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On a computation of rank two Donaldson–Thomas invariants

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For a Calabi–Yau three-fold X, we explicitly compute the Donaldson–Thomas-type invariant counting pairs (F, V), where F is a zero-dimensional coherent sheaf on X and $V \subset F$ is a two-dimensional linear subspace, which satisfy a certain stability condition. This is a rank two version of the Donaldson–Thomas (DT)-invariant of rank one, studied by Li, Behrend-Fantechi and Levine-Pandharipande. We use the wall-crossing formula of DT-invariants established by Joyce-Song, Kontsevich-Soibelman.

1. Introduction

The purpose of this article is to write down a closed formula of the generating series of certain rank two Donaldson–Thomas (DT)-type invariants on Calabi–Yau three-folds. The DT-invariant is a counting invariant of stable coherent sheaves on X, and it is introduced in [23] in order to give a holomorphic analogue of the Casson invariant on real three-manifolds. It is now conjectured by Maulik–Nekrasov–Okounkov–Pandharipande (MNOP) [21] that Gromov–Witten invariants and rank one DT-invariants are related in terms of generating functions. So far, rank one DT-invariants have been studied in several papers, e.g., [2, 3, 18, 19].

On the other hand, it seems that higher rank DT-invariants have not been explicitly calculated yet in any example. Although the rank one case is important in connection with MNOP conjecture, there is also some motivation of studying higher rank DT-invariants. For instance, the rank of a coherent sheaf is not preserved under Fourier–Mukai transformations, e.g., the Pfaffian–Grassmannian derived equivalence established in [5]. Hence in order to compare DT-invariants under Fourier–Mukai transformations, it seems that we also have to work with higher rank DT-invariants.

Recently, the wall-crossing formula of DT-invariants has been developed by Joyce-Song [15] and Kontsevich-Soibelman [16]. As pointed out in [16, Paragraph 6.5], certain higher rank DT-type invariants are in principle calculated by the wall-crossing formula, if we are given data for the DT-invariants of rank one. In this article, we work out the wall-crossing formula established by Joyce–Song [15], and write down the explicit formula of DT-type invariants counting rank two D0–D6 bound state, discussed in [16, Paragraph 6.5]. We also give an evidence of the integrality conjecture proposed by Kontsevich–Soibelman [16, Conjecture 6].

1.1. Rank one DT-invariant

Let X be a smooth projective Calabi–Yau three-fold over $\mathbb{C}$, i.e., $K_X = \wedge^3 T_X^*$ is trivial and $H^1(\mathcal{O}_X) = 0$. For $n \in \mathbb{Z}$, let $\mathrm{Hilb}^n(X)$ be the Hilbert scheme of n-points in X,

$$\begin{aligned}\mathrm{Hilb}^n(X) &= \{Z \subset X : \dim Z = 0, \mathrm{length}\, \mathcal{O}_Z = n\}, \\ &= \left\{(F, v) : \begin{array}{l} F \text{ is a zero-dimensional coherent sheaf on } X \text{ with} \\ \text{length } n, \text{ and } v \in F \text{ generates } F \text{ as an } \mathcal{O}_X\text{-module.}\end{array}\right\}.\end{aligned}$$

The moduli space $\mathrm{Hilb}^n(X)$ is projective and has a symmetric obstruction theory [23]. By integrating the associated zero-dimensional virtual cycle, we can define the rank one DT-invariant,

$$\mathrm{DT}(1, n) = \int_{[\mathrm{Hilb}^n(X)]^{\mathrm{vir}}} 1 \in \mathbb{Z}.$$

Another way of defining DT-invariant is to use Behrend's constructible function [1],

$$\nu \colon \mathrm{Hilb}^n(X) \longrightarrow \mathbb{Z}.$$

In [1], Behrend shows that $\mathrm{DT}(1, n)$ is also written as

$$\mathrm{DT}(1, n) = \int_{\mathrm{Hilb}^n(X)} \nu \, d\chi := \sum_{k \in \mathbb{Z}} k\chi(\nu^{-1}(k)),$$

where $\chi(*)$ is the topological Euler characteristic. Let $\mathrm{DT}(1)$ be the generating series,

$$\mathrm{DT}(1) = \sum_{n \in \mathbb{Z}} \mathrm{DT}(1, n) q^n.$$

The series $\mathrm{DT}(1)$ is computed by Li [19], Behrend-Fantechi [3] and Pandharipande-Levine [18].

Theorem 1.1 [3, 18, 19]. *We have the following formula:*

$$\mathrm{DT}(1) = M(-q)^{\chi(X)}.$$

Here $M(q)$ is the MacMahon function,

$$M(q) = \prod_{k \geq 1} \frac{1}{(1-q)^k}.$$

1.2. Rank two DT-invariant

In this article, we consider a rank two analogue of the invariant $\mathrm{DT}(1, n)$. Let F be a zero-dimensional coherent sheaf on X with length n, and $V \subset F$ is a two-dimensional $\mathbb{C}$-vector subspace. We call the pair (F, V) *semistable* (resp. *stable*) if it satisfies the following stability condition:

- The subspace $V \subset F$ generates F as an $\mathcal{O}_X$-module.
- For any non-zero $v \in V$, the subsheaf $F_v := \mathcal{O}_X \cdot v \subset F$ satisfies

$$\operatorname{length} F_v \geq n/2 \quad (\text{resp. } \operatorname{length} F_v > n/2).$$

We denote by $M^{(2,n)}$ the moduli space of semistable (F, V) with length $F = n$. If n is odd, the space $M^{(2,n)}$ is an algebraic space of finite type, and the integration of the Behrend function yields the DT-type invariant,

$$\mathrm{DT}(2, n) = \int_{M^{(2,n)}} \nu \, d\chi. \tag{1.1}$$

When n is even, the space $M^{(2,n)}$ is an algebraic stack, and the integration such as (1.1) does not make sense. However, we are also able to define the DT-type invariant,

$$\mathrm{DT}(2, n) \in \mathbb{Q},$$

when n is even by using the technique of the Hall-algebra. The existence of the above $\mathbb{Q}$-valued invariant is one of the big achievement of the recent work of Joyce–Song [15]. We will give a brief introduction of the definition of $\mathrm{DT}(2, n)$ in Section 3. Let us consider the generating series,

$$\mathrm{DT}(2) = \sum_{n \in \mathbb{Z}} \mathrm{DT}(2, n) q^n.$$

Applying the wall-crossing formula of DT-invariants [15, 16], we show the following formula.

Theorem 1.2. *We have the following formula:*

$$\mathrm{DT}(2) = \frac{1}{4}M(q)^{2\chi(X)} - \frac{\chi(X)}{2}\{M(q)^{\chi(X)} \cdot M(q)^{\chi(X)} \cdot N(q)\}_{\Delta}, \tag{1.2}$$

where $\Delta \subset \mathbb{Z}^3_{\geq 0}$ *is*

$$\Delta = \{(m_1, m_2, m_3) \in \mathbb{Z}^3_{\geq 0} : -m_3 \leq m_1 - m_2 < m_3\}.$$

Let us explain the notation. The series $N(q)$ is defined by

$$\begin{aligned} N(q) &:= q\frac{d}{dq}\log M(q) \\ &= \sum_{r,n\geq 0, r|n} r^2 q^n, \end{aligned}$$

and for $f_1, f_2, \dots, f_N \in \mathbb{Q}[\![q]\!]$ given by

$$f_i = \sum_{n\geq 0} a_n^{(i)} q^n, \quad 1 \leq i \leq N,$$

and a subset $\Delta \subset \mathbb{Z}^N_{\geq 0}$, the series $\{f_1 \cdot f_2 \cdots f_N\}_{\Delta}$ is defined by

$$\{f_1 \cdot f_2 \cdots f_N\}_{\Delta} = \sum_{(n_1, n_2, \dots, n_N)\in\Delta} a_{n_1}^{(1)} a_{n_2}^{(2)} \cdots a_{n_N}^{(N)} q^{n_1+n_2+\cdots+n_N}.$$

In formula (1.2), we set $N = 3$, $f_1 = f_2 = M(q)^{\chi(X)}$ and $f_3 = N(q)$.

1.3. Integrality property

Following [16], we introduce the invariant

$$\Omega(2, n) = \begin{cases} \mathrm{DT}(2, n), & n \text{ is odd}, \\ \mathrm{DT}(2, n) - \frac{1}{4}\mathrm{DT}(1, \frac{n}{2}), & n \text{ is even}. \end{cases}$$

We also prove an evidence of the integrality conjecture by Kontsevich–Soibelman [16, Conjecture 6].

Theorem 1.3. *We have* $\Omega(2, n) \in \mathbb{Z}$ *for any* $n \in \mathbb{Z}$.

A first few terms of $\Omega(2,n)$ are calculated as follows:

$$\Omega(2,0)=\Omega(2,1)=0,\ \Omega(2,2)=-\chi,$$
$$\Omega(2,3)=-\frac{1}{6}(\chi^3+15\chi^2+20\chi),$$
$$\Omega(2,4)=-\frac{1}{12}(\chi^4+30\chi^3+119\chi^2+102\chi).$$

1.4. Comment and the strategy for the computation

Note that the pair (F,V) satisfying the stability condition determines a surjection,

$$\mathcal{O}_X^{\oplus 2}\stackrel{s}{\twoheadrightarrow} F,$$

and the kernel of s is a trivial vector bundle of rank two except at isolated point singularities. Hence, the invariants $\mathrm{DT}(2,n)$ are more or less "classical" in the sense that they count coherent sheaves, due to Thomas [23] if n is odd and Joyce–Song [15] if n is even.

However the way we compute them steps out of the abelian category of coherent sheaves, using another abelian subcategory of the bounded derived category of coherent sheaves on X. We will consider the triangulated subcategory

$$\mathcal{D}_X\subset D^b(\mathrm{Coh}(X)),$$

generated by $\mathcal{O}_X$ and zero-dimensional coherent sheaves, and do a change of stability conditions in $\mathcal{D}_X$ to transform between the family of stable sheaves we want to count with stability condition Z_+, and some families of two-term complexes semistable with respect to another stability condition Z_-. It will turn out that the Z_--semistable objects are much more simple to understand, and their DT-type counting invariants are explicitly computed. Then we will use Joyce–Song's wall-crossing formula to deduce the Z_+-semistable DT-type counting invariants, which are just $\mathrm{DT}(2,n)$ we want to compute. In Theorem 4.6, we will show that Joyce–Song's wall-crossing formula yields a combinatorial description of the invariant $\mathrm{DT}(2,n)$ in terms of certain graphs with some additional structures.

In order to make Joyce–Song's wall-crossing formula work in the derived category rather than the abelian category of coherent sheaves, we need to prove an additional condition, that is the derived category version of [15, Theorem 5.3] on the local description of the moduli stack of complexes. Namely, we need to prove that the moduli stack of complexes in $\mathcal{D}_X$ can be written locally in the complex analytic topology as the critical locus for some holomorphic function on the smooth complex manifold. It is hoped that this result will follow from the general results announced by Behrend–Getzler [4], however in Proposition 2.12, we will check this result by hand in this special situation.

1.5. Relation to other works

The category $\mathcal{D}_X$ is introduced in [16, Paragraph 6.5] as the triangulated category of D0–D6 bound states. There is a list of the numbers counting semistable objects in $\mathcal{D}_X$ in [16, Paragraph 6.5], while almost all of the numbers in this list remain '?'. We will see in Remark 2.10 that $\Omega(2, n)$ are numbers which fill a part of the marks '?' in [16, Paragraph 6.5].

In the recent paper by Stoppa [22], the invariants have also been computed up to rank three. Especially, he computed the invariants both using Kontsevich–Soibelman formula and Joyce–Song formula. He also show the integrality of Kontsevich–Soibelman's BPS invariant up to rank three, using a different method from ours.

In the very recent paper [7] by Chuang, Diaconescu and Pan, the related invariants counting D0D2D6 bound states on local $(-1, -1)$-curve and $(0, -2)$-curve have been computed up to rank two.

1.6. Notation and convention

In this paper, all the varieties are defined over $\mathbb{C}$. For a variety X, the abelian category of coherent sheaves on X is denoted by $\mathrm{Coh}(X)$. The bounded derived category of coherent sheaves on X, which forms a triangulated category, is denoted by $D^b(\mathrm{Coh}(X))$. For a triangulated category $\mathcal{D}$, the shift functor is denoted by Behrend [1]. For a set of objects $S \subset \mathcal{D}$, we denote by $\langle S \rangle_{\mathrm{tr}} \subset \mathcal{D}$ the smallest triangulated subcategory of $\mathcal{D}$ which contains S. Also we denote by $\langle S \rangle_{\mathrm{ex}} \subset \mathcal{D}$ the smallest extension closed subcategory of $\mathcal{D}$ which contains S. For an abelian category $\mathcal{A}$ and a set of objects $S \subset \mathcal{A}$, the subcategory $\langle S \rangle_{\mathrm{ex}} \subset \mathcal{A}$ is also defined to be the smallest extension closed subcategory of $\mathcal{A}$ which contains S.

2. Triangulated category of D0–D6 bound states

Let X be a smooth projective Calabi–Yau three-fold over $\mathbb{C}$, i.e.,

$$K_X = \wedge^3 T_X^* \cong \mathcal{O}_X, \quad H^1(\mathcal{O}_X) = 0.$$

We denote by $\mathrm{Coh}_0(X)$ the subcategory of $\mathrm{Coh}(X)$, defined by

$$\mathrm{Coh}_0(X) = \{E \in \mathrm{Coh}(X) : \dim \mathrm{Supp}(E) = 0\}.$$

In this section, we study the triangulated subcategory of $D^b(\mathrm{Coh}(X))$ generated by $\mathcal{O}_X$ and objects in $\mathrm{Coh}_0(X)$,

$$\mathcal{D}_X := \langle \mathcal{O}_X, \mathrm{Coh}_0(X)\rangle_{\mathrm{tr}} \subset D^b(\mathrm{Coh}(X)).$$

The triangulated category $\mathcal{D}_X$ is called the category of *D0–D6 bound states* in [16, Paragraph 6.5].

2.1. t-Structure on $\mathcal{D}_X$

Here we construct the heart of a bounded t-structure on $\mathcal{D}_X$. The readers can refer [8, Section 4] for the notion of bounded t-structures and their hearts.

Lemma 2.1. *There is the heart of a bounded t-structure $\mathcal{A}_X \subset \mathcal{D}_X$, written as*

$$\mathcal{A}_X = \langle \mathcal{O}_X, \mathrm{Coh}_0(X)[-1]\rangle_{\mathrm{ex}}. \tag{2.1}$$

Proof. Let $\mathcal{F}$ be the subcategory of $\mathrm{Coh}(X)$, defined by

$$\mathcal{F} := \{E \in \mathrm{Coh}(X) : \mathrm{Hom}(F, E) = 0 \text{ for any } F \in \mathrm{Coh}_0(X)\}.$$

Then $(\mathrm{Coh}_0(X), \mathcal{F})$ is a torsion pair on $\mathrm{Coh}(X)$ (cf. [9]). Let $\mathcal{A}^{\dagger} \subset D^b$ $(\mathrm{Coh}(X))$ be the associated tilting,

$$\mathcal{A}^{\dagger} = \langle \mathcal{F}, \mathrm{Coh}_0(X)[-1]\rangle_{\mathrm{ex}}.$$

Note that $\mathcal{A}^{\dagger}$ is the heart of a bounded t-structure on $D^b(\mathrm{Coh}(X))$ (cf. [9, Proposition 2.1]). It is easy to see the following:

- We have

$$\mathcal{A}^{\dagger} \cap D^b(\mathrm{Coh}_0(X)) = \mathrm{Coh}_0(X)[-1], \tag{2.2}$$

  in $D^b(\mathrm{Coh}(X))$. In particular the LHS of (2.2) is the heart of a bounded t-structure on $D^b(\mathrm{Coh}_0(X))$.

- For any $F \in \mathrm{Coh}_0(X)$, we have

$$\mathrm{Hom}(\mathcal{O}_X, F[-1]) = \mathrm{Hom}(F[-1], \mathcal{O}_X) = 0,$$

by the Serre duality.

Then we can apply [24, Proposition 3.3], and conclude that $\mathcal{A}_X := \mathcal{A}^\dagger \cap \mathcal{D}_X$ is the heart of a bounded t-structure on $\mathcal{D}_X$, satisfying (2.1). ☐

The abelian category $\mathcal{A}_X \subset \mathcal{D}_X$ is described in a simpler way, as follows.

Proposition 2.2. *The abelian category $\mathcal{A}_X$ given by (2.1) is equivalent to the abelian category of triples*

$$\left(\mathcal{O}_X^{\oplus r}, F, s\right), \tag{2.3}$$

where r is an integer, $F \in \mathrm{Coh}_0(X)$ and $s\colon \mathcal{O}_X^{\oplus r} \to F$ is a morphism in $\mathrm{Coh}(X)$. The set of morphisms from $(\mathcal{O}_X^{\oplus r}, F, s)$ to $(\mathcal{O}_X^{\oplus r'}, F', s')$ is given by the commutative diagrams,

$$\begin{array}{ccc} \mathcal{O}_X^{\oplus r} & \xrightarrow{s} & F \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ \mathcal{O}_X^{\oplus r'} & \xrightarrow{s'} & F'. \end{array} \tag{2.4}$$

The equivalence is given by sending a triple $E = (\mathcal{O}_X^{\oplus r}, F, s)$ to the two term complex

$$\Phi(E) = \cdots \longrightarrow 0 \longrightarrow \mathcal{O}_X^{\oplus r} \stackrel{s}{\to} F \to 0 \to \cdots \in \mathcal{A}_X, \tag{2.5}$$

where $\mathcal{O}_X^{\oplus r}$ is located in degree zero.

