+j) \quad \text{and} \quad F_{\mathbf{f},1'}(\tau) = \frac{1}{N}\sum_{j=0}^{N-1}\frac{F_{\mathbf{f}}(\frac{\tau}{j\tau+1})}{(j\tau+1)^{\ell+2}}.$$

Writing

(8.2)
$$(\rho_*\widehat{\varphi}_{\mathbf{f}})(m,n) := \sum_j \widehat{\varphi}_{\mathbf{f}}(m, n-mj), \quad (\rho_*{}'\widehat{\varphi}_{\mathbf{f}})(m,n) := \sum_j \widehat{\varphi}_{\mathbf{f}}(m-nj, n)$$

we get

$$F_{\mathbf{f},1^{(\prime)}}(\tau) = \frac{-(\ell+1)}{(2\pi\mathrm{i})^{\ell+2}}\sum_{m,n}{}' \frac{\frac{1}{N}(\rho_*^{(\prime)}\widehat{\varphi}_{\mathbf{f}})(m,n)}{(m\tau+n)^{\ell+2}}.$$

Using Corollary 8.3(ii) for $\Gamma_1^{(\prime)}(N)$ and surjectivity of $\kappa(N) \to \kappa_1^{(\prime)}(N)$, this implies

Proposition 8.3. $\left(\mathcal{P}^{[\ell]}_{\Gamma_1^{(\prime)}(N)}\right)^*$ *of any class in* $F^{\ell+1} \cap H^{\ell+1}(\mathcal{E}^{[\ell]}_{\Gamma_1^{(\prime)}(N)}, \mathbb{Q}(\ell+1))$ *is* $(2\pi\mathrm{i})^{\ell+1}\Omega_F$ *for* $F = E_\varphi^{[\ell]}$, $\varphi \in \Phi_2^{\mathbb{Q}}(N)$ *with* $\widehat{\varphi} = \frac{1}{N}\rho_*^{(\prime)}\widehat{\varphi}$.

The effect of ρ_* on the q-expansion is especially simple:

$$F_{\mathbf{f}}(\tau) = \sum_{M\geq 0}\alpha_M q_0^M$$
$$\implies F_{\mathbf{f},1}(\tau) = \frac{1}{N}\sum_{M\geq 0}\alpha_M\sum_{j=0}^{N-1}(\xi_N(j)\,q_0)^M = \sum_{m\geq 0}\alpha_{mN}q^m,$$

which makes sense since q is the local coordinate at $[\mathrm{i}\infty]$ on $\overline{Y}_1(N)$.

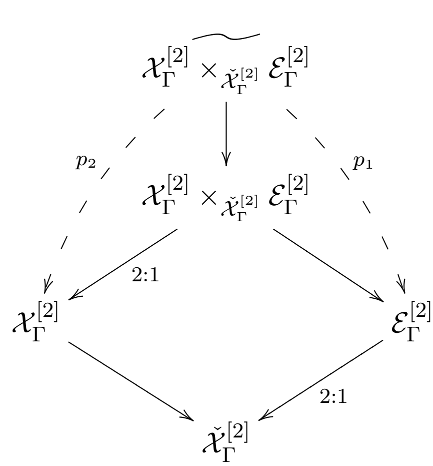
We are interested in Eisenstein symbols with their only residue at $[\mathrm{i}\infty]$, in analogy to Sections 3 and 4. If $F_{\mathbf{f}} = \tilde{E}^{[\ell]}_{[\mathrm{i}\infty]}$, then clearly

$$F_{\mathbf{f},1} = \tilde{E}^{[\ell]}_{[\mathrm{i}\infty]}, \quad \text{while}$$
$$F_{\mathbf{f},1'} = \frac{1}{N}\sum_{j=0}^{N-1} \tilde{E}^{[\ell]}_{[\frac{1}{j}]} = \frac{1}{N} \sum_{\substack{(\alpha,\beta)\in\mathbb{Z}^2 \\ \gcd(1+N\alpha,\beta)=1}} \frac{1}{(1+N\alpha+\beta\tau)^{\ell+2}}.$$

Once Γ and ℓ are specified, such symbols (or rather, their cycle classes) are unique (up to scaling), so for $\Gamma_1(N)$ and $\Gamma_1^{(\prime)}(N)$ this is it!

8.2.2. Eisenstein symbols for $K3$ surfaces and CY three-fold families. Given a cycle $\mathfrak{Z} \in CH^{\ell+1}(\mathcal{E}^{[\ell]}_{\Gamma}, \ell+1)$ (e.g., $\Gamma = \Gamma(N)$ or $\Gamma_1^{(\prime)}(N)$), we have $\Omega_{\mathfrak{Z}} = (2\pi\mathrm{i})^{\ell+1} F_{\mathfrak{Z}}(\tau)\, dz_1 \wedge \cdots \wedge dz_\ell \wedge d\tau$ ($F_{\mathfrak{Z}} \in \mathbb{E}^{\mathbb{Q}}_{\ell+2}(\Gamma)$), which we assume $\neq 0$. If $\ell = 2$, then there is an involution $I : (\tau; z_1, z_2) \mapsto (\tau; -z_1, -z_2)$, with $I^*\Omega_{\mathfrak{Z}} = \Omega_{\mathfrak{Z}}$. Set $\check{\mathcal{X}}^{[2]}_{\Gamma} := \frac{\mathcal{E}^{[2]}_{\Gamma}}{I}$, and let $\mathcal{X}^{[2]}_{\Gamma} \to \check{\mathcal{X}}^{[2]}_{\Gamma}$ be the (smooth) Kummer $K3$ family over $Y_\Gamma \backslash \varepsilon_\Gamma$ obtained by blowing up the two-torsion multisections. Using the diagram

(8.3)



we define a (nontrivial) cycle by $\mathfrak{Z}_{\mathcal{X}} := \frac{1}{2} p_{2*} p_1^* \mathfrak{Z} \in CH^3(\mathcal{X}^{[2]}_{\Gamma}, 3)$. (This will have the same regulator periods and higher normal function as $\mathfrak{Z}$ by the monodromy argument below. Note also that if we take $\Gamma = \Gamma_1(N)$, then quotienting $\mathcal{E}^{[2]}_{\Gamma}$ by the action of $\Gamma_0(N)/\Gamma_1(N)$ and blowing up also yields — due to the presence of $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ — a family of Kummer $K3$ surfaces over $Y_0(N) \backslash \cdots$ and a nontrivial cycle.) There is a fiberwise involution $I' : \mathcal{X}^{[2]}_{\Gamma} \to$

$\mathcal{X}_\Gamma^{[2]}$ induced by $(z_1, z_2) \mapsto (z_1, -z_2)$ [or equivalently $(-z_1, z_2)$], sending $dz_1 \wedge dz_2 \mapsto -dz_1 \wedge dz_2$ and fixing the exceptional divisors.

Passing to $\ell = 3$, and taking $\mathfrak{Z} \in CH^4(\mathcal{E}_\Gamma^{[3]}, 4)$, we can apply the process above to the first two fiber-factors to obtain $\mathfrak{Z}' \in CH^4(\mathcal{X}_\Gamma^{[2]} \times_{Y_\Gamma \backslash \varepsilon_\Gamma} \mathcal{E}_\Gamma, 4)$. Writing $I'' : \mathcal{E}_\Gamma \to \mathcal{E}_\Gamma$ $[z \mapsto -z]$, we have an involution $I' \times I''$ on $\mathcal{X}_\Gamma^{[2]} \times_{\dots} \mathcal{E}_\Gamma$ evidently fixing $\Omega_{\mathfrak{Z}'}$. Blowing up along the singular set (in each fiber this looks like a disjoint union of 64 rational curves) and applying a process similar to the $\ell = 2$ case, yields a family $\mathcal{X}_\Gamma^{[3]}$ of Borcea–Voisin (CY) threefolds over $Y_\Gamma \backslash \varepsilon_\Gamma$, and a nontrivial cycle $\mathfrak{Z}_{\mathcal{X}} \in CH^4(\mathcal{X}_\Gamma^{[3]}, 4)$. (Again, this will have the same regulator periods as $\mathfrak{Z}$.)

Here is a more interesting construction, which yields a K_3-class on a $K3$ surface family over $Y_1(N)^{+N}$. Recall that the Fricke involution $\iota_N \in SL_2(\mathbb{R})$ acts on $\mathfrak{H}$ by $\tau \mapsto -\frac{1}{N\tau}$; this yields an action of $\Gamma_1(N)^{+N}$ on $\mathfrak{H}^*$ with $\overline{Y}_1(N)^{+N}$ as quotient. By normality of $\Gamma_1(N) \trianglelefteq \Gamma_1(N)^{+N}$, ι_N also acts on $\overline{Y}_1(N)$ with quotient map $\rho_{+N} : \overline{Y}_1(N) \twoheadrightarrow \overline{Y}_1(N)^{+N}$.

Set ${}'\mathcal{E}_1(N) := \mathcal{E}(N) \times_{\iota_N} Y_1(N)$, representing points by $(\tau; [z]_{\frac{-1}{N\tau}})$, and consider the relative N-isogeny (*not* an involution!) $J_N : {}'\mathcal{E}_1(N) \to \mathcal{E}_1(N)$ induced by $(\tau; z) \mapsto (\tau; -N\tau z)$. Writing ${}'\mathcal{E}_1^{[2]}(N) := \mathcal{E}_1(N) \times_{Y_1(N)} {}'\mathcal{E}_1(N)$, we have $\mathrm{id} \times J_N =: J_N^{[2]} : {}'\mathcal{E}_1^{[2]}(N) \to \mathcal{E}_1^{[2]}(N)$; given $F \in M_4^{\mathbb{Q}}(\Gamma_1(N))$, ${}'\Omega_F := -\frac{1}{N}(J_N^{[2]})^* \Omega_F = \tau \Omega_F$. Also write $\tilde{J}_N^{[2]} : \mathcal{E}_1^{[2]}(N) \to {}'\mathcal{E}_1^{[2]}(N)$ for $(\tau; z_1, z_2) \mapsto (\tau; z_1, \frac{z_2}{\tau})$.

Now we are ready to consider the *involution*

$$\begin{array}{ccc} {}'\mathcal{E}_1^{[2]}(N) & \xrightarrow{I_N^{[2]}} & {}'\mathcal{E}_1^{[2]}(N) \\ \downarrow \pi & & \downarrow \pi \\ \mathfrak{H} & \xrightarrow{\iota_N} & \mathfrak{H} \end{array}$$

induced by exchanging factors: $(\tau; [z_1]_\tau, [z_2]_{\frac{-1}{N\tau}}) \mapsto (\frac{-1}{N\tau}; [z_2]_{\frac{-1}{N\tau}}, [z_1]_\tau)$. We have

$$\begin{aligned} (I_N^{[2]})^*({}'\Omega_F) &= \frac{-1}{N\tau} F\left(\frac{-1}{N\tau}\right) dz_2 \wedge dz_1 \wedge d\left(\frac{-1}{N\tau}\right) \\ &= \tau \left(\frac{1}{N^2 \tau^4} F\left(\frac{-1}{N\tau}\right)\right) dz_1 \wedge dz_2 \wedge d\tau \\ &= {}'\Omega_{F|^4_{\iota_N}}, \end{aligned}$$

where $F|^k_{\iota_N}(\tau) := \frac{F(\iota_N(\tau))}{(\sqrt{N}\tau)^k}$. Set

$$F^+ := \tfrac{1}{2}\left(F + F|^4_{\iota_N}\right). \tag{8.4}$$

Taking the quotient by $I_N^{[2]}$

$$\mathcal{E}_1^{[2]}(N)^{+N} := \frac{{}'\mathcal{E}_1^{[2]}(N)\backslash\pi^{-1}(\mathrm{i}/\sqrt{N})}{I_N^{[2]}} \overset{\mathcal{P}_{+N}}{\twoheadleftarrow} {}'\mathcal{E}_1^{[2]}(N)\backslash\pi^{-1}(\mathrm{i}/\sqrt{N})$$

and replacing $\mathcal{E}_\Gamma^{[2]}$ in (8.3) by this, we get a family $\mathcal{X}_1^{[2]}(N)^{+N}$ of (smooth) Kummer $K3$ surfaces over $Y_1(N)^{+N}\backslash\{i/\sqrt{N}\}$. It may be more desirable to try to construct cycles on a Shioda–Inose $K3$ family, especially one over $Y_0(N)^{+N}$ — but this seems difficult to do canonically. If $\mathfrak{Z} \in CH^3(\mathcal{E}_1^{[2]}(N), 3)$ with $\theta_4(\Omega_{\mathfrak{Z}}) =: F_{\mathfrak{Z}}$, we may define a cycle

$$\mathfrak{Z}_{+N} := \frac{-1}{4N} p_{2*}p_1^*(\mathcal{P}_{+N})_*(J_N^{[2]})^*\mathfrak{Z} \in CH^3(\mathcal{X}_1^{[2]}(N)^{+N}, 3). \tag{8.5}$$

Also take $W \in CH^3(\mathcal{X}_1^{[2]}(N)^{+N}, 3)$ to be an arbitrary cycle.

Proposition 8.4. (i) ${}'\Omega_F$ *descends to a holomorphic three-form with* $\mathbb{Q}(3)$ *periods on* $\mathcal{X}_1^{[2]}(N)^{+N}$ *if and only if* $F \in M_4^{\mathbb{Q}}\left(\Gamma_1(N)^{+N}\right) := [M_4^{\mathbb{Q}}(\Gamma_1(N))]^+$.

(ii) $\widetilde{W} := (\tilde{J}_N^{[2]})^*(\mathcal{P}_{+N})^* p_{1*}p_2^* W$ *(on* $\mathcal{E}_1^{[2]}(N)$*) has "cycle-class"* $\theta_4(\Omega_{\tilde{W}}) \in M_4^{\mathbb{Q}}\left(\Gamma_1(N)^{+N}\right)$.

(iii) $\theta_4(\Omega_{\widetilde{\mathfrak{Z}_{+N}}}) = F_{\mathfrak{Z}}^+$.

Because ${}'\mathcal{E}_1^{[2]}(N)^{+N}$ is not a Kuga variety, we no longer have that pullbacks $\Omega_{\widetilde{W}}$ to $\mathcal{E}_1^{[2]}(N)$ have equal residues at cusps $\in \kappa_1(N)$ mapping to the same cusps $\in \kappa(N)^{+N}$. Consider for simplicity the residues at[22] [0]

[22]Note: the residues of F (hence F^+) at all $[j]$ ($j \in \mathbb{Z}$) are the same (as the residue at [0]).

and $[\mathrm{i}\infty]$, which are exchanged by the involution on $\mathcal{E}_1^{[2]}(N)$ induced by $\gamma_0 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, and assume $F \in M_4^{\mathbb{Q}}(\Gamma_1(N)^{+N})$ (which implies that $N^{-2}\tau^{-4}F\left(\frac{-1}{N\tau}\right) = F(\tau)$). Then

$$\begin{aligned}\mathfrak{R}_{[0]}(F) &= \lim_{\tau\to\mathrm{i}\infty} F|^4_{\gamma_0}(\tau) \\ &= \lim_{\tau\to\mathrm{i}\infty} \tau^{-4}F\left(-\frac{1}{\tau}\right) \xlongequal[\tau_0:=\frac{\tau}{N}]{} \lim_{\tau_0\to\mathrm{i}\infty} N^{-4}\tau_0^{-4}F\left(\frac{-1}{N\tau_0}\right) \\ &= N^{-2}\lim_{\tau_0\to\mathrm{i}\infty} F(\tau_0) = \frac{\mathfrak{R}_{[\mathrm{i}\infty]}(F)}{N^2}.\end{aligned}$$

If we assume only $F \in M_4^{\mathbb{Q}}(\Gamma_1(N))$, then

$$\begin{aligned}\lim_{\tau\to\mathrm{i}\infty} N^{-2}\tau^{-4}F\left(\frac{-1}{N\tau}\right) &\xlongequal[\tau_1:=N\tau]{} N^2\lim_{\tau_1\to\mathrm{i}\infty}\tau_1^{-4}F\left(-\frac{1}{\tau_1}\right) = N^2\lim_{\tau_1\to\mathrm{i}\infty} F|^4_{\gamma_0}(\tau_1) \\ &= N^2\mathfrak{R}_{[0]}(F).\end{aligned}$$

So

$$\begin{aligned}\mathfrak{R}_{[\mathrm{i}\infty]}(F^+) &= \frac{1}{2}\left\{\mathfrak{R}_{[\mathrm{i}\infty]}(F) + N^2\mathfrak{R}_{[0]}(F)\right\}, \\ \mathfrak{R}_{[0]}(F^+) &= \frac{1}{2}\left\{\frac{1}{N^2}\mathfrak{R}_{[\mathrm{i}\infty]}(F) + \mathfrak{R}_{[0]}(F)\right\}.\end{aligned} \tag{8.6}$$

This calculation shows $\left\langle \widetilde{\mathfrak{Z}_{+N}} \right\rangle$ is nontrivial if one picks $\mathfrak{Z}$ so that $\mathfrak{R}_{[\mathrm{i}\infty]}(F_{\mathfrak{Z}}) \neq -N^2\mathfrak{R}_{[0]}(F_{\mathfrak{Z}})$ (obviously possible by Section 8.1.2).

Remark 8.3. If we replace $I_N^{[2]}$ by the order 4 automorphism $'I_N^{[2]}(\tau; [z_1]_\tau, [z_2]_{\frac{-1}{N\tau}}) = (\frac{-1}{N\tau}; [-z_2]_{\frac{-1}{N\tau}}, [z_1]_\tau)$, then the corresponding quotient $'\mathcal{P}_{+N}$ yields a family of singular Kummer surfaces which is then resolved to yield a smooth $K3$ family $'\mathcal{X}_1^{[2]}(N)^{+N} \xrightarrow{\pi} Y_1(N)^{+N}$. Reworking this in analogy to (8.3) (so as not to pass through a singular variety), one constructs a cycle $'\mathfrak{Z}_{+N}$ and most of the exposition goes through as above with the crucial replacement of $F|^4_{\iota_N}$ by $-F|^4_{\iota_N}$ (and N^2 by $-N^2$ in (8.6)). In some sense this is the more natural construction (as the examples in Section 10 will suggest).

9. Regulator periods and higher normal functions (bis)

9.1. Setup for the fiberwise AJ computation

We restrict once more to $\Gamma = \Gamma(N)$ and the Kuga modular varieties $\mathcal{E}^{[\ell]}(N) \xrightarrow{\pi^{[\ell]}(N)} Y(N)$, and write their middle relative cohomology groups: $\mathbb{H}_N^{[\ell]} := R^\ell \pi^{[\ell]}(N)_* \mathbb{Z}$, $\mathcal{H}_N^{[\ell]} := \mathbb{H}_N^{[\ell]} \otimes \mathcal{O}_{Y(N)}$, $\mathcal{H}_N^{[\ell],\infty} := \mathbb{H}_N^{[\ell]} \otimes \mathcal{O}_{Y(N)^\infty}$, etc. — dropping the "$N$" to work on $\mathcal{E}^{[\ell]}/\mathfrak{H}$, and flipping super/sub-scripts for homology. One has the subsheaves of $\mathcal{G}^*(\implies \tilde{\mathcal{G}}^*)$-invariants $\mathrm{Sym}^\ell \mathbb{H}_{N,\mathbb{Q}}^{[1]} \subset \mathbb{H}_{N,\mathbb{Q}}^{[\ell]}$, $\mathrm{Sym}^\ell \mathcal{H}_N^{[1]} \subset \mathcal{H}_N^{[\ell]}$; as well as $\mathcal{G}^*$-coinvariants $\mathbb{H}_{[\ell]}^{N,\mathbb{Q}} \twoheadrightarrow \mathrm{Sym}_\ell \mathbb{H}_{[1]}^{N,\mathbb{Q}} \xrightarrow[\cong]{\mathcal{G}^* \circ \mathrm{P.D.}} \mathrm{Sym}^\ell \mathbb{H}_{N,\mathbb{Q}}^{[1]}$. There are the following well-defined sections $/\mathfrak{H}$ (multivalued $/Y(N)$):

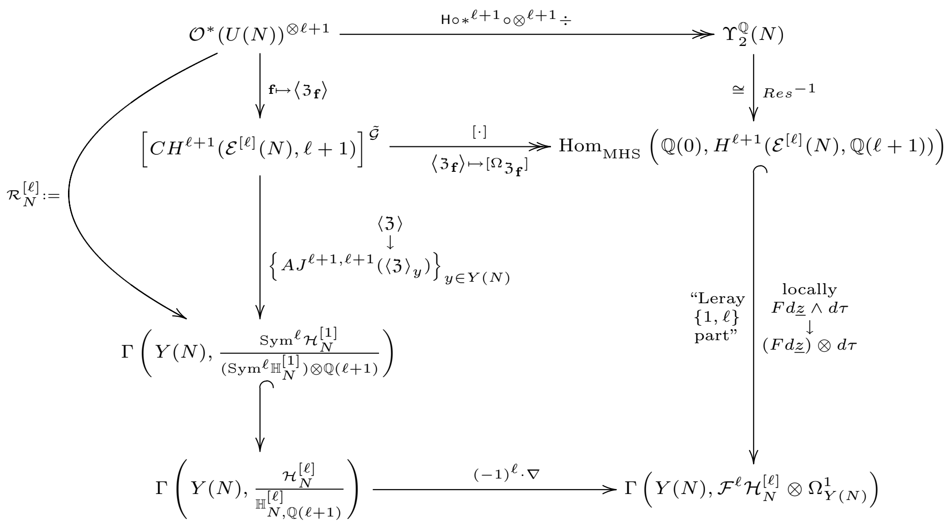
$$\begin{aligned}
\alpha &= \overrightarrow{[0,1]},\ \beta = \overrightarrow{[0,\tau]} \in \Gamma(\mathfrak{H}, \mathbb{H}_{[1]}),\\
\gamma_k^{[\ell]} &:= \alpha^{\ell-k}\beta^k \in \Gamma(\mathfrak{H}, \mathrm{Sym}_\ell \mathbb{H}_{[1]}^{\mathbb{Q}}),\\
\tilde{\gamma}_k^{[\ell]} &:= \mathcal{G}^*(\alpha_1 \times \cdots \times \alpha_{\ell-k} \times \beta_{\ell-k+1} \times \cdots \times \beta_\ell) \in \Gamma(\mathfrak{H}, \mathrm{Sym}^\ell \mathbb{H}_{\mathbb{Q}}^{[1]}),\\
\eta_{\ell-k}^{[\ell]} &:= \mathcal{G}^*(dz_1 \wedge \cdots \wedge dz_{\ell-k} \wedge d\bar{z}_{\ell-k+1} \wedge \cdots \wedge d\bar{z}_\ell)\\
&\qquad \in \Gamma(\mathfrak{H}, \mathcal{F}^{\ell-k}\mathrm{Sym}^\ell \mathcal{H}^{[1],\infty}),
\end{aligned}$$

where one should think of $\mathcal{G}^*$ as reordering the $dz/d\bar{z}$'s or α/β's in all possible ways and dividing by $\binom{\ell}{k}$. Writing $[\cdot]_k =$"term of homogeneous degree k in $\tau, \bar{\tau}$",

$$\left\langle \gamma_k^{[\ell]}, \eta_{\ell-j}^{[\ell]} \right\rangle = \binom{\ell}{k}^{-1} \left[(1+\tau)^{\ell-j}(1+\bar{\tau})^j \right]_k = \frac{\sum_{a=0}^k \binom{\ell-j}{a}\binom{j}{k-a}\tau^a \bar{\tau}^{k-a}}{\binom{\ell}{k}} =: \mathfrak{P}_{jk}^{[\ell]} \tag{9.1}$$

Viewed as the monodromy transformation corresponding to an element of $\pi_1(Y(N))$, $\gamma \in \Gamma(N)$ acts on $(\gamma_0^{[\ell]}, \ldots, \gamma_\ell^{[\ell]})$ from the right, as $\mathrm{Sym}^\ell \gamma$; we think of the $\gamma_i^{[\ell]}$ as degree-ℓ homogeneous polynomials in α and β, with $\mu_{\mathrm{i}\infty} := \left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right) : \beta \mapsto \beta + N\alpha$, $\alpha \mapsto \alpha$ and $\mu_0 := \left(\begin{smallmatrix} 1 & 0 \\ N & 1 \end{smallmatrix}\right) : \beta \mapsto \beta$, $\alpha \mapsto \alpha + N\beta$. (Also, γ sends $\eta_{\ell-k}^{[\ell]} \mapsto \frac{\eta_{\ell-k}^{[\ell]}}{(c\tau+d)^{\ell-k}(c\bar{\tau}+d)^k}$; note that the $\{\eta_{\ell-k}^{[\ell]}\}$ and $\gamma_0^{[\ell]}$ are well-defined over an analytic neighborhood of $[\mathrm{i}\infty]$ in $Y(N)$.)

