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Explicit evaluation of certain Jacquet integrals on $SU(2,2)$

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We give explicit formulas for certain Jacquet integrals on some standard principal series representations of the group $SU(2,2)$.

0. Introduction

The main object of this paper is to obtain explicit integral expressions of some Whittaker functions on $G = SU(2,2)$. More specifically we evaluate the Jacquet integrals with certain K-types belonging to a principal series representation, parabolically induced by the minimal parabolic subgroup of G.

The Whittaker models are one of the main ingredients in the theory of Fourier expansions of automorphic forms at some cusps. In this sense, explicit knowledge of Whittaker functions is very important for deeper studies of automorphic forms.

Jacquet [7] introduced a functional on the space of differentiable vectors in a given representation π of G, which defines an intertwiner from its representation space to the space of smooth functions f on G satisfying $f(ng) = \eta(n)f(g)$ for all $(n, g) \in N \times G$, where η is a unitary character of the standard maximal unipotent subgroup N of G. The image of this intertwiner is a Whittaker model of π. The local multiplicity one theorem of Shalika [13] at the archimedean place implies the uniqueness of such kind of functionals when the representation π is irreducibly admissible. Note also that Wallach [16, §8] reformulated this result in a slightly different but useful manner, i.e., in terms of "moderate growth condition". When π is given by a standard model on $L^2(K)$, the unique functional is realized by the Jacquet integral. We want to compute for special vectors in $L^2(K)$.

Our method of evaluation of Jacquet integral is based on that of Proskurin [12], similarly as in Ishii [6]. Main results of the paper, described in Theorems 3.2, 3.3 and 4.2, show that the Whittaker function corresponding to certain K-type of π is expressed in terms of the modified Bessel function and hence we obtain its Mellin–Barnes integral representation. Since the restricted root system of $SU(2,2)$ is the same type as that of $Sp(2,\mathbb{R})$ except for multiplicities, our results resemble to those of [6]. But because our group is non-split, it is much more involved from technical viewpoints.

In this paper we discuss only “very small” K-types in some standard principal series representations of G. But combined with results of the other paper [3], we can expect to handle other K-types in the same representation.

We want to refer to the meaning in physics of the group $SU(2,2)$, which is locally isomorphic to the conformal group $SO(4,2)$: this group was the group of symmetry of massless free particles [17]; also the Lie algebra $\mathfrak{su}(2,2)$ was the spectrum generating algebra of the hydrogen atom. Related to these topics, there is a very general result on the minimal representation of $O(p,q)$ by Kobayashi–Ørsted [8].

However the group $SO(4,2)$ now becomes fundamental in the conjecture of AdS/CFT correspondence [2]. Though the situation is not clear, our result is very rare on special functions in *“two variables”* related to spherical functions on $SO(4,2)$ in the literature. So this might bring some new aspects that were not found in the case of the minimal representations.

For other Lie groups, there are related works by Bump [4] on $GL(3)$, Stade [14] on $GL(n)$ and Vinogradov and Tahtajan [15] on $SL(3)$.

1. Basic notions

1.1. The group $SU(2,2)$

Let G denote the special unitary group of signature $(+2,-2)$ and K be the maximal compact subgroup of G associated to the Cartan involution $\theta(g) =^t \bar{g}^{-1}, g \in G$:

$$K = S(U(2) \times U(2)) = \left\{ \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} : \ k_1, k_2 \in U(2), \det(k_1 k_2) = 1 \right\}.$$

The associated Lie algebras are

$$\mathfrak{g} = \mathfrak{su}(2,2) = \{X \in M_4(\mathbb{C}) \mid I_{2,2}X +^t \bar{X} I_{2,2} = 0,\ Tr(X) = 0\}$$

and

$$\mathfrak{k} = \left\{ \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \in \mathfrak{g} : -^t \bar{X}_i = X_i \in M_2(\mathbb{C}), i = 1,2 \right\}.$$

Denoting by $\mathfrak{p}$ the (-1)-eigenspace of the differential of θ, we have a Cartan (symmetric) decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