Proof. For a triple $E = (\mathcal{O}_X^{\oplus r}, F, s)$ as in (2.3), note that the two term complex $\Phi(E)$ given by (2.5) fits into the exact sequence in $\mathcal{A}_X$,

$$0 \longrightarrow F[-1] \longrightarrow \Phi(E) \longrightarrow \mathcal{O}_X^{\oplus r} \longrightarrow 0.$$

Let us consider a diagram (2.4). Since $\mathrm{Hom}(\mathcal{O}_X^{\oplus r}, F'[-1]) = 0$, there is a unique morphism $\gamma\colon \Phi(E) \to \Phi(E')$ which fits into the commutative diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F[-1] & \longrightarrow & \Phi(E) & \longrightarrow & \mathcal{O}_X^{\oplus r} & \longrightarrow & 0 \\ & & \downarrow {\scriptstyle \beta[-1]} & & \downarrow {\scriptstyle \gamma} & & \downarrow {\scriptstyle \alpha} & & \\ 0 & \longrightarrow & F[-1] & \longrightarrow & \Phi(E) & \longrightarrow & \mathcal{O}_X^{\oplus r} & \longrightarrow & 0. \end{array} \tag{2.6}$$

Hence $E \mapsto \Phi(E)$ is a functor from the category of triples (2.3) to $\mathcal{A}_X$. Using diagram (2.6) and $\mathrm{Hom}(F[-1], \mathcal{O}_X^{\oplus r'}) = 0$, it is easy to see that Φ is fully faithful. Hence it suffices to show that Φ is essentially surjective.

Let us take an object $M \in \mathcal{A}_X$. By (2.1), there is a filtration in $\mathcal{A}_X$,

$$M_0 \subset M_1 \subset \cdots \subset M_k = M,$$

such that each subquotient $N_i = M_i/M_{i-1}$ is isomorphic to $\mathcal{O}_X$ or an object in $\mathrm{Coh}_0(X)[-1]$. We show that each M_j is quasi-isomorphic to a two term complex (2.5) by the induction on j. The case of $j = 0$ is obvious. Suppose that M_{j-1} is isomorphic to a two term complex $(\mathcal{O}_X^{\oplus r} \stackrel{s}{\to} F)$ for $F \in \mathrm{Coh}_0(X)$. There are two cases.

Case 2.1. N_j is isomorphic to $\mathcal{O}_X$.

In this case, we have the commutative diagram

$$\begin{array}{ccccc} & & \mathcal{O}_X[-1] & & \\ & & \downarrow & \searrow^{0} & \\ F[-1] & \longrightarrow & M_{j-1} & \longrightarrow & \mathcal{O}_X^{\oplus r}, \end{array}$$

since $H^1(\mathcal{O}_X) = 0$. Taking the cones, we obtain the distinguished triangle,

$$F[-1] \longrightarrow M_j \longrightarrow \mathcal{O}_X^{\oplus r+1}.$$

Therefore, M_j is quasi-isomorphic to a two term complex $(\mathcal{O}_X^{\oplus r+1} \to F)$.

Case 2.2. N_j is isomorphic to $F'[-1]$ for $F' \in \mathrm{Coh}_0(X)$.

In this case, we have the commutative diagram

$$\begin{array}{ccccc} & & F'[-2] & & \\ & \swarrow & \downarrow & & \\ F[-1] & \longrightarrow & M_{j-1} & \longrightarrow & \mathcal{O}_X^{\oplus r}, \end{array}$$

since $\operatorname{Hom}(F'[-2], \mathcal{O}_X^{\oplus r}) = 0$. Taking the cones, we obtain the distinguished triangle,

$$F''[-1] \longrightarrow M_j \longrightarrow \mathcal{O}_X^{\oplus r}.$$

Here F'' fits into the exact sequence of sheaves $0 \to F \to F'' \to F' \to 0$, hence $F'' \in \operatorname{Coh}_0(X)$. Then M_j is quasi-isomorphic to a two term complex $(\mathcal{O}_X^{\oplus r} \to F'')$. □

In what follows, we write an object $E \in \mathcal{A}_X$ as a two-term complex $(\mathcal{O}_X^{\oplus r} \to F)$ occasionally. We set $S_0, S_x \in \mathcal{A}_X$ for $x \in X$ as follows:

$$S_0 = (\mathcal{O}_X \longrightarrow 0), \quad S_x = (0 \longrightarrow \mathcal{O}_x). \tag{2.7}$$

The following lemma is obvious.

Lemma 2.3. *An object $E \in \mathcal{A}_X$ is simple if and only if E is isomorphic to S_0 or S_x for $x \in X$. Any object in $\mathcal{A}_X$ is written as successive extensions of these simple objects.*

2.2. Stability condition on $\mathcal{A}_X$

Here we discuss stability conditions on $\mathcal{A}_X$, and the associated (semi)stable objects in $\mathcal{A}_X$. The stability condition discussed here is based on the notion of stability conditions on triangulated categories by Bridgeland [6].

Let $\mathcal{A}_X \subset \mathcal{D}_X$ be the abelian category given by (2.1). We set $\Gamma = \mathbb{Z} \oplus \mathbb{Z}$ and a group homomorphism

$$\mathrm{cl}\colon K(\mathcal{A}_X) \longrightarrow \Gamma,$$

by the following,

$$\mathrm{cl}\colon (\mathcal{O}_X^{\oplus r} \longrightarrow F) \longmapsto (r, \operatorname{length} F).$$

Also we denote by $\mathbb{H} \subset \mathbb{C}$ the upper half plane,

$$\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}.$$

Definition 2.4. A *stability condition* on $\mathcal{A}_X$ is a group homomorphism $Z\colon \Gamma \to \mathbb{C}$, which satisfies

$$Z(\mathrm{cl}(E)) \in \mathbb{H},$$

for any non-zero object $E \in \mathcal{A}_X$.

In what follows, we write $Z(\mathrm{cl}(E))$ as $Z(E)$ for simplicity.

Remark 2.5. By Lemma 2.3, a group homomorphism $Z\colon \Gamma \to \mathbb{C}$ is a stability condition on $\mathcal{A}_X$ if and only if

$$Z(1,0) \in \mathbb{H}, \quad Z(0,1) \in \mathbb{H}.$$

In particular the set of stability conditions is parameterized by points in $\mathbb{H}^2$.

Remark 2.6. For a stability condition $Z\colon \Gamma \to \mathbb{C}$ on $\mathcal{A}_X$, the pair $(Z, \mathcal{A}_X)$ determines a stability condition on $\mathcal{D}_X$ in the sense of Bridgeland [6].

For a non-zero object $E \in \mathcal{A}_X$ and a stability condition Z on $\mathcal{A}_X$, we have the well-defined argument

$$\arg Z(E) \in (0, \pi].$$

The notion of (semi)stable objects are defined as follows.

Definition 2.7. Let $Z\colon \Gamma \to \mathbb{C}$ be a stability condition on $\mathcal{A}_X$. We say $E \in \mathcal{A}_X$ is *Z-semistable* (resp. *stable*) if for any non-zero proper subobject $0 \subsetneq F \subsetneq E$ in $\mathcal{A}_X$, the following inequality holds:

$$\arg Z(F) < \arg Z(E) \quad (\text{resp. } \arg Z(F) \le \arg Z(E)).$$

2.3. Semistable objects in $\mathcal{A}_X$

We fix three stability conditions on $\mathcal{A}_X$,

$$Z_* \colon \Gamma \longrightarrow \mathbb{C}, \quad * = \pm, 0 \tag{2.8}$$

satisfying the following:

$$\begin{aligned} &\arg Z_+(1,0) > \arg Z_+(0,1), \\ &\arg Z_0(1,0) = \arg Z_0(0,1), \\ &\arg Z_-(1,0) < \arg Z_-(0,1). \end{aligned}$$

The set of Z_*-(semi)stable objects are characterized as follows.

Proposition 2.8. (i) *An object* $E \in \mathcal{A}_X$ *is* Z_-*-(semi)stable if and only if* E *is isomorphic to*

$$(\mathcal{O}_X^{\oplus r} \longrightarrow 0) \quad \textit{or} \quad (0 \longrightarrow F), \tag{2.9}$$

for $r \in \mathbb{Z}$ *and* $F \in \mathrm{Coh}_0(X)$ *(resp. isomorphic to* S_0 *or* S_x *for* $x \in X$*, given in (2.7)).*

(ii) *Any object in* $\mathcal{A}_X$ *is* Z_0*-semistable, and* $E \in \mathcal{A}_X$ *is* Z_0*-stable if and only if* E *is isomorphic to* S_0 *or* S_x *for* $x \in X$*.*

(iii) *An object* $E \in \mathcal{A}_X$ *is* Z_+*-(semi)stable if and only if* E *is isomorphic to (2.9), (resp.* S_0 *or* S_x *for* $x \in X$*,) or isomorphic to* $(\mathcal{O}_X^{\oplus r} \stackrel{s}{\to} F)$ *with* $r > 0$*,* $F \neq 0$*, satisfying the following.*

- *The image of the induced morphism between global sections,*

$$V = \mathrm{Im}\{H^0(s) \colon \mathbb{C}^{\oplus r} \longrightarrow H^0(F)\}, \tag{2.10}$$

is r*-dimensional and generates* F *as an* $\mathcal{O}_X$*-module.*

- *For any non-zero proper subvector space* $0 \subsetneq W \subsetneq V$*, the subsheaf* $F_W := \mathcal{O}_X \cdot W \subset F$ *satisfies*

$$\frac{\mathrm{length}\, F_W}{\dim W} \geq \frac{\mathrm{length}\, F}{r} \quad \left(\textit{resp.}\ \frac{\mathrm{length}\, F_W}{\dim W} > \frac{\mathrm{length}\, F}{r} \right). \tag{2.11}$$

Proof. (i) Take a non-zero object $E \in \mathcal{A}_X$, which is isomorphic to $(\mathcal{O}_X^{\oplus r} \stackrel{s}{\to} F)$ for $F \in \mathrm{Coh}_0(X)$. We have the exact sequence in $\mathcal{A}_X$,

$$0 \longrightarrow F[-1] \longrightarrow E \longrightarrow \mathcal{O}_X^{\oplus r} \longrightarrow 0. \tag{2.12}$$

If $r \neq 0$ and $F \neq 0$, then we have

$$\arg Z_-(F[-1]) > \arg Z_-(E),$$

hence (2.12) destabilizes E. Therefore if E is Z_--semistable, we have $r = 0$ or $F = 0$. Furthermore if E is Z_--stable, $r = 1$ or $\mathrm{length}\, F = 1$ must hold. Hence E is isomorphic to S_0 or S_x for $x \in X$. Conversely it is easy to see that objects in (2.9) (resp. S_0, S_x for $x \in X$) are Z_--semistable. (resp. Z_--stable).

(ii) The proof of (ii) is obvious.

(iii) Let us take a non-zero object $E = (\mathcal{O}_X^{\oplus r} \stackrel{s}{\to} F) \in \mathcal{A}_X$. If $r = 0$ or $F = 0$, it is easy to see that E is Z_+-semistable, and it is furthermore Z_+-stable if and only if E is isomorphic to S_0 or S_x for $x \in X$. Therefore we assume that $r \neq 0$ and $F \neq 0$.

Suppose that E is Z_+-(semi)stable, and take $V \subset H^0(F)$ as in (2.10). If $\dim V < r$, then there is an injection $\mathcal{O}_X \hookrightarrow E$ in $\mathcal{A}_X$. Then we have

$$\arg Z_+(\mathcal{O}_X) > \arg Z_+(E).$$

This contradicts to that E is Z_+-semistable, hence V is r-dimensional. Furthermore if V does not generate F as an $\mathcal{O}_X$-module, there is a closed point $x \in X$ and a surjection $E \twoheadrightarrow \mathcal{O}_x[-1]$ in $\mathcal{A}_X$. Since

$$\arg Z_+(E) > \arg Z_+(\mathcal{O}_x),$$

this is a contradiction. Also take a subvector space $0 \subsetneq W \subsetneq V$ and the subsheaf of F, $F_W = \mathcal{O}_X \cdot W \subset F$. Then there is an injection in $\mathcal{A}_X$,

$$(\mathcal{O}_X \otimes_{\mathbb{C}} W \twoheadrightarrow F_W) \hookrightarrow E,$$

hence the Z_+-(semi)stability implies the desired inequality (2.11).

Conversely suppose that V is r-dimensional, V generates F as an $\mathcal{O}_X$-module and inequality (2.11) holds. Since V generates F, the morphism $s \colon \mathcal{O}_X^{\oplus r} \to F$ is surjective, and E is a coherent sheaf. Take an injection in $\mathcal{A}_X$,

$$E' = (\mathcal{O}_X^{\oplus r'} \stackrel{s'}{\to} F') \hookrightarrow E. \tag{2.13}$$

If $r' = r$, then (2.13) is an isomorphism since $\mathcal{O}_X^{\oplus r} \stackrel{s}{\to} F$ is surjective. If $r' = 0$, then $\arg Z_+(E') < \arg Z(E)$ is obviously satisfied. Let us assume $0 < r' < r$, and take $F'' = \operatorname{Im} s' \subset F'$. Note that there are injections in $\mathcal{A}_X$,

$$E'' = (\mathcal{O}_X^{\oplus r'} \twoheadrightarrow F'') \hookrightarrow E' \hookrightarrow E.$$

Since the cokernel of $E'' \hookrightarrow E'$ lies in $\mathrm{Coh}_0(X)[-1]$, we have

$$\arg Z_+(E') \leq \arg Z_+(E''). \tag{2.14}$$

Also since V is r-dimensional, inequality (2.11) implies

$$\arg Z_+(E) > \arg Z_+(E'') \quad (\text{resp. } \arg Z_+(E) \geq \arg Z_+(E'')). \tag{2.15}$$

By (2.14) and (2.15), the object E is Z_+-(semi)stable. □

Remark 2.9. By Proposition 2.8(iii), giving a Z_+-semistable $E \in \mathcal{A}_X$ is equivalent to giving a pair (F, V), where $F \in \mathrm{Coh}_0(X)$ and V is a linear subspace $V \subset H^0(F)$ which generates F as an $\mathcal{O}_X$-module, and satisfying the stability condition (2.11). The notion of such pairs (F, V) also makes sense for non-projective Calabi–Yau three-fold X.

Remark 2.10. The stability conditions constructed in this paper and those in [16, Paragraph 6.5] are related as follows. Let us take a stability condition Z_+ on $\mathcal{A}_X$ as in (2.8), and set $\phi_1 = \frac{1}{\pi}\arg Z_+(1,0)$ and $\phi_2 = \frac{1}{\pi}\arg Z_+(0,1)$. We can write the pair $(Z_+, \mathcal{A}_X)$ as $(Z_+, \mathcal{P})$ for the family of subcategories $\{\mathcal{P}(\phi)\}_{\phi\in\mathbb{R}}$, which is called the slicing as in [6, Definition 5.1]. Namely $\mathcal{P}(\phi)$ for $0 < \phi \leq 1$ is the category of Z_+-semistable $E \in \mathcal{A}_X$ with $Z_+(E) \in \mathbb{R}_{>0}\exp(\mathrm{i}\pi\phi)$, and $\mathcal{P}(\phi)$ for other ϕ is determined by the rule $\mathcal{P}(\phi+1) = \mathcal{P}(\phi)[1]$. Then as in Proposition 2.8(iii), any object in $\mathcal{P}(\phi)$ with $\phi_2 < \phi \leq \phi_1$ is written as $(\mathcal{O}_X^{\oplus r} \stackrel{s}{\to} F)$ such that s is surjective. The associated object in $\mathcal{D}_X$ is the kernel of s, hence it is a coherent sheaf. This implies that the subcategory $\mathcal{P}((\phi_2, \phi_2+1]) \subset \mathcal{D}_X$, that is the smallest extension-closed subcategory which contains $\mathcal{P}(\phi)$ for $\phi_2 < \phi \leq \phi_2 + 1$, is contained in $\mathcal{D}_X \cap \mathrm{Coh}(X)$. Since $\mathcal{P}((\phi_2, \phi_2+1])$ and $\mathcal{D}_X \cap \mathrm{Coh}(X)$ are both hearts of bounded t-structures, both are indeed equal.

In [16, Paragraph 6.5], Kontsevich–Soibelman deals with a stability condition on $\mathcal{D}_X \cap \mathrm{Coh}(X)$, which is denoted by σ. The above argument shows that the stability conditions Z_+ and σ are related by the $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-action on the space of stability conditions on $\mathcal{D}_X$ (cf. [6, Lemma 8.2]). In particular, the sets of semistable objects and the relevant DT-type invariants in the next section with respect to both stability conditions are same.

Example 2.11. (i) If $r = 1$, then (F, V) gives a Z_+-semistable object if and only if V generates F as an $\mathcal{O}_X$-module. Suppose that $X = \mathbb{C}^3$. The torus $T = \mathbb{G}_m^3$ acts on X, and the T-invariant pairs (F, V) with length $F = n$ bijectively corresponds to three-dimensional partitions.

