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Rank two ADHM invariants and wallcrossing

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Generalized Donaldson–Thomas invariants corresponding to local D6–D2–D0 configurations are defined applying the formalism of Joyce and Song to ADHM sheaves on curves. A wallcrossing formula for invariants of D6-rank two is proven and shown to agree with the wallcrossing formula of Kontsevich and Soibelman. Using this result, the asymptotic D6-rank two invariants of $(-1, -1)$ and $(0, -2)$ local rational curves are computed in terms of the D6-rank one invariants.

1. Introduction

Motivated by string theory considerations, ADHM invariants of curves were introduced in [6] as an alternative construction for the local stable pair theory of curves of Pandharipande and Thomas [17]. They have been subsequently generalized in [5] employing a natural variation of the stability condition. An important feature of this construction resides in its compatibility with the Joyce–Song theory of generalized Donaldson–Thomas invariants [14]. Explicit wallcrossing formulas for ADHM invariants have been derived and proven in [2] using Joyce theory [9–11, 13] and Joyce–Song theory [14].

The purpose of the present paper is to study a further generalization of ADHM invariants allowing higher rank framing sheaves. This generalization is motivated in part by recent work of Toda [21] and Stoppa [20] on rank two generalized Donaldson–Thomas invariants of Calabi–Yau three-folds. In contrast to [20, 21], the invariants constructed here count local objects with nontrivial D2-rank, in physics terminology. Similar rank two Donaldson–Thomas invariants of Calabi–Yau three-folds are defined and computed in [18, 19] using both wallcrossing and direct virtual localization methods.

Local invariants with higher D6-rank are also interesting on physical grounds. Explicit results for such invariants are required in order to test the OSV conjecture [16] for magnetically charged black holes. In particular, such results would be needed in order to extend the work of [1] to local D-brane configuration with nonzero D6-rank. According to [4], counting invariants with higher D6-rank are also expected to determine certain subleading corrections to the OSV formula [16]. Moreover, walls of marginal stability for BPS states with nontrivial D6-charge in a local conifold model have been studied from a supergravity point of view in [8]. The construction presented below should be viewed as a rigorous mathematical framework for the microscopic theory of such BPS states. A detailed comparison will appear elsewhere.

From the point of view of six-dimensional gauge theory dynamics, the invariants constructed in this paper can be thought of as a higher rank generalization of local Donaldson–Thomas invariants of curves. It should be noted however that they are not the same as the higher rank local DT invariants defined in [6], which, from a gauge theoretic point of view, are Coulomb branch invariants (see also [3,8] for a noncommutative gauge theory approach.) Instead, employing a different treatment of boundary conditions in the six-dimensional gauge theory, the approach presented below yields Higgs branch invariants.

The geometric setup of the present construction is specified by a triple $\mathcal{X} = (X, M_1, M_2)$ where X is a smooth projective curve of X over $\mathbb{C}$ of genus g, and M_1, M_2 are line bundles on X so that $M = M_1 \otimes_X M_2$ is isomorphic to the anticanonical bundle K_X^{-1}. The data $\mathcal{X}$ determine an abelian category $\mathcal{C}_{\mathcal{X}}$ of quiver sheaves on X constructed in [5, Section 3].

Section 2 consists of a step-by-step construction of counting invariants for objects of $\mathcal{C}_{\mathcal{X}}$ following [14]. The required stability conditions, chamber structure and moduli stacks are presented in Sections 2.1, 2.2 and 2.4, respectively. Some basic homological algebra results are provided in Section 2.3. The construction is concluded in Section 2.5. Given a stability parameter $\delta \in \mathbb{R}$ the geometric data $\mathcal{X}$ determines a function $A_\delta : \mathbb{Z}^{\times 3} \to \mathbb{Q}$, which assigns to any triple $\gamma = (r, e, v)$ the virtual number of δ-semistable ADHM sheaves on X of type γ. This function is supported on $\mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}$. In physics terms, the integers (r, e, v) correspond to D2-, D0- and D6-brane charges, respectively. In the derivation of wallcrossing formulas, it is more convenient to use the alternative notation $\gamma = (\alpha, v)$, $\alpha = (r, e) \in \mathbb{Z} \times \mathbb{Z}$. Moreover, the invariants $A_\delta(\alpha, 0)$ are manifestly independent of δ, and will be denoted by $H(\alpha)$ since they are counting invariants for Higgs sheaves on X.

Note that for a fixed type γ there is a finite set $\Delta(\gamma) \subset \mathbb{R}$ of critical stability parameters dividing the real axis in stability chambers (see Lemma 2.6). The invariants $A_\delta(\gamma)$ are constant when δ varies within a stability chamber. The chamber $\delta > \max \Delta(\gamma)$ will be referred to as the asymptotic chamber, and the corresponding invariants will be also denoted by $A_\infty(\gamma)$. The main result of this paper is a wallcrossing formula for $v = 2$ ADHM invariants at a critical stability parameter $\delta_{\rm c} > 0$ of type $(\alpha, 2)$, for arbitrary $\alpha = (r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$. Certain preliminary definitions will be needed in the formulation of this result, as follows.

For any integer $l \in \mathbb{Z}_{\geq 1}$, and any $v \in \{1, 2\}$ let $\mathcal{HN}_-(\alpha, v, \delta_{\rm c}, l, l-1)$ denote the set of ordered sequences $((\alpha_i))_{1 \leq i \leq l}$, $\alpha_i \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$ satisfying the following conditions:

$$\alpha_1 + \cdots + \alpha_l = \alpha \tag{1.1}$$

and

$$\frac{e_1}{r_1} = \cdots = \frac{e_{l-1}}{r_{l-1}} = \frac{e_l + v\delta_{\rm c}}{r_l} = \frac{e + v\delta_{\rm c}}{r}. \tag{1.2}$$

For any integer $l \in \mathbb{Z}_{\geq 2}$, let $\mathcal{HN}_-(\alpha, 2, \delta_{\rm c}, l, l-2)$ denote the set of ordered sequences $((\alpha_i))_{1\leq i\leq l}$, $\alpha_i \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$ satisfying condition (1.1),

$$\frac{e_1}{r_1} = \cdots = \frac{e_{l-2}}{r_{l-2}} = \frac{e_{l-1} + \delta_{\rm c}}{r_{l-1}} = \frac{e_l + \delta_{\rm c}}{r_l} = \frac{e + 2\delta_{\rm c}}{r} \tag{1.3}$$

and

$$1/r_{l-1} < 1/r_l. \tag{1.4}$$

Let $0 < \delta_- < \delta_{\rm c} < \delta_+$ be stability parameters so that there are no critical stability parameters of type $(\alpha, 2)$ in the intervals $[\delta_-,\ \delta_{\rm c})$, $(\delta_{\rm c},\ \delta_+]$. For any triple (β, v), $\beta \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$, $v \in \{1, 2\}$, the invariants $A_{\delta_\pm}(\beta, v)$ will be denoted by $A_\pm(\beta, v)$. Then the following result holds for δ_-, δ_+ sufficiently close to $\delta_{\rm c}$.

Theorem 1.1. *The $v = 2$ ADHM invariants satisfy the following wallcrossing formula:*

(1.5)

$$\begin{aligned}
&A_-(\alpha, 2) - A_+(\alpha, 2) \\
&\quad = \sum_{l\geq 2} \frac{1}{(l-1)!} \sum_{(\alpha_i)\in\mathcal{HN}_-(\alpha,2,\delta_{\rm c},l,l-1)} A_+(\alpha_l, 2) \prod_{i=1}^{l-1} f_2(\alpha_i) H(\alpha_i) \\
&\qquad - \frac{1}{2}\sum_{l\geq 1} \frac{1}{(l-1)!} \sum_{(\alpha_i)\in\mathcal{HN}_-(\alpha,2,\delta_{\rm c},l+1,l-1)} g(\alpha_{l+1}, \alpha_l) A_+(\alpha_l, 1) A_+(\alpha_{l+1}, 1) \\
&\qquad \times \prod_{i=1}^{l-1} f_2(\alpha_i) H(\alpha_i) + \frac{1}{2} \sum_{(\alpha_1,\alpha_2)\in\mathcal{HN}_-(\alpha,2,\delta_{\rm c},2,0)} \sum_{l_1\geq 1}\sum_{l_2\geq 1} \frac{1}{(l_1-1)!}\frac{1}{(l_2-1)!}
\end{aligned}$$

$$\times \sum_{(\alpha_{1,i})\in\mathcal{HN}_-(\alpha_1,1,\delta_c,l_1,l_1-1)} \sum_{(\alpha_{2,i})\in\mathcal{HN}_-(\alpha_2,1,\delta_c,l_2,l_2-1)}$$

$$\times\, g(\alpha_1,\alpha_2)A_+(\alpha_{1,l_1},1)A_+(\alpha_{2,l_2},1)\prod_{i=1}^{l_1-1} f_1(\alpha_{1,i})H(\alpha_{1,i})$$

$$\times \prod_{i=1}^{l_2-1} f_1(\alpha_{2,i})H(\alpha_{2,i})$$

where

$$f_v(\alpha) = (-1)^{v(e-r(g-1))}v(e-r(g-1)), \qquad v=1,2$$
$$g(\alpha_1,\alpha_2) = (-1)^{e_1-e_2-(r_1-r_2)(g-1)}(e_1-e_2-(r_1-r_2)(g-1))$$

for any $\alpha=(r,e)$*, respectively,* $\alpha_i=(r_i,e_i)$*,* $i=1,2$*, and the sum on the right-hand side of Equation* (1.5) *is finite.*

Theorem 1.1 is proven in Section 3.2 using certain stack function identities established in Section 3.1. Formula (1.5) is shown to agree with the wallcrossing formula of Kontsevich and Soibelman in Section 4.

An application of Theorem 1.1 to genus zero invariants is presented in Section 5. Consider the following generating functions:

$$Z_{\mathcal{X},v}(u,q) = \sum_{r\geq 1}\sum_{n\in\mathbb{Z}} u^r q^n A_\infty(r,n-r,v) \tag{1.6}$$

where $v=1,2$. Using the wallcrossing formula (1.5) and the comparison result of Section 4, the following closed formulas are proven in Section 5.

Corollary 1.1. *Suppose* X *is a genus* 0 *curve and* $M_1\simeq\mathcal{O}_X(d_1)$*,* $M_2\simeq\mathcal{O}_X(d_2)$ *where* $(d_1,d_2)=(1,1)$ *or* $(0,2)$*. Then*

$$Z_{\mathcal{X},1}(u,q) = \prod_{n=1}^{\infty}(1-u(-q)^n)^{(-1)^{d_1-1}n},$$

$$\begin{aligned} Z_{\mathcal{X},2}(u,q) = {} & \frac{1}{4}\prod_{n=1}^{\infty}(1-uq^n)^{2(-1)^{d_1-1}n} \\ & -\frac{1}{2}\sum_{\substack{r_1>r_2\geq 1,\ n_1,n_2\in\mathbb{Z} \\ \text{or } r_1=r_2\geq 1,\ n_2>n_1 \\ \text{or } r_1\geq 1,\ n_1\in\mathbb{Z},\ r_2=n_2=0}} (n_1-n_2)(-1)^{(n_1-n_2)} \\ & \times A_\infty(r_1,n_1-r_1,1)A_\infty(r_2,n_2-r_2,1)u^{r_1+r_2}q^{n_1+n_2}. \end{aligned} \tag{1.7}$$

Remark 1.1. The computations in Section 5 based on the Kontsevich–Soibelman wallcrossing formula can be generalized to invariants of arbitrary rank $v \geq 2$. Then it follows that the rank v invariants of local $(-1,-1)$ and $(0,-2)$ curves are recursively determined by the invariants of lower rank $1 \leq v' \leq v$. The resulting formulas are quite complicated and will be omitted.

2. Higher rank ADHM invariants

2.1. Definitions and basic properties

Let X be a smooth projective curve of genus $g \in \mathbb{Z}_{\geq 0}$ over an infinite field K of characteristic 0 equipped with a very ample line bundle $\mathcal{O}_X(1)$. Let M_1, M_2 be fixed line bundles on X equipped with a fixed isomorphism $M_1 \otimes_X M_2 \simeq K_X^{-1}$. Set $M = M_1 \otimes_X M_2$. For fixed data $\mathcal{X} = (X, M_1, M_2)$, let $\mathcal{Q}_{\mathcal{X}}$ denote the abelian category of (M_1, M_2)-twisted coherent ADHM quiver sheaves. An object of $\mathcal{Q}_{\mathcal{X}}$ is given by a collection $\mathcal{E} = (E, E_\infty, \Phi_1, \Phi_2, \phi, \psi)$ where

- E, E_∞ are coherent $\mathcal{O}_X$-modules;
- $\Phi_i : E \otimes_X M_i \to E$, $i = 1, 2$, $\phi : E \otimes_X M_1 \otimes_X M_2 \to E_\infty$, $\psi : E_\infty \to E$ are morphisms of $\mathcal{O}_X$-modules satisfying the ADHM relation

$$\Phi_1 \circ (\Phi_2 \otimes 1_{M_1}) - \Phi_2 \circ (\Phi_1 \otimes 1_{M_2}) + \psi \circ \phi = 0. \tag{2.1}$$

The morphisms are natural morphisms of quiver sheaves i.e., collections $(\xi, \xi_\infty) : (E, E_\infty) \to (E', E'_\infty)$ of morphisms of $\mathcal{O}_X$-modules satisfying the obvious compatibility conditions with the ADHM data.

Let $\mathcal{C}_{\mathcal{X}}$ be the full abelian subcategory of $\mathcal{Q}_{\mathcal{X}}$ consisting of objects with $E_\infty = V \otimes \mathcal{O}_X$, where V is a finite-dimensional vector space over K (possibly trivial). Note that given any two objects $\mathcal{E}, \mathcal{E}'$ of $\mathcal{C}_{\mathcal{X}}$, the morphisms $\xi_\infty : V \otimes \mathcal{O}_X \to V' \otimes \mathcal{O}_X$ must be of the form $\xi_\infty = f \otimes 1_{\mathcal{O}_X}$, where $f : V \to V'$ is a linear map.

An object $\mathcal{E}$ of $\mathcal{C}_{\mathcal{X}}$ will be called locally free if E is a coherent locally free $\mathcal{O}_X$-module. Given a coherent $\mathcal{O}_X$-module E we will denote by $r(E)$, $d(E)$, $\mu(E)$ the rank, degree slope of E, respectively, if $r(E) \neq 0$. Recall that any coherent locally free sheaf E on X has a unique Harder–Narasimhan filtration $0 \subset E_1 \subset \cdots \subset E_h = E$, $h \in \mathbb{Z}_{\geq 1}$, with respect to slope stability. In the following set $\mu_{\max}(E) = \mu(E_1)$.

The numerical type of an object $\mathcal{E}$ of $\mathcal{C}_{\mathcal{X}}$ is the collection $(r(\mathcal{E}), d(\mathcal{E}), v(\mathcal{E})) = (r(E), d(E), \dim(V)) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}$. An object of $\mathcal{C}_{\mathcal{X}}$ will be called an ADHM sheaf in the following. Throughout this paper, the integer $v(\mathcal{E})$ will be called the rank of $\mathcal{E}$, as opposed to the terminology used in [2,5,6], where the rank of $\mathcal{E}$ was defined to be $r(\mathcal{E})$. Note that the objects of $\mathcal{C}_{\mathcal{X}}$ with $v(\mathcal{E}) = 0$ form a full abelian category which is naturally equivalent to the abelian category of Higgs sheaves on X with coefficient bundles (M_1, M_2) (see, e.g., [5, Appendix A] for a brief summary of the relevant definitions).

Let $\delta \in \mathbb{R}$ be a stability parameter. The δ-degree of an object $\mathcal{E}$ of $\mathcal{C}_{\mathcal{X}}$ is defined by

$$\deg_\delta(\mathcal{E}) = d(\mathcal{E}) + \delta v(\mathcal{E}). \tag{2.2}$$

If $r(\mathcal{E}) \neq 0$, the δ-slope of $\mathcal{E}$ is defined by

$$\mu_\delta(\mathcal{E}) = \frac{\deg_\delta(\mathcal{E})}{r(\mathcal{E})}. \tag{2.3}$$

Definition 2.1. Let $\delta \in \mathbb{R}$ be a stability parameter. A nontrivial object $\mathcal{E}$ of $\mathcal{C}_{\mathcal{X}}$ is δ-(semi)stable if

$$r(E) \deg_\delta(\mathcal{E}') \ (\leq)\ r(E') \deg_\delta(\mathcal{E}) \tag{2.4}$$

for any proper nontrivial subobject $0 \subset \mathcal{E}' \subset \mathcal{E}$.

The following lemmas summarize some basic properties of δ-semistable ADHM sheaves. The proofs are either standard or very similar to those of [5, Lemmas 2.4, 3.7] and will be omitted.

Lemma 2.1. *Suppose $\mathcal{E}$ is a δ-semistable framed ADHM sheaf with $r(\mathcal{E}) > 0$ for some $\delta \in \mathbb{R}$. Then*

(i) *E is locally free;*

(ii) *if $\delta > 0$, there is no nontrivial linear subspace $0 \subset V' \subseteq V$ so that $\psi|_{V' \otimes \mathcal{O}_X}$ is identically zero. Similarly, if $\delta < 0$, there is no proper linear subspace $0 \subseteq V' \subset V$ so that $\mathrm{Im}(\phi) \subseteq V' \otimes \mathcal{O}_X$; and*

(iii) *if $\mathcal{E}$ is δ-stable any endomorphism of $\mathcal{E}$ in $\mathcal{C}_{\mathcal{X}}$ is either trivial or an isomorphism. If the ground field K is algebraically closed, the endomorphism ring of $\mathcal{E}$ is canonically isomorphic to K.*

Lemma 2.2. *For fixed $(r,e,v) \in \mathbb{Z}_{>0} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}$ there is a constant $c \in \mathbb{R}$ (depending only on $\mathcal{X}$ and (r,e,v)) so that for any $\delta \in \mathbb{R}$, any δ-semistable framed ADHM sheaf of type (r,e,v) satisfies*

$$\mu_{\max}(\mathrm{E}) < c.$$

In particular, the set of isomorphism classes of framed ADHM sheaves of fixed type (r,e,v) which are δ-semistable for some $\delta \in \mathbb{R}$ is bounded.