Let $H_i = E_{i,2+i} + E_{2+i,i} (i = 1, 2)$, where $E_{i,j}$ is the matrix unit with 1 in the (i, j)-entry and zero elsewhere. A subalgebra $\mathfrak{a}$ of $\mathfrak{p}$ spanned by H_1, H_2 over $\mathbb{R}$ is maximally abelian and any element a of its Lie group $A = \exp(\mathfrak{a})$ can be expressed by $a = a(t_1, t_2) = \exp(t_1 H_1 + t_2 H_2)$ for some $t_1, t_2 \in \mathbb{R}$. Thus,

$$a(t_1, t_2) = \sum_{i=1}^{2} \Big\{ \cosh(t_i)(E_{i,i} + E_{i+2,i+2}) + \sinh(t_i)(E_{i,i+2} + E_{i+2,i}) \Big\}.$$

Let $\{\lambda_1, \lambda_2\}$ be a basis of the dual space $\mathfrak{a}^*$ such that $\lambda_i(H_j) = \delta_{ij}$ (the Kronecker symbol). Then the restricted root system $\Phi(\mathfrak{g}, \mathfrak{a})$ is of type C_2:

$$\Phi(\mathfrak{g}, \mathfrak{a}) = \{\pm\lambda_1 \pm \lambda_2, \pm 2\lambda_1, \pm 2\lambda_2\}.$$

Choose $\lambda_1 - \lambda_2$ and $2\lambda_2$ as simple roots of $\Phi(\mathfrak{g}, \mathfrak{a})$. Put

$$\begin{aligned} E_0 &= \kappa^{-1}(E_{12} - E_{43})\kappa, \quad & E_1 &= i\kappa^{-1}(E_{12} + E_{43})\kappa, \quad & E_2 &= \kappa^{-1} E_{24}\kappa, \\ F_0 &= \kappa^{-1}(E_{14} + E_{23})\kappa, \quad & F_1 &= i\kappa^{-1}(E_{14} - E_{23})\kappa, \quad & F_2 &= \kappa^{-1} E_{13}\kappa, \end{aligned}$$

by setting

$$\kappa = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -\mathrm{i} & 0 & \mathrm{i} & 0 \\ 0 & -\mathrm{i} & 0 & \mathrm{i} \end{pmatrix}$$

with $\mathrm{i} = \sqrt{-1}$. Then the corresponding root spaces of positive roots in $\Phi(\mathfrak{g}, \mathfrak{a})$ are given by

$$\begin{aligned} \mathfrak{g}_{\lambda_1 - \lambda_2} &= E_0 \cdot \mathbb{R} \oplus E_1 \cdot \mathbb{R}, \quad & \mathfrak{g}_{2\lambda_2} &= E_2 \cdot \mathbb{R}, \\ \mathfrak{g}_{\lambda_1 + \lambda_2} &= F_0 \cdot \mathbb{R} \oplus F_1 \cdot \mathbb{R}, \quad & \mathfrak{g}_{2\lambda_1} &= F_2 \cdot \mathbb{R}. \end{aligned}$$

Let $\mathfrak{n}$ be a subalgebra defined by $\mathfrak{n} = \sum_{\alpha \in \Phi_+} \mathfrak{g}_\alpha$. We now describe elements of a maximal unipotent subgroup N of G given by $N = \exp(\mathfrak{n})$.

Lemma 1.1. *Let E_i, F_i be as above and set $X = x_0 E_0 + y_0 E_1$ and $Y = x_2 F_0 + y_2 F_1 + x_1 F_2 + x_3 E_2$ for $x_i, y_j \in \mathbb{R}$ $(i = 0, 1, 2, 3,\ j = 0, 2)$. Then*

$$\exp(X + Y) = \exp(X) \exp(Y - \frac{1}{2}[X, Y] - \frac{1}{3} XYX).$$

Proof. To see this, it suffices to verify relations $X^2 = Y^2 = YXY = 0$. $\square$

The Killing form $B(X,Y) = \mathrm{tr}(\mathrm{ad}\, X \cdot \mathrm{ad}\, Y)$, $(X, Y \in \mathfrak{g})$ and Cartan involution θ of $\mathfrak{g}$ induce an inner product $\langle , \rangle$ of $\mathfrak{g}$ via

$$\langle X, Y \rangle = -B(X, Y^\theta), \ (X, Y \in \mathfrak{g}).$$

Then one has that $\langle \mathfrak{g}_\alpha, \mathfrak{g}_\beta \rangle = 0$ if $\alpha \neq \beta$, because of the involution θ.

