COMMUNICATIONS IN
NUMBER THEORY AND PHYSICS
Volume 4, Number 1, 103–159, 2010

A modern fareytail

Jan Manschot and Gregory W. Moore

We revisit the “fareytail expansions” of elliptic genera which have been used in discussions of the AdS_3/CFT_2 correspondence and the OSV conjecture. We show how to write such expansions without the use of the problematic “fareytail transform.” In particular, we show how to write a general vector-valued modular form of non-positive weight as a convergent sum over cosets of $SL(2, \mathbb{Z})$. This sum suggests a new regularization of the gravity path integral in AdS_3, resolves the puzzles associated with the “fareytail transform,” and leads to several new insights. We discuss constraints on the polar coefficients of negative weight modular forms arising from modular invariance, showing how these are related to Fourier coefficients of positive weight cusp forms. In addition, we discuss the appearance of holomorphic anomalies in the context of the fareytail.

1. Introduction

The AdS/CFT correspondence [1–4] plays a central role in string theory. While it has yet to be given a concise and precise mathematical definition, it seems clear that part of the formulation involves an equality of partition functions

$$\mathcal{Z}_{\mathrm{String}} = \mathcal{Z}_{\mathrm{CFT}}, \tag{1.1}$$

where $\mathcal{Z}_{\mathrm{String}}$ is the partition function of a string theory (or M-theory) on a spacetime (or sum over spacetimes) with asymptotics of the form $\mathrm{AdS}_n \times K$, for a compact space K, and $\mathcal{Z}_{\mathrm{CFT}}$ is the partition function of a "holographically dual" conformal field theory defined on the conformal boundary of AdS_n. The present paper discusses partition functions in the context of $\mathrm{AdS}_3/\mathrm{CFT}_2$, in which case Equation (1.1) can be investigated with a high degree of precision.

We consider Euclidean AdS_3 geometries whose conformal boundary geometry is a torus. Thus, the partition functions depend on the complex structure parameter τ of the torus. The Fourier expansion of the partition function, given by

$$\mathcal{Z} = \sum_{n=0}^{\infty} c(n) q^{n-\Delta} \tag{1.2}$$

with $q = e^{2\pi i\tau}$, contains a pole when $\mathrm{Im}(\tau) \to \infty$, corresponding to the light states with $n - \Delta < 0$. The partition function is uniquely specified by the polar degeneracies using holomorphy and modular invariance. The main result of this paper is the description of a sum which completes the polar terms to the full partition function $\mathcal{Z}$. The sum is roughly a sum of the polar terms over a coset of the modular group $\mathrm{SL}(2,\mathbb{Z})$,[1] which is known as a Poincaré series. One of the important novel insights of [5] is the connection between Poincaré series and sums over different AdS_3 geometries with fixed asymptotic boundary conditions. This led to the proposal that $\mathcal{Z}_{\mathrm{CFT}}$, written as a Poincaré series, has the interpretation as a sum over partition functions of string theory on different spacetimes with fixed conformal boundary conditions. Such an expansion of $\mathcal{Z}_{\mathrm{CFT}}$ has acquired the name "fareytail expansion" in the physics literature.[2]

A closer inspection shows that the naive Poincaré series for the relevant partition functions are divergent and must be regularized. Dijkgraaf *et al.* [5] proposed a certain regularization which unfortunately does not equal $\mathcal{Z}_{\mathrm{CFT}}$, but rather equals a related function. This function, $\tilde{\mathcal{Z}}_{\mathrm{CFT}}$, the so-called "fareytail transform" of $\mathcal{Z}_{\mathrm{CFT}}$, is of the form $\tilde{\mathcal{Z}}_{\mathrm{CFT}} = \mathcal{O}\mathcal{Z}_{\mathrm{CFT}}$ where $\mathcal{O}$ is a certain pseudo-differential operator. Therefore, the Poincaré series could not be directly interpreted as a confirmation of Equation (1.1).

An important achievement of this paper is a regularized version of the naive Poincaré series which is equal to $\mathcal{Z}_{\mathrm{CFT}}$ and not $\tilde{\mathcal{Z}}_{\mathrm{CFT}}$. Since we no longer need to transform $\mathcal{Z}_{\mathrm{CFT}}$, we have obtained an interpretation of $\mathcal{Z}_{\mathrm{CFT}}$ as a sum over partition functions with fixed conformal boundary conditions. This new version is therefore much more appealing from the point of view of the AdS/CFT correspondence.

---

[1]In the following we will abbreviate $\mathrm{SL}(2,\mathbb{Z})$ to Γ.

[2]The name refers to the fact that the sum over $\Gamma_\infty \backslash \Gamma$ may be identified with a sum over fractions d/c in lowest terms. These define Farey series. In the context of black hole state counting the terms with $c > 1$ are exponentially small and thus represent the tail of the micro-canonical distribution of states associated with the black hole geometry.

This new regularization is an application of a beautiful paper by Niebur [6], following up on earlier work of Knopp [7]. Niebur's regularization reduces to the one proposed by Denef and Moore [8] in the context of the OSV conjecture [9] for Calabi–Yau manifolds with $b_2(X)$ even, and to the one used in [10] for the partition function of pure AdS_3 gravity. Historically, these methods go back to Rademacher's expression for the partition function $p(n)$ of integers as an infinite sum of Bessel functions (see [11] for a modern account) and to his work [12] expressing the modular invariant j-function as a sum over $\Gamma_\infty \setminus \Gamma$.

The new regularization is not only justified by stating that it is more appealing from the point of view of AdS/CFT. It also solves some fundamental problems related to the fareytail transform. These problems recently came to light in the course of some discussions initiated by Hirosi Ooguri, during which Don Zagier pointed out that in fact $\tilde{\mathcal{Z}}_{\mathrm{CFT}}$ is not modular in general. We give a simple explanation of this in Section 5.1 below. Thus, the reliance on the mathematical properties of the fareytail transform in [5] was a mistake and is erroneous.[3] Section 5.1 explains the problems of the fareytail transform in more detail.

The fareytail transform has no strong support from physics either. In particular, other studies of Equation (1.1) did not confirm the need for a modification to $\tilde{\mathcal{Z}}_{\mathrm{CFT}}$. For example, the first terms of the Fourier expansions in Equation (1.1) match in the case of the D1–D5 system without the need for the fareytail transform [13, 14]. The fareytail expansions used in attempts [8, 15, 16] to put the OSV conjecture on a firm footing require Equation (1.1) without application of the fareytail transform. More recently, the study of pure gravity in AdS_3 did not indicate any need for a fareytail transform [17, 18]. Finally, tests in four dimensions involving the singleton modes in $\mathrm{AdS}_5/\mathrm{CFT}_4$ supported Equation (1.1) without the need for modification [19, 20].

Once we have regularized the naive Poincaré series, we have to reexamine modular invariance. We find that, in general, the regularized Poincaré series do not transform covariantly under modular transformations. The partition functions still transform in a controlled way, which can be made precise using the so-called period functions and Eichler cohomology. Thus the choice of polar degeneracies is not arbitrary, as discussed in depth

[3] In the case of negative half-integer weight Jacobi forms, or negative integer weight vectors of modular forms the fareytail transform does preserve modularity. In the application to the OSV conjecture used in [8] this is the reason the authors restricted attention to Calabi–Yau manifolds X with even $b_2(X)$.

in Section 4. Alternatively, one can obtain modular invariance by addition of a suitable non-holomorphic term, as discussed in Section 6.

The regularization does not spoil the semi-classical interpretation of the Poincaré series. The modern fareytail is therefore well suited for use in the original applications, in particular AdS/CFT and phase transitions. In the context of the tests of the OSV conjecture the modern fareytail does not invalidate the previous arguments in the regime of strong topological string coupling, although it does lead to further corrections in the problematic regime of weak topological string coupling. In Section 5.4 we comment on the "entropy enigma" of [8], showing, in the context of a toy model for the polar degeneracies, how in the Rademacher expansion the extreme polar states give the dominant contribution to degeneracies close to the cosmic censorship bound.

In the remaining part of the introduction we will review briefly the connection between Poincaré series and sums over asymptotically AdS_3 geometries. Also the new regularization will be motivated heuristically. The connection between elements in Γ and AdS_3 geometries was suggested in [21] and refined somewhat in [5]. It is reviewed for example in [15, 22, 23] from a supergravity perspective. Three-dimensional gravity has no local degrees of freedom, so different geometries arise from different global identifications. Euclidean AdS_3 is topologically equal to a solid (filled-in) torus. The asymptotic metric is given by

$$ds^2 \sim r^2|d\phi + \mathrm{i}dt/l|^2 + \frac{dr^2}{r^2} \tag{1.3}$$

for $r \to \infty$, where ϕ and t are, respectively, a spatial angular coordinate and periodic time, l is related to the cosmological constant. ϕ and t satisfy the periodicities $\phi + \mathrm{i}t/l \sim \phi + \mathrm{i}t/l + n + m\tau$. A homology basis of the torus is given by two primitive cycles A and B with unit intersection $A \cap B = 1$. We choose the A-cycle to be contractible in case of the solid torus. A choice of A determines the filling of the torus and therefore the AdS_3 geometry. The choice determines B up to multiples of A since $A \cap A = 0$. The choice of A is made with respect to a distinguished homology basis α and β, with $\alpha \cap \beta = 1$. The periods of a holomorphic one form ω over α and β are given by $\int_\alpha \omega = 1$ and $\int_\beta \omega = \tau$. A and B are integer linear combinations of α and β preserving the intersection number. This determines that the two oriented bases are related by an element of Γ. The complex structure parameter of the torus is then defined by

$$\tau' = \frac{\int_B \omega}{\int_A \omega} = \frac{a\tau + b}{c\tau + d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma. \tag{1.4}$$

Since a choice of A determines B only up to a multiple of A we find that AdS_3 geometries are related to elements of $\Gamma_\infty \setminus \Gamma$. Γ_∞ is the parabolic subgroup of Γ of elements

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

for $n \in \mathbb{Z}$. Note that the τ''s correspond to equivalent asymptotic tori, but that they represent different fillings of the tori. We can see what different choices of A correspond to in gravity. For example when the primitive contractible cycle is $\Delta(\phi + \mathrm{i}t) \sim 1$, the spatial circle is contractible and we have periodic time, this is thermal AdS_3. In case we take $\Delta(\phi + \mathrm{i}t) \sim \tau$, the spatial circle is non-contractible and thus we have a black hole geometry, this is the BTZ black hole [24]. The Einstein–Hilbert action can be renormalized to obtain a finite answer [25, 26]. We find for the action of both geometries

$$(1.5) \qquad S_{\mathrm{thermal}} = -\frac{2\pi\mathrm{i}}{24}\left(c_{\mathrm{L}}\tau - c_{\mathrm{R}}\bar{\tau}\right), \qquad S_{\mathrm{BTZ}} = -\frac{2\pi\mathrm{i}}{24}\left(-\frac{c_{\mathrm{L}}}{\tau} + \frac{c_{\mathrm{R}}}{\bar{\tau}}\right),$$

where $c_{\mathrm{L}} = c_{\mathrm{R}} = \frac{3l}{2G}$. These actions naturally generalize to actions of other geometries represented by $\Gamma_\infty \setminus \Gamma$. Eventually, we are interested in the description of supersymmetric geometries, where the right moving part of the boundary SCFT is in the ground state. States are therefore weighted by the exponent of the holomorphic part of the action in the path integral. Such a holomorphic action can be realized by adding an appropriate gravitational Chern–Simons term. Our heuristic Ansatz for the gravity path integral is

$$(1.6) \qquad \mathcal{Z}_{\mathrm{grav}}(\tau) = \sum_{\Gamma_\infty \setminus \Gamma} \mathrm{e}^{-\frac{2\pi\mathrm{i}c_{\mathrm{L}}}{24}\left(\frac{a\tau+b}{c\tau+d}\right)}.$$

This sum is already similar to one of the main results of this paper, (3.4). The partition function is not convergent, so a suitable regularization is necessary. We will determine the divergence and subtract that from the path integral. We can rewrite the exponent for $c \neq 0$ as

$$(1.7) \qquad \mathrm{e}^{-2\pi\mathrm{i}\left(\frac{c_{\mathrm{L}}}{24}\frac{a}{c} - \frac{\frac{c_{\mathrm{L}}}{24}}{c(c\tau+d)}\right)} = \mathrm{e}^{\left(-2\pi\mathrm{i}\frac{c_{\mathrm{L}}}{24}\frac{a}{c}\right)}\left(\sum_{l=0}^{\infty} \frac{\left(2\pi\mathrm{i}\frac{\frac{c_{\mathrm{L}}}{24}}{c(c\tau+d)}\right)^l}{l!}\right).$$

Convergence of the sum over (c, d) can be shown for all but the term with $l = 0$. We thus have to subtract the term with $l = 0$ from the sum. We arrive at

(1.8)
$$\mathcal{Z}_{\mathrm{grav}}(\tau) = \sum_{\Gamma_\infty\backslash\Gamma} \mathrm{e}^{-\frac{2\pi \mathrm{i} c_{\mathrm{L}}}{24}\left(\frac{a\tau+b}{c\tau+d}\right)} - r(a,c), \qquad r(a,c) = \begin{cases} \mathrm{e}^{-\frac{2\pi \mathrm{i} c_{\mathrm{L}}}{24}\frac{a}{c}}, & c \neq 0, \\ 0, & c = 0. \end{cases}$$

This is the regularization suggested in [10] for the partition function of pure gravity in AdS_3. In case of negative integer weight more terms need to be subtracted. This was proposed earlier in [8]. Equations (4.8) to (4.10) explain a very natural generalization of this idea to non-integer weight. We propose that this is the proper way to regularize the gravity path integral in AdS_3 because in contrast to the fareytail transform the degeneracies are not changed with respect to the CFT partition function and it holds for general weights depending on the matter content of the theory.

As indicated earlier, our main interest lies in the study of supergravity in AdS_3 with a supersymmetric boundary theory. Dijkgraaf *et al.* [5] considered the case of type II string theories on $\mathrm{AdS}_3 \times K$ whose holographic dual is an $\mathcal{N} = (4,4)$ superconformal field theory. A second application is to the AdS_3 supergravities with $(0,4)$ supersymmetry. These arise in the context of M-theory black holes. The relevant partition function of the SCFT is the so-called elliptic genus. This is an index[4] and therefore one might hope to find an exact semi-classical expansion of these functions. This gives some motivation for expecting a fareytail expansion.

We denote the elliptic genus by $\chi(\tau, z)$, where z is a vector in a complex vector space. Standard properties of superconformal field theory show that $\chi(\tau, z)$ transforms as a (generalized) Jacobi form. In the case of the $\mathcal{N} = (2,2)$ elliptic genus, z is one dimensional. The dependence on z arises from the presence of gauge fields in the bulk of AdS_3. Applying the reasoning as before, we expect an expansion of the form

$$\chi(\tau, z) \sim \sum_{\Gamma_\infty\backslash\Gamma} \chi^-\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right). \tag{1.9}$$

$\chi^-(\tau, z)$ is a truncation of the Fourier expansion of $\chi(\tau, z)$. This truncation corresponds to states that are not sufficiently massive to form black holes. The partition function $\chi(\tau, z)$, written as in Equation (1.9), is a sum of the light excitations over all the geometries given by $\Gamma_\infty \setminus \Gamma$. Section 3 presents the mathematically rigorous fareytail for the elliptic genera. We refer to

[4]It is the character-valued index of the right-moving Dirac–Ramond operator.

Section 5 for more details on the physical interpretation and the special role played by the constant term in the Fourier expansion.

We conclude the introduction by giving the outline of the paper. In Section 2 we review relevant aspects of partition functions in CFT's. Section 3 presents the modern fareytail, including the expressions for elliptic genera, relevant for the D1–D5 systems and $\mathcal{N} = 2$ black holes. The derivations are relegated to the Appendix. Section 4 discusses possible modular anomalies arising from the regularization, together with the constraints imposed on the polar terms. We discuss applications of the fareytail expansion in Section 5 and indicate novel aspects of the modern fareytail. Section 6 discusses potential holomorphic anomalies in the partition functions. We finish with some concluding remarks in Section 7.

2. Modular invariance and elliptic genera

We review very briefly invariance under Γ of conformal field theory partition functions on a torus, and point out aspects that are important for our discussion. A torus is conveniently represented as the quotient of the complex plane by a lattice Λ, spanned by generators $\vec{\alpha}$ and $\vec{\beta}$. A conformal field theory on a torus does not depend on the size of the torus nor on any absolute direction of the lattice vectors, so it naturally depends only on $\tau = \left(\vec{\alpha}\cdot\vec{\beta} + i|\vec{\alpha}\times\vec{\beta}|\right)/|\vec{\alpha}|^2$. The theory should furthermore be invariant under large orientation preserving reparametrizations which leave the lattice invariant. This is the famous group $\Gamma = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{Z},\ ad-bc=1\right\}$.

The partition function of a bosonic conformal field theory on a torus is defined by

$$\mathcal{Z}(\tau) = \mathrm{Tr}\left(q^{L_0-\frac{c_{\mathrm{L}}}{24}}\bar{q}^{\bar{L}_0-\frac{c_{\mathrm{R}}}{24}}\right). \tag{2.1}$$

A factor of $(-1)^F$ must be included depending on the boundary conditions (Neveu-Schwarz or Ramond) when a partition function with fermions is considered. $\mathcal{Z}(\tau)$ must be regular in the upper half plane $\mathcal{H} : \mathrm{Im}(\tau) > 0$. Possible poles occur only at $i\infty \cup \mathbb{Q}$. Modular invariance has important consequences for the content of holomorphic and anti-holomorphic sectors.

Elliptic genera are distinguished partition functions of supersymmetric CFTs because they contain important topological information. We briefly review now elliptic genera in $\mathcal{N} = (4,4)$ and $(0,4)$ SCFTs. Both appear as boundary conformal field theory of certain supergravities in AdS_3. $\mathcal{N} = (4,4)$ SCFTs arise in the context of D1–D5 systems, the SCFT is a sigma model with target space $\mathrm{Sym}^m(X)$ at the orbifold point in moduli space [27]. X is a two complex dimensional Ricci flat manifold. $\mathcal{N} = (0, 4)$ SCFTs arise in the study of four dimensional $\mathcal{N} = 2$ black holes, which can be described by wrapped M5 branes with fluxes after an uplift to M-theory [28].

