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# Archimedean $L$-factors and topological field theories II

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In the first part of this series of papers, we propose a functional integral representation for local Archimedean $L$-factors given by products of the $\Gamma$-functions. In particular, we derive a representation of the $\Gamma$-function as a properly regularized equivariant symplectic volume of an infinite-dimensional space. The corresponding functional integral arises in the description of a type $A$ equivariant topological linear sigma model on a disk. In this paper, we provide a functional integral representation of the Archimedean $L$-factors in terms of a type $B$ topological sigma model on a disk. This representation leads naturally to the classical Euler integral representation of the $\Gamma$-functions. These two integral representations of $L$-factors in terms of $A$ and $B$ topological sigma models are related by a mirror map. The mirror symmetry in our setting should be considered as a local Archimedean Langlands correspondence between two constructions of local Archimedean $L$-factors.

## 0. Introduction

In [8] we propose a framework of topological quantum field theory as a proper way to describe arithmetic geometry of Archimedean places of the compactified spectrum of global number fields. In particular, we provide a functional integral representation of local Archimedean $L$-factors as correlation functions in two-dimensional type $A$ equivariant topological sigma models. This representation implies that local Archimedean $L$-factors are equal to properly defined equivariant symplectic volumes of spaces of holomorphic maps of a disk into complex vector spaces. Thus, the equivariant infinite-dimensional symplectic geometry (in the framework of a topological quantum field theory) appears as the Archimedean counterpart of the geometry over non-Archimedean local fields [1, 5, 21].

The construction of local Archimedean $L$-factors in terms of type $A$ equivariant topological sigma models should be considered as an analog of "arithmetic" construction of local non-Archimedean $L$-factors in terms of representations of local non-Archimedean Galois group. There is another, "automorphic" construction of the non-Archimedean $L$-factors, which uses a theory of infinite-dimensional representations of reductive groups. For Archimedean places, this provides a representation of the corresponding $L$-factors as products of classical Euler's integral representations of the $\Gamma$-functions. In [8], we conjecture that this finite-dimensional integral representation of $L$-factors naturally arises in a type $B$ topological sigma model which is mirror dual to the type $A$ topological sigma model considered in [8]. This would lead to an identification of local Archimedean Langlands correspondence between "arithmetic" and "automorphic" constructions of $L$-functions with a mirror symmetry between corresponding type $A$ and type $B$ equivariant topological sigma models. In this note, we propose the type $B$ topological sigma model dual to the one considered in [8] and identify a particular set of correlation functions on a disk with Archimedean $L$-functions. As expected the resulting functional integral representation of the $L$-factors is reduced to a product of the Euler integral representations of $\Gamma$-functions.

The type $B$ equivariant topological sigma model considered below is an $S^1$-equivariant sigma model on a disk $D$ with the target space $X = \mathbb{C}^{\ell+1}$ and a non-trivial superpotential $W$. We imply that $S^1$ acts by rotations of the disk $D$. A particular superpotential $W$ corresponding to the mirror dual to the type $A$ equivariant topological sigma model with target space $\mathbb{C}^{\ell+1}$ is well-known [17]. However our considerations have some new interesting features. At first, the $S^1$-equivariance provides a new solution of the so-called Warner problem in topological theories on non-compact manifolds. The standard way to render the theory consistent is to introduce a non-trivial boundary interaction corresponding to a collection of $D$-branes in the target space [18, 20, 23]. We show that in the case of $S^1$-equivariant sigma model on the disk $D$ there is a universal boundary term leading to a consistent topological theory. Another not quite standard feature of our approach is a choice of a real structure on the space of fields of the topological theory. One can construct a topological quantum field theory starting with an $\mathcal{N} = 2$ SUSY quantum field theory [24] and using a twisting procedure (see e.g. [25, 26]). This provides a particular real structure on the space of fields. Another approach is to construct directly topological theory combining (equivariant) topological multiplets of quantum fields. Although this approach produces topological field theories closely related with those obtained by the twisting procedure the resulting real structure may be different (for a discussion of an example see e.g. [25]). In our considerations we use a real structure which is different from the one appeared in twisted $\mathcal{N} = 2$ SUSY two-dimensional sigma models.

We also comment on an explicit mirror map of type $A$ and type $B$ topological sigma models. We provide a heuristic derivation of the $B$-model superpotential $W$ by applying Duistermaat–Heckman localization formula to an infinite-dimensional projective space. The sum over fixed points can be related to the sum over instantons used in the previous derivations of the superpotential [17]. We also consider an explicit change of variables in the functional integral transforming $A$-model into $B$-model. Although these considerations are heuristic they reveal interesting features of the topological theories discussed in this note and in [8].

Finally note that pairs of Langlands dual Lie groups already appear in various instances of mirror symmetry (see e.g. [15]). The most relevant to our discussion is the appearance of the Langlands dual groups in the construction of a mirror dual description of type $A$ topological sigma models associated with flag spaces $G/B$ in terms of eigenfunctions of the quantum Toda chains associated with the dual Lie groups $G^\vee$ [12, 13]. One shall also note that the global geometric Langlands correspondence due to [4] allows an interpretation in terms of $S$-duality in four-dimensional topological Yang–Mills theories [19] (and in turn can be also considered as a mirror symmetry of associated moduli spaces following an old idea of [14]). We shall stress however that in this paper we are dealing with the local arithmetic (Archimedean) Langlands correspondence and the proposed quantum field theory approach to this problem seems new. A quest of possible relations with a (generalization of) geometric Langlands correspondence [4, 19] we leave for a future.

The plan of the paper is as follows. In Section 1, we provide a construction of a $S^1$-equivariant type $B$ topological sigma model on a disk $D$. In Section 2, we identify a particular correlation function of the topological sigma model with a product of $\Gamma$-functions thus providing a new functional integral representation of local Archimedean $L$-factors. In Section 3 we give heuristic constructions of a mirror map of type $A$ topological sigma model considered in [8] to a type $B$ topological sigma model considered in Section 2. In Section 4, we conclude with some general remarks and discuss further directions of research.

## 1. Type $B$ Topological sigma-models

We start by recalling the standard construction of a topological sigma model associated with a Kähler manifold with trivial canonical class supplied with holomorphic superpotential. For general discussion of the two-dimensional topological sigma models see e.g., [6] and reference therein.

Let $X$ be a Kähler manifold of complex dimension $(\ell+1)$ with trivial canonical class and let $W \in H^0(X,\mathcal{O})$. Let $\mathcal{M}(\Sigma,X)=\mathrm{Map}(\Sigma,X)$ be the space of maps $\Phi:\,\Sigma\to X$ of a compact Riemann surface $\Sigma$ into $X$. Let $(z,\bar{z})$ be local complex coordinates on $\Sigma$. We pick a hermitian metric $h$ on $\Sigma$ and denote $\sqrt{h}\,d^2z$ the corresponding measure on $\Sigma$. The complex structure on $\Sigma$ defines a decomposition $d=\partial+\overline{\partial}$, $\partial=dz\,\partial_z$, $\overline{\partial}=d\bar{z}\,\partial_{\bar{z}}$ of the differential $d$ acting on the differential forms on $\Sigma$. Let $K$ and $\bar{K}$ be canonical and anti-canonical bundles over $\Sigma$. Let $\omega$ and $g$ be the Kähler form and the Kähler metric on $X$ and $T_{\mathbb{C}}X=T^{1,0}X\oplus T^{0,1}X$ be a decomposition of the complexified tangent bundle of $X$. We choose local complex coordinates $(\phi^j,\bar{\phi}^{j})$ on $X$. Locally Levi–Civita connection $\Gamma$ and the corresponding Riemann curvature tensor $R$ are given by

$$\Gamma^i_{jk}=g^{i\bar{n}}\partial_j g_{k\bar{n}},\qquad R_{i\bar{j}k\bar{l}}=g_{m\bar{j}}\partial_{\bar{l}}\Gamma^m_{ik}. \tag{1.1}$$

Now let us specify the standard field content of the type $B$ topological sigma model associated with a pair $(X,W)$. Denote $\Pi$ the parity change functor. Thus, $\Pi\mathcal{E}$ is a bundle $\mathcal{E}$ with the opposite parity of the fibers. Let $\eta$, $\theta$ be sections of $\Phi^*(\Pi T^{0,1}X)$, $\rho$ be a section of $(K\oplus\bar{K})\otimes\Phi^*(\Pi T^{1,0}X)$. We also introduce the fields $\bar{G}$ and $G$ given by sections of $\Phi^*(T^{0,1}X)$ and $K\otimes\bar{K}\otimes\Phi^*(T^{1,0}X)$ respectively. The BRST [24] transformation $\delta$ is defined as follows:

(1.2)
$$\delta\bar{\phi}^{\bar{i}}=\bar{\eta}^{\bar{i}},\qquad \delta\bar{\eta}^{\bar{i}}=0,\qquad \delta\theta^{\bar{i}}=\bar{G}^{\bar{i}}-\Gamma^{\bar{i}}_{\bar{j}\bar{k}}\bar{\eta}^{\bar{j}}\theta^{\bar{k}},\qquad \delta\bar{G}^{\bar{i}}=-\Gamma^{\bar{i}}_{\bar{j}\bar{k}}\bar{G}^{\bar{j}}\bar{\eta}^{\bar{k}},$$
$$\delta\rho^i=-d\phi^i,\qquad \delta\phi^i=0,\qquad \delta G^i=d\rho^i+\Gamma^i_{jk}d\phi^j\wedge\rho^k+\tfrac{1}{2}R^i_{jk\bar{l}}\bar{\eta}^{\bar{l}}\rho^j\wedge\rho^k.$$

Straightforward calculations show that $\delta^2=0$. One can define new variables

$$\bar{\mathcal{G}}^{\bar{i}}=\bar{G}^{\bar{i}}-\Gamma^{\bar{i}}_{\bar{j}\bar{k}}\bar{\eta}^{\bar{j}}\theta^{\bar{k}},\qquad \mathcal{G}^i=G^i+\frac{1}{2}\Gamma^i_{jk}\rho^j\wedge\rho^k, \tag{1.3}$$

such that the action of $\delta$ has the following canonical form:

$$\begin{gathered}\delta\bar{\phi}^{\bar{i}} = \bar{\eta}^{\bar{i}}, \qquad \delta\bar{\eta}^{\bar{i}} = 0, \qquad \delta\theta^{\bar{i}} = \bar{\mathcal{G}}^{\bar{i}}, \qquad \delta\bar{\mathcal{G}}^{\bar{i}} = 0, \\ \delta\rho^{i} = -d\phi^{i}, \qquad \delta\phi^{i} = 0, \qquad \delta\mathcal{G}^{i} = d\rho^{i}.\end{gathered} \tag{1.4}$$

Here the property $\delta^2 = 0$ is obvious. The advantage of (1.2) is that the fields $G^i$ and $\bar{G}^{\bar{j}}$ are covariant with respect to diffeomorphisms of the target space $X$.