Lemma 1.2. *The vectors E_i, F_i $(i = 0, 1, 2)$ of the subspace $\mathfrak{n}$ of $\mathfrak{g}$ defined above are an orthogonal basis of $\mathfrak{n}$ with respect to the inner product $\langle , \rangle$.*

Proof. For the orthogonality of the basis of $\mathfrak{n}$, it suffices to show that

$$\langle E_0, E_1 \rangle = \langle F_0, F_1 \rangle = 0.$$

Recall that $\mathrm{ad}\, E_0 \cdot \mathrm{ad}\, E_1^\theta$ sends the subspace $\mathfrak{g}_\lambda$ $(\lambda \in \Phi(\mathfrak{g}, \mathfrak{a}))$ into itself. By setting $A = -\mathrm{ad}\, E_0 \cdot \mathrm{ad}\, E_1^\theta$, we give the list of all non-zero restrictions of A to the subspaces $\mathfrak{g}_\lambda$ of $\mathfrak{g}$:

$$A|_{\mathfrak{g}_{\lambda_1+\lambda_2}} = A|_{\mathfrak{g}_{-\lambda_1-\lambda_2}} = \frac{1}{2^3}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad A|_{\mathfrak{a}+\mathfrak{m}} = \frac{1}{2^4}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}.$$

Hence $tr(\mathrm{ad}\, E_0 \cdot \mathrm{ad}\, E_1^\theta) = 0$ which follows that E_0 and E_1 are orthogonal. Similarly F_0 is orthogonal to F_1. $\square$

We may regard $\mathfrak{n}$ as the vector space $\mathbb{R}^6$. Define a map $\phi : \mathbb{R}^6 \to \mathbb{R}^6$ by

$$\phi(x) = (x_1, x_2, x_3 - \frac{x_1x_4 + x_2x_5}{2} + \frac{(x_1^2 + x_2^2)x_6}{3}, x_4 - x_1x_6, x_5 - x_2x_6, x_6)$$

for $x = (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6$.

Then ϕ is a diffeomorphism and its Jacobian determinant is 1. We now denote ith coordinate function of ϕ by ϕ_i for $1 \leq i \leq 6$ and put

$$n_0 = \phi_1(x) + \sqrt{-1}\phi_2(x), \quad n_1 = \phi_3(x),$$
$$n_2 = \phi_4(x) + \sqrt{-1}\phi_5(x), \quad n_3 = \phi_6(x).$$

Then any element n in the maximal unipotent group N of G takes the form

$$\kappa^{-1}\begin{pmatrix} 1 & n_0 & & \\ & 1 & & \\ & & 1 & \\ & & -\bar{n}_0 & 1 \end{pmatrix}\begin{pmatrix} 1 & & n_1 & n_2 \\ & 1 & \bar{n}_2 & n_3 \\ & & 1 & \\ & & & 1 \end{pmatrix}\kappa$$

for some $n_1, n_3 \in \mathbb{R}$ and $n_0, n_2 \in \mathbb{C}$, and denote it by $n(n_0, n_1, n_2, n_3)$. Since

$$\mathfrak{g}_{\lambda_1+\lambda_2} = [\mathfrak{g}_{\lambda_1-\lambda_2}, \mathfrak{g}_{2\lambda_2}] \quad \text{and} \quad \mathfrak{g}_{2\lambda_1} = [\mathfrak{g}_{\lambda_1-\lambda_2}, \mathfrak{g}_{\lambda_1+\lambda_2}],$$

any character η of N is uniquely determined by the values of $E_i (i = 0, 1, 2)$. Put

$$c_0 = \sqrt{-1}\eta(E_0), \quad c_1 = \sqrt{-1}\eta(E_1) \text{ and } c_2 = \sqrt{-1}\eta(E_2)$$

with $c_0, c_1, c_2 \in \mathbb{C}$. Then these numbers are real when η is unitary and therefore such η is given by

$$\eta(n) = \exp(2\sqrt{-1}(\mathrm{Re}(\bar{c}n_0) + c_2 n_3)), \quad n = n(n_0, n_1, n_2, n_3) \in N$$

for a real number c_2 and $c = c_0 + \sqrt{-1}c_1 \in \mathbb{C}$.