For instance, the case of $n = 3$ is as follows,

$$F = \begin{cases} \mathbb{C} \oplus \mathbb{C}x \oplus \mathbb{C}x^2 \\ \mathbb{C} \oplus \mathbb{C}y \oplus \mathbb{C}y^2 \\ \mathbb{C} \oplus \mathbb{C}z \oplus \mathbb{C}z^2 \\ \mathbb{C} \oplus \mathbb{C}x \oplus \mathbb{C}y \\ \mathbb{C} \oplus \mathbb{C}y \oplus \mathbb{C}z \\ \mathbb{C} \oplus \mathbb{C}y \oplus \mathbb{C}z \end{cases} \supset V = \begin{cases} \mathbb{C} \\ \mathbb{C} \\ \mathbb{C} \\ \mathbb{C} \\ \mathbb{C} \\ \mathbb{C} \end{cases}$$

Here x, y, z are coordinates of $\mathbb{C}^3$:

(ii) Suppose that $X = \mathbb{C}^3$ and $(r, n) = (2, 3)$. In the notation of (i), the T-fixed Z_+-semistable (F, V) are classified as follows:

$$F = \begin{cases} \mathbb{C} \oplus \mathbb{C}x \oplus \mathbb{C}x^2 \\ \mathbb{C} \oplus \mathbb{C}y \oplus \mathbb{C}y^2 \\ \mathbb{C} \oplus \mathbb{C}z \oplus \mathbb{C}z^2 \\ \mathbb{C}x \oplus \mathbb{C}y \oplus \mathbb{C}xy \\ \mathbb{C}y \oplus \mathbb{C}z \oplus \mathbb{C}yz \\ \mathbb{C}x \oplus \mathbb{C}z \oplus \mathbb{C}xz \end{cases} \supset V = \begin{cases} \mathbb{C} \oplus \mathbb{C}x \\ \mathbb{C} \oplus \mathbb{C}y \\ \mathbb{C} \oplus \mathbb{C}z \\ \mathbb{C}x \oplus \mathbb{C}y \\ \mathbb{C}y \oplus \mathbb{C}z \\ \mathbb{C}x \oplus \mathbb{C}z \end{cases}$$

2.4. Moduli stacks

Here we discuss the moduli stack of objects in $\mathcal{A}_X$ and its substack of semistable object. For the notion of stacks, the readers can refer to [17].

Let $\mathcal{O}bj(\mathcal{A}_X)$ be the two-functor,

$$\mathcal{O}bj(\mathcal{A}_X)\colon \operatorname{Sch}/\mathbb{C} \longrightarrow \text{groupoid},$$

which sends a $\mathbb{C}$-scheme S to the groupoid of objects $\mathcal{E} \in D^b(X \times S)$, which is relatively perfect over S and satisfies $\mathbb{L}i_s^*\mathcal{E} \in \mathcal{A}_X$ for any closed point $s \in S$. (See [20].) Here $i_s\colon X \times \{s\} \hookrightarrow X \times S$ is the inclusion. The two-functor $\mathcal{O}bj(\mathcal{A}_X)$ forms a stack, and we have the decomposition

$$\mathcal{O}bj(\mathcal{A}_X) = \coprod_{(r,n)\in\Gamma} \mathcal{O}bj^{(r,n)}(\mathcal{A}_X),$$

where $\mathcal{O}bj^{(r,n)}(\mathcal{A}_X) \subset \mathcal{O}bj(\mathcal{A}_X)$ is the substack of objects $E \in \mathcal{A}_X$ with $\operatorname{cl}(E) = (r, n)$.

Let us show that $\mathcal{O}bj^{(r,n)}(\mathcal{A}_X)$ is an algebraic stack of finite type by describing it as a global quotient stack of the Quot scheme. For $(r,n) \in \Gamma$, recall that the Grothendieck's Quot scheme [10] parameterizes isomorphism classes of quotients,

$$\mathrm{Quot}^{(n)}(\mathcal{O}_X^{\oplus r}) = \{\mathcal{O}_X^{\oplus r} \stackrel{s}{\twoheadrightarrow} F : F \in \mathrm{Coh}_0(X), \mathrm{length}\, F = n\}/\cong .$$

Here two quotients $\mathcal{O}_X^{\oplus r} \stackrel{s}{\twoheadrightarrow} F$ and $\mathcal{O}_X^{\oplus r} \stackrel{s'}{\twoheadrightarrow} F'$ are isomorphic if and only if there is a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_X^{\oplus r} & \xrightarrow{s} & F \\ \downarrow \mathrm{id} & & \downarrow \cong \\ \mathcal{O}_X^{\oplus r} & \xrightarrow{s'} & F'. \end{array}$$

In particular there are no non-trivial automorphisms, and the resulting moduli space $\mathrm{Quot}^{(n)}(\mathcal{O}_X^{\oplus r})$ is a projective fine moduli scheme. Note that there is a natural right $\mathrm{GL}(r,\mathbb{C})$-action on $\mathrm{Quot}^{(n)}(\mathcal{O}_X^{\oplus r})$, given by

$$(\mathcal{O}_X^{\oplus r} \stackrel{s}{\twoheadrightarrow} F) \cdot g = (\mathcal{O}_X^{\oplus r} \stackrel{s \cdot g}{\twoheadrightarrow} F).$$

We set

$$U^{(n)} = \{(\mathcal{O}_X^{\oplus n} \stackrel{s}{\twoheadrightarrow} F) \in \mathrm{Quot}^{(n)}(\mathcal{O}_X^{\oplus n}) \mid H^0(s)\colon \mathbb{C}^n \stackrel{\cong}{\to} H^0(F)\}.$$

It is easy to see that $U^{(n)}$ is an open substack of $\mathrm{Quot}^{(n)}(\mathcal{O}_X^{\oplus n})$. For an object $F \in \mathrm{Coh}_0(X)$ with $\mathrm{length}\, F = n$, let us choose an isomorphism $\mathbb{C}^n \cong H^0(F)$. By applying $\otimes_{\mathbb{C}} \mathcal{O}_X$ and composing the natural surjection,

$$\mathcal{O}_X^{\oplus n} \cong H^0(F) \otimes_{\mathbb{C}} \mathcal{O}_X \twoheadrightarrow F,$$

we obtain a point in $U^{(n)}$. Such a point is obtained up to a choice of an isomorphism $\mathbb{C}^n \cong H^0(F)$, hence $\mathcal{O}bj^{(0,n)}(\mathcal{A}_X)$ is constructed as the quotient stack,

$$\mathcal{O}bj^{(0,n)}(\mathcal{A}_X) = [U^{(n)}/\,\mathrm{GL}(n,\mathbb{C})].$$

For $r > 0$, the moduli stack $\mathcal{O}bj^{(r,n)}(\mathcal{A}_X)$ is constructed as follows. Let $\mathcal{Q} \in \mathrm{Coh}(U^{(n)} \times X)$ be an universal quotient sheaf restricted to $U^{(n)}$, and $\pi_U\colon U^{(n)} \times X \to U^{(n)}$ the projection. We construct the affine bundle

$U^{(r,n)} \to U^{(n)}$ as

$$U^{(r,n)} = \mathcal{S}pec_{\mathcal{O}_{U^{(n)}}} \mathrm{Sym}^{\bullet}(\pi_{U*}\mathcal{Q}^{\oplus r})^{*} \to U^{(n)}. \tag{2.16}$$

It is easy to see that $U^{(r,n)}$ represents the functor sending a $\mathbb{C}$-scheme S to the set of isomorphism classes of the diagram,

$$\mathcal{O}_{S\times X}^{\oplus n} \twoheadrightarrow \mathcal{F} \longleftarrow \mathcal{O}_{S\times X}^{\oplus r}, \tag{2.17}$$

where $\mathcal{F}$ is a coherent sheaf on $S \times X$ flat over S, and the induced quotient $\mathcal{O}_X^{\oplus n} \to \mathcal{F}|_{\{s\}\times X}$ for each closed point $s \in S$ determines a point in $U^{(n)}$. There is a right $\mathrm{GL}(r,\mathbb{C})$-action on $U^{(r,n)}$ along the fibers of the morphism (2.16), acting on the right arrow of (2.17). Also the right $\mathrm{GL}(n,\mathbb{C})$-action on $U^{(n)}$ naturally lifts to the right action on $U^{(r,n)}$, and these actions commute. Hence there is a right $G^{(r,n)} := \mathrm{GL}(r,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C})$-action on $U^{(r,n)}$, and the moduli stack $\mathcal{O}bj^{(r,n)}(\mathcal{A}_X)$ can be constructed as

$$\mathcal{O}bj^{(r,n)}(\mathcal{A}_X) = [U^{(r,n)}/G^{(r,n)}]. \tag{2.18}$$

In particular $\mathcal{O}bj^{(r,n)}(\mathcal{A}_X)$ is an algebraic stack of finite type over $\mathbb{C}$.

Proposition 2.12. *For any $p \in U^{(r,n)}$, there is a $G^{(r,n)}$-invariant analytic open subset $p \in U_p \subset U^{(r,n)}$, a $G^{(r,n)}$-equivariant embedding $U_p \subset M_p$ for a complex manifold with a right $G^{(r,n)}$-action, and a $G^{(r,n)}$-invariant holomorphic function $f_p \colon M_p \to \mathbb{C}$ such that*

$$U_p = \{z \in M_p : df_p(z) = 0\}.$$

Proof. Suppose that $p \in U^{(r,n)}$ corresponds to a diagram

$$\mathcal{O}_X^{\oplus n} \twoheadrightarrow F \longleftarrow \mathcal{O}_X^{\oplus r},$$

such that $F \in \mathrm{Coh}_0(X)$ decomposes as

$$F = \bigoplus_{i=1}^{k} F_i, \quad \mathrm{Supp}(F_i) = \{x_i\}, \quad \mathrm{length}\, F_i = n_i,$$

for distinct closed points $x_1, x_2, \dots, x_i \in X$ and $n_i \in \mathbb{Z}$. Let us take an analytic small open neighborhood $x_i \in V_i \subset X$ such that each V_i is isomorphic to $\mathbb{C}^3$ as a complex manifold, and $V_i \cap V_j = \emptyset$ for $i \neq j$. Note that we have

$$p \in \left\{ (\mathcal{O}_X^{\oplus n} \twoheadrightarrow F' \leftarrow \mathcal{O}_X^{\oplus r}) \in U^{(r,n)} : \mathrm{Supp}(F') \subset \coprod_i V_i \right\}, \tag{2.19}$$

and define $p \in U_p \subset U^{(r,n)}$ to be the connected component of the RHS of (2.19), which contains p. Obviously, U_p is $G^{(r,n)}$-invariant analytic open subset of $U^{(r,n)}$. Restricting to each V_i, giving a point on U_p is equivalent to giving a collection of diagrams

$$\mathcal{O}_{V_i}^{\oplus n} \twoheadrightarrow F_i' \leftarrow \mathcal{O}_{V_i}^{\oplus r}, \quad \mathrm{length}\, F_i' = n_i, \tag{2.20}$$

for each $1 \leq i \leq k$ such that the induced morphism

$$\mathbb{C}^n = H^0(\mathcal{O}_X^{\oplus n}) \longrightarrow \bigoplus_{i=1}^k H^0(\mathcal{O}_{V_i}^{\oplus n}) \longrightarrow \bigoplus_{i=1}^k H^0(F_i'),$$

is an isomorphism. Since $V_i \cong \mathbb{C}^3$, giving such a collection of data (2.20) is equivalent to giving a point

(2.21)

$$\{(X_i, Y_i, Z_i, \{v_i^{(j)}\}_{j=1}^n, \{s_i^{(j)}\}_{j=1}^r)\}_{i=1}^k \in \prod_{i=1}^k M_{n_i}(\mathbb{C})^{\times 3} \times (\mathbb{C}^{n_i})^n \times (\mathbb{C}^{n_i})^r,$$

satisfying

$$X_i Y_i = Y_i X_i, \quad X_i Z_i = Z_i X_i, \quad Y_i Z_i = Z_i Y_i, \quad 1 \leq i \leq k, \tag{2.22}$$

$$\det\left(v^{(1)}, v^{(2)}, \dots, v^{(n)}\right) \neq 0. \tag{2.23}$$

Here X_i, Y_i, Z_i are elements of $M_{n_i}(\mathbb{C})$, $v_i^{(j)}$, $s_i^{(j)}$ are elements of $\mathbb{C}^{n_i}$, and we have regarded

$$v^{(j)} := \sum_{i=1}^k v_i^{(j)} \in \bigoplus_{i=1}^k \mathbb{C}^{n_i} = \mathbb{C}^n,$$

as a column vector of $M_n(\mathbb{C})$. We set M_p to be an open subset of the RHS of (2.21), satisfying only (2.23). Then the zero set of Equation (2.22) is the critical locus of the holomorphic function $f_p\colon M_p \to \mathbb{C}$,

$$f_p\left(\{(X_i, Y_i, Z_i, \{v_i^{(j)}\}_{j=1}^n, \{s_i^{(j)}\}_{j=1}^r)\}_{i=1}^k\right) = \sum_{i=1}^k \mathrm{tr}(X_iY_iZ_i - Z_iY_iX_i).$$

Obviously $G^{(r,n)}$ acts on M_p from the right, f_p is $G^{(r,n)}$-invariant, and there is a $G^{(r,n)}$-equivariant isomorphism between U_p and $\{df_p = 0\} \subset M_p$. □

Let $Z\colon \Gamma \to \mathbb{C}$ be a stability condition on $\mathcal{A}_X$. Let

$$\mathcal{M}^{(r,n)}(Z) \subset \mathcal{O}bj^{(r,n)}(\mathcal{A}_X), \tag{2.24}$$

be the substack of Z-semistable objects $E \in \mathcal{A}_X$ with $\mathrm{cl}(E) = (r, n)$. By Proposition 2.8, we have

$$\mathcal{M}^{(r,n)}(Z_-) = \begin{cases} \mathcal{O}bj^{(r,n)}(\mathcal{A}_X), & r = 0 \text{ or } n = 0, \\ \emptyset, & \text{otherwise}, \end{cases}$$
$$\mathcal{M}^{(r,n)}(Z_0) = \mathcal{O}bj^{(r,n)}(\mathcal{A}_X).$$

Here Z_* is given by (2.8). The moduli stack $\mathcal{M}^{(r,n)}(Z_+)$ is described as follows.

Lemma 2.13. *There is a* $\mathrm{GL}(r, \mathbb{C})$*-invariant Zariski open subset* $Q^{(r,n)} \subset \mathrm{Quot}^{(n)}(\mathcal{O}_X^{\oplus r})$ *such that*

$$\mathcal{M}^{(r,n)}(Z_+) = [Q^{(r,n)} / \mathrm{GL}(r, \mathbb{C})].$$

Proof. Let $\widetilde{U}^{(r,n)} \subset U^{(r,n)}$ be the open subset corresponding to diagrams

$$\mathcal{O}_X^{\oplus n} \twoheadrightarrow F \xleftarrow{s} \mathcal{O}_X^{\oplus r},$$

such that s is surjective. Then the action of the subgroup $\{\mathrm{id}\} \times \mathrm{GL}(n, \mathbb{C}) \subset G^{(r,n)}$ on $U^{(r,n)}$ is free, and the quotient space is

$$\widetilde{U}^{(r,n)} / \mathrm{GL}(n, \mathbb{C}) = \mathrm{Quot}^{(n)}(\mathcal{O}_X^{\oplus r}).$$

We set $Q^{(r,n)} \subset \mathrm{Quot}^{(n)}(\mathcal{O}_X^{\oplus r})$ to be the subset corresponds to quotients $\mathcal{O}_X^{\oplus r} \stackrel{s}{\twoheadrightarrow} F$ such that the associated two-term complex $(\mathcal{O}_X^{\oplus r} \stackrel{s}{\twoheadrightarrow} F) \in \mathcal{A}_X$ is Z_+-semistable. The subset $Q^{(r,n)}$ is $\mathrm{GL}(r, \mathbb{C})$-invariant, and it is straightforward to see that $Q^{(r,n)}$ is open in $\mathrm{Quot}^{(n)}(\mathcal{O}_X^{\oplus r})$ (e.g., use the arguments of the openness of stability in [25, Theorem 3.20]). By (2.18), the quotient stack of $Q^{(r,n)}$ by the action of $\mathrm{GL}(r,\mathbb{C})$ coincides with the desired stack $\mathcal{M}^{(r,n)}(Z_+)$. $\square$

3. Hall algebras and DT-invariants

In this section, we review the result of Joyce–Song [15] applied in our abelian category $\mathcal{A}_X$.

3.1. Notation

In this subsection, we introduce some notation on algebraic groups, following [13]. Let G be an affine algebraic group over $\mathbb{C}$ with maximal torus T^G. We say G is *special* if every principal G-bundle over $\mathbb{C}$ is locally trivial in the Zariski topology. For a subset $S \subset G$, the *normalizer* $N_G(S)$ and the *centralizer* $C_G(S)$ of S in G are

$$N_G(S) = \{g \in G : g^{-1}Sg = S\},$$
$$C_G(S) = \{g \in G : sg = gs \text{ for all } s \in S\},$$

and the centre of G is $C(G) := C_G(G)$. For a subset $S \subset T^G$, note that $S \subset T^G \cap C(C_G(S))$.

Definition 3.1 [13, Definition 5.5]. We define the set $\mathcal{Q}(G, T^G)$ to be the set of closed $\mathbb{C}$-subgroups S of T^G, satisfying

$$S = T^G \cap C(C_G(S)).$$

We say G is very special if any $S \in \mathcal{Q}(G, T^G)$ and $C_G(S)$ are special.

It is shown in [13, Lemma 5.6] that $\mathcal{Q}(G, T^G)$ is a finite set, and any $S \in \mathcal{Q}(G, T^G)$ is written as an intersection of T^G and $C_G(\{t_i\})$ for a finite set of points $t_1, \ldots, t_k \in G$.

Example 3.2. Suppose that $G = \mathrm{GL}(2,\mathbb{C})$, and $\mathbb{G}_m^2 \cong T^G \subset G$ is the subgroup of diagonal matrices. Then $\mathcal{Q}(G, T^G)$ consists of T^G and the following subgroup (cf. [13, Example 5.7]):

$$\mathbb{G}_m \cong \left\{ \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} : t \in \mathbb{C}^* \right\} \subset T^G. \tag{3.1}$$

In particular $\mathrm{GL}(2,\mathbb{C})$ is a very special algebraic group.

In [13], Joyce introduces an important rational number $F(G, T^G, S)$ for a very special algebraic group G and $S \in \mathcal{Q}(G, T^G)$, as follows.