Given a locally free ADHM sheaf $\mathcal{E} = (E, \Phi_1, \Phi_2, \phi, \psi)$ on X of type $(r,e,v) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}$, the data

$$\begin{aligned} \widetilde{E} &= E^\vee \otimes_X M^{-1} \\ \widetilde{\Phi}_i &= (\Phi_i^\vee \otimes 1_{M_i}) \otimes 1_{M^{-1}} : \widetilde{E} \otimes M_i \to \widetilde{E} \\ \widetilde{\phi} &= \psi^\vee \otimes 1_{M^{-1}} : \widetilde{E} \otimes_X M \to V^\vee \otimes \mathcal{O}_X \\ \widetilde{\psi} &= \phi^\vee : V^\vee \otimes \mathcal{O}_X \to \widetilde{E} \end{aligned} \tag{2.5}$$

with $i = 1,2$, determine a locally free ADHM sheaf $\widetilde{\mathcal{E}}$ of type $(r, -e + 2r(g-1), v)$ where g is the genus of X. $\widetilde{\mathcal{E}}$ will be called the dual of $\mathcal{E}$ in the following. Then the following lemma is straightforward.

Lemma 2.3. *Let $\delta \in \mathbb{R}$ be a stability parameter and let $\mathcal{E}$ be a locally free ADHM sheaf on X. Then $\mathcal{E}$ is δ-(semi)stable if and only if $\widetilde{\mathcal{E}}$ is $(-\delta)$-(semi)stable.*

2.2. Chamber structure

This subsection summarizes the main properties of δ-stability chambers.

Definition 2.2. An ADHM sheaf $\mathcal{E}$ of type $(r,e,v) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}$ is asymptotically (semi)stable if the following conditions hold:

(i) E is locally free, $\psi : V \otimes \mathcal{O}_X \to E$ is not identically zero, and there is no saturated proper nontrivial subobject $0 \subset \mathcal{E}' \subset \mathcal{E}$ in $\mathcal{C}_{\mathcal{X}}$ so that $v(\mathcal{E}')/r(\mathcal{E}') > v/r$.

(ii) Any proper nontrivial subobject $0 \subset \mathcal{E}' \subset \mathcal{E}$ with $v(\mathcal{E}')/r(\mathcal{E}') = v/r$ satisfies the slope inequality $\mu(E')\ (\leq)\ \mu(E)$.

Here a subobject $\mathcal{E}' \subset \mathcal{E}$ is called saturated if the underlying coherent sheaf E' is saturated in E. Note that according to [5, Lemma 3.10], any proper subobject $0 \subset \mathcal{E}' \subset \mathcal{E}$ admits a canonical saturation $\overline{\mathcal{E}'} \subset \mathcal{E}$.

Lemma 2.4. *The set of isomorphism classes of asymptotically semistable ADHM sheaves of fixed type* $(r, e, v) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 1}$ *is bounded.*

Proof. The proof is based on Maruyama's boundedness theorem. Suppose $\mathcal{E}$ is asymptotically semistable of type (r, e, v), and the underlying coherent sheaf E is not semistable. Then there is a nontrivial Harder–Narasimhan filtration

$$0 \subset E_1 \subset \cdots \subset E_h = E$$

with $h \geq 2$ so that $\mu(E_j) > \mu(E)$ and $r(E_j) < r$ for all $1 \leq j \leq h-1$. Suppose E_j is Φ_i-invariant, $i = 1, 2$, and $\mathrm{Im}(\psi) \subseteq E_j$ for some $1 \leq j \leq h-1$. Then the data $\mathcal{E}_j = (E_j, \Phi_i|_{E_j \otimes_X M_i}, \phi|_{E_j \otimes_X M}, \psi)$ are subobject of $\mathcal{E}$ with

$$v(\mathcal{E}_j)/r(\mathcal{E}_j) = \frac{v}{r(\mathcal{E}_j)} > \frac{v}{r}.$$

Since $E_j \subset E$ is saturated, it follows that $\mathcal{E}_j$ violates condition (i) in Definition 2.2. Therefore, for any $1 \leq j \leq h$, E_j is either not preserved by some Φ_i, $i = 1, 2$, or it does not contain the image of ψ. From this point on the proof is identical to the proof of [6, Proposition 2.7]. □

Definition 2.3. Let $\delta \in \mathbb{R}_{>0}$. A δ-semistable ADHM sheaf $\mathcal{E}$ of type $(r, e, v) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}$ is generic if it is either δ-stable or any proper nontrivial subobject $0 \subset \mathcal{E}' \subset \mathcal{E}$ of type $(r', e', v') \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}$ satisfies

$$\frac{e'}{r'} = \frac{e}{r}, \qquad \frac{v'}{r'} = \frac{v}{r}. \tag{2.6}$$

The stability parameter $\delta \in \mathbb{R}_{>0}$ is called generic of type (r, e, v) if any δ-semistable ADHM sheaf of type (r, e, v) is generic. The stability parameter $\delta \in \mathbb{R}_{>0}$ is called critical of type (r, e, v) if there exists a nongeneric δ-semistable ADHM sheaf of type (r, e, v).

Lemma 2.2 implies the following.

Lemma 2.5. *For fixed* $(r, e, v) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 1}$ *there exists* $\delta_\infty \in \mathbb{R}_{>0}$ *so that for all* $\delta \geq \delta_\infty$ *an ADHM sheaf* $\mathcal{E}$ *of type* (r, e, v) *is* δ*-(semi)stable if and only if it is asymptotically (semi)stable.*

Proof. The proof is similar to the proof of lemma [5, Lemma 4.7]. Some details will be provided for convenience. It is straightforward to prove that asymptotic stability implies δ-stability for sufficiently large δ using

Lemma 2.2. The converse is slightly more involved. First note that given any nontrivial locally free ADHM sheaf $\mathcal{E}$, any linear subspace $V' \subset V$ determines a canonical subobject $\mathcal{E}_{V'} \subset \mathcal{E}$. $\mathcal{E}_{V'}$ is the saturation of the subobject of $\mathcal{E}$ generated by $V' \otimes \mathcal{O}_X$ by successive applications of the ADHM morphisms ψ, Φ_i, ϕ. Since $\mathcal{E}_{V'}$ is canonically determined by V' and $\mathcal{E}$, Lemma 2.2 implies that the set of isomorphism classes of subobjects $\mathcal{E}_{V'}$, where $\mathcal{E}$ is a δ-semistable ADHM sheaf of type (r, e, v) for some $\delta > 0$ is bounded. Moreover, by construction, any subobject $0 \subset \mathcal{E}' \subset \mathcal{E}$ contains the canonical subobject $\mathcal{E}_{V'}$.

Now suppose that for any $\delta > 0$ there exists a δ-semistable ADHM sheaf $\mathcal{E}$ of type (r, e, v) which is not asymptotically stable. Let $0 \subset \mathcal{E}' \subset \mathcal{E}$ be a saturated nontrivial proper saturated subobject violating the asymptotic stability conditions. Note that $\mathcal{E}'$ cannot violate condition (ii) in Definition 2.2 since $\mathcal{E}$ is δ-semistable. Therefore, it must violate condition (i) i.e., $v'/r' > v/r$ where $r' = r(\mathcal{E}')$. In particular $v' = v(\mathcal{E}') > 0$. Then the subobject $\mathcal{E}_{V'}$ also violates condition (i) since

$$\frac{v(\mathcal{E}_{V'})}{r(\mathcal{E}_{V'})} = \frac{v'}{r(\mathcal{E}_{V'})} \geq \frac{v'}{r'} > v/r.$$

Since $\mathcal{E}$ is δ-semistable $\mu_\delta(\mathcal{E}_{V'}) \leq \mu_\delta(\mathcal{E})$. However, as noted above, the set of isomorphism classes of all $\mathcal{E}_{V'}$ is bounded, therefore the set of all types $(r(\mathcal{E}_{V'}), d(\mathcal{E}_{V'}), v(\mathcal{E}_{V'}))$ is finite. Taking δ sufficiently large, this leads to a contradiction. □

By analogy with [5, Lemmas 4.4, 4.6], Lemmas 2.5 and 2.3 imply the following.

Lemma 2.6. *Let $(r, e, v) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 1}$ be a fixed type. Then there is a finite set $\Delta(r, e, v) \subset \mathbb{R}$ of critical stability parameters of type (r, e, v). Given any two stability parameters $\delta, \delta' \in \mathbb{R}$, $\delta < \delta'$ so that $[\delta,\ \delta'] \cap \Delta(r, e, v) = \emptyset$, the set of δ-semistable ADHM sheaves of type (r, e, v) is identical to the set of δ'-semistable ADHM sheaves of type (r, e, v).*

Remark 2.1. It is straightforward to check that $\Delta(1, e, v) = \{0\}$ for any $v \geq 1$.

Lemma 2.7. *Let $(r, e, v) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 1}$ and let $\delta_c > 0$ be a critical stability parameter of type (r, e, v). Let $\delta_\pm > 0$ be stability parameters so that $\delta_- < \delta_c < \delta_+$ and $[\delta_-,\ \delta_c) \cap \Delta(r, e, v) = \emptyset$, $(\delta_c,\ \delta_+] \cap \Delta(r, e, v) = \emptyset$. If $\mathcal{E}$ is a $\delta_\pm$-semistable ADHM sheaf of type (r, e, v), then $\mathcal{E}$ is also δ_c-semistable.*

Definition 2.4. Let $(r, v) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$.

(a) A positive admissible configuration of type (r, v) is an ordered sequence of integral points: $(\rho_i = (r_i, v_i) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 0})_{1 \leq i \leq h,\, h \geq 1}$ satisfying the following conditions:
- $\rho_1 + \cdots + \rho_h = (r, v)$.
- $r_{i+1} v_i > r_i v_{i+1}$ for all $i = 1, \ldots, h-1$.

(b) A negative admissible configuration of type (r, v) is an ordered sequence of integral points: $(\rho_i = (r_i, v_i) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 0})_{1 \leq i \leq h,\, h \geq 1}$ satisfying the following conditions:
- $\rho_1 + \cdots + \rho_h = (r, v)$.
- $r_{i+1} v_i < r_i v_{i+1}$ for all $i = 1, \ldots, h-1$.

Remark 2.2. (i) It is straightforward to prove that for fixed $(r, v) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$ the set of positive, respectively negative, admissible configurations is finite. These sets will be denoted by $\mathcal{HN}_{\pm}(r, v)$.

(ii) The only positive, respectively negative admissible configuration of type (r, v) with $h = 1$ is $(\rho = (r, v))$.

Lemma 2.8. *Let $\delta_c \in \mathbb{R}_{>0}$ be a critical stability parameter of type $(r, e, v) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 1}$. Then the following hold:*

(i) *There exists $\epsilon_+ > 0$, so that $(\delta_c,\ \delta_c + \epsilon_+] \cap \Delta(r, e, v) = \emptyset$ and the following holds for any $\delta_+ \in (\delta_c,\ \delta_c + \epsilon_+)$. A locally free ADHM sheaf $\mathcal{E}$ of type (r, e, v) on X is δ_c-semistable if and only if it is either δ_+-semistable or there exists a unique filtration of the form*

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_h = \mathcal{E} \tag{2.7}$$

with $h \geq 2$ satisfying the following conditions:
- *The successive quotients $\mathcal{F}_i = \mathcal{E}_i / \mathcal{E}_{i-1}$, $i = 1, \ldots, h$ of filtration (2.7) are locally free ADHM sheaves with numerical types $(r_i, e_i, v_i) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}$. δ_+ is noncritical of type (r_i, e_i, v_i), $\mathcal{F}_i$ is δ_+-semistable and $\mu_{\delta_c}(\mathcal{F}_i) = \mu_{\delta_c}(\mathcal{E})$ for all $i = 1, \ldots, h$.*
- *The sequence $\rho_i = (r_i, v_i)$, $i = 1, \ldots, h$ is a positive admissible configuration of type (r, e, v).*

(ii) *There exists $\epsilon_- > 0$, so that $[\delta_c - \epsilon_-, \delta_c) \cap \Delta(r, e, v) = \emptyset$ and the following holds for any $\delta_- \in (\delta_c - \epsilon_-,\ \delta_c)$. A locally free ADHM sheaf*

$\mathcal{E}$ of type (r,e,v) on X is δ_c-semistable if and only if it is either δ_--semistable or there exists a unique filtration of the form

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_h = \mathcal{E} \tag{2.8}$$

with $h \geq 2$ satisfying the following conditions:

- *The successive quotients $\mathcal{F}_i = \mathcal{E}_i/\mathcal{E}_{i-1}$, $i = 1,\ldots,h$ of filtration (2.8) are locally free ADHM sheaves with numerical types $(r_i, e_i, v_i) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}$. δ_- is noncritical of type (r_i, e_i, v_i), $\mathcal{F}_i$ is δ_--semistable and $\mu_{\delta_c}(\mathcal{F}_i) = \mu_{\delta_c}(\mathcal{E})$ for all $i = 1,\ldots,h$.*
- *The sequence $\rho_i = (r_i, v_i)$, $i = 1,\ldots,h$ is a negative admissible configuration of type (r,e,v).*

Proof. The proof is similar to the proof of [5, Lemma 4.13]. Details are included below for completeness. Note that it suffices to prove statement (i) since the proof of (ii) is analogous.

Let $\delta_+ > \delta_c$ be an arbitrary noncritical stability parameter of type (r,e,v) so that $(\delta_c, \delta_+] \cap \Delta(r,e,v) = \emptyset$. Suppose $\mathcal{E}$ is a δ_c-semistable ADHM sheaf on X. Then $\mathcal{E}$ is either δ_+-stable or there is a Harder–Narasimhan filtration of $\mathcal{E}$ with respect to δ_+-semistability

$$0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_h = \mathcal{E} \tag{2.9}$$

where $h \geq 2$. It is straightforward to check that $\mathcal{E}_l$, $1 \leq l \leq h$ must have $r(\mathcal{E}_l) \geq 1$ and the successive quotients $\mathcal{F}_l$, $0 \leq l \leq h-1$ must also have $r_l \geq 1$. Then by the general properties of Harder–Narasimhan filtrations

$$\mu_{\delta_+}(\mathcal{E}_1) > \mu_{\delta_+}(\mathcal{E}_2/\mathcal{E}_1) > \cdots > \mu_{\delta_+}(\mathcal{E}_h/\mathcal{E}_{h-1}) \tag{2.10}$$

and

$$\mu_{\delta_+}(\mathcal{E}_l) > \mu_{\delta_+}(\mathcal{E}) \tag{2.11}$$

for all $1 \leq l \leq h-1$. Since $\mathcal{E}$ is δ_c-semistable by assumption, inequalities (2.11) imply that

$$v(\mathcal{E}_l)/r(\mathcal{E}_l) > v/r \tag{2.12}$$

for all $l = 1,\ldots,h$. Note that $v(\mathcal{E}_l) = v_1 + \cdots + v_l$, $r(\mathcal{E}_l) = r_1 + \cdots + r_l$ for any $l = 1,\ldots,h$.

Moreover, using the δ_c-semistability condition and inequalities (2.11) we have

$$\delta_+ \left(\frac{v}{r} - \frac{v(\mathcal{E}_l)}{r(E_l)} \right) < \mu(E_l) - \mu(E) \leq \delta_c \left(\frac{v}{r} - \frac{v(\mathcal{E}_l)}{r(E_l)} \right) \tag{2.13}$$

for all $l = 1, \ldots, h$.

Now let $\gamma > \delta_c$ be a fixed stability parameter so that $(\delta_c, \gamma] \cap \Delta(r, e, v) = \emptyset$. Using Grothendieck's lemma and Lemma 2.2, inequalities (2.13) imply that the set of isomorphism classes of locally free ADHM sheaves $\mathcal{E}'$ on X satisfying condition $(\star)$ below is bounded.

- $(\star)$ There exists a δ_c-semistable ADHM sheaf $\mathcal{E}$ of type (r, e, v) and a stability parameter $\delta_+ \in (\delta_c\ \gamma]$ so that $\mathcal{E}' \simeq \mathcal{E}_l$ for some $l \in \{1, \ldots, h\}$, where $0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_h = \mathcal{E}$, $h \geq 1$, is the Harder–Narasimhan filtration of $\mathcal{E}$ with respect to δ_+-semistability.

Then it follows that the set of numerical types (r', e', v') of locally free ADHM sheaves $\mathcal{E}'$ satisfying property $(\star)$ is finite. This implies that there exists $0 < \epsilon_+ < \gamma - \delta_c$ so that for any $\delta_+ \in (\delta_c,\ \ \delta_c + \epsilon_+)$, and any δ_c-semistable ADHM sheaf $\mathcal{E}$ of type (r, e, v) inequalities (2.13) can be satisfied only if

$$\mu_{\delta_c}(\mathcal{E}_l) = \mu_{\delta_c}(\mathcal{E}) \tag{2.14}$$

for all $l = 1, \ldots, h$. Hence also

$$\mu_{\delta_c}(\mathcal{E}_l/\mathcal{E}_{l-1}) = \mu_{\delta_c}(\mathcal{E})$$

for all $l = 2, \ldots, h$. Then inequalities (2.10) and (2.12) imply that the sequence $\rho_l = (r_l, v_l)$, $l = 1, \ldots, h$ is a positive admissible configuration. Therefore, for all $\delta_+ \in (\delta_c,\ \ \delta_c + \epsilon_+)$, any locally free δ_c-semistable ADHM sheaf $\mathcal{E}$ of type (r, e, v) is either δ_+-stable or has a Harder–Narasimhan filtration with respect to δ_+-semistability as in Lemma 2.8(i).