Consider a topological sigma model with the action given by

$$S = S_0 + S_{\bar{W}} + S_W, \tag{1.5}$$

where

$$\begin{aligned} S_0 = \int_\Sigma (g_{i\bar{j}} d\phi^i \wedge *d\bar{\phi}^{\bar{j}} + g_{i\bar{j}}\rho^i \wedge *D\bar{\eta}^{\bar{j}} - \, g_{i\bar{j}}\theta^{\bar{j}} D\rho^i + g_{i\bar{j}} G^i \bar{G}^{\bar{j}} \\ - \frac{1}{2} R_{i\bar{l}k\bar{j}} \bar{\eta}^{\bar{l}}\, \theta^{\bar{j}}\, \rho^i \wedge \rho^k), \end{aligned} \tag{1.6}$$

$$S_{\bar{W}} = \int_\Sigma d^2z\sqrt{h}\left(D_{\bar{i}}\partial_{\bar{j}}\bar{W}(\bar{\phi})\ \bar{\eta}^{\bar{i}}\theta^{\bar{j}} + \bar{G}^{\bar{i}}\,\partial_{\bar{i}}\bar{W}(\bar{\phi})\right), \tag{1.7}$$

$$S_W = \int_\Sigma \left(-\frac{1}{2} D_i\partial_j W(\phi)\,\rho^i \wedge \rho^j + G^i\,\partial_i W(\phi)\right), \tag{1.8}$$

and

$$D_i\partial_j W(\phi) = \partial_i\partial_j W - \Gamma^k_{ij}\partial_k W, \qquad D\bar{\eta}^{\bar{j}} = d\bar{\eta}^{\bar{j}} + \Gamma^{\bar{j}}_{\bar{k}\bar{\ell}} d\bar{\phi}^{\bar{k}} \bar{\eta}^{\bar{\ell}}.$$

The Hodge $*$-operator acts on one forms as follows $*dz = \imath dz$, $*d\bar{z} = -\imath d\bar{z}$.

The parts $S_0$ and $S_{\bar{W}}$ are $\delta$-exact as it follows from $\delta^2 = 0$ and the following representation:

$$S_0 = \int_\Sigma \delta\mathcal{V}_0, \qquad S_{\bar{W}} = \int_\Sigma d^2z\,\sqrt{h}\,\delta\mathcal{V}_{\bar{W}},$$

where

$$\mathcal{V}_0 = -g_{i\bar{j}}\rho^i \wedge *d\bar{\phi}^{\bar{j}} + G^i\theta_i, \qquad \mathcal{V}_{\bar{W}} = \theta^{\bar{j}}\partial_{\bar{j}}\bar{W}(\bar{\phi}), \tag{1.9}$$

and $\theta_i = g_{i\bar{j}}\theta^{\bar{j}}$. The variation of $S_W$ is given by

$$\delta S_W = \int_\Sigma d(\rho^i\partial_i W(\phi)), \tag{1.10}$$

and thus is trivial on a compact surface $\Sigma$. Note that the action $S_W$ is $\delta$-closed but does not $\delta$-exact.

In this paper, we consider a particular case of an equivariant type $B$ topological sigma model on a non-compact two-dimensional manifold $\Sigma$. Let $\Sigma$ be a disk $D=\{z\in\mathbb{C}|\ |z|\leq 1\}$. We fix a flat metric $h$ on $D$

$$h=\tfrac{1}{2}(dzd\bar{z}+d\bar{z}\,dz)=(dr)^2+r^2(d\sigma)^2,\qquad r\in[0,1],\quad \sigma\in[0,2\pi], \tag{1.11}$$

where $z=re^{\imath\sigma}$. This metric is obviously invariant with respect to the rotation group $S^1$ acting by $\sigma\to\sigma+\alpha$.

We would like to consider an $S^1$-equivariant version of the type $B$ topological linear sigma model on a disk $D$ with a superpotential $W$. To construct an $S^1$-equivariant extension of the topological field theory we modify the $\delta$-transformations taking into account an interpretation of $\delta$ as the de Rham differential in the infinite-dimensional setting. Let us first recall a construction of an algebraic model of $S^1$-equivariant cohomology. Let $M$ be a $2(\ell+1)$-dimensional manifold supplied with an action of $S^1$. The Cartan algebraic model of $S^1$-equivariant de Rham cohomology $H^*_{S^1}(M)$ is the following equivariant extension $(\Omega^*_{S^1}(M),d_{S^1})$ of the standard de Rham complex $(\Omega^*(M),d)$:

$$\Omega^*_{S^1}(M)=(\Omega^*(M))^{S^1}\otimes\mathbb{C}[\hbar],\qquad d_{S^1}=d+\hbar\iota_{v_0}, \tag{1.12}$$

where $(\Omega^*(M))^{S^1}$ is an $S^1$-invariant part of $\Omega^*(M)$, $\hbar$ is a generator of the ring $H^*(BS^1)$ and $v_0$ is a vector field on $M$ corresponding to a generator of $\mathrm{Lie}(S^1)$. We have

$$d^2_{S^1}=\hbar\mathcal{L}_{v_0},\qquad \mathcal{L}_{v_0}=d\,\iota_{v_0}+\iota_{v_0}\,d, \tag{1.13}$$

where $\mathcal{L}_{v_0}$ is the Lie derivative along the vector field $v_0$. The equivariant differential $d_{S^1}$ satisfies $d^2_{S^1}=0$ when acting on $\Omega^*_{S^1}(M)$. The cohomology groups $H^*_{S^1}(M)$ of the complex (1.12) have a natural structure of modules over $H^*_{S^1}(\mathrm{pt})=\mathbb{C}[\hbar]$.

The $S^1$-equivariant version of the BRST transformations (1.4) is a direct generalization of the expression (1.12) for the equivariant differential to the infinite-dimensional setting. Taking into account an induced action of $S^1$ on the space of fields we have

$$\delta_{S^1}\bar{\phi}^{\bar{i}}=\bar{\eta}^{\bar{i}},\qquad \delta_{S^1}\bar{\eta}^{\bar{i}}=\hbar\iota_{v_0}d\bar{\phi}^{\bar{i}},\qquad \delta_{S^1}\theta^{\bar{i}}=\bar{\mathcal{G}}^{\bar{i}},\qquad \delta_{S^1}\bar{\mathcal{G}}^{\bar{i}}=\hbar\iota_{v_0}d\theta^{\bar{i}},$$
$$\delta_{S^1}\mathcal{G}^i=d\rho^i,\qquad \delta_{S^1}\rho^i=-d\phi^i-\hbar\iota_{v_0}\mathcal{G}^i,\qquad \delta_{S^1}\phi^i=\hbar\iota_{v_0}\rho^i.$$

Obviously, we have $\delta^2_{S^1}=\hbar\mathcal{L}_{v_0}$.

In terms of the variables $G^i$ and $\bar{G}^i$ we have the following transformations:

$$\delta_{S^1}\bar{\phi}^{\bar{i}} = \bar{\eta}^{\bar{i}}, \quad \delta_{S^1}\bar{\eta}^{\bar{i}} = \hbar\iota_{v_0}d\bar{\phi}^{\bar{i}}, \quad \delta_{S^1}\theta^{\bar{i}} = \bar{G}^{\bar{i}} - \Gamma^{\bar{i}}_{\bar{j}\bar{k}}\bar{\eta}^{\bar{j}}\theta^{\bar{k}},$$
$$\delta_{S^1}\bar{G}^{\bar{i}} = -\Gamma^{\bar{i}}_{\bar{j}\bar{k}}\bar{\eta}^{\bar{j}}\bar{G}^{\bar{k}} + \hbar\iota_{v_0}\big(D\theta^{\bar{i}}\big) + \hbar\partial_l\Gamma^{\bar{i}}_{\bar{j}\bar{k}}(\iota_{v_0}\rho^l)\bar{\eta}^{\bar{j}}\theta^{\bar{k}},$$
$$\delta_{S^1}G^i = d\rho^i + \Gamma^i_{jk}d\phi^j \wedge \rho^k + \frac{1}{2}R^i_{jk\bar{l}}\bar{\eta}^{\bar{l}}\rho^j \wedge \rho^k + \hbar\Gamma^i_{jk}(\iota_{v_0}G^j) \wedge \rho^k,$$
$$\delta_{S^1}\rho^i = -d\phi^i - \hbar\iota_{v_0}G^i - \hbar\Gamma^i_{jk}(\iota_{v_0}\rho^j)\rho^k, \quad \delta_{S^1}\phi^i = \hbar\iota_{v_0}\rho^i.$$

Now the $S^1$-equivariant version of (1.6) and (1.7) on a disk $\Sigma = D$ is obtained by applying modified $\delta_{S^1}$ to $\mathcal{V}_0$ and $\mathcal{V}_{\bar{W}}$ given by (1.9). The action $S_W$ given by (1.8) is not $\delta_{S^1}$ invariant on the disk and needs a correction boundary term.

**Proposition 1.1.** *The following modified action functional of a type B topological sigma model*

$$S = \int_D \Big(g_{i\bar{j}}\left(d\phi^j + \hbar\iota_{v_0}G^j\right) \wedge *d\bar{\phi}^{\bar{j}} + g_{i\bar{j}}\rho^i \wedge *D\bar{\eta}^{\bar{j}}$$
$$-g_{i\bar{j}}\theta^{\bar{j}}D\rho^i + g_{i\bar{j}}G^i\bar{G}^{\bar{j}} - \frac{1}{2}R_{i\bar{l}k\bar{j}}\bar{\eta}^{\bar{l}}\theta^{\bar{j}}\rho^i \wedge \rho^k\Big)$$
$$+ \int_D d^2z\sqrt{h}\left(D_{\bar{i}}\partial_{\bar{j}}\bar{W}(\bar{\phi})\bar{\eta}^{\bar{i}}\theta^{\bar{j}} + \partial_{\bar{i}}\bar{W}(\bar{\phi})\bar{G}^{\bar{i}}\right)$$
$$+ \int_D \left(-\frac{1}{2}D_i\partial_jW(\phi)\rho^i \wedge \rho^j + \partial_iW(\phi)G^i\right)$$
$$- \frac{1}{\hbar}\int_{S^1=\partial D} d\sigma W(\phi) \tag{1.14}$$

*is $\delta_{S^1}$-invariant.*

*Proof.* Direct calculation shows that $\delta_{S^1}$-variation of the sum of the integrals over $D$ in (1.14) is given by the boundary term

$$\delta_{S^1}S = \int_{\partial D} \rho^i\partial_iW(\phi).$$

The $\delta_{S^1}$-variation of the boundary term in (1.14) precisely cancels this contribution. □

**Remark 1.1.** The action (1.14) does not have a smooth limit $\hbar \to 0$. This is a so called "Warner problem" in the type $B$ topological sigma model with a non-trivial superpotential $W \in H^0(X, \mathcal{O})$ on non-compact surface $\Sigma$. In non-equivariant setting it is resolved by imposing special boundary conditions corresponding to a collection of $D$-branes on the target space $X$ [18, 20, 23]. Remarkably the $S^1$-equivariant setting discussed above allows a construction of a universal $\delta_{S^1}$-invariant boundary condition by adding boundary term in (1.14).

**Remark 1.2.** The relation between the boundary term in (1.14) and the variation (1.10) is a particular instance of a general descent relation between various observables in topological field theories.