Definition 3.3 [13, Definitions 5.8 and 6.7]. Let G be a very special algebraic group. For $S \subset S'$ in $\mathcal{Q}(G, T^G)$, we define $n^G_{T^G}(S, S') \in \mathbb{Z}$ to be

$$n^G_{T^G}(S, S') = \sum_{S' \in A \subseteq \{S'' \in \mathcal{Q}(G, T^G): S'' \subset S'\},\ \cap_{S'' \in A} S'' = S} (-1)^{|A|-1},$$

and for $S \in \mathcal{Q}(G, T^G)$, define $F(G, T^G, S) \in \mathbb{Q}$ by

$$F(G, T^G, S) = \lim_{t \to 1} \sum_{\substack{S' \in \mathcal{Q}(G, T^G) \\ S \subset S'}} \left| \frac{N_G(T^G)}{C_G(S') \cap N_G(T^G)} \right|^{-1} \cdot n^G_{T^G}(S, S') \frac{P_t(S)}{P_t(C_G(S'))}.$$

Here for a quasi-projective $\mathbb{C}$-variety Y, the *virtual Poincaré polynomial* $P_t(Y) \in \mathbb{Q}[t]$ is defined by

$$P_t(Y) = \sum_{j,k \geq 0} \dim(-1)^k W_j(H^k_c(Y, \mathbb{C})) t^j,$$

where $W_*(H^k_c(Y, \mathbb{C}))$ is the weight filtration on the compact support cohomology group $H^k_c(Y, \mathbb{C})$ introduced by Deligne. The existence of the limit $t \to 1$ is proved in [13, Theorem 6.6].

Example 3.4. For $G = \mathrm{GL}(2, \mathbb{C})$, it is easy to calculate $F(G, T^G, S)$ as follows (cf. Example 3.2, [13, Paragraph 6.2]):

$$F(G, T^G, T^G) = \frac{1}{2}, \quad F(G, T^G, \mathbb{G}_m) = -\frac{3}{4}.$$

Here $\mathbb{G}_m \subset T^G$ is given by (3.1).

3.2. Hall algebra

Let X be a smooth projective Calabi–Yau three-fold over $\mathbb{C}$, and $\mathcal{A}_X \subset D_X$ the abelian subcategory given by (2.1). Here we introduce the Hall algebra based on the algebraic stack $\mathcal{O}bj(\mathcal{A}_X)$, following [13, Definition 6.8]. Let us consider the symbol

$$[\mathcal{X} \stackrel{f}{\to} \mathcal{O}bj(\mathcal{A}_X)], \tag{3.2}$$

where $\mathcal{X}$ is an algebraic stack of finite type over $\mathbb{C}$ with affine geometric stabilizers, and f is a one-morphism of stacks. We say two such symbols

$[\mathcal{X}_i \xrightarrow{f_i} \mathcal{O}bj(\mathcal{A}_X)]$ for $i=1,2$ are *isomorphic* if there is a one-isomorphism $g\colon \mathcal{X}_1 \xrightarrow{\cong} \mathcal{X}_2$ which two-commutes with f_1 and f_2.

Definition 3.5. We define the $\mathbb{Q}$-vector space $\mathcal{H}(\mathcal{A}_X)$ to be spanned by the isomorphism classes of symbols (3.2): with relations as follows.

- For a closed substack $\mathcal{Y} \subset \mathcal{X}$ and $\mathcal{U} = \mathcal{X} \setminus \mathcal{Y}$, we have
$$[\mathcal{X} \xrightarrow{f} \mathcal{O}bj(\mathcal{A}_X)] = [\mathcal{Y} \xrightarrow{f|_{\mathcal{Y}}} \mathcal{O}bj(\mathcal{A}_X)] + [\mathcal{U} \xrightarrow{f|_{\mathcal{U}}} \mathcal{O}bj(\mathcal{A}_X)].$$
- For a quasi-projective $\mathbb{C}$-variety U, we have
$$[\mathcal{X} \times U \xrightarrow{\pi_{\mathcal{X}} \circ f} \mathcal{O}bj(\mathcal{A}_X)] = \chi(U)[\mathcal{X} \xrightarrow{f} \mathcal{O}bj(\mathcal{A}_X)].$$
Here $\pi_{\mathcal{X}}\colon \mathcal{X} \times U \to \mathcal{X}$ is the projection, and $\chi(U) = P_t(U)|_{t=1} \in \mathbb{Z}$.
- Let U be a quasi-projective $\mathbb{C}$-variety and G a very special algebraic group, which acts on U with maximal torus T^G. Then we have

(3.3)
$$[[U/G] \xrightarrow{f} \mathcal{O}bj(\mathcal{A}_X)] = \sum_{S \in \mathcal{Q}(G,T^G)} F(G,T^G,S)[[U/S] \xrightarrow{f \circ \tau^S} \mathcal{O}bj(\mathcal{A}_X)].$$

Here $\tau^S\colon [U/S] \to [U/G]$ is a natural morphism.

We denote by $\mathcal{E}x(\mathcal{A}_X)$ the stack of short exact sequences in $\mathcal{A}_X$. There are morphisms of stacks
$$p_i\colon \mathcal{E}x(\mathcal{A}_X) \longrightarrow \mathcal{O}bj(\mathcal{A}_X) \quad (i=1,2,3),$$
sending a short exact sequence $0 \to A_1 \to A_2 \to A_3 \to 0$ to objects A_i, respectively. There is an associative product on $\mathcal{H}(\mathcal{A}_X)$ based on Ringel-Hall algebras, defined by
$$[\mathcal{X} \xrightarrow{f} \mathcal{O}bj(\mathcal{A}_X)] * [\mathcal{Y} \xrightarrow{g} \mathcal{O}bj(\mathcal{A}_X)] = [\mathcal{Z} \xrightarrow{p_2 \circ h} \mathcal{O}bj(\mathcal{A}_X)],$$
where the morphism h fits into the Cartesian square
$$\begin{array}{ccccc} \mathcal{Z} & \xrightarrow{h} & \mathcal{E}x(\mathcal{A}_X) & \xrightarrow{p_2} & \mathcal{O}bj(\mathcal{A}_X). \\ \downarrow & & \downarrow{\scriptstyle (p_1,p_3)} & & \\ \mathcal{X} \times \mathcal{Y} & \xrightarrow{f \times g} & \mathcal{O}bj(\mathcal{A}_X)^{\times 2}. & & \end{array}$$

We have the following.

Theorem 3.6 [11, Theorem 5.2]. *The $*$-product is well-defined and associative with unit given by* $[\mathrm{Spec}\,\mathbb{C} \to \mathcal{O}bj(\mathcal{A}_X)]$ *which corresponds to* $0 \in \mathcal{A}_X$.

3.3. DT-invariant

Let $Z\colon \Gamma \to \mathbb{C}$ be a stability condition on $\mathcal{A}_X$. The embedding of the algebraic stack (2.24) defines an element

$$\delta^{(r,n)}(Z) = [\mathcal{M}^{(r,n)}(Z) \subset \mathcal{O}bj(\mathcal{A}_X)] \in \mathcal{H}(\mathcal{A}_X).$$

In order to define counting invariants of Z-semistable objects, we want to take a (weighted) Euler characteristic of the moduli stack $\mathcal{M}^{(r,n)}(Z)$. However in general, geometric points on the moduli stack $\mathcal{M}^{(r,n)}(Z)$ have non-trivial stabilizers, hence its Euler characteristic does not make sense. Instead we take the 'logarithm' of $\delta^{(r,n)}(Z)$ in $\mathcal{H}(\mathcal{A}_X)$ to kill non-trivial stabilizers.

Definition 3.7 [14, Definition 3.18]. We define $\epsilon^{(r,n)}(Z) \in \mathcal{H}(\mathcal{A}_X)$ to be

(3.4)

$$\epsilon^{(r,n)}(Z) = \sum_{\substack{l\geq 0,\ (r_1,n_1)+\cdots+(r_l,n_l)=(r,n), \\ Z(r_i,n_i)\in\mathbb{R}_{>0}Z(r,n) \text{ for all } i.}} \frac{(-1)^{l-1}}{l}\delta^{(r_1,n_1)}(Z) * \cdots * \delta^{(r_l,n_l)}(Z).$$

Since $\delta^{(r,n)}(Z)$ is non-zero only if $r \geq 0$ and $n \geq 0$, the sum (3.4) is a finite sum. Also if r and n are coprime, then $\epsilon^{(r,n)}(Z) = \delta^{(r,n)}(Z)$. The important fact [11, Corollary 5.10; 12, Theorem 8.7] is that $\epsilon^{(r,n)}(Z)$ is supported on 'virtual indecomposable objects', and written as

$$\epsilon^{(r,n)}(Z) = \sum_{i=1}^{m} c_i[U_i \times [\mathrm{Spec}\,\mathbb{C}/\mathbb{G}_m] \xrightarrow{f_i} \mathcal{O}bj(\mathcal{A}_X)], \tag{3.5}$$

for quasi-projective $\mathbb{C}$-varieties $U_1, \dots, U_m$, and $c_1, \dots, c_m \in \mathbb{Q}$. Now the (weighted) Euler characteristic of $\epsilon^{(r,n)}(Z)$ makes sense.

Definition 3.8. Suppose that $\epsilon \in \mathcal{H}(\mathcal{A}_X)$ is written as

$$\epsilon = \sum_{i=1}^{m} c_i[U_i \times [\mathrm{Spec}\,\mathbb{C}/\mathbb{G}_m] \xrightarrow{f_i} \mathcal{O}bj(\mathcal{A}_X)]. \tag{3.6}$$

For a constructible function $\mu\colon \mathcal{O}bj(\mathcal{A}_X) \to \mathbb{Z}$, we define $\chi(\epsilon, \mu) \in \mathbb{Q}$ to be

$$\chi(\epsilon, \mu) = \sum_{i=1}^{m} c_i \sum_{k \in \mathbb{Z}} k \cdot \chi(f_i^{-1}\mu^{-1}(k)).$$

Next recall that for any $\mathbb{C}$-scheme U, Behrend [1] associates a canonical locally constructible function $\nu\colon U \to \mathbb{Z}$, satisfying the following:

- For $p \in U$, suppose that there is an analytic open neighborhood $p \in U_p$, a complex manifold M_p with $U_p \subset M_p$, and a holomorphic function $f_p\colon M_p \to \mathbb{C}$ such that $U_p = \{df_p = 0\}$. Then

$$\nu(p) = (-1)^{\dim M_p}(1 - \chi(M_p(f_p))).$$

  Here $M_p(f_p)$ is the Milnor fiber of f_p at p.

- If U has a symmetric perfect obstruction theory with zero-dimensional virtual cycle U^{vir}, we have

$$\int_{U^{\mathrm{vir}}} 1 = \int_U \nu \, d\chi.$$

The notion of Behrend's locally constructible function can be easily extended to an arbitrary algebraic stack (cf. [15, Proposition 4.4]). Hence we have the Behrend locally constructible function

$$\nu\colon \mathcal{O}bj(\mathcal{A}_X) \longrightarrow \mathbb{Z}.$$

Explicitly using the notation of (2.18) and Proposition 2.12, we have

$$\nu(p) = (-1)^{n+r+nr}(1 - \chi(M_p(f_p))), \tag{3.7}$$

for $p \in U^{(r,n)}$. We then define $\mathrm{DT}(r,n) \in \mathbb{Q}$ as follows (cf. [15, Definition 5.13]).

Definition 3.9. We define $\mathrm{DT}(r,n) \in \mathbb{Q}$ to be

$$\mathrm{DT}(r,n) = \chi(\epsilon^{(r,n)}(Z_+), -\nu).$$

Here we need to change the sign of the Behrend function. This is basically because that the Behrend functions on the variety M and on the stack $M \times [\operatorname{Spec}\mathbb{C}/\mathbb{G}_m]$ have the different sign.

Remark 3.10. (i) If $r = 1$, then $\mathrm{DT}(1, n)$ coincides with the DT-invariant counting points, studied and calculated in [3, 18, 19, 21]. The result is

$$\sum_{n \geq 0} \mathrm{DT}(1, n) q^n = M(-q)^{\chi(X)},$$

where $M(q)$ is the MacMahon function

$$M(q) = \prod_{k \geq 1} \frac{1}{(1 - q^k)^k}. \tag{3.8}$$

(ii) For $n = 0$, the invariant $\mathrm{DT}(r, 0)$ is easily shown to be (cf. [15, Example 6.2; 16, Paragraph 6.5]),

$$\mathrm{DT}(r, 0) = \frac{1}{r^2}. \tag{3.9}$$

(iii) For $r = 0$, the invariant $\mathrm{DT}(0, n)$ is computed in [15, Paragraph 6.3; 16, Paragraph 6.5; 24, Remark 8.13] using the wall-crossing formula. The result is

$$\exp\left(\sum_{n \geq 0} (-1)^{n-1} n \, \mathrm{DT}(0, n) q^n \right) = M(-q)^{\chi(X)}, \tag{3.10}$$

hence

$$\mathrm{DT}(0, n) = -\chi(X) \sum_{m \geq 1, m | n} \frac{1}{m^2}. \tag{3.11}$$

We can similarly define the invariant,

$$\mathrm{DT}(r, n)_- = \chi(\epsilon^{(r,n)}(Z_-), -\nu). \tag{3.12}$$

By Proposition 2.8(i), we have

$$\mathrm{DT}(r, n)_- = \begin{cases} \mathrm{DT}(r, 0), & n = 0, \\ \mathrm{DT}(0, n), & r = 0, \\ 0, & \text{otherwise}. \end{cases} \tag{3.13}$$

By (3.13), (3.9) and (3.11), we completely know the invariant $\mathrm{DT}(r, n)_-$.

3.4. Euler characteristic version

In Section 5, we will also use the Euler characteristic version of counting invariants of Z_+-semistable objects in $\mathcal{A}_X$, defined as follows, (cf. [14, Section 6.5]).

Definition 3.11. We define $\mathrm{Eu}(r,n) \in \mathbb{Q}$ to be

$$\mathrm{Eu}(r,n) = \chi(\epsilon^{(r,n)}(Z_+), 1).$$

Here 1 is the constant locally constructible function on $\mathcal{O}bj(\mathcal{A}_X)$ which takes the value 1.

Similarly to $\mathrm{DT}(r,n)$, the invariant $\mathrm{Eu}(r,n)$ is already computed when $r=0$ or $n=0$. The result is (cf. [15, Example 6.2; 24, Remark 5.14])

$$\mathrm{Eu}(r,0) = \frac{(-1)^{r-1}}{r^2}, \tag{3.14}$$

$$\mathrm{Eu}(0,n) = \chi(X) \sum_{m\ge 1, m|n} \frac{1}{m^2}. \tag{3.15}$$

Similarly to $\mathrm{DT}(r,n)_-$, the invariant

$$\mathrm{Eu}(r,n)_- = \chi(\epsilon^{(r,n)}(Z_-), 1)$$

satisfies the following by Proposition 2.8(i):

$$\mathrm{Eu}(r,n)_- = \begin{cases} \mathrm{Eu}(r,0), & n=0, \\ \mathrm{Eu}(0,n), & r=0, \\ 0, & \text{otherwise}. \end{cases} \tag{3.16}$$

By (3.16), (3.14) and (3.15), we completely know the invariant $\mathrm{Eu}(r,n)_-$.

4. Computation of DT(2, n)

In this section, we deduce the generating series of $\mathrm{DT}(2,n)$ using the wall-crossing formula of DT-invariants by Joyce–Song [15]. The wall-crossing formula provides a transformation formula between DT-type counting invariants of Z_+-semistable objects $\mathrm{DT}(r,n)$, which we want to compute, and those of Z_--semistable objects $\mathrm{DT}(r,n)_-$, which we completely know by (3.13). The transformation formula involves certain combinatorial coefficients, which we recall in Section 4.1 below.

4.1. Combinatorial coefficients

In this subsection, we introduce some notation which will be used in describing the wall-crossing formula. For $\Gamma = \mathbb{Z} \oplus \mathbb{Z}$, we set

$$C(\Gamma) = \{(r,n) \in \Gamma \setminus \{0\} : r \geq 0,\ n \geq 0\}.$$

Define $\mu\colon C(\Gamma) \to \mathbb{Q} \cup \{\infty\}$ to be $\mu(r,n) = n/r$.

Definition 4.1. For $l \geq 1$, we define the map

$$s_l \colon C(\Gamma)^l \longrightarrow \{0, \pm 1\},$$

as follows. Suppose that $v_1, \dots, v_l \in C(\Gamma)^l$ saisfies one of (a) or (b) for each i,

(a) $\mu(v_i) > \mu(v_{i+1})$ and $\mu(v_1 + \cdots + v_i) \geq \mu(v_{i+1} + \cdots + v_l)$.

(b) $\mu(v_i) \leq \mu(v_{i+1})$ and $\mu(v_1 + \cdots + v_i) < \mu(v_{i+1} + \cdots + v_l)$.

Then $s_l(v_1, \dots, v_l) = (-1)^k$, where k is the number of $i = 1, \dots, l-1$ satisfying (b). Otherwise $s_l(v_1, \dots, v_l) = 0$.

Definition 4.2. For $l \geq 1$, we define the map

$$u_l \colon C(\Gamma)^l \longrightarrow \mathbb{Q}$$

as follows:

$$u_l(v_1, \dots, v_l) = \sum_{1 \leq l'' \leq l' \leq l} \sum_{\substack{\psi\colon \{1,\dots,l\} \to \{1,\dots,l'\},\ \xi\colon \{1,\dots,l'\} \to \{1,\dots,l''\}, \\ \psi, \xi \text{ are non-decreasing surjective maps}, \\ \mu(v_i) = \mu(v_j) \text{ if } \psi(i) = \psi(j), \\ \mu(\sum_{k \in (\xi\circ\psi)^{-1}(i)} v_k) = \mu(\sum_{k \in (\xi\circ\psi)^{-1}(j)} v_k) \text{ for any } i,j.}}$$

$$\prod_{a=1}^{l''} s_{|\xi^{-1}(a)|} \left(\left\{ \sum_{k \in \psi^{-1}(j)} v_k \right\}_{j \in \xi^{-1}(a)} \right) \frac{(-1)^{l''+1}}{l''} \prod_{b=1}^{l'} \frac{1}{|\psi^{-1}(b)|!}. \tag{4.1}$$

Remark 4.3. The maps s_l and u_l are specializations of [14, Definitions 4.2 and 4.4], respectively.