Next note that the set of numerical types

$$\begin{aligned} &\mathsf{S}_{\delta_c}(r, e, v)\{(r', e', v') \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0} | 0 < r' \leq r, \\ &\quad 0 \leq v' \leq v, r(e' + \delta_c v') = r'(e + \delta_c v)\} \end{aligned} \tag{2.15}$$

is finite. Therefore, $0 < \epsilon_+ < \gamma - \delta_i$ above may be chosen so that there are no critical stability parameters of type (r', e', v') in the interval $(\delta_c,\ \delta_c + \epsilon_+)$ for any $(r', e', v') \in \mathsf{S}_{\delta_c}(r, e, v)$. In particular, δ_+ is noncritical of type (r_i, e_i, v_i), $i = 1, \ldots, h$ for any Harder–Narasihan filtration as above.

Conversely, suppose $\mathcal{E}$ is a locally free ADHM sheaf of type (r, e, v) on X which has a filtration of the form (2.7) with $\mathcal{E}'$ δ_+-stable and satisfying the conditions of Lemma 2.8(i) for some $\delta_+ \in (\delta_c,\ \delta_c + \epsilon_+)$. By the above choice of ϵ_+, there are no critical stability parameters of type (r_i, e_i, v_i) in the interval $(\delta_c,\ \delta_c + \epsilon_+)$, for any $i = 1, \ldots, h$. Since $\mathcal{F}_i$ are δ_+-semistable, Lemma 2.7 implies that $\mathcal{F}_i$ is also δ_c-semistable, for any $i = 1, \ldots, h$. Hence, $\mathcal{E}$ is also δ_c-semistable since the $\mathcal{F}_i$ have equal δ_c-slopes. □

2.3. Extension groups

Let $\mathcal{E}', \mathcal{E}''$ be nontrivial locally free objects in $\mathcal{C}_{\mathcal{X}}$ of types (r', e', v'), $(r'', e'', v'') \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}$. Let $\mathcal{C}(\mathcal{E}'', \mathcal{E}')$ be the three term complex

(2.16)
$$0 \to \mathcal{H}om_X(E'', E') \xrightarrow{d_1} \begin{array}{c} \mathcal{H}om_X(E'' \otimes_X M_1, E') \\ \oplus \\ \mathcal{H}om_X(E'' \otimes_X M_2, E') \\ \oplus \\ \mathcal{H}om_X(E'' \otimes_X M, V' \otimes \mathcal{O}_X) \\ \oplus \\ \mathcal{H}om_X(V'' \otimes \mathcal{O}_X, E') \end{array} \xrightarrow{d_2} \mathcal{H}om_X(E'' \otimes_X M, E') \to 0$$

where

$$d_1(\alpha) = (-\alpha \circ \Phi_1'' + \Phi_1' \circ (\alpha \otimes 1_{M_1}), -\alpha \circ \Phi_2'' + \Phi_2' \circ (\alpha \otimes 1_{M_2}), \phi' \circ (\alpha \otimes 1_M), -\alpha \circ \psi'')$$

for any local sections (α, α_∞) of the first term and

$$d_2(\beta_1, \beta_2, \gamma, \delta) = \beta_1 \circ (\Phi_2'' \otimes 1_{M_1}) - \Phi_2' \circ (\beta_1 \otimes 1_{M_2}) - \beta_2 \circ (\Phi_1'' \otimes 1_{M_2}) + \Phi_1' \circ (\beta_2 \otimes 1_{M_1}) + \psi' \circ \gamma + \delta \circ \phi''$$

for any local sections $(\beta_1, \beta_2, \gamma, \delta)$ of the middle term. The degrees of the three terms in (2.16) are 0,1 and 2, respectively.

Let $C(\mathcal{C}(\mathcal{E}'', \mathcal{E}'))$ be the double complex obtained from $\mathcal{C}(\mathcal{E}'', \mathcal{E}')$ by taking Čech resolutions and let $D(\mathcal{E}', \mathcal{E}'')$ be the diagonal complex of $C(\mathcal{C}(\mathcal{E}'', \mathcal{E}'))$. Note that there is a canonical linear map

$$\begin{aligned} \mathrm{Hom}(V'', V') &\to D^1(\mathcal{E}', \mathcal{E}'') = C^0(\mathcal{C}^1(\mathcal{E}'', \mathcal{E}')) \oplus C^1(\mathcal{C}^0(\mathcal{E}'', \mathcal{E}')) \\ f &\to \begin{bmatrix} {}^t(0, 0, -(f \otimes 1_{\mathcal{O}_X}) \circ \phi'', \psi' \circ (f \otimes 1_{\mathcal{O}_X})) \\ 0 \end{bmatrix}. \end{aligned}$$

Given the above expressions for the differentials d_1, d_2 it is straightforward to check that this map yields a morphism of complexes

$$\varrho : \mathrm{Hom}(V'', V')[-1] \to D(\mathcal{E}'', \mathcal{E}').$$

Let $\widetilde{D}(\mathcal{E}'', \mathcal{E}')$ denote the cone of ϱ. Then the lemma below follows either by explicit Čech cochain computations as in [6, Section 4] or using the methods of [7].

Lemma 2.9. *The extension groups* $\mathrm{Ext}^k_{\mathcal{C}_{\mathcal{X}}}(\mathcal{E}'', \mathcal{E}')$, $k = 0, 1$ *are isomorphic to the cohomology groups* $H^k(\widetilde{D}(\mathcal{E}'', \mathcal{E}'))$, $k = 0, 1$. *Moreover there is an exact sequence*

$$\begin{aligned} 0 \longrightarrow & \mathbb{H}^0(\mathcal{C}(\mathcal{E}'', \mathcal{E}')) \longrightarrow \mathrm{Ext}^0_{\mathcal{C}_{\mathcal{X}}}(\mathcal{E}'', \mathcal{E}') \longrightarrow Hom(V'', V') \\ \longrightarrow & \mathrm{Ext}^1_{\mathcal{C}_{\mathcal{X}}}(\mathcal{E}'', \mathcal{E}') \longrightarrow \mathbb{H}^1(\mathcal{C}(\mathcal{E}'', \mathcal{E}')) \longrightarrow 0 \end{aligned} \tag{2.17}$$

where $\mathbb{H}^k(\mathcal{C}(\mathcal{E}'', \mathcal{E}'))$, $k = 0, 1$ *are hypercohomology groups of the complex* $\mathcal{C}(\mathcal{E}'', \mathcal{E}')$.

Corollary 2.1. *Given any two locally free objects* $\mathcal{E}', \mathcal{E}''$

$$\begin{aligned} & \dim(\mathrm{Ext}^0_{\mathcal{C}_{\mathcal{X}}}(\mathcal{E}'', \mathcal{E}')) - \dim(\mathrm{Ext}^1_{\mathcal{C}_{\mathcal{X}}}(\mathcal{E}'', \mathcal{E}')) - \dim(\mathrm{Ext}^0_{\mathcal{C}_{\mathcal{X}}}(\mathcal{E}', \mathcal{E}'')) \\ & + \dim(\mathrm{Ext}^1_{\mathcal{C}_{\mathcal{X}}}(\mathcal{E}', \mathcal{E}'')) = v'e'' - v''e' - (v'r'' - v''r')(g-1) \end{aligned} \tag{2.18}$$

Proof. Follows from the exact sequence (2.17) and the fact that the hypercohomology groups of the complex $\mathcal{C}(\mathcal{E}'', \mathcal{E}')$ satisfy the duality relation

$$\mathbb{H}^k(\mathcal{C}(\mathcal{E}'', \mathcal{E}')) \simeq \mathbb{H}^{3-k}(\mathcal{C}(\mathcal{E}', \mathcal{E}''))^\vee$$

for $k = 0, \ldots, 3$. □

2.4. Moduli stacks

In the following, let the ground field K be $\mathbb{C}$. Let $\mathfrak{Ob}(\mathcal{X})$ denote the moduli stack of all objects of the abelian category $\mathcal{C}_{\mathcal{X}}$ and let $\mathfrak{Ob}(\mathcal{X}, r, e, v)$ denote the open and closed component of type $(r, e, v) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}$. Standard arguments analogous to [9, Sections 9, 10] prove that $\mathfrak{Ob}(\mathcal{X})$ is an algebraic stack locally of finite type and it satisfies conditions [9, Assumptions 7.1, 8.1]. Given the boundedness result (2.2), the following is also standard.

Proposition 2.1. *For fixed type* $(r,e,v) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}$ *and fixed* $\delta \in \mathbb{R}_{>0}$ *there is an algebraic moduli stack of finite type* $\mathfrak{M}^{ss}_\delta(\mathcal{X}, r, e, v)$ *of* δ*-semistable objects of type* (r,e,v) *of* $\mathcal{C}_\mathcal{X}$*. If* $\delta < \delta'$ *are two stability parameters so that* $[\delta, \ \delta'] \cap \Delta(r,e,v) = \emptyset$*, the corresponding moduli stacks are canonically isomorphic. Moreover, for any* $\delta \in \mathbb{R}$ *there are canonical open embeddings*

$$\mathfrak{M}^{ss}_\delta(\mathcal{X}, r, e, v) \hookrightarrow \mathfrak{Ob}(\mathcal{X}, r, e, v) \hookrightarrow \mathfrak{Ob}(\mathcal{X}). \tag{2.19}$$

2.5. ADHM invariants

ADHM invariants will be defined applying the formalism of Joyce and Song [14] to δ-semistable ADHM sheaves on X. Given Corollary 2.1, the required results on Behrend constructible functions are a straightforward generalization of the analogous statements proven in [5, Section 7] for ADHM sheaves with $v = 1$. Therefore, the construction of generalized Donaldson–Thomas invariants via Behrend's constructible functions [14] applies to the present case.

Recall that the central element in the construction of Joyce and Song is the stack function algebra $\mathsf{SF}(\mathfrak{Ob}(\mathcal{X}))$, which is Grothendieck group generated over $\mathbb{Q}$ by isomorphism classes of pairs $(\mathfrak{X}, \rho)$ where $\mathfrak{X}$ is an algebraic stack of finite type over $\mathbb{C}$ and $\rho : \mathfrak{X} \to \mathfrak{Ob}(\mathcal{X})$ is a representable morphism of stacks. The associative algebra structure is naturally determined by extensions in the abelian category $\mathcal{C}_\mathcal{X}$. One then defines a Lie subalgebra $\mathsf{SF}^{\mathsf{ind}}_{\mathsf{alg}}(\mathfrak{Ob}(\mathcal{X}))$ imposing certain conditions on the stabilizers of closed points $\mathfrak{x}$ of the stacks $\mathfrak{X}$. Namely, the subscript alg stands for "algebra stabilizers", which requires each such stabilizer $Stab(\mathfrak{x})$ to be identified with the group of invertible elements in a certain subring of the endomorphism ring $\mathrm{End}_{\mathcal{C}_\mathcal{X}}(\rho(\mathfrak{x}))$. The upperscript ind stands for "virtually indecomposable" stack functions, which requires the closed points $\mathfrak{x}$ to have virtual rank one stabilizers. The definition of virtual rank is very technical and will not be reviewed here in detail (see [12]).

Next let $\mathsf{L}(\mathcal{X})$ be the Lie algebra over $\mathbb{Q}$ spanned by $\{\lambda(\gamma) \,|\, \gamma \in \mathbb{Z}^3\}$ with Lie bracket

$$[\lambda(\gamma'), \lambda(\gamma'')] = (-1)^{\chi(\gamma',\gamma'')}\chi(\gamma',\gamma'')\lambda(\gamma'+\gamma'')$$

where

$$\chi(\gamma',\gamma'') = v''e' - v'e'' - (v''r' - v'r'')(g-1)$$

for any $\gamma' = (r', e', v')$, $\gamma'' = (r'', e'', v'')$. Then there is a Lie algebra morphism

$$\Psi : \mathsf{SF}^{\mathsf{ind}}_{\mathsf{alg}}(\mathfrak{Ob}(\mathcal{X})) \to \mathsf{L}(\mathcal{X}) \tag{2.20}$$

so that for any stack function of the form $[(\mathfrak{X}, \rho)]$, with $\rho : \mathfrak{X} \hookrightarrow \mathfrak{Ob}(\mathcal{X}, \gamma) \hookrightarrow \mathfrak{Ob}(\mathcal{X})$ an open embedding, and $\mathfrak{X}$ a $\mathbb{C}^\times$-gerbe over an algebraic space X,

$$\Psi([(\mathfrak{X}, \rho)]) = -\chi^B(\mathsf{X}, \rho^*\nu)\lambda(\gamma)$$

where ν is Behrend's constructible function of the stack $\mathfrak{Ob}(\mathcal{X})$.

In order to define ADHM invariants note that for any $\delta \in \mathbb{R}$, the canonical open embedding stack $\mathfrak{M}^{ss}_\delta(\mathcal{X}, \gamma) \hookrightarrow \mathfrak{Ob}(\mathcal{X})$ determines a stack function $\mathfrak{d}_\delta(\gamma) \in \underline{\mathsf{SF}}(\mathfrak{Ob}(\mathcal{X}))$. For $v = 0$, the resulting stack functions are independent of stability parameters and will be denoted by $\mathfrak{h}(\gamma)$.

According to [11, Theorem 8.7] the associated log stack function

$$\mathfrak{e}_\delta(\gamma) = \sum_{l \geq 1} \frac{(-1)^{l-1}}{l} \sum_{\substack{\gamma_1 + \cdots + \gamma_l = \gamma \\ \mu_\delta(\gamma_i) = \mu_\delta(\gamma),\ 1 \leq i \leq l}} \mathfrak{d}_\delta(\gamma_1) * \cdots * \mathfrak{d}_\delta(\gamma_l) \tag{2.21}$$

belongs to $\mathsf{SF}^{\mathsf{ind}}_{\mathsf{alg}}(\mathfrak{Ob}(\mathcal{X}))$ and is supported in $\mathfrak{Ob}(\mathcal{X}, \gamma)$. Note that for fixed γ and δ the sum on the right-hand side is finite, and therefore there are no convergence issues in the present case.

Then, for $\gamma \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}$, the δ-ADHM invariant $A_\delta(\gamma)$ is defined by

$$\Psi(\mathfrak{e}_\delta(\gamma)) = -A_\delta(\gamma)\lambda(\gamma). \tag{2.22}$$

Note that $\mathfrak{e}_\delta(\gamma)$ is independent of δ if $v = 0$. Then the corresponding invariants will be denoted by $H(\gamma)$.

By analogy with [14], define the invariants $\overline{A}_\delta(r, e, v)$ by the multicover formula

$$A_\delta(r, e, v) = \sum_{\substack{m \geq 1 \\ m|r,\ m|e,\ m|v}} \frac{1}{m^2} \overline{A}_\delta(r/m, e/m, v/m). \tag{2.23}$$

Conjecturally, $\overline{A}_\delta(r/m, e/m, v/m)$ are integral for noncritical δ. Obviously, the alternative notation $\overline{H}(r, e)$ will be used for $v = 0$.

3. Wallcrossing formulas

3.1. Stack function identities

Let $\gamma = (r, e, v) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 1}$ and let $\delta_{\rm c} > 0$ be a critical stability parameter of type γ. Let $\delta_- < \delta_{\rm c}$, $\delta_+ > \delta_{\rm c}$ be stability parameters as in Lemma 2.8. Recall that $\mathcal{HN}_\pm(r, v)$ denote the set of positive, respectively, negative admissible configurations of type (r, v) introduced in Definition 2.4. For any $h \in \mathbb{Z}_{\geq 2}$ let $\mathcal{HN}_\pm(\gamma, \delta_{\rm c}, h)$ denote the set of ordered sequences of triples $(\gamma_i = (r_i, e_i, v_i) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0})_{1\leq i\leq h}$ so that $(\rho_i = (r_i, v_i))_{1\leq i\leq h} \in \mathcal{HN}_\pm(r, v)$,

$$e_1 + \cdots + e_h = e \qquad \text{and} \qquad \frac{e_i + v_i\delta_{\rm c}}{r_i} = \frac{e + v\delta_{\rm c}}{r} \qquad \text{for all } 1 \leq i \leq h.$$

More generally, given $h \in \mathbb{Z}_{\geq 2}$, for any $0 \leq k \leq h-1$ let $\mathcal{HN}_+(\gamma, \delta_{\rm c}, h, k)$ denote the set of ordered sequences $(\gamma_i = (r_i, e_i, v_i) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0})_{1\leq i\leq h}$ so that

- $\gamma_1 + \cdots + \gamma_h = \gamma$, $v_{h-k+1} = \cdots = v_h = 0$, $v_i > 0$ for $1 \leq i \leq h-k$, and
$$\frac{e_1 + v_1\delta_{\rm c}}{r_1} = \cdots = \frac{e_{h-k} + v_{h-k}\delta_{\rm c}}{r_{h-k}} = \frac{e_{h-k+1}}{r_{h-k+1}} = \cdots = \frac{e_h}{r_h} = \frac{e + v\delta_{\rm c}}{r}.$$
- The sequence $(\rho_j = (r_j, v_j))_{1\leq j\leq h-k}$ belongs to $\mathcal{HN}_+\Big(r - \sum_{i=1}^k r_i, v\Big)$.