## 2. Linear sigma model on a disk

In this section we calculate a particular correlation function of the $S^1$-equivariant type $B$ linear sigma model on the disk $D$ with the target space $\mathbb{C}^{\ell+1}$ and a generic superpotential $W$. The $\delta_{S^1}$-transformations in the case of $X = \mathbb{C}^{\ell+1}$ are given by

(2.1)
$$\delta_{S^1}\bar{\phi}^{\bar{i}} = \bar{\eta}^{\bar{i}}, \qquad \delta_{S^1}\bar{\eta}^{\bar{i}} = \hbar\iota_{v_0}d\bar{\phi}^{\bar{i}}, \qquad \delta_{S^1}\theta^{\bar{i}} = \bar{G}^{\bar{i}}, \qquad \delta_{S^1}\bar{G}^{\bar{i}} = \hbar\iota_{v_0}d\theta^{\bar{i}},$$
$$\delta_{S^1}\rho^i = -d\phi^i - \hbar\iota_{v_0}G^i, \qquad \delta_{S^1}\phi^i = \hbar\iota_{v_0}\rho^i, \qquad \delta_{S^1}G^i = d\,\rho^i.$$

The action (1.14) in this case is reduced to

$$S = \sum_{j=1}^{\ell+1}\int_D \big((d\phi^j + \hbar\iota_{v_0}G^j)\wedge *d\bar{\phi}^j + \rho^j\wedge *d\bar{\eta}^j - \theta_j d\rho^j + G^j\bar{G}^j\big)$$
$$+\sum_{i,j=1}^{\ell+1}\int_D d^2z\sqrt{h}\,\big(\bar{\partial}_i\bar{\partial}_j\bar{W}(\bar{\phi})\bar{\eta}^i\bar{\theta}^j + \bar{\partial}_i\bar{W}(\bar{\phi})\,\bar{G}^i\big)$$
$$+\int_D\left(-\frac{1}{2}\partial_i\partial_j W\,\rho^i\wedge\rho^j + \partial_i W\,G^i\right)$$
$$-\frac{1}{\hbar}\int_{S^1=\partial D} d\sigma W(\phi). \tag{2.2}$$

Topological linear sigma model (2.2) allows a non-standard real structure. This means the following. Let us consider the fields $\phi^i$, $\bar{\phi}^i$, $\theta_i$, $\bar{\theta}_i$, $\bar{\eta}^i$, $\eta^i$, $\rho^i$ $\bar{\rho}^i$, $G^i$ and $\bar{G}^i$ as independent complex fields. The subspace of the fields entering the description of the topological theory with the action (2.2) is defined as a subspace invariant with respect to an involution acting as follows:

(2.3)
$$(\phi^i)^\dagger = \bar{\phi}^i, \qquad (\theta_i)^\dagger = \bar{\theta}_i, \qquad (\bar{\eta}^i)^\dagger = \eta^i, \qquad (\rho^i)^\dagger = \bar{\rho}^i, \qquad (G^i)^\dagger = \bar{G}^i.$$

The involution defines a real structure on the space of fields. One can, however, consider another real structure defined by the reality conditions

(2.4)
$$\begin{aligned} &(\phi^i)^\dagger = \phi^i, \qquad (\bar{\phi}^i)^\dagger = -\bar{\phi}^i, \qquad (\theta_i)^\dagger = -\theta_i, \\ &(\bar{\eta}^i)^\dagger = -\bar{\eta}^i, \qquad (\rho^i)^\dagger = \rho^i, \qquad (G^i)^\dagger = G^i, \qquad (\bar{G}^i)^\dagger = -\bar{G}^i. \end{aligned}$$

Thus for example the fields $\phi^i$ and $\imath\bar{\phi}^i$ are real independent fields. To distinguish the real fields in the sense (2.4) let us introduce new notations $\phi^i_+$, $\phi^i_-$, $G^i_+$, $G^i_-$ for $\phi^i$, $\imath\bar{\phi}^i$, $G^i$, $\imath\bar{G}^i$. Similarly we redefine the fields $\bar{\eta}$ and $\theta$ by multiplying them on $\imath$ and considering the resulting fields as real ones. The $S^1$-equivariant BRST operator can be defined on the new set of real fields as follows:

(2.5)
$$\begin{aligned} &\delta_{S^1}\phi^i_- = \eta^i, \qquad \delta_{S^1}\eta^i = \hbar\iota_{v_0} d\phi^i_-, \qquad \delta_{S^1}\theta^i = G^i_-, \qquad \delta_{S^1}G^i_- = \hbar\iota_{v_0} d\theta^i, \\ &\qquad \delta_{S^1}\rho^i = -d\phi^i_+ - \hbar\iota_{v_0} G^i_+, \qquad \delta_{S^1}\phi^i_+ = \hbar\iota_{v_0}\rho^i, \qquad \delta_{S^1}G^i_+ = d\,\rho^i, \end{aligned}$$

where now the fields $\eta^i$ and $\theta^i$ are odd real zero-form valued fields, $\rho^i$ are odd real one-form valued fields, $G^i_-$ are even real zero-form valued fields and $G^i_+$ are even real two-form valued fields. The action of the sigma model for the new real structure is now given by

(2.6)
$$\begin{aligned} S = &-\imath\sum_{j=1}^{\ell+1}\int_D \Big((d\phi^j_+ + \hbar\iota_{v_0}G^j_+)\wedge *d\phi^j_- + \rho^j\wedge *d\eta^j - \theta_j d\rho^j + G^j_+G^j_-\Big) \\ &+\sum_{i,j=1}^{\ell+1}\int_D d^2z\sqrt{h}\left(-\frac{\partial^2 W_-(\phi_-)}{\partial\phi^i_-\partial\phi^j_-}\eta^i\theta^j - \imath\frac{\partial W_-(\phi_-)}{\partial\phi^i_-}\,G^i_-\right) \\ &+\sum_{i,j=1}^{\ell+1}\int_D\left(-\frac{1}{2}\frac{\partial^2 W_+(\phi_+)}{\partial\phi^i_+\partial\phi^j_+}\rho^i\wedge\rho^j + \frac{\partial W_+(\phi_+)}{\partial\phi^i_+}\,G^i_+\right) \\ &-\frac{1}{\hbar}\int_{S^1=\partial D} d\sigma W_+(\phi_+). \end{aligned}$$

Here $W_+$ and $W_-$ are arbitrary independent regular functions on $\mathbb{R}^{\ell+1}$. Thus defined action is $\delta_{S^1}$-closed.

**Remark 2.1.** Our choice of the real structure is such that the constructed type $B$ topological sigma model is a mirror dual to the type $A$ topological sigma model considered in [8]. In Section 3.3 we demonstrate that the mirror correspondence applied to the type $A$ topological sigma models from [8] leads to the real structure of type (2.4). Note also that the construction of the topological Yang–Mills theories using an equivariant setting [25] also leads to the non-standard real structure analogous to the one we use.

In the following, we consider the case of $W_-(\phi_-)=0$. Thus we have

(2.7)
$$\begin{aligned}S=&-\imath\sum_{j=1}^{\ell+1}\int_D\Big((d\phi_+^j+\hbar\iota_{v_0}G_+^j)\wedge *d\phi_-^j+\rho^j\wedge *d\eta^j-\theta_j d\rho^j+G_+^jG_-^j\Big)\\&+\sum_{i,j=1}^{\ell+1}\int_D\left(-\frac{1}{2}\frac{\partial^2W_+(\phi_+)}{\partial\phi_+^i\partial\phi_+^j}\rho^i\wedge\rho^j+\frac{\partial W_+(\phi_+)}{\partial\phi_+^i}\,G_+^i\right)\\&-\frac{1}{\hbar}\int_{S^1=\partial D}d\sigma W_+(\phi_+).\end{aligned}$$

Given an observable $\mathcal{O}(z,\bar z)$ on the disk $D$ we define its correlation function as a functional integral below:

$$\begin{aligned}\big\langle\mathcal{O}(z,\bar z)\big\rangle_{W_+}&:=\int D\mu\;\mathcal{O}(z,\bar z)\,e^{-S}\\ D\mu&=\prod_{i=1}^{\ell+1}[D\phi_+^i][D\phi_-^i][D\eta^i][D\theta^i][D\rho^i][DG_+^i][DG_-^i].\end{aligned}\tag{2.8}$$

**Lemma 2.1.** *The following observable inserted at the center $z=0$ of the disk $D$*

$$\mathcal{O}_*(0):=\mathcal{O}_*(z,\bar z)|_{z=0}=\prod_{i=1}^{\ell+1}\delta(\phi_-^i(z,\bar z))\,\eta^i(z,\bar z)|_{z=0}\tag{2.9}$$

*is $\delta_{S^1}$-invariant.*

*Proof.* We have

$$\delta_{S^1}\mathcal{O}_*(z,\bar z)=\sum_{m=1}^{\ell+1}\eta^m(z,\bar z)\prod_{j\neq m}\delta(\phi_-^j)\prod_{i=1}^{\ell+1}\eta^i(z,\bar z)$$
$$+\sum_{m=1}^{\ell+1}\prod_j\delta(\phi_-^j)(-1)^m\eta^1\ldots\eta^{m-1}(\hbar\iota_{v_0}\eta^m)\eta^{m+1}\ldots.\eta^{\ell+1}.$$

The first term is equal to zero since for odd variables $\eta^2=0$. The second term vanishes since the center of the disk $z=0$ is a fixed point of the $S^1$-action so that $\iota_{v_0}(\eta^m)\big|_{z=0}=0$. □

**Theorem 2.1.** *The correlation function of the observable* (2.9) *in the type B topological $S^1$-equivariant linear sigma model* (2.7) *is given by*

$$\langle\mathcal{O}_*(0)\rangle_{W_+}=\int_{\mathbb{R}^{\ell+1}}\prod_{j=1}^{\ell+1}dt^j\ e^{\frac{1}{\hbar}W_+(t)}. \tag{2.10}$$

*Proof.* Firstly, we make an integration over $G_-^i$:

$$\int[DG_-]\,\exp\left\{\imath\int_D\sum_{i=1}^{\ell+1}G_+^i(z)G_-^i(z)\right\}=\prod_{i=1}^{\ell+1}\delta(G_+^i).$$

The integration over $G_+^j$ is then equivalent to the substitution of $G_+^j=0$. Thus we should calculate the following functional integral:

$$Z=\int[D\phi_+]\,[D\phi_-]\,\mathcal{O}_1(0)\,\exp\left\{\imath\int_D\sum_{i=1}^{\ell+1}d\phi_+^i\wedge *d\phi_-^i\right.$$
$$\left.-\frac{1}{\hbar}\int_{S^1}d\sigma W_+(\phi_+)\right\}\,Z_f(\phi_+), \tag{2.11}$$

where

$$Z_f(\phi_+)=\int[D\rho]\,[D\theta][D\eta]\ \mathcal{O}_2(0)\,\exp\left\{\imath\int_D\sum_{i=1}^{\ell+1}(\rho^i\wedge *d\eta^i-\theta^i d\rho^i)\right.$$
$$\left.+\frac{1}{2}\int_D\sum_{i,j=1}^{\ell+1}\frac{\partial^2W_+}{\partial\phi_+^i\partial\phi_+^j}\,\rho^i\wedge\rho^j\right\},$$

and

$$\mathcal{O}_1(0)=\prod_{j=1}^{\ell+1}\delta(\phi_-^i(0)),\qquad \mathcal{O}_2(0)=\prod_{j=1}^{\ell+1}\eta^i(0).$$