Now refer to the cycle-construction of Section 7.3.4, denote the fiberwise "slices" (pullbacks) of $\langle \mathfrak{Z}_{\mathbf{f}} \rangle$ by $\langle \mathfrak{Z}_{\mathbf{f}} \rangle_{y \text{ (or } \tau)}$, etc.; and consider the diagram



in which the upper square commutes by the proof of Corollary 8.3. Write simply $\mathcal{R}_{\mathbf{f}}(y)$ for the $\mathcal{R}_N^{[\ell]}$-image of $\mathbf{f}$; if we pull this back to $\mathfrak{H}$, we may choose a well-defined lift $\tilde{\mathcal{R}}_{\mathbf{f}}(\tau) \in \Gamma(\mathfrak{H}, \mathrm{Sym}^\ell \mathcal{H}^{[1]})$.

Lemma 9.1. (i) *The bottom square commutes.*

(ii) ∇ *is surjective.*

Proof. (i) $\langle \mathfrak{Z} \rangle \in CH^{\ell+1}(\mathcal{E}^{[\ell]}(N), \ell+1)$ has $T_{\mathfrak{Z}} \overset{\text{hom}}{\equiv} 0$ on $(\pi^{[\ell]}(N))^{-1}(\text{disk})$; so locally we may write $R'_{\mathfrak{Z}} := R_{\mathfrak{Z}} + (2\pi \mathrm{i})^{\ell+1} \delta_{\partial^{-1} T_{\mathfrak{Z}}}$ and compute $\nabla [R'_{\mathfrak{Z}}]_y = (d[R'_{\mathfrak{Z}}])^{\{1,\ell\}} = \Omega_{\mathfrak{Z}}^{\{1,\ell\}}$.

(ii) follows from irreducibility of the monodromy action on $\mathrm{Sym}^{[\ell]} \mathbb{H}_N^{[1]}$ and consequent vanishing of the space of (∇-)flat $\mathcal{G}^*$-symmetric normal functions $\Gamma \left(Y(N), \frac{(\mathrm{Sym}^\ell \mathbb{H}_N^{[1]}) \otimes \mathbb{C}}{(\mathrm{Sym}^\ell \mathbb{H}_N^{[1]}) \otimes \mathbb{Q}(\ell+1)} \right)$. Explicitly, given any $\Gamma = \sum_{k=0}^{\ell} \epsilon_k \tilde{\gamma}_k^{[\ell]}$ ($\{\epsilon_k\} \in \mathbb{C}$), the coefficients of $\tilde{\gamma}_j^{[\ell]}$ in $\mu_{\mathrm{i}\infty}(\Gamma) - \Gamma = \sum_{j=0}^{\ell-1} \left(\sum_{k=j+1}^{\ell} \binom{k}{j} \epsilon_k N^{k-j} \right) \tilde{\gamma}_j^{[\ell]}$ must belong to $\mathbb{Q}(\ell+1)$; inductively one has $\epsilon_\ell, \epsilon_{\ell-1}, \ldots, \epsilon_1 \in \mathbb{Q}$. To show $\epsilon_0 \in \mathbb{Q}$, similarly apply $\mu_0 - \mathrm{id}$. □

Corollary 9.1. *$\mathcal{R}_{\boldsymbol{f}}(y)$ depends only on $\{\mathsf{H}_\sigma^{[\ell]}(\varphi_{\boldsymbol{f}})\} \in \Upsilon_2^{\mathbb{Q}}(N)$ (or on $\varphi_{\boldsymbol{f}} \in \Phi_2^{\mathbb{Q}}(N)$).*

According to Sections 7.2.4 and 7.2.5, it therefore suffices to compute $\mathcal{R}_{\mathbf{f}}$ for $\mathbf{f}\in\mathbb{Q}\left[\mathfrak{F}(N)^{\times(\ell+1)}_{\left(\begin{smallmatrix}p & q\\ -s & r\end{smallmatrix}\right)}\right]$ for "each" (p,q). (In fact, it suffices to do so for $(p,q)=(0,1)$ and $(1,0)$, but it is computationally *convenient* to consider at least our choices of $\left(\begin{smallmatrix}p & q\\ -s & r\end{smallmatrix}\right)$ for each *cusp* $\sigma\in\kappa(N)$.)

For a fixed choice of lift $\tilde{\mathcal{R}}^{[\ell]}_{\mathbf{f}}$ (to be discussed), write

$$\tilde{\mathcal{R}}^{[\ell]}_{\mathbf{f}}(\tau) =: \sum_{k=0}^{\ell} R^{[\ell]}_{\mathbf{f},j}(\tau)\left[\eta^{[\ell]}_{\ell-j}\right]. \tag{9.2}$$

We then define regulator periods

$$\Psi^{[\ell]}_{\mathbf{f},k}(\tau) := \left\langle \gamma^{[\ell]}_k, \tilde{\mathcal{R}}^{[\ell]}_{\mathbf{f}}(\tau)\right\rangle \quad (k=0,\ldots,\ell) \tag{9.3}$$

and a higher normal function[23]

$$V^{[\ell]}_{\mathbf{f}}(\tau) := \left\langle \tilde{\mathcal{R}}^{[\ell]}_{\mathbf{f}}(\tau), \eta^{[\ell]}_{\ell}\right\rangle = (-1)^{\binom{\ell+1}{2}}\nu^{\ell} R^{[\ell]}_{\mathbf{f},\ell}(\tau). \tag{9.4}$$

These are the objects which we aim (in the next subsection) to compute with the [50] formula; first we can derive a number of their properties by "pure thought".

Holomorphicity: Since $\nabla_{\partial_\tau}\tilde{\mathcal{R}}_{\mathbf{f}}(\tau)=0\in\Gamma(\mathfrak{H},\mathcal{H}^{[\ell]})$, $V^{[\ell]}_{\mathbf{f}}$ and the $\{\Psi^{[\ell]}_{\mathbf{f},k}\}$ belong to $\mathcal{O}(\mathfrak{H})$. The $\{R^{[\ell]}_{\mathbf{f},j}\}$ are *not* holomorphic since the $[\eta^{[\ell]}_j]$ are not (except for $\eta^{[\ell]}_\ell$):

$$\nabla\eta^{[\ell]}_j = j\,\frac{[\eta^{[\ell]}_{j-1}]-[\eta^{[\ell]}_j]}{\nu}\otimes d\tau - (\ell-j)\frac{[\eta^{[\ell]}_{j+1}]-[\eta^{[\ell]}_j]}{\nu}\otimes d\bar{\tau}. \tag{9.5}$$

Picard–Fuchs equations: Let $\nabla^{\mathbf{f}}_{\mathrm{PF}}=\nabla^{\ell+1}_{\partial_\tau}+\cdots$ denote the PF operator for $\Omega^{[\ell]}_{\mathbf{f}}(\tau):=(2\pi\mathrm{i})^{\ell+1}F_{\mathbf{f}}(\tau)[\eta^{[\ell]}_\ell]\in\Gamma(\mathfrak{H},\mathcal{F}^\ell\mathcal{H}^{[\ell]})$. Writing $\bar{\nabla}_{\partial_\tau}:{}^{\mathcal{F}^j}\!/\!_{\mathcal{F}^{j+1}}\to{}^{\mathcal{F}^{j-1}}\!/\!_{\mathcal{F}^j}$, (9.5) $\implies$ $\bar{\nabla}_{\partial_\tau}\eta^{[\ell]}_j=\frac{j}{\nu}[\eta^{[\ell]}_{j-1}]$ $\implies$ $\bar{\nabla}^{\ell}_{\partial_\tau}\eta^{[\ell]}_\ell=\frac{\ell!}{\nu^\ell}[\eta^{[\ell]}_0]$, which yields the "stupid

[23]It would make more sense on $Y(N)$ to take $V(\tau)=\left\langle\tilde{\mathcal{R}},\mathsf{F}\eta_\ell\right\rangle$ for some $\mathsf{F}\in M_\ell(\Gamma(N))$; we will essentially do this later.

Yukawa coupling"

$$\begin{aligned} Y_{\tau^\ell}(\tau) &:= \left\langle \eta_\ell^{[\ell]},\, \nabla^\ell_{\partial_\tau}\eta_\ell^{[\ell]} \right\rangle \\ &= (-1)^{\binom{\ell}{2}} \frac{\ell!}{\nu^\ell} \int dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_\ell \wedge d\bar{z}_\ell = (-1)^{\binom{\ell}{2}} \ell!. \end{aligned}$$

Moreover, $\nabla^{\ell+1}_{\partial_\tau}\eta_\ell^{[\ell]} = 0$ as $\eta_\ell^{[\ell]}$ has periods $1, \tau, \ldots, \tau^\ell$.

Proposition 9.1. (i) *The* $\{\Psi^{[\ell]}_{\boldsymbol{f},k}\}$ *satisfy the homogeneous equation* $(D^{\boldsymbol{f}}_{PF} \circ \partial_\tau)(\cdot) = 0$. *More precisely,* $\frac{d\Psi^{[\ell]}_{\boldsymbol{f},k}}{d\tau} = (-1)^\ell (2\pi \mathrm{i})^{\ell+1} \tau^k F_{\boldsymbol{f}}(\tau)$.

(ii) $V^{[\ell]}_{\boldsymbol{f}}$ *satisfies, for any lift* $\tilde{\mathcal{R}}_{\boldsymbol{f}}$, *the inhomogenous equation*

$$\partial_\tau^{\ell+1}(\cdot) = (-1)^{\binom{\ell+1}{2}} (2\pi\mathrm{i})^{\ell+1} \ell! F_{\boldsymbol{f}}(\tau); \tag{9.6}$$

i.e., the higher normal function is (const. $\times$*) an Eichler integral of* $F_{\boldsymbol{f}}$. *The various* $\{V^{[\ell]}_{\boldsymbol{f}}\}$ *resulting from the different lifts yield a basis of solutions for* (9.6).

Proof. (i) Lemma 9.1(i) says $\nabla_{\partial_\tau}\tilde{\mathcal{R}}^{[\ell]}_{\mathbf{f}} = (-1)^\ell \Omega^{[\ell]}_{\mathbf{f}}$; the result follows.

(ii) There are two ways to do this, both instructive:

Method I:

$$\begin{aligned} &\partial_\tau^{\ell+1} \left\langle \tilde{\mathcal{R}}_{\mathbf{f}}, \eta_\ell \right\rangle \\ &= \partial_\tau^\ell \left\langle \tilde{\mathcal{R}}_{\mathbf{f}}, \nabla_{\partial_\tau}\eta_\ell \right\rangle = \cdots [\text{using } \langle \eta_\ell, \nabla^p_{\partial_\tau}\eta_\ell \rangle = 0 \quad \forall p < \ell] \cdots \\ &= \partial_\tau \left\langle \tilde{\mathcal{R}}_{\mathbf{f}}, \nabla^\ell_{\partial_\tau}\eta_\ell \right\rangle = (-1)^\ell (2\pi\mathrm{i})^{\ell+1} \left\langle F_{\mathbf{f}}\eta_\ell, \nabla^\ell_{\partial_\tau}\eta_\ell \right\rangle + \left\langle \tilde{\mathcal{R}}_{\mathbf{f}}, \nabla^{\ell+1}_{\partial_\tau}\eta_\ell \,[=0] \right\rangle \\ &= (-1)^\ell (2\pi\mathrm{i})^{\ell+1} \left\langle F_{\mathbf{f}}\eta_\ell^{[\ell]}, \frac{\ell!}{\nu^\ell}\eta_0^{[\ell]} + \mathcal{F}^1 \right\rangle = (-1)^{\ell+\binom{\ell}{2}} (2\pi\mathrm{i})^{\ell+1} \ell! \frac{F_{\mathbf{f}}}{\nu^\ell}\nu^\ell. \end{aligned}$$

Method II: Note that $\log(\mu_{\mathrm{i}\infty})\tilde{\gamma}^{[\ell]}_j = j\tilde{\gamma}^{[\ell]}_{j-1} (= 0$ if $j = 0)$. Taking the privileged extension basis (single-valued on $\overline{Y}(N)$, in a neighborhood of $[\mathrm{i}\infty]$)

$$\hat{\gamma}^{[\ell]}_j := \mathrm{e}^{-\tau \log(\mu_{\mathrm{i}\infty})}\tilde{\gamma}^{[\ell]}_j \overset{\nabla_{\partial_\tau}}{\mapsto} -\mathrm{e}^{-\tau\log(\mu_{\mathrm{i}\infty})}\log(\mu_{\mathrm{i}\infty})\tilde{\gamma}^{[\ell]}_j = -j\hat{\gamma}^{[\ell]}_{j-1},$$

we write $\tilde{\mathcal{R}}_{\mathbf{f}}^{[\ell]} = \sum \hat{\psi}_j \hat{\gamma}_j^{[\ell]}$. Applying ∇_{∂_τ}, and using $\hat{\gamma}_\ell^{[\ell]} \equiv \eta_\ell^{[\ell]}$, yields

$$\left(\sum_{j=0}^{\ell-1} \left\{ \frac{\partial \hat{\psi}_j}{d\tau} - (j+1)\hat{\psi}_{j+1} \right\} \hat{\gamma}_j^{[\ell]} + \frac{d\hat{\psi}_\ell}{d\tau} \hat{\gamma}_\ell^{[\ell]} \right) \otimes d\tau$$
$$= (-1)^\ell \Omega_{\mathbf{f}}^{[\ell]} \otimes d\tau$$
$$= (-1)^\ell (2\pi \mathrm{i})^{\ell+1} F_{\mathbf{f}} \hat{\gamma}_\ell^{[\ell]} \otimes d\tau.$$

So

$$\text{(9.7)} \qquad \begin{cases} \hat{\psi}_\ell = (-1)^\ell (2\pi\mathrm{i})^{\ell+1} \int F_{\mathbf{f}} d\tau \\ \hat{\psi}_j = (j+1) \int \hat{\psi}_{j+1} d\tau \quad (j = 0, \dots, \ell-1); \end{cases}$$

while $V_{\mathbf{f}}^{[\ell]} = \sum \hat{\psi}_j \langle \hat{\gamma}_j, \hat{\gamma}_\ell \rangle = (-1)^{\binom{\ell}{2}} \hat{\psi}_0$. To see the "basis" assertion: modifying $\tilde{\mathcal{R}}_{\mathbf{f}}$ changes $V_{\mathbf{f}}$ by a polynomial in τ (coefficients $\in \mathbb{Q}(\ell+1)$) of degree $\leq \ell$. □

Remark. If we notate $\tilde{\mathcal{R}}_{\mathbf{f}}^{[\ell]} = \sum \psi_j \tilde{\gamma}_j$, then $\begin{pmatrix} \psi_\ell \\ \vdots \\ \psi_0 \end{pmatrix} = \mathrm{e}^{\tau \log[\mu_{\mathrm{i}\infty}]_\gamma} \begin{pmatrix} \hat{\psi}_\ell \\ \vdots \\ \hat{\psi}_0 \end{pmatrix}$ and this may be used to "compute" $\Psi_{\mathbf{f},k}^{[\ell]} = \langle \tilde{\gamma}_k, \tilde{\gamma}_{\ell-k} \rangle \psi_{\ell-k} = \frac{(-1)^{k+\binom{\ell}{2}}}{\binom{\ell}{k}} \psi_{\ell-k}$.

Monodromy and special values at $[\mathrm{i}\infty]$*:* (This cusp will play a distinguished role later.) If $F_{\mathbf{f}}(\tau) \to 0$ as $\tau \to \mathrm{i}\infty$, then integrating $(-1)^\ell (2\pi\mathrm{i})^\ell F_{\mathbf{f}}(q) \hat{\gamma}_\ell^{[\ell]} \otimes \frac{dq}{q} = \nabla \tilde{\mathcal{R}}_{\mathbf{f}}^{[\ell]}$ yields on a disk $\Delta \subset Y(N)$ (containing $\{y = 0\} = [\mathrm{i}\infty]$):

$$\text{(9.8)} \qquad (2\pi\mathrm{i})^{\ell+1} \sum_{j=0}^{\ell} (Q_j + qP_j(\tau)) \tilde{\gamma}_j^{[\ell]}, \quad Q_j \in \mathbb{C} \text{ and } P_j \in \mathcal{O}(\Delta)[X].$$

Since $(\mu_{\mathrm{i}\infty} - \mathrm{id}) \tilde{\mathcal{R}}_{\mathbf{f}}^{[\ell]}$ is of the form $(2\pi\mathrm{i})^{\ell+1} \sum_{j=0}^{\ell} Q'_j \tilde{\gamma}_j^{[\ell]}$, we deduce that the $Q_j \in \mathbb{Q}$ for $j \neq 0$. A change of lift $\tilde{\mathcal{R}}_{\mathbf{f}}$ merely changes the $\{Q_j\}$ (including Q_0) by rational numbers.

Proposition 9.2. *Suppose* $\mathsf{H}_{[\mathrm{i}\infty]}^{[\ell]}(\varphi_{\boldsymbol{f}}) = 0$*, and set* $\mathfrak{K}_i := \lim_{\tau \to \mathrm{i}\infty} \Psi_{\boldsymbol{f},i}^{[\ell]}(\tau)$*.*

(i) $\mathfrak{K}_i \in \mathbb{Q}(\ell+1)$ *for* $0 \leq i < \ell$*.*

(ii) *The value of* $\mathfrak{K}_\ell \in \mathbb{C}/\mathbb{Q}(\ell+1)$ *is independent of the lift (i.e., depends only on the other* $\{\mathsf{H}_\sigma^{[\ell]}(\varphi_{\boldsymbol{f}})\}_{(\sigma \neq \mathrm{i}\infty)}$*).*

(iii) *Lift* $\tilde{\mathcal{R}}_{\boldsymbol{f}}^{[\ell]}$ *chosen so that* $\{\mathfrak{K}_i\}_{i=0}^{\ell-1}$ *vanish* $\iff$ $\mathfrak{K} := \lim_{\tau\to i\infty} V_{\boldsymbol{f}}^{[\ell]}(\tau)$ *defined. In this case,* $\mathfrak{K} = (-1)^{\ell}\mathfrak{K}_{\ell}$ *and*

$$V_{\boldsymbol{f}}^{[\ell]}(q) = \mathfrak{K} + (-1)^{\binom{\ell+1}{2}}\ell! \int_0 F_{\boldsymbol{f}}(q)\frac{dq}{q} \circ \cdots \circ \frac{dq}{q}. \tag{9.9}$$

Proof. Conditions (i) and (ii) are clear from (9.8). For (iii) (except (9.9)), plug (9.8) into $\left\langle \cdot, \eta_{\ell}^{[\ell]} \right\rangle$. (9.9) follows from ($\tau \to$ i∞) $\{\Psi_{\mathbf{f},i}^{[\ell]} \to 0$ for $0 \le i < \ell\}$ if and only if $\{\psi_i \to 0$ for $0 < i \le \ell\}$ if and only if $\{\hat{\psi}_i \to 0$ for $0 < i \le \ell\}$ if and only if every $\int$ but the last in (9.7) is taken from $\tau = $ i∞. □

Remark 9.1. (a) $\mathsf{H}_{[i\infty]}^{[\ell]}(\varphi_{\mathbf{f}}) = 0$ means that (an AJ-trivial modification of) $\langle \mathfrak{Z}_{\mathbf{f}} \rangle$ extends across the Néron N-gon $\hat{E}_{[i\infty]}^{[\ell]}(N)$, and $\mathfrak{K}_{\ell}$ is essentially AJ of its restriction (in $H^{\ell}(\hat{E}_{[i\infty]}^{[\ell]}(N), \mathbb{C}/\mathbb{Q}(\ell+1))$). Even with this being well-defined, and even if $\tilde{\mathcal{R}}_{\mathbf{f}}^{[\ell]}$ is normalized as in (iii) above, it *need not* be free of monodromy about $y = 0$! (Of course, when it *is* monodromy-free, the $\{R_{\mathbf{f},k}\}$, $V_{\mathbf{f}}$, and $\Psi_{\mathbf{f},0}$ all follow suit.) This issue has to do with $\pi^{[\ell]}(N)\,(|T_{\mathfrak{Z}_{\mathbf{f}}}|) \subset Y(N)$ and is related to Proposition 4.1.

(b) The lifts used below are chosen for computability rather than vanishing of $\{\mathfrak{K}_i\}$.

(c) One reason we have to do the AJ computation below is to find $\mathfrak{K}_{\ell}$, if $\mathsf{H}_{[i\infty]}^{[\ell]}(\varphi_{\mathbf{f}}) = 0$ (though we are most interested in the case $\mathsf{H}_{[i\infty]}^{[\ell]}(\varphi_{\mathbf{f}}) \neq 0$).

For an arbitrary $\mathbf{f}$, here is the "lift" we use to apply KLM:

- break it up in $\mathcal{O}^*(U(N))^{\otimes(\ell+1)}$ into $\sum_{\alpha} \mathbf{f}^{\alpha}$, with each $\varphi_{\mathbf{f}^{\alpha}} \in \Phi_2^{\mathbb{Q}}(N)_{(p,q)}^{\circ}$ for some (p,q) as in Section 7.2.4. This step is not well-defined w.r.t. the final outcome. Next,
- break each $\mathbf{f}^{\alpha}$ into $\sum_{\beta} \mathbf{f}^{\alpha\beta}$, with each $\mathbf{f}^{\alpha\beta} = (f_1^{\alpha\beta}, \ldots, f_{\ell+1}^{\alpha\beta}) \in \mathfrak{F}(N)^{\times(\ell+1)}_{\left(\begin{smallmatrix} p & q \\ -s & r \end{smallmatrix}\right)}$ for some $(-s, r)$ as in Section 7.2.5; then
- construct $\tilde{\mathcal{R}}_{\mathbf{f}^{\alpha\beta}}$ as in the next section, and apply KLM.

The last two steps will yield a well-defined map

$$\Phi_2^{\mathbb{Q}}(N)_{(p,q)}^{\circ} \to \Gamma(\mathfrak{H}, \mathrm{Sym}^{\ell}\mathcal{H}^{[1]}),$$

as will be clear from the computations.

Remark. $\mathsf{H}_\sigma(\varphi_{\mathbf{f}^\alpha})$ (or $\mathsf{H}_\sigma(\varphi_{\mathbf{f}^{\alpha\beta}})$) is 0 for those $\sigma \longleftrightarrow (-s_0, r_0) \in \langle (p,q) \rangle \subset (\mathbb{Z}/N\mathbb{Z})^2$, but not necessarily for any other $\sigma \in \kappa(N)$.