Similarly, for any $0 \leq k \leq h-1$ let $\mathcal{HN}_-(\gamma, \delta_{\rm c}, h, k)$ denote the set of ordered sequences $(\gamma_i = (r_i, e_i, v_i) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0})_{1\leq i\leq h}$ so that

- $\gamma_1 + \cdots + \gamma_h = \gamma$, $v_1 = \cdots = v_k = 0$, $v_i > 0$ for $k+1 \leq i \leq h$, and
$$\frac{e_1}{r_1} = \cdots = \frac{e_k}{r_k} = \frac{e_{k+1} + v_{k+1}\delta_{\rm c}}{r_{k+1}} = \cdots = \frac{e_h + v_h\delta_{\rm c}}{r_h} = \frac{e + v\delta_{\rm c}}{r}.$$
- The sequence $(\rho_j = (r_{k+j}, v_{k+j}))_{1\leq j\leq h-k}$ belongs to $\mathcal{HN}_-\Big(r - \sum_{i=1}^k r_i, v\Big)$.

Remark 3.1. (i) Obviously, in both cases $v_i > 0$ for all $1 \leq i \leq h$ if $k = 0$. Moreover,

$$\mathcal{HN}_\pm(\gamma, \delta_{\rm c}, h) = \mathcal{HN}_\pm(\gamma, \delta_{\rm c}, h, 0) \cup \mathcal{HN}_\pm(\gamma, \delta_{\rm c}, h, 1).$$

If $k = h - 1$, the condition that the sequence $(\rho_j)_{1\leq j\leq h-k}$ belong to $\mathcal{HN}_\pm(r - \sum_{i=1}^k r_i, v)$ is empty.

(ii) For fixed γ and $\delta_c > 0$ it straightforward to check whether the following set is finite

$$\bigcup_{h\geq 2}\bigcup_{0\leq k\leq h-1} \mathcal{HN}_\pm(\gamma, \delta_c, h, k)$$

i.e., the set $\mathcal{HN}_\pm(\gamma, \delta_c, h, k)$ is nonempty only for a finite set of pairs (h, k).

For any triple $\gamma' = (r', e', v') \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 1}$ let $\mathfrak{d}_\pm(\gamma'), \mathfrak{d}_c(\gamma')$ be the stack functions determined by the open embeddings $\mathfrak{M}^{ss}_{\delta_\pm}(\mathcal{X}, r', e', v') \hookrightarrow \mathfrak{Ob}(\mathcal{X})$, respectively $\mathfrak{M}^{ss}_{\delta_c}(\mathcal{X}, r', e', v') \hookrightarrow \mathfrak{Ob}(\mathcal{X})$. The alternative notation $\mathfrak{h}(\gamma')$ will be used if $v' = 0$.

Lemma 3.1. *The following relations hold in the stack function algebra* $\underline{\mathrm{SF}}(\mathfrak{Ob}(\mathcal{X}))$*:*

$$\mathfrak{d}_c(\gamma) = \mathfrak{d}_\pm(\gamma) + \sum_{h\geq 2}\sum_{(\gamma_i)\in\mathcal{HN}_\pm(\gamma,\delta_c,h)} \mathfrak{d}_\pm(\gamma_1) * \cdots * \mathfrak{d}_\pm(\gamma_h) \tag{3.1}$$

(3.2)
$$\mathfrak{d}_-(\gamma) + \sum_{h\geq 2}\sum_{(\gamma_i)\in\mathcal{HN}_-(\gamma,\delta_c,h,0)} \mathfrak{d}_-(\gamma_1) * \cdots * \mathfrak{d}_-(\gamma_h)$$
$$= \mathfrak{d}_c(\gamma) + \sum_{h\geq 2}(-1)^{h-1}\sum_{(\gamma_i)\in\mathcal{HN}_-(\gamma,\delta_c,h,h-1)} \mathfrak{h}(\gamma_1) * \cdots * \mathfrak{h}(\gamma_{h-1}) * \mathfrak{d}_c(\gamma_h).$$

Proof. Equation (3.1) follows directly from Lemma 2.8. In order to prove formula (3.2), it will be first proven by induction that the following formula holds for any $l \in \mathbb{Z}_{\geq 1}$:

(3.3)
$$\mathfrak{d}_-(\gamma) + \sum_{h\geq 2}\sum_{(\gamma_i)\in\mathcal{HN}_-(\gamma,\delta_c,h,0)} \mathfrak{d}_-(\gamma_1) * \cdots * \mathfrak{d}_-(\gamma_h)$$
$$= \mathfrak{d}_c(\gamma) + \sum_{k=2}^{l}(-1)^{k-1}\sum_{(\gamma_i)\in\mathcal{HN}_-(\gamma,\delta_c,k,k-1)} \mathfrak{h}(\gamma_1) * \cdots * \mathfrak{h}(\gamma_{k-1}) * \mathfrak{d}_c(\gamma_k)$$
$$+ (-1)^l \sum_{h\geq l+1}\sum_{(\gamma_i)\in\mathcal{HN}_-(\gamma,\delta_c,h,l)} \mathfrak{h}(\gamma_1) * \cdots * \mathfrak{h}(\gamma_l) * \mathfrak{d}_-(\gamma_{l+1}) * \cdots * \mathfrak{d}_-(\gamma_h).$$

First note that Remark 3.1 (ii) implies that all sums in Equation (3.3) are finite for any $l \geq 1$.

Next, if $l = 1$, Equation (3.3) is equivalent to (3.1). Suppose it holds for some $l \geq 1$. Then note that Equation (3.1) is valid for any triple $\gamma = (r, e, v)$ and any stability parameter δ_c. If δ_c is not critical of type γ as assumed above, it reduces to a trivial identity. In particular setting $\gamma = \gamma_{l+1}$ in Equation (3.1) yields

$$\mathfrak{d}_-(\gamma_{l+1}) = \mathfrak{d}_c(\gamma_{l+1}) - \sum_{m\geq 2} \sum_{(\eta_i)\in \mathcal{HN}_-(\gamma_{l+1},\delta_c,m,1)} \mathfrak{h}(\eta_1) * \mathfrak{d}_-(\eta_2) * \cdots * \mathfrak{d}_-(\eta_m)$$
$$- \sum_{m\geq 2} \sum_{(\eta_i)\in \mathcal{HN}_-(\gamma_{l+1},\delta_c,m,0)} \mathfrak{d}_-(\eta_1) * \mathfrak{d}_-(\eta_2) * \cdots * \mathfrak{d}_-(\eta_m).$$

Using this expression, the third term on the right-hand side of Equation (3.3) can be rewritten as follows:

(3.4)
$$(-1)^l \sum_{h\geq l+1} \sum_{(\gamma_i)\in \mathcal{HN}_-(\gamma,\delta_c,h,l)} \mathfrak{h}(\gamma_1) * \cdots * \mathfrak{h}(\gamma_l) * \mathfrak{d}_-(\gamma_{l+1}) * \cdots * \mathfrak{d}_-(\gamma_h)$$
$$= (-1)^l \sum_{(\gamma_i)\in \mathcal{HN}_-(\gamma,\delta_c,l+1,l)} \Big[\mathfrak{h}(\gamma_1) * \cdots * \mathfrak{h}(\gamma_l) * \mathfrak{d}_c(\gamma_{l+1})$$
$$\times \sum_{m\geq 2} \sum_{(\eta_i)\in \mathcal{HN}_-(\gamma_{l+1},\delta_c,m,1)} \mathfrak{h}(\gamma_1) * \cdots * \mathfrak{h}(\gamma_l) * \mathfrak{h}(\eta_1) * \mathfrak{d}_-(\eta_2) * \cdots * \mathfrak{d}_-(\eta_m)$$
$$- \sum_{m\geq 2} \sum_{(\eta_i)\in \mathcal{HN}_-(\gamma_{l+1},\delta_c,m,0)}$$
$$\times \mathfrak{h}(\gamma_1) * \cdots * \mathfrak{h}(\gamma_l) * \mathfrak{d}_-(\eta_1) * \mathfrak{d}_-(\eta_2) * \cdots * \mathfrak{d}_-(\eta_m)\Big]$$
$$+ (-1)^l \sum_{h\geq l+2} \sum_{(\gamma_i)\in \mathcal{HN}_-(\gamma,\delta_c,h,l)} \mathfrak{h}(\gamma_1) * \cdots * \mathfrak{h}(\gamma_l) * \mathfrak{d}_-(\gamma_{l+1}) * \cdots * \mathfrak{d}_-(\gamma_h).$$

By construction

$$\bigcup_{(\gamma_i)\in \mathcal{HN}_-(\gamma,\delta_c,l+1,l)} \mathcal{HN}_-(\gamma_{l+1}, \delta_c, m, j) = \mathcal{HN}_-(\gamma, \delta_c, l+m, l+j)$$

for any $m \in \mathbb{Z}_{\geq 2}$, $j \in \{0,1\}$. Therefore, the last two terms on the right-hand side of Equation (3.4) cancel, and formula (3.4) reduces to

(3.5)
$$\begin{aligned}(-1)^l \sum_{h\geq l+1} \sum_{(\gamma_i)\in\mathcal{HN}_-(\gamma,\delta_c,h,l)} & \mathfrak{h}(\gamma_1) * \cdots * \mathfrak{h}(\gamma_l) * \mathfrak{d}_-(\gamma_{l+1}) * \cdots * \mathfrak{d}_-(\gamma_h) \\ = (-1)^l & \sum_{(\gamma_i)\in\mathcal{HN}_-(\gamma,\delta_c,l+1,l)} \mathfrak{h}(\gamma_1) * \cdots * \mathfrak{h}(\gamma_l) * \mathfrak{d}_c(\gamma_{l+1}) \\ & + (-1)^{l+1} \sum_{h\geq l+2} \sum_{(\gamma_i)\in\mathcal{HN}_-(\gamma,\delta_c,h,l+1)} \\ & \times \mathfrak{h}(\gamma_1) * \cdots * \mathfrak{h}(\gamma_{l+1}) * \mathfrak{d}_-(\gamma_{l+2}) * \cdots * \mathfrak{d}_-(\gamma_h).\end{aligned}$$

Substituting (3.5) in (3.3) it follows that formula (3.3) also holds if l is replaced by $(l+1)$. This concludes the inductive proof of formula (3.3).

In order to conclude the proof of Equation (3.2), it suffices to observe that for sufficiently large l, Equation (3.3) stabilizes to Equation (3.2) using Remark 3.1 (ii). □

Now note that Equations (3.1) and (3.2) yield a recursive algorithm expressing $\mathfrak{d}_-(\gamma)$ in terms of $\mathfrak{d}_+(\gamma_i)$, $1 \leq i \leq h$, $h \geq 1$. This follows observing that on the left-hand side of (3.2) $0 < v_i < v$ for all stack functions $\mathfrak{d}_-(\gamma_i)$ occurring in the sum

$$\sum_{h\geq 2} \sum_{(\gamma_i)\in\mathcal{HN}_-(\gamma,\delta_c,h,0)} \mathfrak{d}_-(\gamma_1) * \cdots * \mathfrak{d}_-(\gamma_h).$$

Therefore, once a formula for the difference $\mathfrak{d}_-(\gamma) - \mathfrak{d}_+(\gamma)$ has been derived for triples of the form $\gamma = (r,e,v)$, one can recursively derive an analogous formula for triples of the form $\gamma = (r,e,v+1)$. For $v = 1$, Equations (3.1) and (3.2) easily imply

(3.6)
$$\mathfrak{d}_-(\gamma) = \mathfrak{d}_+(\gamma) + \sum_{l\geq 2} (-1)^l \sum_{(\gamma_i)\in\mathcal{HN}_-(\gamma,\delta_c,l,l-1)} \mathfrak{h}(\gamma_1) * \cdots * [\mathfrak{d}_+(\gamma_l), \mathfrak{h}(\gamma_{l-1})].$$

Employing the above recursive algorithm, one can determine in principle analogous formulas for $v \geq 2$. Since the resulting expressions quickly become cumbersome, explicit formulas will be given below only for $v = 2$.

Corollary 3.1. *Suppose $\gamma = (r, e, 2)$ with $(r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$. The following relations hold in the stack function algebra $\underline{\mathsf{SF}}(\mathfrak{Ob}(\mathcal{X}))$:*

(3.7)
$$\begin{aligned}\mathfrak{d}_-(\gamma) = \mathfrak{d}_+(\gamma) + \sum_{l\geq 2}(-1)^l \sum_{(\gamma_i)\in \mathcal{HN}_-(\gamma,\delta_c,l,l-1)} \mathfrak{h}(\gamma_1) * \cdots * [\mathfrak{d}_+(\gamma_l), \mathfrak{h}(\gamma_{l-1})] \\ + \sum_{(\gamma_1,\gamma_2)\in \mathcal{HN}_+(\gamma,\delta_c,2,0)} \mathfrak{d}_+(\gamma_1) * \mathfrak{d}_+(\gamma_2) \\ - \sum_{(\gamma_1,\gamma_2)\in \mathcal{HN}_-(\gamma,\delta_c,2,0)} \mathfrak{d}_-(\gamma_1) * \mathfrak{d}_-(\gamma_2) \\ + \sum_{l\geq 2}(-1)^l \sum_{(\gamma_i)\in \mathcal{HN}_-(\gamma,\delta_c,l+1,l-1)} \mathfrak{h}(\gamma_1) * \cdots * \\ \times [\mathfrak{d}_+(\gamma_{l+1}) * \mathfrak{d}_+(\gamma_l), \mathfrak{h}(\gamma_{l-1})]\end{aligned}$$

where $\mathfrak{d}_-(\gamma_1), \mathfrak{d}_-(\gamma_2)$ are given by Equation (3.6).

3.2. Wallcrossing for $v = 2$ invariants

Let $\gamma = (r, e, 2)$, $(r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$, $\delta_c > 0$ a critical stability parameter of type γ, and $\delta_\pm$ two noncritical stability parameters as in Lemma 2.8. The main goal of this section is to convert the stack function relation (3.7) to a wall-crossing formula for generalized Donaldson–Thomas invariants of ADHM sheaves.

As mentioned in the introduction the alternative notation $\alpha = (r, e)$ will be used for pairs $(r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$. Using this notation, the sets $\mathcal{HN}_-(\alpha, v, \delta_c, h, k)$, $v \in \{1, 2\}$, $k \in \{0, h-2, h-1\}$ can be identified with sets of ordered sequences $(\alpha_i)_{1\leq i\leq h}$ satisfying the conditions listed above Theorem 1.1. For convenience, recall that $\mathcal{HN}_-(\alpha, v, \delta_c, l, l-1)$, $l \in \mathbb{Z}_{\geq 1}$, $v \in \{1, 2\}$, denotes the set of ordered sequences $((\alpha_i))_{1\leq i\leq l}$, $\alpha_i \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$, $1 \leq i \leq l$ so that

$$\alpha_1 + \cdots + \alpha_l = \alpha \tag{3.8}$$

and

$$\frac{e_1}{r_1} = \cdots = \frac{e_{l-1}}{r_{l-1}} = \frac{e_l + v\delta_c}{r_l} = \frac{e + v\delta_c}{r}. \tag{3.9}$$

Similarly, $\mathcal{HN}_-(\alpha, v, \delta_c, l, l-2)$, $l \in \mathbb{Z}_{\geq 2}$, denotes the set of ordered sequences $((\alpha_i))_{1\leq i\leq l}$, $\alpha_i \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$, $1 \leq i \leq l$ satisfying condition (1.1),

$$\frac{e_1}{r_1} = \cdots = \frac{e_{l-2}}{r_{l-2}} = \frac{e_{l-1}+\delta_c}{r_{l-1}} = \frac{e_l+\delta_c}{r_l} = \frac{e+2\delta_c}{r} \tag{3.10}$$

and $1/r_{l-1} < 1/r_l$.

Note that the sets $\mathcal{HN}_-(\alpha, 2, \delta_c, h, 0)$ are nonempty if and only if $h = 2$, in which case they consist of ordered pairs (α_1, α_2) so that $\alpha_1 + \alpha_2 = \alpha$, $1/r_1 < 1/r_2$, and

$$\frac{e_1+\delta_c}{r_1} = \frac{e_2+\delta_c}{r_2} = \frac{e+2\delta_c}{r}$$

Moreover, the set $\mathcal{HN}_-(\alpha, 2, \delta_c, 1, 0)$ consists of only the element (α).

It is straightforward to check that for fixed $\alpha = (r, e)$ and δ_c, the union

$$\begin{aligned} &\bigcup_{l\geq 1} [\mathcal{HN}_-(\alpha, 2, \delta_c, l, l-1) \cup \mathcal{HN}_-(\alpha, 2, \delta_c, l+1, l-1)] \\ &\bigcup_{(\alpha_1,\alpha_2)\in\mathcal{HN}_-(\alpha,2,\delta_c,2,0)} \bigcup_{l_1\geq 1} \bigcup_{l_2\geq 1} [\mathcal{HN}_-(\alpha_1, 1, \delta_c, l_1, l_1-1) \\ &\qquad\qquad \times \mathcal{HN}_-(\alpha_2, 1, \delta_c, l_2, l_2-1)] \end{aligned} \tag{3.11}$$

is a finite set.

Now let $0 < \delta_- < \delta_c < \delta_+$ be stability parameters so that there are no critical stability parameters of type $(\alpha, 2)$ in the intervals $[\delta_-,\ \delta_c)$, $(\delta_c,\ \delta_+]$. Since the set (3.11) is finite δ_-, δ_+ can be chosen so that the same holds for all numerical types (α_i, v_i) in all ordered sequences in (3.11). Then the following lemma holds.