Let us first integrate over $\theta$ in $Z_f$. We have

$$Z_f(\phi_+)=\int[D\rho]\,[D\eta]\;\mathcal{O}_2(0)\prod_{j=1}^{\ell+1}\delta(d\rho^j)\exp\left\{\imath\int_D\sum_{j=1}^{\ell+1}\rho^j\wedge *d\eta^j\right.$$
$$\left.+\frac{1}{2}\int_D\sum_{i,j=1}^{\ell+1}\frac{\partial^2W_+}{\partial\phi_+^i\partial\phi_+^j}\,\rho^i\wedge\rho^j\right\}.$$

One-forms allow the following decomposition:

$$\rho^i=df_1^i+*df_2^i=\partial_z\bar{F}^idz+\partial_{\bar{z}}F^id\bar{z},\qquad F^i=f_1^i-\imath f_2^i. \tag{2.12}$$

It is easy to check (using for example series expansions) that for given $\rho^i$ the solutions $f_1$, $f_2$ of (2.12) always exist and are unique up to addition to $F^i$ a holomorphic function. Therefore, we make the following change of variables $\rho^i\to(f_1^i,f_2^i)/\sim$ where the equivalence relation is generated by addition to $f_1^j$ and $f_2^j$ of real and imaginary parts of a holomorphic function $g(z)$

$$f_1^i\,\sim\,f_1^i+\mathrm{Re}(g^i(z)),\qquad f_2^i\,\sim\,f_2^i+\mathrm{Im}(g^i(z)). \tag{2.13}$$

Thus we have

$$[D\rho]=\frac{[Df_1]\,[Df_2]}{[Dg]}\,\mathrm{Jac}_1^{-1},$$

where Jacobian is given by the determinant of the operator

$$(d\oplus *d):\;(f_1^i,f_2^i)\to\rho^i=df_1^i+*df_2^i,$$

acting $\mathcal{A}^0_{\mathrm{orth}}\subset\mathcal{A}^0(D)$ orthogonal to its kernel. We define a determinant of an operator acting between different spaces as a square root of the determinant of the product of the operator and its conjugated

$$\begin{aligned}\mathrm{Jac}_1&=|\,{\det}'_{\mathcal{A}^0_{\mathrm{orth}}\oplus\mathcal{A}^0_{\mathrm{orth}}}(d+*d)|\\&:=\left({\det}'_{\mathcal{A}^0_{\mathrm{orth}}\oplus\mathcal{A}^0_{\mathrm{orth}}}(d+*d)^2\right)^{\frac{1}{2}}={\det}'_{\mathcal{A}^0_{\mathrm{orth}}}\Delta_0,\end{aligned}$$

where $\Delta_0=(d+d^*)^2$ acting in the space of functions $\mathcal{A}_0$. We have

$$\delta(d\rho^i)=\delta(d(df_1^i+*df_2^i))=\delta(d*df_2^i),$$

and thus

$$Z_f(\phi_+) = \int [D\eta] \frac{[Df_1]\,[Df_2]}{[Dg]} \, \frac{1}{\det'_{\mathcal{A}^0_{\mathrm{orth}}} \Delta_0} \, \mathcal{O}_2(0) \prod_{i=1}^{\ell+1} \delta(d * d \; f_2^i)$$

$$\times \, \exp \left\{ \imath \int_D \sum_{i=1}^{\ell+1} (df_1^i + *d \; f_2^i) \wedge *d\eta^i \right.$$

$$\left. + \frac{1}{2} \int_D \sum_{i,j=1}^{\ell+1} \frac{\partial^2 W_+}{\partial \phi_+^i \partial \phi_+^j} \, (df_1^i + *d \; f_2^i) \wedge (df_1^j + *d \; f_2^j) \right\}.$$

Let us fix a representative for the equivalence relation (2.13) by the condition that $f_2^i$ is in the subspace orthogonal to the space of harmonic functions on the disk. This leaves a freedom to add to $f_1^i$ a real constant (indeed $\mathrm{Im}(g^i(z)) = 0$ implies $g^i(z) = a^i \in \mathbb{R}$). We denote by $[Df_1]'$ the induced measure on this subspace. The integration over $f_2^i$ gives

$$Z_f(\phi_+) = \int [D\eta]\,[Df_1]' \;\; \mathcal{O}_2(0) \, \exp \left\{ \imath \int_D \sum_{i=1}^{\ell+1} df_1^i \wedge *d\eta^i \right.$$

$$\left. + \frac{1}{2} \int_D \sum_{i,j=1}^{\ell+1} \frac{\partial^2 W_+}{\partial \phi_+^i \partial \phi_+^j} \, df_1^i \wedge df_1^j \right\},$$

where the determinant in the denominator is canceled by the determinant appearing from the integration of the delta-function.

We split the space of functions $\mathcal{A}^0(D)$ on a disk on the space $\mathcal{A}^0_h$ of harmonic functions and the space $\mathcal{A}^0_N$ of functions that have zero normal derivative on the boundary:

$$f^i = f_h^i + f_N^i, \qquad f_h^i \in \mathcal{A}^0_h, \quad f_N^i \in \mathcal{A}^N$$
$$\Delta_0 f_h^i = 0, \quad \partial_n f_N^i|_{S^1} = 0.$$

The subspace $\mathcal{A}^0_h$ can be identified with the space $\mathrm{Fun}(S^1)$ of functions on the boundary $S^1 = \partial D$. This is not an orthogonal decomposition with respect to the natural scalar product on the space of functions on the disk. Thus we have a non-trivial Jacobian in the integration measure:

$$[Df] = [Df_h]\,[Df_N]\,\mathrm{Jac}_2^{-1},$$

which is a some constant. Note that the following relation holds:

$$\int_D \sum_{i=1}^{\ell+1} df_1^i \wedge *d\eta^i = \int_D \sum_{i=1}^{\ell+1} \eta_N^i * \Delta f_{1,N}^i - \int_{S^1} \sum_{i=1}^{\ell+1} \eta_h^i * df_{1,h}^i.$$

Taking integral over $\eta_{1,N}^i$ and $\eta_{1,h}^i$, we obtain

$$\begin{aligned} Z_f(\phi_+) &= \frac{1}{\mathrm{Jac}_2} \int [Df_1]' \prod_{i=1}^{\ell+1} \delta(\Delta_0 f_{1,N}^i)\, \delta(*df_{1,h}^i) \\ &\quad \times \exp\left\{ \frac{1}{2} \int_D \sum_{i,j=1}^{\ell+1} \frac{\partial^2 W_+}{\partial \phi_+^i \partial \phi_+^j}\, d(f_{1,N}^i + f_{1,h}^i) \wedge d(f_{1,N}^j + f_{1,h}^j) \right\} \\ &= \frac{1}{\mathrm{Jac}_2^2} \det{}'_{\mathcal{A}_N^0} \Delta_0 \ \det{}'_{\mathrm{Fun}(S^1)}(*d). \end{aligned}$$

Now let us calculate the functional integral (2.11). The calculation is basically the same as in the case of $Z_f$. The only difference (apart of the fact that Jacobins and determinants appear inverse) is that the integral over constant mode of $\phi_-^j$ is present and is eaten up by the delta-function insertion. On the other hand the integral over constant mode of $\phi_+^j$ remains. Taking into account the cancelation of the Jacobians and determinants for fermions and bosons the total integral is equal to

$$Z = \int_{\mathbb{R}^{\ell+1}} \prod_{j=1}^{\ell+1} dt^j \ \mathrm{e}^{\frac{1}{\hbar} W_+(t)},$$

where $t^j$ are constant modes of the fields $\phi_+^j$. □

**Corollary 2.1.** *The correlation function of the observable* (2.9) *in the type B topological* $S^1$*-equivariant linear sigma model* (2.7) *with the superpotential*

$$W_+^{(0)}(\phi_+) = \sum_{j=1}^{\ell+1} (\lambda_j \phi_+^j - e^{\phi_+^j}), \ \ \lambda_j \in \mathbb{R}_+ \tag{2.14}$$

*is given by the following product of the* $\Gamma$*-functions*

$$\langle \mathcal{O}_*(0) \rangle_{W_+^{(0)}} = \prod_{j=1}^{\ell+1} \hbar^{\frac{\lambda_j}{\hbar}}\, \Gamma\left(\frac{\lambda_j}{\hbar}\right). \tag{2.15}$$

*Proof.* Using the result of the previous Theorem for the superpotential (2.14) we straightforwardly have

$$\langle \mathcal{O}_*(0)\rangle_{W_+^{(0)}} = \int_{\mathbb{R}^{\ell+1}} \prod_{j=1}^{\ell+1} dt^j \quad e^{\frac{1}{\hbar}\sum_{j=1}^{\ell+1}(\lambda_j t^j - e^{t^j})} = \prod_{j=1}^{\ell+1} \hbar^{\frac{\lambda_j}{\hbar}}\, \Gamma\left(\frac{\lambda_j}{\hbar}\right).$$

□

The expression (2.15) is equivalent to the one obtained in type $A$ topological sigma model considered in [8]. The coincidence of a particular correlation functions in type $A$ model considered in [8] and the correlation function from Corollary 2.1 is a manifestation of the mirror symmetry between two underlying sigma models. Without taking into account the involved $S^1$-equivariance, the mirror correspondence between the two models follows from the results of [17]. In particular the exponential terms in the superpotential (2.14) are attributed to the summation over instantons in type $A$ sigma model. In the following section we provide heuristic arguments for the mirror symmetry between the topological theory considered in this note and the one considered in [8].

## 3. On equivalence of *A* and *B* topological sigma models

As it was demonstrated in the previous section the Euler integral representation of the Γ-function

$$\Gamma(s) = \int_{-\infty}^{+\infty} dx\, e^{xs}\, e^{-e^x}, \qquad \mathrm{Re}(s) > 0, \tag{3.1}$$

naturally arises as a particular correlation function in a certain $S^1$-equivariant type $B$ topological sigma model on the disk $D$. In [8] it was argued that this integral representation is dual to the representation of the Γ-function as an equivariant symplectic volume of an infinite-dimensional space. The natural framework for this duality is a mirror symmetry. Below we establish a direct relation of the Euler integral representation (3.1) of the Γ-function with the representation of the Γ-function as an equivariant symplectic volume of an infinite-dimensional space proposed in [8]. We also discuss an explicit mirror map between the type $A$ equivariant topological linear sigma model considered in [8] and the type $B$ equivariant topological sigma model considered in the previous sections. Finally we elucidate the appearance of the non-standard real structure (2.4) in a simple example of the mirror map for a sigma model on $\mathbb{P}^1$ with the target space being an infinite cylinder $\mathbb{C}^* = \mathbb{R} \times S^1$.

### 3.1. Fixed point calculation of equivariant volume

In this subsection we derive the Euler integral representation of the Gamma-function (3.1) applying the Duistermaat–Heckman fixed point formula to the infinite-dimensional integral representation for the Gamma function proposed in [8]. The main step of the derivation is a calculation (see Lemma 3.2) of the infinite-dimensional determinant entering the stationary phase evaluation of the relevant functional integral.