9.2. Applying the KLM formula

This will take place on (subsets of) $\mathcal{E}^{[\ell]}$ rather than $\mathcal{E}^{[\ell]}(N)$; instead of writing $\mathcal{P}_N^*$ constantly to pull functions and cycles back to $\mathcal{E}$ ($\xrightarrow{\pi} \mathfrak{H}$), we will take this to be understood.

Fix a choice of $p, q \in \mathbb{Z}$ such that $\langle (\bar{p}, \bar{q}) \rangle \cong \mathbb{Z}/N\mathbb{Z} \subset (\mathbb{Z}/N\mathbb{Z})^2$. Taking any r, s "completing" this to an element $M = \left(\begin{smallmatrix} p & q \\ -s & r \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, we consider $\mathbf{f} = (f_1, \ldots, f_{\ell+1}) \in \mathfrak{F}(N)_M^{\times(\ell+1)}$, and compute the $\{R_{\mathbf{f},k}^{[\ell]}(\tau)\}$ for a particular choice of lift $\tilde{\mathcal{R}}_{\mathbf{f}}^{[\ell]}(\tau)$ over $(\tau \in) \, \mathcal{A}_M$, with $\mathfrak{F}(N)_M$ and $\mathcal{A}_M$ as in Section 7.2.5. We then use this to compute the $\Psi_{\mathbf{f},j}^{[\ell]}$ over $\mathcal{A}_M$, analytically continue these to $\mathfrak{H}$, and employ the result to find the (nonholomorphic) $\{R_{\mathbf{f},k}^{[\ell]}(\tau)\}$ over all of $\mathfrak{H}$.

The choice of lift over $\mathcal{A}_M$ must be dealt with in two cases, according as whether for the Pontryagin product of (p,q)-vertical sets

$$0 \notin |T_{f_1}| * \cdots * |T_{f_{\ell+1}}| \text{ on } \pi^{-1}(\mathcal{A}_M) \subset \mathcal{E}. \tag{9.10}$$

If this is true, then (on all of $\mathcal{E}$) $\{0\} \notin |(f_1)| * \cdots * |(f_{\ell+1})|$ and (on $\mathcal{E}^{[\ell]}$) we can take $\mathfrak{Z}_{\mathbf{f}} :=$ Zariski closure of $Z_{\mathbf{f}} = \tilde{\mathcal{G}}^* \iota^* \{\mathbf{f}\}$ (see Section 7.3.4). With this understood, we have

Lemma 9.2. *Equation* (9.10) $\iff$ $|T_{\mathfrak{Z}_f}| = \emptyset$ *on* $\mathcal{E}_{\mathcal{A}_M}^{[\ell]} := (\pi^{[\ell]})^{-1}(\mathcal{A}_M) \subset \mathcal{E}^{[\ell]}$.

Proof. Since $\iota\ (E_\tau^{[\ell]}) = \{u_1 + \cdots + u_{\ell+1} = 0\} \subset E_\tau^{[\ell+1]}$, $0 \in |T_{f_1}| * \cdots * |T_{f_{\ell+1}}| \subset E_\tau \iff 0 \equiv u_1 + \cdots + u_{\ell+1}$ for some $(u_1, \ldots, u_{\ell+1}) \in |T_{f_1}| \cap \cdots \cap |T_{f_{\ell+1}}| \subset E_\tau^{[\ell+1]} \iff \exists (u_1, \ldots, u_{\ell+1}) \in T_{f_1} \cap \cdots \cap T_{f_{\ell+1}} \cap \iota(E_\tau^{[\ell]}) \iff |T_{\iota^*\{\mathbf{f}\}}|$ nonempty. □

As a consequence we can take as our lift

$$\tilde{\mathcal{R}}_{\mathbf{f}}^{[\ell]}(\tau) := [R_{\mathfrak{Z}_{\mathbf{f},\tau}}] \in H^\ell(E_\tau^{[\ell]}, \mathbb{C}) \text{ for } \tau \in \mathcal{A}_M,$$

since (on each fiber) $dR_{\mathfrak{Z}_{\mathbf{f},\tau}} = (2\pi \mathrm{i})^{\ell+1} \delta_{T_{\mathfrak{Z}_{\mathbf{f},\tau}}} = 0$.

Informal remarks on well-definedness: Given $\mathbf{f}\in\mathfrak{F}(N)^{\times(\ell+1)}_{\left(\begin{smallmatrix} p & q\\ -s & r\end{smallmatrix}\right)}$, $\mathbf{g}\in\mathfrak{F}(N)^{\times(\ell+1)}_{\left(\begin{smallmatrix} p & q\\ -s' & r'\end{smallmatrix}\right)}$, with $\varphi_{\mathbf{f}}=\varphi_{\mathbf{g}}\in\Phi_2^{\mathbb{Q}}(N)^{\circ}_{(p,q)}$ and satisfying (9.10), taking limits along $\mathcal{A}_M$ resp. $\mathcal{A}_{M'}$ one finds that $\lim_{\tau\to-\frac{q}{p}}\tilde{\mathcal{R}}^{[\ell]}_{\mathbf{f}}$, $\lim_{\tau\to-\frac{q}{p}}\tilde{\mathcal{R}}^{[\ell]}_{\mathbf{g}}$ yield classes in $H^{\ell}(\hat{E}^{[\ell]}_{-\frac{q}{p}},\mathbb{C})$ (the $\{\mathfrak{K}^{(')}_i\}_{i=0}^{\ell-1}$ vanish). Also, by Proposition 9.2(ii) these classes are equal up to $H^{\ell}(\hat{E}^{[\ell]}_{-\frac{q}{p}},\mathbb{Q}(\ell+1))$; hence the lifts differ at most by $\mathbb{Q}(\ell+1)\left\langle p[\beta]+q[\alpha]\right\rangle$ on $\mathfrak{H}$. That they are in fact equal may be argued from Lemma 8.4, but the computations below will bear witness to all of this (including the irrelevancy of $(-s,r)$).

Now we compute the $\{R^{[\ell]}_{\mathbf{f},j}\}$ for our lift. the diagram (8.1) is replaced for this purpose by

$$E^{\ell}_{\tau}\overset{\iota}{\hookrightarrow}E^{\ell+1}_{\tau}\overset{P}{\twoheadrightarrow}E_{\tau},\quad \tau\in\mathcal{A}_M,$$

with resp. coordinates $z_1,\ldots,z_\ell$; $u_1,\ldots,u_{\ell+1}$; u, and the π's by integration. Write $\Gamma:=H^1(E_\tau,\mathbb{Z})=\mathbb{Z}\left\langle[\alpha],[\beta]\right\rangle$, $\gamma=m[\beta]+n[\alpha]=(m,n)\in\Gamma$.

Remarks on currents: (i) The fact that $\mathfrak{Z}_{\mathbf{f}}=\overline{Z_{\mathbf{f}}}$ means that if $\bar{\mathsf{U}}_{N,\epsilon}\subset E_\tau$ denotes the complement of ϵ-disks about the N-torsion points, then $\left\langle[R_{\mathfrak{Z}_{\mathbf{f}}}],\eta^{[\ell]}_j\right\rangle=\lim_{\epsilon\to0}\int_{\bar{\mathsf{U}}^{\ell}_{N,\epsilon}}R_{Z_{\mathbf{f}}}\wedge\eta^{[\ell]}_j$ — but we will just view $R_{Z_{\mathbf{f}}}$ as an L^1-form on E^{ℓ}_{τ} (rather than write this).

(ii) $R_{\{\mathbf{f}\}}=\sum_{j=1}^{\ell+1}(2\pi\mathrm{i})^{j-1}(-1)^{\ell(j-1)}\log f_j(u_j)\,d\log f_{j+1}(u_{j+1})\wedge\cdots\wedge d\log f_{\ell+1}(u_{\ell+1})\cdot\delta_{T_{f_1(u_1)}}\cdot\cdots\cdot\delta_{T_{f_{j-1}(u_{j-1})}}$ is a normal current (of intersection type with respect to $\iota(E^{\ell}_{\tau})$) on $E^{\ell+1}_{\tau}$, so admits pullback $\iota^*R_{\{\mathbf{f}\}}=R_{\iota^*\{\mathbf{f}\}}$ to E^{ℓ}_{τ} (see Section 8 of [49]). We also note that the "singularities" of $P_*(R_{\{\mathbf{f}\}}\wedge\tilde{\eta}^{[\ell]}_j)$ are contained in $|T_{f_1}|*\cdots*|T_{f_{\ell+1}}|\subset E_\tau$, and so are as in Lemma 8.1(ii). Write $\hat{\sum}_{\gamma\in\Gamma}$ for the $\sum_k\sum_j^{P.V.}$ described there (and depending on (p,q)).

Writing

$$\begin{gathered}E^{\ell+1}_{\tau}\overset{\pi_{\widehat{\ell+1}}}{\twoheadrightarrow}E^{\ell}_{\tau}\\(u_1,\ldots,u_\ell,u_{\ell+1})\mapsto(u_1,\ldots,u_\ell),\end{gathered}$$

let

$$\tilde{\eta}^{[\ell]}_j:=(-1)^{\ell}\pi^*_{\widehat{\ell+1}}\eta^{[\ell]}_j=(-1)^{\ell}\binom{\ell}{j}^{-1}\sum_{\substack{|J|=j\\J\subseteq\{1,\ldots,\ell\}}}du_1^{\{J\}}\wedge\cdots\wedge du_\ell^{\{J\}}\in A^{\ell-k,k}(E^{\ell+1}_{\tau}),$$

where $du_i^{\{J\}} := \begin{cases} du_i, & i \in J \\ d\bar{u}_i, & i \notin J \end{cases}$. We then have $\iota^* \tilde{\eta}_j^{[\ell]} = \eta_j^{[\ell]}$, and so:

$$\begin{aligned}
&\frac{(-1)^{\binom{\ell+1}{2}}(-1)^{\ell-j}\nu^\ell}{\binom{\ell}{j}} R_{\mathbf{f},j}^{[\ell]}(\tau) \\
&= R_{\mathbf{f},j}^{[\ell]}(\tau) \int_{E_\tau^\ell} \eta_{\ell-j}^{[\ell]} \wedge \eta_j^{[\ell]} \\
&= \left\langle \tilde{\mathcal{R}}_{\mathbf{f}}^{[\ell]}, \eta_j^{[\ell]} \right\rangle = \int_{E_\tau^\ell} R_{Z_{\mathbf{f}}} \wedge \eta_j^{[\ell]} = \int_{E_\tau^\ell} \tilde{\mathcal{G}}^* R_{\iota^*\{\mathbf{f}\}} \wedge \tilde{\mathcal{G}}^* \eta_j^{[\ell]} \\
&= \int_{E_\tau^\ell} R_{\iota^*\{\mathbf{f}\}} \wedge \eta_j^{[\ell]} = \int_{\iota(E_\tau^\ell)} R_{\{\mathbf{f}\}} \wedge \tilde{\eta}_j^{[\ell]} = \left\{ P_* \left(R_{\{\mathbf{f}\}} \wedge \tilde{\eta}_j^{[\ell]} \right) \right\}(0) \\
&= \hat{\sum_{\gamma\in\Gamma}} \widehat{P_*(R_{\{\mathbf{f}\}} \wedge \tilde{\eta}_j^{[\ell]})}(\gamma) = \nu^{-1} \hat{\sum_{\gamma\in\Gamma}} \int_{E_\tau} \overline{\chi_\gamma} P_*(R_{\{\mathbf{f}\}} \wedge \tilde{\eta}_j^{[\ell]}) \wedge du \wedge d\bar{u} \\
&= \nu^{-1} \hat{\sum_{\gamma\in\Gamma}} \int_{E_\tau^{\ell+1}} P^* \overline{\chi_\gamma} \cdot R_{\{\mathbf{f}\}} \wedge \tilde{\eta}_j^{[\ell]} \wedge P^*(du \wedge d\bar{u}) \\
&= \nu^{-1} \binom{\ell}{j}^{-1} \sum_{j_0=1}^{\ell+1} (2\pi \mathrm{i})^{j_0-1} (-1)^{\ell j_0} \sum_{\substack{|J|=j \\ J\subseteq\{1,\ldots,\ell\}}} \hat{\sum_{\gamma\in\Gamma}} \\
&\quad \times \int_{E_\tau^{\ell+1}} \begin{array}{c} P^* \overline{\chi_\gamma} \cdot \begin{pmatrix} \log f_{j_0} d\log f_{j_0+1} \wedge \cdots \wedge d\log f_{\ell+1} \\ \cdot \delta_{T_{f_1}} \cdot \cdots \cdot \delta_{T_{f_{j_0-1}}} \end{pmatrix} \wedge \\ du_1^{\{J\}} \wedge \cdots \wedge du_\ell^{\{J\}} \wedge (du_1 + \cdots + du_{\ell+1}) \wedge (d\bar{u}_1 + \cdots + d\bar{u}_{\ell+1}) \end{array} \\
&= \nu^{-1} \binom{\ell}{j}^{-1} \sum_{j_0=1}^{\ell+1} (2\pi \mathrm{i})^{j_0-1} (-1)^{(\ell+1)(j_0+1)} \sum_{\substack{|J_0|=j \\ J_0\subseteq\{1,\ldots,j_0-1\}}} \hat{\sum_{\gamma\in\Gamma}} \\
&\quad \times \int_{E_\tau^{\ell+1}} \begin{array}{c} P^* \overline{\chi_\gamma} \begin{pmatrix} \log f_{j_0} d\log f_{j_0+1} \wedge \cdots \wedge d\log f_{\ell+1} \\ \cdot \delta_{T_{f_1}} \cdot \cdots \cdot \delta_{T_{f_{j_0-1}}} \end{pmatrix} \wedge \\ du_1^{\{J_0\}} \wedge \cdots \wedge du_{j_0-1}^{\{J_0\}} \wedge du_{j_0} \wedge d\bar{u}_{j_0} \wedge d\bar{u}_{j_0+1} \wedge \cdots \wedge d\bar{u}_{\ell+1} \end{array} \\
&= (-1)^{\binom{\ell}{2}} \nu^{-1} \binom{\ell}{j}^{-1} \sum_{j_0=j+1}^{\ell+1} (2\pi \mathrm{i})^{j_0-1} \\
&\quad \times \sum_{\substack{|J_0|=j \\ J_0\subseteq\{1,\ldots,j_0-1\}}} \hat{\sum_{\gamma\in\Gamma}} \left(\prod_{m=1}^{j_0-1} \int_{T_{f_m}} \overline{\chi_\gamma} du_m^{\{J\}} \right) \left(\int_{E_\tau} \overline{\chi_\gamma} \log f_{j_0} du_{j_0} \wedge d\bar{u}_{j_0} \right) \\
&\qquad \times \left(\prod_{m=j_0+1}^{\ell+1} \int_{E_\tau} \overline{\chi_\gamma} d\log f_m \wedge d\bar{u}_m \right)
\end{aligned}$$

$$\overset{\text{Lemmas}}{\underset{8.3\text{-}4}{=\!=}} (-1)^{\binom{\ell}{2}}\nu^{-1}\binom{\ell}{j}^{-1}\sum_{j_0=j+1}^{\ell+1}(2\pi\mathrm{i})^{j_0-1}(-1)^{\ell+1-j_0}\binom{j_0-1}{j}$$
$$\times \hat{\sum_{\gamma\in\Gamma}}{}' \frac{(p\tau+q)^{j+1}(p\bar{\tau}+q)^{j_0-j-1}\nu^{\ell-j_0+2}\prod_{m=1}^{\ell+1}\widehat{\varphi_{f_m}}(\gamma)}{(2\pi\mathrm{i})^{j_0}(mq-np)^{j_0}\omega(\gamma)^{\ell-j_0+2}}$$
$$= \frac{(-1)^{\binom{\ell+1}{2}}\nu^{\ell}}{2\pi\mathrm{i}\binom{\ell}{j}}\sum_{j_0=j+1}^{\ell+1}(-1)^{j_0-1}\binom{j_0-1}{j}\frac{(p\tau+q)^{j+1}(p\bar{\tau}+q)^{j_0-j-1}}{\nu^{j_0-1}}$$
$$\times \hat{\sum_{\gamma\in\Gamma}}{}' \frac{\widehat{\varphi}_{\mathbf{f}}(m,n)}{(m\tau+n)^{\ell-M-j+1}(mq-np)^{M+j+1}},$$

where the primed sum means to omit terms with $mq-np=0$. Taking $M=j_0-j-1$ as summation index, we have therefore

$$R^{[\ell]}_{\mathbf{f},j}(\tau)=\frac{(-1)^{\ell}}{2\pi\mathrm{i}}\sum_{M=0}^{\ell-j}(-1)^{M}\binom{M+j}{j}\frac{(p\tau+q)^{j+1}(p\bar{\tau}+q)^{M}}{\nu^{M+j}} \times \hat{\sum_{(m,n)\in\mathbb{Z}^2}}{}' \frac{\widehat{\varphi}_{\mathbf{f}}(m,n)}{(m\tau+n)^{\ell-M-j+1}(mq-np)^{M+j+1}}. \tag{9.11}$$

We now treat the second case, where

$$\{0\}\in|T_{f_1}|*\cdots*|T_{f_{\ell+1}}| \quad \text{over } \mathcal{A}_M$$

so that $|T_{\mathfrak{Z}_{\mathbf{f}}}|\neq\emptyset$ there. Without loss of generality, the reader can have in mind the case where each T_{f_i} (hence $|(f_i)|$) lies in the connected component of $W^{(p,q)}_{\tau}(N)$ containing $\{0\}$. Let $(\varepsilon_1,\ldots,\varepsilon_{\ell+1})\in\{|x|<\varepsilon\,|\,x\in\mathbb{R}\}^{\times(\ell+1)}$ be a very general point in a small polycylinder; we sketch a deformation argument which shows a lift of $\mathcal{R}^{[\ell]}_{\mathbf{f}}(\tau)$ ($\tau\in\mathcal{A}_M$) is still given by (9.11).

Begin by replacing each f_j by $f_j e^{i\varepsilon_j}$ globally on $\mathcal{E}(N)$, denoting the resulting cycles (from Section 7.3.4) by $\{\mathbf{f}^{\underline{\varepsilon}}\}$, $Z^{\underline{\varepsilon}}_{\mathbf{f}}=\tilde{\mathcal{G}}^*\iota^*\{\mathbf{f}^{\underline{\varepsilon}}\}$; and note that $\overline{Z^{\underline{\varepsilon}}_{\mathbf{f}}}$ is still closed, and now in real good position, on the complement $\bar{U}^{[\ell]}(N)$ of the $N^{2\ell}$ N-torsion sections. To obtain $\mathfrak{Z}^{\underline{\varepsilon}}_{\mathbf{f}}$, we must "move and complete" $\overline{Z^{\underline{\varepsilon}}_{\mathbf{f}}}$; that is,

$$\mathfrak{Z}^{\underline{\varepsilon}}_{\mathbf{f}}\big|_{\bar{U}^{[\ell]}(N)}=\overline{Z^{\underline{\varepsilon}}_{\mathbf{f}}}+\partial_{\mathcal{B}}\mathcal{W}^{\underline{\varepsilon}}_{\mathbf{f}}$$

for some $\mathcal{W}^{\underline{\varepsilon}}_{\mathbf{f}}\in Z^{\ell+1}_{\mathbb{R}}(\bar{U}^{[\ell]}(N),\ell+2)$. Since obviously $\varphi_{\mathbf{f}}=\varphi_{\mathbf{f}^{\underline{\varepsilon}}}$, we have $\Omega_{\mathfrak{Z}^{\underline{\varepsilon}}_{\mathbf{f}}}=\Omega_{\mathfrak{Z}_{\mathbf{f}}}$ (Theorem 8.1) and therefore $\mathcal{R}^{[\ell]}_{\mathbf{f}^{\underline{\varepsilon}}}\equiv\mathcal{R}^{[\ell]}_{\mathbf{f}}$ (Corollary 9.1). So it suffices to calculate a lift $\tilde{\mathcal{R}}^{[\ell]}_{\mathbf{f}^{\underline{\varepsilon}}}$ for any $\underline{\varepsilon}$, or $\lim_{\underline{\varepsilon}\to\underline{0}} \tilde{\mathcal{R}}_{\mathbf{f}^{\underline{\varepsilon}}}$ — which is in fact what we shall do, working henceforth over a point $\tau \in \mathcal{A}_M$.

Inside $E^{[\ell]}_{\tau}$ we have the open sets

$$\begin{aligned} \bar{U}^{[\ell]}_{N,\epsilon} \subset \bar{U}^{[\ell]}_{N} &:= \text{ complement of } N^{2\ell}\ N\text{-torsion points,} \\ \hat{U}^{[\ell]}_{N,\epsilon} \subset \hat{U}^{[\ell]}_{N} &:= \text{ complement of the } \{z_i = 0, z_j, -z_j\}, \end{aligned}$$

where the ϵ-subscript denotes removing a closed ϵ-ball/tube neighborhood. We want to compute (compatible lift-components)

$$\begin{aligned} \frac{(-1)^{\binom{\ell}{2}+j}\nu^{\ell}}{\binom{\ell}{j}} R^{[\ell]}_{\mathbf{f}^{\underline{\varepsilon}},j}(\tau) &= \int_{E^{\ell}_{\tau}} R_{\mathfrak{Z}^{\underline{\varepsilon}}_{\mathbf{f}}} \wedge \eta^{[\ell]}_j, \\ \lim_{\epsilon\to 0} \int_{\bar{U}^{[\ell]}_{N,\epsilon}} R_{\mathfrak{Z}^{\underline{\varepsilon}}_{\mathbf{f}}} \wedge \eta^{[\ell]}_j &= \lim_{\epsilon\to 0} \int_{\bar{U}^{[\ell]}_{N,\epsilon}} \left(R_{\overline{Z^{\underline{\varepsilon}}_{\mathbf{f}}}} + d[R_{\mathcal{W}^{\underline{\varepsilon}}_{\mathbf{f}}}] + (2\pi \mathrm{i})^{\ell+1} \delta_{\mathcal{S}^{\underline{\varepsilon}}_{\mathbf{f}}} \right) \wedge \eta^{[\ell]}_j \\ &= \lim_{\epsilon\to 0} \int_{\hat{U}^{[\ell]}_{N,\epsilon}} R_{\overline{Z^{\underline{\varepsilon}}_{\mathbf{f}}}} \wedge \eta^{[\ell]}_j \,+\, \lim_{\epsilon\to 0} \int_{\partial\bar{U}^{[\ell]}_{N,\epsilon}} R_{\mathcal{W}^{\underline{\varepsilon}}_{\mathbf{f}}} \wedge \eta^{[\ell]}_j \\ &\quad + (2\pi \mathrm{i})^{\ell+1} \int_{\mathcal{S}^{\underline{\varepsilon}}_{\mathbf{f}}} \eta^{[\ell]}_j, \end{aligned} \tag{9.12}$$

where $\mathcal{S}^{\underline{\varepsilon}}_{\mathbf{f}}$ is an ℓ-chain with $\partial(\mathcal{S}^{\underline{\varepsilon}}_{\mathbf{f}}) = T_{\overline{Z^{\underline{\varepsilon}}_{\mathbf{f}}}} + \mathcal{N}$ (with $|\mathcal{N}| \subset N$-torsion points, and nonzero only for $\ell = 1$). One can show that the middle term of (9.12) goes to zero (with $\epsilon \to 0$) at worst like $\epsilon \log^{\kappa} \epsilon$.