We use $\mathcal{N} = (2, 2)$ notation to calculate the elliptic genus of $\mathcal{N} = (4, 4)$ SCFT. The elliptic genus of an $\mathcal{N} = (2, 2)$ SCFT is defined as a trace over the Ramond–Ramond sector by

$$\chi(\tau, z)_X = \mathrm{Tr}_{\mathrm{RR}}(-)^F y^{J_0} q^{L_0 - \frac{c_L}{24}} \bar{q}^{\bar{L}_0 - \frac{c_R}{24}}. \tag{2.2}$$

F is the fermion number and given by $\frac{1}{2}(J_0 - \tilde{J}_0)$. $\chi(\tau, z)_X$ is independent of $\bar{q}$, because the insertion of $(-)^F$ projects to right moving ground states. When the SCFT is a sigma model, the elliptic genus can be shown to equal an integral of a Chern character times the Todd class over X. This point of view leads to the following explicit expression for the elliptic genus [30, 31]:

$$\chi(\tau, z)_X = \int_X \prod_{i=1}^{d/2} \frac{\theta_1(\tau, z + \xi_i)}{\theta_1(\tau, \xi_i)} 2\pi \mathrm{i} \xi_i, \tag{2.3}$$

where the ξ_i are defined by

$$c(T_X) = 1 + c_1(T_X) + \cdots + c_{d/2}(T_X) = \prod_{i=1}^{d/2} (1 + 2\pi \mathrm{i} \xi_i). \tag{2.4}$$

$\chi(\tau, z)_X$ reduces for different values of the parameter z to the Euler number, Hirzebruch signature or $\hat{A}$ genus. $\theta_1(\tau, z)$ is the odd Jacobi theta function. For the definition see the appendix of [30].

The elliptic genus for a two complex dimensional Kähler manifold X with Euler number χ and Hirzebruch signature σ can straightforwardly be calculated:

$$\chi(\tau, z)_X = -\frac{\sigma}{16} \chi(\tau, z)_{K3} + \frac{3}{8\pi^2} \left(\sigma + \frac{2}{3} \chi \right) \frac{(\partial_z \theta_1(\tau, z))^2}{\eta(\tau)^6}, \tag{2.5}$$

with

$$\chi(\tau, z)_{K3} = 24 \frac{\theta_3(\tau, z)^2}{\theta_3(\tau)^2} - 2 \frac{\theta_4(\tau)^4 - \theta_2(\tau)^4}{\eta(\tau)^4} \frac{\theta_1(\tau, z)^2}{\eta(\tau)^2}. \tag{2.6}$$

Dijkgraaf *et al.* [32] explains how to write a generating function for the elliptic genera of $\mathrm{Sym}^m(X)$, starting from the elliptic genus of X.

Transformation properties of the elliptic genus under Γ can be deduced from the CFT and as well from Equation (2.3) [30]. Most important is the case when $c_1(T_X)=0$. The elliptic genus transforms in this case as a weak Jacobi form of weight $k=0$ and index $m=c_{\rm L}/6=d/4$. Jacobi forms with weight k and index m transform in the following way:[5]

(2.7)
$$\phi\left(\gamma(\tau),\frac{z}{j(\gamma,\tau)}\right)=j(\gamma,\tau)^k e\left(\frac{mcz^2}{j(\gamma,\tau)}\right)\phi(\tau,z),\qquad \gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma,$$
$$\phi\left(\tau,z+\lambda\tau+\mu\right)=(-1)^{2m(\lambda+\mu)}e\left(-m(\lambda^2\tau+2\lambda z)\right)\phi(\tau,z),\qquad (\lambda,\ \mu)\in\mathbb{Z}^2.$$

The transformation property in the second line follows from the invariance of the SCFT under spectral flow. Spectral flow is a symmetry of the algebra; the bosonic generators transform as

$$L_n\to L_n+\lambda J_n+\frac{c}{6}\lambda^2\delta_{n,0},\qquad J_n\to J_n+\frac{c}{3}\lambda\delta_{n,0}. \tag{2.8}$$

Integer spectral flow maps Ramond states to Ramond states and Neveu–Schwartz to Neveu–Schwartz states, whereas half-integer spectral flow exchanges the states in the two sectors. The elliptic genus does not transform as a Jacobi form when $c_1(T_X)\neq 0$, but instead transforms with a shift.

We describe now some important properties of Jacobi forms. Proofs can be found in [33]. We expand a weak Jacobi form as a Fourier series

$$\phi(\tau,z)=\sum_{n\geq 0,l\in\mathbb{Z}}c(n,l)q^n y^l. \tag{2.9}$$

The transformation property that is based on spectral flow determines $c(n,l)$ to be a function only of $4mn-l^2$ and $l \mod 2m$. A Jacobi form is called a "weak" Jacobi form when $c(n,l)$ is only non-zero when $4mn-l^2\geq -m^2$. Furthermore, we can deduce that $\phi(\tau,z)$ can be decomposed into a vector-valued modular form and theta functions

$$\phi(\tau,z)=\sum_{\mu \mod 2m}h_\mu(\tau)\theta_{m,\mu}(\tau,z), \tag{2.10}$$

[5]Throughout the paper we use the convention common in the math literature that $e(x):=\mathrm{e}^{2\pi i x}$. We will also frequently use the notation $\gamma(\tau)=\frac{a\tau+b}{c\tau+d}$ and $j(\gamma,\tau)=c\tau+d$ where a,b,c,d are the familiar elements of γ when written as a 2×2 matrix. Warning: the use of $j(\gamma,\tau)$ in the mathematics literature is not consistent, it is also sometimes used to denote $(c\tau+d)^{\frac{1}{2}}$ multiplied with the appropriate unitary factor.

where μ is a coset representative $\mathbb{Z}/2m\mathbb{Z}$. $h_\mu(\tau,z)$ and $\theta_{m,\mu}(\tau,z)$ are given by

(2.11)
$$h_\mu(\tau) = \sum_{n=-\mu^2 \mod 4m} c_\mu(n) q^{n/4m}, \qquad \theta_{m,\mu}(\tau,z) = \sum_{\substack{l\in\mathbb{Z} \\ l=\mu \mod 2m}} q^{l^2/4m} y^l,$$

with $c_\mu(n) = (-1)^{2ml} c(\frac{n+l^2}{4m}, l)$, $l=\mu \mod 2m$. All the information concerning the Fourier coefficients of $\phi(\tau,z)$ is thus captured in $h_\mu(\tau)$. The theta functions transform as a modular vector under modular transformations. The generators S and T of Γ transform $\theta_{m,\mu}(\tau)$ to

$$\theta_{m,\mu}\left(\frac{-1}{\tau}, \frac{z}{\tau}\right) = \sqrt{\frac{\tau}{2m\mathrm{i}}} e\left(\frac{mz^2}{\tau}\right) \sum_{\nu \mod 2m} e\left(-\frac{\mu\nu}{2m}\right) \theta_{m,\nu}(\tau,z),$$
$$\theta_{m,\mu}(\tau+1,z) = e\left(\frac{\mu^2}{4m}\right)\theta_{m,\mu}(\tau,z). \tag{2.12}$$

For an unambiguous value of the square root, we define $\log z$ to be given by $\log z := \log|z| + \mathrm{i}\arg(z)$ with $-\pi < \arg(z) \leq \pi$. For general transformations under Γ, we define a matrix $M(\gamma)^\mu_\nu$ by

$$\theta_{m,\mu}\left(\gamma(\tau), \frac{z}{c\tau+d}\right) = j(\gamma,\tau)^{\frac{1}{2}} e\left(\frac{mcz^2}{c\tau+d}\right) M^{-1}(\gamma)^\mu_\nu \theta_{m,\nu}(\tau,z). \tag{2.13}$$

Such that we have for $h_\mu(\gamma(\tau))$ by Equation (2.7)

$$h_\mu(\gamma(\tau)) = j(\gamma,\tau)^{k-\frac{1}{2}} M(\gamma)^\nu_\mu h_\nu(\tau). \tag{2.14}$$

The introduction of $M(\gamma)^\nu_\mu$ is convenient for a generalization to similar partition functions, as for example elliptic genera of $\mathcal{N}=(0,4)$ SCFTs.

We will very briefly review the elliptic genera of $\mathcal{N}=(0,4)$ SCFT arising in the study of $\mathcal{N}=2$ M-theory black holes. We refer to [15, 23, 28, 29, 34] for the precise details. Denef and Moore [8] perform a similar analysis which results in the same partition function from the point of view of IIA string theory. Elliptic genera in an $\mathcal{N}=(0,4)$ SCFT are defined in a similar manner to those in $\mathcal{N}=(2,2)$ SCFT. However, we need to insert a factor of F^2 in order to obtain a non-zero answer, because of the cancellation between bosonic and fermionic degrees of freedom on the supersymmetric side of the $\mathcal{N}=(0,4)$ SCFT. This sum projects on half-BPS states on the supersymmetric side. The CFT arises after reducing the degrees of freedom from an M5-brane with world volume $\Sigma \times T^2$ to T^2 where Σ is an ample divisor Poincaré dual to $P \in H^2(X,\mathbb{Z})$ in a Calabi–Yau three-fold X [28,29]. We will often write P in place of Σ for quantities that only depend on the homology class of Σ. The $\mathcal{N}=(0,4)$ elliptic genus of this SCFT is given by [15, 34]

$$\begin{aligned}\chi(\tau,z)_P = \mathrm{Tr}_{\mathrm{R}}\Bigg[&\frac{1}{2}F^2(-)^F e(P\cdot Q/2)\\ &\times e\left(\tau\left(L_0-\frac{c_{\mathrm{L}}}{24}\right)-\bar\tau\left(\bar L_0-\frac{c_{\mathrm{R}}}{24}\right)+z\cdot Q\right)\Bigg],\end{aligned} \tag{2.15}$$

where $Q \in H^4(X;\mathbb{Z})$ are M2 brane charges of the black hole (generated by fluxes on the M5 brane) and $z \in H^2(X;\mathbb{C})$.

A spectral flow exists in this SCFT similar to the spectral flow in $\mathcal{N}=(2,2)$ SCFT allowing one to give an analogous "singleton" decomposition in terms of theta functions. In order to write this out we need to introduce some notation. The lattice $L_X := \iota_P^*(H^2(X;\mathbb{Z})) \subset H^2(P;\mathbb{Z})$ has signature $(+^1,-^{b_2-1})$, where $b_2=\dim H_2(X)$. The integral quadratic form on L_X can be written in terms of the intersection numbers d_{abc} of X by introducing an integral basis D_a for $H_4(X,\mathbb{Z})$ and writing $v^2 = d_{abc}P^a v^b v^c$. The sublattice $L_X \oplus L_X^\perp \subset H^2(P,\mathbb{Z})$ is of index $\det D_{ab}$ where $D_{ab} := d_{abc}P^c$. We choose a set of glue vectors, μ, i.e., a rule for lifting elements of the discriminant group $[\mu] \in \mathcal{D} = H^2(P,\mathbb{Z})/(L_X \oplus L_X^\perp)$ to $\mu \in H^2(P,\mathbb{Z})$ so that any vector $v \in H^2(P,\mathbb{Z})$ can be written $v = v^\parallel + \mu + v^\perp$, with $v^\parallel \in L_X, v^\perp \in L_X^\perp$. Now $H^2(P;\mathbb{Z})\otimes\mathbb{Q}$ has a projection to the negative and positive definite subspaces and we denote this projection by $v \to v_+ \oplus v_-$. If $X, Y \in H^2(P;\mathbb{Z})\otimes\mathbb{Q}$ and f is holomorphic introduce the notation

$$E[f(\tau)X\cdot Y] := \mathrm{e}^{-2\pi\mathrm{i}\, f(\tau)\, X_-\cdot Y_- -2\pi i\, f(\bar\tau)\, X_+\cdot Y_+}, \qquad E[A+B] := E[A]\, E[B]. \tag{2.16}$$

We now introduce the Siegel–Narain theta function for the lattice L_X:

$$\Theta_\mu(\tau,z) := \sum_{v\in L_X} E\left[\frac{\tau}{2}\left(\frac{P}{2}+\mu^\parallel+v\right)^2+\left(\frac{P}{2}+\mu^\parallel+v\right)\cdot\left(z+\frac{P}{2}\right)\right], \tag{2.17}$$

where $z \in L_X\otimes\mathbb{C}$ and the projection to (z_+,z_-) is extended $\mathbb{C}$-linearly. Note that Θ_μ is non-holomorphic in τ. In terms of these theta functions we have the decomposition:

$$\chi(\tau,z)_P = \sum_\mu h_\mu(\tau)\Theta_\mu(\tau,z), \tag{2.18}$$

Here the functions $h_\mu(\tau)$ are holomorphic in τ and have no singularities in the upper half plane.

Modular transformations act on the argument of the theta function according to

$$\gamma \cdot (\tau, z_+, z_-) := \left(\frac{a\tau + b}{c\tau + d}, \frac{z_+}{c\bar{\tau} + d}, \frac{z_-}{c\tau + d} \right). \tag{2.19}$$

We will abbreviate (2.19) as $\gamma \cdot (\tau, z)$. Now, for generic SU(3) holonomy Calabi–Yau, duality symmetries in string theory imply

$$\chi(\gamma \cdot (\tau, z)) = \tilde{M}(\gamma)\,(c\tau + d)^{-3/2}(c\bar{\tau} + d)^{1/2}\,E\left[\frac{c}{c\tau + d}\frac{z^2}{2} \right] \chi(\tau, z), \tag{2.20}$$

where $\tilde{M}$ is a multiplier system given in [8]. From this one deduces that the vector of modular forms $h_\mu(\tau)$ transforms with weight $\frac{-b_2}{2} - 1$. These functions have a Fourier expansion

$$h_\mu(\tau) = \sum_{n \geq 0} H_\mu(n) e((n - \Delta_\mu)\tau), \tag{2.21}$$

where

$$\Delta_\mu = \frac{c_{\mathrm{L}}}{24} + \mathrm{Max}_{v \in L_X^\perp} \frac{1}{2}(v + \mu^\perp)^2, \tag{2.22}$$

and $c_{\mathrm{L}} = \chi(P) = P^3 + c_2(X) \cdot P$ is the Euler character of a generic smooth divisor in the linear system $|P|$. (In taking the maximum note that the quadratic form on $L_X^\perp$ is negative definite.) For $\mu = 0$ the leading coefficient $H_{\mu=0}(0) = (-1)^{I_P - 1} I_P$ where $I_P = \frac{P^3}{6} + \frac{c_2(X) \cdot P}{12}$ is the Euler character of the linear system $|P|$.

There is also a supergravity viewpoint on the decomposition Equation (2.18). It can be regarded as the singleton decomposition of the M5-brane partition function. The general singleton decomposition of the M5-brane partition function was given in [35], where it was explained that the discriminant group $\mathcal{D}$ is the group of Page charges in the presence of G-flux.

Summarizing, we have seen the relevance of vector-valued modular forms in the study of partition functions; the weight and multiplier system are determined by the content and symmetries of the theory.

3. The modern fareytail

The previous section introduced elliptic genera and some of their properties. It motivated the study of vector-valued modular forms $f_\mu(\tau)$ of non-positive weight w. This section describes a fareytail expansion for vector-valued modular forms and subsequently for elliptic genera. The novel aspect of our discussion is the absence of the "fareytail transform." A summary of the derivation of the result is given in Appendix A. Section 4 examines how the regularization preserves the modular properties.

3.1. Vector-valued modular forms

This section states the fareytail expansion of vector-valued modular forms in detail. Let us, then, consider a vector-valued modular form $f_\mu(\tau)$ transforming under Γ, as

$$f_\mu(\gamma(\tau)) = j(\gamma,\tau)^w M(\gamma)_\mu^\nu f_\nu(\tau). \tag{3.1}$$

We will be concerned with forms of weight $w \leq 0$, where w is not necessarily integral. For example, for the elliptic genus $w = -1/2$. For the OSV conjecture $w = -1 - b_2/2$. We therefore must choose a branch of the log to define $j(\gamma,\tau)^w$ and we take $\log z := \log|z| + \mathrm{i}\arg(z)$ with $-\pi < \arg(z) \leq \pi$. For the $(2,2)$ and $(0,4)$ elliptic genus the multiplier system $M(\gamma)$ will turn out to be unitary matrices. See Appendix C.

We assume $M(T)$ is diagonalizable, and hence f_μ has a Fourier expansion

$$f_\mu(\tau) = \sum_{m=0}^{\infty} F_\mu(m) q^{m-\Delta_\mu}, \tag{3.2}$$

where $F_\mu(0) \neq 0$.[6] Poles of $f_\mu(\tau)$ occur only at the cusps, i.e., $\gamma(i\infty)$, $\gamma \in \Gamma$. The pole at $\tau = \mathrm{i}\infty$ arises from the polar part $f^-(\tau)$ of the partition function

$$f_\mu^-(\tau) := \sum_{m-\Delta_\mu<0} F_\mu(m) q^{m-\Delta_\mu}. \tag{3.3}$$

The Fourier coefficients $F_\mu(m)$ can be calculated by the Rademacher circle method [5, 11, 36]. Sufficient information to calculate them are the Fourier

[6] In general, we follow the notation of [5]. However, we have changed the sign of Δ_μ relative to this reference. Also, following [33] we denote the index of a Jacobi form by m, whereas k is used in [5]. In this paper, we use w for the weight of a vector-valued modular form; k is the weight of a Jacobi form.

coefficients $F_\mu(m)$ for $m-\Delta_\mu<0$, the weight w and the multiplier system. Starting from the Fourier coefficients for general m, we can derive the fareytail expansion of the partition function as a sum over the limit coset: $\lim_{K\to\infty}(\Gamma_\infty\setminus\Gamma)_K=\lim_{K\to\infty}\sum_{|c|\le K}\sum_{\substack{|d|\le K\\(c,d)=1}}$.

Some details are given in Appendix A. The result is a sum over the polar part

$$f_\mu(\tau)=\frac{1}{2}F_\mu(\Delta_\mu)+\frac{1}{2}\sum_{n-\Delta_\nu<0}\lim_{K\to\infty}\sum_{(\Gamma_\infty\setminus\Gamma)_K} \times j(\gamma,\tau)^{-w}M^{-1}(\gamma)_\mu^\nu F_\nu(n)e((n-\Delta_\nu)\gamma(\tau))R\left(\frac{2\pi\mathrm{i}|n-\Delta_\nu|}{c(c\tau+d)}\right). \tag{3.4}$$

Here $R(x)$ is the function

$$R(x):=1-\frac{1}{\Gamma(1-w)}\int_x^\infty \mathrm{e}^{-z}z^{-w}dz=\frac{1}{\Gamma(1-w)}\int_0^x \mathrm{e}^{-z}z^{-w}dz. \tag{3.5}$$

The expression $F_\mu(\Delta_\mu)$ vanishes except when $\Delta_\mu\in\mathbb{N}$, in which case it is given by Equation (A.7). We stress that Equation (3.4) is derived for general non-positive weight w, including integer and half-integer cases. The exclusion of positive weight is a consequence of the bound $p\ge 1$ in (B.1). Of course, the well-known technique of Poincaré series is applicable for $w>2$, since the sum is convergent in that case. Naive application of this technique for the reconstruction of a modular form with $w\le 0$ from its polar part would not have the first term in (3.4) and would not have the regularizing factor $R(x)$. Note that the first integral expression in (3.5) shows that $R(x)$ approaches 1 exponentially fast for $\mathrm{Re}(x)\to\infty$, while the second shows that $R(x)\sim\frac{x^{1-w}}{\Gamma(2-w)}$ for $x\to 0$. Using these simple estimates, convergence of the sum for $w\le 0$ is established in Appendix A.