Let us start with recalling the functional integral representation of the $\Gamma$-function as an equivariant symplectic volume from [8]. Let $\mathcal{M}(D, \mathbb{C})$ be a space of holomorphic maps of the disk $D = \{z \in \mathbb{C} |\ |z| \leq 1\}$ into the complex plane $\mathbb{C}$. An element of $\mathcal{M}(D, \mathbb{C})$ can be described as a complex function $\varphi(z, \bar{z})$ on $D$, satisfying the equation

$$\partial_{\bar{z}} \varphi(z, \bar{z}) = 0. \tag{3.2}$$

We denote the complex conjugated function by $\bar{\varphi}(z, \bar{z})$. Define a symplectic form on the space $\mathcal{M}(D, \mathbb{C})$ as follows:

$$\Omega = \frac{\imath}{4\pi} \int_0^{2\pi} \delta\varphi(\sigma) \wedge \delta\bar{\varphi}(\sigma)\, d\sigma, \tag{3.3}$$

where $\varphi(\sigma)$, $\bar{\varphi}(\sigma)$ are restrictions of $\varphi(z, \bar{z})$, $\bar{\varphi}(z, \bar{z})$ to the boundary $\partial D = S^1$ and $\sigma$ is a coordinate on the boundary such that $\sigma \sim \sigma + 2\pi$. The symplectic form (3.3) is invariant with respect to the action of the group $S^1$ of loop rotations and to the action of $U(1)$ induced from the standard action of $U(1)$ on $\mathbb{C}$

$$\varphi(z) \longrightarrow \mathrm{e}^{\mathrm{i}\alpha} \varphi(z), \qquad \bar{\varphi}(\bar{z}) \longrightarrow \mathrm{e}^{-\mathrm{i}\alpha} \bar{\varphi}(\bar{z}), \qquad \mathrm{e}^{\mathrm{i}\alpha} \in U(1), \tag{3.4}$$

$$\varphi(z) \longrightarrow \varphi(\mathrm{e}^{\imath\beta} z), \qquad \bar{\varphi}(\bar{z}) \longrightarrow \bar{\varphi}(\mathrm{e}^{-\imath\beta} \bar{z}), \qquad \mathrm{e}^{\imath\beta} \in S^1. \tag{3.5}$$

Let $\hbar$ and $\lambda$ be generators of the Lie algebras of $S^1$ and $U(1)$ correspondingly. The action of $S^1 \times U(1)$ on $(\mathcal{M}(D, \mathbb{C}), \Omega)$ is Hamiltonian and the corresponding momenta are given by

$$H_{S^1} = -\frac{\imath}{4\pi} \int_0^{2\pi} \bar{\varphi}(\sigma) \partial_\sigma \varphi(\sigma)\, d\sigma, \qquad H_{U(1)} = \frac{1}{4\pi} \int_0^{2\pi} |\varphi(\sigma)|^2\, d\sigma. \tag{3.6}$$

The $S^1 \times U(1)$-equivariant volume of $\mathcal{M}(D, \mathbb{C})$ is defined formally as follows [8]. Let $\chi(z, \bar{z})$ and $\bar{\chi}(z, \bar{z})$ be a pair of complex conjugated odd functions satisfying the equations

$$\partial_{\bar{z}} \chi(z, \bar{z}) = 0, \qquad \partial_z \bar{\chi}(z, \bar{z}) = 0. \tag{3.7}$$

The functions $(\chi(z, \bar{z}), \bar{\chi}(z, \bar{z}))$ can be considered as a section of the odd tangent bundle $\Pi T\mathcal{M}(D, \mathbb{C})$ to $\mathcal{M}(D, \mathbb{C})$. Using the standard correspondence between differential forms on a manifold $X$ and the functions on the odd tangent bundle $\Pi TX$ one can write down the symplectic form (3.3) as follows:

$$\Omega = \frac{\imath}{4\pi} \int_0^{2\pi} d\sigma \, \chi(\sigma) \, \bar{\chi}(\sigma).$$

Below we freely use the equivalence between differential forms and functions on superspaces without further notice.

The $S^1 \times U(1)$-equivariant volume of the space of holomorphic maps $\mathcal{M}(D, \mathbb{C})$ is given by the following functional integral:

$$Z(\lambda, \hbar, \mu) = \int_{\Pi T\mathcal{M}(D,\mathbb{C})} dm(\varphi, \chi) \;\; e^{\mu(\lambda H_{U(1)} + \hbar H_{S^1} + \Omega)}, \qquad \mathrm{Re}(\mu) < 0, \tag{3.8}$$

where $H_{S^1}$, $H_{U(1)}$ are given by (3.6), and $dm(\varphi, \chi)$ is a canonical integration measure on the superspace $\Pi T\mathcal{M}(D, \mathbb{C})$ defined in [8]. The integral (3.8) is an infinite-dimensional Gaussian integral and is understood using the zeta-function regularization. Note that in general, regularized infinite-dimensional integrals depend on auxiliary parameters defined by a particular choice of a regularization scheme. For the integral (3.8) this leads to the following general dependence on a regularization scheme [8]:

$$Z(\lambda, \hbar, \mu) = A(\mu) \, B(\mu)^{\frac{\lambda}{\hbar}} \; \Gamma\left(\frac{\lambda}{\hbar}\right), \tag{3.9}$$

where $A(\mu)$ and $B(\mu)$ are some $\lambda$-independent functions. Thus taking into account the dependence on a choice of a regularization scheme it is natural to consider the $S^1 \times U(1)$-equivariant volume of the space of holomorphic maps $\mathcal{M}(D, \mathbb{C})$ (and thus in particular the gamma-function) as a $\mathbb{R}^* \times \mathbb{R}_+$-torsor. The regularization scheme we use below leads to a particular choice of $A$ and $B$.

In [8] the integral (3.8) was expressed in terms of infinite-dimensional determinant and no obvious relation with the Euler integral representation

(3.1) was given. Below we consider a heuristic derivation of (3.9) using an infinite-dimensional version of the Duistermaat–Heckman fixed point formula [7]. In this derivation the Euler integral representation (3.1) appears in a natural way.

To proceed let us first recall a construction of a projective space $\mathbb{P}^N$ as the Hamiltonian reduction of a symplectic manifold $(\mathbb{C}^{N+1}, \omega_{\mathbb{C}^{N+1}})$ by the Hamiltonian action of the group $U(1)$. Here the symplectic form $\omega_{\mathbb{C}^{N+1}}$ is given by

$$\omega_{\mathbb{C}^{N+1}} = \frac{\imath}{2}\sum_{j=1}^{N+1} dz_j \wedge d\bar{z}_j, \tag{3.10}$$

and the $U(1)$ action

$$\mathrm{e}^{\imath\alpha}:\ z_j \longrightarrow \mathrm{e}^{\imath\alpha}\, z_j, \qquad \mathrm{e}^{\imath\alpha}\in U(1), \qquad j=1,\ldots,N+1 \tag{3.11}$$

is generated by the vector field

$$v = \sum_{i=1}^{N+1} \imath\left\{z_i\frac{\partial}{\partial z_i} - \bar{z}_i\frac{\partial}{\partial \bar{z}_i}\right\}.$$

The momentum $H_{U(1)}$ corresponding to the Hamiltonian action (3.11) is defined by the equation $\iota_v\omega = -dH_{U(1)}$ and is given by $H_{U(1)} = \frac{1}{2}\sum_{j=1}^{N+1} |z_j|^2$. Projective space $\mathbb{P}^N$ can be realized as a Hamiltonian reduction of $(\mathbb{C}^{N+1}, \omega_{\mathbb{C}^{N+1}})$ by $U(1)$

$$\mathbb{P}^N = \left\{z\in\mathbb{C}^{N+1}\,\Big|\, H_{U(1)}(z,\bar{z}) = \frac{1}{2}r^2\right\}\Big/U(1), \qquad r\in\mathbb{R}. \tag{3.12}$$

Thus constructed $\mathbb{P}^N$ has a canonical symplectic structure $\omega_{\mathbb{P}^N}$ proportional to the Fubini–Study form. In terms of inhomogeneous coordinates $w_j = z_j/z_{N+1}$, $z_{N+1}\neq 0$ it is given by

$$\omega_{\mathbb{P}^N} = \frac{\imath r^2}{2}\,\frac{(1+\sum_{i=1}^N |w_i|^2)\sum_{j=1}^N dw_j\wedge d\bar{w}_j - \sum_{i,j}^N w_i\bar{w}_j dw_j\wedge d\bar{w}_i}{(1+\sum_{i=1}^N |w_i|^2)^2}. \tag{3.13}$$

The symplectic space $(\mathbb{C}^{N+1}, \omega_{\mathbb{C}^{N+1}})$ allows also the Hamiltonian action of the group $U(1)^{N+1}$

$$z_i \longmapsto z_i e^{\imath\alpha_i}, \qquad e^{\imath\alpha_i}\in U(1)_i, \qquad i=1,\ldots,N+1, \tag{3.14}$$

generated by vector fields

$$v_i = \imath\left\{z_i\frac{\partial}{\partial z_i} - \bar{z}_i\frac{\partial}{\partial \bar{z}_i}\right\}, \qquad i = 1,\ldots,N+1.$$

Solving the equations $\iota_{v_i}\omega_{\mathbb{C}^{\ell+1}} = -dH_i$ we find the corresponding momenta

$$H_i = \frac{1}{2}|z_i|^2, \qquad i = 1,\ldots,N+1.$$

The action of $U(1)^{N+1}$ descents to the Hamiltonian action on $(\mathbb{P}^N, \omega^{\mathbb{P}^N})$ with the corresponding momenta

$$H_j^{\mathbb{P}^N} = \frac{r^2}{2}\,\frac{|w_j|^2}{1+\sum_{j=1}^N |w_j|^2}, \qquad j = 1,\ldots N, \tag{3.15}$$

and

$$H_{N+1}^{\mathbb{P}^N} = \frac{r^2}{2}\,\frac{1}{1+\sum_{j=1}^N |w_j|^2}. \tag{3.16}$$

**Lemma 3.1.** *The following identity holds:*

(3.17)

$$\frac{1}{2\pi\mu}\int_{\mathbb{C}^{N+1}} \delta\Big(H_{U(1)} - r^2/2\Big)\mathrm{e}^{\mu(\omega_{\mathbb{C}^{N+1}}+\sum_{j=1}^{N+1}\lambda_j H_j)} = \int_{\mathbb{P}^N} \mathrm{e}^{\mu(\omega_{\mathbb{P}^N}+\sum_{j=1}^{N+1}\lambda_j H_j^{\mathbb{P}^N})},$$

*where $\omega_{\mathbb{P}^N}$ is given by* (3.13) *and the reduced Hamiltonians $H_j^{\mathbb{P}^N}$ are given by* (3.15) *and* (3.16)*.*