In our situation, it is convenient to describe the wall-crossing formula in terms of certain graphs with some additional structures. First, we introduce the notion of bi-colored weighted ordered vertex, as follows.

Definition 4.4. We call a data

$$\Lambda = (V, \pi, v, \leq), \tag{4.2}$$

bi-colored weighted ordered vertex if it satisfies the following:

- V is a finite set.
- $\pi\colon V \to \{\bullet, \circ\}$ is a map, where $\{\bullet, \circ\}$ is a set with two elements.
- v is a map $v\colon V \to \mathbb{Z}_{\geq 1}$.
- $\leq$ is a total order on V.

Let Λ be a data (4.2) with $l = |V|$. The total order $\leq$ on V gives an identification between V and $\{1, \ldots, l\}$. We set $V_\bullet$ and $V_\circ$ to be

$$V_\bullet = \{v \in V : \pi(v) = \bullet\},$$
$$V_\circ = \{v \in V : \pi(v) = \circ\}.$$

We set $v_i \in C(\Gamma)$ to be

$$v_i = \begin{cases} (v(i), 0), & \text{if } i \in V_\bullet, \\ (0, v(i)), & \text{if } i \in V_\circ. \end{cases}$$

We set $s(\Lambda) \in \{0, \pm 1\}$ and $u(\Lambda) \in \mathbb{Q}$ to be

$$s(\Lambda) = s_l(v_1, \ldots, v_l), \quad u(\Lambda) = u_l(v_1, \ldots, v_l).$$

Also we set

$$r(\Lambda) = \sum_{i \in V_\bullet} v(i), \quad n(\Lambda) = \sum_{i \in V_\circ} v(i).$$

We define $\mathrm{DT}(\Lambda) \in \mathbb{Q}$ and $\mathrm{Eu}(\Lambda) \in \mathbb{Q}$ to be

$$\mathrm{DT}(\Lambda) = \prod_{i \in V_\bullet} \mathrm{DT}(v(i), 0) \prod_{i \in V_\circ} \mathrm{DT}(0, v(i)),$$
$$\mathrm{Eu}(\Lambda) = \prod_{i \in V_\bullet} \mathrm{Eu}(v(i), 0) \prod_{i \in V_\circ} \mathrm{Eu}(0, v(i)).$$

Definition 4.5. Let $\Lambda = (V, \pi, v, \leq)$ be a bi-colored weighted ordered vertex. We define the set $\mathcal{E}(\Lambda)$ to be the set of data

$$(E, s, t),$$

satisfying the following:

- E is a finite set and s, t are maps $E \to V$, i.e., the data (V, E, s, t) determines a quiver. The geometric realization of this quiver is connected and simply connected.
- For any $e \in E$, we have $\pi s(e) \neq \pi t(e)$.
- For any $e \in E$, we have $s(e) < t(e)$ with respect to the total order $\leq$ on V.

For $(E, s, t) \in \mathcal{E}(\Lambda)$, we set $E_{\bullet \to \circ}$ to be

$$E_{\bullet \to \circ} = \{e \in E : \pi s(e) = \bullet\}.$$

4.2. Combinatorial descriptions of $\mathrm{DT}(r,n)$, $\mathrm{Eu}(r,n)$

Using the combinatorial data given in the previous subsection, we can describe the invariant $\mathrm{DT}(r, n)$ as follows.

Theorem 4.6. *We have the following formula:*

$$\begin{aligned}\mathrm{DT}(r,n) = &\sum_{\substack{\Lambda=(V,\pi,v,\leq) \text{ is a bi-colored} \\ \text{weighted ordered vertex with} \\ r(\Lambda)=r,\ n(\Lambda)=n.}} (-1)^{rn} u(\Lambda)\, \mathrm{DT}(\Lambda) \\ &\times \left(-\frac{1}{2}\right)^{|V|-1} \sum_{(E,s,t)\in\mathcal{E}(\Lambda)} (-1)^{|E_{\bullet\to\circ}|} \prod_{e\in E} v(s(e))v(t(e)).\end{aligned} \tag{4.3}$$

Proof. Let $\chi \colon \Gamma \times \Gamma \to \mathbb{Z}$ be

$$\chi((r,n),(r',n')) = rn' - r'n.$$

For $E, F \in \mathcal{A}_X$, we have

$$\begin{aligned}\chi(\mathrm{cl}(E), \mathrm{cl}(F)) = &\dim \mathrm{Hom}(E,F) - \dim \mathrm{Ext}^1(E,F) \\ &+ \dim \mathrm{Ext}^1(F,E) - \dim \mathrm{Hom}(F,E),\end{aligned} \tag{4.4}$$

by the Riemann–Roch theorem and the Serre duality. Equation (4.4) provides an analog of [15, Equation (39)] and Proposition 2.12 provides an analog of [15, Theorem 5.3]. The proof of the Behrend function identity given in [15, Theorem 5.9] depends on these two properties, hence the analog of [15, Theorem 5.9] also holds for our abelian category $\mathcal{A}_X$. Then we can apply the proof of [15, Theorem 5.16] for stability conditions $Z_{\pm}\colon \Gamma \to \mathbb{C}$, and obtain the same wall-crossing formula given in [15, Theorem 5.6]. It gives a following transformation formula between DT-type counting invariants of Z_+-semistable objects $\mathrm{DT}(r,n)$, and those of Z_--semistable objects $\mathrm{DT}(r,n)_-$ given by (3.12),

$$\begin{aligned} \mathrm{DT}(r,n) = & \sum_{\substack{l\geq 1,\ v_1+\cdots+v_l=(r,n), \\ v_i=(r_i,n_i)\in\Gamma.}} (-1)^{\sum_{1\leq i<j\leq l}\chi(v_i,v_j)} u_l(v_1,\ldots,v_l) \prod_{i=1}^{l} \mathrm{DT}(r_i,n_i)_- \\ & \times \left(-\frac{1}{2}\right)^{l-1} \sum_{\substack{\text{connected and simply connected} \\ \text{oriented graph } G \text{ with vertex } 1,\ldots,l, \\ \overset{i}{\bullet}\to\overset{j}{\bullet} \text{ implies } i<j.}} \prod_{\substack{\overset{i}{\bullet}\to\overset{j}{\bullet} \\ \text{in } G}} \chi(v_i,v_j). \end{aligned} \tag{4.5}$$

Noting that (3.13) and

$$\chi((r,0),(r',0))=0, \quad \chi((0,n),(0,n'))=0,$$

formula (4.5) immediately implies formula (4.3). □

The formula for $\mathrm{Eu}(r,n)$ is similarly obtained by using [14, Theorem 6.28] instead of [15, Theorem 5.16].

Theorem 4.7. *We have the following formula:*

$$\begin{aligned} \mathrm{Eu}(r,n) = & \sum_{\substack{\Lambda=(V,\pi,v,\leq) \text{ is a bi-colored} \\ \text{weighted ordered vertex with} \\ r(\Lambda)=r,\ n(\Lambda)=n.}} u(\Lambda)\,\mathrm{Eu}(\Lambda) \\ & \times \left(\frac{1}{2}\right)^{|V|-1} \sum_{(E,s,t)\in\mathcal{E}(\Lambda)} (-1)^{|E_{\bullet\to\circ}|} \prod_{e\in E} v(s(e))v(t(e)). \end{aligned} \tag{4.6}$$

As a corollary, we have the following.

Corollary 4.8. *We have*

$$\mathrm{DT}(r,n) = (-1)^{rn+r-1}\,\mathrm{Eu}(r,n). \tag{4.7}$$

Proof. By formulas (3.9), (3.11), (3.14) and (3.15), we have

$$\mathrm{DT}(\Lambda) = (-1)^{|V|+r}\,\mathrm{Eu}(\Lambda),$$

for a bi-colored weighted ordered vertex $\Lambda = (V, \pi, v, \leq)$ with $r(\Lambda) = r$. Applying formulas (4.3) and (4.6), we obtain (4.7). □

Remark 4.9. If $X = \mathbb{C}^3$, it seems likely that the value of the Behrend function at a $\mathbb{G}_m^3$-fixed point $p \in \mathcal{O}bj^{(r,n)}(\mathcal{A}_X)$ is $(-1)^{rn+r-1}$. If it is true, then $\mathrm{DT}(r,n)$ and $\mathrm{Eu}(r,n)$ differs by the sign $(-1)^{rn+r-1}$ by the localization method. In general, the combination of the above argument and taking the stratification on the moduli space as in [3, Theorem 4.11] should yield a geometric proof of Corollary 4.8.

4.3. Computation of $s(\Lambda)$

In this subsection, we compute $s(\Lambda)$ for a data (4.2) with $r(\Lambda) = 2$. Let us take a data (4.2) with $|V| = l$ and

$$r(\Lambda) = 2, \quad n(\Lambda) = n. \tag{4.8}$$

We fix an identification between V and $\{1, \dots, l\}$ induced by the total order $\leq$. We denote by $\pi(\Lambda)$ the sequence of $\bullet$ and $\circ$, given by

$$\pi(\Lambda) = \pi(1)\pi(2)\dots\pi(l).$$

Note that we have $|V_\bullet| \leq 2$. We first have the following lemma.

Lemma 4.10. *Suppose that $\pi(1) = \pi(2) = \circ$, i.e. $\pi(\Lambda)$ is*

$$\overset{1}{\circ}\,\overset{2}{\circ}\ \cdots\ \circ\ \bullet \cdots .$$

Then $s(\Lambda) = 0$.

Proof. Since $\mu(v_1) = \mu(v_2)$ and $\infty = \mu(v_1) > \mu(v_2 + \cdots + v_l)$, $(v_1, \dots, v_l)$ does not satisfy (a) nor (b) in Definition 4.1. □

Next, we compute the case of $|V_\bullet| = 1$.

Lemma 4.11. *Suppose that $|V_\bullet| = 1$ with $s(\Lambda) \neq 0$. Then the value $s(\Lambda)$ is computed as follows:*

- *Suspect that* $\pi(1) = \circ$, $\pi(2) = \bullet$ *and* $\pi(i) = \circ$ *for all* $i \geq 3$, *i.e.,* $\pi(\Lambda)$ *is*

$$\overset{1}{\circ}\ \overset{2}{\bullet}\ \circ\ \cdots\ \overset{l}{\circ}\,. \tag{4.9}$$

Then $s(\Lambda) = (-1)^l$.

- *Suppose that* $\pi(1) = \bullet$ *and* $\pi(i) = \circ$ *for all* $i \geq 2$, *i.e.* $\pi(\Lambda)$ *is*

$$\overset{1}{\bullet}\ \overset{2}{\circ}\ \circ\ \cdots\ \overset{l}{\circ}\,. \tag{4.10}$$

Then $s(\Lambda) = (-1)^{l-1}$.

Proof. By Lemma 4.10, the sequence $\{\pi(1), \pi(2), \dots, \pi(l)\}$ is either (4.9) or (4.10). In case (4.9) (resp. (4.10)), condition (a) or (b) in Definition 4.1 is satisfied and the number of $1 \leq i \leq l-1$ in which (b) holds is $l-2$. (resp. $l-1$). □

The case of $|V_\bullet| = 2$ is computed as follows.

Lemma 4.12. *Suppose that* $|V_\bullet| = 2$ *with* $s(\Lambda) \neq 0$. *Then* $l \geq 3$ *and* $s(\Lambda)$ *is computed as follows:*

- *Suppose that* $V_\bullet = \{1, 2\}$, *i.e.* $\pi(\Lambda)$ *is*

$$\overset{1}{\bullet}\ \overset{2}{\bullet}\ \circ\ \cdots\ \overset{l}{\circ}\,. \tag{4.11}$$

Then $s(\Lambda) = (-1)^{l-1}$.

- *Suppose that* $V_\bullet = \{1, a\}$ *for* $a \geq 3$, *i.e.,* $\pi(\Lambda)$ *is*

$$\overset{1}{\bullet}\ \overset{2}{\circ}\ \cdots\ \overset{a-1}{\circ}\ \overset{a}{\bullet}\ \overset{a+1}{\circ}\ \cdots\cdots\ \overset{l}{\circ}\,. \tag{4.12}$$

Then we have

$$v(2) + v(3) + \cdots + v(a-2) < v(a-1) + v(a+1) + \cdots + v(l),$$
$$v(2) + v(3) + \cdots + v(a-2) + v(a-1) \geq v(a+1) + \cdots + v(l),$$

and $s(\Lambda) = (-1)^l$.

- *Suppose that* $V_\bullet = \{2, 3\}$, *i.e.,* $\pi(\Lambda)$ *is*

$$\overset{1}{\circ}\ \overset{2}{\bullet}\ \overset{3}{\bullet}\ \overset{4}{\circ}\ \cdots\ \overset{l}{\circ}\,. \tag{4.13}$$

Then we have $v(1) < v(4) + \cdots + v(l)$ *and* $s(\Lambda) = (-1)^l$.

- *Suppose that* $V_\bullet = \{2, a\}$ *for* $a \geq 4$, *i.e.,* $\pi(\Lambda)$ *is*

$$\overset{1}{\circ}\ \overset{2}{\bullet}\ \overset{3}{\circ}\ \cdots\ \overset{a-1}{\circ}\ \overset{a}{\bullet}\ \overset{a+1}{\circ}\ \cdots\cdots\ \overset{l}{\circ}\,. \tag{4.14}$$

Then we have

$$v(1) + v(3) + \cdots + v(a-2) < v(a-1) + v(a+1) + \cdots + v(l), \tag{4.15}$$
$$v(1) + v(3) + \cdots + v(a-2) + v(a-1) \geq v(a+1) + \cdots + v(l), \tag{4.16}$$

and $s(\Lambda) = (-1)^{l-1}$.

Proof. By Lemma 4.10, the sequence $\pi(\Lambda)$ is one of (4.11) to (4.14). In each case, $s(\Lambda)$ is easily computed by Definition 4.1. For instance, let us consider the case (4.14). Since $\mu(v_{a-2}) \leq \mu(v_{a-1})$ and $\mu(v_{a-1}) > \mu(v_a)$, we have

$$\mu(v_1 + v_2 + \cdots + v_{a-2}) < \mu(v_{a-1} + \cdots + v_l), \tag{4.17}$$
$$\mu(v_1 + v_2 + \cdots + v_{a-2} + v_{a-1}) \geq \mu(v_a + \cdots + v_l). \tag{4.18}$$

Since $v_2 = v_a = (1, 0)$, conditions (4.17), (4.18) are equivalent to (4.15), (4.16) respectively. Conversely if conditions (4.15), (4.16) are satisfied it is easy to check that one of (a) or (b) in Definition 4.1 holds at each $1 \leq i \leq l-1$. In this case, the number of $1 \leq i \leq i-1$ in which $\mu(v_i) \leq \mu(v_{i+1})$ holds is $l-3$, hence $s(\Lambda) = (-1)^{l-1}$. □

4.4. Computation of $u(\Lambda)$

In this subsection, we compute $u(\Lambda)$ for a data (4.2) satisfying (4.8). We fix an identification between V and $\{1, 2, \ldots, l\}$ via $\leq$. Let us take $1 \leq l' \leq l$ and a map

$$\psi\colon \{1, 2, \ldots, l\} \twoheadrightarrow \{1, 2, \ldots, l'\}, \tag{4.19}$$

which appears in (4.1). Note that $\pi(i) = \pi(j)$ if $\psi(i) = \psi(j)$, hence the map π descends to the map

$$\pi'\colon \{1, \ldots, l'\} \longrightarrow \{\bullet, \circ\}, \tag{4.20}$$

via ψ. We set $v'\colon \{1, \ldots, l'\} \to \mathbb{Z}_{\geq 1}$ to be

$$v'(i) = \sum_{j \in \psi^{-1}(i)} v(j).$$

Then the data

$$\Lambda' = (\{1, \dots, l'\}, \pi', v', \leq)$$

is a bi-colored weighted ordered vertex. The map ψ descends to the map of the sequences $\pi(\psi)\colon \pi(\Lambda) \to \pi'(\Lambda')$. First we compute the case of $|V_\bullet| = 1$.

Lemma 4.13. *Suppose that $V_\bullet = \{a\}$ for $1 \leq a \leq l$. Then we have*

$$u(\Lambda) = \frac{(-1)^{l-a}}{(a-1)!(l-a)!}. \tag{4.21}$$

Proof. Formula (4.21) follows from [15, Proposition 15.8]. □

Next we compute $u(\Lambda)$ when $|V_\bullet| = 2$. We write $V_\bullet = \{a, b\}$ for $1 \leq a < b \leq l$. Note that we have

$$v(a) = v(b) = 1, \quad l'' \leq 2.$$

Here l'' is a number which appears in (4.1). When $b - a \geq 3$, the coefficient $u(\Lambda)$ does not contribute to the sum (4.3) by the following lemma.