Equation (3.4) can be rewritten in the following form:

$$f_\mu(\tau)=\frac{1}{2}F_\mu(\Delta_\mu)+\frac{1}{2}\sum_{n-\Delta_\nu<0}\lim_{K\to\infty}\sum_{(\Gamma_\infty\setminus\Gamma)_K} \times M^{-1}(\gamma)_\mu^\nu F_\nu(n)\left\{\frac{e((n-\Delta_\nu)\gamma(\tau))}{j(\gamma,\tau)^w}-r(\gamma,\tau,n-\Delta_\nu)\right\}. \tag{3.6}$$

For integer weight $r(\gamma,\tau,n-\Delta_\nu)$ can be simplified to

$$r(\gamma,\tau,n-\Delta_\nu)=\begin{cases}\dfrac{e\left((n-\Delta_\nu)\frac{a}{c}\right)\sum_{l=0}^{|w|}\frac{1}{l!}\left(\frac{2\pi\mathrm{i}|n-\Delta_\nu|}{c(c\tau+d)}\right)^l}{j(\gamma,\tau)^w}, & c\neq 0,\\ 0, & c=0.\end{cases} \tag{3.7}$$

This is the subtraction used in [8, 10] to write a non-positive weight partition function directly as a fareytail. The same regularization had been previously used in the math literature in [7]. The generalization $R(x)$ is due to Niebur [6].

It is natural to ask if one can turn things around, that is: starting with a projective representation $M(\gamma)$, and a non-positive weight w, can one choose arbitrary coefficients $F_\mu(n)$ with $n - \Delta_\mu < 0$ and use Equation (3.4) to construct a corresponding modular form with specified polar part? In general, this is not possible. We discuss this in detail in Section 4, drawing on the technical results of Appendix A.

3.2. Application to elliptic genera

As explained in Section 2 elliptic genera may be expressed as sums of theta functions with coefficients $h_\mu(\tau)$ forming a vector of modular forms. The theta functions used in case of $\mathcal{N} = (2,2)$ elliptic genera, transform as

$$\theta_{m,\mu}(\tau, z) = M(\gamma)_\nu^\mu \frac{e\left(-m\frac{cz^2}{c\tau+d}\right)}{j(\gamma,\tau)^{\frac{1}{2}}} \theta_{m,\nu}\left(\gamma(\tau), \frac{z}{c\tau+d}\right). \tag{3.8}$$

We will insert Equations (3.4) and (3.8) in Equation (2.10). The coefficients $F_\mu(n)$ are in this case the Fourier coefficients of the elliptic genus, $c(n,\ell) = c_\mu(4mn - \ell^2)$ with $\ell = \mu \mod 2m$. Note that in this case Δ_μ is given by $\frac{\mu^2}{4m}$ mod $\mathbb{Z}$ and $F_\mu(\Delta_\mu)$ is only non-zero when $\Delta_\mu \in \mathbb{N}$.

Thus we find for the elliptic genus of a Ricci flat manifold

$$\begin{aligned}\chi(\tau, z)_X = &\sum_{\mu \bmod 2m} \frac{1}{2} c_\mu(0)\theta_{m,\mu}(\tau, z) \\ &+ \frac{1}{2} \sum_{n-\frac{l^2}{4m}<0} \lim_{K\to\infty} \sum_{(\Gamma_\infty\backslash\Gamma)_K} c_\mu(4mn - l^2) \\ &\times e\left(n\gamma(\tau) + l\frac{z}{c\tau+d} - m\frac{cz^2}{c\tau+d}\right) R\left(\frac{2\pi \mathrm{i}|n - \frac{l^2}{4m}|}{c(c\tau+d)}\right).\end{aligned} \tag{3.9}$$

Note that we cannot write $c_\mu(0)\theta_{m,\mu}(\tau)$ as a sum of simple exponential factors over $\Gamma_\infty \setminus \Gamma$ but it could, in principle, be written as such a sum over $\Gamma_\infty \setminus \Gamma / \Gamma_\infty$ by Equation (A.7). Since the weight of the vector-valued modular forms is $-\frac{1}{2}$ in this case, $R(x)$ can be expressed as

$$R(x) = \mathrm{erf}(\sqrt{x}) - 2\sqrt{\frac{x}{\pi}}\mathrm{e}^{-x}, \tag{3.10}$$

where $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \mathrm{e}^{-t^2}dt$, which is the error function.

Analogously, the $(0,4)$ elliptic genus can be written, using the notation introduced below Equation (2.15):

$$\begin{aligned}\chi(\tau,z)_P = &\sum_\mu \frac{1}{2}H_\mu(\Delta_\mu)\Theta_\mu(\tau,z) + \frac{1}{2}\sum_{(\Gamma_\infty\backslash\Gamma)} \tilde{M}^{-1}(\gamma)j(\gamma,\tau)^{\frac{3}{2}}j(\gamma,\bar{\tau})^{-\frac{1}{2}}E\\ &\times\left[-\frac{c}{c\tau+d}\frac{z^2}{2}\right]\sum_{n,\mu:n-\Delta_\mu<0} H_\mu(n)R\left(\frac{2\pi\mathrm{i}|n-\Delta_\mu|}{c(c\tau+d)}\right)\\ &\times e((n-\Delta_\mu)\gamma(\tau))\sum_{q\in L_X+\mu^{\|}+P/2} E\left[\frac{1}{2}\gamma(\tau)q^2 + q\cdot\left(\frac{z}{c\tau+d}+\frac{P}{2}\right)\right].\end{aligned} \tag{3.11}$$

The exponentials of $\gamma(\tau)$ are weighted by the quantity

$$n - \Delta_\mu - \frac{1}{2}q_-^2. \tag{3.12}$$

In the type IIA setting discussed in [8] this quantity is the denoted $-\hat{q}_0$ and it can be written in terms of D0- and D2-charges (q_0, Q_a) using

$$\hat{q}_0 = q_0 - \frac{1}{2}D^{ab}Q_aQ_b, \tag{3.13}$$

where D^{ab} is the matrix inverse of $D_{ab} = d_{abc}P^c$. In this form, the polarity condition $\hat{q}_0 > 0$ is analogous to the condition $n - l^2/4m < 0$ in the $(2,2)$ case.

4. Anomalies and period functions

Let us now return to the question asked at the end of Section 3.1. We have seen that the physical considerations motivate the following problem in mathematics:

Suppose we are given a weight $w \leq 0$ and a rank r multiplier system $M(\gamma)$ on Γ. We wish to construct a vector-valued modular form, transforming with weight w and multiplier system M with a prescribed polar part. That is, the coefficients $F_\mu(m)$ in Equation (3.2) with $m - \Delta_\mu < 0$ are prescribed. Note that consistency of this data requires $M(T^\ell)_\mu^\nu = e(-\delta_\mu\ell)\delta_\mu^\nu$.

In general, there is an obstruction to finding such a vector-valued form. We will show that the obstruction is measured by the non-vanishing of a certain vector-valued cusp form of weight $2 - w$ and multiplier system $M(\gamma)^*$.

Let us begin by choosing a vector δ with components δ_μ, $\mu = 1, \ldots, r$, some of whose components are positive. We will attempt to construct a vector-valued modular form which behaves like

$$f(\tau) = \varepsilon(-\delta\tau) + \text{regular}, \tag{4.1}$$

as $q \to 0$. Here $\varepsilon(-\delta\tau)$ is a vector with components

$$\varepsilon(-\delta\tau)_\mu = \begin{cases} e(-\delta_\mu\tau), & \delta_\mu > 0, \\ 0, & \delta_\mu \leq 0 \end{cases} \tag{4.2}$$

and "regular" means there is a q-expansion with non-negative (possibly fractional) powers of q.

At first, it would appear to be straightforward to construct $f(\tau)$ by the method of images. Introduce the vector of functions

$$s_\gamma^{(\delta)}(\tau) := j(\gamma,\tau)^{-w} M(\gamma)^{-1} \varepsilon(-\delta\gamma(\tau)).$$

Then it is elementary to check that

$$s_{\gamma\tilde{\gamma}}^{(\delta)}(\tau) = j(\tilde{\gamma},\tau)^{-w} M(\tilde{\gamma})^{-1} s_\gamma^{(\delta)}(\tilde{\gamma}\tau), \tag{4.3}$$

and hence $s_{\gamma\tilde{\gamma}}^{(\delta)}(\tau) = s_{\tilde{\gamma}}^{(\delta)}(\tau)$ for $\gamma \in \Gamma_\infty$. Accordingly, we attempt to average:

$$S^{(\delta)}(\tau) \stackrel{?}{=} \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} s_\gamma^{(\delta)}(\tau). \tag{4.4}$$

Formally, from Equation (4.3) we find $S^{(\delta)}(\tilde{\gamma}\tau) = j(\tilde{\gamma},\tau)^w M(\gamma) S^{(\delta)}(\tau)$. Moreover, the cosets $[\pm 1]$ lead to the prescribed polar term and the remaining terms in the sum are regular for $\tau \to i\infty$. It would thus appear that we have succeeded, but in fact we have not.

The problem with the naive attempt (4.4) is that for $c \to \infty$ we have $|s_\gamma^{(\delta)}(\tau)| \sim |c\tau|^{-w}$ and since we must have weight $w \leq 0$, the series does not converge. We therefore must regularize the series.

To motivate our regularization let us suppose for the moment that $-w \in \mathbb{N}$. We use the identity

$$\gamma(\tau) = \frac{a}{c} - \frac{1}{c(c\tau+d)}, \tag{4.5}$$

which is valid for $c \neq 0$. This allows us to write

$$e(-\delta\gamma(\tau)) = \mathrm{e}^{-2\pi \mathrm{i}\delta\frac{a}{c}}\mathrm{e}^{2\pi\mathrm{i}\frac{\delta}{c(c\tau+d)}}. \tag{4.6}$$

An evident regularization would be to subtract the first $|w|$ terms from the Taylor series expansion of $\mathrm{e}^{2\pi\mathrm{i}\frac{\delta}{c(c\tau+d)}}$ around zero. Thus we introduce the regularized sum:

$$S^{(\delta)}_{\mathrm{Reg}}(\tau) := \frac{1}{2}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}(s^{(\delta)}_\gamma(\tau) + t^{(\delta)}_\gamma(\tau)), \tag{4.7}$$

with

$$t^{(\delta)}_\gamma(\tau) := -j(\gamma,\tau)^{-w}M^{-1}(\gamma)\sum_{j=0}^{|w|}\frac{1}{j!}\left(\frac{1}{c(c\tau+d)}\right)^j (2\pi\mathrm{i}\delta)^j\,\mathrm{e}^{-2\pi\mathrm{i}\delta\frac{a}{c}}. \tag{4.8}$$

Here and in what follows, we understand expressions like $(2\pi\mathrm{i}\delta)^j\mathrm{e}^{-2\pi\mathrm{i}\delta\frac{a}{c}}$ to be vectors whose μth component is zero if $\delta_\mu \leq 0$ and is $(2\pi\mathrm{i}\delta_\mu)^j\mathrm{e}^{-2\pi\mathrm{i}\delta_\mu\frac{a}{c}}$ if $\delta_\mu > 0$, as in Equation (4.2). Note that $t^{(\delta)}_\gamma(\tau)$ is a polynomial in τ. Moreover, the sum in Equation (4.7) is convergent.[7]

Now, the regularization has been carried out for w integral. Remarkably, it may be generalized to non-integral w as follows. Returning to the expression for $t^{(\delta)}_\gamma(\tau)$ we recognize a truncated exponential series. The latter can be written in terms of the incomplete Gamma function using the identity (see Equation (A.16) below):

$$\sum_{k=0}^{\infty}\frac{x^{k+1-w}}{\Gamma(k+2-w)} = \mathrm{e}^x\left(1 - \frac{1}{\Gamma(1-w)}\int_x^\infty \mathrm{e}^{-z}z^{-w}dz\right). \tag{4.9}$$

[7]The convergence is actually a little delicate. One must group together terms with positive and negative values of d to avoid a logarithmic divergence in the sum over d. Once this is done, convergence can be shown for $w \leq 0$. See Appendix A for more details.

Using this we may write $t_\gamma^{(\delta)}(\tau) = 0$ for $c = 0$, while for $c \neq 0$,

$$t_\gamma^{(\delta)}(\tau) := -j(\gamma,\tau)^{-w} M^{-1}(\gamma)\varepsilon(-\delta\gamma(\tau)) \frac{1}{\Gamma(1-w)} \int_{x(\gamma,\delta)}^{\infty} \mathrm{e}^{-z} z^{-w} dz, \tag{4.10}$$

where the factor multiplying $M^{-1}(\gamma)$ on the right is the vector whose μth component is zero for $\delta_\mu \leq 0$ and

$$e(-\delta_\mu \gamma(\tau)) \frac{1}{\Gamma(1-w)} \int_{\frac{2\pi \mathrm{i}\delta_\mu}{c(c\tau+d)}}^{\infty} \mathrm{e}^{-z} z^{-w} dz, \tag{4.11}$$

for $\delta_\mu > 0$. In this form the regularization (4.10) still makes sense for w non-integral, and the regularized sum is again convergent. This follows from the $x \to 0$ asymptotics of $R(x)$.

Of course, now our regularization has spoiled the formal covariance under modular transformations! However, it turns out that it has spoiled it in a controlled way because $t_\gamma^{(\delta)}(\tau)$ is related to certain *period integrals*. For any function $h(\tau)$ on $\mathcal{H}$ decaying sufficiently rapidly at $\mathrm{Im}(\tau) \to \infty$, we can define its period function

$$p(\tau, \bar{y}, \bar{h}) := \frac{1}{\Gamma(1-w)} \int_{\bar{y}}^{-i\infty} \overline{h(z)} (\bar{z} - \tau)^{-w} d\bar{z}. \tag{4.12}$$

Then we claim that

$$t_\gamma^{(\delta)}(\tau) = p(\tau, \gamma^{-1}(-\mathrm{i}\infty), \overline{g_\gamma^{(\delta)}}), \tag{4.13}$$

where

$$g_\gamma^{(\delta)}(z) := j(\gamma, z)^{w-2} \overline{M^{-1}(\gamma)} (-2\pi \mathrm{i}\delta)^{1-w} \varepsilon(\delta\gamma(z)). \tag{4.14}$$

Now, $g_\gamma^{(\delta)}(z)$ transforms simply, and from this one can verify that

$$t_\gamma^{(\delta)}(\tilde{\gamma}\tau) = j(\tilde{\gamma}, \tau)^w M(\tilde{\gamma}) \left[t_{\gamma\tilde{\gamma}}^{(\delta)}(\tau) - p(\tau, \tilde{\gamma}^{-1}(-\mathrm{i}\infty), \overline{g_{\gamma\tilde{\gamma}}^{(\delta)}}) \right]. \tag{4.15}$$

Because of the second term in Equation (4.15) our regularized sum does not transform covariantly. Rather we have

$$S_{\mathrm{Reg}}^{(\delta)}(\tilde{\gamma}\tau) = j(\tilde{\gamma}, \tau)^w M(\tilde{\gamma}) S_{\mathrm{Reg}}^{(\delta)}(\tau) - j(\tilde{\gamma}, \tau)^w M(\tilde{\gamma}) \frac{1}{2} \times \sum_{\Gamma_\infty \backslash \Gamma} p(\tau, \tilde{\gamma}^{-1}(-\mathrm{i}\infty), \overline{g_{\gamma\tilde{\gamma}}^{(\delta)}}). \tag{4.16}$$

Now we would like to simplify the "anomalous" second term on the right-hand side of Equation (4.16). To this end we would like to exchange the summation with the integration in the definition of the period function. Although the second term involves an absolutely convergent sum, we must be very careful about exchanging the sum and integration as well as redefining the sum by $\gamma \to \gamma\tilde{\gamma}^{-1}$. Using results of Niebur [6], which are further explained in the appendix, we have

$$\begin{aligned}&\frac{1}{2}\sum_{\Gamma_\infty\backslash\Gamma} p\left(\tau, \tilde{\gamma}^{-1}(-\mathrm{i}\infty), \overline{g^{(\delta)}_{\gamma\tilde{\gamma}}}\right)\\ &\quad = p\left(\tau, \tilde{\gamma}^{-1}(-\mathrm{i}\infty), \overline{G^{\delta}}\right) + j(\tilde{\gamma},\tau)^{-w} M(\tilde{\gamma})^{-1} F(\delta) - F(\delta),\end{aligned} \tag{4.17}$$

where

$$G^{(\delta)}(\tau) := \frac{1}{2}\sum_{\Gamma_\infty\backslash\Gamma} g^{(\delta)}_{\gamma}(\tau), \tag{4.18}$$

and $F(\delta)$ is a vector of constants given by

$$F(\delta)_\mu = \begin{cases} \pi \displaystyle\sum_{\delta_\nu>0} \frac{(2\pi\delta_\nu)^{1-w}}{\Gamma(2-w)} \sum_{c=1}^{\infty} c^{w-2} K_c(0_\mu, -\delta_\nu), & \delta_\mu \in \mathbb{N}, \\ 0, & \delta_\mu \notin \mathbb{N}, \end{cases} \tag{4.19}$$

where 0_μ is the vector all of whose components are zero and K_c is the generalized Kloosterman sum of Equation (A.5).

The net result of all of this is that in our attempt to construct the weight w modular vector with polar term (4.1) the method of images leads us — more or less uniquely — to define a vector of functions $\hat{S}^{(\delta)}_{\mathrm{Reg}}(\tau) := F(\delta) + S^{(\delta)}_{\mathrm{Reg}}(\tau)$. As $\tau \to \mathrm{i}\infty$ this vector indeed behaves as

$$\hat{S}^{(\delta)}_{\mathrm{Reg}}(\tau) = \varepsilon(-\delta\tau) + \text{regular}. \tag{4.20}$$

However, it satisfies the transformation law:

$$\hat{S}^{(\delta)}_{\mathrm{Reg}}(\tilde{\gamma}\tau) = j(\tilde{\gamma},\tau)^{w} M(\tilde{\gamma}) \left[\hat{S}^{(\delta)}_{\mathrm{Reg}}(\tau) - p^{(\delta)}(\tau,\tilde{\gamma})\right], \tag{4.21}$$

where

$$p^{(\delta)}(\tau,\tilde{\gamma}) := p(\tau,\tilde{\gamma}^{-1}(\infty),\overline{G^{(\delta)}}) = \frac{1}{\Gamma(1-w)}\int_{-\tilde{d}/\tilde{c}}^{-\mathrm{i}\infty}\overline{G^{(\delta)}(z)}(\bar{z}-\tau)^{-w}d\bar{z} \tag{4.22}$$

is a vector of functions defined by

$$G^{(\delta)}_{\mu}(z) = \frac{1}{2}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j(\gamma,z)^{w-2}\sum_{\delta_\nu>0}\left(M^{-1}(\gamma)^{\nu}_{\mu}\right)^{*}(-2\pi\mathrm{i}\delta_\nu)^{1-w}e(\delta_\nu\gamma(z)). \tag{4.23}$$

The vector of functions $p^{(\delta)}$ is an obstruction to the existence of $f(\tau)$.