*Proof.* Let us introduce new variables $w_j = z_j/z_{N+1}$, $j = 1,\ldots,N$ and $t = |z_{N+1}|^2$, $\theta = \frac{1}{2\imath}\ln\frac{z_{N+1}}{\bar{z}_{N+1}}$, so that $z_{N+1} = \sqrt{t}\,\mathrm{e}^{\imath\theta}$. Then we have

$$\begin{aligned}
&\frac{\mu^N}{2\pi}\Big(\frac{\imath}{2}\Big)^{N+1}\int_{\mathbb{C}^{N+1}}\bigwedge_{i=1}^{N+1} dz_i\wedge d\bar{z}_i\,\delta\Big(\frac{1}{2}\sum_{i=1}^{N+1}|z_i|^2 - \frac{r^2}{2}\Big)\mathrm{e}^{\mu\sum_{j=1}^{N+1}\lambda_j H_j}\\
&= \frac{\mu^N}{2\pi}\Big(\frac{\imath}{2}\Big)^{N}\int_0^{2\pi}d\theta\int_0^\infty dt\,t^N\int_{\mathbb{C}^N}\frac{\bigwedge_{n=1}^N (dw_n\wedge d\bar{w}_n)}{1+\sum|w_n|^2}\\
&\quad\times\delta\Big(t - \frac{r^2}{1+\sum|w_n|^2}\Big)\mathrm{e}^{\mu\sum_{j=1}^{N+1}\lambda_j H_j}\\
&= \mu^N r^{2N}\Big(\frac{\imath}{2}\Big)^N\int_{\mathbb{C}^N}\frac{\bigwedge_{n=1}^N (dw_n\wedge d\bar{w}_n)}{\big(1+\sum|w_n|^2\big)^{N+1}}\,\mathrm{e}^{\mu\sum_{j=1}^{N+1}\lambda_j H_j^{\mathbb{P}^N}}.
\end{aligned}$$

Taking into account that

$$\frac{\omega_{\mathbb{P}^N}^N}{N!}=r^{2N}\left(\frac{\imath}{2}\right)^N\frac{\bigwedge\limits_{n=1}^{N}(dw_n\wedge d\bar{w}_n)}{\left(1+\sum|w_n|^2\right)^{N+1}},$$

we obtain the identity (3.17). □

We shall use an infinite-dimensional analog of the identity (3.17) to calculate the integral (3.8). Let us rewrite the integral (3.8) as follows:

$$\begin{aligned}Z(\lambda,\hbar,\mu)&=\int_{-\infty}^{+\infty}dt\,\mathrm{e}^{\mu\lambda t}\,Z_t(\hbar,\mu),\\ Z_t(\hbar,\mu)&=\int_{\mathcal{M}(D,\mathbb{C})}\mathrm{e}^{\mu(\hbar H_{S^1}+\Omega)}\;\delta(t-H_{U(1)}).\end{aligned}\tag{3.18}$$

Now taking into account (3.17), we can interpret $Z_t(\hbar,\mu)$ as an integral over the infinite-dimensional projective space $\mathbb{P}\mathcal{M}(D,\mathbb{C})$

$$Z_t(\hbar,\mu)=2\pi\mu\int_{\mathbb{P}\mathcal{M}(D,\mathbb{C})}\mathrm{e}^{\mu(\hbar\tilde{H}_{S^1}+\Omega(t))},\tag{3.19}$$

where $\Omega(t)$ is an induced symplectic form on $\mathbb{P}\mathcal{M}(D,\mathbb{C})$ and $\tilde{H}_{S^1}$ is a momentum corresponding to the $S^1$-action on $\mathbb{P}\mathcal{M}(D,\mathbb{C})$. We should stress that the integral in (3.19) is an infinite-dimensional one and thus requires a proper regularization which will be discussed below.

To calculate the integral (3.19) we use an infinite-dimensional version of the Duistermaat–Heckman formula [7], [2] (for a detailed introduction into the subject see e.g. [3]). Let $M$ be a $2N$-dimensional symplectic manifold with the Hamiltonian action of $S^1$ having only isolated fixed points. Let $H$ be the corresponding momentum. The tangent space $T_{p_k}M$ to a fixed point $p_k\in M^{S^1}$ has a natural action of $S^1$. Let $v$ be a generator of $\mathrm{Lie}(S^1)$ and let $\hat{v}$ be its action on $T_{p_k}M$. Then the following identity holds:

$$\int_M \mathrm{e}^{\mu(\hbar H+\omega)}=\sum_{p_k\in M^{S^1}}\frac{\mathrm{e}^{\mu\hbar H(p_k)}}{\det_{T_{p_k}M}\hbar\hat{v}/2\pi}\,.\tag{3.20}$$

Let us formally apply (3.20) to the integral (3.19). A set of fixed points of $S^1$ acting on $\mathbb{P}\mathcal{M}(D,\mathbb{C})$ can be easily found using linear coordinates on $\mathcal{M}(D,\mathbb{C})$ (considered as homogeneous coordinates on $\mathbb{P}\mathcal{M}(D,\mathbb{C})$). Let

$\varphi(z)$ be a holomorphic map of $D$ to $\mathbb{C}$. It represents an $S^1$-fixed point on $\mathbb{P}\mathcal{M}(D,\mathbb{C})$ if rotations by $S^1$ can be compensated by an action of $U(1)$

$$e^{\imath\alpha(\beta)}\varphi(e^{\imath\beta}z) = \varphi(z), \qquad \beta \in [0, 2\pi]. \tag{3.21}$$

It is easy to see that solutions of (3.21) are enumerated by non-negative integers and are given by

$$\varphi^{(n)}(z) = \varphi_n z^n, \qquad \varphi_n \in \mathbb{C}^* \quad n \in \mathbb{Z}_{\geq 0}. \tag{3.22}$$

The tangent space to $\mathcal{M}(D,\mathbb{C})$ at an $S^1$-fixed point $\varphi^{(n)}$ has natural linear coordinates $\varphi_m/\varphi_n$, $m \in \mathbb{Z}_{\geq 0}, m \neq n$, where coordinates $\varphi_k$, $k \in \mathbb{Z}_{\geq 0}$ are defined by the series expansion of $\varphi \in \mathcal{M}(D,\mathbb{C})$

$$\varphi(z) = \sum_{k=0}^{\infty} \varphi_k z^k.$$

After identification of $\hbar$ in (3.19) with a generator of Lie$(S^1)$ its action on the tangent space at the fixed point is given by a multiplication of each $\varphi_m/\varphi_n$ on $(m-n)$. Thus to define an analog of the denominator in the right-hand side of the Duistermaat–Heckman formula (3.20), one should provide a meaning to the infinite product $\prod_{m=0,m\neq n}^{\infty} \hbar(m-n)/2\pi$. We use a $\zeta$-function regularization (see e.g. [16] and also Appendix in [8])

$$\begin{aligned} &\ln\Big[\prod_{m\in\mathbb{Z}_{\geq 0},,m\neq n} \frac{\hbar}{2\pi}(m-n)\Big]_a \\ &:= -\frac{\partial}{\partial s}\left(\sum_{m=1}^{n} \frac{e^{-\imath\pi s}}{(a\hbar m/2\pi)^s} + \sum_{m=1}^{\infty} \frac{1}{(a\hbar m/2\pi)^s}\right)\Bigg|_{s\to 0}, \end{aligned} \tag{3.23}$$

where $a$ is a normalization multiplier. The introduction of $a$ is to take into account a multiplicative anomaly $\det(AB) \neq \det A \cdot \det B$ appearing for generic operators $A$ and $B$. We specify $a$ at the final step of the calculation of (3.19).

**Lemma 3.2.** *The regularized product* (3.23) *is given by*

$$\frac{1}{\Big[\prod_{m\in\mathbb{Z}_{\geq 0},m\neq n} \hbar(m-n)/2\pi\Big]_a} = (-1)^n \frac{(a\hbar/2\pi)^{-n}}{n!} \frac{\sqrt{a\hbar}}{2\pi}. \tag{3.24}$$

*Proof.* Using the Riemann $\zeta$-function

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s},$$

one can express the right hand side of (3.23) as follows:

$$\ln\Big[\prod_{m\in\mathbb{Z}_{\geq 0},,m\neq n}\frac{\hbar}{2\pi}(m-n)\Big]_a=(\zeta(0)+n)\ln a\hbar/2\pi+\ln n!-\zeta'(0)+\imath\pi n.$$

Taking into account $\zeta(0)=-\frac{1}{2}$ and $\zeta(0)'=-\frac{1}{2}\ln 2\pi$, we obtain (3.24). $\Box$

Let us now calculate the difference of the values of $S^1$-momentum map $\tilde{H}_{S^1}$ at two $S^1$-fixed points $\varphi^{(n)},\varphi^{(0)}\in\mathbb{P}\mathcal{M}(D,\mathbb{C})$. Consider an embedded projective line $\mathbb{P}^1\subset\mathbb{P}\mathcal{M}(D,\mathbb{C})$, containing $\varphi^{(n)}$ and $\varphi^{(0)}$. Let us choose homogeneous coordinates $[z_0:z_1]$ on $\mathbb{P}^1$ such that $\varphi^{(0)}=[1:0]$ and $\varphi^{(n)}=[0:1]$. The action of $S^1$ on $\mathbb{P}\mathcal{M}(D,\mathbb{C})$ descends to the embedded $\mathbb{P}^1$ via the vector field

$$V=\imath n\left\{w\frac{\partial}{\partial w}-\bar{w}\frac{\partial}{\partial\bar{w}}\right\},\qquad w=z_1/z_0. \tag{3.25}$$

The pull back of the symplectic form $\Omega(t)$ is given by

$$\omega_{\mathbb{P}^1}=\imath t\frac{dw\wedge d\bar{w}}{(1+|w|^2)^2}.$$

The action of the vector field (3.25) on $\mathbb{P}^1$ is the Hamiltonian one. Let $H_{S^1}^{(n)}$ be the corresponding momentum given by a restriction of the momentum $\tilde{H}_{S^1}$ for $S^1$-action $\mathbb{P}\mathcal{M}(D,\mathbb{C})$. From the definition of the momentum map we have

$$H_{S^1}^{(n)}(\varphi^{(n)})-H_{S^1}^{(n)}(\varphi^{(0)})=\int_{[1:0]}^{[0:1]}dH_{S^1}^{(n)}=-\int_{[1:0]}^{[0:1]}\iota_V\omega_{\mathbb{P}^1}. \tag{3.26}$$

A momentum defined as a solution of the equation $i_V\omega=-dH$ is unique up an additive constant. To fix this constant we normalize the momentum

$\tilde{H}_{S^1}(\varphi)$ so that $H_{S^1}(\varphi^{(0)}) = 0$. Thus we obtain the following:

$$H_{S^1}^{(n)}(\varphi^{(n)}) = nt \int_{[1:0]}^{[0:1]} \frac{wd\bar{w} + \bar{w}dw}{(1+|w|^2)^2} = -nt \Big[ \frac{1}{(1+|w|^2)} \Big]_0^\infty = nt. \tag{3.27}$$

Substituting (3.27) and (3.24) into (3.20) for $M = \mathbb{P}\mathcal{M}(D, \mathbb{C})$ we obtain