Lemma 3.2. *The following relations hold in the stack function algebra* $\mathsf{SF}(\mathfrak{Ob}(\mathcal{X}))$*:*

$$\begin{aligned} &\mathfrak{d}_-(\alpha, 1) \\ &\quad = \sum_{l\geq 1} \frac{(-1)^{l-1}}{(l-1)!} \sum_{(\alpha_i)\in\mathcal{HN}_-(\alpha,1,\delta_c,l,l-1)} [\mathfrak{g}(\alpha_1), \ldots, [\mathfrak{g}(\alpha_{l-1}), \mathfrak{d}_+(\alpha_l, 1)] \cdots] \end{aligned} \tag{3.12}$$

(3.13)

$$
\begin{aligned}
&\mathfrak{d}_-(\alpha,2)\\
&\quad=\sum_{l\geq 1}\frac{(-1)^{l-1}}{(l-1)!}\sum_{(\alpha_i)\in\mathcal{HN}_-(\alpha,2,\delta_c,l,l-1)}[\mathfrak{g}(\alpha_1),\ldots,[\mathfrak{g}(\alpha_{l-1}),\mathfrak{d}_+(\alpha_l,2)]\cdots]\\
&\qquad+\sum_{l\geq 1}\frac{(-1)^{l-1}}{(l-1)!}\sum_{(\alpha_i)\in\mathcal{HN}_-(\alpha,2,\delta_c,l+1,l-1)}\\
&\qquad\times[\mathfrak{g}(\alpha_1),\ldots,[\mathfrak{g}(\alpha_{l-1}),\mathfrak{d}_+(\alpha_{l+1},1)*\mathfrak{d}_+(\alpha_l,1)]\cdots]\\
&\qquad-\sum_{(\alpha_1,\alpha_2)\in\mathcal{HN}_-(\alpha,2,\delta_c,2,0)}\sum_{l_1\geq 1}\sum_{l_2\geq 1}\frac{(-1)^{l_1-1}}{(l_1-1)!}\frac{(-1)^{l_2-1}}{(l_2-1)!}\\
&\qquad\times\sum_{(\alpha_{1,i})\in\mathcal{HN}_-(\alpha_1,1,\delta_c,l_1,l_1-1)}\sum_{(\alpha_{2,i})\in\mathcal{HN}_-(\alpha_2,1,\delta_c,l_2,l_2-1)}\\
&\qquad\times\big([\mathfrak{g}(\alpha_{1,1}),\ldots,[\mathfrak{g}(\alpha_{1,l_1-1}),\mathfrak{d}_+(\alpha_{1,l_1},1)]\cdots]\\
&\qquad\quad*[\mathfrak{g}(\alpha_{2,1}),\ldots,[\mathfrak{g}(\alpha_{2,l_2-1}),\mathfrak{d}_+(\alpha_{2,l_2},1)]\cdots]\big)
\end{aligned}
$$

where for any $\beta\in\mathbb{Z}_{\geq 1}\times\mathbb{Z}$, $\mathfrak{g}(\beta)$ *denotes the log stack function* $\mathfrak{e}_\delta(\beta,0)$ *defined in Equation* (2.21) *(which is independent of* δ*). For any* $k\geq 1$*, and any collection of stack functions* $(\mathfrak{f}_1,\ldots\mathfrak{f}_k)$, $[\mathfrak{f}_1,\ldots,[\mathfrak{f}_{k-1},\mathfrak{f}_k]\cdots]$ *stands for the successive commutator* $[\mathfrak{f}_1,\ldots,[\mathfrak{f}_{k-1},\mathfrak{f}_k\underbrace{]\cdots]}_{k-1}$.

Proof. Formulas (3.12) and (3.13) follow from Equations (3.7) and (3.6) by repeating the computations in the proof of [2, Lemm. 2.6] in the present context. □

Proof of Theorem 1.1. The proof consists of two steps. First the stack function identities (3.12) and (3.13) must be converted into similar identities for log stack functions in the Lie stack function algebra $\mathsf{SF}^{\mathsf{ind}}_{\mathsf{alg}}(\mathcal{C}_{\mathcal{X}})$. Then wall-crossing formulas for generalized Donaldson–Thomas invariants are derived applying the Lie algebra morphism (2.20) to the resulting log stack function identities.

It will be proved below that Equation (3.15) yields the following log stack function relation:

(3.14)

$$
\begin{aligned}
&\mathfrak{e}_-(\alpha,2)\\
&\quad=\sum_{l\geq 1}\frac{(-1)^{l-1}}{(l-1)!}\sum_{(\alpha_i)\in\mathcal{HN}_-(\alpha,2,\delta_c,l,l-1)}[\mathfrak{g}(\alpha_1),\ldots,[\mathfrak{g}(\alpha_{l-1}),\mathfrak{e}_+(\alpha_l,2)]\cdots]\\
&\qquad+\frac{1}{2}\sum_{l\geq 1}\frac{(-1)^{l-1}}{(l-1)!}\sum_{(\alpha_i)\in\mathcal{HN}_-(\alpha,2,\delta_c,l+1,l-1)}\\
&\qquad\times[\mathfrak{g}(\alpha_1),\ldots,[\mathfrak{g}(\alpha_{l-1}),[\mathfrak{e}_+(\alpha_{l+1},1),\mathfrak{e}_+(\alpha_l,1)]\cdots]\\
&\qquad-\frac{1}{2}\sum_{(\alpha_1,\alpha_2)\in\mathcal{HN}_-(\alpha,2,\delta_c,2,0)}\sum_{l_1\geq 1}\sum_{l_2\geq 1}\frac{(-1)^{l_1-1}}{(l_1-1)!}\frac{(-1)^{l_2-1}}{(l_2-1)!}\\
&\qquad\times\sum_{(\alpha_{1,i})\in\mathcal{HN}_-(\alpha_1,1,\delta_c,l_1,l_1-1)}\ \sum_{(\alpha_{2,i})\in\mathcal{HN}_-(\alpha_2,1,\delta_c,l_2,l_2-1)}\\
&\qquad\times\Big[\ [\mathfrak{g}(\alpha_{1,1}),\ldots,[\mathfrak{g}(\alpha_{1,l_1-1}),\mathfrak{e}_+(\alpha_{1,l_1},1)]\cdots],\\
&\qquad[\mathfrak{g}(\alpha_{2,1}),\ldots,[\mathfrak{g}(\alpha_{2,l_2-1}),\mathfrak{e}_+(\alpha_{2,l_2},1)]\cdots]\ \Big].
\end{aligned}
$$

Since the right-hand side of Equation (3.14) is written as a linear combination of successive commutators, the wallcrossing formula for the invariants $A_\pm(\alpha,l)$ follows by applying the Lie algebra morphism (2.20). Given that

$$
\begin{aligned}
[\lambda((r_1,e_1,0)),\lambda((r_2,e_2,v))] &= (-1)^{v(e_1-r_1(g-1))}v\lambda(r_1+r_2,e_1+e_2,v)\\
&= f_v(r_1,e_1)\lambda(r_1+r_2,e_1+e_2,v)\\
[\lambda((r_1,e_1,1)),\lambda((r_2,e_2,1))] &= (-1)^{e_1-e_2-(r_1-r_2)(g-1)}\lambda(r_1+r_2,e_1+e_2,2)\\
&= g((r_1,e_1),(r_2,e_2))\lambda(r_1+r_2,e_1+e_2,2)
\end{aligned}
$$

for any $(r_1,e_1),(r_2,e_2)\in\mathbb{Z}_{\geq 1}\times\mathbb{Z}$, $v=1,2$, a straightforward computation yields Equation (1.5).

The proof of identity (3.14) is presented below. Using Equation (2.21), formula (3.13) yields

(3.15)
$$\begin{aligned}
\mathfrak{e}_-(\alpha,2) &= \sum_{l\geq 1}\frac{(-1)^{l-1}}{(l-1)!}\sum_{(\alpha_i)\in\mathcal{HN}_-(\alpha,2,\delta_c,l,l-1)}[\mathfrak{g}(\alpha_1),\ldots,[\mathfrak{g}(\alpha_{l-1}),\mathfrak{e}_+(\alpha_l,2)]\cdots]\\
&\quad+\sum_{l\geq 1}\frac{(-1)^{l-1}}{(l-1)!}(F(l,\alpha)+\frac{1}{2}F'(l,\alpha))
\end{aligned}$$

where

(3.16)
$$\begin{aligned}
F(l,\alpha) &= \sum_{(\alpha_i)\in\mathcal{HN}_-(\alpha,2,\delta_c,l+1,l-1)}[\mathfrak{g}(\alpha_1),\ldots,[\mathfrak{g}(\alpha_{l-1}),\mathfrak{e}_+(\alpha_{l+1},1)*\mathfrak{e}_+(\alpha_l,1)]\cdots]\\
&\quad-\sum_{(\alpha_1,\alpha_2)\in\mathcal{HN}_-(\alpha,2,\delta_c,2,0)}\ \sum_{\substack{l_1,l_2\geq 1\\ l_1+l_2=l+1}}(l-1,l_1-1)\sum_{(\alpha_{1,i})\in\mathcal{HN}_-(\alpha_1,1,\delta_c,l_1,l_1-1)}\\
&\quad\times\sum_{(\alpha_{2,i})\in\mathcal{HN}_-(\alpha_2,1,\delta_c,l_2,l_2-1)}\big([\mathfrak{g}(\alpha_{1,1}),\ldots,[\mathfrak{g}(\alpha_{1,l_1-1}),\mathfrak{e}_+(\alpha_{1,l_1},1)]\cdots]\\
&\qquad*[\mathfrak{g}(\alpha_{2,1}),\ldots,[\mathfrak{g}(\alpha_{2,l_2-1}),\mathfrak{e}_+(\alpha_{2,l_2},1)]\cdots]\big).
\end{aligned}$$

(3.17)
$$\begin{aligned}
F'(l,\alpha) &= \sum_{(\alpha_i)\in\mathcal{HN}_-(\alpha,2,\delta_c,l,l-1)}[\mathfrak{g}(\alpha_1),\ldots,[\mathfrak{g}(\alpha_{l-1}),\mathfrak{e}_+(\alpha_l/2,1)*\mathfrak{e}_+(\alpha_l/2,1)]\cdots]\\
&\quad-\sum_{\substack{l_1,l_2\geq 1\\ l_1+l_2=l+1}}(l-1,l_1-1)\sum_{(\alpha_{1,i})\in\mathcal{HN}_-(\alpha/2,1,\delta_c,l_1,l_1-1)}\\
&\quad\times\sum_{(\alpha_{2,i})\in\mathcal{HN}_-(\alpha/2,1,\delta_c,l_2,l_2-1)}[\mathfrak{g}(\alpha_{1,1}),\ldots,[\mathfrak{g}(\alpha_{1,l_1-1}),\mathfrak{e}_+(\alpha_{1,l_1},1)]\cdots]\\
&\qquad*[\mathfrak{g}(\alpha_{2,1}),\ldots,[\mathfrak{g}(\alpha_{2,l_2-1}),\mathfrak{e}_+(\alpha_{2,l_2},1)]\cdots]
\end{aligned}$$

where by convention $\epsilon_+(\beta/2,1)=0$ and $\mathcal{HN}_-(\beta/2,1,\delta_c,k,k-1)=\emptyset$ if $\beta\in\mathbb{Z}_{\geq 1}\times\mathbb{Z}$ is not divisible by 2. In the above equations, (n,k) denotes the binomial coefficient $n!/(k!(n-k)!)$ for any $n,k\in\mathbb{Z}_{\geq 0}$, $n\geq k$. In the following, it will be convenient to formally define (n,k) for any $n\in\mathbb{Z}_{\geq 0}$, $k\in\mathbb{Z}$ adopting the convention that $(n,k)=0$ whenever $k<0$ or $k>n$.

Recall that $\mathcal{HN}_-(\alpha, 2, \delta_c, l, l-2)$, $l \in \mathbb{Z}_{\geq 2}$ is the set of all ordered sequences $(\alpha_i)_{1\leq i\leq l}$, $\alpha_i \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$ so that

$$\alpha_1 + \cdots + \alpha_l = \alpha \tag{3.18}$$

$$\frac{e_1}{r_1} = \cdots = \frac{e_{l-2}}{r_{l-2}} = \frac{e_{l-1}+\delta_c}{r_{l-1}} = \frac{e_l+\delta_c}{r_l} = \frac{e+2\delta_c}{r} \tag{3.19}$$

$$r_{l-1} > r_l. \tag{3.20}$$

Let $\mathcal{HN}'_-(\alpha, 2, \delta_c, l, l-2)$, $\mathcal{HN}''_-(\alpha, 2, \delta_c, l, l-2)$ denote the set of all ordered sequences $(\alpha_i)_{1\leq i\leq l}$ satisfying conditions (3.18) and (3.19), condition (3.20) being replaced by $r_{l-1} = r_l$, respectively, $r_{l-1} < r_l$. Note that the union

$$\begin{aligned}\mathcal{S}(\alpha, \delta_c, l) = \mathcal{HN}_-(\alpha, 2, \delta_c, l, l-2) \cup \mathcal{HN}'_-(\alpha, 2, \delta_c, l, l-2)\\ \cup \mathcal{HN}''_-(\alpha, 2, \delta_c, l, l-2)\end{aligned} \tag{3.21}$$

is the set of all ordered sequences $(\alpha_i)_{1\leq i\leq l}$ satisfying only conditions (3.18) and (3.19). Moreover, condition $r_{l-1} = r_l$ imposed simultaneously with (3.18) and (3.19) implies $\alpha_l = \alpha_{l+1}$. This shows that $F'(l, \alpha)$ can be rewritten as

$$\begin{aligned}&F'(l,\alpha)\\ &= \sum_{(\alpha_i)\in\mathcal{HN}'_-(\alpha,2,\delta_c,l+1,l-1)} [\mathfrak{g}(\alpha_1), \ldots, [\mathfrak{g}(\alpha_{l-1}), \mathfrak{e}_+(\alpha_l, 1) * \mathfrak{e}_+(\alpha_{l+1}, 1)] \cdots]\\ &\quad - \sum_{\substack{l_1,l_2\geq 1\\ l_1+l_2=l+1}} (l-1, l_1-1) \sum_{(\alpha_{1,i})\in\mathcal{HN}_-(\alpha/2,1,\delta_c,l_1,l_1-1)}\\ &\quad \times \sum_{(\alpha_{2,i})\in\mathcal{HN}_-(\alpha/2,1,\delta_c,l_2,l_2-1)} [\mathfrak{g}(\alpha_{1,1}), \ldots, [\mathfrak{g}(\alpha_{1,l_1-1}), \mathfrak{e}_+(\alpha_{1,l_1}, 1)] \cdots]\\ &\quad * [\mathfrak{g}(\alpha_{2,1}), \ldots, [\mathfrak{g}(\alpha_{2,l_2-1}), \mathfrak{e}_+(\alpha_{2,l_2}, 1)] \cdots].\end{aligned} \tag{3.22}$$

Define also

$$\begin{aligned}&F''(l,\alpha)\\ &= \sum_{(\alpha_i)\in\mathcal{HN}''_-(\alpha,2,\delta_c,l+1,l-1)} [\mathfrak{g}(\alpha_1), \ldots, [\mathfrak{g}(\alpha_{l-1}), \mathfrak{e}_+(\alpha_{l+1}, 1) * \mathfrak{e}_+(\alpha_l, 1)] \cdots]\end{aligned} \tag{3.23}$$

$$- \sum_{(\alpha_1,\alpha_2)\in\mathcal{HN}''_{-}(\alpha,2,\delta_c,2,0)} \sum_{\substack{l_1,l_2\geq 1\\ l_1+l_2=l+1}} (l-1,l_1-1) \sum_{(\alpha_{1,i})\in\mathcal{HN}_{-}(\alpha_1,1,\delta_c,l_1,l_1-1)}$$
$$\times \sum_{(\alpha_{2,i})\in\mathcal{HN}_{-}(\alpha_2,1,\delta_c,l_2,l_2-1)} \big([\mathfrak{g}(\alpha_{1,1}),\ldots,[\mathfrak{g}(\alpha_{1,l_1-1}),\mathfrak{e}_+(\alpha_{1,l_1},1)]\cdots] * [\mathfrak{g}(\alpha_{2,1}),\ldots,[\mathfrak{g}(\alpha_{2,l_2-1}),\mathfrak{e}_+(\alpha_{2,l_2},1)]\cdots]\big).$$

Next note that the right-hand side of Equation (3.14) can then be written as

(3.24)
$$\sum_{l\geq 1} \frac{(-1)^{l-1}}{(l-1)!}\Bigg(\sum_{(\alpha_i)\in\mathcal{HN}_{-}(\alpha,2,\delta_c,l,l-1)} [\mathfrak{g}(\alpha_1),\ldots,[\mathfrak{g}(\alpha_{l-1}),\mathfrak{e}_+(\alpha_l,2)]\cdots] + \frac{1}{2}(F(l,\alpha)-F''(l,\alpha))\Bigg)$$

Comparing Equations (3.15) and (3.24) it follows that in order to prove formula (3.14) it suffices to prove the identity

$$F(l,\alpha)+F'(l,\alpha)+F''(l,\alpha)=0 \tag{3.25}$$

This will be done below by induction on $l\in\mathbb{Z}_{\geq 1}$. Let $S(l,\alpha)=F(l,\alpha)+F'(l,\alpha)+F''(l,\alpha)$. Note that Equations (3.16), (3.22) and (3.23) imply

(3.26)
$$S(l,\alpha) = \sum_{(\alpha_i)\in\mathcal{S}(\alpha,2,\delta_c,l+1,l-1)} [\mathfrak{g}(\alpha_1),\ldots,[\mathfrak{g}(\alpha_{l-1}),\mathfrak{e}_+(\alpha_{l+1},1)*\mathfrak{e}_+(\alpha_l,1)]\cdots]$$
$$- \sum_{(\alpha_1,\alpha_2)\in\mathcal{S}(\alpha,2,\delta_c,2,0)} \sum_{\substack{l_1,l_2\geq 1\\ l_1+l_2=l+1}} (l-1,l_1-1) \sum_{(\alpha_{1,i})\in\mathcal{HN}_{-}(\alpha_1,1,\delta_c,l_1,l_1-1)}$$
$$\times \sum_{(\alpha_{2,i})\in\mathcal{HN}_{-}(\alpha_2,1,\delta_c,l_2,l_2-1)} \big([\mathfrak{g}(\alpha_{1,1}),\ldots,[\mathfrak{g}(\alpha_{1,l_1-1}),\mathfrak{e}_+(\alpha_{1,l_1},1)]\cdots] * [\mathfrak{g}(\alpha_{2,1}),\ldots,[\mathfrak{g}(\alpha_{2,l_2-1}),\mathfrak{e}_+(\alpha_{2,l_2},1)]\cdots]\big)$$

where $\mathcal{S}(\alpha,2,\delta_c,l+1,l-1)$ is the set union (3.21).