In contrast to Equation (4.4), the series (4.23) for $G^{(\delta)}(z)$ is nicely convergent. It therefore follows that $G^{(\delta)}(\tau)$ is a vector-valued modular form of weight $2-w$ transforming according to

$$G^{(\delta)}(\gamma\tau) = j(\gamma,\tau)^{2-w}\overline{M(\gamma)}G^{(\delta)}(\tau). \tag{4.24}$$

In fact, $G^{(\delta)}$ is a vector-valued *cusp form*, that is, the components vanish for $\tau\to\mathrm{i}\infty\cup\Gamma(\mathrm{i}\infty)$. This follows since it is clear from the series expansion that $G^{(\delta)}$ vanishes for $\tau\to\mathrm{i}\infty$. We give an explicit formula for the Fourier coefficients in Equation (4.33) below.

Lemma 3.2 of [6] shows that the period integral vanishes if and only if $G^{(\delta)}$ vanishes. Therefore, our cusp form $G^{(\delta)}$, if non-vanishing, forms an obstruction to constructing the vector valued form with prescribed polar term δ. The shift in Equation (4.21) by $p^{(\delta)}$ represents an anomaly under modular transformations. This is a familiar situation in quantum field theory: a divergent quantity is formally invariant, the regularized quantity breaks the invariance, but in a controlled way. Thus the problem of constructing a true modular form with negative weight and specified polar part is a kind of anomaly cancellation problem: one must form linear combinations $\sum_\delta \Omega_\delta \hat{S}^{(\delta)}_{\mathrm{Reg}}$ so that the associated cusp form cancels. The coefficients Ω_δ are exactly the "polar degeneracies" that play a crucial role in the physical discussions of the fareytail transform and the OSV conjecture.

In fact, the analogy goes deeper, since the anomaly is in fact related to a cohomology theory known as Eichler cohomology. It follows from the definition of the period vector that we have the transformation law given in

Equation (A.20). Therefore

$$p^{(\delta)}(\tau,\tilde{\gamma}) - p^{(\delta)}(\tau,\gamma\tilde{\gamma}) + j(\tilde{\gamma},\tau)^{-w}M(\tilde{\gamma})^{-1}p^{(\delta)}(\tilde{\gamma}\tau,\gamma) = 0. \tag{4.25}$$

Defining the standard slash operator on functions $f(\tau,\gamma)$:

$$f(\cdot,\gamma)|_w^M\tilde{\gamma} := j(\tilde{\gamma},\tau)^{-w}M(\tilde{\gamma})^{-1}f(\tilde{\gamma}\tau,\gamma), \tag{4.26}$$

we see that the obstruction to modularity lies in the space of functions satisfying

$$f(\cdot,\tilde{\gamma}) - f(\cdot,\gamma\tilde{\gamma}) + f(\cdot,\gamma)|_w^M\tilde{\gamma} = 0. \tag{4.27}$$

If we interpret $f(\tau,\gamma)$ as a cochain on the group Γ with values in functions of τ then (4.27) is the statement that f is a one-cocycle. A one-coboundary is a function of the form $f(\cdot,\gamma) = b(\cdot) - b(\cdot)|_w^M\gamma$ where $b(\tau)$ is a single function of τ. We would like to define a cohomology group as one-cocycles modulo one-coboundaries. Of course, the transformation law (4.21) shows that $\hat{S}^{(\delta)}_{\mathrm{Reg}}$ trivializes $p^{(\delta)}$, so to get an interesting theory we need to restrict the Γ module of functions in which we compute cohomology.

When the weight w is a negative integer, $p^{(\delta)}(\tau,\gamma)$ is a vector of polynomials of degree $\leq |w|$. In the scalar case the space of obstructions to constructing a modular form with prescribed polar part is $H^1(\Gamma, V_{|w|})$ where $V_{|w|}$ is the vector space of polynomials of degree $\leq |w|$. For $|w| \notin \mathbb{N}$, we are forced to work in a larger space of functions, those with at most polynomial growth at the cusps. We refer to [7, 37, 38] for more details.

We conclude by giving some more explicit conditions on the polar degeneracies Ω_δ for anomaly cancellation. Note first that $p^{(\delta)}(\tau,T) = 0$ so it suffices to check

$$\sum_{\delta} \Omega_\delta p^{(\delta)}(\tau,S) = 0 \tag{4.28}$$

since S, T generate Γ. In the case of $-w \in \mathbb{N}$ the coefficients of such a period polynomial are calculated by $\int_0^{i\infty} G^{(\delta)}(z)z^{s-1}dz$, $s \in \mathbb{N}$. Such integrals are known as Mellin transforms. When the Fourier expansion of $G^{(\delta)}(\tau)$ is given by

$$G^{(\delta)}(\tau)_\mu = \sum_{n+\alpha_\mu>0} u^{(\delta)}(n)_\mu q^{n+\alpha_\mu}, \qquad \alpha_\mu = \delta_\mu - \lfloor \delta_\mu \rfloor, \tag{4.29}$$

then the Mellin transform $\mathcal{M}(G^{(\delta)},s)$ can be calculated to be

(4.30)
$$\mathcal{M}(G^{(\delta)},s)=\frac{\Gamma(s-1)}{(-2\pi\mathrm{i})^s}\sum_{n+\alpha>0}^{\infty}\frac{u^{(\delta)}(n)}{(n+\alpha)^s}.$$

These quantities can be analytically continued to general values of s. Series like $\sum_{n+\alpha>0}^{\infty}\frac{a(n)}{(n+\alpha)^s}$ are known as L-series. Thus, anomaly cancellation can be expressed in terms of L-series.

In the case of w half-integral the period functions are much more complicated than polynomial, but can be expressed in terms of error functions. For example, for $w=-1/2$

(4.31)
$$\overline{p^{(\delta)}(\tau)_\mu}=\frac{\mathrm{e}^{3\pi\mathrm{i}/4}}{\Gamma(3/2)}\sum_{n+\alpha_\mu>0}\frac{u_\mu^{(\delta)}(n)}{(2\pi(n+\alpha_\mu))^{3/2}}e((n+\alpha_\mu)\bar\tau)\Gamma\left(\frac{3}{2},2\pi\mathrm{i}(n+\alpha_\mu)\bar\tau\right).$$

The upper incomplete Gamma function can be written as

(4.32)
$$\Gamma(3/2,x)=x^{1/2}\mathrm{e}^{-x}+\frac{\sqrt{\pi}}{2}\mathrm{erfc}(\sqrt{x}),$$

where erfc is the complementary error function, $\mathrm{erfc}(x)=1-\mathrm{erf}(x)$.

Returning to the case of general weight, for completeness we give the Fourier decomposition of $G^{(\delta)}$:

(4.33)
$$\begin{aligned}G_\mu^{(\delta)}(\tau)&=(-2\pi\mathrm{i}\delta_\mu)^{1-w}e(\delta_\mu\tau)\theta(\delta_\mu>0)\\&\quad+\mathrm{i}(-2\pi\mathrm{i})^{2-w}\sum_{\ell+\delta_\mu>0}e((\ell+\delta_\mu)\tau)\left\{\sum_{c=1}^{\infty}\sum_{\delta_\nu>0}\frac{1}{c}\tilde K_c(\ell+\delta_\mu,\delta_\nu)\right.\\&\quad\left.\times(\delta_\nu(\ell+\delta_\mu))^{(1-w)/2}J_{1-w}\left(\frac{4\pi}{c}\sqrt{(\ell+\delta_\mu)\delta_\nu}\right)\right\},\end{aligned}$$

with generalized Kloosterman sum

(4.34)
$$\tilde K_c(\ell+\delta_\mu,\delta_\nu)=\mathrm{e}^{-\mathrm{i}\pi(2-w)/2}\sum_{0\le d<c;(d,c)=1}e\left((\ell+\delta_\mu)\frac{d}{c}\right)(M^{-1}(\gamma_{c,d})_\mu^\nu)^*e\left(\delta_\nu\frac{a}{c}\right).$$

This is a straightforward application of the Poisson summation formula.

Besides calculation of the Fourier coefficients of $G^{(\delta)}(\tau)$ directly, a decomposition of $G^{(\delta)}(\tau)$ in terms of a basis of cusp forms is instructive as well. This is potentially useful since we have learned that the obstruction to forming a good modular form with prescribed polar term lies in a space isomorphic to the space of vector-valued cusp forms $S(2-w,\overline{M})$. Let us restrict attention to the scalar case for simplicity. We denote an orthonormal basis of the appropriate cusp forms by $H^j(\tau)$, with $j = 1 \ldots \dim[S(2-w,\overline{M})]$. The Fourier coefficients of $H^j(\tau)$ are defined by

$$H^j(\tau) = \sum_{n\geq 0} h^j(n) q^{n+\alpha}$$

with $\alpha = \delta - \lfloor \delta \rfloor$. The Petersson inner product calculates the coefficients of $G^{(\delta)}(\tau)$ with respect to this basis. By unfolding of the integration domain we find

$$\int_{\Gamma\backslash\mathcal{H}} G^{(\delta)}(\tau)\overline{H^j(\tau)} y^{2-w} \frac{dxdy}{y^2} = \frac{\Gamma(1-w)}{(2\mathrm{i})^{1-w}} \overline{h^j(\lfloor\delta\rfloor)}, \tag{4.35}$$

where x and y are, respectively, the real and imaginary parts of τ. The question whether a given set of polar terms gives rise to a vector-valued modular form is now reduced to the finite set of conditions:

$$\forall j \quad \sum_{\delta>0} \Omega_\delta \overline{h^j(\lfloor\delta\rfloor)} = 0. \tag{4.36}$$

This is a difficult question to analyze in general, but is potentially tractable for the cases when a concrete basis of $S(2-w,\overline{M})$ is known.[8]

In the case of $(2,2)$ elliptic genera, we have to consider vector-valued cusp forms. These vector-valued cusp forms can be mapped to scalar cusp forms of congruence subgroups [33] with weight $2-w$. The dimension of the spaces of these cusp forms is expected to grow linearly in m [39]. A more precise study shows that the space of obstructions can be related to a proper subspace of the space of cusp forms known as the Kohnen $+$-space [40].

In the case of $(0,4)$ elliptic genera as we scale $P \to \lambda P$ a rough estimate suggests the number of polar terms scales as λ^{b_2+3}, whereas the dimension

[8]As a measure of the difficulty involved suppose the weight $w=-10$. In this case $h^i(\lfloor\delta\rfloor) = \tau(\delta)$ are the famous Ramanujan functions. We are trying to construct integral linear combinations of these coefficients which vanish.

of the space of relevant cusp forms scales only as λ^{b_2}. We refer to [41], were a more precise calculation of these quantities is performed.

5. Applications of the fareytail expansion

5.1. The fareytail transform revisited

We now put into the present perspective the discussions of the fareytail transform which have appeared previously in [5, 22].

First, the transformation law (3.1) makes clear why the fareytail transformation is flawed in general. In the present context, we would use the operator $\mathcal{O} = \left(q\frac{d}{dq}\right)^{1-w}$ which formally transforms modular forms of weight w to modular forms of weight $2-w$. Being a (pseudo-)differential operator it cannot change the multiplier system $M(\gamma)$. On the other hand, substituting $\gamma = -1$ in Equation (3.1) we find $f_\mu(\tau) = \mathrm{e}^{-\mathrm{i}\pi w} M(-1)_\mu^\nu f_\nu(\tau)$. Since the $f_\mu(\tau)$ are independent functions of τ we conclude that $M(-1)_\mu^\nu = \mathrm{e}^{\mathrm{i}\pi w}\delta_\mu^\nu$. Since the multiplier system does not change under the fareytail transform we must have $\mathrm{e}^{\mathrm{i}\pi w} = \mathrm{e}^{\mathrm{i}\pi(2-w)}$ implying $\mathrm{e}^{2\pi \mathrm{i} w} = 1$ implying that w is integral.[9]

On the other hand, the fareytail transform is valid in the case of non-positive integer weight. We summarize the arguments from [5, 22]. In this case the operator $\left(q\frac{d}{dq}\right)^{1-w}$ really does map a modular form of weight w to a modular form of weight $2-w$ thanks to Bol's identity

$$L^n\left[(c\tau+d)^{-1+n} f\left(\frac{a\tau+b}{c\tau+d}\right)\right] = (c\tau+d)^{-1-n}(L^n f)\left(\frac{a\tau+b}{c\tau+d}\right), \tag{5.1}$$

where $L := q\frac{d}{dq}$. Bol's identity is valid for any non-negative integer n and any suitably differentiable function $f(\tau)$. If $f(\tau)$ is a modular form of weight w and with a pole for $q \to 0$, we define $\tilde{f}(\tau) := \mathcal{O}f(\tau)$. Using a regularized Petersson inner product one shows that $\tilde{f}(\tau)$ is orthogonal to non-singular modular forms and is hence uniquely determined by its polar part [22]. Therefore, the convergent Poincaré series of weight $2-w$ obtained by averaging the polar part of $\tilde{f}$ must in fact be equal to $\tilde{f}$.

[9]The reason adduced by Don Zagier for the failure of the transform for w half-integral was based on results concerning the field of definition of the Fourier coefficients of modular forms.

Consider for simplicity the case of a trivial multiplier system, such that we have

$$\begin{aligned}\mathcal{O}\sum_{\Gamma_\infty\backslash\Gamma}\sum_{\delta>0}\Omega_\delta(s_\gamma^{(\delta)}+t_\gamma^{(\delta)}) &= \sum_{\Gamma_\infty\backslash\Gamma}\sum_{\delta>0}\Omega_\delta\mathcal{O}\left[(c\tau+d)^{-w}\exp\left(-2\pi\mathrm{i}\delta\frac{a\tau+b}{c\tau+d}\right)\right]\\ &= \sum_{\Gamma_\infty\backslash\Gamma}\sum_{\delta>0}\Omega_\delta\left(\mathcal{O}e(-\delta\tau)\right)\Big|_{2-w}\gamma,\end{aligned} \tag{5.2}$$

using Bol's identity to write the second line. Note that in the first line we can exchange summation and differentiation on the left-hand side, but not on the right-hand side. The second line is indeed the claimed Poincaré series expansion of the polar part of $\tilde{f}$. Thus, we have recovered the previous story. Clearly, the operator $\mathcal{O}$ annihilates the constant term in the Poincaré series as well as the regularizing term $t_\gamma^{(\delta)}$ (since the latter is a polynomial in τ of order $|w|$). Also modular anomalies would be annihilated by $\mathcal{O}$, thereby removing the any constraints on the polar terms.

5.2. AdS/CFT interpretation

The introduction motivated the Poincaré series as a sum over classical geometries. We have seen that this semi-classical expansion is remarkably accurate for the partition functions of BPS states. The sums given by Equations (3.4), (3.9) and (3.11) are however more involved than the gravity path integral described in the introduction. The elliptic genera contain a theta function and the polar part can possibly consist of many terms. We will briefly discuss these aspects here and point out a subtlety with respect to the constant term of the partition function. This subtlety is new since the fareytail transform, present in previous discussions, would annihilate the constant term.

The dependence on z in Equation (3.9) is a consequence of the fact that we are not dealing with pure gravity but with a reduction of Type IIB string theory to AdS_3. The parameter z arises since the bulk contains SU(2) gauge fields. It corresponds to a Wilson line from the three-dimensional point of view [5]. States in the bulk are also well described in six-dimensional supergravity on $\mathrm{AdS}_3 \otimes S^3$. The z variable couples then to the momentum of spinning particles on the S^3. In the $(0,4)$ elliptic genus the parameters y arise similarly from the presence of a number of $U(1)$ gauge fields in the bulk.

Equation (3.9) contains a sum over $n - \frac{l^2}{4m} < 0$. The contribution of these states in thermal AdS_3 to the full elliptic genus, is given by

$$\chi(\tau, z)^- = \sum_{\substack{-m+1 \leq \mu \leq m \\ 4mn - \mu^2 < 0}} c_\mu(4mn - \mu^2) q^{n - \frac{\mu^2}{4m}} \theta_{m,\mu}(\tau, z). \tag{5.3}$$

This partition function counts only the "light" excitations of thermal AdS_3. These excitations are typically Kaluza–Klein modes or (charged) point particles. The charged point particles can be branes wrapping cycles in an orthogonal compact manifold. The theta function arises from the singleton modes. The cut-off on the contributing states appears to be equal to the cosmic censorship bound for black holes. This bound is given by $4mM - J_0^2 \geq 0$ with $M = L_0 - \frac{c_L}{24}$ [42]. The "light" excitations are thus exactly those states which do not collapse to a black hole in thermal AdS_3. This is the regime where counting of the degeneracies in supergravity could be reliable. For a meaningful comparison between supergravity and CFT, we apply spectral flow to transform the trace over the R–R sector to the NS–NS sector. To avoid confusion we will denote the eigenvalues of $L_0 - \frac{c_L}{24}$ in the NS sector by n_{NS}. de Boer and Maldacena *et al.* [13,14] have shown that the supergravity degeneracies indeed match with the CFT degeneracies for small values of n_{NS}, in particular $n_{\mathrm{NS}} < 0$. The computations on either side of the correspondence do not match for states with a higher energy. This suggests that gravitational degrees of freedom start contributing at this level. Since $n_{\mathrm{NS}} = 0$ is the smallest value of n_{NS} which satisfies the cosmic censorship bound this is not surprising [5]. The fareytail expansion of the elliptic genus (3.9) is a sum of the light excitations in all the black hole geometries. The excitations that would collapse into the black hole are excluded, since those states are counted by another classical black hole geometry in the sum.

The exponent of the classical action is multiplied by $R\left(\frac{2\pi \mathrm{i}|n - \frac{l^2}{4m}|}{c(c\tau + d)}\right)$. As explained in depth in previous sections, this factor is indispensable for a proper convergence of the gravity path integral. Moreover, it has the effect of a smooth cut-off on the contributions of the light excitations in thermal AdS_3 to the geometries with $c \neq 0$, since $R\left(\frac{2\pi \mathrm{i}|n - \frac{l^2}{4m}|}{c(c\tau + d)}\right)$ is exponentially close to 1 for $|n - \frac{l^2}{4m}| \gg 1$, and is zero for $|n - \frac{l^2}{4m}| = 0$. The geometries with complicated topologies (c and/or $d \gg 1$) are similarly cut-off.

We would like to draw attention now to the contribution to the elliptic genus of states with $4mn - l^2 = 0$. Half of these states are counted by the term

$$\sum_{\mu \bmod 2m} \frac{1}{2} c_\mu(0)\theta_{m,\mu}(\tau, z),$$

which appears separately in Equation (3.9). Comparison with the Fourier series of the elliptic genus, (2.9) and (2.11), shows that the sum over $\Gamma_\infty \setminus \Gamma$ contains an equal term. This suggests that half of the states at $n - \frac{l^2}{4m} = 0$ correspond to black holes, whereas the other half are stable states in thermal AdS_3. Since these stable states in thermal AdS_3 do not contribute to the black hole states, their interpretation is more subtle than the states with $4mn - l^2 < 0$. The way the states at the threshold appear in the partition function leads us to suggest that these excitations are so close to a collapse in thermal AdS_3, that they would collapse into the black hole when added to a black hole geometry. A more quantitative description of this phenomenon is highly desirable.