(3.28)

$$Z_t(\hbar, \mu) = 2\pi\mu \sqrt{\frac{a\hbar}{(2\pi)^2}} \sum_{n=0}^{\infty} (-1)^n \frac{e^{nt\mu\hbar}}{(a\hbar/2\pi)^n \, n!} = \mu\sqrt{a\hbar} \exp\left\{ -\frac{2\pi}{a\hbar} e^{\mu\hbar t} \right\},$$

where the dependence on the normalization constant $a$ reflects an ambiguity of the regularized infinite-dimensional integral. Taking into account (3.18), the regularized $S^1 \times U(1)$-equivariant symplectic volume of $\mathcal{M}(D, \mathbb{C})$ is given by

$$\begin{aligned} Z_{\mathrm{reg}}(\lambda,\, \hbar,\, \mu) &= \int_0^\infty dt \, e^{\mu\lambda t} Z_t(\hbar,\, \mu) = \mu\sqrt{a\hbar} \int_0^\infty dt \, e^{\mu\lambda t} \, e^{-\frac{2\pi}{a\hbar} e^{\mu\hbar t}} \\ &= \Big(\frac{a}{\hbar}\Big)^{1/2} \Big(\frac{a\hbar}{2\pi}\Big)^{\frac{\lambda}{\hbar}} \int_{-\ln(a\hbar/2\pi)}^{+\infty} du \, e^{\frac{\lambda}{\hbar}u} \, e^{-e^u}, \end{aligned} \tag{3.29}$$

where $u = \mu\hbar t - \ln(a\hbar/2\pi)$. To get rid of the renormalization ambiguity we take the limit $a \to +\infty$ in the following way:

$$Z(\lambda, \hbar) = \lim_{a \to +\infty} \Big(\frac{a}{\hbar}\Big)^{-1/2} \Big(\frac{a}{2\pi}\Big)^{-\frac{\lambda}{\hbar}} Z_{\mathrm{reg}}(\mathcal{M};\, \lambda,\, \hbar) = \hbar^{\frac{\lambda}{\hbar}} \, \Gamma\Big(\frac{\lambda}{\hbar}\Big). \tag{3.30}$$

Thus, we show that the formal application of the Duistermaat–Heckman formula to the infinite-dimensional integral (3.8) in the form (3.18) leads to the Euler integral representation (3.1) of the Γ-function and reproduces the results of Section 2.

### 3.2. On explicit mirror map for the target space $\mathbb{C}$

In this subsection we consider an explicit mirror map of the type $A$ topological sigma model considered in [8] to the type $B$ topological sigma model considered in Section 1.

In the previous sections, we take into account the action (3.4) of $U(1)$ on the symplectic space $(\mathcal{M}(D, \mathbb{C}), \Omega)$ of holomorphic maps of the disk $D$ into the complex plane $\mathbb{C}$. Now we introduce a larger infinite-dimensional group acting on $(\mathcal{M}(D,\mathbb{C}),\Omega)$ in a Hamiltonian way. The space $(\mathcal{M}(D,\mathbb{C}),\Omega)$ supports the Hamiltonian action of a commutative Lie algebra $\mathcal{G}=\mathrm{Map}(S^1,\mathbb{R})$ of real functions on $S^1$ given by

$$\alpha\cdot\varphi(\sigma)=\imath\,[\alpha(\sigma)\varphi(\sigma)]_+\,,\qquad \alpha\cdot\bar{\varphi}(\sigma)=-\imath\,[\alpha(\sigma)\bar{\varphi}(\sigma)]_-\,,$$

where $\alpha(\sigma)\in\mathcal{G}$ and $\varphi(\sigma)$, $\bar{\varphi}(\sigma)$ are restrictions of $\varphi(z)$, $\bar{\varphi}(\bar{z})$ to the boundary $S^1=\partial D$. The projectors $[\ ]_\pm$ are defined as follows:

$$[\mathrm{e}^{\imath n\sigma}]_+=\mathrm{e}^{\imath n\sigma},\quad n\geq 0,\quad [\mathrm{e}^{\imath n\sigma}]_+=0,\quad n<0,\quad [\mathrm{e}^{\imath n\sigma}]_-=\mathrm{e}^{\imath n\sigma}-[\mathrm{e}^{\imath n\sigma}]_+.$$

Given a Hamiltonian action of $\mathcal{G}$ one can define corresponding momentum map of $\mathcal{M}(D,\mathbb{C})$ into the dual to the Lie algebra $\mathcal{G}$. The value of the momentum on the element $\alpha(\sigma)$ of the Lie algebra $\mathcal{G}$ is given by

$$H_{\mathcal{G}}(\alpha)=\int_0^{2\pi}d\sigma\ \alpha(\sigma)\,H_{\mathcal{G}}(\bar{\varphi}(\sigma),\varphi(\sigma)),\qquad H_{\mathcal{G}}(\bar{\varphi}(\sigma),\varphi(\sigma))=\frac{1}{4\pi}|\varphi(\sigma)|^2. \tag{3.31}$$

Note that the subalgebra $\mathfrak{u}(1)\subset\mathcal{G}$ corresponding to $\alpha(\sigma)=\mathrm{const}$ coincides with the Lie algebra of the group $U(1)$ considered in the previous subsection. The momenta (3.31) motivate an introduction of a new parametrization of $\mathcal{M}(D,\mathbb{C})$

$$\varphi(\sigma)=\tau^{1/2}(\sigma)\,\mathrm{e}^{\imath\phi(\sigma)},\qquad \bar{\varphi}(\sigma)=\tau^{1/2}(\sigma)\,\mathrm{e}^{-\imath\phi(\sigma)},$$

and thus

$$\tau(\sigma)=|\varphi(\sigma)|^2,\qquad \phi(\sigma)=-\frac{\imath}{2}\ln\left(\frac{\varphi(\sigma)}{\bar{\varphi}(\sigma)}\right). \tag{3.32}$$

Note that thus defined $\tau(\sigma)$ is constrainted by the condition to be a restriction to the boundary $S^1$ of the square module of a holomorphic function on $D$. Also let us stress that $\phi(\sigma)$ given by (3.32) is not single-valued. Indeed let $\varphi^{(n)}(z)=p_n(z)\varphi^{(0)}(z)$ be a holomorphic function on $D$ such that $p_n(z)=\prod_{j=1}^n(z-a_j)$, $a_j\in D$ is a polynomial of degree $n$ and $\varphi^{(0)}(z)$ is a holomorphic function without zeroes inside $D$. Then we have for the corresponding function $\phi(\sigma)$

$$\phi^{(n)}(\sigma+2\pi)=\phi^{(n)}(\sigma)+2\pi n,\qquad n\in\mathbb{Z}_{\geq 0}. \tag{3.33}$$

Hence the space of holomorphic maps has the following decomposition (modulo subspaces of non-zero codimension):

$$\mathcal{M}(D,\mathbb{C}) = \cup_{n=0}^{\infty}\mathcal{M}^{(n)}(D,\mathbb{C}), \tag{3.34}$$

where $\mathcal{M}^{(n)}(D,\mathbb{C})$ includes holomorphic maps $\varphi(z)$ such that for the corresponding function $\phi$ the relation (3.33) holds. We would like to reformulate the integral (3.8) using new variables (3.32) and the decomposition (3.34). Let us decompose the space of fields $\tau(\sigma)$ on the subspace of constant modes $\tau(\sigma)=2t$ and the orthogonal subspace of $\tau_*(\sigma)$ such that $\int_{S^1} d\sigma\,\tau_*(\sigma)=0$.

For $\varphi\in\mathcal{M}^{(n)}(D,\mathbb{C})$ the momenta (3.31) for $U(1)$- and $S^1$-actions in the new variables $(\tau,\phi)$ are given by

$$H_{U(1)} = \frac{1}{4\pi}\int_0^{2\pi} d\sigma\,\tau(\sigma) = t, \qquad H_{S^1} = \frac{1}{4\pi}\int_0^{2\pi} d\sigma\,\tau(\sigma)\partial_\sigma\phi(\sigma)$$
$$= -\frac{1}{4\pi}\int_0^{2\pi} d\sigma\,\partial_\sigma\tau(\sigma)\phi(\sigma) + nt,$$

where we take into account (3.33). Thus we have the following equivalent representation for (3.8):

(3.35)
$$Z(\lambda,\hbar,\mu) = \sum_{n=0}^{+\infty}\int_{\mathcal{M}^{(n)}(D,\mathbb{C})} dt\,[D\tau_*]\,[D\phi]\,J(\tau_*+t)\,e^{-\frac{\mu}{4\pi}\int_{S^1} d\sigma\,\hbar\partial_\sigma\tau_*\,\phi+\mu t(\hbar n+\lambda)},$$

where $J(\tau_*+t)$ is a Jacobain of the transformation from the variables $(\varphi,\bar{\varphi})$ to the variables $(\tau,\phi)$. The integration over $\phi$ leads to a delta-function with a support on the space of solutions of the equation

$$\partial_\sigma\tau(\sigma)=0, \qquad \tau(\sigma)=|\varphi(z)|^2|_{z=e^{\imath\sigma}}, \tag{3.36}$$

where $\varphi(z)$ is a holomorphic function on the disk $D$. The solutions of (3.36) are given by

$$\varphi^{(n)}(z) = \varphi_n z^n, \qquad n\in\mathbb{Z}_{\geq 0} \tag{3.37}$$

and coincide with the fixed points (3.22) of the $S^1$-action on $\mathbb{P}\mathcal{M}(D,\mathbb{C})$. Thus the sum over $n$ for a fixed $t$ is an analog of the sum over $S^1$-fixed points entering Duistermaat–Heckman formula applied to $\mathbb{P}\mathcal{M}(D,\mathbb{C})$. It remains to integrate the delta-function $\delta(\partial_\sigma\tau)$ in the vicinity of each solution (3.37) taking into account that $\tau(\sigma)$ is a square of a holomorphic function such that the integral $\int_0^{2\pi} d\sigma\tau(\sigma) = 2t$ is fixed. Actually we already evaluated this integral which is equivalent to the regularized product (3.24) entering the Duistermaat–Heckman formula. Thus we obtain

$$
\begin{aligned}
(3.38)\qquad Z(\lambda,\hbar,\mu)_{reg} &= \mu\sqrt{a\hbar}\sum_{n=0}^{\infty}\int_0^{+\infty} dt\, \frac{(-1)^n}{(a\hbar/2\pi)^n n!} e^{t\mu(\hbar n+\lambda)} \\
&= \mu\sqrt{a\hbar}\int_0^{+\infty} dt\, e^{\mu t\lambda - \frac{2\pi}{a\hbar}e^{\mu\hbar t}}.
\end{aligned}
$$

Note that to make the integral (3.38) well-defined we should sum the series for an appropriate range of the variables $\mu$ and $a$. The integral (3.38) reproduces the regularized integral (3.29). Taking appropriate limit (3.30) we recover the expression obtained using the Duistermaat–Heckman formula.