Now take the (previously very general) $\varepsilon_2, \ldots, \varepsilon_{\ell+1} \to \underline{0}$; then $|T_{\iota^*\{\mathbf{f}^{\underline{\varepsilon}}\}}|$ limits into $\{z_1 \equiv 0\}$ and so $|T_{\overline{Z^{\underline{\varepsilon}}_{\mathbf{f}}}}|$ limits into $\hat{W}^{[\ell]}_N$ (while $R_{\overline{Z^{\underline{\varepsilon}}_{\mathbf{f}}}}$ still makes sense on the complement). Since $\overline{Z^{\underline{\varepsilon}}_{\mathbf{f}}}$ is $\tilde{\mathcal{G}}^*$-invariant by construction, everything else in (9.12) — $\mathcal{W}^{\underline{\varepsilon}}_{\mathbf{f}}$, $\mathcal{S}^{\underline{\varepsilon}}_{\mathbf{f}}$, etc. — can be taken to be $\tilde{\mathcal{G}}^*$-invariant as well. But if $\mathcal{S}^{(\varepsilon_1,0,\ldots,0)}_{\mathbf{f}}$ is $\tilde{\mathcal{G}}^*$-invariant and bounds on $\hat{W}^{[\ell]}_N$ it must in fact be a cycle on E^{ℓ}_{τ}. This means that in constructing our lift, the third term of (9.12) can simply be thrown out (which must be done $(\forall j)$). Finally, taking the limit as $\varepsilon_1 \to 0$ and using $\tilde{\mathcal{G}}^*$-invariance of $\eta^{[\ell]}_j$, the first term of (9.12) becomes $\lim_{\epsilon\to 0} \int_{\hat{U}^{[\ell]}_{N,\epsilon}} R_{\iota^*\{\mathbf{f}\}} \wedge \eta^{[\ell]}_j$ which puts us back at the start of the computation which led to (9.11).

9.3. Regulator periods and analytic continuation

The computations using (9.11) that follow may be justified by appealing to absolute convergence of the series of the form

(9.13)
$$\hat{\sum_{(m,n)\in\mathbb{Z}^2}}{}' := \sum_{\substack{\varkappa\in\mathbb{Z}\\ \varkappa\neq 0}} \lim_{J\to\infty}\sum_{\jmath=-J}^{J} \qquad \left\{\begin{matrix} m = \jmath p - \varkappa s \\ \\ n = \jmath q + \varkappa r \end{matrix} \quad \longleftrightarrow \quad \begin{matrix} \varkappa = np - mq \\ \\ \jmath = ns + mr \end{matrix}\right\}$$

if $\pm\jmath$ terms are added first (replacing the "$\lim\sum$" by $\sum_{\jmath\geq 0}$). Moreover, the series of this form which occur do not actually depend on the choice of (r,s).

We start by computing the $\Psi^{[\ell]}_{\mathbf{f},k}(\tau)$ for the lifts $\tilde{\mathcal{R}}^{[\ell]}_{\mathbf{f}}(\tau)$ $(\tau\in\mathcal{A}_M)$ of the last section. Recycling "ϵ", we let it now denote a formal variable, and work in $\mathbb{C}[[\epsilon]]$. Referring to (9.1), if we write

$$\gamma^{[\ell]} := \sum_{k=0}^{\ell}\epsilon^k\binom{\ell}{k}\gamma^{[\ell]}_k,$$

then $\left\langle \gamma^{[\ell]}, \eta^{[\ell]}_{\ell-j}\right\rangle = (1+\tau\epsilon)^{\ell-j}(1+\bar{\tau}\epsilon)^j$, so that

(9.14)
$$\sum_{k=0}^{\ell}\Psi^{[\ell]}_{\mathbf{f},k}(\tau)\binom{\ell}{k}\epsilon^k = \left\langle\gamma^{[\ell]},\tilde{\mathcal{R}}^{[\ell]}_{\mathbf{f}}\right\rangle = \sum_{j=0}^{\ell} R^{[\ell]}_{\mathbf{f},j}(1+\tau\epsilon)^{\ell-j}(1+\bar{\tau}\epsilon)^j$$

(9.15)
$$= \frac{(-1)^\ell}{2\pi\mathrm{i}}(1+\tau\epsilon)^\ell(p\tau+q)\hat{\sum}{}'_{m,n}\frac{\widehat{\varphi}_{\mathbf{f}}(m,n)}{(m\tau+n)^{\ell+1}(mq-np)}$$
$$\times\sum_{j=0}^{\ell}\sum_{M=0}^{\ell-j}\left(\frac{(m\tau+n)(p\bar{\tau}+q)}{(np-mq)\nu}\right)^{M+j}$$
$$\times\binom{M+j}{j}\left(-\frac{(1+\bar{\tau}\epsilon)(p\tau+q)}{(1+\tau\epsilon)(p\bar{\tau}+q)}\right)^j.$$

Replacing $M+j$ by K and $\sum_j\sum_M$ by $\sum_{K=0}^{\ell}\sum_{j=0}^{K}$, and using

$$\sum_{j=0}^{K}\binom{K}{j}\left(-\frac{(1+\bar{\tau}\epsilon)(p\tau+q)}{(1+\tau\epsilon)(p\bar{\tau}+q)}\right)^j = \left(1-\frac{(1+\bar{\tau}\epsilon)(p\tau+q)}{(1+\tau\epsilon)(p\bar{\tau}+q)}\right)^K$$
$$= \left(\frac{\nu(p-\epsilon q)}{(1+\tau\epsilon)(p\bar{\tau}+q)}\right)^K$$

the double sum in (9.15) becomes

$$\sum_{K=0}^{\ell}\left(\frac{(m\tau+n)(p-\epsilon q)}{(np-mq)(1+\tau\epsilon)}\right)^K = \frac{(np-mq)^{\ell+1}(1+\tau\epsilon)^{\ell+1}-(m\tau+n)^{\ell+1}(p-\epsilon q)^{\ell+1}}{(np-mq)^{\ell}(1+\tau\epsilon)^{\ell}[(np-mq)(1+\tau\epsilon)-(m\tau+n)(p-\epsilon q)]}.$$

Simplifying the expression in square brackets to $(p\tau+q)(n\epsilon-m)$, (9.15) becomes

$$\frac{(-1)^{\ell+1}}{2\pi\mathrm{i}}\hat{\sum_{m,n}}{}' \frac{\widehat{\varphi}_{\mathbf{f}}(m,n)\left\{(np-mq)^{\ell+1}(1+\tau\epsilon)^{\ell+1}-(m\tau+n)^{\ell+1}(p-\epsilon q)^{\ell+1}\right\}}{(np-mq)^{\ell+1}(m\tau+n)^{\ell+1}(n\epsilon-m)}$$

— a "zipped" formula for the $\{\Psi^{[\ell]}_{\mathbf{f},k}\}$ which is obviously holomorphic in τ, and hence yields the analytic continuation to $\mathfrak{H}$. Since it was substituting (9.11) in (9.14) which yielded this continuation, (9.11) is the correct lift over *all of* $\mathfrak{H}$ (not just $\mathcal{A}_M$).

To get explicit formulas for the regulator periods, we reverse the last step to get (9.15) =

$$\frac{(-1)^{\ell+1}}{2\pi\mathrm{i}}\hat{\sum_{m,n}}{}' \widehat{\varphi}_{\mathbf{f}}(m,n)(p\tau+q)\sum_{\mu=0}^{\ell}\frac{(1+\tau\epsilon)^{\mu}(p-q\epsilon)^{\ell-\mu}}{(np-mq)^{\ell-\mu+1}(n+m\tau)^{\mu+1}},$$

and take coefficients of $\{\epsilon^k\}_{k=0}^{\ell}$ (and divide by $\binom{\ell}{k}$) to find

(9.16)

$$\boxed{\Psi^{[\ell]}_{\mathbf{f},k}(\tau) = \frac{(-1)^{\ell+1}}{2\pi\mathrm{i}}(p\tau+q)\hat{\sum_{m,n}}{}' \widehat{\varphi}_{\mathbf{f}}(m,n)\sum_{\mu=0}^{\ell}\sum_{a=\max\{0,k-\mu\}}^{\min\{k,\ell-\mu\}}\frac{\left\{(-1)^a\binom{\ell-\mu}{a}\binom{\mu}{k-a}\times\binom{\ell}{k}^{-1}p^{\ell-\mu-a}q^a\tau^{k-a}\right\}}{\left\{(np-mq)^{\ell-\mu+1}\times(m\tau+n)^{\mu+1}\right\}}.}$$

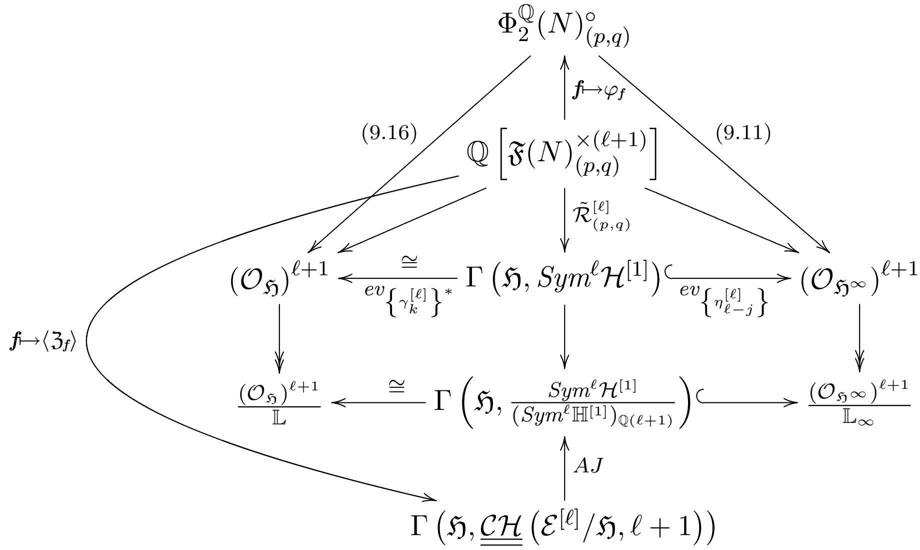
One can check that this is compatible with Proposition 9.1(i).

Now if we write

$$\mathfrak{F}(N)_{(p,q)} := \bigcup_{\substack{(r,s)\,:\\ \begin{pmatrix} p & q\\ -s & r\end{pmatrix}\in SL_2(\mathbb{Z})}} \mathfrak{F}(N)_{\begin{pmatrix} p & q\\ -s & r\end{pmatrix}},$$

then (9.11) and (9.16) extend linearly in an obvious way to *sums* of "monomials" $\in \mathfrak{F}(N)^{\times(\ell+1)}_{(p,q)}$ (we did this for $\mathbf{f} \mapsto \varphi_{\mathbf{f}}$ in Section 8.1.2).

Theorem 9.1. *Formulas* (9.11) *and* (9.16) *yield an abelian group homomorphism* $\tilde{\mathcal{R}}^{[\ell]}_{(p,q)}$ *inducing AJ on "*(p,q)*-vertical Eisenstein symbols", as described in the diagram*



*where "*ev*" means to write a vector with respect to the given basis,* $\{\ \}^*$ *is the dual basis, while* $\mathbb{L} = \mathbb{Q}(\ell+1)\left\langle \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right\rangle$ *and*

$$\mathbb{L}_{\infty} \overset{(7.1)}{=\!=\!=} \mathbb{Q}(\ell+1)\left\langle \begin{pmatrix} \mathfrak{P}^{[\ell]}_{00} \\ \vdots \\ \mathfrak{P}^{[\ell]}_{\ell 0} \end{pmatrix}, \ldots, \begin{pmatrix} \mathfrak{P}^{[\ell]}_{0\ell} \\ \vdots \\ \mathfrak{P}^{[\ell]}_{\ell\ell} \end{pmatrix} \right\rangle.$$

The two "extreme" periods are of special interest. For the α^{ℓ}-period, (9.16) yields

$$\Psi^{[\ell]}_{\mathbf{f},0}(\tau) = (-1)^{\ell}(2\pi \mathrm{i})^{\ell+1}(\tau + \tfrac{q}{p})\mathsf{H}^{[\ell]}_{[i\infty]}(\varphi_{\mathbf{f}}) + \frac{(-1)^{\ell+1}}{2\pi \mathrm{i}} \hat{\sum_{\substack{m,n \\ m \neq 0}}}' \widehat{\varphi}_{\mathbf{f}}(m,n) \frac{(m\tau+n)^{\ell+1}p^{\ell+1} - (np-mq)^{\ell+1}}{m(m\tau+n)^{\ell+1}(np-mq)^{\ell+1}} \tag{9.17}$$

if $p \neq 0$, and

(9.18)
$$\Psi^{[\ell]}_{\mathbf{f},0}(\tau) = \frac{(-1)^\ell}{2\pi \mathrm{i}} \hat{\sum}'_{m,n} \frac{\widehat{\varphi}_{\mathbf{f}}(m,n)}{m(m\tau+n)^{\ell+1}}$$

if $p = 0$ ($q = 1$). For the β^ℓ-period, we have

(9.19)
$$\begin{aligned}\Psi^{[\ell]}_{\mathbf{f},\ell} &= (-1)^{\ell+1}(2\pi\mathrm{i})^{\ell+1}(\tfrac{1}{\tau}+\tfrac{p}{q})\mathsf{H}^{[\ell]}_{[0]}(\varphi_{\mathbf{f}}) \\ &\quad + \tfrac{(-1)^{\ell+1}}{2\pi\mathrm{i}} \hat{\sum}'_{\substack{m,n \\ n\neq 0}} \widehat{\varphi}_{\mathbf{f}}(m,n) \tfrac{(np-mq)^{\ell+1}\tau^{\ell+1}+(-1)^\ell(m\tau+n)^{\ell+1}q^{\ell+1}}{n(m\tau+n)^{\ell+1}(np-mq)^{\ell+1}}\end{aligned}$$

if $q \neq 0$ and

(9.20)
$$\Psi^{[\ell]}_{\mathbf{f},\ell}(\tau) = \frac{(-1)^{\ell+1}}{2\pi\mathrm{i}}\tau^{\ell+1} \hat{\sum}'_{m,n} \frac{\widehat{\varphi}_{\mathbf{f}}(m,n)}{n(m\tau+n)^{\ell+1}}$$

if $q = 0$ ($p = 1$). We also record the higher normal function for convenience: using (9.4) and (9.11), this is

(9.21)
$$V^{[\ell]}_{\mathbf{f}}(\tau) = \frac{(-1)^{\binom{\ell}{2}}}{2\pi\mathrm{i}}(p\tau+q)^{\ell+1} \hat{\sum}'_{(m,n)\in\mathbb{Z}^2} \frac{\widehat{\varphi}_{\mathbf{f}}(m,n)}{(m\tau+n)(mq-np)^{\ell+1}}.$$

By the monodromy argument (Lemma 9.1(ii)) together with Section 8.1.2, AJ factors through $\Upsilon^{\mathbb{Q}}_2(N)$. That is, for *any* $\mathbf{f} \in \mathcal{O}^*(U(N))^{\otimes(\ell+1)}$

(9.22)
$$\Psi^{[\ell]}_{\mathbf{f},k}(\tau) = \sum_{\sigma\in\kappa(N)} \mathsf{H}^{[\ell]}_{\sigma}(\varphi_{\mathbf{f}})\tilde{\Psi}^{[\ell]}_{\sigma,k}(\tau) \mod \mathbb{Q}(\ell+1),$$

where (using our chosen $\left(\begin{smallmatrix} p & q \\ -s & r\end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ for each $\sigma = [\frac{r}{s}]$) $\tilde{\Psi}^{[\ell]}_{\sigma,k} = \Psi^{[\ell]}_{\mathbf{f}_\sigma,k}$ for some $\mathbf{f}_\sigma \in \mathbb{Q}\left[\mathfrak{F}(N)^{\times(\ell+1)}_{\left(\begin{smallmatrix} p & q \\ -s & r\end{smallmatrix}\right)}\right]$ satisfying $\mathsf{H}^{[\ell]}_{\sigma'}(\varphi_{\mathbf{f}_\sigma}) = \delta_{\sigma\sigma'}$. We take $\varphi_{\mathbf{f}_\sigma} = \frac{1}{N}\pi^*_\sigma\varphi^{[\ell]}_N$, so that (9.16) yields

(9.23)
$$\tilde{\Psi}^{[\ell]}_{\sigma,k}(\tau) := \frac{(-1)^{\ell+1}}{\ell+1}(2\pi\mathrm{i})^{\ell+1}(p\tau+q) \hat{\sum_{\substack{\alpha,\beta\in\mathbb{Z}^2 \\ \gcd(1+N\alpha,N\beta)=1}}} \sum_{\mu=0}^{\ell} \times \frac{\sum_{a=\max\{0,k-\mu\}}^{\min\{k,\ell-\mu\}}(-1)^a\binom{\ell-\mu}{a}\binom{\mu}{k-a}\binom{\ell}{k}^{-1}p^{\ell-\mu-a}q^a\tau^{k-a}}{(1+N\alpha)^{\ell-\mu+1}\{(1+N\alpha)(r-s\tau)+N\beta(q+p\tau)\}^{\mu+1}}.$$

Here the choice of (p,q) in (9.16) is different for each σ, we have computed as in Section 8.1.3 with $(m,n) =: \mathfrak{z}(m_0,n_0)$, $(m_0,n_0) =: (r+N(\beta q+\alpha r), -s+N(\beta p-\alpha r))$ and where $\hat{\sum}$ means to sum $\pm\beta$ first. A similar result holds for $V_{\mathbf{f}}^{[\ell]}(\tau)$, only modulo polynomials (of degree $\leq \ell$ with $\mathbb{Q}(\ell+1)$ coefficients).

Also as in Section 8.1.3 one can do the Fourier expansions in some cases (and we need these for the examples below). For instance, for $(p,q)=(1,0)$ and $k=0$, (9.16) becomes

(9.24)

$$(-1)^{\ell}(2\pi \mathrm{i})^{\ell+1}\tau \mathsf{H}^{[\ell]}_{[i\infty]}(\varphi_{\mathbf{f}}) + \frac{(-1)^{\ell+1}}{2\pi\mathrm{i}} \hat{\sum_{\substack{m,n\\ m\neq 0}}}' \hat{\varphi}_{\mathbf{f}}(m,n) \frac{(m\tau+n)^{\ell+1}-n^{\ell+1}}{m(m\tau+n)^{\ell+1}n^{\ell+1}},$$

where $\hat{\sum}'_{m,n}$ means $\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \lim_{M\to\infty}\sum_{m=-M}^{M}$. Assuming additionally that $\varphi_{\mathbf{f}}(m,n)=\varphi_{\mathbf{f}}(m,-n)$ $[\Longleftrightarrow \hat{\varphi}_{\mathbf{f}}(m,n)=\hat{\varphi}_{\mathbf{f}}(-m,n)]$, the $\hat{\sum}'_{\substack{m,n\\ m\neq 0}} \frac{\hat{\varphi}_{\mathbf{f}}(m,n)}{mn^{\ell+1}}=0$ and the second term of (9.24) becomes

$$\frac{(-1)^{\ell}}{2\pi\mathrm{i}} \sum_{(m,n)\in(\mathbb{Z}\backslash 0)^2} \frac{\hat{\varphi}_{\mathbf{f}}(m,n)}{m(m\tau+n)^{\ell+1}}. \tag{9.25}$$

Proposition 9.3. *If $\varphi_{\boldsymbol{f}} = \frac{1}{N}\pi^*_{[\mathrm{i}\infty]}\varphi$ ($\varphi\in\Phi^{\mathbb{Q}}(N)^{\circ}$) then $\hat{\varphi}_{\boldsymbol{f}} = \iota_{[\mathrm{i}\infty]_*}\hat{\varphi}$ and we have*

$$\Psi^{[\ell]}_{\boldsymbol{f},0}(\tau) = \frac{(2\pi\mathrm{i})^{\ell}(\ell+1)N}{(\ell+2)!}\left(\sum_{b=0}^{N-1}\varphi(b)B_{\ell+2}\left(\frac{b}{N}\right)\right)\log q_0 - \frac{(2\pi\mathrm{i})^{\ell}}{\ell! N^{\ell+1}}\sum_{M\geq 1}\frac{\left(\sum_{r|M} r^{\ell+1}\, {}^{\ell}\varphi(r)\right)}{M} q_0^{MN}, \tag{9.26}$$

where ${}^{\ell}\varphi(r)=\varphi(r)+(-1)^{\ell}\varphi(-r)$.