At a heuristic level the factor $R\left(\frac{2\pi i |n - \frac{l^2}{4m}|}{c(c\tau+d)}\right)$ can be understood in a similar way as the "fraction" of light excitations with a given value of $4mn - l^2$ in thermal AdS_3, which can exist as a stable excitation of the black hole given by (c, d). The other states are unstable and will collapse into the black hole. Note that this quantity is in general complex so such an interpretation is heuristic, at best.

Finally, we comment on another less understood aspect of the Poincaré series and AdS/CFT correspondence. We have argued that the states counted by the theta function are pure gauge in the bulk and only dynamical on the boundary. Therefore, these states should not be summed over all different bulk geometries. This interpretation implies that all non-polar states are black hole states. This statement might be questioned for the following reason. The singleton degrees of freedom are not just given by the theta function, since these enumerate only the primaries. The descendants of the primaries should also be included, since they are also excitations on the boundary, and not to be summed over all geometries. In addition, Witten [17] explains that the descendants of primaries should not be considered as black hole states. Since the descendants are not black hole states, one should sum these descendants over all geometries. In other words, in the Poincaré series for $f_\mu(\tau)$ one would like to remove the condition $n - \Delta_\mu < 0$ and include also the descendants of the polar primaries.

Except for a special case, this does not seem to be allowed by the analysis of this paper, since the non-polar terms lead to non-vanishing obstruction forms with a polar part. However in the case of weight 0, and trivial multiplier system, meromorphic obstruction forms can be written as the derivative of a meromorphic weight zero form, such that the integrand of the period function is a total derivative. Since the boundary of the integration domain are two equivalent cusps under Γ, the modular anomaly vanishes. Also non-polar terms can therefore be included in the Poincaré series without affecting modularity. Unfortunately, we are not aware of a generalization to the vector-valued case. A possible way out might be that the non-polar terms should not be taken into account in the sum, since the contributions of states with less energy is already cut-off, $R(x) = 0$.

The polar states in the case of the $\mathcal{N} = (0, 4)$ elliptic genus have a similar interpretation of states which are not massive enough to form black holes. They include massless supergravity modes as well as M2-branes and anti-M2-branes [16]. In addition there are other exotica such as M5-black rings, $\mathbb{Z}_r$ quotients of $\mathrm{AdS}_3 \times S^2$ and even more complicated geometries. We expect these are all dual to the multi-centered D6 anti-D6 configurations that played a crucial role in [8].

5.3. Phase transitions

One attractive feature of the fareytail expansion is that it is well-suited to deduce phase transitions between different AdS_3 geometries [5]. Such phase transitions were first described in four dimensions by Hawking and Page [43] and interpreted in the AdS/CFT context by Witten [44]. We can understand the phase transformations by determining which term in the sum (1.6) contributes most to the partition function. We have

$$|Z_{\mathrm{grav}}(\tau)| \leq \sum_{\Gamma_\infty \backslash \Gamma} \mathrm{e}^{\frac{2\pi c_{\mathrm{L}}}{24} \frac{\mathrm{Im}(\tau)}{|c\tau+d|^2}}. \tag{5.4}$$

So the combination of (c, d) which maximizes $\frac{\mathrm{Im}(\tau)}{|c\tau+d|^2}$ determines the term that contributes most to the path integral. This (c, d) describes the dominant classical geometry. Phase transitions occur between geometries by variation of τ. The regularizing factor $R\left(\frac{2\pi\mathrm{i}|n-\Delta_\nu|}{c(c\tau+d)}\right)$ does not change this conclusion. To see this we estimate $\left|R\left(\frac{2\pi\mathrm{i}|n-\Delta_\nu|}{c(c\tau+d)}\right) - 1\right|$:

$$\left|R\left(\frac{2\pi\mathrm{i}|n-\Delta_\nu|}{c(c\tau+d)}\right) - 1\right| \leq \frac{1}{\Gamma(1-w)} \left(\frac{2\pi\frac{c_{\mathrm{L}}}{24}}{|c(c\tau+d)|}\right)^{w-1} \mathrm{e}^{-2\pi\frac{c_{\mathrm{L}}}{24}\frac{\mathrm{Im}(\tau)}{|c\tau+d|^2}}, \tag{5.5}$$

where we assumed that $\frac{2\pi|n-\Delta_\nu|}{|c(c\tau+d)|} \gg 1$. We observe that the correction is typically exponentially smaller than the exponent of the classical action, and we can conclude that the new fareytail predicts as well phase transitions parametrized by $\Gamma_\infty \setminus \Gamma$.

5.4. The OSV conjecture

The fareytail expansion of $(0, 4)$ elliptic genera has been used in recent attempts to prove a refined version of the OSV conjecture [8, 9, 15]. The regularization factor $R(x)$ does not alter the discussion when the black hole charges are such that the saddle point topological string coupling is strong. In the notation of [8] we have

$$g_s \sim \sqrt{\frac{-\hat{q}_0}{P^3}} \gg 1. \tag{5.6}$$

The dominant term in the evaluation of $\Omega(\mathcal{Q})$, where $\mathcal{Q} = P + Q + q_0 dV$ is the charge of a D4–D2–D0 brane system on a Calabi–Yau manifold X, is the $c = \pm 1, d = 0$ term in the fareytail expansion of the $(0, 4)$ elliptic genus for $\tau \cong i\sqrt{P^3/|\hat{q}_0|}$. Therefore, for strong topological string coupling $\mathrm{Re}(x) \to \infty$ in the argument of $R(x)$. Thus the regularization factor introduces exponentially small corrections in this regime. In this way the artificial restriction to $b_2(X)$ even, imposed in [8], may be removed.

On the other hand, in the more interesting regime of *weak* topological string coupling, $P^3 \gg |\hat{q}_0|$ the value of x goes to zero for the $c = \pm 1, d = 0$ terms in the fareytail expansion and the effects of our regularization become significant, introducing further corrections to the OSV formula in this regime.

An interesting phenomenon described in [8, 45] is the “entropy enigma.” This refers to the fact that for charges corresponding to weak topological string coupling, semi-classical multicentered states exist which contribute to the “large radius BPS degeneracies” $\Omega(\mathcal{Q})$ with entropies which grow exponentially in P^3 for $P \to \infty$. In particular, they dominate the single centered entropy, the latter growing like $\sqrt{-\hat{q}_0 P^3}$. A growth of $\log|\Omega(\mathcal{Q})| \sim P^3$ for $P \to \infty$ would be a sharp counterexample to the OSV conjecture, and would have other interesting implications. As discussed at length in [8, 46], since $\Omega(\mathcal{Q})$ is an index it is conceivable that the exponentially large contributions might cancel, leaving asymptotics $\log|\Omega(\mathcal{Q})| \sim \sqrt{-\hat{q}_0 P^3}$. Denef and Moore [8] argued that such cancellations are unlikely, but left this central question unanswered.

It is interesting to consider this central question in the light of the present paper. One way to approach this problem is via the behavior of "barely polar degeneracies," that is, the coefficients Ω_δ for δ of order 1 or smaller (compared to P^3). The entropy enigma suggests that these barely polar degeneracies grow like $\exp[kP^3]$ as $P \to \infty$ for some constant k. We are thus led to ask what constraints are imposed by modular invariance on polar degeneracies, and whether the existence of terms with large poles $\sim q^{-P^3/24}$ implies, through anomaly cancellation, that the coefficients of terms with small or order one poles $\sim q^{-1/|P|}, \dots, q^{-1}, \dots, q^{-2}, \dots$ are large. It is convenient to apply the anomaly cancellation condition in the form (4.36). The Fourier coefficients $h(n)$ of cusp forms (for Γ, with trivial multiplier system) of weight k grow as $n^{k/2}$. Although modular invariance therefore bounds the growth of the polar degeneracies, a lot of freedom remains for these degeneracies. From these heuristic arguments, it is clear that we must look elsewhere for an explanation of exponentially large barely polar degeneracies.

In the following, we will refine a suggestion made in [8, p. 117]. We make a toy model of the polar terms of the $(0, 4)$ elliptic genus by considering a modular form for Γ with trivial multiplier system (for symplicity) and considering the polar terms of the negative weight form $\Phi\eta^{-\chi}$ where $\chi = P^3 + c_2(X) \cdot P$ and Φ is a non-singular modular form for Γ of positive weight $w_\Phi = \frac{1}{2}\chi - 1 - \frac{1}{2}b_2$. As we remarked above, the leading coefficient $H_{\mu=0}(0)$ is, up to a sign, $I_P \sim P^3/6$ and therefore in our toy model Φ will have a non-zero Petersson inner product with the Eisenstein series.

To begin, let us sharpen the comments made in [8] about the barely polar degeneracies of $\eta^{-\chi}$ for large χ. For simplicity, we assume χ is a positive integer divisible by 24. Let us define Fourier coefficients by

$$\eta^{-\chi}(\tau) = q^{-\chi/24} \sum_{n=0}^{\infty} p_\chi(n) q^n. \tag{5.7}$$

We are considering degeneracies for $n = \frac{\chi}{24} + \ell$ with ℓ fixed as $\chi \to \infty$ (and of either sign) so the usual Hardy–Ramanujan analysis ("Cardy formula") is slightly altered. A naive saddle-point analysis proceeds by writing

$$p_\chi(n) = \int_{\tau_0}^{\tau_0+1} \mathrm{e}^{-2\pi \mathrm{i}(n-\chi/24)\tau} \frac{1}{\eta^\chi} d\tau \cong \int_{\tau_0}^{\tau_0+1} \mathrm{e}^{-2\pi \mathrm{i}(n-\chi/24)\tau + \frac{\chi}{2}\log(-\mathrm{i}\tau) + \frac{\mathrm{i}\pi\chi}{12\tau}} d\tau. \tag{5.8}$$

In contrast to the usual estimate, it is now the second and third terms in the exponential which dominate the saddle point. In this way, we estimate

$$p_\chi\left(\frac{\chi}{24} + \ell\right) \sim_{\chi\to\infty} \text{const.}\chi^{-1/2} \exp\left(\frac{\chi}{2}\left(1 + \log\frac{\pi}{6}\right) + \frac{\pi^2}{3}\ell\right). \tag{5.9}$$

This agrees very well with a numerical analysis of $\log p_\chi(\chi/24)$ in [8, p. 117]. Moreover, we see that although the degeneracies grow exponentially with ℓ, the proportionality between $p_\chi\left(\frac{\chi}{24}+\ell\right)$ and $p_\chi\left(\frac{\chi}{24}+\ell+1\right)$ is not exponential in χ. This agrees with the earlier statement that the anomaly cancellation bounds the growth of the polar degeneracies.

It is interesting to compare with the Rademacher formula for $p_\chi(\chi/24)$:

(5.10)
$$p_\chi\left(\frac{\chi}{24}\right) = 2\pi \sum_{0\le n<\frac{\chi}{24}} p_\chi(n)\frac{\left(2\pi|n-\frac{\chi}{24}|\right)^{1+\chi/2}}{\Gamma(2+\chi/2)}\sum_{c=1}^{\infty} c^{-2-\chi/2}K_c\left(0,n-\frac{\chi}{24}\right).$$

We can use a beautiful formula of Ramanujan:[10]

$$\sum_{c=1}^{\infty} c^{-s}K_c(0,n) = \frac{\sigma_{1-s}(n)}{\zeta(s)} \tag{5.11}$$

to simplify our formula to:

$$p_\chi\left(\frac{\chi}{24}\right) = 2\pi \sum_{0\le n<\frac{\chi}{24}} p_\chi(n)\frac{\left(2\pi|n-\frac{\chi}{24}|\right)^{1+\chi/2}}{\Gamma(2+\chi/2)}\frac{\sigma_{-1-\chi/2}(\frac{\chi}{24}-n)}{\zeta(2+\chi/2)}. \tag{5.12}$$

Now, note for large χ there is a very large denominator from the Gamma function. The factor $\left(2\pi|n-\frac{\chi}{24}|\right)^{1+\chi/2}$ starts very large for $n=0$ and falls exponentially rapidly. Meanwhile, note that since the index on the divisor sum is negative the factor $\sigma_{-1-\chi/2}(\frac{\chi}{24}-n)$ is a slowly varying function of n, and strictly smaller than $\frac{\chi}{24}-n$. Thus, the sum is dominated by the terms $n=0$. Using Stirling's formula we find that the contribution of the $n=0$ term is

$$\sigma_{-1-\chi/2}\left(\frac{\chi}{24}\right)\times \text{const.}\times \chi^{-1/2}\exp\left(\frac{\chi}{2}\left(1+\log\frac{\pi}{6}\right)\right) \tag{5.13}$$

in agreement with the naive evaluation. Thus we learn that the contribution of the *extreme polar states* in the Rademacher expansion gives the dominant contribution to the constant term.

[10] To show this we first relate the relevant Kloosterman sum to the Möbius function $\mu(n)$: $\sum_{\substack{a=1\\(a,c)=1}}^{c} e(n\frac{a}{c}) = \sum_{m|(c,n)}\mu(\frac{c}{m})m$ [47, p. 160]. We substitute this identity on the left-hand side of Equation (5.11). Application of $\zeta(s)\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}=1$ leads then to the claimed identity.

Now let us turn to the numerator Φ. A similar discussion applies to the contributions of Φ to the barely polar degeneracies. If Φ is a non-singular modular form of weight w with $\Phi(\tau) = \sum_{n\geq 0} \hat{\phi}(n) q^n$ then a naive saddle point evaluation of the Fourier coefficients $\hat{\phi}(n)$ gives

$$\hat{\phi}(n) \sim \pm \frac{\hat{\phi}(0)}{\sqrt{2\pi}} w^{-w+\frac{1}{2}} \mathrm{e}^{w(1+\log(2\pi))} n^{w-1} \left(1 + \mathcal{O}(\mathrm{e}^{-4\pi^2 n/w})\right) \tag{5.14}$$

(Although this is naive, numerical checks indicate it is valid.) To estimate the biggest contribution of the Fourier coefficients of Φ to the constant term in $\eta^{-\chi}\Phi$ we apply this to $w = w_\Phi = \frac{1}{2}\chi - \frac{1}{2}b_2 - 1$ and $n = \frac{\chi}{24}$ yielding, remarkably,

$$\text{const.}\chi^{-1/2} \exp\left[\frac{\chi}{2}\left(1 + \log\frac{\pi}{6}\right)\right] \tag{5.15}$$

having the same order of exponential growth as the barely polar terms of $\eta^{-\chi}$. Thus, in our model for polar degeneracies the barely polar degeneracies are indeed expected to grow exponentially in χ.

It is conceivable that this kind of estimate could be rigorously applied to estimate the coefficients near the cosmic censorship bound in the $(0,4)$ elliptic genus, and it would be very interesting to do so.

5.5. Enumerative geometry

As a final application of the fareytail expansion, we would like to point out its potential relevance to problems in enumerative geometry. The Fourier coefficients Ω_δ of the $\mathcal{N} = (0,4)$ elliptic genera are the degeneracies of bound states of D4-, D2-, and D0-branes on a Calabi–Yau manifold X. From a more mathematical perspective, these are (generalized) Donaldson–Thomas invariants, which count the stable coherent sheaves on X with given Chern classes. The BPS degeneracies (or equivalently Donaldson–Thomas invariants) are subject to wall-crossing behavior, since the BPS-states are not stable for all values of the (complexified) Kähler moduli t (specified at spatial infinity in the black hole solution). The complexified Kähler moduli are given by $t = B + \mathrm{i}J$, where B is the anti-symmetric tensor field and J is the Kähler class. The generating function of the BPS-degeneracies has only an interpretation as an $\mathcal{N} = (0,4)$ elliptic genus [8, 28] in the large Kähler limit. de Boer *et al.* [48] argues more precisely that the $(0,4)$ SCFT analysis is only valid if the t^a are chosen such that $t^a = d^{ab}q_b + \mathrm{i}\lambda p^a$ with $\lambda \to \infty$.

The fact that a class of DT-invariants are enumerated by a modular form has interesting consequences. For example, Section 4 discussed how a modular anomaly arises if the polar coefficients do not satisfy certain constraints. These constraints are such that a linear combination of cuspidal Poincaré series vanishes. The constraints are given in the form

$$\forall j \qquad \sum_{\delta>0} \Omega_\delta \, \overline{h^j(\lfloor\delta\rfloor)} = 0, \tag{5.16}$$

where the $h^j(n)$ are Fourier coefficients of an orthonormal basis of cusp forms. Therefore, we see that interesting relations exist among the coefficients of cusp forms and DT-invariants in a specific chamber of the moduli space. Generically, it is very difficult to find such relations among cusp forms. A concrete example where this phenomenon occurs, is the case where the M5-brane wraps the hyperplane section of the bicubic in $\mathbb{CP}^5$. Gaiotto and Yin [49] compute explicitly the elliptic genus of this configuration (and several others) by a determination of the polar degeneracies using algebraic geometry and Gromov–Witten invariants. Interestingly, a relation among the polar coefficients was found, which was explained in [41] as a consequence of the existence of a (vector-valued) cusp form with the relevant properties.

In some respects, the $(0,4)$ elliptic genus can be seen as a generalization of the partition function of bound states of D4–D2–D0 branes on K3. For example, if the 11-dimensional geometry is chosen to be $\mathbb{R}^5 \times T^2 \times$K3, and a single M5-brane wraps $T^2 \times$K3, then the $\mathcal{N} = (0,4)$ elliptic genus becomes

$$\chi(\tau, z)_{\mathrm{K3}} = \frac{\Theta_{\Gamma_{3,19}}(\tau, \bar{\tau}, z)}{\eta(\tau)^{24}}. \tag{5.17}$$

Note that since this geometry preserves more supersymmetry a factor F^4 needs to be inserted in the trace (2.15), instead of F^2. $\Gamma_{3,19}$ is the lattice of the second cohomology of K3. We observe that $\eta(\tau)^{-24}$ provides us the number of BPS-degeneracies of D0-branes as well as D2-branes on K3. The D0-branes are the physical equivalent of the Hilbert scheme of points. This partition function is earlier computed from this perspective in [50]. Recently, the interpretation of $\eta(\tau)^{-24}$ as a generating function for D2-branes wrapping cycles in K3 has been put on a firmer mathematical basis [51]. It provides the (reduced) Gromov–Witten invariants of K3. The (0,4) elliptic genus in the case of a proper Calabi–Yau three-fold X and possibly multiple M5-branes, is a major generalization of (5.17). We expect that it can play an important role in problems of enumerative geometry related to Calabi–Yau three-folds.

6. Non-holomorphic partition functions

This section explains how the anomalous transformation property of $\hat{S}^{(\delta)}_{\text{Reg}}(\tau)$ under Γ in Equation (4.21), can be corrected by the addition of a non-holomorphic term to produce a covariant object. Section 4 shows that a proper choice of polar degeneracies can result in the vanishing of the shift in Equations (4.21) or (A.30). However, physics might prescribe a set of polar degeneracies that cannot be consistently extended to a holomorphic modular form with the required transformation properties. Holomorphy is useful, but diffeomorphism invariance is fundamental, hence in such a situation there is necessarily a holomorphic anomaly. We now explore what can be said about such holomorphic anomalies from the viewpoint of this paper.