For $l=1$ Equation (3.25) reduces trivially to $0=0$ using formula (3.26). Suppose Equation (3.25) holds for some $l\in\mathbb{Z}_{\geq 2}$, for all $\alpha\in\mathbb{Z}_{\geq 1}\times\mathbb{Z}$. Then, using the identity

$$(l,l_1-1)=(l-1,l_1-1)+(l-1,l_1-2),$$

it follows that

(3.27)

$$S(l+1,\alpha) = \sum_{(\alpha_i)\in\mathcal{S}(\alpha,2,\delta_c,l+2,l)} [\mathfrak{g}(\alpha_1),\ldots,[\mathfrak{g}(\alpha_l),\mathfrak{e}_+(\alpha_{l+2},1)*\mathfrak{e}_+(\alpha_{l+1},1)]\cdots] + G(l+1,\alpha)$$

where

$$\begin{aligned} &G(l+1,\alpha)\\ &= -\sum_{(\alpha_1,\alpha_2)\in\mathcal{S}(\alpha,2,\delta_c,2,0)} \sum_{\substack{l_1,l_2\geq 1\\ l_1+l_2=l+2}} (l-1,l_1-1) \sum_{(\alpha_{1,i})\in\mathcal{HN}_-(\alpha_1,1,\delta_c,l_1,l_1-1)}\\ &\quad\times \sum_{(\alpha_{2,i})\in\mathcal{HN}_-(\alpha_2,1,\delta_c,l_2,l_2-1)} [\mathfrak{g}(\alpha_{1,1}),\ldots,[\mathfrak{g}(\alpha_{1,l_1-1}),\mathfrak{e}_+(\alpha_{1,l_1},1)]\cdots]\\ &\quad * [\mathfrak{g}(\alpha_{2,1}),\ldots,[\mathfrak{g}(\alpha_{2,l_2-1}),\mathfrak{e}_+(\alpha_{2,l_2},1)]\cdots]\\ &\quad - \sum_{(\alpha_1,\alpha_2)\in\mathcal{S}(\alpha,2,\delta_c,2,0)} \sum_{\substack{l_1,l_2\geq 1\\ l_1+l_2=l+2}} (l-1,l_1-2) \sum_{(\alpha_{1,i})\in\mathcal{HN}_-(\alpha_1,1,\delta_c,l_1,l_1-1)}\\ &\quad\times \sum_{(\alpha_{2,i})\in\mathcal{HN}_-(\alpha_2,1,\delta_c,l_2,l_2-1)} [\mathfrak{g}(\alpha_{1,1}),\ldots,[\mathfrak{g}(\alpha_{1,l_1-1}),\mathfrak{e}_+(\alpha_{1,l_1},1)]\cdots]\\ &\quad * [\mathfrak{g}(\alpha_{2,1}),\ldots,[\mathfrak{g}(\alpha_{2,l_2-1}),\mathfrak{e}_+(\alpha_{2,l_2},1)]\cdots]. \end{aligned}$$

In order to keep the formulas short, set

$$[\alpha_1,\ldots,\alpha_k] = [\mathfrak{g}(\alpha_1),\ldots,[\mathfrak{g}(\alpha_{k-1}),\mathfrak{e}_+(\alpha_k,1)]\cdots]$$

for any $k\geq 1$, and any (α_i), $1\leq i\leq k$. Then by changing the summation variable $l_1\to l_1-1$ in the second multiple sum in the above expression, $G(l+1,\alpha)$ can be written as follows:

$$\begin{aligned} &G(l+1,\alpha)\\ &= -\sum_{(\alpha_1,\alpha_2)\in\mathcal{S}(\alpha,2,\delta_c,2,0)} \sum_{l_1=1}^{l} (l-1,l_1-1) \sum_{(\alpha_{1,i})\in\mathcal{HN}_-(\alpha_1,1,\delta_c,l_1,l_1-1)}\\ &\quad\times \sum_{(\alpha_{2,i})\in\mathcal{HN}_-(\alpha_2,1,\delta_c,l-l_1+2,l-l_1+1)} [\alpha_{1,1},\ldots,\alpha_{1,l_1}]*[\alpha_{2,1},\ldots,\alpha_{2,l-l_1+2}] \end{aligned}$$

$$- \sum_{(\alpha_1,\alpha_2)\in\mathcal{S}(\alpha,2,\delta_c,2,0)} \sum_{l_1=1}^{l} (l-1,l_1-1) \sum_{(\alpha_{1,i})\in\mathcal{HN}_-(\alpha_1,1,\delta_c,l_1+1,l_1)}$$
$$\times \sum_{(\alpha_{2,i})\in\mathcal{HN}_-(\alpha_2,1,\delta_c,l-l_1+1,l-l_1)} [\alpha_{1,1},\ldots,\alpha_{1,l_1+1}] * [\alpha_{2,1},\ldots,\alpha_{2,l-l_1+1}].$$

Next note that

$$\begin{aligned}
&[\alpha_{1,1},\ldots,\alpha_{1,l_1}] * [\alpha_{2,1},\ldots,\alpha_{2,l-l_1+2}] \\
&\quad = [\alpha_{1,1},\ldots,\alpha_{1,l_1}] * \mathfrak{g}(\alpha_{2,1}) * [\alpha_{2,2},\ldots,\alpha_{2,l-l_1+2}] \\
&\qquad - [\alpha_{1,1},\ldots,\alpha_{1,l_1}] * [\alpha_{2,2},\ldots,\alpha_{2,l-l_1+2}] * \mathfrak{g}(\alpha_{2,1}) \\
&[\alpha_{1,1},\ldots,\alpha_{1,l_1+1}] * [\alpha_{2,1},\ldots,\alpha_{2,l-l_1+1}] \\
&\quad = \mathfrak{g}(\alpha_{1,1}) * [\alpha_{1,2},\ldots,\alpha_{1,l_1+1}] * [\alpha_{2,1},\ldots,\alpha_{2,l-l_1+1}] \\
&\qquad - [\alpha_{1,2},\ldots,\alpha_{1,l_1+1}] * \mathfrak{g}(\alpha_{1,1}) * [\alpha_{2,1},\ldots,\alpha_{2,l-l_1+1}].
\end{aligned}$$

The terms of the form $[\ \ ] * \mathfrak{g} * [\ \ ]$ cancel pairwise when summing over all ordered pairs (α_1,α_2) and all ordered sequences $(\alpha_{1,i})$, $(\alpha_{2,i})$ satisfying the summation conditions:

$$\alpha_1+\alpha_2=\alpha, \qquad \sum_{i=1}\alpha_{1,i}=\alpha_1, \qquad \sum_{i=1}\alpha_{2,i}=\alpha_2$$
$$\frac{e_{1,i}}{r_{1,i}} = \frac{e_{2,j}}{r_{2,j}} = \frac{e_1+\delta_c}{r_1} = \frac{e_2+\delta_c}{r_2} = \frac{e+2\delta-s}{r}$$

for all i,j in the appropriate range. For any $\beta=(r,e)\in\mathbb{Z}_{\geq 1}\times\mathbb{Z}$, $v\in\mathbb{Z}_{\geq 1}$, and $\delta\in\mathbb{R}$ set $\mu(\beta)=e/r$, $\mu_\delta(\beta,v)=(e+v\delta)/r$. Then, by a simple change of summation variables, the sum over the terms of the form $[\ \ ]*[\ \ ]*\mathfrak{g}$, $\mathfrak{g}*[\ \ ]*[\ \ ]$ reduces to

$$\begin{aligned}
&G(l+1,\alpha) \\
&\quad = \sum_{\substack{\alpha',\beta\in\mathbb{Z}_{\geq 1}\times\mathbb{Z} \\ \alpha'+\beta=\alpha,\ \mu_{\delta_c}(\alpha',2)=\mu(\beta)}} \sum_{(\alpha_1,\alpha_2)\in\mathcal{S}(\alpha',2,\delta_c,2,0)} \sum_{\substack{l_1,l_2\geq 1 \\ l_1+l_2=l+1}} (l-1,l_1-1) \\
&\qquad \times \sum_{(\alpha_{1,i})\in\mathcal{HN}_-(\alpha_1,1,\delta_c,l_1,l_1-1)} \sum_{(\alpha_{2,i})\in\mathcal{HN}_-(\alpha_2,1,\delta_c,l_2,l_2-1)} \\
&\qquad \times [\mathfrak{g}(\alpha_{1,1}),\ldots,[\mathfrak{g}(\alpha_{1,l_1-1}),\mathfrak{e}_+(\alpha_{1,l_1},1)]\cdots] \\
&\qquad * [\mathfrak{g}(\alpha_{2,1}),\ldots,[\mathfrak{g}(\alpha_{2,l_2-1}),\mathfrak{e}_+(\alpha_{2,l_2},1)]\cdots] * \mathfrak{g}(\beta)
\end{aligned}$$

$$- \sum_{\substack{\alpha',\beta\in\mathbb{Z}_{\geq 1}\times\mathbb{Z} \\ \alpha'+\beta=\alpha,\ \mu_{\delta_c}(\alpha',2)=\mu(\beta)}} \sum_{(\alpha_1,\alpha_2)\in\mathcal{S}(\alpha',2,\delta_c,2,0)} \sum_{\substack{l_1,l_2\geq 1\\ l_1+l_2=l+1}} (l-1,l_1-1)$$
$$\times \sum_{(\alpha_{1,i})\in\mathcal{HN}_-(\alpha_1,1,\delta_c,l_1,l_1-1)} \sum_{(\alpha_{2,i})\in\mathcal{HN}_-(\alpha_2,1,\delta_c,l_2,l_2-1)}$$
$$\times\, \mathfrak{g}(\beta) * [\mathfrak{g}(\alpha_{1,1}),\ldots,[\mathfrak{g}(\alpha_{1,l_1-1}),\mathfrak{e}_+(\alpha_{1,l_1},1)]\cdots]$$
$$* [\mathfrak{g}(\alpha_{2,1}),\ldots,[\mathfrak{g}(\alpha_{2,l_2-1}),\mathfrak{e}_+(\alpha_{2,l_2},1)]\cdots].$$

A similar change of variables in the first sum on the right-hand side of Equation (3.27) yields

$$\sum_{(\alpha_i)\in\mathcal{S}(\alpha,2,\delta_c,l+2,l)} [\mathfrak{g}(\alpha_1),\ldots,[\mathfrak{g}(\alpha_l),\mathfrak{e}_+(\alpha_{l+2},1)*\mathfrak{e}_+(\alpha_{l+1},1)]\cdots]$$
$$= \sum_{\substack{\alpha',\beta\in\mathbb{Z}_{\geq 1}\times\mathbb{Z} \\ \alpha'+\beta=\alpha,\ \mu_{\delta_c}(\alpha',2)=\mu(\beta)}} \sum_{(\alpha_i)\in\mathcal{S}(\alpha',2,\delta_c,l+1,l-1)}$$
$$\times [\mathfrak{g}(\beta),[\mathfrak{g}(\alpha_1),\ldots,[\mathfrak{g}(\alpha_l),\mathfrak{e}_+(\alpha_{l+1},1)*\mathfrak{e}_+(\alpha_l,1)]\cdots].$$

Collecting the all terms, it follows that

$$S(l+1,\alpha) = \sum_{\substack{\alpha',\beta\in\mathbb{Z}_{\geq 1}\times\mathbb{Z} \\ \alpha'+\beta=\alpha,\ \mu_{\delta_c}(\alpha',2)=\mu(\beta)}} [\mathfrak{g}(\beta),S(l,\alpha)].$$

Obviously, the inductive hypothesis now implies that $S(l+1,\alpha)=0$. □

4. Comparison with Kontsevich–Soibelman formula

The goal of this section is to prove that formula (1.5) is in agreement with the wallcrossing formula of Kontsevich and Soibelman [15], which will be referred to as the KS formula in the following.

As in Section 3.2, numerical types of ADHM sheaves will be denoted by $\gamma=(\alpha,v)$, $\alpha=(r,e)\in\mathbb{Z}_{\geq 1}\times\mathbb{Z}$, $v\in\mathbb{Z}_{\geq 0}$. In order to streamline the computations, let $\mathsf{L}(\mathcal{X})_{\leq 2}$ denote the truncation of the Lie algebra $\mathsf{L}(\mathcal{X})$ defined by

$$(4.1)\quad [\lambda(\alpha_1,v_1),\lambda(\alpha_2,v_2)]_{\leq 2} = \begin{cases} [\lambda(\alpha_1,v_1),\lambda(\alpha_2,v_2)] & \text{if } v_1+v_2\leq 2 \\ 0 & \text{otherwise.}\end{cases}$$

Furthermore, it will be more convenient to use the alternative notation $\mathsf{e}_\alpha = \lambda(\alpha,0)$, $\mathsf{f}_\alpha=\lambda(\alpha,1)$, and $\mathsf{g}_\alpha=\lambda(\alpha,2)$.

Given a critical stability parameter $\delta_{\rm c}$ of type $(r,e,2)$, $(r,e)\in \mathbb{Z}_{\geq 1}\times \mathbb{Z}$, there exist two pairs $\alpha=(r_\alpha,e_\alpha)$ and $\beta=(r_\beta,e_\beta)$ with

$$\frac{e_\alpha+\delta_{\rm c}}{r_\alpha}=\frac{e_\beta}{r_\beta}=\mu_{\delta_{\rm c}}(\gamma)$$

so that any $\eta\in \mathbb{Z}_{\geq 1}\times\mathbb{Z}\times\mathbb{Z}_{\geq 0}$ with $\mu_{\delta_{\rm c}}(\eta)=\mu_{\delta_{\rm c}}(\gamma)$ can be uniquely written as $\eta=(q\beta,0),(\alpha+q\beta,1)$, or $(2\alpha+q\beta,2)$, with $q\in\mathbb{Z}_{\geq 0}$.

For any $q\in\mathbb{Z}_{\geq 0}$ the following formal expressions will be needed in the KS formula:

$$\begin{aligned} &U_{\alpha+q\beta}=\exp\left(\mathsf{f}_{\alpha+q\beta}+\frac{1}{4}\mathsf{g}_{2\alpha+2q\beta}\right), \quad U_{2\alpha+q\beta}=\exp(\mathsf{g}_{2\alpha+q\beta}),\\ &U_{q\beta}=\exp\left(\sum_{m\geq 1}\frac{\mathsf{e}_{mq\beta}}{m^2}\right). \end{aligned} \tag{4.2}$$

Moreover, let

$$\mathbb{H}=\sum_{q\geq 0}H(q\beta)\mathsf{e}_{q\beta},$$

where the invariants $H(\alpha)$ are defined in (2.22). Then the wallcrossing formula of Kontsevich and Soibelman reads

$$\begin{aligned} &\exp(\mathbb{H})\prod_{q\geq 0,\ q\downarrow}U_{2\alpha+q\beta}^{\overline{A}_+(2\alpha+q\beta,2)}\prod_{q\geq 0,\ q\downarrow}U_{\alpha+q\beta}^{A_+(\alpha+q\beta,1)}\\ &\quad=\prod_{q\geq 0,\ q\uparrow}U_{\alpha+q\beta}^{A_-(\alpha+q\beta,1)}\prod_{q\geq 0,\ q\uparrow}U_{2\alpha+q\beta}^{\overline{A}_-(2\alpha+q\beta,2)}\exp(\mathbb{H}) \end{aligned} \tag{4.3}$$

where an up, respectively, down, arrow means that the factors in the corresponding product are taken in increasing, respectively, decreasing, order of q. Note that $\overline{A}_\pm(2\alpha+q\beta,2)$ are the invariants defined in Section (2.5) by the multicover formula (2.23). In this case Equation (2.23) reduces to

$$A_\pm(2\alpha+q\beta,2)=\overline{A}_\pm(2\alpha+q\beta,2)+\frac{1}{4}A_\pm(\alpha+q\beta/2,1).$$

Expanding the right-hand side, Equation (4.3) yields

$$\begin{aligned} &\exp\Bigg(\sum_{q\geq 0}A_-(2\alpha+q\beta,2)\mathsf{g}_{2\alpha+q\beta}\\ &\quad+\sum_{q_2>q_1\geq 0}\frac{1}{2}g(q_1\beta,q_2\beta)A_-(\alpha+q_1\beta,1)A_-(\alpha+q_2\beta,1)\mathsf{g}_{2\alpha+(q_1+q_2)\beta}\Bigg) \end{aligned} \tag{4.4}$$

$$= \exp(\mathbb{H}) \exp\left(\sum_{q\geq 0} A_+(2\alpha+q\beta,2)\mathsf{g}_{2\alpha+q\beta} + \sum_{q_1>q_2\geq 0} \frac{1}{2} g(q_1\beta,q_2\beta)A_+(\alpha+q_1\beta,1)A_+(\alpha+q_2\beta,1)\mathsf{g}_{2\alpha+(q_1+q_2)\beta}\right) \times \exp(-\mathbb{H})$$

modulo terms involving f_γ. These terms are omitted since they enter $v=1$ wallcrossing formula derived in [2]. The BCH formula

$$\begin{aligned} \exp(A)\exp(B)\exp(-A) &= \exp\left(\sum_{n=0} \frac{1}{n!}(Ad(A))^n B\right) \\ &= \exp\left(B + [A,B] + \frac{1}{2}[A,[A,B]] + \cdots\right) \end{aligned} \tag{4.5}$$

yields

$$\begin{aligned} &\exp(\mathbb{H}) \exp(\mathsf{g}_{2\alpha+q\beta}) \exp(-\mathbb{H}) \\ &\quad = \exp(\mathsf{g}_{2\alpha+q\beta} + \sum_{q_1>0} f_2(q_1\beta)H(q_1\beta)\mathsf{g}_{2\alpha+(q+q_1)\beta} \\ &\qquad + \frac{1}{2!}\sum_{q_1>0,q_2>0} f_2(q_1\beta)H(q_1\beta)f_2(q_2\beta)H(q_2\beta)\mathsf{g}_{2\alpha+(q+q_1+q_2)\beta} + \cdots) \\ &\quad = \exp\left(\sum_{l\geq 0,q_i>0} \frac{1}{l!}\left(\prod_{i=1}^{l} f_2(q_i\beta)H(q_i\beta)\right)\mathsf{g}_{2\alpha+(q+q_1+\cdots+q_l)\beta}\right). \end{aligned} \tag{4.6}$$