Proof. Let $\xi\in\{1,2,\ldots,N-1\}$, and $m_0\in\mathbb{N}$. Using the product expansion of $\sin(\pi(\alpha+z))$ from [1, Section 2.3, Example 2], we have

(9.27)

$$\frac{d^{\ell+1}}{d\tau^{\ell+1}}\log\left\{\sin\left(\frac{\pi\xi}{N}+\pi m_0\tau\right)\right\}$$

$$= \frac{d^{\ell+1}}{d\tau^{\ell+1}} \left\{ \pi m_0 \tau \cot\left(\frac{\pi\xi}{N}\right) + \sum_{n_0\in\mathbb{Z}} \left[\log\left(1 + \frac{Nm_0\tau}{Nn_0+\xi}\right) - \frac{Nn_0\tau}{Nn_0+\xi} \right] \right\}$$
$$= -\frac{d^{\ell}}{d\tau^{\ell}} \left\{ \sum_{n_0\in\mathbb{Z}} \frac{N^2 m_0^2 \tau}{(Nm_0+\xi)(Nn_0+\xi+Nm_0\tau)} \right\}$$
$$= (-1)^{\ell} \ell! N^{\ell+1} m_0^{\ell+1} \sum_{n_0\in\mathbb{Z}} \frac{1}{(Nn_0+\xi+Nm_0\tau)^{\ell+1}}.$$

On the other hand using the Taylor expansion for log, (9.27) becomes

$$\frac{d^{\ell+1}}{d\tau^{\ell+1}} \log\left\{ \frac{1}{2\mathrm{i}} \left(\mathrm{e}^{\frac{\pi\mathrm{i}}{N}(\xi+m_0N\tau)} - \mathrm{e}^{-\frac{\pi\mathrm{i}}{N}(\xi+m_0N\tau)} \right) \right\}$$
$$= \frac{d^{\ell+1}}{d\tau^{\ell+1}} \log\left(1 - \mathrm{e}^{2\pi\mathrm{i}m_0\tau}\mathrm{e}^{\frac{2\pi\mathrm{i}}{N}\xi}\right) = -\frac{d^{\ell+1}}{d\tau^{\ell+1}} \sum_{r\geq 1} \frac{1}{r} \mathrm{e}^{\frac{2\pi\mathrm{i}r\xi}{N}} \mathrm{e}^{2\pi\mathrm{i}m_0 r\tau}$$
$$= -(2\pi\mathrm{i})^{\ell+1} m_0^{\ell+1} \sum_{r\geq 1} r^{\ell} \mathrm{e}^{\frac{2\pi\mathrm{i}r\xi}{N}} q_0^{rm_0N};$$

hence we have (for $m_0 > 0$) $\alpha(\xi, m_0) :=$

$$\sum_{n_0\in\mathbb{Z}} \frac{1}{(Nn_0+\xi+Nm_0\tau)^{\ell+1}} = \frac{(-1)^{\ell+1}(2\pi\mathrm{i})^{\ell+1}}{\ell! N^{\ell+1}} \sum_{r\geq 1} r^{\ell} \mathrm{e}^{\frac{2\pi\mathrm{i}\xi r}{N}} q_0^{rm_0N}.$$

Substituting $\widehat{\varphi}_{\mathbf{f}} = \iota_{[\mathrm{i}\infty]_*}\widehat{\varphi}$ in (9.25) therefore yields

$$\frac{(-1)^{\ell}}{2\pi\mathrm{i}} \sum_{(n,m_0)\in(\mathbb{Z}\setminus\{0\})^2} \frac{\widehat{\varphi}(n)}{Nm_0(n+Nm_0\tau)^{\ell+1}}$$
$$= \frac{(-1)^{\ell}}{2\pi\mathrm{i}N} \sum_{\xi=1}^{N-1} \widehat{\varphi}(\xi) \sum_{m_0'\geq 1} \frac{1}{m_0'} \left\{ \alpha(\xi, m_0') + (-1)^{\ell}\alpha(-\xi, m_0') \right\}$$
$$= \frac{-(2\pi\mathrm{i})^{\ell}}{\ell! N^{\ell+2}} \sum_{M\geq 1} q_0^{MN} \sum_{r|M} \frac{r^{\ell+1}}{M} \sum_{\xi\in\mathbb{Z}/N\mathbb{Z}} \widehat{\varphi}(\xi) \left\{ \mathrm{e}^{\frac{2\pi\mathrm{i}\xi r}{N}} + (-1)^{\ell}\mathrm{e}^{-\frac{2\pi\mathrm{i}\xi r}{N}} \right\}$$
$$= \frac{-(2\pi\mathrm{i})^{\ell}}{\ell! N^{\ell+1}} \sum_{M\geq 1} q_0^{MN} \sum_{r|M} \frac{r^{\ell+1}}{M} {}^{\ell}\varphi(r),$$

where we have reindexed $M = m_0' r$. The first term of (9.26) is much easier. □

We turn briefly to the higher normal function. In analogy to (9.24), for $(p,q)=(1,0)$ Equation (9.21) becomes

(9.28) $$V_{\mathbf{f}}^{[\ell]}(\tau)=\frac{(-1)^{\binom{\ell+1}{2}}(2\pi\mathrm{i})^{\ell+1}}{\ell+1}\tau^{\ell+1}\mathsf{H}_{[\mathrm{i}\infty]}^{[\ell]}(\varphi_{\mathbf{f}}) - \frac{(-1)^{\binom{\ell+1}{2}}}{2\pi\mathrm{i}}\tau^{\ell+1}\hat{\sum_{\substack{m,n\\ m\neq 0}}}' \frac{\widehat{\varphi}_{\mathbf{f}}(m,n)}{(m\tau+n)n^{\ell+1}},$$

and if $\varphi_{\mathbf{f}}=\frac{1}{N}\pi^*_{[\mathrm{i}\infty]}\varphi$ we can calculate its q_0-expansion as follows. Using

$$\frac{\tau^{\ell+1}}{(Nm_0\tau+n)n^{\ell+1}}=\sum_{j=1}^{\ell}\frac{(-1)^{j-1}\tau^{\ell-j+1}}{(Nm_0)^j m^{\ell-j+2}}+\frac{(-1)^{\ell}\tau}{(Nm_0\tau+n)(Nm_0)^{\ell}n},$$

the second term of (9.28) becomes

$$\frac{(-1)^{\binom{\ell+1}{2}}}{2\pi\mathrm{i}}\sum_{J=1}^{\lfloor\frac{\ell}{2}\rfloor}\frac{\tau^{\ell-2J+1}}{N^{2J}}\sum_{(m_0,n)\in(\mathbb{Z}\setminus\{0\})^2}\frac{\widehat{\varphi}(n)}{m_0^{2J}n^{\ell-2J+2}} - \frac{(-1)^{\binom{\ell}{2}}}{2\pi\mathrm{i}N^{\ell+2}}\sum_{\xi=1}^{N-1}\widehat{\varphi}(\xi)\sum_{m_0\in\mathbb{Z}}{}'\frac{1}{m_0^{\ell+1}}\sum_{n_0\in\mathbb{Z}}\frac{N^2m_0\tau}{(\xi+Nn_0)(\xi+Nn_0+Nm_0\tau)}.$$

For $m_0>0$ the $\sum_{n_0\in\mathbb{Z}}$ is

$$\pi\left(\mathrm{i}+\cot\left(\frac{\pi\xi}{N}\right)\right)+2\pi\mathrm{i}\sum_{r\geq 1}\mathrm{e}^{\frac{2\pi\mathrm{i}r\xi}{N}}q_0^{m_0Nr}$$

by an argument like that in the above proof. Writing

$$\Theta_\ell(\varphi):=\begin{cases}-\frac{\mathrm{i}}{N}\sum_{\xi\in\mathbb{Z}/N\mathbb{Z}}\widehat{\varphi}(\xi)\cot\left(\frac{\pi\xi}{N}\right), & \ell\text{ odd},\\ \varphi(0), & \ell\text{ even}\end{cases}$$

and noting $\zeta(2J)=\frac{-(2\pi\mathrm{i})^{2J}}{2(2J)!}B_{2J}$, we eventually arrive at this expression for the higher normal function (associated to our lift):

(9.29)
$$\frac{(-1)^{\binom{\ell}{2}}N^{\ell+1}}{(\ell+2)!}\left\{\begin{matrix}\left(\sum_{a=0}^{N-1}\varphi(a)B_{\ell+2}\left(\frac{a}{N}\right)\right)\log^{\ell+1}q_0+\sum_{J=1}^{\lfloor\frac{\ell}{2}\rfloor}\frac{(-2\pi\mathrm{i})^{2J}}{N^{4J}}\binom{\ell+2}{2J}\\ \times B_{2J}\left(\sum_{a=0}^{N-1}\varphi(a)B_{\ell-2J+2}\left(\frac{a}{N}\right)\right)\log^{\ell-2J+1}q_0\end{matrix}\right\} - \frac{(-1)^{\binom{\ell}{2}}}{N^{\ell+1}}\left\{\zeta(\ell+1)\Theta_\ell(\varphi)+\sum_{M\geq 1}q_0^{MN}\left(\frac{\sum_{r|M}r^{\ell+1}\cdot{}^{\ell}\varphi(r)}{M^{\ell+1}}\right)\right\}.$$

The first big braced expression in (9.29) is a polynomial in τ with $\mathbb{Q}(\ell+1)$-coefficients. Both (9.29) and (9.26) check against Proposition 9.1 and Corollary 8.4, as the reader may verify.

Finally, one can evaluate the regulator periods at cusps where $\Omega_{\mathfrak{Z}_{\mathrm{f}}}$ has no residue. We demonstrate this for the $\alpha^{\times \ell}$-period.

Proposition 9.4. *Assume that* $\mathsf{H}^{[\ell]}_{[\frac{r}{s}]}(\varphi_{\boldsymbol{f}})\ \left[= \frac{-(\ell+1)}{(2\pi\mathrm{i})^{\ell+2}}\tilde{L}(\widehat{\varphi}_{\boldsymbol{f}}, \ell+2)\right] = 0$*; then*

$$\lim_{\tau \to \frac{r}{s}} \Psi^{[\ell]}_{\boldsymbol{f},0}(\tau) \equiv \frac{-s^{\ell}}{2N}\tilde{L}_{-}(\pi_{[\frac{r}{s}]_*}\widehat{\varphi}_{\boldsymbol{f}}, \ell+1) \bmod \mathbb{Q}(\ell+1),$$

where $\tilde{L}_{-}(\phi, \ell+1) := \sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{\phi(m)\cdot\frac{|m|}{m}}{m^{\ell+1}}$.

Proof. Will proceed by *first* showing that

$$\lim_{\tau\to \mathrm{i}\infty} \Psi^{[\ell]}_{\mathbf{f},\ell}(\tau) \equiv \frac{-1}{2N}\tilde{L}_{-}(\pi_{[\mathrm{i}\infty]_*}\widehat{\varphi}_{\mathbf{f}}, \ell+1) \tag{9.30}$$

when $\mathsf{H}^{[\ell]}_{[\mathrm{i}\infty]}(\varphi_{\mathbf{f}}) = 0$. We can write $\varphi_{\mathbf{f}} = \varphi_{\mathbf{f}'} + \varphi_{\mathbf{f}''}$ where $\varphi_{\mathbf{f}'} \in \pi^*_{[0]}\Phi^{\mathbb{Q}}(N)^{\circ} \subset \Phi_2^{\mathbb{Q}}(N)^{\circ}_{(0,1)}$ and $\varphi_{\mathbf{f}''} \in \Phi_2^{\mathbb{Q}}(N)^{\circ}_{(1,0)}$, then apply (9.19) [with $(p,q) = (0,1)$] resp. (9.20) to conclude

(9.31)

$$\lim_{\tau\to\mathrm{i}\infty} \Psi^{[\ell]}_{\mathbf{f},\ell}(\tau) \equiv \lim_{\tau\to\mathrm{i}\infty} \frac{(-1)^{\ell+1}}{2\pi\mathrm{i}} \sum_{(m,n)\in(\mathbb{Z}\setminus 0)^2} \frac{\widehat{\varphi}_{\mathbf{f}}(m,n)}{n(m+\frac{n}{\tau})^{\ell+1}} \bmod \mathbb{Q}(\ell+1)$$

after "reassembling" the results. (In (9.19) the sum becomes

$$\frac{1}{N}\hat{\sum_{\substack{m,\, n_0\\ n_0\neq 0}}}{}' \left(\frac{\widehat{\varphi}_{\mathbf{f}}(m,0)}{n_0(m+\frac{Nn_0}{\tau})^{\ell+1}} - \frac{\widehat{\varphi}_{\mathbf{f}}(m,0)}{n_0 m^{\ell+1}} \right),$$

where the $\hat{\sum}$ means to sum $\pm n_0$ first, so that one can delete the second term inside the sum. Then one can remove the "$\widehat{\ }$", in both (9.19) and (9.20),[24] since the double-sum is now absolutely convergent.) The r.h.s. of (9.31) is

[24]Where it means to sum $\pm m$ first.

now (summing $\pm n$ first)

$$\lim_{\tau\to \mathrm{i}\infty}\frac{(-1)^{\ell+1}}{2\pi\mathrm{i}}\sum_{\xi=0}^{N-1}\sideset{}{'}\sum_{m\in\mathbb{Z}}\widehat{\varphi}_{\mathbf{f}}(m,\xi)$$
$$\times\sum_{n\geq 1}\left(\frac{1}{(n_0N-\xi)\left(M+\frac{n_0N-\xi}{\tau}\right)^{\ell+1}}-\frac{1}{(n_0N-\xi)\left(m-\frac{n_0N-\xi}{\tau}\right)^{\ell+1}}\right),$$

where we have made the (unnecessary) assumption that $\widehat{\varphi}_{\mathbf{f}}(m,-n)=\widehat{\varphi}_{\mathbf{f}}(m,n)$ to simplify the exposition. This becomes (writing $\tau=\mathrm{i}t$)

$$\frac{2(-1)^{\ell+1}i^{\ell+1}}{2\pi\mathrm{i}N}\sideset{}{'}\sum_{m\in\mathbb{Z}}\sum_{\xi=0}^{N-1}\widehat{\varphi}_{\mathbf{f}}(m,\xi)$$
$$\times\sum_{k=0}^{\ell}(-1)^k\left\{\lim_{t\to\infty}\sum_{n_0\geq 1}\frac{N/t}{\left(\frac{n_0N-\xi}{t}+\mathrm{i}m\right)^{\ell-k+1}\left(\frac{n_0N-\xi}{t}-\mathrm{i}m\right)^{k+1}}\right\},$$

where the limit in braces is the Riemann sum for

$$\int_0^\infty\frac{dX}{(X+\mathrm{i}m)^{\ell-k+1}(X-\mathrm{i}m)^{k+1}}=\frac{1}{2}(2\pi\mathrm{i})(-1)^{\ell+k}\frac{|m|}{m}\binom{\ell}{k}\frac{1}{(2m\mathrm{i})^{\ell+1}}$$

(using residues), and so we get

$$-\frac{\sum_{k=0}^{\ell}\binom{\ell}{k}}{2^{\ell+1}N}\sideset{}{'}\sum_{m\in\mathbb{Z}}\frac{|m|}{m^{\ell+2}}\sum_{\xi=0}^{N-1}\widehat{\varphi}_{\mathbf{f}}(m,\xi)$$

which is just the r.h.s. of (9.30).

Now let $\mathbf{f}$ be as in the statement of the proposition:

$$\lim_{\tau\to\frac{r}{s}}\Psi^{[\ell]}_{\mathbf{f},0}(\tau)=\left\langle[\alpha^{\times\ell}],\,\lim_{\tau\to\frac{r}{s}}\mathcal{R}^{[\ell]}_{\mathbf{f}}(\tau)\right\rangle=\left\langle[\alpha^{\times\ell}],\,\left(\begin{smallmatrix}p&q\\-s&r\end{smallmatrix}\right)^*\mathcal{R}^{[\ell]}_{\left(\begin{smallmatrix}r&-q\\s&p\end{smallmatrix}\right)^*\mathbf{f}}(\tau)\right\rangle.$$

By (9.30) this is

$$-\frac{(-1)^{\binom{\ell+1}{2}}}{2N}\tilde{L}_-\left(\pi_{[\mathrm{i}\infty]_*}\begin{pmatrix}r&-q\\s&p\end{pmatrix}^*\widehat{\varphi}_{\mathbf{f}},\,\ell+1\right)\left\langle[\alpha^{\times\ell}],\,\left(\begin{smallmatrix}p&q\\-s&r\end{smallmatrix}\right)^*[\alpha^{\times\ell}]\right\rangle$$

$$= -\frac{(-1)^{\binom{\ell+1}{2}}}{2N}\tilde{L}_{-}\left(\pi_{[\frac{r}{s}]_*}\widehat{\varphi}_{\mathbf{f}},\,\ell+1\right)\left\langle [\alpha^{\times\ell}],\,[(r\alpha-s\beta)^{\times\ell}]\right\rangle$$

which yields the result. □

Remark. In fact, Proposition 9.3 leads to a more general result when combined with results from previous sections:

Corollary 9.2. *For any* $\boldsymbol{f}\in\mathcal{O}^*(U(N))^{\otimes(\ell+1)}$,

$$\Psi^{[\ell]}_{\boldsymbol{f},0}(\tau) \overset{\mathbb{Q}(\ell+1)}{\equiv} (-2\pi\mathrm{i})^{\ell}\mathsf{H}^{[\ell]}_{[\mathrm{i}\infty]}(\varphi_{\boldsymbol{f}})N\log q_0 - \frac{(2\pi\mathrm{i})^{\ell}}{N^{\ell+1}\ell!}\sum_{M\geq 1}\frac{q_0^M}{M}\left\{\sum_{r|M} r^{\ell+1}\left(\sum_{n_0\in\mathbb{Z}/N\mathbb{Z}} \mathrm{e}^{\frac{2\pi\mathrm{i}n_0 r}{N}}\cdot {}^{\ell}\widehat{\varphi}_{\boldsymbol{f}}\left(\frac{M}{r},n_0\right)\right)\right\}.$$

Proof. Split $\varphi_{\mathbf{f}}=\varphi_{\mathbf{f}'}+\varphi_{\mathbf{f}''}$ with $\varphi_{\mathbf{f}'}\in\pi^*_{[\mathrm{i}\infty]}(\Phi^{\mathbb{Q}}(N)^{\circ})$, and $\varphi_{\mathbf{f}''}$ $(0,1)$-vertical so that $\mathsf{H}^{[\ell]}_{[\mathrm{i}\infty]}(\varphi_{\mathbf{f}''})=0$. By Proposition 9.2(i), $\lim_{\tau\to i\infty}\Psi^{[\ell]}_{\mathbf{f}'',0}(\tau)=0$ while the constant and divergent terms (as $\tau\to i\infty$) for $\Psi^{[\ell]}_{\mathbf{f}',0}$ (hence $\Psi^{[\ell]}_{\mathbf{f},0}$) are given by Proposition 9.3. Using this together with Propositions 8.2 and 9.1(i) (which says that $\Psi^{[\ell]}_{\mathbf{f},0}=(-1)^{\ell}(2\pi\mathrm{i})^{\ell+1}\int E_{\varphi_{\mathbf{f}}}(\tau)d\tau$) gives the result. □

10. Toric versus Eisenstein: comparing constructions

In this final section we consider the possible coincidence of (push-forwards of) Beilinson's Eisenstein symbol over genus zero modular curves, and the toric symbol on suitably "modular" hypersurface pencils. This will be done on the level of regulator periods and cycle classes, and the general result in Section 10.3 is followed by many examples. To whet the reader's appetite we include two motivating examples in Section 10.1, which come from extending the computations of regulator periods and their special values to the cycles considered in Section 8.2.

10.1. Regulator periods for other congruence subgroups

It is worth mentioning a subtlety that enters into computations for the "push-forward cycles" of Section 8.2.1 $\mathfrak{Z}_{\mathbf{f},1^{(\prime)}} := \frac{1}{N}\left(\mathcal{P}^{[\ell]}_{\Gamma(N)/\Gamma_1^{(\prime)}(N)}\right)_*$ $\mathfrak{Z}_{\mathbf{f}}\in CH^{\ell+1}\left(\mathcal{E}^{[\ell]}_{\Gamma_1^{(\prime)}(N)},\ell+1\right)$ (equivalently one can consider $\widetilde{\mathfrak{Z}_{\mathbf{f},1^{(\prime)}}} :=$

$\left(\mathcal{P}^{[\ell]}_{\Gamma(N)/\Gamma_1^{(\prime)}(N)}\right)^* \mathfrak{Z}_{\mathbf{f},1^{(\prime)}}$ on $\mathcal{E}^{[\ell]}(N)$). Letting $\Psi^{[\ell]}_{\mathbf{f},1^{(\prime)};k}$ denote the period over $\gamma_k^{[\ell]}$ $(=\alpha^{\ell-k}\beta^k)$ for an appropriate lift of the fiberwise AJ of $\mathfrak{Z}_{\mathbf{f},1^{(\prime)}}$ over $Y_1^{(\prime)}(N)$, we have obviously

$$\Psi^{[\ell]}_{\mathbf{f},1;0}(\tau) = \frac{1}{N}\sum_{j=0}^{N-1} \Psi^{[\ell]}_{\mathbf{f},0}(\tau+j) \tag{10.1}$$

but also

$$\Psi^{[\ell]}_{\mathbf{f},1;\ell}(\tau) = \frac{1}{N}\sum_{j=0}^{N-1}\sum_{k=0}^{\ell}\binom{\ell}{k}(-j)^{\ell-k}\Psi^{[\ell]}_{\mathbf{f},k}(\tau+j) \tag{10.2}$$

$$\Psi^{[1]}_{\mathbf{f},1';0}(\tau) = \frac{1}{N}\sum_{j=0}^{N-1}\left\{\Psi^{[1]}_{\mathbf{f},0}\left(\frac{\tau}{j\tau+1}\right) - j\Psi^{[1]}_{\mathbf{f},1}\left(\frac{\tau}{j\tau+1}\right)\right\} \tag{10.3}$$

since (see Section 8.2.1) $\mathcal{J}_{j_*}\beta = \beta - j\alpha$ (resp. $\mathcal{J}'_{j_*}\alpha = \alpha - j\beta$). Likewise, for the "$K_3(K3)$" cycles $\mathfrak{Z}_{\mathbf{f},+N} := \frac{-1}{4N}(p_2)_*(p_1)^*(\mathcal{P}_{+N})_*(J_N^{[2]})^*\mathfrak{Z}_{\mathbf{f},1} \in CH^3(\mathcal{X}_1^{[2]}(N)^{+N},3)$ (resp. ${}'\mathfrak{Z}_{\mathbf{f},+N}$) of Section 8.2.2, we find

$${}^{(\prime)}\Psi^{[2]}_{\mathbf{f},+N;0}(\tau) = \frac{1}{2}\left\{\Psi^{[2]}_{\mathbf{f},1;0}(\tau) \underset{(-)}{+} N\Psi^{[2]}_{\mathbf{f},1;2}\left(\frac{-1}{N\tau}\right)\right\} \tag{10.4}$$

for the periods of $AJ\left(\langle{}^{(\prime)}\mathfrak{Z}_{\mathbf{f},+N}\rangle_{[\tau]\in Y_1(N)^{+N}}\right)$ against $(\mathcal{P}_{+N})_*(J_N^{[2]})_*(\alpha\times\alpha)$. (The latter, it turns out, is divisible by $2N$ in the integral homology of the $K3$ fibers.) To obtain limiting values of (10.1)–(10.4) at a cusp, one could apply the proof of Proposition 9.4 to each term.

An easier approach is to consider the effect of $\mathfrak{Z}_{\mathbf{f}} \mapsto \widetilde{\mathfrak{Z}_{\mathbf{f},1^{(\prime)}}}$ (or ${}'\widetilde{\mathfrak{Z}_{\mathbf{f},+N}}$) on the residues of the cycle-class, transform $\widehat{\varphi_{\mathbf{f}}}$ accordingly (cf. (8.2)), and plug the result into Proposition 9.4. We carry this out in two examples related to toric constructions in this paper.

Example 10.1 ($\ell = 1$, $N = 4$, $\Gamma = \Gamma_1'(4)$). Begin with $\mathbf{f}$ so that $\varphi_{\mathbf{f}} = -\frac{1}{4}\pi^*_{[i\infty]}\varphi_4^{[1]}$ (see Proposition 7.3) and consider $\mathfrak{Z}_{\mathbf{f},1'}$; the corresponding divisor $\varphi_{\mathbf{f},1'}$ has $\widehat{\varphi_{\mathbf{f},1'}} = \frac{1}{4}\rho'_*\widehat{\varphi}_{\mathbf{f}} = -\frac{1}{4}\rho'_*\iota_{[i\infty]_*}\widehat{\varphi_4^{[1]}} = -\frac{1}{4}\pi^*_{[0]}\widehat{\varphi_4^{[1]}}$ where $\widehat{\varphi_4^{[1]}} = 0$,

$2^6\mathrm{i}, 0, -2^6\mathrm{i}$. We have $\pi_{[0]_*}\widehat{\varphi_{\mathbf{f},1'}} = -\widehat{\varphi_4^{[1]}}$ and so

$$\lim_{\tau\to 0}\Psi^{[1]}_{\mathbf{f},1';0}(\tau) \equiv \frac{1}{8}\tilde{L}_-\left(\widehat{\varphi_4^{[1]}}, 2\right) = -16\mathrm{i}G \mod \mathbb{Q}(2);$$

this corresponds exactly to the $D5$ example of Section 6.3.

Example 10.2 ($\ell = 2$, $N = 6$, $\Gamma = \Gamma_1(6)^{+6}$). Start with $\varphi_{\mathbf{f}} = -4\pi^*_{[\mathrm{i}\infty]}\varphi_6^{[2]}$, and consider $\widetilde{'\mathfrak{Z}_{\mathbf{f},+6}}$: from (8.6) (and Remark 8.3) we know that if $\mathsf{H}_\sigma(\varphi_{\mathbf{f}}) = -24\delta_{\sigma,[\mathrm{i}\infty]}$ then $\mathsf{H}_{[\mathrm{i}\infty]}('\varphi_{\mathbf{f},+6}) = -12$ and $\mathsf{H}_{[j]}('\varphi_{\mathbf{f},+6}) = \frac{1}{3}$ $(\forall j \in \mathbb{Z})$. As $\widehat{\varphi_6^{[2]}} = 0, -\frac{6^4}{5}, 0, 0, 0, -\frac{6^4}{5}$, this leads to

$$\text{(10.5)}\qquad '\widehat{\varphi_{\mathbf{f},+6}}(m,n) = \begin{cases} \frac{2\cdot 6^5}{5}, & (m,n) \overset{(6)}{\equiv} \pm(0,1),\\ -\frac{2\cdot 6^3}{5}, & m \overset{(6)}{\equiv} \pm 1,\\ 0 & \text{otherwise,}\end{cases}$$

and $\pi_{[\frac{-1}{2}]_*}{}'\widehat{\varphi_{\mathbf{f},+6}} = -\frac{8\cdot 6^3}{5}\cdot\{0,1,-9,1,-9,1;\ldots\}$ so that

$$\begin{aligned}\lim_{\tau\to -\frac{1}{2}} {}'\Psi^{[2]}_{\mathbf{f},+6;0}(\tau) &\overset{\mathbb{Q}(3)}{\equiv} -\frac{4}{12}\cdot\frac{-8\cdot 6^3}{5}\cdot 2L(\{0,1,-9,1,-9,1;\ldots\},3)\\ &= \frac{2^5\cdot 6^2}{5}\zeta(3)\cdot\left(1-\frac{10}{2^3}+\frac{9}{6^3}\right) = -48\zeta(3).\end{aligned}$$

This means that the AJ class of $\left\langle \widetilde{'\mathfrak{Z}_{\mathbf{f},+6}}\right\rangle_\tau$ limits to $12\zeta(3)\left[(\alpha+2\beta)^{\times 2}\right]$, which is the pullback from the $K3$ family of $2\zeta(3)$ times a vanishing cycle at $[\frac{-1}{2}] \in \overline{Y}_1(6)^{+6}$. This suggests a link to the Apéry–Beukers higher normal function from the introduction; the precise relation will be established in Section 10.5 below.