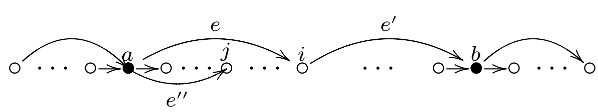
Lemma 4.14. *Suppose that $V_\bullet = \{a, b\}$ with $b - a \geq 3$. Then we have*

$$\sum_{(E,s,t) \in \mathcal{E}(\Lambda)} (-1)^{|E_{\bullet \to \circ}|} \prod_{e \in E} v(s(e))v(t(e)) = 0. \tag{4.22}$$

Proof. Take $(E, s, t) \in \mathcal{E}(\Lambda)$. Since the quiver (V, E, s, t) is connected and simply connected, there is unique $a < i < b$ and $e, e' \in E$ such that

$$s(e) = a,\ t(e) = s(e') = i,\ t(e') = b.$$

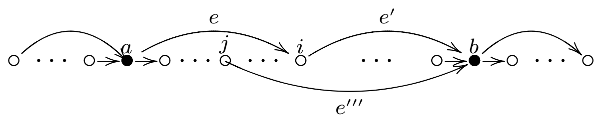
Since $b - a \geq 3$, there is $a < j < b$ such that $j \neq i$. Since (V, E, s, t) is connected, there is $e'' \in E$ such that either $(s(e''), t(e'')) = (a, j)$ or $(s(e''), t(e'')) = (j, b)$ holds. Suppose that $(s(e''), t(e'')) = (a, j)$ holds, i.e., the geometric realization of the quiver (V, E, s, t) is as follows:



Note that by the simply connectedness of (V, E, s, t), there is no $e''' \in E$ which satisfies $(s(e'''), t(e''')) = (j, b)$. We set E' to be the set

$$E' = (E \backslash \{e''\}) \coprod \{e'''\},$$

and define maps $s', t' \colon E' \to V$ so that $s'|_{E\setminus\{e''\}} = s|_{E\setminus\{e''\}}$, $t'|_{E\setminus\{e''\}} = t|_{E\setminus\{e''\}}$, and $(s(e'''), t(e''')) = (j, b)$. The geometric realization of the quiver (V, E', s', t') is as follows:



Since $v(a) = v(b) = 1$, we have

$$(-1)^{|E_{\bullet\to\circ}|} \prod_{e\in E} v(s(e))v(t(e)) + (-1)^{|E'_{\bullet\to\circ}|} \prod_{e\in E'} v(s'(e))v(t'(e)) = 0.$$

Therefore the sum (4.22) vanishes. □

We compute $u(\Lambda)$ when $b - a \le 2$. Let us divide $u(\Lambda)$ into the following sum:

$$u(\Lambda) = u^{(1)}(\Lambda) + u^{(2)}(\Lambda) + u^{(3)}(\Lambda).$$

Each $u^{(i)}(\Lambda)$ is the following:

- $u^{(1)}(\Lambda)$ is defined by the sum (4.1) with $l'' = 1$ and $\psi \colon \{1, \dots, l\} \to \{1, \dots, l'\}$ satisfying $|\pi'^{-1}(\bullet)| = 2$. Here $\pi' \colon \{1, \dots, l'\} \to \{\bullet, \circ\}$ is given by (4.20).
- $u^{(2)}(\Lambda)$ is defined by the sum (4.1) with $l'' = 1$ and $\psi \colon \{1, \dots, l\} \to \{1, \dots, l'\}$ satisfying $|\pi'^{-1}(\bullet)| = 1$.
- $u^{(3)}(\Lambda)$ is defined by the sum (4.1) with $l'' = 2$.

We compute $u^{(1)}(\Lambda)$ as follows.

Lemma 4.15. (i) *Suppose that $V_\bullet = \{a, a+1\}$ for $1 \le a \le l-1$. Then $u^{(1)}(\Lambda)$is non-zero if and only if*

$$v(1) + \cdots + v(a-1) < v(a+2) + \cdots + v(l).$$

In this case, we have

$$u^{(1)}(\Lambda) = \frac{(-1)^{l-a}}{(a-1)!(l-a-1)!}.$$

(ii) *Suppose that* $V_\bullet = \{a, a+2\}$ *for* $1 \le a \le l-2$. *Then* $u^{(1)}(\Lambda)$ *is non-zero if and only if*

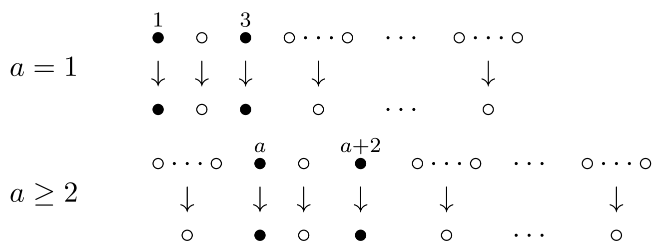
$$v(1) + \cdots + v(a-1) < v(a+1) + v(a+3) + \cdots + v(l), \tag{4.23}$$

$$v(1) + \cdots + v(a-1) + v(a+1) \ge v(a+3) + \cdots + v(l). \tag{4.24}$$

In this case, we have

$$u^{(1)}(\Lambda) = \frac{(-1)^{l-a-1}}{(a-1)!(l-a-2)!}. \tag{4.25}$$

Proof. The computations of (i) and (ii) are identical, so we only check (ii). Let $\psi\colon \{1, \dots, l\} \to \{1, \dots, l'\}$ be a map which appears in (4.1). By Lemma 4.10, the map $\pi(\psi)\colon \pi(\Lambda) \to \pi'(\Lambda')$ is one of the following forms:



For simplicity, we calculate the case of $a \ge 2$. By Lemma 4.12, we see that $u^{(1)}(\Lambda)$ is non-zero only if (4.23) and (4.24) hold. By Lemma 4.12 and (4.1), we have

$$u^{(1)}(\Lambda) = \frac{1}{(a-1)!} \sum_{\substack{\psi\colon \{a+3, \dots, l\} \to \{5, \dots, l'\}, \\ \psi \text{ is a non-decreasing} \\ \text{surjective map.}}} (-1)^{l'-1} \prod_{i=5}^{l'} \frac{1}{|\psi^{-1}(i)|!}.$$

Applying Lemma 4.16 below, we obtain (4.25). □

We have used the following lemma, whose proof is written in [14, Proposition 4.9].

Lemma 4.16. *For any* $l \ge 1$, *we have*

$$\sum_{\substack{l' \ge 0,\ \psi\colon \{1, \cdots, l\} \to \{1, \cdots, l'\}, \\ \psi \text{ is a non-decreasing surjective map.}}} (-1)^{l-l'} \prod_{i=1}^{l'} \frac{1}{|\psi^{-1}(i)|!} = \frac{1}{l!}.$$

We can similarly compute $u^{(2)}(\Lambda)$ and $u^{(3)}(\Lambda)$. The proofs of the following Lemmas 4.17 and 4.18 are identical to Lemma 4.15, so we leave the detail to the reader and omit the proof.

Lemma 4.17. *We have $u^{(2)}(\Lambda) \neq 0$ if and only if $V_\bullet = \{a, a+1\}$ for some $1 \leq a \leq l-1$. In this case, we have*

$$u^{(2)}(\Lambda) = \frac{(-1)^{l-a-1}}{2(a-1)!(l-a-1)!}. \tag{4.26}$$

Lemma 4.18. (i) *Suppose that $V_\bullet = \{a, a+1\}$ for $1 \leq a \leq l-1$. Then $u^{(3)}(\Lambda)$ is non-zero if and only if the following condition holds:*

$$v(1) + v(2) + \cdots + v(a-1) = v(a+2) + \cdots + v(l).$$

In this case, we have

$$u^{(3)}(\Lambda) = \frac{(-1)^{l-a}}{2(a-1)!(l-a-1)!}.$$

(ii) *Suppose that $V_\bullet = \{a, a+2\}$ for $1 \leq a \leq l-2$. Then $u^{(3)}(\Lambda)$ is non-zero either one of the following conditions holds:*

$$v(1) + \cdots + v(a-1) = v(a+1) + \cdots + v(l), \tag{4.27}$$

$$v(1) + \cdots + v(a-1) + v(a+1) = v(a+2) + \cdots + v(l). \tag{4.28}$$

If (4.27) (resp. (4.28)) holds, then we have

$$u^{(3)}(\Lambda) = \frac{(-1)^{l-a-1}}{2(a-1)!(l-a-1)!}, \quad \left(\textit{resp.}\ \frac{(-1)^{l-a}}{2(a-1)!(l-a-1)!}\right). \tag{4.29}$$

4.5. Generating series of $\mathrm{DT}(2, n)$

Combining the calculations in the previous subsections, we compute $\mathrm{DT}(2,n)$. We divide $\mathrm{DT}(2,n)$ into the following four parts:

$$\mathrm{DT}(2,n) = \mathrm{DT}^{(0)}(2,n) + \mathrm{DT}^{(1)}(2,n) + \mathrm{DT}^{(2)}(2,n) + \mathrm{DT}^{(3)}(2,n).$$

Each $\mathrm{DT}^{(i)}(2,n)$ is the following:

- $\mathrm{DT}^{(0)}(2,n)$ is defined by the sum (4.3) for bi-colored weighted ordered vertices $\Lambda = (V, \pi, v, \leq)$ with $r(\Lambda) = 2$, $n(\Lambda) = n$ and $|V_\bullet| = 1$.

- For $1 \leq i \leq 3$, $\mathrm{DT}^{(i)}(2,n)$ is defined by the sum (4.3) for bi-colored weighted ordered vertices $\Lambda = (V, \pi, v, \leq)$ with $r(\Lambda) = 2$, $n(\Lambda) = n$, $|V_{\bullet}| = 2$, and $u(\Lambda)$ is replaced by $u^{(i)}(\Lambda)$.

We define the generating series $\mathrm{DT}^{(i)}(2)$ by

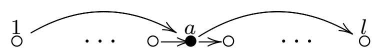
$$\mathrm{DT}^{(i)}(2) = \sum_{n \geq 0} \mathrm{DT}^{(i)}(2,n) q^n.$$

In what follows, we compute $\mathrm{DT}^{(i)}(2)$. Recall the definition of the MacMahon function $M(q)$ given in (3.8).

Lemma 4.19. *We have the following formula:*

$$\mathrm{DT}^{(0)}(2) = \frac{1}{4} M(q)^{2\chi(X)}. \tag{4.30}$$

Proof. Let $\Lambda = (V, \pi, v, \leq)$ be a bi-colored weighted ordered vertex with $|V| = l$ and $V_{\bullet} = \{a\}$ for $1 \leq a \leq l$. Obviously the set $\mathcal{E}(\Lambda)$ consists of one element $(E, s, t) \in \mathcal{E}(\Lambda)$, whose geometric realization is as follows:



Note that we have $|E_{\bullet \to \circ}| = l - a$. By Remark 3.10, Theorem 4.6 and Lemma 4.13, we have

$$\begin{aligned}
\mathrm{DT}^{(0)}(2) = & \sum_{\substack{l \geq 1,\ 1 \leq a \leq l, \\ v \colon \{1,\dots,l\} \to \mathbb{Z}_{\geq 1}, \\ v(a) = 2.}} \frac{(-1)^{l-a}}{(a-1)!(l-a)!} \cdot \frac{1}{4} \prod_{i \neq a} \mathrm{DT}(0, v(i)) q^{v(i)} \\
& \times \left(-\frac{1}{2}\right)^{l-1} \cdot (-1)^{l-a} \prod_{i \neq a} 2v(i) \\
= & \frac{1}{4} \sum_{\substack{l \geq 0, \\ v \colon \{1,\dots,l\} \to \mathbb{Z}_{\geq 1}.}} \frac{1}{l!} \prod_{i=1}^{l} (-2v(i))\, \mathrm{DT}(0, v(i)) q^{v(i)} \qquad (4.31) \\
= & \frac{1}{4} \exp\left(\sum_{n \geq 0} -2n\, \mathrm{DT}(0,n) q^n \right) \\
= & \frac{1}{4} M(q)^{2\chi(X)}. \qquad (4.32)
\end{aligned}$$

Here we have used the following in (4.31):

$$\sum_{1\le a\le l} \frac{1}{(a-1)!(l-a)!}\cdot\frac{1}{2^{l-1}}=\frac{1}{(l-1)!},$$

and formula (3.10) in (4.32). □

Next let us compute $\mathrm{DT}^{(1)}(2)$. We introduce the following notation. We define the series $N(q)$ to be

$$\begin{aligned} N(q) &:= q\frac{d}{dq}\log M(q) \\ &= \sum_{n\ge 0,\ r|n} r^2 q^n. \end{aligned}$$

For series $f_1, f_2, \dots, f_N \in \mathbb{Q}[\![q]\!]$ given by

$$f_i=\sum_{n\ge 0} a_n^{(i)} q^n, \quad 1\le i\le N,$$

and a subset $\Delta\subset\mathbb{Z}_{\ge 0}^N$, we define the series $\{f_1\cdot f_2\cdots f_N\}_\Delta$ to be

$$\{f_1\cdot f_2\cdots f_N\}_\Delta=\sum_{(n_1,n_2,\cdots,n_N)\in\Delta} a_{n_1}^{(1)}a_{n_2}^{(2)}\cdots a_{n_N}^{(N)} q^{n_1+n_2+\cdots+n_N}. \tag{4.33}$$

Lemma 4.20. *We have the following formula:*

$$\mathrm{DT}^{(1)}(2)=-\frac{\chi(X)}{2}\{M(q)^{\chi(X)}\cdot M(q)^{\chi(X)}\cdot N(q)\}_\Delta. \tag{4.34}$$

Here $\Delta\subset\mathbb{Z}_{\ge 0}^3$ *is*

$$\Delta=\{(m_1,m_2,m_3)\in\mathbb{Z}_{\ge 0}^3 : -m_3\le m_1-m_2<m_3\}. \tag{4.35}$$

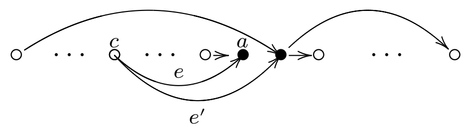
Proof. Let $\Lambda=(V,\pi,v,\le)$ be a bi-colored weighted ordered vertex with $r(\Lambda)=2$, $n(\Lambda)=n$ and $|V_\bullet|=2$. Let $|V|=l$ and we identify V and $\{1,\cdots,l\}$ via $\le$. By Lemma 4.14, the data Λ contributes to (4.3) only if one of the following conditions hold:

- We have $V_\bullet=\{a,a+1\}$ for $1\le a\le l-1$. In this case, there are two types for $(E,s,t)\in\mathcal{E}(\Lambda)$.

Type A. There is unique $1 \le c \le a-1$ and $e, e' \in E$ such that

$$s(e) = s(e') = c, \quad t(e) = a, \quad t(e') = a+1.$$

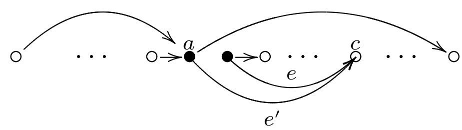
In this case, we have $|E_{\bullet \to \circ}| = l - a - 1$. If we fix such c, there are 2^{l-3}-choices of such $(E, s, t) \in \mathcal{E}(\Lambda)$. One of their geometric realizations is as follows:



Type B. There is unique $a+2 \le c \le l$ and $e, e' \in E$ such that

$$t(e) = t(e') = c, \quad s(e) = a+1, \quad s(e') = a.$$

In this case, we have $|E_{\bullet \to \circ}| = l - a$. If we fix such c, there are 2^{l-3}-choices of such $(E, s, t) \in \mathcal{E}(\Lambda)$. One of their geometric realizations is as follows:

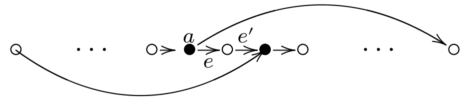


- We have $V_\bullet = \{a, a+2\}$ for $1 \le a \le l-2$. In this case, we call an element $(E, s, t) \in \mathcal{E}(\Lambda)$ as Type C.

  Type C. There is $e, e' \in E$ such that

$$s(e) = a, \quad t(e) = s(e') = a+1, \quad t(e') = a+2.$$

There are 2^{l-3}-choices of $(E, s, t) \in \mathcal{E}(\Lambda)$. One of their geometric realizations is as follows:



We write $\mathrm{DT}^{(1)}(2)$ as

$$\mathrm{DT}^{(1)}(2) = \mathrm{DT}^{(1)}_A(2) + \mathrm{DT}^{(1)}_B(2) + \mathrm{DT}^{(1)}_C(2),$$

where $\mathrm{DT}^{(1)}_A(2)$, $\mathrm{DT}^{(1)}_B(2)$ and $\mathrm{DT}^{(1)}_C(2)$ are contributions of $(E, s, t) \in \mathcal{E}(\Lambda)$ of type A, B and C, respectively. Using Lemma 4.15(i) and Theorem 4.6, the series $\mathrm{DT}_A^{(1)}(2)$ is computed as follows:

$$\mathrm{DT}_A^{(1)}(2) = \sum_{\substack{l\ge 1,\ 1\le a\le l-1,\ 1\le c\le a-1,\\ v\colon\{1,\dots,l\}\to\mathbb{Z}_{\ge 1},\ v(a)=v(a+1)=1,\\ v(1)+\cdots+v(a-1)<v(a+2)+\cdots+v(l).}} \frac{(-1)^{l-a}}{(a-1)!(l-a-1)!}$$

$$\times \prod_{i\neq a,a+1} \mathrm{DT}(0,v(i))q^{v(i)}\left(-\frac{1}{2}\right)^{l-1} \cdot (-1)^{l-a-1}\cdot 2^{l-3}\prod_{i\neq c} v(i)\cdot v(c)^2$$

$$= \frac{1}{4} \sum_{\substack{a\ge 0,\ b\ge 0,\ k\ge 1,\\ v\colon\{1,\dots,a\}\to\mathbb{Z}_{\ge 1},\ v'\colon\{1,\dots,b\}\to\mathbb{Z}_{\ge 1},\\ v(1)+\cdots+v(a)+k<v'(1)+\cdots+v'(b).}} \frac{1}{a!}\prod_{i=1}^{a}(-v(i))\,\mathrm{DT}(0,v(i))q^{v(i)}$$

$$\cdot\frac{1}{b!}\prod_{i=1}^{b}(-v'(i))\,\mathrm{DT}(0,v(i))q^{v'(i)}\cdot(-k^2)\,\mathrm{DT}(0,k)q^k$$

$$= \frac{\chi(X)}{4}\{M(q)^{\chi(X)}\cdot M(q)^{\chi(X)}\cdot N(q)\}_{\Delta_A}. \tag{4.36}$$

Here Δ_A is defined by

$$\Delta_A = \{(m_1,m_2,m_3)\in\mathbb{Z}_{\ge 0}^3 : m_1+m_3<m_2\},$$

and we have used formula (3.11) in (4.36). Using Lemma 4.15, similar computations show that

$$\mathrm{DT}_B^{(1)}(2) = -\frac{\chi(X)}{4}\{M(q)^{\chi(X)}\cdot M(q)^{\chi(X)}\cdot N(q)\}_{\Delta_B},$$
$$\mathrm{DT}_C^{(1)}(2) = -\frac{\chi(X)}{4}\{M(q)^{\chi(X)}\cdot M(q)^{\chi(X)}\cdot N(q)\}_{\Delta},$$

where Δ_B is defined by

$$\Delta_B = \{(m_1,m_2,m_3)\in\mathbb{Z}_{\ge 0}^3 : m_1<m_2+m_3\},$$

and Δ is defined by (4.35). Noting that

$$\Delta_B = \Delta_A \coprod \Delta,$$

we obtain formula (4.34). □

Finally, we show that $\mathrm{DT}^{(i)}(2)$ vanish for $i=2,3$.