Substituting (4.6) in (4.4) results in

$$\begin{aligned} &\exp\left(\sum_{q\geq 0} A_-(2\alpha+q\beta,2)\mathsf{g}_{2\alpha+q\beta} + \sum_{q_2>q_1\geq 0} \frac{1}{2} g(q_1\beta,q_2\beta)A_-(\alpha+q_1\beta,1)A_-(\alpha+q_2\beta,1)\mathsf{g}_{2\alpha+(q_1+q_2)\beta}\right) \\ &= \exp\left(\sum_{\substack{q\geq 0,l\geq 0\\ q_i>0}} A_+(2\alpha+q\beta,2)\frac{1}{l!}\left(\prod_{i=1}^{l} f_2(q_i\beta)H(q_i\beta)\right)\mathsf{g}_{2\alpha+(q+q_1+\cdots+q_l)\beta}\right. \end{aligned} \tag{4.7}$$

$$
+ \sum_{\substack{q_1'>q_2'\geq 0\\ l\geq 0, q_i>0}} \frac{1}{2} g(q_1'\beta, q_2'\beta) A_+(\alpha+q_1'\beta,1) A_+(\alpha+q_2'\beta,1)\frac{1}{l!} \\
\times \left(\prod_{i=1}^{l} f_2(q_i\beta)H(q_i\beta)\right) \mathsf{g}_{2\alpha+(q_1'+q_2'+q_1+\cdots+q_l)\beta} \Bigg).
$$

In order to further simplify the notation, let

$$
A_\pm(v\alpha+q\beta, v) \equiv A_\pm(q,v), \qquad \mathsf{g}_{2\alpha+q\beta} \equiv \mathsf{g}_q.
$$

Comparing the coefficients of g_Q in (4.4) yields

$$
\begin{aligned}
A_-(Q,2) = & \sum_{\substack{q'\geq 0,\ l\geq 0,\ q_i>0\\ q'+q_1+\cdots+q_l=Q}} A_+(q',2)\frac{1}{l!}\left(\prod_{i=1}^{l} f_2(q_i\beta)H(q_i\beta)\right) \\
& + \frac{1}{2} \sum_{\substack{q_1'>q_2'\geq 0\\ l\geq 0,\ q_i>0\\ q_1'+q_2'+q_1+\cdots+q_l=Q}} g(q_1'\beta, q_2'\beta) A_+(q_1',1)A_+(q_2',1) \\
& \times \frac{1}{l!}\left(\prod_{i=1}^{l} f_2(q_i\beta)H(q_i\beta)\right) \\
& - \frac{1}{2} \sum_{q_2'>q_1'\geq 0,\ q_1'+q_2'=Q} g(q_1'\beta, q_2'\beta) A_-(q_1',1)A_-(q_2',1).
\end{aligned} \tag{4.8}
$$

Using the $v=1$ wallcrossing formula [2, Theorem 1.1], the last term in (4.8) becomes

$$
\begin{aligned}
& -\frac{1}{2} \sum_{q_2>q_1\geq 0,\ q_1+q_2=Q} g(q_1\beta, q_2\beta) A_-(q_1,1)A_-(q_2,1) \\
& = -\frac{1}{2} \sum_{\substack{q_2>q_1\geq 0\\ q_1+q_2=Q\\ l\geq 0,\ \tilde{l}\geq 0\\ q_1'\geq 0,\ q_2'\geq 0\\ n_i>0,\ \tilde{n}_i>0\\ q_1'+n_1+\cdots+n_l=q_1\\ q_2'+\tilde{n}_1+\cdots+\tilde{n}_{\tilde{l}}=q_2}} g(q_1\beta, q_2\beta) A_+(q_1',1)A_+(q_2',1)
\end{aligned}
$$

$$\times \frac{1}{l!}\left(\prod_{i=1}^{l} f_1(n_i\beta)H(n_i\beta)\right)\frac{1}{\tilde{l}!}\left(\prod_{i=1}^{\tilde{l}} f_1(\tilde{n}_i\beta)H(\tilde{n}_i\beta)\right). \tag{4.9}$$

Therefore, the final wallcrossing formula for $v = 2$ invariants is

$$\begin{aligned}
A_-(Q,2) = & \sum_{\substack{q'\geq 0,\ l\geq 0,\ q_i>0 \\ q'+q_1+\cdots+q_l=Q}} A_+(q',2)\frac{1}{l!}\left(\prod_{i=1}^{l} f_2(q_i\beta)H(q_i\beta)\right) \\
& + \frac{1}{2}\sum_{\substack{q_1'>q_2'\geq 0 \\ l\geq 0,\ q_i>0 \\ q_1'+q_2'+q_1+\cdots+q_l=Q}} \frac{1}{2} g(q_1'\beta, q_2'\beta)A_+(q_1',1)A_+(q_2',1) \\
& \times \frac{1}{l!}\left(\prod_{i=1}^{l} f_2(q_i\beta)H(q_i\beta)\right) \\
& - \frac{1}{2}\sum_{\substack{q_2>q_1\geq 0 \\ q_1+q_2=Q \\ l\geq 0,\ \tilde{l}\geq 0 \\ q_1'\geq 0,\ q_2'\geq 0 \\ n_i>0,\ \tilde{n}_i>0 \\ q_1'+n_1+\cdots+n_l=q_1 \\ q_2'+\tilde{n}_1+\cdots+\tilde{n}_{\tilde{l}}=q_2}} g(q_1\beta, q_2\beta)A_+(q_1',1)A_+(q_2',1) \\
& \times \frac{1}{l!}\left(\prod_{i=1}^{l} f_1(n_i\beta)H(n_i\beta)\right)\frac{1}{\tilde{l}!}\left(\prod_{i=1}^{\tilde{l}} f_1(\tilde{n}_i\beta)H(\tilde{n}_i\beta)\right).
\end{aligned} \tag{4.10}$$

This formula agrees with (1.5), since the bilinear function $g(\ \ ,\ \ )$ is antisymmetric.

5. Asymptotic invariants in the $g = 0$ theory

In this subsection, X will be a smooth genus 0 curve over a $\mathbb{C}$-field K. and $M_1 \simeq \mathcal{O}_X(d_1)$, $M_2 \simeq \mathcal{O}_X(d_2)$, with $(d_1, d_2) = (1,1)$ or $(d_1, d_2) = (0,2)$. In this case, any coherent locally free sheaf E on X is isomorphic to a direct sum of line bundles. Let $E_{\geq 0}$ denote the direct sum of all summands of nonnegative degree and $E_{<0}$ denote the direct sum of all summands of negative degree.

Lemma 5.1. *Let $\mathcal{E} = (E, V, \Phi_1, \Phi_2, \phi, \psi)$ be a nontrivial δ-semistable ADHM sheaf of type $(r,e,v) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 1}$, for some $\delta > 0$. Then $E_{<0} = 0$ and ϕ is identically zero.*

Proof. Since $\delta > 0$, Lemma 2.1(ii) implies that ψ is not identically zero. Then obviously $E_{\geq 0}$ must be nontrivial and $\mathrm{Im}(\psi) \subseteq E_{\geq 0}$. Since $M \simeq K_X^{-1} \simeq \mathcal{O}_X(2)$, $E_{\geq 0} \otimes_X M \subseteq \mathrm{Ker}(\phi)$. Moreover, since $\deg(M_1) \geq 0$, $\deg(M_2) \geq 0$, $\Phi_i(E_{\geq 0} \otimes_X M_i) \subseteq E_{\geq 0}$. It follows that the data

$$\mathcal{E}_{\geq 0} = (E_{\geq 0}, V \otimes \mathcal{O}_X, \Phi_i|_{E_{\geq 0} \otimes_X M_i}, 0, \psi)$$

are nontrivial subobjects of $\mathcal{E}$. If $E_{<0}$ is not the zero sheaf, $\mathcal{E}_{\geq 0}$ is a proper subobject of $\mathcal{E}$. Then δ-semistability condition implies $r(\mathcal{E}_{\geq 0}) < r(\mathcal{E})$, and hence

$$\frac{d(\mathcal{E}_{\geq 0}) + v(\mathcal{E}_{\geq 0})\,\delta}{r(\mathcal{E}_{\geq 0})} \leq \frac{e + v\,\delta}{r}. \tag{5.1}$$

However $e < d(\mathcal{E}_{\geq 0})$ and $0 < r(\mathcal{E}_{\geq 0}) < r$ under the current assumptions. Since also $v(\mathcal{E}_{\geq 0}) = v$ and $\delta, d(\mathcal{E}_{\geq 0}) > 0$, inequality (5.1) leads to a contradiction. Therefore, $E_{<0} = 0$ and ϕ must be identically zero. □

Now let the ground field K be $\mathbb{C}$. Let $\mathcal{C}^0_{\mathcal{X}}$ be the full abelian subcategory of $\mathcal{C}_{\mathcal{X}}$ consisting of ADHM sheaves $\mathcal{E}$ with $\phi = 0$. For any $\delta \in \mathbb{R}$, an object $\mathcal{E}$ of $\mathcal{C}^0_{\mathcal{X}}$ will be called δ-semistable if it is δ-semistable as an object of $\mathcal{C}_{\mathcal{X}}$. Note that given an object $\mathcal{E}$ of $\mathcal{C}^0_{\mathcal{X}}$, any subobject $\mathcal{E}' \subset \mathcal{E}$ must also belong to $\mathcal{C}^0_{\mathcal{X}}$. In particular, all test subobjects in Definition 2.1 also belong to $\mathcal{C}^0_{\mathcal{X}}$, and one obtains a stability condition on the abelian category $\mathcal{C}^0_{\mathcal{X}}$. Then the properties of δ-stability and moduli stacks of semistable objects in $\mathcal{C}^0_{\mathcal{X}}$ are analogous to those of $\mathcal{C}_{\mathcal{X}}$. In particular, for fixed $(r, e, v) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 1}$ there are finitely many critical stability parameters of type (r, e, v) dividing the real axis into stability chambers. The main difference between $\mathcal{C}^0_{\mathcal{X}}$ and $\mathcal{C}_{\mathcal{X}}$ is the presence of an empty chamber, as follows.

Lemma 5.2. *For any $(r, e, v) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 1}$ the moduli stack of δ-semistable objects of $\mathcal{C}^0_{\mathcal{X}}$ of type (r, e, v) is empty if $\delta < 0$.*

Proof. Given an ADHM sheaf $\mathcal{E} = (E, V, \Phi_i, \psi)$ of type (r, e, v), it is straightforward to check that for $\delta < 0$ the proper nontrivial object $(E, 0, \Phi_i, 0)$ is always destabilizing if $\delta < 0$. □

Lemma 5.3. *Let $\mathcal{E}$ be a δ-semistable object of $\mathcal{C}^0_{\mathcal{X}}$ of type $(r, e, v) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}$ for some $\delta \geq 0$. If $e \geq 0$, then $E_{<0} = 0$.*

Proof. For $\delta > 0$ and $v > 0$, this obviously follows from Lemma 5.1. If $\delta = 0$ or $v = 0$ note that $E_{\geq 0}$ cannot be the zero sheaf, since $e \geq 0$. Then the proof of Lemma 5.1 also applies to this case as well. □

Lemma 5.4. *Let $\mathcal{E} = (E, 0, \Phi_i, 0, 0)$ be a semistable object of $\mathcal{C}^0_{\mathcal{X}}$ of type $(r, e, 0)$, $(r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$. If $(d_1, d_2) = (1, 1)$, E must be isomorphic to $\mathcal{O}_X(n)^{\oplus r}$ for some $n \in \mathbb{Z}$ and $\Phi_i = 0$ for $i = 1, 2$. If $(d_1, d_2) = (0, 2)$, E must be isomorphic to $\mathcal{O}_X(n)^{\oplus r}$ for some $n \in \mathbb{Z}$, and $\Phi_2 = 0$.*

Proof. In both cases, let $E \simeq \oplus_{s=1}^r \mathcal{O}_X(n_s)$ for some $n_s \in \mathbb{Z}$ so that $n_1 \leq n_2 \leq \cdots \leq n_r$. Since $d_1, d_2 \geq 0$, any subsheaf of the form

$$\oplus_{s=s_0}^r \mathcal{O}_X(n_s)$$

for some $1 \leq s_0 \leq r$ must be Φ_i-invariant, $i = 1, 2$. Therefore, the semistability condition implies

$$\frac{n_{s_0} + \cdots + n_r}{r - s_0 + 1} \leq \frac{n_1 + \cdots + n_r}{r}$$

for any $1 \leq s_0 \leq r$. Then it is straightforward to check that $n_1 = \cdots = n_r = n$. The rest is obvious. □

Corollary 5.1. *Under the same conditions as in Lemma 5.4,*

$$H(r, e) = \begin{cases} \dfrac{(-1)^{d_1 - 1}}{r^2} & \text{if } e = rn, \ n \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases} \tag{5.2}$$

Proof. If $(d_1, d_2) = (1, 1)$, Lemma 5.4 implies that the moduli stack $\mathfrak{M}^{ss}(\mathcal{X}, r, e, 0)$ is isomorphic to the quotient stack $[*/GL(r)]$ if $e = rn$ for some $n \in \mathbb{Z}$, and empty otherwise. Alternatively, if $e = rn$, the moduli stack $\mathfrak{M}^{ss}(\mathcal{X}, r, e, 0)$ can be identified with the moduli stack of trivially semistable representations of dimension r of a quiver consisting of only one vertex and no arrows. Recall that the trivial semistability condition for quiver representations is King stability with all stability parameters associated to the vertices set to zero [14, Example 7.3].

If $(d_1, d_2) = (0, 2)$, Lemma 5.4 implies that the moduli stack $\mathfrak{M}^{ss}(\mathcal{X}, r, rn, 0)$, $n \in \mathbb{Z}$, is isomorphic to the moduli stack of trivially semistable representations of dimension r of a quiver consisting of one vertex and one arrow joining the unique vertex with itself. If e is not a multiple of r, the moduli stack $\mathfrak{M}^{ss}(\mathcal{X}, r, e, 0)$ is empty.

Then Corollary 5.1 follows by a computation very similar to [14, Section 7.5.1]. □

Remark 5.1. The same arguments as in the proof of Corollary 5.1 imply that for any $\delta > 0$,

$$A_\delta(0,0,1) = 1 \quad A_\delta(0,0,2) = \frac{1}{4}. \tag{5.3}$$

Extension groups in $\mathcal{C}^0_{\mathcal{X}}$ can be determined by analogy with those of $\mathcal{C}_{\mathcal{X}}$. Given two locally free objects $\mathcal{E}'', \mathcal{E}'$ of $\mathcal{C}^0_{\mathcal{X}}$, let $\widetilde{\mathcal{C}}(\mathcal{E}'', \mathcal{E}')$ be the three term complex of locally free $\mathcal{O}_X$-modules

$$0 \to \begin{array}{c} \mathcal{H}om_X(E'', E') \\ \oplus \\ \mathcal{H}om_X(V'' \otimes \mathcal{O}_X, V' \otimes \mathcal{O}_X) \end{array} \xrightarrow{d_1} \begin{array}{c} \mathcal{H}om_X(E'' \otimes_X M_1, E') \\ \oplus \\ \mathcal{H}om_X(E'' \otimes_X M_2, E') \\ \oplus \\ \mathcal{H}om_X(V'' \otimes \mathcal{O}_X, E') \end{array} \xrightarrow{d_2} \mathcal{H}om_X(E'' \otimes_X M, E') \to 0 \tag{5.4}$$

where

$$\begin{aligned} d_1(\alpha, f) = (&-\alpha \circ \Phi''_1 + \Phi'_1 \circ (\alpha \otimes 1_{M_1}), -\alpha \circ \Phi''_2 + \Phi'_2 \circ (\alpha \otimes 1_{M_2}), \\ &- \alpha \circ \psi'' + \psi' \circ f) \end{aligned}$$

for any local sections (α, f) of the first term and

$$\begin{aligned} d_2(\beta_1, \beta_2, \gamma) = \beta_1 \circ (\Phi''_2 \otimes 1_{M_1}) - \Phi'_2 \circ (\beta_1 \otimes 1_{M_2}) - \beta_2 \circ (\Phi''_1 \otimes 1_{M_2}) \\ + \Phi'_1 \circ (\beta_2 \otimes 1_{M_1}) \end{aligned}$$

for any local sections $(\beta_1, \beta_2, \gamma)$ of the middle term. The degrees of the three terms in (2.16) are $0, 1, 2$, respectively. By analogy with Lemma 2.9, the following holds.

Lemma 5.5. *Under the current assumptions,* $\mathrm{Ext}^k_{\mathcal{C}^0_{\mathcal{X}}}(\mathcal{E}'', \mathcal{E}') \simeq \mathbb{H}^k(\widetilde{\mathcal{C}}(\mathcal{E}'', \mathcal{E}'))$ *for* $k = 0, 1$.