Using the evaluation of the integral (3.35) near the solutions (3.37) and summing the series one can rewrite (3.35) in the following form:

$$
(3.39)\quad Z(\lambda,\hbar,\mu)_{\rm reg} = \int_0^{\infty} dt \int [D\tau_*]\ \det\Delta\ \delta(\Delta\tau_*)\,\delta(\partial_\sigma\tau_*|_{S^1})e^{\mu t\lambda-\frac{2\pi}{a\hbar}e^{\mu\hbar t}},
$$

where $\Delta$ is a Laplace operator on the disk $D$ and now the functional integral is taken over the space of real functions on the disk orthogonal to the subspace of constant functions. It is easy to see that the integral over $\tau_*$ reduces to an additional $t$-independent factor for $Z(\lambda,\hbar,a)_{\rm reg}$. Combining the variables $t$ and $\tau_*$ into a new variable $\tau = \tau_* + t - \hbar^{-1}\ln(a\hbar/2\pi)$ and taking the limit $a \to +\infty$ we obtain the following:

$$
\begin{aligned}
(3.40)\quad Z(\lambda,\hbar,\mu) &= \frac{1}{\hbar}\lim_{a\to\infty} C(a,\hbar)\, a^{-\lambda/\hbar} Z(\lambda,\hbar,a)_{\rm reg} \\
&= \int [D\tau]\ \det\Delta\ \delta(\Delta\tau)\,\delta(\partial_\sigma\tau|_{S^1})\, e^{\frac{1}{2\pi}\int_0^{2\pi} d\sigma(\mu\lambda\tau(\sigma)-e^{\hbar\mu\tau(\sigma)})},
\end{aligned}
$$

where $C(a,\hbar)$ is an appropriate function. Let us note that the integral representation (3.40) can be directly derived from (3.35) in the limit $a \to +\infty$. Indeed, in the limit $a\to\infty$ (taking into account the shift $t \to t - \hbar^{-1}\ln(a\hbar/2\pi)$) the Jacobain becomes field independent and the condition on the function $\tau$ to be the square of a holomorphic function reduces to the harmonicity condition on $\tau$ due to the expansion

$$
\Delta\ln(\tau - \hbar^{-1}\ln a\hbar/2\pi) = -\frac{\hbar}{\ln a\hbar/2\pi}\Delta\tau_* + \cdots, \qquad a\to+\infty.
$$

The summation over $n$ with the weight factor obtained by a proper integration over $n$ zeroes of $\tau$ leads to the exponential term in (3.40).

To make a contact with the representation of the equivariant volume integral (3.8) in terms of type $B$ topological sigma model described in Section 1 we note that the condition $\partial_\sigma \tau|_{S^1} = 0$ imposed on restrictions of harmonic functions to the boundary $S^1 = \partial D$ is equivalent to the condition $\partial_n \tau|_{S^1} = 0$ where $\partial_n$ is a normal derivative to the boundary of $D$. Therefore we have

$$(3.41)\quad Z(\lambda, \hbar, \mu) = \int [D\tau]\ \det \Delta\ \delta(\Delta\tau)\,\delta(\partial_n\tau|_{S^1})\, e^{\frac{1}{2\pi}\int_0^{2\pi} d\sigma(\mu\lambda\tau(\sigma) - e^{\mu\hbar\tau(\sigma)})}.$$

The $\delta$-functions can be replace by an integral over an auxiliary field $\kappa(\sigma)$. Thus we obtain the following integral representation:

(3.42)
$$Z(\lambda, \hbar, \mu) = \int [D\tau]\,[D\kappa]\ \det \Delta\, e^{\int_D \imath\, d\kappa \wedge * d\tau + \int_{S^1} d\sigma(\mu\lambda\tau(\sigma) - e^{\hbar\mu\tau(\sigma)})}\,\delta(\kappa(0)).$$

This functional integral is equivalent to the one entering the formulation of the Corollary 2.1 for $\ell = 0$ with $\tau = \phi_+$ and $\kappa = \phi_-$. This can be demonstrated by integrating over the fields $\eta$, $\theta$ and $\rho$ in the type $B$ model considered in previous section.

### 3.3. *T*-duality for target space $\mathbb{C}^*$

Finally we clarify the appearance of the non-standard real structure in the topological type $B$-model proposed in Section 1 as a mirror dual to the topological type $A$-model considered in [8]. To elucidate this issue we consider a simple example of the bosonic sigma model on $\mathbb{P}^1$ with the target space $\mathbb{C}^* = \mathbb{R} \times S^1$. The mirror symmetry in this case is straightforwardly realized as a $T$-duality with respect to $S^1$. We will demonstrate below that starting with a sigma model similar to the one considered in [8] we obtain after $T$-duality the topological sigma model with the real structure on the space of fields considered in Section 1.

Let us given the following action functional:

$$(3.43)\qquad S = \int_{\mathbb{P}^1} \left(\frac{t}{2} F \wedge *F \,+\, F \wedge \partial\bar{\varphi} \,-\, F \wedge \overline{\partial}\varphi\right)$$

$$(3.44)\qquad = \int_{\mathbb{P}^1} \left(\frac{t}{2} F \wedge *F \,-\, \imath F \wedge * d\tau \,-\, \imath F \wedge d\phi\right),$$

where $\varphi=\tau+\imath\phi$ is a complex coordinate on the cylinder $\mathbb{R}\times S^1$, $\phi\sim\phi+2\pi$ and $F=\bar{F}_z dz+F_{\bar{z}}d\bar{z}$ is a real valued one-form. We imply that $\mathbb{P}^1$ is supplied with the Kähler metric associated with the standard Kähler form

$$\omega=\frac{\imath}{2}\frac{dz\wedge d\bar{z}}{(1+|z|^2)^2}.$$

Note that (in the classical theory) the action (3.43) does not depend on the choice of the two-dimensional Kähler metric. This action (3.43) is a part of an action of the topological sigma model consider in [8] adopted to the case of the target space $\mathbb{C}^*$. Indeed, the integration over $F$ gives the standard functional integral for the sigma-model

(3.45)
$$\begin{aligned} Z &= \int [DF][D\varphi]\,\mathrm{e}^{-S} \\ &= \int [DF_{\bar{z}}][D\bar{F}_z][D\varphi]\,\exp\left\{-\int_{\mathbb{P}^1} d^2z\,\left(t\bar{F}_zF_{\bar{z}}\,-\,\imath F_{\bar{z}}\partial_z\bar{\varphi}\,-\,\imath\bar{F}_z\partial_{\bar{z}}\varphi\,\right)\right\} \\ &= C(t)\int [D\varphi]\,\exp\left\{-t^{-1}\int_{\mathbb{P}^1} d^2z\,\partial_z\bar{\varphi}\,\partial_{\bar{z}}\varphi\right\}, \end{aligned}$$

where $d^2z=\imath dz\wedge d\bar{z}$ and $C(t)$ is a function of $t$.

The standard way to implement $T$-duality is to introduce an auxiliary field $B=B_zdz+B_{\bar{z}}d\bar{z}$ and $\kappa$ and consider a theory with the following action:

$$S=\imath\int_{\mathbb{P}^1} d\kappa\wedge B\,+\,\int_{\mathbb{P}^1}\left(\frac{t}{2}F\wedge *F-\imath F\wedge *d\tau-\imath F\wedge B\right). \tag{3.46}$$

Indeed integrating over $\kappa$ leads to a constraint $B=d\phi$, where $\phi$ is a real valued field and thus we come back to the action (3.44). On the other hand, integration over $B$ leads to the action

$$S=\int_{\mathbb{P}^1}\left(\frac{t}{2}F\wedge *F\,-\,\imath F\wedge *d\tau\right),$$

with the constraint

$$F=d\kappa. \tag{3.47}$$

Thus, the integration over $F$ with the constraint (3.47) gives

$$S=\int_{\mathbb{P}^1}\left(\frac{t}{2}d\kappa\wedge *d\kappa\,-\,\imath d\kappa\wedge *d\tau\right). \tag{3.48}$$

In [8], we consider a sigma-model without $F \wedge *F$-term (i.e., we imply that $t = 0$). Taking $t = 0$ in (3.48) we obtain

$$S = -\imath \int_{\mathbb{P}^1} d\kappa \wedge *d\tau. \tag{3.49}$$

This action is precisely the two-derivative term in (2.2) where the role of $\kappa$ and $\tau$ is played by the fields $\phi_+$ and $\phi_-$. Thus the non-standard real structure on the fields in (2.2) is a consequence of taking a limit $t \to 0$ in the mirror dual model discussed in [8]. Note that the action (3.49) can be straightforwardly obtained by taking $t = 0$ in (3.44) and integrating out $\phi$. Let us finally note that the action (3.49) arising in the limit $t \to 0$ is analogous to the action functionals describing discrete light-cone quantization. This relation will be discussed elsewhere.

## 4. Conclusion

To conclude this paper we briefly outline some directions for future research. The constructions of [8] and of this note allow several straightforward generalizations. For instance, one can consider an equivariant type $A$ topological sigma model on a disk $D$ with a compact target space being (partial) flag manifolds. Their mirror dual type $B$ topological theories are also known [12]. Simple examples are provided by projective spaces $\mathbb{P}^\ell$ and more generally Grassmannian spaces $Gr(m, \ell+1)$. Such topological sigma models can be described in terms of a twisting of $\mathcal{N} = 2$ SUSY gauged linear sigma models [22, 27]. For instance in the case of the target space $X = \mathbb{P}^\ell$ the corresponding linear sigma model has target space $\mathbb{C}^{\ell+1}$ gauged by the diagonal action of $U(1)$. For its mirror dual see for example [17]. An analog of the correlation functions considered in [8] but for the target space $\mathbb{P}^\ell$ should be equal to a degenerate $\mathfrak{gl}_{\ell+1}$-Whittaker function given by

$$\Psi_{\lambda_1,\dots,\lambda_{j+1}}(x) = \int_{\mathcal{C}} d\gamma \, e^{\imath\gamma x} \prod_{j=1}^{\ell+1} \Gamma\left(\frac{\gamma - \lambda_j}{\hbar}\right). \tag{4.1}$$

For a detailed discussion of the relation of (4.1) to Toda chains see [9]. The same expression should be equal to an analog of the correlation function in mirror dual type $B$ equivariant topological sigma model with the target space $\mathbb{C}^\ell$ and a superpotential $W(\phi) = \sum_{j=1}^{\ell} \left((\lambda_j - \lambda_{\ell+1})\phi^j - e^{\phi_j}\right) - e^{x - \sum_{k=1}^{\ell} \phi_k}$. The structure of the integral (4.1) is quite transparent. The product of $\Gamma$-functions is a correlation function in the type $A$ topological sigma model with the target space $\mathbb{C}^{\ell+1}$ of the type considered in [8] (as well as a correlation function in the mirror dual type $B$ theory) and the integral over $\gamma$ is a projection corresponding to an integration over the fields in the topological $U(1)$-gauge multiplet (over dual scalar topological multiplet in the mirror dual type $B$ theory). Similar reasoning can be applied to the case of the Grassmannian target space [28]. We will provide a detailed discussion of these cases in [10]. The case of general partial flag manifolds is a bit more complicated but accessible by the technique developed in [11] and will be discussed elsewhere.

Let us stress that the discussed examples of explicit calculations of particular correlation functions in topological theories on non-compact manifolds is not restricted to the case of dimension two. The three- and four-dimensional examples of such calculations have an interesting interpretation (see e.g. [8]). These higher dimensional examples should provide additional insights on the conjectural relation between local Archimedean Langlands correspondence and the mirror symmetry. We are going to pursue these directions elsewhere.

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