Lemma 4.21. *We have* $\mathrm{DT}^{(i)}(2,n)=0$ *for any* $n\geq 0$ *and* $i=2,3$.

Proof. Let $\Lambda=(V,\pi,v,\leq)$ be a bi-colored weighted ordered vertex with $r(\Lambda)=2$, $|V|=l$, and take $(E,s,t)\in\mathcal{E}(\Lambda)$. By Lemma 4.14, we may assume that $V_{\bullet}=\{a,a+1\}$ or $V_{\bullet}=\{a,a+2\}$ for some $1\leq a\leq l-1$. Let us consider the following data:

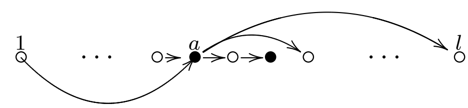
$$\Lambda^{*}=(V,\pi,v,\leq^{*}),\quad (E,s^{*},t^{*}),$$

by setting $\leq^{*}$, s^{*} and t^{*} to be

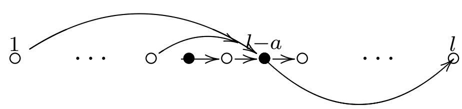
$$\alpha\leq^{*}\beta \text{ if and only if } \alpha\geq\beta,\quad s^{*}=t,\quad t^{*}=s.$$

Then it is obvious that $(E,s^{*},t^{*})\in\mathcal{E}(\Lambda^{*})$. For instance, the relationship between geometric realizations is as follows:

(Λ,E,s,t) :



$(\Lambda^{*},E,s^{*},t^{*})$:



Note that if $V_{\bullet}=\{a,a+1\}$, then (E,s,t) is of type A (resp. B) in the proof of Lemma 4.20 if and only if (E^{*},s^{*},t^{*}) is of type B (resp. A). Also if $V_{\bullet}=\{a,a+2\}$, then Λ satisfies (4.27) (resp. (4.28)) if and only if Λ^{*} satisfies (4.28) (resp. (4.27)). Hence the map

$$(\Lambda,(E,s,t))\mapsto(\Lambda^{*},(E,s^{*},t^{*})),$$

is a free involution on the set of pairs $(\Lambda,(E,s,t))$ for data (4.2) satisfying $V_{\bullet}=\{a,b\}$ with $0<b-a\leq 2$ and $(E,s,t)\in\mathcal{E}(\Lambda)$. Using the computations of $u^{(2)}(\Lambda)$, $u^{(3)}(\Lambda)$ in Lemmas 4.17 and 4.18, it is easy to check that

$$(-1)^{|E_{\bullet\to\circ}|}u^{(i)}(\Lambda)+(-1)^{|E^{*}_{\bullet\to\circ}|}u^{(i)}(\Lambda^{*})=0,$$

for $i=2,3$. Therefore $\mathrm{DT}^{(i)}(2,n)=0$ for any $n\geq 0$ and $i=2,3$. □

Summarizing Lemmas 4.19 to 4.21, we obtain the following.

Theorem 4.22. *We have the following formula:*

$$\mathrm{DT}(2) = \frac{1}{4}M(q)^{2\chi(X)} - \frac{\chi(X)}{2}\{M(q)^{\chi(X)} \cdot M(q)^{\chi(X)} \cdot N(q)\}_{\Delta}, \tag{4.37}$$

for $\Delta = \{(m_1, m_2, m_3) \in \mathbb{Z}^3_{\geq 0} : -m_3 \leq m_1 - m_2 < m_3\}$.

Remark 4.23. By Corollary 4.8 and Theorem 4.22, we have

$$\sum_{n \geq 0} \mathrm{Eu}(2, n) q^n = -\frac{1}{4}M(q)^{2\chi(X)} + \frac{\chi(X)}{2}\{M(q)^{\chi(X)} \cdot M(q)^{\chi(X)} \cdot N(q)\}_{\Delta},$$

for $\Delta = \{(m_1, m_2, m_3) \in \mathbb{Z}^3_{\geq 0} : -m_3 \leq m_1 - m_2 < m_3\}$.

5. Integrality property

In this section, we study the invariant $\Omega(2, n) \in \mathbb{Q}$, defined as follows.

Definition 5.1. We define $\Omega(2, n) \in \mathbb{Q}$ to be

$$\Omega(2, n) = \begin{cases} \mathrm{DT}(2, n), & n \text{ is odd}, \\ \mathrm{DT}(2, n) - \dfrac{1}{4}\mathrm{DT}\left(1, \dfrac{n}{2}\right), & n \text{ is even}. \end{cases}$$

By Corollary 4.8, the invariant $\Omega(2, n)$ is also written as

$$\Omega(2, n) = \begin{cases} -\mathrm{Eu}(2, n), & n \text{ is odd}, \\ -\mathrm{Eu}(2, n) - \dfrac{(-1)^{\frac{n}{2}}}{4}\mathrm{Eu}\left(1, \dfrac{n}{2}\right), & n \text{ is even}. \end{cases} \tag{5.1}$$

In this section, we show the following result, which is an evidence of the integrality conjecture by Kontsevich–Soibelman [16, Conjecture 6].

Theorem 5.2. *We have* $\Omega(2, n) \in \mathbb{Z}$.

It seems that Theorem 5.2 is not obvious from the explicit formula (4.37). Instead of using (4.37), we give a geometric proof of Theorem 5.2 using the definition of $\mathrm{DT}(2, n)$. (See [22] for the proof using a number theoretic method.)

Let $Q^{(2,n)} \subset \mathrm{Quot}^{(n)}(\mathcal{O}_X^{\oplus 2})$ be a $\mathrm{GL}(2,\mathbb{C})$-invariant Zariski open subset constructed in Lemma 2.13. By Lemma 2.13, there is a smooth morphism

$$f \colon Q^{(2,n)} \to \mathcal{O}bj^{(2,n)}(\mathcal{A}_X).$$

For $p \in Q^{(2,n)}$, we denote by $E_p \in \mathcal{A}_X$ the object corresponding to $f(p) \in \mathcal{O}bj^{(2,n)}(\mathcal{A}_X)$.

By the definition of $\mathrm{DT}(2,n)$, it is obvious that $\Omega(2,n) \in \mathbb{Z}$ when n is odd. Therefore, in what follows, we set $n = 2m$ for $m \in \mathbb{Z}$. We take a $\mathrm{GL}(2,\mathbb{C})$-invariant stratification of $Q^{(2,2m)}$,

$$Q^{(2,2m)} = Q_0^{(2,2m)} \amalg Q_1^{(2,2m)} \amalg Q_2^{(2,2m)} \amalg Q_3^{(2,2m)} \amalg Q_4^{(2,2m)},$$

as follows:

- $Q_0^{(2,2m)}$ corresponds to $p \in Q^{(2,2m)}$ such that $E_p \in \mathcal{A}_X$ is Z_+-stable.
- $Q_1^{(2,2m)}$ corresponds to $p \in Q^{(2,2m)}$ such that $E_p \in \mathcal{A}_X$ fits into a non-split exact sequence

$$0 \longrightarrow E_1 \longrightarrow E_p \longrightarrow E_2 \longrightarrow 0, \tag{5.2}$$

  for Z_+-stable $E_i \in \mathcal{A}_X$ with $\mathrm{cl}(E_i) = (1,m)$ and E_1 is not isomorphic to E_2.
- $Q_2^{(2,2m)}$ corresponds to $p \in Q^{(2,2m)}$ such that $E_p \in \mathcal{A}_X$ is isomorphic to $E_1 \oplus E_2$ for Z_+-stable $E_i \in \mathcal{A}_X$ with $\mathrm{cl}(E_i) = (1,m)$ and E_1 is not isomorphic to E_2.
- $Q_3^{(2,2m)}$ corresponds to $p \in Q^{(2,2m)}$ such that $E_p \in \mathcal{A}_X$ fits into a non-split exact sequence (5.2) such that $E_1 \cong E_2$.
- $Q_4^{(2,2m)}$ corresponds to $p \in Q^{(2,2m)}$ such that $E_p \in \mathcal{A}_X$ is isomorphic to $E_1^{\oplus 2}$ for a Z_+-stable $E_1 \in \mathcal{A}_X$.

Then we can write $\delta^{(2,2m)}(Z_+) \in \mathcal{H}(\mathcal{A}_X)$ as

$$\delta^{(2,2m)}(Z_+) = \sum_{i=0}^{4} \delta_i,$$

where δ_i is

$$\begin{aligned}\delta_i &= \left[\left[\frac{Q_i^{(2,2m)}}{\mathrm{GL}(2,\mathbb{C})}\right] \to \mathcal{O}bj(\mathcal{A}_X)\right] \\ &= \frac{1}{2}\left[\left[\frac{Q_i^{(2,2m)}}{\mathbb{G}_m^2}\right] \to \mathcal{O}bj(\mathcal{A}_X)\right] - \frac{3}{4}\left[\left[\frac{Q_i^{(2,2m)}}{\mathbb{G}_m}\right] \to \mathcal{O}bj(\mathcal{A}_X)\right]. \qquad (5.3)\end{aligned}$$

Here we have used relation (3.3) and Example 3.4.

Lemma 5.3. *The element* $\delta^{(1,m)}(Z_+) * \delta^{(1,m)}(Z_+) \in \mathcal{H}(\mathcal{A}_X)$ *is written as*

$$\delta^{(1,m)}(Z_+) * \delta^{(1,m)}(Z_+) = \sum_{i=1}^{4} \tilde{\delta}_i, \qquad (5.4)$$

where each $\tilde{\delta}_i$ *is as follows:*

$$\tilde{\delta}_1 = \int_{(p_1,p_2)\in Q^{(1,m)}\times Q^{(1,m)}\setminus D} \left[\left[\frac{\mathbb{P}(\mathrm{Ext}^1(E_{p_2},E_{p_1}))}{\mathbb{G}_m}\right] \to \mathcal{O}bj(\mathcal{A}_X)\right] d\mu, \qquad (5.5)$$

$$\tilde{\delta}_2 = \left[\left[\frac{(Q^{(1,m)}\times Q^{(1,m)})\setminus D}{\mathbb{G}_m^2}\right] \to \mathcal{O}bj(\mathcal{A}_X)\right], \qquad (5.6)$$

$$\tilde{\delta}_3 = \int_{p\in Q^{(1,m)}} \left[\left[\frac{\mathbb{P}(\mathrm{Ext}^1(E_p,E_p))}{\mathbb{A}^1\times\mathbb{G}_m}\right] \to \mathcal{O}bj(\mathcal{A}_X)\right] d\mu, \qquad (5.7)$$

$$\tilde{\delta}_4 = \left[\left[\frac{Q^{(1,m)}}{\mathbb{A}^1\rtimes\mathbb{G}_m^2}\right] \to \mathcal{O}bj(\mathcal{A}_X)\right]. \qquad (5.8)$$

Here $D \subset Q^{(1,m)}\times Q^{(1,m)}$ *is a diagonal, the algebraic groups in the denominators act on the varieties in the numerators trivially. The measure* μ *for the integrations (5.5), (5.7) sends constructible sets on* $Q^{(1,m)}\times Q^{(1,m)}$ *or* $Q^{(1,m)}$ *to the associated elements of the Grothendieck group of varieties.*

Proof. Recall that $\delta^{(1,m)}(Z_+) * \delta^{(1,m)}(Z_+)$ is defined by taking the fiber product of the following diagram:

$$\begin{array}{ccc} & & \mathcal{E}x(\mathcal{A}_X) \\ & & \big\downarrow{\scriptstyle (p_1,p_3)} \\ [Q^{(1,m)}/\mathbb{G}_m]\times[Q^{(1,m)}/\mathbb{G}_m] & \longrightarrow & \mathcal{O}bj(\mathcal{A}_X)\times\mathcal{O}bj(\mathcal{A}_X). \end{array} \qquad (5.9)$$

Here $\mathbb{G}_m$ acts on $Q^{(1,m)}$ trivially. Take $\mathbb{C}$-valued points $\rho_i\colon \operatorname{Spec}\mathbb{C} \to Q^{(1,m)}$ for $i = 1, 2$, which corresponds to $E_i \in \mathcal{A}_X$. We have the associated elements in the Hall-algebra,

$$f_i = \left[[\operatorname{Spec}\mathbb{C}/\mathbb{G}_m] \xrightarrow{\rho_i} [Q^{(1,m)}/\mathbb{G}_m] \to \mathcal{O}bj(\mathcal{A}_X) \right].$$

Then $f_1 * f_2$ is as follows:

$$f_1 * f_2 = \left[\left[\frac{\operatorname{Ext}^1(E_2, E_1)}{\operatorname{Hom}(E_2, E_1) \rtimes \mathbb{G}_m^2} \right] \to \mathcal{O}bj(\mathcal{A}_X) \right]. \tag{5.10}$$

Here $(t_1, t_2) \in \mathbb{G}_m^2$ acts on $\operatorname{Ext}^1(E_2, E_1)$ and $\operatorname{Hom}(E_2, E_1)$ via multiplying $t_1 t_2^{-1}$, and $\operatorname{Hom}(E_2, E_1)$ acts on $\operatorname{Ext}^1(E_2, E_1)$ trivially. For $u \in \operatorname{Ext}^2(E_2, E_1)$, the stabilizer group of the $\mathbb{G}_m^2$-action on $\operatorname{Ext}^1(E_2, E_1)$ at u is $\mathbb{G}_m^2$ if $u = 0$ and the diagonal subgroup $\mathbb{G}_m \subset \mathbb{G}_m^2$ if $u \neq 0$. Therefore we have

$$\begin{aligned} f_1 * f_2 = &\left[\left[\frac{\mathbb{P}(\operatorname{Ext}^1(E_2, E_1))}{\operatorname{Hom}(E_2, E_1) \times \mathbb{G}_m} \right] \to \mathcal{O}bj(\mathcal{A}_X) \right] \\ &+ \left[\left[\frac{\operatorname{Spec}\mathbb{C}}{\operatorname{Hom}(E_2, E_1) \rtimes \mathbb{G}_m^2} \right] \to \mathcal{O}bj(\mathcal{A}_X) \right]. \end{aligned} \tag{5.11}$$

Here the algebraic groups in the denominators act trivially on the varieties in the numerators. Since $E_i \in \mathcal{A}_X$ are Z_+-stable, we have

$$\operatorname{Hom}(E_1, E_2) = \begin{cases} \mathbb{A}^1 & \text{if } \rho_1 = \rho_2, \\ \operatorname{Spec}\mathbb{C} & \text{if } \rho_1 \neq \rho_2. \end{cases}$$

Taking the integration of (5.11) over points on $(Q^{(1,m)} \times Q^{(1,m)}) \setminus D$ and $D \cong Q^{(1,m)}$, we obtain decomposition (5.4). □

Lemma 5.4. *The element $\delta_0 \in \mathcal{H}(\mathcal{A}_X)$ is written as (3.6) such that $\chi(\delta_0, 1) \in \mathbb{Z}$.*

Proof. For a point $p \in Q_0^{(2,2m)}$, the object $E_p \in \mathcal{A}_X$ satisfies $\operatorname{Aut}(E_p) = \mathbb{G}_m$ since E_p is Z_+-stable. Hence the diagonal subgroup $\mathbb{G}_m \subset \operatorname{GL}(2, \mathbb{C})$ acts on $Q_0^{(2,2m)}$ trivially, and the quotient group $\operatorname{GL}(2, \mathbb{C})/\mathbb{G}_m = \operatorname{PGL}(2, \mathbb{C})$ acts freely on $Q_0^{(2,2m)}$. Hence δ_0 is written as $[[M/\mathbb{G}_m] \to \mathcal{O}bj(\mathcal{A}_X)]$ for an algebraic space $M = Q_0^{(2,2m)} / \operatorname{PGL}(2, \mathbb{C})$, and $\mathbb{G}_m$ acts on M trivially. Since any algebraic space is written as a disjoint union of quasi-projective varieties, δ_0 is written as (3.6) with each $c_i \in \mathbb{Z}$. Therefore $\chi(\delta_0, 1) \in \mathbb{Z}$ follows. □

For $1 \le i \le 4$, we set $\epsilon_i \in \mathcal{H}(\mathcal{A}_X)$ as follows:

$$\epsilon_i = \delta_i - \frac{1}{2}\tilde{\delta}_i.$$

Lemma 5.5. *The element* $\epsilon_1 \in \mathcal{H}(\mathcal{A}_X)$ *is written as* (3.6) *such that* $\chi(\epsilon_1, 1) \in \mathbb{Z}$.