10.2. Uniformizing the genus zero case

Let $\Gamma \subset SL_2(\mathbb{Z})$ be a congruence subgroup in the sense of Section 7.1.1 ($\{-\mathrm{id}\} \notin \Gamma$, $\Gamma \supset \Gamma(N)$ for some $N \geq 3$), and assume $\overline{Y}_\Gamma \cong \mathbb{P}^1$. To fix a uniformizing parameter, note first that $\overline{Y}_\Gamma$ has local coordinate $q_0 := q^{\frac{1}{N_\Gamma}} = \mathrm{e}^{\frac{2\pi\mathrm{i}\tau}{N_\Gamma}}$ in a neighborhood of $[\mathrm{i}\infty]$, e.g., $N_\Gamma = N$ for $\Gamma = \Gamma(N)$ or $\Gamma_1'(N)$, while $N_\Gamma = 1$ for $\Gamma = \Gamma_1(N)$ (or $\Gamma_1(N)^{+N}$, though we do not treat this yet). Then let $H \in \check{M}_0(\Gamma)$ be the (unique) Hauptmodul with Fourier expansion $H(q_0) =$

const. $\cdot\, q_0$ + h.o.t. We will assume H is normalized so that this constant is a root of unity. Given an "Eisenstein symbol" $\mathfrak{Z} \in CH^{\ell+1}(\mathcal{E}_{\Gamma}^{[\ell]}, \ell+1)$ (with $(\mathcal{P}^{[\ell]}_{\Gamma(N)/\Gamma})^{*}\mathfrak{Z} \equiv \mathfrak{Z}_{\mathbf{f}} \in CH^{\ell+1}(\mathcal{E}^{[\ell]}(N), \ell+1)$), writing the data $\{\Omega_{\mathfrak{Z}_{\mathbf{f}}}, \Psi^{[\ell]}_{\mathbf{f},0}, V^{[\ell]}_{\mathbf{f}}$, PF-equations, etc.$\}$ in terms of $t := H(\tau)$ yields expressions resembling those of Sections 3 and 4 arising from the "toric symbols".

While there are intersections between the two constructions (systematically developed in Sections 10.3–10.6), neither one includes the other. Let $\omega^{\Gamma}_{\mathcal{E}/Y} := K_{\overline{\mathcal{E}}^{[\ell]}_{\Gamma}} \otimes \overline{\pi}^{-1}(\theta^{1}_{\overline{Y}_{\Gamma}})$ denote the relative dualizing sheaf; if $\deg(\overline{\pi}_{\Gamma *}\omega^{\Gamma}_{\mathcal{E}/Y})$ (always ≥ 1) is >1, then $\overline{\mathcal{E}}_{\Gamma}$ cannot be birational to a Fano $n\,(=\ell+1)$-fold $\mathbb{P}_{\Delta}$. Conversely, the construction of Theorem 3.1 need not yield a modular family — e.g., the E_7 and E_8 families of elliptic curves (cf. Section 6.3) have marked *non*torsion points (which are used in the construction of the toric symbol); other examples will be given in Sections 10.4–10.6.

To begin "uniformizing" the data, let $\{\sigma_j\} \subset \kappa_{\Gamma}$ be the cusps *other* than $[\mathrm{i}\infty]$ where $\mathfrak{Z}$ has nonvanishing residue, and differentiate the AJ class over $\mathbb{P}^1$ to get

$$\omega_{\mathbf{f}} := \nabla_{\delta_t}\tilde{\mathcal{R}}^{[\ell]}_{\mathbf{f}} \in \Gamma\left(\overline{Y}_{\Gamma},\, \omega^{\Gamma}_{\mathcal{E}/Y} \otimes \mathcal{O}_{\overline{Y}_{\Gamma}}\left(\sum \sigma_j\right)\right).$$

Pulling this back to $(\mathcal{E}^{[\ell]} \to)\,\mathfrak{H}$ yields

$$(-2\pi\mathrm{i})^{\ell} A_{\mathbf{f}}(\tau)\eta^{[\ell]}_{\ell}, \quad A_{\mathbf{f}}(\tau) \in \check{M}_{\ell}(\Gamma),$$

here $A_{\mathbf{f}}$ may have "poles" (as an automorphic form) at elliptic points, nonunipotent cusps, and the $\{\sigma_j\}$. Similarly, writing $H' := \frac{dH}{dq_0}$, $\frac{dt}{t}$ pulls back to $2\pi\mathrm{i}B_{\mathbf{f}}(\tau)d\tau$, where

$$B_{\mathbf{f}}(\tau) := \frac{d\log t}{d\log q} = \frac{q_0}{N_{\Gamma}} \cdot \frac{H'}{H} \in M_2^{\mathbb{Q}}(\Gamma).$$

Pulling back the cycle class $\Omega_{\mathfrak{Z}_{\mathbf{f}}} = (-1)^{\ell}\nabla_{\delta_t}\tilde{\mathcal{R}}^{[\ell]}_{\mathbf{f}} \wedge \frac{dt}{t}$, we see that

$$F_{\mathbf{f}}(\tau) = A_{\mathbf{f}}(\tau) \cdot B_{\mathbf{f}}(\tau) \; (\in M^{\mathbb{Q}}_{\ell+2}(\Gamma)).$$

Now we can write down a power-series expansion for the period of $\omega_{\mathbf{f}}$ over the (locally defined) family of topological cycles $\alpha^{\times\ell} \in H_{\ell}(E^{[\ell]}_{\Gamma,t}, \mathbb{Z})$ vanishing at $t=0$. Using Proposition 8.2 and inverting the Fourier expansion of H, one has

$$\int_{\alpha^{\times\ell}} \omega_{\mathbf{f}}(t) = (-2\pi \mathrm{i})^{\ell} \left(\frac{F_{\mathbf{f}}}{B_{\mathbf{f}}} \circ H^{-1} \right)(t) = (-2\pi \mathrm{i})^{\ell} N_{\Gamma} \frac{t(H^{-1})'(t)}{H^{-1}(t)} \cdot F_{\mathbf{f}}(H^{-1}(t))$$
$$=: (2\pi \mathrm{i})^{\ell} \sum_{m \geq 0} a_m t^m,$$

where $(H^{-1})' = \frac{dq_0}{dt}$. Moreover $a_0 = (-1)^{\ell} N_{\Gamma} \cdot \mathsf{H}^{[\ell]}_{[\mathrm{i}\infty]}(\varphi_{\mathbf{f}})$, and

$$\Psi^{\Gamma}_{\mathbf{f}}(t) := \int_{\alpha^{\times\ell}} R_{\left(\mathfrak{Z}|_{E^{[\ell]}_{\Gamma,t}}\right)} \overset{\mathbb{Q}(\ell+1)}{\equiv} \Psi^{[\ell]}_{\mathbf{f},0}(H^{-1}(t)) = (2\pi \mathrm{i})^{\ell} \left\{ a_0 \log t + \sum_{m \geq 1} \frac{a_m}{m} t^m \right\}$$

(compare Theorem 4.1).

A key observation is that $A_{\mathbf{f}}(\tau)\eta^{[\ell]}_{\ell}$ descends to $\mathcal{E}_{\Gamma}$, whereas the relative differentials ($\eta^{[\ell]}_{\ell}$ or $F_{\mathbf{f}}(\tau)\eta^{[\ell]}_{\ell}$) used in previous sections did not. This leads to a higher normal function and PF equations which make sense over Y_{Γ}. Recalling $\nabla^{\mathbf{f}}_{\mathrm{PF}} = \nabla^{\ell+1}_{\partial_\tau} + \text{l.o.t.}$ from Section 9.1,

$$\nabla^{\omega}_{\mathrm{PF}} := \frac{1}{(2\pi \mathrm{i} B_{\mathbf{f}}(\tau))^{\ell+2}} \circ \nabla^{\mathbf{f}}_{\mathrm{PF}} \circ (2\pi \mathrm{i} B_{\mathbf{f}}(\tau)) = \nabla^{\ell+1}_{\delta_t} + \text{l.o.t.}$$

descends to $\mathbb{P}^1$, yielding the homogeneous equation

$$(D^{\omega}_{\mathrm{PF}} \circ \delta_t)\Psi^{\Gamma}_{\mathbf{f}} = 0.$$

Writing

$$\nu_{\mathbf{f}}(\tau) := \left\langle \tilde{\mathcal{R}}^{[\ell]}_{\mathbf{f}}, \omega_{\mathbf{f}} \right\rangle = (-2\pi \mathrm{i})^{\ell} V^{[\ell]}_{\mathbf{f}}(\tau) \cdot A_{\mathbf{f}}(\tau),$$

we have the inhomogeneous equation

$$D^{\omega}_{\mathrm{PF}} \nu_{\mathbf{f}} = \left\langle \nabla_{\delta_t} \tilde{\mathcal{R}}^{[\ell]}_{\mathbf{f}}, \nabla^{\ell}_{\delta_t} \omega_{\mathbf{f}} \right\rangle = \left\langle \omega_{\mathbf{f}}, \nabla^{\ell}_{\delta_t} \omega_{\mathbf{f}} \right\rangle =: \mathcal{Y}^{[\ell]}_{\mathbf{f}}(t),$$

where the Yukawa coupling

$$\mathcal{Y}^{[\ell]}_{\mathbf{f}}(H(\tau)) = (-2\pi \mathrm{i})^{2\ell} A^2_{\mathbf{f}}(\tau) \left\langle \eta^{[\ell]}_{\ell}, \frac{1}{(2\pi \mathrm{i} B_{\mathbf{f}})^{\ell}} \nabla^{\ell}_{\partial_\tau} \eta^{[\ell]}_{\ell} \right\rangle = (2\pi \mathrm{i})^{\ell} \frac{A^2_{\mathbf{f}}}{B^{\ell}_{\mathbf{f}}} Y_{\tau^{\ell}}(\tau)$$
$$= (-1)^{\binom{\ell}{2}} \ell! \left(\frac{2\pi \mathrm{i}}{B_{\mathbf{f}}(\tau)} \right)^{\ell} (A_{\mathbf{f}}(\tau))^2.$$

Obviously the weights cancel so that $\mathcal{Y}^{[\ell]}_{\mathbf{f}} \circ H \in \check{M}_0(\Gamma)$, i.e., $\mathcal{Y}^{[\ell]}_{\mathbf{f}}$ yields a rational function on $\mathbb{P}^1$.

Suppose $\mathsf{H}^{[\ell]}_{[\mathrm{i}\infty]}(\varphi_{\mathbf{f}}) \neq 0$ and $|\kappa^{[\ell]}_{\Gamma}| > 1$, so that one can choose $\mathbf{g} \in \Phi_2^{\mathbb{Q}}(N)^{\circ}$ (such that $\mathfrak{Z}_{\mathbf{g}}$ also descends to $\mathcal{E}^{[\ell]}_{\Gamma}$) with $\mathsf{H}^{[\ell]}_{[\mathrm{i}\infty]}(\varphi_{\mathbf{g}}) = 0$ but $\mathsf{H}^{[\ell]}_{\sigma}(\varphi_{\mathbf{g}}) \neq 0$ (for some $\sigma \neq [\mathrm{i}\infty]$). Then one can consider $A_{\mathbf{f}} \cdot V^{[\ell]}_{\mathbf{g}} = \frac{1}{(-2\pi\mathrm{i})^{\ell}} \left\langle \tilde{\mathcal{R}}^{[\ell]}_{\mathbf{g}}, \omega_{\mathbf{f}} \right\rangle$, where $\tilde{\mathcal{R}}^{[\ell]}_{\mathbf{g}}$ is a lift with all $\mathfrak{K}_{\mathbf{g},i} = 0$ $(0 \leq i < \ell)$; cf. Proposition 9.2: in this case $\mathfrak{K}_{\mathbf{g}} := \lim_{\tau \to \mathrm{i}\infty} V^{[\ell]}_{\mathbf{g}}(\tau) = (-1)^{\ell} \lim_{\tau \to \mathrm{i}\infty} V^{[\ell]}_{\mathbf{g}}(\tau)$. This is the more general type of higher normal function implicit in the Apéry–Beukers irrationality proofs (cf. Introduction). (The general idea is this: one must show the radius of convergence of its t-series expansion to be "much larger" than that for either $A_{\mathbf{f}}$ or $A_{\mathbf{f}} \cdot (V^{[\ell]}_{\mathbf{g}} - \mathfrak{K}_{\mathbf{g}})$, while the latter expansions must satisfy certain integrality properties.) The story will be related from a less "modular" perspective in [48].

10.3. Identifying pullbacks of toric symbols

If (in oversimplified terms) the idea of Section 10.2 was to pull back the Eisenstein construction along H^{-1} (when it exists), here we pull back a given toric symbol (if possible) along some H, and try to recognize the result as an Eisenstein symbol. This leads to motivic proofs of several of the Mahler measure computations in [9, 10, 77].

We begin with an "anticanonical pencil" $\tilde{\mathcal{X}} = \overline{\{1 - t\phi(\underline{x}) = 0\}} \subset \mathbb{P}^1 \times \mathbb{P}_{\tilde{\Delta}}$ satisfying the assumptions of Theorem 3.1, with its attendant cycle $\tilde{\Xi} \in H^n_{\mathcal{M}}(\tilde{\mathcal{X}}_{-}, \mathbb{Q}(n))$ for $n = 2, 3, 4$. We also require ϕ to have root-of-1 vertex coefficients so that Theorem 4.1 holds. Set $\ell := n - 1$, and restrict/refine this family in several steps:

- (1) $\ell = 3$: assume that $\mathbb{P}_{\tilde{\Delta}}$ is smooth (so that $t = 0$ is a point of maximal unipotent monodromy).
- (2) If ϕ is regular, define[25] $\mathcal{X}$ $(\xrightarrow{\pi} \mathbb{P}^1)$ to be the (smooth) proper transform of $\tilde{\mathcal{X}}$ under successive blow-up of the components of the base locus $\mathbb{P}^1 \times (\tilde{X}_{\eta} \cap \tilde{\mathbb{D}}) \subset \mathbb{P}^1 \times \mathbb{P}_{\tilde{\Delta}}$, where X_{η} denotes a very general fiber. This accomplishes semistable reduction at $t = 0$. When ϕ is *not* regular this must be combined with the desingularization of $\tilde{\mathcal{X}}_{-}$ from the proof of Theorem 3.1 (to produce $\mathcal{X}$). Denote that pulled-back cycle by $\Xi \in CH^{\ell+1}(\mathcal{X} \backslash X_0, \ell + 1)$.

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[25] Preferring inconsistent notation to writing everywhere $\tilde{\tilde{\mathcal{X}}}$. We retain this convention for the rest of the paper.

(In what follows, one could also replace $\mathcal{X}$ by a [desingularized] quotient — if one exists — over a $t \mapsto t^{\kappa}$ quotient of the base preserving unipotency at $t = 0$, and Ξ by the push-forward cycle.)

- (3) $\ell = 2:$ assume $\mathrm{rk}(Pic(X_\eta)) = 19$,
  $\ell = 3:$ assume $h^{2,1}(X_\eta) = 1$, and that the VHS has no "instanton corrections" (cf. [32])

  Then $H^\ell(X_t)$ (or $H^2_{tr}(X_t)$ for $\ell = 2$) is the symmetric ℓth power of a weight 1 (rank 2) VHS; likewise for the PF equation of the section of $\omega_{\mathcal{X}/\mathbb{P}^1} := K_{\mathcal{X}} \otimes \pi^{-1}\theta^1_{\mathbb{P}^1}$ given by $\omega := \nabla_{\delta_t}\mathcal{R}_t$ (cf. Sections 4.2 and 4.3).

In fact, ω is (up to scaling) the unique section of $\omega_{\mathcal{X}/\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}(-[\infty]) \cong \mathcal{O}_{\mathbb{P}^1}$.

Now let $U \subset \mathbb{P}^1$ be a small neighborhood of $t = 0$. Working over U^*, denote by $W_\bullet$ the weight monodromy filtration on $H^\ell(X_t, \mathbb{Q})$ (H^2_{tr} if $\ell = 2$) and set $W^{\mathbb{Z}}_\bullet := W_\bullet \cap H^\ell_{(\mathrm{tr})}(X_t, \mathbb{Z})$. There are unique generating sections $\varphi_0 \in \Gamma(U, W^{\mathbb{Z}}_0)$, $-\overline{\varphi_1} \in \Gamma(U^*, W^{\mathbb{Z}}_2/W^{\mathbb{Z}}_0)$ positively oriented as topological cycles; the latter lifts to a multivalued section of $W^{\mathbb{Z}}_2$ with monodromy $\varphi_1 \mapsto \varphi_1 + N_{\mathcal{X}}\varphi_0$. The mirror map

$$(q =)\, \mathcal{M}(t) = \exp\left\{2\pi\mathrm{i}\frac{\int_{\varphi_1(t)}\omega_t}{\int_{\varphi_0(t)}\omega_t}\right\} \tag{10.6}$$

is well-defined on U^*; its logarithm $\mu = \frac{\log \mathcal{M}}{2\pi\mathrm{i}}$ extends to a multivalued map $\mathbb{P}^1 \rightsquigarrow \mathfrak{H}^*$. Recall $A(t) := \int_{\varphi_0(t)}\omega_t$, $\Psi(t) := \int_{\varphi_0(t)}\mathcal{R}_t$ (with $\partial_t\Psi = A$).

- (4) Assume the mirror map is "modular": that is, $\exists\, \tilde{N} \geq 3$ such that $\mu^{-1} =: \tilde{H}(\tau)$ is a well-defined automorphic function for $\Gamma(\tilde{N})$ ($H \in \check{M}_0(\Gamma(\tilde{N}))$); for odd ℓ, we also demand that $\{-\mathrm{id}\} \notin$monodromy group of $R^\ell\pi_*\mathbb{Z}$. (Obviously, this implies $N_{\mathcal{X}}|\tilde{N}$ and $\tilde{H}(\tau) = C \cdot \tilde{q}_0 + \text{h.o.t.}$ where $\tilde{q}_0 = q^{\frac{1}{N_{\mathcal{X}}}}$.) Then

  $$A(\tilde{H}(\tau)) \in \check{M}_\ell(\Gamma(\tilde{N})),$$

  where the "poles" come from non-unipotent singular fibers and are canceled by $\tilde{H}^*\frac{dt}{t}$ to yield

$$\mathsf{F}(t) := \frac{(-1)^\ell}{(2\pi\mathrm{i})^{\ell+1}}\partial_\tau\Psi(\tilde{H}(\tau)) = (-1)^\ell\frac{d\log\tilde{H}}{d\tau}\cdot\frac{A(\tilde{H}(\tau))}{(2\pi\mathrm{i})^{\ell+1}} \in M_{\ell+2}(\Gamma(\tilde{N})).$$

Now we want to force F to be an Eisenstein series; the following stronger assumption (which for $\ell = 1$ follows from the previous) does the job after a slight adjustment to $\tilde{H}$ (and $\tilde{N}$).

- (5) Assume $\mathcal{X}$ is "modular": That is in addition to assumptions (1)–(3), $\exists\, N \geq 3$, $H \in \check{M}_0(\Gamma(N))$, and a (surjective) rational map $\theta : \overline{\mathcal{E}}^{[\ell]}(N) \dashrightarrow \mathcal{X}$ over $H : \overline{Y}(N) \twoheadrightarrow \mathbb{P}^1_t$ (which can include e.g., a fiberwise Kummer- or Borcea–Voisin-type construction). While there are plenty of examples for $\ell = 1, 2$, we will see that for $\ell = 3$ there are *no* modular anticanonical families of this form; the problem already arises in hypothesis (3). However, there are relaxations of the hypotheses that *are* likely to produce examples. See Section 10.6. Define $\theta^*\Xi \in CH^{\ell+1}(\mathcal{E}^{[\ell]}(N), \ell+1)$ by pulling back (to an appropriate blow-up of $\mathcal{E}^{[\ell]}(N)$) and pushing forward. Then

$$\Omega_{\theta^*\Xi} = (2\pi \mathrm{i})^{\ell+1} F_{\theta^*\Xi}(\tau)\eta_\ell^{[\ell]} \wedge d\tau \in F^{\ell+1} \cap H^{\ell+1}(\mathcal{E}^{[\ell]}(N), \mathbb{Q}(\ell+1)), \tag{10.7}$$

where $F_{\theta^*\Xi} \in M^{\mathbb{Q}}_{\ell+2}(\Gamma(N))$. If we know the divisor

$$\theta^*(X_0) =: (-1)^\ell \sum_{\sigma \in \kappa(N)} r_\sigma(\Xi) \cdot \overline{\pi}^{-1}_{\Gamma(N)}(\sigma), \tag{10.8}$$

then taking $\mathbf{f} \in \mathcal{O}^*(U(N))^{\otimes(\ell+1)}$ with $\mathsf{H}^{[\ell]}_\sigma(\varphi_{\mathbf{f}}) = r_\sigma(\Xi)$ $(\forall \sigma \in \kappa(N))$, $\Omega_{\mathfrak{Z}_{\mathbf{f}}}$ and $\Omega_{\theta^*\Xi}$ have the same residues. By Section 7.1.5 they are equal (i.e., $F_{\theta^*\Xi} = F_{\mathbf{f}}$) hence (by Lemma 9.1(ii)) so are the fiberwise AJ classes.