Equation (A.20) shows that if we add a non-holomorphic term as in

$$\tilde{S}^{(\delta)}_{\text{Reg}}(\tau,\bar{\tau}) = \hat{S}^{(\delta)}_{\text{Reg}}(\tau) - p(\tau,\bar{\tau},\overline{G^{(\delta)}}), \tag{6.1}$$

then the new function $\tilde{S}^{(\delta)}_{\text{Reg}}(\tau,\bar{\tau})$ transforms covariantly. In this way, we can trade the modular anomaly for a holomorphic anomaly. To study its properties more precisely, we rewrite $p(\tau,\bar{\tau},\overline{G^{(\delta)}})$ as

$$\begin{aligned} &\frac{1}{\Gamma(1-w)}\int_{\bar{\tau}}^{-\mathrm{i}\infty}\overline{G^{(\delta)}(z)}(\bar{z}-\tau)^{-w}d\bar{z} \\ &= \frac{(-2\mathrm{i}\tau_2)^{1-w}}{\Gamma(1-w)}\int_1^{\infty}\overline{G^{(\delta)}(\bar{\tau}+2u\mathrm{i}\tau_2)}u^{-w}du. \end{aligned} \tag{6.2}$$

From the first expression it is clear that $\tilde{S}^{(\delta)}_{\text{Reg}}(\tau,\bar{\tau})$ satisfies the holomorphic anomaly equation

$$\frac{\partial}{\partial\bar{\tau}}\tilde{S}^{(\delta)}_{\text{Reg}}(\tau,\bar{\tau}) = \frac{(-2\mathrm{i}\tau_2)^{-w}}{\Gamma(1-w)}\overline{G^{(\delta)}(\tau)}. \tag{6.3}$$

Of course, such a non-holomorphic correction is far from being unique! The above choice is distinguished by the fact that $\tilde{S}^{(\delta)}_{\text{Reg}}(\tau,\bar{\tau})$ is annihilated by a Laplacian given by $\Delta = \frac{\partial}{\partial\tau}\tau_2^w\frac{\partial}{\partial\bar{\tau}}$. Note that it also reduces to a polynomial in τ for $-w \in \mathbb{N}$.

The holomorphic anomaly described here is similar to the one appearing for the $w = \frac{3}{2}$ modular forms discussed in [52, 53]. In physics, such holomorphic anomalies arise in the partition function of $\mathcal{N} = 4$ topologically twisted

Yang–Mills theory on $\mathbb{CP}^2$ with gauge group SO(3) [54], and also in the context of Donaldson invariants [55]. Now, as reviewed in Section 2, if we consider an M5-brane partition function on $\Sigma \times T^2$ then for small T^2 we would expect the partition function to be related to the four-dimensional gauge theory computations of [54]. On the other hand in the limit when the Kähler class of the T^2 is much larger than those of Σ, and Σ is embedded in a Calabi–Yau manifold, a $(0,4)$ conformal field theory analysis analogous to that of [28] should be applicable. This suggests that there might be holomorphic anomalies in the $(0,4)$ elliptic genus.[11]

As a possible example of this situation consider wrapping an M5-brane on a rigid divisor equal to $\mathbb{CP}^2$ in a suitable Calabi–Yau (e.g. the Calabi–Yau elliptic fibration over $\mathbb{CP}^2$). Vafa and Witten [54] calculate the partition function of the twisted gauge theory. The coefficients of this partition function are the Euler numbers of the moduli space of instantons. In the case of $\mathbb{CP}^2$ with gauge group SO(3), Vafa and Witten [54] give two partition functions, $Z_0(\tau,\bar\tau)$ and $Z_1(\tau,\bar\tau)$, related to the two different possibilities for the second Stiefel–Whitney class w_2 of SO(3) bundles on $\mathbb{CP}^2$. $Z_0(\tau,\bar\tau)$ and $Z_1(\tau,\bar\tau)$ transform as a modular vector under Γ. The holomorphic anomaly for $Z_\mu(\tau,\bar\tau)$, given in [54], is

$$\frac{\partial}{\partial\bar\tau} Z_\mu(\tau,\bar\tau) = \frac{3}{16\pi \mathrm{i}\tau_2^{3/2}}\frac{1}{\eta(\tau)^6}\sum_{n\in\mathbb{Z}+\frac{\mu}{2}} \bar q^{n^2} = \frac{3}{16\pi \mathrm{i}\tau_2^{3/2}}\frac{1}{\eta(\tau)^6}\overline{\theta_{3-\mu}(2\tau)}, \tag{6.4}$$

where $\theta_{3-\mu}(\tau)$ are the standard Jacobi theta functions. From this one can derive the modular transformations of the purely holomorphic partition function:

$$\begin{aligned} Z_\mu(\gamma(\tau)) &= j(\gamma,\tau)^{-\frac{3}{2}} M(\gamma)_\mu^\nu \\ &\times \left[Z_\nu(\tau) + \frac{3e(-\frac{1}{8})}{2\sqrt{2\pi}\,\eta(\tau)^6} p\left(\tau, \gamma^{-1}(-\mathrm{i}\infty), \overline{\theta_{3-\nu}(2\cdot)}\right)\right], \end{aligned} \tag{6.5}$$

where $M(\gamma)$ is the multiplier system generated from

$$M(T) = \begin{pmatrix} e(-1/4) & 0 \\ 0 & -1 \end{pmatrix}, \qquad M(S) = e(-1/8)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{6.6}$$

To compare these partition functions with a dual supergravity partition function we must recall that the gauge theory dual to the string theory

[11] Exactly this suggestion has been made previously by D. Gaiotto in a seminar at Princeton, October 13, 2006.

will include singleton degrees of freedom leading to extra $U(1)$ factors in the gauge group. (See [56], Appendix B, or [20, 57].) In the present case, we should presumably compare to a theory with gauge group $U(2)$. After inclusion of the $U(1)$ degrees of freedom, we obtain

$$\chi(\tau,\bar{\tau},\bar{z}) = Z_0(\tau,\bar{\tau})\overline{\theta_2(2\tau,2z)} - Z_1(\tau,\bar{\tau})\overline{\theta_3(2\tau,2z)}. \tag{6.7}$$

$\chi(\tau,\bar{\tau},\bar{z})$ transforms under Γ with weight $(-\frac{3}{2},\frac{1}{2})$ and multiplier system. This clearly resembles an elliptic genus of a $(0,4)$ SCFT as given in Equation (2.18).

Let us therefore contrast these formulae with what would be expected from the viewpoint of this paper. We might expect to be able to construct the partition function — in the AdS_3 regime — from a Poincaré series based on its polar part. A priori, this partition function does not need to equal $\chi(\tau,\bar{\tau},\bar{z})$ since we might not be able to rely on modular invariance and/or holomorphy. Therefore, we distinguish the fareytail partition function and denote it by $\chi^{\mathrm{FT}}(\tau,\bar{\tau},\bar{z})$. The theta functions in Equation (6.7) can be derived from this point of view as a specialization of Equation (2.17). Note that $\mu^{\|}$ is 0 when the second Stiefel–Whitney class w_2 of the SO(3) bundle is trivial, and equal to 1 when w_2 is non-trivial.

The comparison reduces now to a comparison of the holomorphic part of $Z_\mu(\tau,\bar{\tau})$, $Z_\mu(\tau)$, with the vector-valued modular form constructed by the Poincaré series. We label the constructed vector-valued modular form by "FT": $Z_\mu^{\mathrm{FT}}(\tau)$. The polar part of $Z_\mu^{\mathrm{FT}}(\tau)$ is equal to the polar part of $Z_\mu(\tau)$, if we assume that the polar part is not renormalized as we continue to the AdS_3 regime. $Z_0(\tau)$ has a polar term equal to $-\frac{1}{4}q^{-\frac{1}{4}}$ while $Z_1(\tau)$ does not contain a polar term. Therefore, we attempt to construct with the fareytail a modular form of weight $-3/2$, with multiplier system given by Equation (6.6) and polar term given by $\delta = \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix}$. The obstruction to the construction of a holomorphic modular form with these properties is given by a space of vector-valued cusp forms as discussed extensively in previous sections. The space of these cusp forms turns out to be non-vanishing in this case. A vector-valued cusp form of weight $7/2$ and the appropriate multiplier system is given by

$$\eta(\tau)^6 \begin{pmatrix} \theta_3(2\tau) \\ \theta_2(2\tau) \end{pmatrix}. \tag{6.8}$$

Using the dimension formulas for vector-valued modular forms, one can show that this form is the unique cusp form with the required properties. See [41] for more details and illustrations of dimension formulas. Then we find the following transformation law for $Z_\mu^{\mathrm{FT}}(\tau)$:

(6.9)
$$Z_\mu^{\mathrm{FT}}(\gamma(\tau)) = j(\gamma,\tau)^{-\frac{3}{2}} M(\gamma)_\mu^\nu \left[Z_\nu^{\mathrm{FT}}(\tau) + \frac{1}{4} p\left(\tau, \gamma^{-1}(-i\infty), \overline{\eta^6 \theta_{3-\nu}(2\,\cdot)}\right)\right].$$

The factor $\frac{1}{4}$ in front of the period function is a consequence of the coefficient of the polar term.

A simple check whether the fareytail can reproduce the gauge theory partition function is a comparison of the anomalies under modular transformations. Even without a detailed analysis, we can observe qualitative differences between the shifts. An important difference is the behavior for $\mathrm{Im}(\tau) \to \infty$. In this limit the shift in Equation (6.5) grows exponentially whereas the period function in Equation (6.9) vanishes. This shows clearly that the holomorphic fareytail does not equal the generating function of the Euler numbers of instanton moduli spaces.

As a consequence of the different modular anomalies, the associated holomorphic anomalies are different. The holomorphic anomaly given by Equation (6.4) is not annihilated by the Laplacian Δ. Another difference is that for $\mathrm{Im}(\tau) \to \infty$, the right-hand side of Equation (6.4) grows exponentially (for $\mu = 0$).

This raises the question of what the elliptic genus of the $\mathcal{N} = (0, 4)$ SCFT on the boundary of AdS_3 really is. The results of this section are clearly inconclusive. We are considering several possible resolutions and we hope to address them in future work.

7. Conclusion

In this paper we have revisited the "fareytail expansion" of [5], and have improved on the story in many ways. We have shown how to regularize the relevant Poincaré series so that we have an expansion for the partition function, and not its "fareytail transform." The latter is problematic, and now rendered irrelevant.

The modern fareytail is well-suited to the earlier applications of fareytail expansions. It is relevant for the program of determining the black hole entropy by study of the near horizon microstates. We have argued that the new expansion is consistent with the OSV conjecture at strong topological string coupling.

In addition, the modern fareytail contains a number of interesting new aspects. This includes new wrinkles on the interpretation of the expansion in the AdS/CFT context, as well as new corrections to the OSV formula at weak coupling. Moreover, we have given an extended discussion how the regularization can give rise to a modular or holomorphic anomaly. The modular anomalies can be described in terms of period functions of positive weight cusp forms. The holomorphic anomaly is compared with a similar anomaly appearing in the partition function of $\mathcal{N} = 4$ Yang–Mills on $\mathbb{CP}^2$.

There are further implications of the new fareytail, not discussed in this paper, which might prove fruitful for future study. One of these questions concerns the spaces of obstructions to the construction of the modular forms. We would like to sharpen our understanding by computing, for example, the precise dimension of the space of obstructions. Another point which deserves further study is the possibility of holomorphic anomalies in the elliptic genus. A better understanding of the relation of the holomorphic anomalies to those of topological $\mathcal{N} = 4$ Yang–Mills is desirable.

Finally, we mention a more speculative connection to arithmetic varieties. Arithmetic varieties appeared earlier in the context of black holes in [22, 58, 59]. It is possible to associate arithmetic varieties in two distinct ways to a polar term. On the one hand, a polar term corresponds to several split attractor flows [8]. The split attractor flows of Denef end on regular attractor points. The conjectures in [22, 58, 59] state that the Calabi–Yau at a regular attractor point is an arithmetic variety. On the other hand, arithmetic varieties can also appear in an alternative way via the cusp form which is associated to the polar term. The cusp form can be decomposed into Hecke eigenforms. The Hecke eigenforms can be related to arithmetic Calabi–Yau manifolds (usually with dimension larger than 3), generalizing the celebrated case of the elliptic curve. For a review see, for example, [60]. Thus we have two different ways to relate a polar term to an arithmetic manifold. It would be quite interesting if this correspondence turns out to have any arithmetic significance.

Acknowledgments

J.M. would like to thank the Department of Physics & Astronomy of Rutgers University and the School of Natural Sciences of the Institute for Advanced Study for hospitality during the completion of this work. He is grateful to E. Verlinde for discussions and encouragement to survey the problems related to the fareytail transform.

G.M. would like to thank H. Ooguri, whose crucial question led to the discovery of the problems with the fareytail transform. He also thanks D. Zagier for important discussions and correspondence and he thanks S. Gukov, H. Ooguri, and C. Vafa, for collaboration on closely related matters.

In addition, we thank J. de Boer, F. Denef, G. van der Geer, E. Diaconescu, J. Maldacena, S. Miller, P. Sarnak, B. van Rees and E. Witten for helpful discussions. The research of J.M. is supported by the Foundation of Fundamental Research on Matter (FOM). The work of G.M. is supported by the US DOE under grant DE-FG02-96ER40949.

Appendix A. Technicalities of the modern fareytail

A.1. Derivation

This appendix derives Equation (3.4). The derivation is in some sense a reversed version of the analysis in [6]. We start with a vector-valued modular, and derive Equation (3.4) based on its Fourier coefficients, which are calculated by the Rademacher circle method. Whereas Niebur [6] basically starts at the other end, and determines its Fourier coefficients together with its transformation properties. We take the opportunity to generalize the result to vector-valued modular forms.

To start, we state the transformation properties of a vector-valued modular form

$$f_\mu(\gamma(\tau)) = M(\gamma)_\mu^\nu (c\tau + d)^w f_\nu(\tau). \tag{A.1}$$

with $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. We take $w \leq 0$ and use $-\pi < \arg(z) \leq \pi$ as domain for the argument of a complex variable z. The Fourier expansion of the modular vector is given by

$$f_\mu(\tau) = \sum_{m=0}^{\infty} F_\mu(m) q^{m-\Delta_\mu}, \tag{A.2}$$

where $F_\mu(0) \neq 0$ is the lowest non-zero coefficient. The part of $f_\mu(\tau)$ with $m - \Delta_\mu < 0$ is denoted as its polar part $f_\mu^-(\tau)$, because of the divergence of these terms when $\tau \to i\infty$. The series with $m - \Delta_\mu \geq 0$ is correspondingly called the non-polar part, $f_\mu^+(\tau)$. Note that for transformations $\gamma_n(\tau) = \tau + n$, $M(\gamma)_\mu^\nu$ is given by $\delta_\mu^\nu e(-\Delta_\mu n)$. The Fourier coefficients (with $m -$

$\Delta_\mu \geq 0$) are determined by the Rademacher circle method or Farey fractions [36]. This method is beautifully applied to $1/\eta(\tau)$ in [11] and generalized to vector-valued modular forms in [5]. The Fourier coefficients are given by the infinite series

$$\begin{aligned} F_\mu(m) = 2\pi \sum_{n-\Delta_\nu<0} F_\nu(n) \sum_{c=1}^{\infty} \frac{1}{c} K_c(m-\Delta_\mu, n-\Delta_\nu) \\ \times \left(\frac{|n-\Delta_\nu|}{m-\Delta_\mu}\right)^{(1-w)/2} I_{1-w}\left(\frac{4\pi}{c}\sqrt{(m-\Delta_\mu)|n-\Delta_\nu|}\right), \end{aligned} \tag{A.3}$$

where $I_\nu(z)$ is the modified Bessel function of the first kind. $I_\nu(z)$ is given as an infinite sum by

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}z^2\right)^k}{k!\Gamma(\nu+k+1)}. \tag{A.4}$$

$K_c(m-\Delta_\mu, n-\Delta_\nu)$ is a generalized version of the Kloosterman sum

(A.5)
$$K_c(m-\Delta_\mu, n-\Delta_\nu) := \mathrm{i}^{-w} \sum_{\substack{-c\leq d<0 \\ (c,d)=1}} M^{-1}(\gamma)_\mu^\nu e\left((n-\Delta_\nu)\frac{a}{c} + (m-\Delta_\mu)\frac{d}{c}\right),$$

with $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, thus $ad = 1 \mod c$. We have taken a specific domain for d in the Kloosterman sum. This is necessary since Δ_μ is in general not an integer. The dependence on a in the exponent and in $M^{-1}(\gamma)_\mu^\nu$ via γ combine such that the product with the generalized Kloosterman sum is independent of a. The factor of i^{-w} in front of the sum is a consequence of the definition of $M(\gamma)_\mu^\nu$ in Equation (A.1). Finally, if $m - \Delta_\mu = 0$ we should take a limit as $m - \Delta_\mu \to 0$.

Since $M(\gamma)$ is unitary, the generalized Kloosterman sum is bounded above by the Euler totient function $\phi(c) \leq c$. For later use, we need an estimate of the generalized Kloosterman sum. Weil has derived a particularly strong bound for $K_c(m,n)$ when $m, n \in \mathbb{Z}$ and a trivial multiplier system. He estimated that $K_c(m,n)$ is bounded above by $\mathcal{O}(c^{\frac{1}{2}+\epsilon})$. We do not need such a strong bound. For our applications with $w < 0$, the upperbound of the Kloosterman sum by c suffices. For the example in the introduction with $w = 0$ and a trivial multiplier system (Equation (1.8)), an estimate $c^{1-\epsilon}$ with $\epsilon > 0$ is necessary. Such a bound can be established in an elementary way, see for example [61]. We do not attempt to establish a non-trivial bound for

Kloosterman sums arising from modular forms with $w = 0$ and a non-trivial multiplier system.