Proof. For $p \in Q_1^{(2,2m)}$, it is easy to see that the object $E_p \in \mathcal{A}_X$ satisfies $\mathrm{Aut}(E_p) = \mathbb{G}_m$ by using the exact sequence (5.2). Hence $\mathrm{PGL}(2, \mathbb{C})$ acts freely on $Q_1^{(2,2m)}$ as in the proof of Lemma 5.4, and the quotient space $Q_1^{(2,2m)}/\mathrm{PGL}(2, \mathbb{C})$ is an algebraic space over $\mathbb{C}$. Also it is easy to see that the objects $E_i \in \mathcal{A}_X$ which appear in (5.2) are uniquely determined up to isomorphisms for a given $p \in Q_1^{(2,2m)}$. Hence there is a map of algebraic spaces

$$\gamma\colon Q_1^{(2,2m)}/\mathrm{PGL}(2, \mathbb{C}) \to (Q^{(1,m)} \times Q^{(1,m)}) \setminus D,$$

such that if $\gamma(p) = (p_1, p_2)$, there is an exact sequence in $\mathcal{A}_X$,

$$0 \longrightarrow E_{p_1} \longrightarrow E_p \longrightarrow E_{p_2} \longrightarrow 0. \tag{5.12}$$

By the construction, closed points of the fiber of γ at (p_1, p_2) bijectively correspond to isomorphism classes of objects $E_p \in \mathcal{A}_X$ which fit into an exact sequence (5.12), which also bijectively correspond to closed points in $\mathbb{P}(\mathrm{Ext}^1(E_{p_2}, E_{p_1}))$. Therefore, we have

$$\begin{aligned}
\chi(\epsilon_1, 1) &= \int_{(p_1,p_2)\in(Q^{(1,m)}\times Q^{(1,m)})\setminus D} \chi(\mathbb{P}(\mathrm{Ext}^1(E_{p_2}, E_{p_1})))d\chi \\
&\quad - \frac{1}{2}\int_{(p_1,p_2)\in(Q^{(1,m)}\times Q^{(1,m)})\setminus D} \chi(\mathbb{P}(\mathrm{Ext}^1(E_{p_2}, E_{p_1})))d\chi, \\
&= \frac{1}{2}\int_{(p_1,p_2)\in(Q^{(1,m)}\times Q^{(1,m)})\setminus D} \dim \mathrm{Ext}^1(E_{p_2}, E_{p_1})d\chi \\
&= \int_{(p_1,p_2)\in\mathrm{Sym}^2(Q^{(1,m)})\setminus D} \dim \mathrm{Ext}^1(E_{p_2}, E_{p_1})d\chi \in \mathbb{Z}. \qquad (5.13)
\end{aligned}$$

In (5.13), we have used the fact that

$$\dim \mathrm{Ext}^1(E_{p_2}, E_{p_1}) = \dim \mathrm{Ext}^1(E_{p_1}, E_{p_2}),$$

for $(p_1, p_2) \in (Q^{(1,m)} \times Q^{(1,m)}) \setminus D$, which follows from formula (4.4) and

$$\mathrm{Hom}(E_{p_1}, E_{p_2}) = \mathrm{Hom}(E_{p_2}, E_{p_1}) = 0.$$

□

Lemma 5.6. *The element* $\epsilon_2 \in \mathcal{H}(\mathcal{A}_X)$ *is written as* (3.6) *such that* $\chi(\epsilon_2, 1) = 0$.

Proof. Let $T^G = \mathbb{G}_m^2 \subset \mathrm{GL}(2, \mathbb{C})$ be the subgroup of diagonal matrices, and consider the associated $\mathbb{G}_m^2$-action on $Q_2^{(2,2m)}$. Since the subgroup $\mathbb{G}_m \subset T^G$ given by (3.1) acts on $Q_2^{(2,2m)}$ trivially, the quotient group $T^G/\mathbb{G}_m \cong \mathbb{G}_m$ acts on $Q_2^{(2,2m)}$. The set of $T^G/\mathbb{G}_m$-fixed points is the image of the map

$$\iota \colon (Q^{(1,m)} \times Q^{(1,m)}) \setminus D \to Q_2^{(2,2m)},$$

defined by

$$\Big((\mathcal{O}_X \stackrel{s_1}{\twoheadrightarrow} F_1), (\mathcal{O}_X \stackrel{s_2}{\twoheadrightarrow} F_2)\Big) \mapsto (\mathcal{O}_X^{\oplus 2} \stackrel{(s_1,s_2)}{\twoheadrightarrow} F_1 \oplus F_2).$$

It is easy to see that ι is an injection, and $T^G/\mathbb{G}_m$ acts on $Q_2^{(2,2m)} \setminus \mathrm{Im}\, \iota$ freely. We set $\tilde{Q}_2^{(2,2m)}$ to be the quotient algebraic space,

$$\tilde{Q}_2^{(2,2m)} = (Q_2^{(2,2m)} \setminus \mathrm{Im}\, \iota)/(T^G/\mathbb{G}_m).$$

Noting (5.3), we obtain that

$$\begin{aligned}
\epsilon_2 &= \frac{1}{2}\left[\left[\frac{\mathrm{Im}\, \iota}{\mathbb{G}_m^2}\right] \to \mathcal{O}bj(\mathcal{A}_X)\right] + \frac{1}{2}\left[\left[\frac{\tilde{Q}_2^{(2,2m)}}{\mathbb{G}_m}\right] \to \mathcal{O}bj(\mathcal{A}_X)\right] \\
&\quad - \frac{3}{4}\left[\left[\frac{Q_2^{(2,2m)}}{\mathbb{G}_m}\right] \to \mathcal{O}bj(\mathcal{A}_X)\right] \\
&\quad - \frac{1}{2}\left[\left[\frac{(Q^{(1,m)} \times Q^{(1,m)}) \setminus D}{\mathbb{G}_m^2}\right] \to \mathcal{O}bj(\mathcal{A}_X)\right] \\
&= \frac{1}{2}\left[\left[\frac{\tilde{Q}_2^{(2,2m)}}{\mathbb{G}_m}\right] \to \mathcal{O}bj(\mathcal{A}_X)\right] - \frac{3}{4}\left[\left[\frac{Q_2^{(2,2m)}}{\mathbb{G}_m}\right] \to \mathcal{O}bj(\mathcal{A}_X)\right]. \qquad (5.14)
\end{aligned}$$

Hence ϵ_2 is written as (3.6). Let us compute the Euler characteristic of $\tilde{Q}_2^{(2,2m)}$. For a point $p \in Q_2^{(2,2m)}$ and the object $E_p \in \mathcal{A}_X$, take $(p_1, p_2) \in (Q^{(1,m)} \times Q^{(1,m)}) \setminus D$ such that $E_p \cong E_{p_1} \oplus E_{p_2}$. It is easy to see that the pair (E_1, E_2) is uniquely determined up to isomorphisms and a permutation.

Hence $p \mapsto (p_1, p_2)$ defines a well-defined map

$$\gamma \colon Q_2^{(2,2m)} \to \mathrm{Sym}^2(Q^{(1,m)}) \setminus D.$$

For $(p_1, p_2) \in \mathrm{Sym}^2(Q^{(1,m)}) \setminus D$, the $\mathrm{GL}(2,\mathbb{C})$-action on $Q_2^{(2,2m)}$ induces a map

$$\mathrm{GL}(2,\mathbb{C}) \twoheadrightarrow \gamma^{-1}(p_1, p_2),$$

which is a $\mathbb{G}_m^2$-bundle over $\gamma^{-1}(p_1,p_2)$. Restricting to $\gamma^{-1}(p_1,p_2) \setminus \mathrm{Im}\,\iota$, we obtain the $\mathbb{G}_m^2$-bundle over $\gamma^{-1}(p_1,p_2) \setminus \mathrm{Im}\,\iota$,

$$\mathrm{GL}(2,\mathbb{C}) \setminus (T^G \cup \mathrm{i}(T^G)) \twoheadrightarrow \gamma^{-1}(p_1,p_2) \setminus \mathrm{Im}\,\iota.$$

Here $i = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathrm{GL}(2,\mathbb{C})$. Since $\mathbb{G}_m^2$ is a special algebraic group, the above map is Zariski locally trivial. Hence the virtual Poincaré polynomial of $\gamma^{-1}(p_1,p_2) \setminus \mathrm{Im}\,\iota$ is

$$\begin{aligned} P_t(\gamma^{-1}(p_1,p_2) \setminus \mathrm{Im}\,\iota) &= \frac{P_t(\mathrm{GL}(2,\mathbb{C}) \setminus (T^G \cup \mathrm{i}(T^G)))}{P_t(\mathbb{G}_m^2)} \\ &= t^4 + t^2 - 1. \end{aligned} \tag{5.15}$$

The free $T^G/\mathbb{G}_m \cong \mathbb{G}_m$-action on $Q^{(2,2m)} \setminus \mathrm{Im}\,\iota$ restricts to the free $\mathbb{G}_m$-action on $\gamma^{-1}(p_1,p_2) \setminus \mathrm{Im}\,\iota$. By (5.15), we have

$$\begin{aligned} P_t((\gamma^{-1}(p_1,p_2) \setminus \mathrm{Im}\,\iota)/\mathbb{G}_m) &= \frac{t^4 + t^2 - 1}{t^2 - 1} \\ &= t^2 + 2. \end{aligned}$$

By inverting $t = 1$, we obtain

$$\chi((\gamma^{-1}(p_1,p_2) \setminus \mathrm{Im}\,\iota)/\mathbb{G}_m) = 3. \tag{5.16}$$

Now the map γ descends to a map

$$\gamma' \colon \tilde{Q}_2^{(2,2m)} \to \mathrm{Sym}^2(Q^{(1,m)}) \setminus D,$$

such that the Euler characteristic of each fiber of γ' is 3 by (5.16). Therefore we obtain

$$\begin{aligned} \chi(\tilde{Q}_2^{(2,2m)}) &= 3 \cdot \chi(\mathrm{Sym}^2(Q^{(1,m)}) \setminus D) \\ &= \frac{3}{2}\left(\chi(Q^{(1,m)})^2 - \chi(Q^{(1,m)})\right). \end{aligned} \tag{5.17}$$

On the other hand, since the $T^G/\mathbb{G}_m$-fixed points in $Q_2^{(2,2m)}$ coincides with $\operatorname{Im}\iota$, the localization implies

$$\chi(Q_2^{(2,2m)}) = \chi(Q^{(1,m)})^2 - \chi(Q^{(1,m)}). \tag{5.18}$$

By (5.14), (5.17) and (5.18), we obtain $\chi(\epsilon_2, 1) = 0$. □

Lemma 5.7. *The element $\epsilon_3 \in \mathcal{H}(\mathcal{A}_X)$ is written as* (3.6) *such that*

$$\chi(\epsilon_3, 1) \equiv \frac{m}{2}\chi(Q^{(1,m)}), \quad (\text{mod } \mathbb{Z}). \tag{5.19}$$

Proof. For a point $p \in Q_3^{(2,2m)}$, the object $E_p \in \mathcal{A}_X$ satisfies

$$\operatorname{Aut}(E) = \operatorname{Stab}_p(\operatorname{GL}(2,\mathbb{C})) \cong \mathbb{A}^1 \rtimes \mathbb{G}_m,$$

since E_p fits into the exact sequence (5.2) with $E_1 \cong E_2$. Then for the diagonal matrices $T^G \subset \operatorname{GL}(2,\mathbb{C})$, we have $\operatorname{Stab}_p(\operatorname{GL}(2,\mathbb{C})) \cap T^G$ is the subgroup $\mathbb{G}_m \subset T^G$ given by (3.1). Therefore the action of T^G on $Q_3^{(2,2m)}$ descends to the free action of $T^G/\mathbb{G}_m \cong \mathbb{G}_m$. We set $\tilde{Q}_3^{(2,2m)}$ to be the quotient algebraic space

$$\tilde{Q}_3^{(2,2m)} = Q_3^{(2,2m)}/(T^G/\mathbb{G}_m).$$

Using (5.3) and relation (3.3), we have

$$\epsilon_3 = \frac{1}{2}\left[\left[\frac{\tilde{Q}_3^{(2,2m)}}{\mathbb{G}_m}\right] \to \mathcal{O}bj(\mathcal{A}_X)\right] - \frac{3}{4}\left[\left[\frac{Q_3^{(2,2m)}}{\mathbb{G}_m}\right] \to \mathcal{O}bj(\mathcal{A}_X)\right] \tag{5.20}$$

$$- \frac{1}{2}\int_{p \in Q^{(1,m)}} \left[\left[\frac{\mathbb{P}(\operatorname{Ext}^1(E_p, E_p))}{\mathbb{G}_m}\right] \to \mathcal{O}bj(\mathcal{A}_X)\right]. \tag{5.21}$$

Here the algebraic groups in the denominators act on the varieties in the numerators trivially. Therefore ϵ_3 is written as (3.6). Let us calculate the Euler characteristic of $\tilde{Q}_3^{(2,2m)}$. For $p \in Q_3^{(2,2m)}$, let $\gamma(p) \in Q^{(1,m)}$ be the point such that E_p fits into the exact sequence (5.2) with $E_1 \cong E_{\gamma(p)}$. It is easy to see that $p \mapsto \gamma(p)$ is a well-defined morphism of varieties

$$\gamma \colon Q_3^{(2,2m)} \to Q^{(1,m)}.$$

For $p' \in Q^{(1,m)}$, the fiber of γ at p' carries a surjection

$$\gamma' \colon \gamma^{-1}(p') \twoheadrightarrow \operatorname{Ext}^1(E_{p'}, E_{p'}) \setminus \{0\},$$

which sends a point $p \in \gamma^{-1}(p')$ to the extension class of (5.2). For $u \in \mathrm{Ext}^1(E_{p'}, E_{p'}) \setminus \{0\}$, we have the surjective morphism

$$\gamma'' \colon \mathrm{GL}(2,\mathbb{C}) \twoheadrightarrow \gamma'^{-1}(u),$$

induced by the $\mathrm{GL}(2,\mathbb{C})$-action on $Q^{(2,2m)}$. Each fiber of γ'' is isomorphic to the special algebraic group $\mathbb{A}^1 \rtimes \mathbb{G}_m$, hence γ'' is Zariski locally trivial. The free $T^G/\mathbb{G}_m$-action on $Q_3^{(2,2m)}$ restricts to the free $T^G/\mathbb{G}_m \cong \mathbb{G}_m$-action on $\gamma'^{-1}(u)$, and the virtual Poincaré polynomial of the quotient space is

$$\begin{aligned} P_t(\gamma'^{-1}(u)/\mathbb{G}_m) &= \frac{P_t(\mathrm{GL}(2,\mathbb{C}))}{P_t(\mathbb{A}^1 \rtimes \mathbb{G}_m)P_t(T^G/\mathbb{G}_m)} \\ &= t^2+1. \end{aligned} \tag{5.22}$$

Now γ' descends to a morphism

$$\gamma^{-1}(E)/\mathbb{G}_m \to \mathbb{P}(\mathrm{Ext}^1(E,E)),$$

such that the Euler characteristic of each fiber is equal to $P_t(\gamma'^{-1}(u)/\mathbb{G}_m)|_{t=1} = 2$ by (5.22). Therefore $\chi(\tilde{Q}_3^{(2,2m)})$ is

$$\chi(\tilde{Q}_3^{(2,2m)}) = 2\int_{p\in Q^{(1,m)}} \dim \mathrm{Ext}^1(E_p,E_p)\ d\chi. \tag{5.23}$$

Since $\mathbb{G}_m$ acts on $Q_3^{(2,2m)}$ freely, we have $\chi(Q_3^{(2,2m)}) = 0$. By (5.20) and (5.23), we have

$$\chi(\epsilon_3, 1) = \frac{1}{2}\int_{p\in Q^{(1,m)}} \dim \mathrm{Ext}^1(E_p,E_p)\ d\chi. \tag{5.24}$$

On the other hand, the same argument of [3, Theorem 4.11] shows that

$$\int_{p\in Q^{(1,m)}} (-1)^{\dim \mathrm{Ext}^1(E_p,E_p)}\ d\chi = (-1)^m \chi(Q^{1,m}). \tag{5.25}$$

By (5.24) and (5.25), we obtain (5.19). □

Lemma 5.8. *The element $\epsilon_4 \in \mathcal{H}(\mathcal{A}_X)$ is written as* (3.6) *and we have*

$$\chi(\epsilon_4, 1) = -\frac{1}{4}\chi(Q^{(1,m)}).$$

Proof. By (5.3) and noting that $F(G, T^G, T^G) = 1$, $F(G, T^G, \mathbb{G}_m) = -1$ for $G = \mathbb{A}^1 \rtimes \mathbb{G}_m^2$, $T^G = \{0\} \times \mathbb{G}_m^2$ and $\mathbb{G}_m \subset T^G$ given by (3.1), we have

$$\begin{aligned}
\epsilon_4 &= \frac{1}{2}\left[\left[\frac{Q^{(1,m)}}{\mathbb{G}_m^2}\right] \to \mathcal{O}bj(\mathcal{A}_X)\right] - \frac{3}{4}\left[\left[\frac{Q^{(1,m)}}{\mathbb{G}_m}\right] \to \mathcal{O}bj(\mathcal{A}_X)\right] \\
&\quad - \frac{1}{2}\left[\left[\frac{Q^{(1,m)}}{\mathbb{G}_m^2}\right] \to \mathcal{O}bj(\mathcal{A}_X)\right] + \frac{1}{2}\left[\left[\frac{Q^{(1,m)}}{\mathbb{G}_m}\right] \to \mathcal{O}bj(\mathcal{A}_X)\right] \\
&= -\frac{1}{4}\left[\left[\frac{Q^{(1,m)}}{\mathbb{G}_m}\right] \to \mathcal{O}bj(\mathcal{A}_X)\right].
\end{aligned}$$

Here the algebraic groups in the denominators act on the varieties in the numerators trivially. The above formula immediately imply the result. □

Proof of Theorem 5.2.

Proof. By (5.1), Lemmas 5.4 to 5.8, we obtain

$$\begin{aligned}
\Omega(2, 2m) &\equiv -\frac{\chi(Q^{(1,m)})}{4}\{2m - 1 + (-1)^m\} \quad (\bmod \mathbb{Z}) \\
&\equiv 0 \pmod{\mathbb{Z}}.
\end{aligned}$$

□

Remark 5.9. The argument in this section also yields an analog of [15, Conjecture 6.13], that is the integrality of certain constructible function on the coarse moduli space of Z_+-semistable objects, assuming it exists. In general, it might be possible to prove the integrality conjecture [16, Conjecture 6] in this case by proving the integrality of that constructible function.

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