To compute further we need precise information about θ: consider the positive integers $M_\theta := \deg(\theta)$, $m_0 := \frac{\theta_*(\alpha^\ell)}{\varphi_0}$, $m_1 := \frac{\theta_*(\mathcal{G}^*(\alpha^{\ell-1}\beta))}{\varphi_1}$ (see Section 9.1), $m_\theta := \frac{m_0}{m_1}$, and (in suggestive notation) $N_\Gamma := \frac{N_{\mathcal{X}}}{m_\theta}$. For $\ell = 1$ we just have $m_0 = m_1 = m_\theta = 1$ $(\implies N_\Gamma = N_{\mathcal{X}})$, $M_\theta = \kappa$. One easily checks that $H(\tau) = \tilde{H}(m_\theta \tau) = C_0 \cdot q_0 + \text{h.o.t.}$, when $q_0 := q^{\frac{1}{N_\Gamma}}$ (by abuse of notation we will write this $H(q_0)$, and $H'(q_0) := \frac{dH}{dq_0}$). We then have

$$\theta^*\omega = m_0 A(H(q_0))\eta_\ell^{[\ell]} \in \Gamma(\overline{Y}(N), \omega^{\Gamma(N)}_{\mathcal{E}/Y}),$$
$$H^*\frac{dt}{t} = \frac{2\pi \mathrm{i}}{N_\Gamma}\frac{q_0}{H(q_0)}H'(q_0)d\tau \in \Omega^1(\overline{Y}(N))\left\langle \log(H^{-1}(0) \cup H^{-1}(\infty))\right\rangle,$$

$$\begin{aligned}\theta^*\Omega_\Xi &= \theta^*\left(\frac{dt}{t}\wedge\nabla_{\delta_t}\mathcal{R}_t\right) = (-1)^\ell\theta^*\omega\wedge H^*\frac{dt}{t}\\ &= (-1)^\ell\frac{2\pi \mathrm{i} m_0}{N_\Gamma}\frac{q_0}{H(q_0)}H'(q_0)A(H(q_0))\eta_\ell^{[\ell]}\wedge d\tau\\ &\in \Omega^{\ell+1}(\overline{\mathcal{E}}^{[\ell]}(N))\left\langle \log\theta^*(X_0)\right\rangle.\end{aligned}$$

Under pullback the regulator period becomes (for f as above)

$$\begin{aligned}\Psi(H(\tau)) &= \int_{\varphi_0(H(\tau))}\tilde{\mathcal{R}}_{H(\tau)} = \frac{1}{m_0}\int_{\alpha^\ell(\tau)}\tilde{\mathcal{R}}_{\theta^*\Xi}(\tau)\\ &= \frac{1}{m_0}\int_{\alpha^\ell}\tilde{\mathcal{R}}_{\mathbf{f}}^{[\ell]}(\tau) = \frac{1}{m_0}\Psi_{\mathbf{f},0}^{[\ell]}(\tau),\end{aligned} \tag{10.9}$$

so that (by Proposition 9.1(i))

$$\partial_\tau\Psi(H(\tau)) = (-1)^\ell(2\pi\mathrm{i})^{\ell+1}m_0^{-1}F_{\theta^*\Xi}(\tau).$$

That

$$\Psi(H(\tau)) \text{ is of the form (9.17)}$$

is of fundamental importance; if one divides by $(2\pi\mathrm{i})^\ell$ and takes the real parts it essentially says the real regulator period (or Mahler measure, in the region described in Corollary 4.4) pulls back to an Eisenstein–Kronecker–Lerch series (noticed in examples of [9, 10, 69]). Furthermore, this allows us to use Proposition 9.4 to compute its special values at $H\{\text{unipotent cusps}\}$, which therefore must be a sum (with coefficients $\in \mathbb{Q}(e^{\frac{2\pi\mathrm{i}}{N}})$) of $(\ell+1)$th special values of Dirichlet L-functions. This is similar to the case in Section 6 of $L/\mathbb{Q}$ abelian (which however does not imply modularity).

Our last object of interest is the Yukawa coupling $Y(t) = \left\langle \omega_t, \nabla_{\delta_t}^\ell\omega_t\right\rangle$, which becomes

$$\begin{aligned}Y(H(q_0)) &= M_\theta^{-1}\left\langle \theta^*\omega, \theta^*\nabla_{\delta_t}^\ell\omega\right\rangle\\ &= \frac{N_\Gamma^\ell}{(2\pi\mathrm{i})^\ell M_\theta}\cdot\frac{1}{\{H'(q_0)\}^\ell}\left\langle\theta^*\omega, \nabla_{\partial_\tau}^\ell\theta^*\omega\right\rangle\\ &= \frac{N_\Gamma^\ell m_0^2}{(2\pi\mathrm{i})^\ell M_\theta}\cdot\frac{\{A(H(q_0))\}^2}{\{H'(q_0)\}^\ell}\left\langle\eta_\ell^{[\ell]}, \nabla_{\partial_\tau}^\ell\eta_\ell^{[\ell]}\right\rangle\\ &= \frac{(-1)^{\binom{\ell}{2}}\ell! N_\Gamma^\ell m_0^2}{(2\pi\mathrm{i})^\ell M_\theta}\cdot\frac{\{A(H(q_0))\}^2}{\{H'(q_0)\}^\ell},\end{aligned} \tag{10.10}$$

a rational function on $\overline{Y}(N)$. Noting $A(0) = (2\pi\mathrm{i})^{\ell}$ and using (10.7) and Proposition 9.4 gives

Theorem 10.1. *Assuming modularity of a family of CY ℓ-folds $\mathcal{X}$ arising (as described) from the toric construction, we have*

$$\text{(10.11)} \qquad \frac{(-1)^{\ell} m_0}{(2\pi\mathrm{i})^{\ell} N_{\Gamma}} \delta_{q_0} \Psi(H(q_0)) = \frac{(-1)^{\ell} m_0}{(2\pi\mathrm{i})^{\ell} N_{\Gamma}} \frac{q_0}{H(q_0)} H'(q_0) A(H(q_0)) = F_{\theta^* \Xi}(q_0) = \sum_{\sigma \in \kappa(N)} r_{\sigma}(\Xi) \tilde{E}_{\sigma}^{[\ell]}(q_0)$$

for the pulled-back cycle class of the toric symbol, and also

$$\text{(10.12)} \qquad \frac{Y(0)}{(2\pi\mathrm{i})^{\ell}} = \frac{(-1)^{\binom{\ell}{2}} \ell! N_{\Gamma}^{\ell} m_0^2}{M_{\theta} C_0^{\ell}} \in \mathbb{Q}(C_0).$$

Finally, if $X_{t_0 \neq 0}$ is a maximally unipotent singular fiber, then[26] $\mu(t_0) \equiv [\frac{r_0}{s_0}] \in \kappa(N)$ *and*

(10.13)
$$\lim_{t \to t_0} \Psi(t) \overset{\mathbb{Q}(\ell+1)}{\equiv} \frac{(-1)^{\ell+1}}{2N} \sum_{\substack{[\frac{r}{s}] \in \kappa(N) \\ [\frac{r}{s}] \not\equiv [\frac{r_0}{s_0}]}} s^{\ell} r_{[\frac{r}{s}]}(\Xi) \tilde{L}_{-} \left(\pi_{[\frac{r_0}{s_0}]_*} \iota_{[\frac{r}{s}]_*} \widehat{\varphi_N^{[\ell]}}, \ell+1 \right).$$

By comparing values at $[\mathrm{i}\infty]$ (i.e., $q_0 = 0$) in (10.11), we have the interesting

Corollary 10.1. $r_{[\mathrm{i}\infty]}(\Xi) = (-1)^{\ell} \frac{m_0}{N_{\Gamma}}$.

Remark. If the $r_{\sigma}(\Xi)$ are known but the series expansion $t = H(q_0) = C_0 q_0 + \cdots$ for the mirror map is not, one can in principle determine the latter from

$$\Psi(H(\tau)) = \frac{1}{m_0} \Psi_{\mathbf{f},0}^{[\ell]}(\tau)$$

(cf. (10.9)), by using (4.5) for the l.h.s. and Corollary 9.2 for the r.h.s. (In the computations below, we have preferred to take H from other sources, in order to partially vet our formulas.) Since the "log + constant" terms of both sides must agree (mod $\mathbb{Q}(n)$), an immediate consequence is

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[26] The specific choice of representative $\frac{r_0}{s_0}$ of the cusp $\mu(t_0)$ depends on the path along which $\Psi(t)$ has been continued prior to taking $\lim_{t \to t_0}$.

Corollary 10.2. *C_0 (hence $\frac{Y(0)}{(2\pi \mathrm{i})^\ell}$) is a root of unity.*

Clearly one can normalize ϕ (retaining the assumption on vertex coefficients) so that $Y(0) \in \mathbb{Q}(\ell)$.

10.4. The elliptic curve case

Start with a reflexive tempered Laurent polynomial $\phi \in \bar{\mathbb{Q}}[x^{\pm 1}, y^{\pm 1}]$ defining a family of (generically smooth) elliptic curves, $\tilde{\mathcal{X}} \subset \mathbb{P}^1_t \times \mathbb{P}_{\widetilde{\Delta}_\phi}$. Possibly after a finite ($t \mapsto t^\kappa$) quotient, again preserving unipotency at $t = 0$, we desingularize this and blow down all (-1)-curves contained in fibers. The resulting elliptic surface is denoted $\mathcal{X}$, and is relatively minimal in the sense that $\omega_{\mathcal{X}/\mathbb{P}^1} \cong \pi^* \pi_* \omega_{\mathcal{X}/\mathbb{P}^1}$; the singular fibers are therefore of the types described by Kodaira [51]. Clearly $\chi(\mathcal{X}) = 12 \deg(\pi_* \omega_{\mathcal{X}/\mathbb{P}^1})$ is 12, either by looking at zeroes of $\omega = \nabla_{\delta_t} \mathcal{R}_t \in \Gamma(\pi_* \omega_{\mathcal{X}/\mathbb{P}^1})$ or the fact that $\mathcal{X}$ is birational to $\mathbb{P}_{\Delta_\phi}$ hence to $\mathbb{P}^2$. This constrains the possible combinations of singular fibers in light of the table:

Sing. fiber type	Contrib. to $\chi(\mathcal{X})$	Ord. of monodromy	No. of components
$I_{n \geq 1}$	n	∞	n
$I^*_{n \geq 0}$	$n+6$	∞	$n+5$
II	2	6	1
IV*	8	3	7
III	3	4	2
III*	9	4	8
IV	4	3	3
II*	10	6	9

where we have paired those types related by a quadratic transformation ("adding a $*$"). We identify families by the set of fiber types, e.g. I_1^4/I_4^* means 4 I_1's and 1 I_4^*.

Now referring to (10.6), we make a precise

Definition 10.1. $\mathcal{M}$ is *weakly modular* if and only if $\mu^{-1} (=: H)$ is a Hauptmodul for $\Gamma \subset SL_2(\mathbb{Z})$ of finite index. We say $\mathcal{M}$ is *modular* if in addition $\{-\mathrm{id}\} \notin \Gamma$ and $\Gamma \supset \Gamma(N)$ for some $N \geq 3$.

Obviously if $\mathcal{M}$ is modular then one has a canonical quotient $\overline{\mathcal{E}}^{[1]}_{\Gamma(N)} \overset{\theta}{\dashrightarrow} \overline{\mathcal{E}}^{[1]}_{\Gamma} \cong \mathcal{X}$ and $\mathcal{X}$ is modular in the sense of Section 10.3.

Lemma 10.1 [32, Proposition 2]. *$\mathcal{M}$ is weakly modular if and only if the J-invariant $J(\mu(t))$ ramifies only over $J = 0$ (to order 1 or 3), $J = 1$ (to order 1 or 2), and $J = \infty$ (to any order).*

The point is that μ^{-1} cannot possibly be single-valued if $J \circ \mu$ has "excess ramification" (which explains why we wanted to allow order-κ quotients of the base in constructing $\mathcal{X}$). It follows (cf. [32]) that fiber types II^* and IV are not permitted (so no I_1^2/II^*), and neither are certain other combinations (e.g. I_1^6/I_6); in [33, Theorem 4.12] the remaining possibilities are listed (up to "transfer of $*$"). Disallowing those fiber types left which contain $-\mathrm{id}$ in their local monodromy group ($\mathrm{II}, \mathrm{III}, \mathrm{III}^*$), and checking for $-\mathrm{id}$ also in global monodromy, one arrives at the list below.

Proposition 10.1. *Suppose the singular fiber configuration of $\mathcal{X}$ is one of those shown in the table, with fiber $I_{n_\mathcal{X}}$ at $t = 0$. (This gives an additional degree of freedom.) Then $\mathcal{M}$ is modular, $\mathcal{X} \cong \overline{\mathcal{E}}_\Gamma$ (for $\Gamma \supset \Gamma(N)$ as displayed), and*[27]

$$-\frac{1}{n_\mathcal{X}} \sum_{\sigma \in |H^{-1}(0)| \subset \kappa(N)} \tilde{E}_\sigma^{[1]}(q_0) = F_{\theta^* \Xi}(q_0), \tag{10.14}$$

where $|H^{-1}(0)|$ is not counted with multiplicity. Finally, all the configurations below occur in the toric construction.

Configuration	Γ	N
I_3^4	$\Gamma(3)$	3
$I_1/I_3/IV^*$	$\Gamma^{(')}(3)$	3
$I_1/I_1^*/I_4$	$\Gamma_1^{(')}(4)$	4
I_2^2/I_4^2	$\left\langle \Gamma(4), \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \right\rangle$	4
I_2^2/I_2^*	$\widetilde{\Gamma(2)} := \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle$	4
I_1^2/I_5^2	$\Gamma_1^{(')}(5)$	5
$I_1/I_2/I_3/I_6$	$\Gamma_1^{(')}(6)$	6
$I_1^2/I_2/I_8$	$\left\langle \Gamma_1'(8), \begin{pmatrix} -3 & -8 \\ -1 & -3 \end{pmatrix} \right\rangle$	8
I_1^2/I_4^*	$\left\langle \Gamma_1'(8), \begin{pmatrix} -3 & -8 \\ -1 & -3 \end{pmatrix}, \begin{pmatrix} -1 & -4 \\ 0 & -1 \end{pmatrix} \right\rangle$	8
I_1^3/I_9	$\left\langle \Gamma_1'(9), \begin{pmatrix} -4 & -9 \\ 1 & 2 \end{pmatrix} \right\rangle$	9

[27] Here $q_0 = q^{\frac{1}{n_\mathcal{X}}}$.

For computations it is desirable to replace $-\frac{1}{n_{\mathcal{X}}}\sum \tilde{E}_{\sigma}^{[1]}$ by $F_{\mathbf{f}}$ with $\varphi_{\mathbf{f}}$ chosen to have $\mathsf{H}_{\sigma}(\varphi_{\mathbf{f}}) = \begin{cases} \frac{-1}{n_{\mathcal{X}}}, & \sigma \in |H^{-1}(0)| \\ 0, & \text{otherwise} \end{cases}$. Note that by (10.9), for $\tau \in \mathfrak{H}$

$$\Psi(H(\tau)) \equiv \Psi_{\mathbf{f},0}^{[1]}(\tau) \mod \mathbb{Q}(2). \tag{10.15}$$

The two "E_6" examples below both correspond to the second row of the table, and their difference illustrates a technical subtlety. The first computation is essentially that in [77, Example 3]; Examples 4,5,6 in [77] also fall under Proposition 10.1's aegis, and correspond to lines 3,6,7 (resp.) in the table.

Example 10.3. $\phi = x^2y^{-1} + x^{-1}y^2 + x^{-1}y^{-1}$, $\kappa = 3$ (quotient).

This yields $\mathcal{X}$ with fibers $X_t \cong \{1 - t^{\frac{1}{3}}\phi = 0\} \subset \mathbb{P}^2$, $\Gamma = \Gamma_1(3)$, and $n_{\mathcal{X}} = 1$. (This is just the Hesse pencil, which appears as Example 1 in [69] and Example 3 in [77].) The singular fibers occur at $t = 0\,(I_1)$, $\frac{1}{3^3}\,(I_3)$, $\infty\,(IV^*)$; whereas if we had not taken the quotient ($\kappa = 1$), there would be 4 I_3's (at $t = 0, \frac{1}{3}, \frac{\zeta_3}{3}, \frac{\zeta_3^2}{3}$) with $\Gamma = \Gamma(3)$.

From [77],

$$\begin{aligned} H(q) = H_{\Gamma_1(3)}(q) &:= \left(27 + \frac{\eta(q)^{12}}{\eta(q^3)^{12}}\right)^{-1} \\ &= q(1 - 15q + 171q^2 - 1679q^3 + \cdots), \end{aligned}$$

where of course $\eta(q) = q^{\frac{1}{24}}\prod_{n\geq 1}(1-q^n)$, and we have

$$A(t) = 2\pi\mathrm{i}\sum_{m\geq 0}\frac{(3m)!}{(m!)^3}t^m = 2\pi\mathrm{i}(1 + 6t + 90t^2 + 1680t^3 + \cdots).$$

Since $|H^{-1}(0)| = \{[\mathrm{i}\infty]\}$, we put $\varphi_{\mathbf{f}} := -\frac{1}{3}\pi^*_{[\mathrm{i}\infty]}\varphi_3^{[1]}$; by Example 8.1

$$F_{\mathbf{f}}(q) = -1 + 9\sum_{K\geq 1} q^K \sum_{r|K} r^2\chi_{-3}(r) = -1 + 9q - 27q^2 + 9q^3 + \cdots .$$

The proposition says this equals

$$\begin{aligned} &\frac{-q}{H(q)}H'(q)\frac{A(H(q))}{2\pi\mathrm{i}} \\ &\quad = -(1 + 15q + 54q^2 - 76q^3 + \cdots)(1 - 30q + 513q^2 - 6716q^3 + \cdots) \\ &\qquad \times (1 + 6q + 6q^3 + \cdots), \end{aligned}$$

which is clearly plausible from the first three terms of the series. From (9.26) we have

$$\Psi^{[1]}_{\mathbf{f},0}(q) = 2\pi\mathrm{i}\left\{\log q - 9\sum_{K\geq 1}\left(\frac{\sum_{r|K} r^2\chi_{-3}(r)}{K}\right)q^K\right\}$$

while $\Psi(t) = 2\pi\mathrm{i}\left\{\log t + \sum_{m\geq 1}\frac{(3m)!}{(m!)^3}t^m\right\}$; computation again suggests that $\Psi(H(q)) = \Psi^{[1]}_{\mathbf{f},0}(q)$, which is (mod $\mathbb{Q}(2)$) exactly what (10.15) asserts.

Example 10.4. $\phi = x + y + x^{-1}y^{-1}$, $\kappa = 3$.

This gives $\mathcal{X}$ with $\Gamma = \Gamma_1'(3)$, $n_{\mathcal{X}} = 3$, and singular fibers at $t = 0\,(I_3)$, $\frac{1}{3^3}\,(I_1)$, $\infty\,(IV^*)$; before the quotient these are $t=0\,(I_9)$ and $t = \frac{1}{3}, \frac{\zeta_3}{3}, \frac{\zeta_3^2}{3}\,(I_1)$. Put $\mathfrak{g}(u) = 1 - \left(\frac{1-3u}{1+6u}\right)^3$; by considering locations of singular fibers one deduces

$$\begin{aligned} H(q_0) &= H_{\Gamma_1'(3)}(q_0) = \frac{1}{3^3}\mathfrak{g}\left(H_{\Gamma(3)}(q_0)\right) = \frac{1}{3^3}\mathfrak{g}\left[\left(H_{\Gamma_1(3)}(q_0^3)\right)^{\frac{1}{3}}\right] \\ &= q_0(1 - 15q_0 + 171q_0^2 - 5q_0^3 + \cdots). \end{aligned}$$

This is so similar to the previous example that the $A(t)$'s are the same, and

$$-\frac{1}{3}\frac{q_0}{H(q_0)}H'(q_0)\frac{A(H(q_0))}{2\pi\mathrm{i}} = -\frac{1}{3} + 3q_0 - 9q_0^2 + \cdots.$$

We want $\widehat{\varphi}_{\mathbf{f}} = -\frac{1}{3}\rho'_*(\iota_{[\mathrm{i}\infty]_*}\widehat{\varphi_3^{[1]}}) = -\frac{1}{3}\pi^*_{[0]}\widehat{\varphi_3^{[1]}}$ $(\implies \varphi_{\mathbf{f}} = \frac{1}{3}\iota_{[0]_*}\varphi_3^{[1]})$ since $|H^{-1}(0)| = \{[\mathrm{i}\infty], [1], [\frac{1}{2}]\}$. Using Proposition 8.3

$$F_{\mathbf{f}}(q_0) = -\frac{1}{3} + 3\sum_{K\geq 1} q_0^K \sum_{r|K} r^2\chi_{-3}(r),$$

in agreement with the above.

It is interesting to explain why the "E_8" family [69, Example 3; 77, Example 3]

$$\phi = xy^{-1} + x^{-1}y^2 + x^{-1}y^{-1}, \quad \kappa = 6, \quad I_1^2/\mathrm{II}^*$$

and "E_7" family

$$\phi = xy^{-1} + x^{-1}y^3 + x^{-1}y^{-1}, \quad \kappa = 4, \quad I_1/I_2/\mathrm{III}^*$$

fail to yield Eisenstein series (despite nontriviality of $\Xi \in CH^2(\mathcal{X}\backslash X_0, 2)$). More to the point,

$$\frac{q}{\mu^{-1}(q)}(\mu^{-1})'(q)\frac{A(\mu^{-1}(q))}{2\pi \mathrm{i}} =: \sum_{m\geq 0} \alpha_m q^m \tag{10.16}$$

does not even yield a modular form (of any level) since $\limsup_{M\to\infty} \sqrt[M]{|\alpha_M|} =: \gamma > 1$. (At least one infers this from the data $\{b_n\}$ in [77].) It is insufficient to say that the divisors of $\{x|_{X_t}, y|_{X_t}\}$ are not supported on torsion (perhaps this could be fixed by an AJ-equivalence), although this is probably required for instances where Proposition 10.1 fails.

In the E_8 case, $J(\mu(t))$ vanishes to order 2 at $t = \infty$ (the II* fiber), so that μ^{-1} is multivalued at $\tau = \mathrm{e}^{\frac{2\pi \mathrm{i}}{6}}$. As a result (10.16) both is multivalued and blows up there.

According to Lemma 10.7, for the E_7 family μ^{-1} is a Hauptmodul. However, the fact that $\Gamma = \Gamma_1(2) \ni \{-\mathrm{id}\}$ manifests itself in ($\pm$) multivaluedness of $A \circ H$ about $\tau = \frac{1+\mathrm{i}}{2}$ (where $J = 1$ and $t = \infty$).

In neither case does one have $\theta : \mathcal{E}^{[1]}_{\Gamma(N)} \dashrightarrow \mathcal{X}$ along which to pull back Ξ. Perhaps this suggests a study of "generalized Eisenstein symbols" on families over finite covers of $\mathfrak{H}$, with additional (nontorsion) marked structure; the elliptic Bloch groups of Wildeshaus [83] seem quite suitable for this purpose.

10.5. Examples in the $K3$ case

Up to unimodular transformation, there are 4319 reflexive polytopes in $\mathbb{R}^3$ [52]; according to Corollary 3.1ff we immediately get (at least) 358 examples for $\ell = 2$ where the toric symbol completes by taking ϕ = characteristic polynomial of vertices. (Putting "random" roots of unity instead of "1" on each vertex renders all 1071 polytopes from Remark 3.4 usable.) For each $\mathcal{X}/\Xi$ to be a candidate for modularity/Eisenstein-ness, we must have $\mathrm{rk}(\mathrm{Pic}(X_\eta)) = 19$, in which case X_η has the Shioda–Inose structure [60] (and one can then ask whether the underlying family of elliptic curves is suitably modular). Such candidates are nontrivial to produce, but "non-candidates" seem much more elusive.

Problem. Does Theorem 3.1 produce any families of $K3$'s with generic Picard rank ≤ 18? Or does the tempered condition indirectly furnish enough additional divisors to preclude this possibility?