Our strategy to derive Equation (3.4) is fairly straightforward. We substitute the expression for the Fourier coefficients in the Fourier series for the non-polar part of $f_\mu(\tau)$. Then we use the formulas given in Appendices A.2 and B to rewrite $f_\mu(\tau)$ in the form of Equation (3.4). After the substitution of the Fourier coefficients, Equation (A.3), and Kloosterman sum Equation (A.5), we insert the series expansion of the Bessel function Equation (A.4). We obtain

$$\begin{aligned} f_\mu^+(\tau) &= \sum_{m-\Delta_\mu\geq 0} F_\mu(m) q^{m-\Delta_\mu} \\ &= \sum_{n-\Delta_\nu<0} \sum_{c=1}^{\infty} \sum_{\substack{-c\leq d<0 \\ (c,d)=1}} \sum_{k=0}^{\infty} i^{-w} M^{-1}(\gamma)_\mu^\nu F_\nu(n) \left(\frac{2\pi}{c}\right)^{2k+2-w} \\ &\quad \times \frac{|n-\Delta_\nu|^{k+1-w}}{\Gamma(k+2-w)} e\left((n-\Delta_\nu)\frac{a}{c}\right) \sum_{m-\Delta_\mu\geq 0} \frac{(m-\Delta_\mu)^k}{k!} \\ &\quad \times e\left((m-\Delta_\mu)\left(\tau+\frac{d}{c}\right)\right), \end{aligned} \tag{A.6}$$

where we interchanged the sum over m with the other four sums and grouped the terms dependent on m. We apply the Lipschitz summation formula (B.1) to the sum over m, the new summation variable will be denoted by l. The error term $E(\tau, k+1, N+\frac{1}{2})$ vanishes in the limit $N \to \infty$, except when $k = 0$ and $\Delta_\mu \in \mathbb{N}$. When the error term does not vanish, we get an additional constant. This constant is equal to $\frac{1}{2}F_\mu(\Delta_\mu)$ and is given by

$$\frac{1}{2}F_\mu(\Delta_\mu) = \begin{cases} \pi \displaystyle\sum_{n-\Delta_\nu<0} \frac{(2\pi|n-\Delta_\nu|)^{1-w}}{\Gamma(2-w)} F_\nu(n) \\ \quad \displaystyle\sum_{c=1}^{\infty} c^{w-2} K_c(0_\mu, n-\Delta_\nu), & \Delta_\mu \in \mathbb{N}, \\ 0, & \Delta_\mu \notin \mathbb{N}, \end{cases} \tag{A.7}$$

where 0_μ is a vector all of whose components are zero. The fact that the right-hand side of Equation (A.7) is equal to $\frac{1}{2}F_\mu(\Delta_\mu)$ can be shown for example by Equation (A.3) for $F_\mu(\Delta_\mu)$ and the limiting behavior of the Bessel function for $z \to 0$: $\lim_{z\to 0} I_\nu(z) = \left(\frac{z}{2}\right)^\nu \frac{1}{\Gamma(\nu+1)}$. We get after interchanging the sum over k and l

$$\begin{aligned} f_\mu^+(\tau) = \frac{1}{2}F_\mu(\Delta_\mu) + \sum_{n-\Delta_\nu<0} \sum_{c=1}^{\infty} \sum_{\substack{-c\le d<0\\(c,d)=1}} \lim_{N\to\infty} \sum_{l=-N}^{N} M^{-1}(\gamma)_\mu^\nu F_\nu(n) \\ \times e\left((n-\Delta_\nu)\frac{a}{c}\right) \frac{1}{(c\tau+d+cl)^w} e(\Delta_\mu l) \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+2-w)} \\ \times \left(\frac{2\pi \mathrm{i}|n-\Delta_\nu|}{c(c\tau+d+cl)}\right)^{k+1-w}. \end{aligned} \tag{A.8}$$

The exchange of the sum over k and l is allowed because the sums are absolutely convergent for $k > 0$. In case $k = 0$, the sum over l in the limit $N \to \infty$ is as well convergent. This is shown using the weak bound on the Kloosterman sum, to which we referred earlier.

The sums over c and d can be such that they have an equal upperbound. This is clear for $k > 0$, but to show it for $k = 0$ is slightly subtle. First, we incorporate the sum over l in the sum over d. Since the sum over l and d is convergent for finite c, we can choose for $|d|$ an upperbound N for which we take the limit $N \to \infty$. We thus get a sum of the form

$$\sum_{c=1}^{\infty} \lim_{N\to\infty} \sum_{\substack{|d|\le N\\(c,d)=1}} M^{-1}(\gamma)_\mu^\nu \frac{e((n-\Delta_\nu)\frac{a}{c})}{c^{1-w}(c\tau+d)}, \tag{A.9}$$

where we used that $e(\Delta_\mu l)\delta_\mu^\nu = M^{-1}(\gamma_l)_\mu^\nu$ and Equation (C.1) to include $e(\Delta_\mu l)$ in $M^{-1}(\gamma)_\mu^\nu$. Rademacher [12] shows that

$$\lim_{K\to\infty} \sum_{c=1}^{K} \lim_{N\to\infty} \sum_{\substack{K<|d|\le N\\(c,d)=1}} M^{-1}(\gamma)_\mu^\nu \frac{e((n-\Delta_\nu)\frac{a}{c})}{c^{1-w}(c\tau+d)} = 0, \tag{A.10}$$

in case $M(\gamma) = 1$ and $(n - \Delta_\nu) = -1$. We can show in a similar way that the generalization holds as well. To this end define the matrix $g(d)_\mu^\nu$ (with $-\delta_\nu = n - \Delta_\nu$)

$$g(d)_\mu^\nu = \begin{cases} M^{-1}(\gamma)_\mu^\nu e\left(-\delta_\nu \frac{a}{c}\right), & \text{for } (c,d) = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{A.11}$$

Using that $M(\gamma_l)_\mu^\nu = \delta_\mu^\nu e(-\delta_\nu l)$ (where δ_μ^ν should not be confused with δ_ν), we observe that $e(-\delta_\mu \frac{d}{c})g(d)_\mu^\nu$ is periodic in d modulo c. Therefore, $e(-\delta_\mu \frac{d}{c})$

$g(d)^\nu_\mu$ has a Fourier expansion, and we find for $g(d)^\nu_\mu$

$$g(d)^\nu_\mu = \sum_{j=1}^{c} (B_{j,c})^\nu_\mu \, e\left((j+\delta_\mu)\frac{d}{c}\right), \tag{A.12}$$

with

$$(B_{j,c})^\nu_\mu = \frac{1}{c} \sum_{\substack{d'=1 \\ (c,d')=1}}^{c} M^{-1}(\gamma)^\nu_\mu e\left(-\delta_\nu \frac{a}{c} - (j+\delta_\mu)\frac{d'}{c}\right). \tag{A.13}$$

$B_{j,c}$ contains a Kloosterman sum, and with the bound $c^{1-\epsilon}$ on the vector-valued Kloosterman sums (see the discussion below Equation (A.5)), we obtain $\mathcal{O}(c^{-\epsilon})$ as a bound for $B_{j,c}$. The left-hand side of Equation (A.10) can be written as

$$\lim_{K\to\infty} \sum_{c=1}^{K} \frac{1}{c^{1-w}} \sum_{j=1}^{c} (B_{j,c})^\nu_\mu \sum_{|d|=K+1}^{\infty} \frac{e((j+\delta_\nu)\frac{d}{c})}{(c\tau+d)}. \tag{A.14}$$

Rademacher [12] gives estimates for the sum over d which continue to hold for the generalization after minor modifications. We find that in case $(j+\delta_\nu)/c \in \mathbb{Z}$ for some j, the sum over d has an upperbound given by $\mathcal{O}\left(\frac{c\log(K)}{K}\right)$, otherwise the upperbound is $\mathcal{O}\left(K^{-1}\right)$. The estimates for Equation (A.10) become, respectively, $\lim_{K\to\infty} \mathcal{O}(K^{w-\epsilon}\log(K))$ and $\lim_{K\to\infty}$ $\mathcal{O}(K^{w-\epsilon})$, which are indeed zero for ($w<0$, $\epsilon=0$) and ($w=0$, $\epsilon>0$). We therefore have shown that Equation (A.9) is equal to

$$\lim_{K\to\infty} \sum_{c=1}^{K} \sum_{\substack{|d|\leq K \\ (c,d)=1}} M^{-1}(\gamma)^\nu_\mu \frac{e((n-\Delta_\nu)\frac{a}{c})}{c^{1-w}(c\tau+d)}, \tag{A.15}$$

for the cases which are relevant to us.

The sum over k in Equation (A.8) is equal to an exponent minus the first terms of the Fourier expansion: $\sum_{k=0}^{\infty} \frac{z^{k+1-w}}{\Gamma(k+2-w)} = \mathrm{e}^z - \sum_{k=0}^{|w|} z^k/k!$, when w is a negative integer. We recognize the regularization of Equation (3.7). However we want to obtain a closed form for general non-positive weight.

This can be obtained using the equality

$$\begin{aligned} h(z) = \sum_{k=0}^{\infty} \frac{z^{k+1-w}}{\Gamma(k+2-w)} &= \mathrm{e}^z \left(1 - \frac{1}{\Gamma(1-w)} \int_z^{\infty} \mathrm{e}^{-t} t^{-w} dt\right) \\ &= \frac{\mathrm{e}^z}{\Gamma(1-w)} \int_0^z \mathrm{e}^{-t} t^{-w} dt, \end{aligned} \tag{A.16}$$

which is valid for general $w < 1$. One can establish Equation (A.16) by developing the second integral expression in series using successive integration by parts, or by considering the differential equation satisfied by $h(z)$.

We define $R(z) = \mathrm{e}^{-z} h(z)$. Inserting this and the equal upperbound for c and d in Equation (A.8), we obtain

$$\begin{aligned} f_\mu^+(\tau) = \frac{1}{2} F_\mu(\Delta_\mu) + \sum_{n-\Delta_\nu<0} \lim_{K\to\infty} \sum_{c=1}^{K} \sum_{\substack{|d|\leq K \\ (c,d)=1}} \frac{M^{-1}(\gamma)_\mu^\nu F_\nu(n)}{(c\tau+d)^w} \\ \times e\left((n-\Delta_\nu)\gamma(\tau)\right) R(x), \end{aligned} \tag{A.17}$$

where $x = \frac{2\pi \mathrm{i}|n-\Delta_\nu|}{c(c\tau+d)}$. The summand is invariant under $\gamma \to -\gamma$ or equivalently $(c,d) \to (-c,-d)$. We can extend therefore the sum over c to $0 < |c| \leq K$, and divide by two. The polar part can be included by extending the sum with $c = 0$. Note that $\gcd(0,d) = |d|$, thus $c = 0$ adds $(c,d) = (0,1)$ and $(c,d) = (0,-1)$ to the sum, which works out nicely with the overall factor of $\frac{1}{2}$. We obtain finally

$$\begin{aligned} f_\mu(\tau) = \frac{1}{2} F_\mu(\Delta_\mu) + \frac{1}{2} \sum_{n-\Delta_\nu<0} \lim_{K\to\infty} \sum_{\gamma\in(\Gamma_\infty\backslash\Gamma)_K} j(\gamma,\tau)^{-w} M^{-1}(\gamma)_\mu^\nu F_\nu(n) \\ \times e\left((n-\Delta_\nu)\gamma(\tau)\right) R(x), \end{aligned} \tag{A.18}$$

where we have defined $\sum_{|c|\leq K} \sum_{\substack{|d|\leq K \\ (c,d)=1}} = \sum_{\gamma\in(\Gamma_\infty\backslash\Gamma)_K}$.

A.2. Period functions and their transformation properties

This subsection reviews relevant properties of period functions. These properties are necessary for the derivation of the transformation properties of $f_\mu(\tau)$ in Section A.3. For simplicity of exposition we discuss the case of scalar modular forms. Using the notation of Section 4, the discussion generalizes easily to the vector-valued case.

We start with the period function of a cusp form $G(z)$ transforming as $G(\gamma(z)) = M^{-1}(\gamma)(cz+d)^{2-w}G(z)$ under $\gamma \in \Gamma$. The period function of $G(z)$, $p(\tau, \bar{y}, \overline{G})$ is defined by

(A.19)
$$p(\tau, \bar{y}, \overline{G}) = \frac{1}{\Gamma(1-w)} \int_{\bar{y}}^{-i\infty} \overline{G(z)}(\bar{z}-\tau)^{-w} d\bar{z}, \qquad y \in \mathcal{H} \cup \mathbb{Q} \cup \mathrm{i}\infty.$$

Note that in case $-w \in \mathbb{N}$, this expression is a polynomial in τ. Also note that the expression $p(\tau, \bar{y}, \overline{G})$ makes sense for any function $G(z)$ that decays sufficiently rapidly at infinity, e.g., $G(x + \mathrm{i}\rho) \sim_{\rho \to +\infty} \mathrm{const.}\rho^{\alpha}\mathrm{e}^{-A\rho}$ for $A > 0$ will suffice. The constituents of the integrand satisfy simple transformation properties: $\gamma(\bar{z}) - \gamma(\tau) = \frac{\bar{z}-\tau}{j(\gamma,\bar{z})j(\gamma,\tau)}$ and $d\gamma(z) = \frac{dz}{j(\gamma,z)^2}$. Using these equations we obtain for $p(\gamma(\tau), \gamma(\bar{y}), \overline{G(z)})$ the transformation rule

$$p(\gamma(\tau), \gamma(\bar{y}), \overline{G}) = j(\gamma, \tau)^{w} M(\gamma) \left[p(\tau, \bar{y}, \overline{G}) - p(\tau, \gamma^{-1}(\infty), \overline{G}) \right], \tag{A.20}$$

where we have used the fact that $M(\gamma)$ is unitary.

If we choose a constant $\delta > 0$ we can try to construct a cusp form $G^{(\delta)}(z)$ of weight $2-w$ by forming the Poincaré series

$$G^{(\delta)}(z) = \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \frac{M(\gamma)(-2\pi \mathrm{i}\delta)^{1-w} e(\delta\gamma(z))}{j(\gamma, z)^{2-w}} := \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} g_\gamma^{(\delta)}(z), \tag{A.21}$$

where we defined $g_\gamma^{(\delta)}(z)$ by the second equality. The prefactor is chosen for later convenience. We will sometimes drop the superscript δ when the context is clear. For $w < 0$ the series is convergent, although it might vanish.

The period functions are relevant for our discussion of the fareytail expansions as explained in Appendix A.3 and Section 4. In those discussions we make use of the function $t_\gamma(\tau)$ defined by

$$t_\gamma(\tau) := p(\tau, \gamma^{-1}(\mathrm{i}\infty), \overline{g_\gamma}). \tag{A.22}$$

Using the above identities and Equation (C.3) one can check that $t_\gamma(\tau)$ satisfies the transformation rule with $\tilde{\gamma} \in \Gamma$

$$t_\gamma(\tilde{\gamma}(\tau)) = j(\tilde{\gamma}, \tau)^{w} M(\tilde{\gamma}) \left[t_{\gamma\tilde{\gamma}}(\tau) - p(\tau, \tilde{\gamma}^{-1}(\mathrm{i}\infty), \overline{g_{\gamma\tilde{\gamma}}}) \right], \tag{A.23}$$

Note that $t_\gamma(\tau)$ can be rewritten as

$$t_\gamma(\tau) = \frac{-1}{\Gamma(1-w)} j(\gamma, \tau)^{-w} M^{-1}(\gamma) e(-\delta\gamma(\tau)) \int_x^\infty \mathrm{e}^{-z} z^{-w} dz, \tag{A.24}$$

with $x = \frac{2\pi i\delta}{cj(\gamma,\tau)}$ where c is the 21 matrix element of γ. The steps involved are first a transformation of $\bar{z}$ to $\gamma^{-1}(\bar{z})$, then rewriting of the integrand using its modular properties and at last another redefinition of $\bar{z}$.

A.3. Transformation properties of the fareytail

We will deduce the transformation properties of $f_\mu(\tau)$ from the expression given in Equation (A.18). Many intermediate steps are given without rigorous proofs, these can be found in [6]. We discuss the case of scalar modular forms; at the end we simply state the straightforward generalization to vector-valued modular forms. The discussion reverses the logic of Section 4.

We study first the transformation properties of a (scalar) modular form with a single polar term $q^{-\delta}$ ($\delta > 0$) for a clear exposition. Eventually, we will deduce the transformation law for general $f_\mu(\tau)$. We define the function $s_\gamma(\tau) = j(\gamma,\tau)^{-w}M^{-1}(\gamma)e(-\delta\gamma(\tau))$ and use $t_\gamma(\tau)$ as in Equation (A.24). Equation (A.18) is in this case given by

$$f^{(-\delta)}(\tau) = \frac{1}{2}F(\delta) + \frac{1}{2}\lim_{K\to\infty}\sum_{\gamma\in(\Gamma_\infty\backslash\Gamma)_K} s_\gamma(\tau) + t_\gamma(\tau). \tag{A.25}$$

$s_\gamma(\tau)$ satisfies $s_\gamma(\tilde{\gamma}(\tau)) = j(\tilde{\gamma},\tau)^w M(\tilde{\gamma})s_{\gamma\tilde{\gamma}}(\tau)$. We obtain with Equation (A.23)

$$\begin{aligned} f^{(-\delta)}(\tilde{\gamma}(\tau)) = {} & \frac{1}{2}F(\delta) + \frac{1}{2}M(\tilde{\gamma})(\tilde{c}\tau+\tilde{d})^w \lim_{K\to\infty}\sum_{\gamma\in(\Gamma_\infty\backslash\Gamma)_K} s_{\gamma\tilde{\gamma}}(\tau) + t_{\gamma\tilde{\gamma}}(\tau) \\ & - p(\tau, \tilde{\gamma}^{-1}(-i\infty), \overline{g_{\gamma\tilde{\gamma}}}). \end{aligned} \tag{A.26}$$

The invariance under $T = \gamma_1$ is obvious from the Fourier expansion and Equation (A.8). We therefore only need to check the invariance under the other generator of Γ, $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. $(\Gamma_\infty \backslash \Gamma)_K$ is however left invariant under right multiplication of S. Therefore, $\sum_{\gamma\in(\Gamma_\infty\backslash\Gamma)_K} s_{\gamma S}(\tau) + t_{\gamma S}(\tau) = \sum_{\gamma\in(\Gamma_\infty\backslash\Gamma)_K} s_\gamma(\tau) + t_\gamma(\tau)$ holds.

The anomalous terms compared to the usual transformation rule of modular forms are the constant term $\frac{1}{2}F(\delta)$ and the subtraction of period integrals. A careful study of the limit $K \to \infty$ and the period integrals is needed. Lemma 4.4 of [6] shows that for $y \in \mathcal{H}$

$$\lim_{K\to\infty}\sum_{\gamma\in(\Gamma_\infty\backslash\Gamma)_K} p(\tau, \bar{y}, \overline{g_\gamma^{(\delta)}}) = p(\tau, \bar{y}, \overline{G^{(\delta)}}) - F(\delta), \tag{A.27}$$

thus the limit $K \to \infty$ and the integral do not commute. This comes about as follows. Calculation of the Fourier coefficients of $G^{(\delta)}$ gives an error term by the Lipschitz summation formula. This error term tends to zero, however the period integral over the error does not vanish and provides us with the offset.

In Equation (A.26), we however have $y \notin \mathcal{H}$ but $y = \tilde{\gamma}^{-1}(i\infty) \in \mathbb{Q}$. In this case we obtain with Corollary 4.5 of [6]

(A.28)
$$\lim_{K\to\infty} \sum_{\gamma\in(\Gamma_\infty\backslash\Gamma)_K} p(\tau, \tilde{\gamma}^{-1}(\mathrm{i}\infty), \overline{g_\gamma^{(\delta)}}) = p(\tau, \tilde{\gamma}^{-1}(\mathrm{i}\infty), \overline{G^{(\delta)}}) + F(\delta)\left(M^{-1}(\tilde{\gamma})(\tilde{c}\tau + \tilde{d})^{-w} - 1\right).$$

Inserting this result in Equation (A.26) we find the transformation of $f^{(-\delta)}(\tau)$ under γ

$$f^{(-\delta)}(\gamma(\tau)) = j(\gamma,\tau)^w M(\gamma)\left[f^{(-\delta)}(\tau)_\delta - p(\tau, \gamma^{-1}(\mathrm{i}\infty), \overline{G^{(\delta)}})\right]. \tag{A.29}$$

Note that in special cases G is zero. This is for example the case for $\delta \in \mathbb{N}$ and $w = 0, -2, -4, -6, -8$ and -12 [7]. A cusp form with weight $12 = 2 - w$ of Γ exists, which explains that in case $w = -10$, we will find a transformation with a non-zero shift.