Lemma 5.6. *Let* $\mathcal{E}', \mathcal{E}''$ *be two nontrivial locally free objects of* $\mathcal{C}^0_{\mathcal{X}}$ *of types* $(r', e', v'), (r'', e'', v'') \in \mathbb{Z}_{\geq 1} \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}$. *Suppose that* $E'_{<0} = 0$, $E''_{<0} = 0$ *for both underlying locally free sheaves* E', E''. *Then*

$$\begin{aligned} &\dim(\mathrm{Ext}^0_{\mathcal{C}_{\mathcal{X}}}(\mathcal{E}'', \mathcal{E}')) - \dim(\mathrm{Ext}^1_{\mathcal{C}_{\mathcal{X}}}(\mathcal{E}'', \mathcal{E}')) - \dim(\mathrm{Ext}^0_{\mathcal{C}_{\mathcal{X}}}(\mathcal{E}', \mathcal{E}'')) \\ &\quad + \dim(\mathrm{Ext}^1_{\mathcal{C}_{\mathcal{X}}}(\mathcal{E}', \mathcal{E}'')) = v'(e'' + r'') - v''(e' + r'). \end{aligned} \tag{5.5}$$

Proof. Note that complex (5.4) can be written as the cone of a morphism of locally free complexes on X

$$\varrho : \mathcal{H}[-1] \longrightarrow \mathcal{V}$$

where $\mathcal{H}$ is the complex obtained from $\widetilde{\mathcal{C}}(\mathcal{E}'', \mathcal{E}')$ by omitting all direct summands depending on V', V'' (as well as making some obvious changes of signs), and $\mathcal{V}$ is the two term complex

$$\begin{aligned} \mathcal{H}om_X(V'' \otimes \mathcal{O}_X, V' \otimes \mathcal{O}_X) &\longrightarrow \mathcal{H}om_X(V'' \otimes \mathcal{O}_X, E') \\ f &\longrightarrow \psi' \circ f \end{aligned}$$

with degrees $0, 1$. The morphism ϱ is determined by the map

$$\begin{aligned} \mathcal{H}om_X(E'', E') &\longrightarrow \mathcal{H}om_X(V'' \otimes \mathcal{O}_X, E') \\ \alpha &\longrightarrow -\alpha \circ \psi''. \end{aligned}$$

Therefore, there is a long exact sequence of hypercohomology groups

$$(5.6) \qquad \begin{aligned} 0 &\longrightarrow \mathbb{H}^0(\mathcal{V}) \longrightarrow \mathrm{Ext}^0_{\mathcal{C}^0_{\mathcal{X}}}(\mathcal{E}'', \mathcal{E}') \longrightarrow \mathbb{H}^0(\mathcal{H}(\mathcal{E}'', \mathcal{E}')) \\ &\longrightarrow \mathbb{H}^1(\mathcal{V}) \longrightarrow \mathrm{Ext}^1_{\mathcal{C}^0_{\mathcal{X}}}(\mathcal{E}'', \mathcal{E}') \longrightarrow \mathbb{H}^1(\mathcal{H}(\mathcal{E}'', \mathcal{E}')) \\ &\longrightarrow \mathbb{H}^2(\mathcal{V}) \longrightarrow \cdots . \end{aligned}$$

Since $E'_{<0} = 0$ and X is rational, $\mathbb{H}^2(\mathcal{V}) = 0$. Obviously, there is a similar exact sequence with $\mathcal{E}', \mathcal{E}''$ interchanged. Then Equation (5.5) easily follows observing that

$$\mathbb{H}^k(\mathcal{H}(\mathcal{E}'', \mathcal{E}')) \simeq \mathbb{H}^{3-k}(\mathcal{H}(\mathcal{E}', \mathcal{E}''))^\vee$$

for all $0 \le k \le 3$. □

Proof of Corollary 1.1. Lemma 5.1 implies that for any $\delta \in \mathbb{R}_{>0}$, $(r, e, v) \in \mathbb{Z}_{\ge 1} \times \mathbb{Z} \times \mathbb{Z}_{\ge 0}$ there is a canonical isomorphism of moduli stacks of δ-semistable objects of numerical invariants (r, e, v) in the abelian categories $\mathcal{C}_{\mathcal{X}}$, $\mathcal{C}^0_{\mathcal{X}}$. Moreover, using Lemma 5.6, the construction of Joyce and Song summarized in Section 2.5 applies to δ-semistable objects of $\mathcal{C}^0_{\mathcal{X}}$ as well. For $\delta > 0$, the resulting invariants are identical with the invariants $A_\delta(r, e, v)$

defined is Section 2.5. In particular, they satisfy identical wallcrossing formulas for any positive critical stability parameter $\delta_c > 0$.

In order to prove relations (1.7), a wallcrossing formula at $\delta_c = 0$ will be required for counting invariants of semistable objects in $\mathcal{C}^0_{\mathcal{X}}$. The derivation of this wallcrossing formula is analogous with the proof of Theorem 1.1, provided the following facts are taken into account:

(a) Lemma 2.8 holds δ-semistable objects in $\mathcal{C}^0_{\mathcal{X}}$ at $\delta_c = 0$ if Definition 2.4 is modified as follows. In the definition of positive admissible configurations one must allow $(r_1, v_1) = (0, v_1)$ with $v_1 > 0$. All other elements (r_i, v_i), $i > 1$, are still required to satisfy $r_i \geq 1$. Similarly, in the definition of negative admissible configurations one must allow $(r_h, v_h) = (0, v_h)$, $r_i \geq 1$ being still imposed on all other elements (r_i, v_i), $i < h$.

(b) Let $\mathfrak{d}(0,0,1)$ be the stack function determined by the object $O = (0, \mathbb{C}, 0, 0, 0, 0)$. Then $\Psi(\mathfrak{d}(0,0,1)) = -\lambda(0,0,1)$, since the moduli stack of δ-semistable objects with numerical invariants $(0,0,1)$ is isomorphic to the classifying stack of $\mathbb{C}^\times$ for any $\delta \in \mathbb{R}$.

Taking into account (a) and (b) above, Lemmas 3.1 and 3.2 and the proof of Theorem 1.1 carry over with obvious modifications. The resulting wallcrossing formula for counting invariants of semistable objects in $\mathcal{C}^0_{\mathcal{X}}$ at $\delta_c = 0$ is entirely analogous to (1.5) provided that the sets $\mathcal{HN}_-(\alpha, v, \delta_c, l, l-1)$, $l \geq 1$, $\mathcal{HN}_-(\alpha, 2, \delta_c, l, l-2)$, $l \geq 2$, $\alpha \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$, $v = 1, 2$ are replaced by $\mathcal{HN}_-(\alpha, v, 0, l, l-1)$, $\mathcal{HN}_-(\alpha, 2, 0, l, l-2)$ defined below, and one sets $A_+(0,0,1) = 1$.

The set $\mathcal{HN}_-(\alpha, v, \delta_c, l, l-1)$, $l \geq 1$, $\alpha \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$, $v = 1, 2$, consists of ordered sequences $(\alpha_i)_{1 \leq i \leq l} \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}$ so that $r_i \geq 1$ for $i < l$ and

$$\alpha_1 + \cdots + \alpha_l = \alpha, \quad re_i = r_i e, \quad 1 \leq i \leq l. \tag{5.7}$$

The set $\mathcal{HN}_-(\alpha, 2, \delta_c, l, l-2)$, $l \geq 2$, $\alpha \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$ consists of ordered sequences $(\alpha_i)_{1 \leq i \leq l} \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}$ so that $r_i \geq 1$ for $i < l$, $r_l < r_{l-1}$, and (5.7) holds.

Moreover, the resulting formula is again in agreement with the Kontsevich–Soibelman wallcrossing formula by computations identical to those presented in Section 4.

Then the proof of Corollary 1.1 will be based on the KS wallcrossing formula relating δ-invariants for $\delta < 0$ to δ-invariants with $\delta >> 0$. Let $(r, e) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 0}$ and let $\delta_+ \in \mathbb{R}_{>0} \setminus \mathbb{Q}$ an irrational stability parameter so that δ_+ is asymptotic of type (r', e') for all $1 \leq r' \leq r$, $0 \leq e' \leq e$, $1 \leq v \leq 2$.

Moreover, assume that $re < \delta_+$. Then the KS formula reads

$$
\begin{aligned}
(5.8) \qquad & \prod_{(r,n,v)\in \mathbb{Z}_{\geq 1}\times\mathbb{Z}_{\geq 0}\times\{0,1,2\}\cup\{0,0,1\}} U_{\lambda(r,n,v)}^{\overline{A^0}_-(r,n,v)} \\
&= \prod_{(r,n,v)\in \mathbb{Z}_{\geq 1}\times\mathbb{Z}_{\geq 0}\times\{0,1,2\}\cup\{0,0,1\}} U_{\lambda(r,n,v)}^{\overline{A^0}_+(r,n,v)}
\end{aligned}
$$

where in each term the factors are ordered in increasing order of $\delta_\pm$-slopes from left to right. The alternative notation introduced in Section 4 will be used in the following. Then Corollary 5.1 and Equation 5.3 imply that the left-hand side of (5.8) reads

$$
(5.9) \qquad \exp\left(\mathsf{f}_{00} + \frac{1}{4}\mathsf{g}_{00}\right)\prod_{n=0}^{\infty} U_{\mathsf{e}_{1n}}
$$

where

$$
U_{\mathsf{e}_{rn}} = \exp\left((-1)^{d_1-1}\sum_{k=1}^{\infty}\frac{\mathsf{e}_{kr,kn}}{k^2}\right).
$$

Moreover, given the above choice of δ_+,

$$
\begin{aligned}
e < \frac{\delta_+}{r} < \cdots \frac{e+\delta_+}{r} < \frac{\delta_+}{r-1} < \cdots < \frac{e+\delta_+}{r-1} < \cdots < \\
\delta_+ + e < \frac{2\delta_+}{r} < \cdots < 2\delta_+ + e.
\end{aligned}
$$

Therefore, on the right-hand side of Equation (5.8), the factors of the form $U_{\lambda(r',e',v)}^{\overline{A}_+(r',e',v)}$, with $v \in \{0,1,2\}$, and $1 \leq r' \leq r$, $1 \leq e' \leq e$ occur in the following order:

$$
\begin{aligned}
& \prod_{n=0}^{e} U_{\mathsf{e}_{1n}} \prod_{n=0}^{e} U_{\mathsf{f}_{r,n}}^{\overline{A}_+(r,n,1)} \prod_{n=0}^{e} U_{\mathsf{f}_{r-1,n}}^{\overline{A}_+(r-1,n,1)} \cdots \prod_{n=0}^{e} U_{\mathsf{f}_{1,n}}^{\overline{A}_+(1,n,1)} U_{\mathsf{f}_{0,0}}^{\overline{A}_+(0,0,1)} \\
(5.10) \qquad & \prod_{n=0}^{e} U_{\mathsf{g}_{r,n}}^{\overline{A}_+(r,n,2)} \cdots \prod_{n=0}^{e} U_{\mathsf{g}_{r-1,n}}^{\overline{A}_+(r-1,n,2)} \cdots \prod_{n=0}^{e} U_{\mathsf{g}_{1,n}}^{\overline{A}_+(1,n,2)}
\end{aligned}
$$

where

$$
U_{\mathsf{f}_{rn}} = \exp(\mathsf{f}_{rn} + \tfrac{1}{4}\mathsf{g}_{2r,2n}), \quad U_{\mathsf{g}_{rn}} = \exp(\mathsf{g}_{rn}).
$$

In addition, the right-hand side of (5.8) contains of course extra factors of the form $U_{\lambda(r',e',v)}^{\overline{A}_+(r',e',v)}$, with $v \in \{0,1,2\}$, and either $r' > r$ or $e' > e$. Some of these extra factors may in fact occur between the factors listed in (5.10). However, they can be ignored for the purpose of this computation, since commutators involving such factors are again expressed in terms of generators $\lambda(r', e', v)$ with either $r' > r$ or $e' > e$. Therefore, using the BCH formula, (5.8) yields

(5.11)
$$\left(\prod_{n=0}^{e} U_{\mathsf{e}_{1n}}\right)^{-1} \exp\left(\mathsf{f}_{00} + \frac{1}{4}\mathsf{g}_{00}\right) \prod_{n=0}^{\infty} U_{\mathsf{e}_{1n}}$$
$$= \exp\Bigg(\mathsf{f}_{00} + \frac{1}{4}\mathsf{g}_{00} + \sum_{1\leq s\leq r,\ 0\leq n\leq e} A_+(s,n,1)\,\mathsf{f}_{sn}$$
$$+ \sum_{1\leq s\leq r,\ 0\leq n\leq e} A_+(s,n,2)\,\mathsf{g}_{sn}$$
$$+ \sum_{\substack{r_1>r_2\geq 1,\ r_1+r_2\leq r,\ n_1,\ n_2\geq 0,\ n_1+n_2\leq e\\ \text{or } 1\leq r_1=r_2\leq r/2,\ 0\leq n_1<n_2,\ n_1+n_2\leq e\\ \text{or } 1\leq r_1\leq r,\ 0\leq n_1\leq e,\ r_2=n_2=0}} \frac{1}{2}(n_1-n_2+r_1-r_2)$$
$$\times (-1)^{(n_1-n_2+r_1-r_2)} A_+(r_1,n_1,1)A_+(r_2,n_2,1)\,\mathsf{g}_{r_1+r_2,n_1+n_2} + \cdots\Bigg)$$

where $\cdots$ are terms involving generators $\lambda(r', e', v)$ with either $r' > r$ or $e' > e$. For fixed $e \geq 1$, let $\mathcal{H}_e$ be defined by

$$\exp(\mathcal{H}_e) \equiv \prod_{n=0}^{e} U_{\mathsf{e}_{1,n}} = \exp\left((-1)^{d_1-1}\sum_{0\leq n\leq e,\ k\geq 1}\frac{\mathsf{e}_{k,kn}}{k^2}\right). \tag{5.12}$$

Using the BCH formula, the left-hand side of Equation (5.11) becomes

$$\exp\left(\mathsf{f}_{00} + \frac{1}{4}\mathsf{g}_{00} + \sum_{j=1}^{\infty}\frac{1}{j!}[\underbrace{-\mathcal{H}_e,\cdots[-\mathcal{H}_e}_{\text{j times}},\mathsf{f}_{00} + \frac{1}{4}\mathsf{g}_{00}\underbrace{]\cdots]}_{\text{j times}}\right) \tag{5.13}$$

modulo terms involving generators $\lambda(r', e', v)$ with either $r' > r$ or $e' > e$.

Next, the Lie algebra commutators

$$[\mathsf{e}_{r_1,n_1}, \mathsf{f}_{r_2,n_2}] = (-1)^{n_1+r_1}(n_1+r_1)\,\mathsf{f}_{r_1+r_2,n_1+n_2}$$
$$[\mathsf{e}_{r_1,n_1}, \mathsf{g}_{r_2,n_2}] = 2(n_1+r_1)\,\mathsf{g}_{r_1+r_2,n_1+n_2}$$

yield

$$[\underbrace{-\mathcal{H}_e,\cdots[-\mathcal{H}_e}_{\text{j times}},\mathsf{f}_{00}\underbrace{]\cdots]}_{\text{j times}} = \sum_{n_1,\ldots,n_j=0}^{e}\sum_{k_1,\ldots,k_j\geq 1}(-1)^{j(d_1-1)} \prod_{i=1}^{j}\frac{n_i+1}{k_i}(-1)^{(n_i+1)k_i-1}\,\mathsf{f}_{k_1+\cdots+k_j,k_1n_1+\cdots+k_jn_j}$$

and

$$[\underbrace{-\mathcal{H}_e,\cdots[-\mathcal{H}_e}_{\text{j times}},\mathsf{g}_{00}\underbrace{]\cdots]}_{\text{j times}} = \sum_{n_1,\ldots,n_j=0}^{e}\sum_{k_1,\ldots,k_j\geq 1}(-1)^{j(d_1-1)} \prod_{i=1}^{j}(-2)\frac{n_i+1}{k_i}\,\mathsf{g}_{k_1+\cdots+k_j,k_1n_1+\cdots+k_jn_j}$$

Therefore, identifying the coefficients of the generators f_{rn} in (5.11) it follows that the invariant $A_+(r',e',1)$ with $1\leq r'\leq r$ and $0\leq e'\leq e$ equals the coefficient of the monomial $u^{r'}q^{e'+r'}$ in the expression

$$\sum_{j=0}^{\infty}\frac{1}{j!}\left(\ln\left(\prod_{n=0}^{e}(1-u(-q)^{n+1})^{(-1)^{d_1-1}(n+1)}\right)\right)^j = \prod_{n=1}^{e+1}(1-u(-q)^n)^{(-1)^{d_1-1}n}.$$

Similarly, identifying the coefficients of the generators g_{rn} in (5.11) proves that the invariant $A_+(r',e',2)$ with $1\leq r'\leq r$ and $0\leq e'\leq e$ equals the coefficient of the monomial $u^{r'}q^{e'+r'}$ in the expression

$$\frac{1}{4}\prod_{n=1}^{e+1}(1-uq^n)^{2(-1)^{d_1-1}n} - \sum_{\substack{r_1>r_2\geq 1,\ r_1+r_2\leq r,\ n_1,\ n_2\geq 0,\ n_1+n_2\leq e\\ \text{or } 1\leq r_1=r_2\leq r/2,\ 0\leq n_1<n_2,\ n_1+n_2\leq e\\ \text{or } 1\leq r_1\leq r,\ 0\leq n_1\leq e,\ r_2=n_2=0}} \frac{1}{2}(n_1+r_1-n_2-r_2)(-1)^{(n_1+r_1-n_2-r_2)}A_+(r_1,n_1,1) \times A_+(r_2,n_2,1)q^{r_1+r_2}u^{n_1+n_2}.$$

Since this holds for any $(r,e)\in\mathbb{Z}_{\geq 1}\times\mathbb{Z}_{\geq 0}$ (with a suitable choice of δ_+), Corollary 1.1 follows.

□

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