Here are eight Laurent polynomials which satisfy Theorem 3.1 and produce (after desingularization; see Section 10.3 for the definition of $\mathcal{X}$) one-parameter $K3$ families $\mathcal{X}$ provably of generic Picard rank 19 (together with the method of proof).

	family	$\phi(x,y,z)$	$\frac{A(t)}{(2\pi\mathrm{i})^2}$	method
1	Fermat quartic	$\frac{1+x^4+y^4+z^4}{xyz}$	$\sum_{m\geq 0}\frac{(4m)!}{(m!)^4}t^{4m}$	symmetry $\mathfrak{G}\cong(\mathbb{Z}/4\mathbb{Z})^2$
2	quartic mirror	$x+y+z+\frac{1}{xyz}$	same	restrict from $\mathbb{P}_{\tilde{\Delta}}$
3	$\mathbb{WP}(1,1,1,3)$ "Fermat"	$\frac{1+x^6+y^6+z^2}{xyz}$	$\sum_{m\geq 0}\frac{(6m)!}{(m!)^3(3m)!}t^{6m}$	symmetry $\mathfrak{G}\cong\mathbb{Z}/6\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$
4	$\mathbb{WP}(1,1,1,3)$ mirror	$x+y+z+\frac{1}{xyz^3}$	same	restrict from $\mathbb{P}_{\tilde{\Delta}}$
5	"box"	$\frac{(x-1)^2(y-1)^2(z-1)^2}{xyz}$	$\sum_{m\geq 0}\binom{2m}{m}^3 t^m$	Shioda
6	Fermi [68]	$x+\frac{1}{x}+y+\frac{1}{y}+z+\frac{1}{z}$	$\left\{\sum_{m\geq 0}t^{2m}\binom{2m}{m}\times\sum_{k=0}^{m}\binom{m}{k}^2\binom{2k}{k}\right\}$	"double cover" of Apery
7	Apéry	$\frac{\{(x-1)(y-1)(z-1)\times[(x-1)(y-1)-xyz]\}}{xyz}$	$\left\{\sum_{m\geq 0}t^m\times\sum_{k=0}^{m}\binom{m}{k}^2\binom{m+k}{k}^2\right\}$	Shioda
8	Verrill [81]	$\frac{\{(1+x+xy+xyz)\times(1+z+zy+zyx)\}}{xyz}$	$\left\{\sum_{m\geq 0}t^m\times\sum_{p+q+r+s=m}\left(\frac{m!}{p!q!r!s!}\right)^2\right\}$	intersection form

(The "Apéry" family is birational to the one studied in [15,16,67].) Families #1–4 and 6 are instances of Example 3.1 (with Remark 3.4 for #1 and #3). The other three ϕ's are not regular and need Theorem 3.1 with $K=\mathbb{Q}$ (for #5 and #7) or Remark 3.3(iv) (for #8) applied to the equivalent symbol $\{xy,y,z\}$.

We quickly summarize the "methods" in the r.h. column; a study (including most of these examples) can be found in [82]. If $\tilde{X}_\eta$ is nonsingular ($=X_\eta$) then [70]

$$\mathrm{rk}(\mathrm{Pic}(X_\eta))\geq\mathrm{rk}\{\mathrm{im}(\mathrm{Pic}(\mathbb{P}_{\tilde{\Delta}})\to Pic(X_\eta))\}=\ell(\Delta^\circ)-\sum_{\sigma\in\Delta^\circ(1)}\ell^*(\sigma)-4,$$

which $=19$ for families #2 and #4 and $=1$ for #1 and #3. For the latter cases, the action on X_η by a finite subgroup $\mathfrak{G}\subset(\mathbb{C}^*)^3$ augments the Picard rank by

$$\mathrm{rk}\left[\left(H^2(X_\eta,\mathbb{Z})^{\mathfrak{G}}\right)^\perp\right]$$

[64, 82], which turns out to be 18. For #5 (resp #7), X_η is obtained from $\tilde{X}_\eta$ (remember X_η is really $\widetilde{\tilde{X}_\eta}$) by blowing up the 12 (resp. 7) A_1 singularities. The elliptic fibration $X_\eta \to \mathbb{P}^1_z$ has singular fibers $(I_1^*)^2/I_8/I_1^2$ (resp. $I_1^*/I_5/I_8/I_1^4$). By Shioda [74]

$$\mathrm{rk}(\mathrm{Pic}(X_\eta)) = 2 + r + \sum(\mathfrak{M}_i - 1),$$

where r = rank of group of sections = 0 (resp. 1; the existence of a nontorsion section is demonstrated in [16]) and $\mathfrak{M}_i = \#$ of fiber components in each singular fiber; this yields 19. This result is transferred to the Fermi family by observing that its pullback $\{1 - \frac{1}{u+u^{-1}}\phi_{\mathrm{Fermi}} = 0\}$ has a $2:1$ rational map (over $u \mapsto u^2 = t$) onto the Apéry family $\{1 - t\phi_{\mathrm{Apery}} = 0\}$ (see [68]). Finally, to deal with #8, [81] adds some lines to the components of $D \subset X_t$ and shows the rank of the resulting intersection form is 19.

The Fermi, Apéry and Verrill pencils (which are modular) yield an instructive set of examples for Theorem 10.1: $N = 6$ in all three cases but the $\{r_\sigma(\Xi)\}$, hence $\{F_{\theta^*\Xi}\}$, are all different.

Example 10.5. By Peters [67], the Apéry pencil's $\mathbb{Z}$-PVHS is equivalent to that coming from the construction of Remark 8.3 for $N = 6$ (and we will assume the 2 $\mathcal{X}$'s birational). This gives[28] (with $\Gamma = \Gamma_1(6)^{+6}$)

$$m_0 = -12, m_1 = 1, N_\Gamma = 1, M_\theta = 24 \quad \Longrightarrow \quad \frac{Y(0)}{(2\pi \mathrm{i})^2} = -12;$$

moreover, $\widehat{\varphi}_{\mathbf{f}}$ should be a constant multiple of (10.5). Since (by Corollary 10.1) $r_{[i\infty]}(\Xi) = -12$, we take

$$\begin{aligned}\widehat{\varphi}_{\mathbf{f}} &:= (10.5) \\ &= -\frac{2\cdot 6^3}{5}\{\widehat{\varphi_{\{1,1\}}} - \widehat{\varphi_{\{2,1\}}} - \widehat{\varphi_{\{3,1\}}} + \widehat{\varphi_{\{6,1\}}}\} \\ &\quad + \frac{2\cdot 6^5}{5}\{\widehat{\varphi_{\{6,1\}}} - \widehat{\varphi_{\{6,2\}}} - \widehat{\varphi_{\{6,3\}}} + \widehat{\varphi_{\{6,6\}}}\}\end{aligned}$$

where $\widehat{\varphi_{\{a,b\}}}(m,n) := \begin{cases} 1, & a|m \text{ and } b|n \\ 0 & \text{otherwise} \end{cases}$. (See figure 12 for a depiction of $\frac{5}{2\cdot 6^5}\widehat{\varphi}_{\mathbf{f}}$; any places where it takes the value 0 are simply left blank.) By

[28]See below for C_0. Singularities: monodromy is maximally unipotent about 0, $\infty(=t)$, finite (order 2) about $(\sqrt{2}+1)^4$, $(\sqrt{2}-1)^4$.

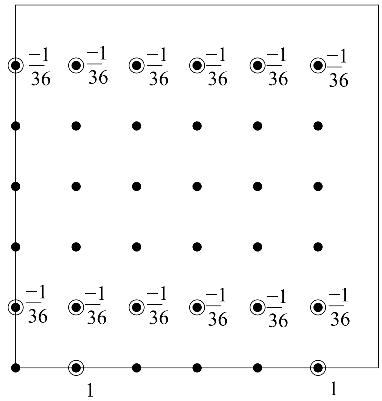

Figure 12: Eisenstein coefficients for toric symbol on Apery pencil.

Proposition 8.2,

$$
\begin{aligned}
E^{[2]}_{\varphi_{\{a,b\}}}(q) &= \frac{-3}{(2\pi\mathrm{i})^4}\tilde{L}\left(\iota^*_{[\mathrm{i}\infty]}\widehat{\varphi_{\{a,b\}}},4\right) \\
&\quad -\frac{1}{6^4}\sum_{M\geq 1} q^{\frac{M}{6}}\left\{\sum_{r|M} r^3\left(\sum_{n_0\in\mathbb{Z}/6\mathbb{Z}} \mathrm{e}^{\frac{2\pi\mathrm{i}n_0 r}{6}}\widehat{\varphi_{\{a,b\}}}\left(\frac{M}{r},n_0\right)\right)\right\} \\
&= \frac{-1}{240b^4}-\frac{1}{b^4}\sum_{K\geq 1} q^{\frac{a}{b}K}\left\{\sum_{\mathfrak{r}|K}\mathfrak{r}^3\right\} \\
&= \frac{-1}{240b^4}E_4(q^{\frac{a}{b}}),
\end{aligned}
$$

using substitutions $M = 6\frac{a}{b}K$ and $r = \frac{6}{b}\mathfrak{r}$. So we have, with $E_4(q) = 1 + 240(q + 9q^2 + 28q^3 + 73q^4 + \cdots)$,

$$
\begin{aligned}
E^{[2]}_{\varphi_{\mathfrak{f}}}(q) &= \tfrac{-12}{240\cdot 5}\{(1-6^2)E_4(q) + (6^2-2^4)E_4(q^2) + (6^2-3^4)E_4(q^3) \\
&\quad + (6^4-6^2)E_4(q^6)\} \\
&= \tfrac{7}{20}E_4(q) - \tfrac{1}{5}E_4(q^2) + \tfrac{9}{20}E_4(q^3) - \tfrac{63}{5}E_4(q^6) \\
&= -12 + 84q + 708q^2 + 2460q^3 + \cdots .
\end{aligned}
$$

On the other hand, from [10] $u = \frac{\eta(\tau)^6\eta(6\tau)^6}{\eta(2\tau)^6\eta(3\tau)^6}$ implies that

$$H(q) = u^2 = q(1 - 12q + 66q^2 - 220q^3 + \cdots),$$

while from the table

$$A(t) = (2\pi \mathrm{i})^2(1 + 5t + 73t^2 + 1445t^3 + \cdots);$$

therefore (from Theorem 10.1)

$$F_{\theta^*\Xi} = \frac{m_0}{(2\pi \mathrm{i})^2 N_\Gamma}\frac{q}{H(q)}H'(q)A(H(q)) = -12 + 84q + 708q^2 + 2460q^3 + \cdots .$$

So here we were able to correctly predict the Eisenstein series; in the remaining examples (where obviously Theorem 10.1 predicts (10.11) is *an* Eisenstein series) we have found $\varphi_{\mathbf{f}}$ essentially by solving for the correct combination of $\varphi_{\{a,b\}}$'s.

Example 10.6. (Compare [10, Example 1].) For the Fermi family, one deduces from Apéry (and the relationship between the two) that

$$\begin{aligned} &m_0 = -12, \quad m_1 = 1, \quad C_0 = 1, \quad N_\Gamma = 2, \quad M_\theta = 24 \\ &\qquad \Longrightarrow \qquad r_{[\mathrm{i}\infty]}(\Xi) = -6,\ \frac{Y(0)}{(2\pi \mathrm{i})^2} = -48; \end{aligned}$$

so $q_0 = q^{\frac{1}{2}}$ and

$$H(q_0) = \frac{1}{u + \frac{1}{u}} = q_0(1 - 7q_o^2 + 34q_0^4 - 204q_0^6 + \cdots).$$

(The family has order 2 monodromy about $t = \pm\frac{1}{2}, \pm\frac{1}{6}$ and maximally unipotent monodromy about $t = 0$.) From the table $A(t) = (2\pi \mathrm{i})^2(1 + 6t^2 + 90t^4 + 1860t^6 + \cdots)$, and by Theorem 10.1

$$F_{\theta^*\Xi}(q_0) = -6\frac{q_0}{H(q_0)}H'(q_0)\frac{A(H(q_0))}{(2\pi \mathrm{i})^2} = -6 + 48q_0^2 + 240q_0^4 + 1776q_0^6 + \cdots .$$

An educated guess for $\widehat{\varphi_{\mathbf{f}}}(m,n)$ is $\frac{6^5}{5}$ times figure 13

$$\begin{aligned} &= \left(\widehat{\varphi_{\{6,1\}}} - \widehat{\varphi_{\{6,2\}}} - \widehat{\varphi_{\{6,3\}}} + \widehat{\varphi_{\{6,6\}}}\right) - \tfrac{1}{36}\left(\widehat{\varphi_{\{1,1\}}} - \widehat{\varphi_{\{2,1\}}} - \widehat{\varphi_{\{3,1\}}} + \widehat{\varphi_{\{6,1\}}}\right) \\ &\quad + \tfrac{1}{9}\left(\widehat{\varphi_{\{2,1\}}} - \widehat{\varphi_{\{2,2\}}} - \widehat{\varphi_{\{6,1\}}} + \widehat{\varphi_{\{6,2\}}}\right) - \tfrac{1}{4}\left(\widehat{\varphi_{\{3,1\}}} - \widehat{\varphi_{\{3,3\}}}\right. \\ &\quad \left. - \widehat{\varphi_{\{6,1\}}} + \widehat{\varphi_{\{6,3\}}}\right), \end{aligned}$$

which yields

$$\begin{aligned} E^{[2]}_{\varphi_{\mathbf{f}}}(q) &= \tfrac{1}{5}E_4(q) - \tfrac{4}{5}E_4(q^2) + \tfrac{9}{5}E_4(q^3) - \tfrac{36}{5}E_4(q^6) \\ &= -6 + 48q + 240q^2 + 1776q^3 + \cdots \end{aligned}$$

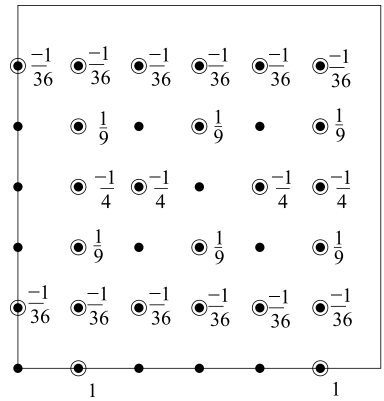

Figure 13: Eisenstein coefficients for toric symbol on Fermi pencil.

in agreement with the above.

Example 10.7. Verrill's pencil has order 2 monodromy at $t = \frac{1}{16}, \frac{1}{4}$ and maximal unipotent monodromy at 0, ∞; it is modular with $\Gamma = \Gamma_1(6)^{+3}$, and presumably a construction analogous to that in Remark 8.3 (with ι_3 replacing ι_6) yields the total space (up to birational equivalence). This implies

$$m_0 = -6,\ m_1 = 1,\ N_\Gamma = 1,\ M_\theta = 12 \implies r_{[\mathrm{i}\infty]} = -6,\ \frac{Y(0)}{(2\pi \mathrm{i})^2} = -6.$$

Verrill's $\Lambda = -\frac{\eta(\tau)^6\eta(3\tau)^6}{\eta(2\tau)^6\eta(6\tau)^6} - 4$ which implies that our $t =$

$$H(q) = \frac{1}{\Lambda+4} = -\frac{\eta(2\tau)^6\eta(6\tau)^6}{\eta(\tau)^6\eta(3\tau)^6} = -9(1+6q+21q^2+68q^3+198q^4+\cdots);$$

together with $\frac{A(t)}{(2\pi\mathrm{i})^2} = 1+4t+28t^2+256t^3 = \cdots$, this gives

$$F_{\theta^*\Xi} = -6\frac{q}{H(q)}H'(q)\frac{A(H(q))}{(2\pi\mathrm{i})^2} = -6-12q+84q^2-228q^3+\cdots.$$

Put $\widehat{\varphi}_{\mathbf{f}} := \frac{6^5}{5}$ times figure 14

$$= \left(\widehat{\varphi_{\{6,1\}}} - \widehat{\varphi_{\{6,2\}}} - \widehat{\varphi_{\{6,3\}}} + \widehat{\varphi_{\{6,6\}}}\right) - \frac{1}{9}\left(\widehat{\varphi_{\{2,1\}}} - \widehat{\varphi_{\{2,2\}}} - \widehat{\varphi_{\{6,1\}}} + \widehat{\varphi_{\{6,2\}}}\right);$$

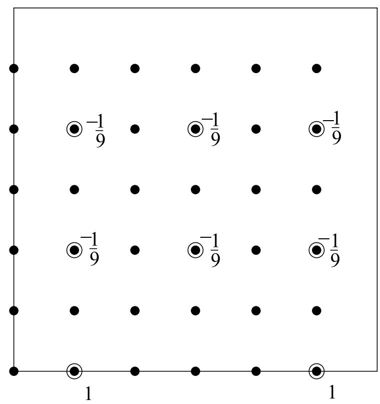

Figure 14: Eisenstein coefficients for toric symbol on Verrill pencil.

then indeed

$$\begin{aligned} E^{[2]}_{\varphi_{\mathfrak{f}}}(q) &= -\tfrac{1}{20}E_4(q) + \tfrac{4}{5}E_4(q^2) + \tfrac{9}{20}E_4(q^3) - \tfrac{36}{5}E_4(q^6) \\ &= -6 - 12q + 84q^2 - 228q^3 + \cdots . \end{aligned}$$

10.6. Remarks on the CY three-fold case

In this subsection we present no further examples of Theorem 10.1, because there are not any (Proposition 10.3). To illustrate what the problem is, we begin by describing a *local* modularity criterion for $\pi : \mathcal{X} \to \mathbb{P}^1$ in terms of the associated limit mixed Hodge structure at $t = 0$. This is a necessary condition for applying that result, and it fails dramatically for the celebrated quintic mirror family (as we shall see).

Let $(H_{\mathbb{Z}}, \mathcal{H}, \mathcal{F}^{\bullet})$ be a weight 3 rank 4 polarized $\mathbb{Z}$-VHS over a punctured disk $U = D^*_\epsilon(0)$ with maximal unipotent monodromy $T \in \mathrm{Aut}(H_{\mathbb{Z}})$ about $t = 0$. The weight monodromy filtration $W_{\bullet}$ can be defined on $H_{\mathbb{Z}}$, with adapted symplectic $\mathbb{Z}$-basis $\{\varphi_i\}_{i=0}^3$:

$$\mathrm{Gr}^W \varphi_i \in \Gamma\left(U, \frac{W_{2\mathrm{i}}}{W_{2\mathrm{i}-2}} H_{\mathbb{Z}}\right), \quad [\langle \varphi_i, \varphi_j \rangle] = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}.$$

Moreover, there is a unique ($\mathcal{O}_U$-) basis $\{\omega_i\}_{i=0}^3$ for $\mathcal{H}$ adapted to the Hodge filtration ($\omega_i \in \Gamma(U, \mathcal{F}^i)$) and satisfying

$$\mathrm{Gr}^W \omega_i = \mathrm{Gr}^W \varphi_i \in \Gamma\left(U, \frac{W_{2\mathrm{i}}}{W_{2\mathrm{i}-2}} \mathcal{H}\right).$$

Replacing t by $q := \exp\left(2\pi\mathrm{i}\frac{\langle\varphi_1,\omega_3\rangle}{\langle\varphi_0,\omega_3\rangle}\right)$, an "integral" basis for the LMHS $(H_{\mathbb{Z}}^{\lim}, W_\bullet, \mathcal{H}_{\lim}, \mathcal{F}_{\lim})$ is $\{e_i := \tilde{\varphi}_i(0)\}$, where $\tilde{\varphi}_i(q) := \exp\left(-\frac{\log q}{2\pi\mathrm{i}}\log T\right)\varphi_i(q)$.

The *period matrix* Ω of $\mathcal{H}_{\lim}$ is given by writing the $\omega_i(0) \in \mathcal{F}^i_{\lim}$ as vectors w.r.t. the basis $\{e_i\}$. If $\mathcal{H} = \mathrm{Sym}^3\mathcal{H}^{[1]}$ as in the beginning of Section 9.1, then since $\tilde{\beta} = \beta - \frac{\log q}{2\pi\mathrm{i}}\alpha$ and $[\tilde{\beta}(0)] = \lim_{q\to 0}[dz] \in \mathcal{H}^{[1]}_{\lim}$, $\Omega = \mathrm{Sym}^3\Omega^{[1]} =$ identity (up to unimodular transformations preserving $W_\bullet$). This leads to (ii) in the following

Proposition 10.2. *(i) [41] In the above situation,*

$$\Omega = \begin{pmatrix} 1 & 0 & \frac{f}{2a} & \xi \\ & 1 & \frac{e}{a} & \frac{f}{2a} \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \quad \textit{with } a, e, f \in \mathbb{Z} \textit{ (but } \xi \in \mathbb{C}\textit{)}.$$

(ii) If $\mathcal{H} = R^3\pi_*\mathbb{C} \otimes \mathcal{O}_U$ comes from a modular family $\pi : \mathcal{X} \to \mathbb{P}^1$ of CY three-folds (in the sense of Section 10.3*), then $\xi \in \mathbb{Q}$.*

In the language of [32], $\xi \in \mathbb{C}/\mathbb{Q}$ detects the presence of *instanton corrections*: in fact ξ is nothing but $-\frac{1}{2}F(0)$ where F is the prepotential. This is considered in [20] for the quintic mirror, which in our setup is

$$\phi = x + y + z + w + \frac{1}{xyzw}.$$

(Obviously this satisfies Corollary 3.1 for $n = 4$.) Indeed, for this most fundamental example (by Green *et al.* [41])

$$\Omega = \begin{pmatrix} 1 & 0 & \frac{25}{12} & -\frac{200\zeta(3)}{(2\pi\mathrm{i})^3} \\ & 1 & \frac{-11}{2} & \frac{25}{12} \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}$$

tells us that $\mathcal{X}$ is *not* modular.

Now consider the five Laurent polynomials

$\phi(\underline{x})$	Corresponding CY family $\{\tilde{X}_t\}$
$x_1 + x_2 + x_3 + x_4 + \frac{1}{x_1 x_2 x_3 x_4}$	Quintic mirror
$x_1 + x_2 + x_3 + x_4 + \frac{1}{x_1^2 x_2 x_3 x_4}$	Sextic mirror
$x_1 + x_2 + x_3 + x_4 + \frac{1}{x_1^4 x_2 x_3 x_4}$	Octic mirror
$x_1 + x_2 + x_3 + x_4 + \frac{1}{x_1^5 x_2^2 x_3 x_4}$	Dectic mirror
$x_1 + x_2 + x_3 + x_1 x_2^2 x_3^3 x_4^5 + \frac{1}{x_1^2 x_2^3 x_3^4 x_4^5}$	Quintic twin mirror

all of which fall under the aegis of Corollary 3.1 ($n = 4$). These are the *only* families of smooth $h^{2,1} = 1$ Calabi–Yau anticanonical hypersurfaces in Gorenstein toric Fano fourfolds, and their Picard–Fuchs equations are all classical generalized hypergeometric equations [34]. In particular, the corresponding polytopes Δ have only six integral points, so the anticanonical hypersurfaces in $\mathbb{P}_{\tilde{\Delta}}$ have one modulus and modifying the monomial coefficients yields isomorphic families. Moreover, none of these is a symmetric cube of a second-order ODE whose projective normal form is the uniformizing differential equation for a modular curve [32]. We conclude:

Proposition 10.3. *There are no anticanonical toric modular families of CY three-folds in the precise sense of* (5) *from Section* 10.3.

There are a couple of ways to relax the toric hypotheses that would likely lead to modular examples. What does *not* work is relaxing the rank 4 ($h^{2,1} = 1$) hypothesis on $H^3(X_t)$ (e.g., to H^3 having a rank 4 level 3 *sub*-Hodge-structure), since the geometric information of $\theta : \overline{\mathcal{E}}^{[\ell]}(N) \dashrightarrow \mathcal{X}$ is crucial and birational (smooth) CY's have equal Hodge numbers [4].

One possibility is to consider a toric four-fold $\mathbb{P}_{\tilde{\Delta}}$ whose anticanonical hypersurfaces have multiple moduli, and choose our one-parameter family $(1 - t\phi = 0)$ to have (fiberwise) crepant singularities on its generic member. Resolving the singularities would then yield a family of CY's with $h^{p,q}$'s distinct from those of the generic (smooth) anticanonical hypersurface. This approach will require a generalization of Theorem 3.1 to treat such singularities. Alternately, one could try to extend the construction of motivic cohomology classes from Section 3 to families of complete intersections in toric $\geq$5-folds. The generation of such families by way of nef-partitions of polytopes [6] yields an as-yet unknown number of $h^{2,1} = 1$ examples.

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