Extending the above to the case of vector-valued modular forms with multiple polar terms is straightforward. The period function should vanish of course in this case. For a general choice of Δ_μ and polar $F_\mu(n)$, we obtain the transformation

$$f_\mu(\gamma(\tau)) = (c\tau + d)^w M(\gamma)_\mu^\nu \left[f_\nu(\tau) - p(\tau, \gamma^{-1}(-\mathrm{i}\infty), \overline{G_\nu})\right]. \tag{A.30}$$

with

(A.31)
$$G_\mu(z) = \frac{1}{2} \sum_{n-\Delta_\nu<0} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \overline{M^{-1}(\gamma)_\mu^\nu} \frac{(2\pi\mathrm{i}(n-\Delta_\nu))^{1-w} F_\nu(n) e(|n-\Delta_\nu|\gamma(z))}{(cz+d)^{2-w}}.$$

Appendix B. Lipschitz summation formula

A crucial ingredient for the derivation in Appendix A is the Lipschitz summation formula for general $p \geq 1$ [6]. Let $\tau \in \mathcal{H}$, $N \in \mathbb{N}$, $0 \leq \alpha < 1$, then

$$\sum_{l=-N}^{N} \frac{e(-l\alpha)}{(\tau+l)^p} = \frac{(-2\pi \mathrm{i})^p}{\Gamma(p)} \sum_{m=0}^{\infty} (m+\alpha)^{p-1} q^{m+\alpha} + E(\tau,p,Q), \tag{B.1}$$

where $Q = N + \frac{1}{2}$ and $E(\tau,p,Q)$ is an error term and given by

(B.2)
$$E(\tau,p,Q) = (\mathrm{i}Q)^{1-p} \int_{-\infty}^{\infty} \frac{h(x-\mathrm{i}) - h(x+\mathrm{i})}{1+\exp(2\pi x Q)} dx, \qquad h(x) = \frac{\exp(2\pi x Q \alpha)}{(x+\frac{\tau}{\mathrm{i}Q})^p}.$$

The error tends to 0 for $Q \to \infty$, except for the case $p = 1$, $\alpha = 0$; then we obtain $\lim_{Q\to\infty} E(\tau, 1, Q) = \pi\mathrm{i}$. The case $p = 1$, $\alpha = 0$ gives the two well-known infinite sums for $\cot \pi\tau$

$$\frac{1}{\tau} + \sum_{l=1}^{\infty} \left(\frac{1}{\tau - l} + \frac{1}{\tau + l} \right) = \pi \cot \pi\tau = \pi\mathrm{i} - 2\pi\mathrm{i} \sum_{m=0}^{\infty} q^m, \tag{B.3}$$

which can be proved by using $\sin \pi\tau = \pi\tau \prod_{n=1}^{\infty} (1 - \tau^2/n^2)$.

The proof of Equation (B.1) uses the function $f(z) = e((z+\tau)\alpha)/(\mathrm{i}z)^p$ $(e(z+\tau)-1)$. This function has poles at $z = -\tau - l$, $l \in \mathbb{Z}$ with residue $(2\pi\mathrm{i})^{-1} e(-l\alpha)/(-\mathrm{i}\tau - \mathrm{i}l)^p$. The right-hand side is obtained by integrating along the boundary of the rectangle $-\mathrm{Re}(\tau) \pm Q \pm \mathrm{i}M$, which is slit along the positive imaginary axis to avoid a branch cut of $(\mathrm{i}z)^p$. The main contribution to the integral comes from this part of the contour. It can be calculated using the Hankel contour integral $\frac{1}{\Gamma(p)} = \frac{1}{2\pi\mathrm{i}} \int_{\mathcal{C}} \mathrm{e}^t t^{-p} dt$, where $\mathcal{C}$ is the contour which begins at $-\infty - \mathrm{i}0^+$, circles the origin in the counterclockwise direction and ends at $-\infty + \mathrm{i}0^+$. The horizontal sides do not contribute when $M \to \infty$, the error is accordingly calculated by the integral along the vertical segments.

Appendix C. Details on multiplier systems

We remark that consistency of Equation (3.1) requires $M(\gamma)$ to satisfy

$$M(\gamma_1)_\mu^\rho M(\gamma_2)_\rho^\nu = c_w(\gamma_1,\gamma_2) M(\gamma_1\gamma_2)_\mu^\nu, \tag{C.1}$$

where

$$c_w(\gamma_1,\gamma_2) := \frac{j(\gamma_1\gamma_2,\tau)^w}{j(\gamma_1,\gamma_2\tau)^w j(\gamma_2,\tau)^w}. \tag{C.2}$$

Using the identity

$$j(\gamma_1\gamma_2,\tau) = j(\gamma_1,\gamma_2\tau)j(\gamma_2,\tau), \tag{C.3}$$

we see that the right-hand side of Equation (C.2) is a phase. On the other hand, it is locally analytic in τ, and hence it does not depend on τ. Indeed $c_w(\gamma_1,\gamma_2)$ is a cocycle on Γ. Then the cocycle is most easily evaluated by taking $\tau = \mathrm{i}\Lambda, \Lambda \to +\infty$. Define $\epsilon(\gamma) = \pm 1$ by

$$\epsilon(\gamma) := \begin{cases} \mathrm{sign}(c), & c \neq 0, \\ \mathrm{sign}(d), & c = 0. \end{cases} \tag{C.4}$$

Then we have with $\epsilon_i = \epsilon(\gamma_i)$

$$c_w(\gamma_1,\gamma_2) = \exp\left[\frac{\mathrm{i}\pi}{2} w(\epsilon_1\epsilon_2\epsilon_3 - \epsilon_1 - \epsilon_2 + \epsilon_3)\right], \tag{C.5}$$

where $\gamma_3 = \gamma_1\gamma_2$. This expression takes values $1, \mathrm{e}^{\pm 2\pi \mathrm{i} w}$.

Note that

1. c_w is symmetric and $c_w(1,\gamma) = c_w(\gamma,1) = 1$,
2. $M(-\gamma)_\mu^\nu = \mathrm{e}^{\mathrm{i}\pi w \epsilon(\gamma)} M(\gamma)_\mu^\nu$,
3. It is perfectly possible to have $(\epsilon_1,\epsilon_2,\epsilon_3) = (-1,-1,+1)$. For example, take

$$\gamma_1 = \begin{pmatrix} N+1 & N \\ -N & 1-N \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 1 & 0 \\ -N & 1 \end{pmatrix},$$

with $N > 2$, thus realizing $c_w(\gamma_1,\gamma_2) = \mathrm{e}^{2\pi \mathrm{i} w}$.

In applications to the elliptic genus it is possible to describe the multiplier system explicitly. In the case of the $(2,2)$ elliptic genus, in order to have a basis of linearly independent functions we should make a unitary transformation to the even and odd level m theta functions and correspondingly define f_μ by expanding with respect to the even level m theta functions, defined by

$$\text{(C.6)} \qquad \theta^+_{\mu,m}(\tau,z) := \begin{cases} \theta_{0,m}(\tau,z), & \mu = 0, \\ \dfrac{1}{\sqrt{2}}(\theta_{\mu,m}(\tau,z) + \theta_{-\mu,m}(\tau,z)), & 1 \le \mu \le m-1, \\ \theta_{m,m}(\tau,z), & \mu = m \end{cases}$$

and defining $\phi(\tau,z) := \sum_{\mu=0}^{m} h^+_\mu(\tau)\theta^+_{\mu,m}(\tau,z)$. Taking $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ we find

(C.7)

$$M(S) = \mathrm{e}^{-\mathrm{i}\pi/4} \begin{pmatrix} S_{00} = \dfrac{1}{\sqrt{2m}} & S_{0,\mu} = \dfrac{1}{\sqrt{m}} & S_{0,m} = \dfrac{1}{\sqrt{2m}} \\ S_{\mu,0} = \dfrac{1}{\sqrt{m}} & S_{\mu\nu} = \sqrt{\dfrac{2}{m}}\cos\left(2\pi\dfrac{\mu\nu}{2m}\right) & S_{\mu,m} = \dfrac{(-1)^\mu}{\sqrt{m}} \\ S_{m,0} = \dfrac{1}{\sqrt{2m}} & S_{m,\mu} = \dfrac{(-1)^\mu}{\sqrt{m}} & S_{m,m} = \dfrac{(-1)^m}{\sqrt{2m}} \end{pmatrix},$$

where $1 \le \mu, \nu \le m-1$ in the above matrix. Of course, we also have

$$\text{(C.8)} \qquad M(T)_\mu^\nu = e\left(-\frac{\mu^2}{4m}\right)\delta_\mu^\nu.$$

Together these generate the multiplier system.

References

[1] J. M. Maldacena, *The large n limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. **2** (1998), 231–252, hep-th/9711200.

[2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *Gauge theory correlators from non-critical string theory,* Phys. Lett. B **428** (1998), 105, arXiv:hep-th/9802109.

[3] E. Witten, *Anti-de Sitter space and holography,* Adv. Theor. Math. Phys. **2** (1998), 253, arXiv:hep-th/9802150.

[4] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, *Large N field theories, string theory and gravity,* Phys. Rept. **323** (2000), 183, arXiv:hep-th/9905111.

[5] R. Dijkgraaf, J. M. Maldacena, G. W. Moore and E. P. Verlinde, *A black hole farey tail*, hep-th/0005003.

[6] D. Niebur, *Construction of automorphic forms and integrals*, Trans. Amer. Math. Soc. **191** (1974), 373–385.

[7] I. Knopp, Marvin, *Rademacher on $j(\tau)$, poincaré series of nonpositive weights and the eichler cohomology*, Not. Amer. Math. Soc. **37** (1990), 385–393.

[8] F. Denef and G. W. Moore, *Split states, entropy enigmas, holes and halos*, hep-th/0702146.

[9] H. Ooguri, A. Strominger and C. Vafa, *Black hole attractors and the topological string*, Phys. Rev. **D70** (2004), 106007, hep-th/0405146.

[10] J. Manschot, *AdS_3 partition functions reconstructed,* JHEP **10** (2007), 103, arXiv:0707.1159 [hep-th].

[11] T. M. Apostol, *Modular Functions and Dirichlet Series in Number Theory*, Springer-Verlag, Berlin, 1976.

[12] H. Rademacher, *The Fourier coefficients and the functional equation of the absolute modular invariant $j(\tau)$*, Amer. J. Math. **61** (1939), 237–248.

[13] J. de Boer, *Large n elliptic genus and ads/cft correspondence*, JHEP **05** (1999), 017, hep-th/9812240.

[14] J. M. Maldacena, G. W. Moore and A. Strominger, *Counting BPS black holes in toroidal type II string theory*, arXiv:hep-th/9903163.

[15] J. de Boer, M. C. N. Cheng, R. Dijkgraaf, J. Manschot and E. Verlinde, *A farey tail for attractor black holes*, JHEP **11** (2006), 024, hep-th/0608059.

[16] D. Gaiotto, A. Strominger and X. Yin, *From ads(3)/cft(2) to black holes/topological strings*, hep-th/0602046.

[17] E. Witten, *Three-dimensional gravity revisited*, 0706.3359.

[18] A. Maloney and E. Witten, *Quantum gravity partition functions in three dimensions*, arXiv:0712.0155 [hep-th].

[19] E. Witten, *AdS/CFT correspondence and topological field theory*, JHEP **9812** (1998), 012, arXiv:hep-th/9812012.

[20] D. Belov and G. W. Moore, *Conformal blocks for AdS(5) singletons*, arXiv:hep-th/0412167.

[21] J. M. Maldacena and A. Strominger, *AdS(3) black holes and a stringy exclusion principle*, JHEP **9812** (1998), 005, arXiv:hep-th/9804085.

[22] G. W. Moore, *Les Houches lectures on strings and arithmetic*, arXiv:hep-th/0401049.

[23] P. Kraus and F. Larsen, *Partition functions and elliptic genera from supergravity*, JHEP **0701** (2007), 002.

[24] M. Banados, C. Teitelboim and J. Zanelli, *The black hole in three-dimensional space-time*, Phys. Rev. Lett. **69** (1992), 1849–1851, hep-th/9204099.

[25] M. Henningson and K. Skenderis, *The holographic weyl anomaly*, JHEP **07** (1998), 023, hep-th/9806087.

[26] K. Skenderis, *Lecture notes on holographic renormalization*, Class. Quant. Grav. **19** (2002), 5849, arXiv:hep-th/0209067.

[27] A. Strominger and C. Vafa, *Microscopic origin of the Bekenstein–Hawking entropy*, Phys. Lett. **B379** (1996), 99–104, hep-th/9601029.

[28] J. M. Maldacena, A. Strominger and E. Witten, *Black hole entropy in m-theory*, JHEP **12** (1997), 002, hep-th/9711053.

[29] R. Minasian, G. W. Moore and D. Tsimpis, *Calabi–Yau black holes and (0,4) sigma models,* Commun. Math. Phys. **209** (2000), 325, arXiv:hep-th/9904217.

[30] T. Kawai, Y. Yamada and S.-K. Yang, *Elliptic genera and* $n = 2$ *superconformal field theory*, Nucl. Phys. **B414** (1994), 191–212, hep-th/9306096.

[31] V. Gritsenko, *Elliptic genus of calabi-yau manifolds and jacobi and siegel modular forms*, math/9906190.

[32] R. Dijkgraaf, G. W. Moore, E. Verlinde and H. Verlinde, *Elliptic genera of symmetric products and second quantized strings*, Commun. Math. Phys. **185** (1997), 197–209, hep-th/9608096.

[33] M. Eichler and D. Zagier, *The Theory of Jacobi Forms*. Birkhäuser, Basel, 1985.

[34] D. Gaiotto, A. Strominger and X. Yin, *The m5-brane elliptic genus: modularity and bps states*, hep-th/0607010.

[35] G. W. Moore, *Anomalies, Gauss laws, and page charges in M-theory,* C. R. Phys. **6** (2005), 251, arXiv:hep-th/0409158.

[36] H. Rademacher and H. S. Zuckerman, *On the fourier coefficients of certain modular forms of positive dimension*, Ann. Math. **39** (2) (1938), 433–462.

[37] M. I. Knopp, *Some new results on the Eichler cohomology of automorphic forms*, Bull. Amer. Math. Soc. **80** (1974), 607–632.

[38] D. Zagier, *Periods of modular forms and Jacobi theta functions,* Invent. Math. **104** (1991), 449.

[39] H. Cohen and J. Oesterlé, *Dimensions des espaces de formes modulaires*, in Lecture Notes in Mathematics, **627**, Springer-Verlag, 1976, 69–78.

[40] W. Kohnen, *New forms of half-integral weight*, J. Reine Angew. Math. **333** (1982), 32–72.

[41] J. Manschot, *On the space of elliptic genera*, Comm. Theory. Phys. **2** (2008), 803, arXiv:0805.4333 [hep-th].

[42] M. Cvetic and F. Larsen, *Near horizon geometry of rotating black holes in five dimensions*, Nucl. Phys. **B531** (1998), 239–255, hep-th/9805097.

[43] S. W. Hawking and D. N. Page, *Thermodynamics of black holes in anti-de sitter space*, Commun. Math. Phys. **87** (1983), 577.

[44] E. Witten, *Anti-de Sitter space, thermal phase transition, and confinement in gauge theories,* Adv. Theor. Math. Phys. **2** (1998), 505, arXiv:hep-th/9803131.

[45] F. Denef and G. W. Moore, *How many black holes fit on the head of a pin?,* Gen. Rel. Grav. **39** (2007), 1539, arXiv:0705.2564 [hep-th].

[46] M. x. Huang, A. Klemm, M. Marino and A. Tavanfar, *Black holes and large order quantum geometry*, arXiv:0704.2440 [hep-th].

[47] T. M. Apostol, *Introduction to analytic number theory*, Springer-Verlag, Berlin, 1976.

[48] J. de Boer, F. Denef, S. El-Showk, I. Messamah and D. Van den Bleeken, *Black hole bound states in* $AdS_3 \times S^2$, `arXiv:0802.2257 [hep-th]`.

[49] D. Gaiotto and X. Yin, *Examples of M5-brane elliptic genera*, JHEP **0711** (2007), 004 `arXiv:hep-th/0702012`.

[50] L. Göttsche, *The Betti numbers of the Hilbert scheme of points on a smooth projective surface*, Math. Ann. **286** (1990), 193.

[51] A. Klemm, D. Maulik, R. Pandharipande and E. Scheidegger, *Noether–Lefschetz theory and the Yau–Zaslow conjecture*, `arXiv:0807.2477`.

[52] D. Zagier, *Nombres de classes et formes modulaires de poids 3/2,* C.R. Acad. Sc. Paris, **281** (1975), 883.

[53] F. Hirzebruch and D. Zagier, *Intersection numbers of curves on Hilbert modular surfaces and modular forms of Nebentypus,* Inv. Math. **36** (1976), 57.

[54] C. Vafa and E. Witten, *A strong coupling test of S duality,* Nucl. Phys. B **431** (1994), 3, `arXiv:hep-th/9408074`.

[55] G. W. Moore and E. Witten, *Integration over the u-plane in Donaldson theory,* Adv. Theor. Math. Phys. **1** (1998), 298, `arXiv:hep-th/9709193`.

[56] J. M. Maldacena, G. W. Moore and N. Seiberg, *D-brane charges in five-brane backgrounds*, JHEP **0110** (2001), 005, `arXiv:hep-th/0108152`.

[57] S. Gukov, E. Martinec, G. W. Moore and A. Strominger, *Chern–Simons gauge theory and the AdS(3)/CFT(2) correspondence*, `arXiv:hep-th/0403225`.

[58] G. W. Moore, *Attractors and arithmetic*, `arXiv:hep-th/9807056`.

[59] G. W. Moore, *Arithmetic and attractors*, `arXiv:hep-th/9807087`.

[60] N. Yui, *Update on the modularity of Calabi–Yau varieties*, Fields Inst. Comm. **38**, AMS, Providence, RI, 2003, 307–362.

[61] D. R. Heath-Brown, *Arithmetic applications of Kloosterman sums,* N. Arch. Wiskunde **1** (2000), 380.

Institute for Theoretical Physics
University of Amsterdam
Valckenierstraat 65
1018 XE Amsterdam
The Netherlands
E-mail address: manschot@physics.rutgers.edu

NHETC and Department of Physics and Astronomy
Rutgers University
Piscataway
NJ 08855-0849
USA
E-mail address: gmoore@physics.rutgers.edu

Received February 27, 2009