Conventions. We say that the character η of N is non-degenerate if both $c_0^2 + c_1^2$ and c_2 are non-zero. Throughout this paper, we shall fix a non-degenerate character η of N.

1.2. Principal series representations

Let P be a minimal parabolic subgroup of G with Langlands decomposition $P = MAN$ with $M = Z_A(K)$. In particularly, the subgroup M of P is given by

$$M = \{[\mathrm{e}^{\sqrt{-1}\theta}]\gamma^j \mid \theta \in \mathbb{R}, j \in \{0, 1\}\},$$

where $\gamma = \mathrm{diag}(1, -1, 1, -1) \in G$ and

$$[e^{\sqrt{-1}\theta}] = \mathrm{diag}(e^{\sqrt{-1}\theta}, e^{-\sqrt{-1}\theta}, e^{\sqrt{-1}\theta}, e^{-\sqrt{-1}\theta}).$$

For a pair $n \in \mathbb{Z}$ and a character ε of the group $\mu_2 = \{\pm 1\}$, we define a unitary character of M as

$$\sigma_{n,\varepsilon}([e^{\sqrt{-1}\theta}]\gamma^j) = \varepsilon(-1)^j e^{\sqrt{-1}n\theta}.$$

Denote by ρ the half sum of the positive restricted roots, i.e., $\rho = 3\lambda_1 + \lambda_2$, and define a quasi-character $e^{\nu+\rho}$ of A:

$$e^{\nu+\rho}(a) = e^{(\nu+\rho)\log(a)} \quad (\nu = (\nu_1, \nu_2) \in (\mathfrak{a}_{\mathbb{C}})^*).$$

We extend it to a character of AN so that the restriction to N is trivial. Define an admissible character of P by tensoring these characters of M

and AN. Then we get the induced representation called the principal series representation of G

$$\pi_\nu = \mathrm{ind}_{\mathrm{P}}^{G}(\sigma_{n,\varepsilon} \otimes e^{\nu+\rho} \otimes 1_N).$$

In this paper we will be dealing with the principal series representations that contain one-dimensional K-types. For an integer u, we define a K-module structure τ_u on $\mathbb{C}$ by

$$\tau_u(k)v = \det(k_2)^u v, \quad k = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \in K, v \in \mathbb{C}$$

and denote by $\mathbb{C}_u$ the underlying one-dimensional K-module. Let $\pi_\nu \mid_K$ be the subspace of all K-finite vectors in π_ν.

Lemma 1.3. *Let $\pi_\nu = \mathrm{ind}_{\mathrm{P}}^{G}(\sigma_{0,\varepsilon} \otimes e^{\nu+\rho} \otimes 1_N)$ and τ_u be as above. Then τ_u is a K-submodule of $\pi_\nu \mid_K$ if and only if $\varepsilon(-1) = (-1)^u$. In this case τ_u occurs exactly once.*

Proof. By Frobenius reciprocity we have that $[\pi_\nu \mid_K : \tau_u] = [\tau_u \mid_M : \sigma_{0,\varepsilon}]$. Hence the multiplicity is at most one. By considering the action of M on $\mathbb{C}_u$ we get the assumption on u as required. □

Assumption. When we consider the principal series representation $\pi_\nu = \mathrm{ind}_{\mathrm{P}}^{G}(\sigma_{0,\varepsilon} \otimes e^{\nu+\rho} \otimes 1_N)$, throughout this paper, we assume that

$$\nu_1 + 1 + e, \quad \nu_2 + 1 + e \text{ and } \nu_1 \pm \nu_2 \text{ are not integers.}$$

1.3. The Jacquet integral

Let $\sigma = \sigma_{n,\varepsilon}$. By definition the principal series representation π_ν of G can be realized on the Hilbert space

$$L^2_\sigma(K) = \{f \in L^2(K) \mid f(mk) = \sigma(m)f(k), m \in M, k \in K\}$$

with G-action defined by

$$(\pi_\nu(g)f)(x) = a(xg)^{\nu+\rho} f(k(xg)), \; x \in K, g \in G,$$

where $xg = n(xg)a(xg)k(xg)$ stands for the Iwasawa decomposition of the element xg.

In [7], Jacquet defined the continuous functional $J_{\sigma,\nu}$ on the space of differentiable functions of $L^2_\sigma(K)$ satisfying $J_{\sigma,\nu}(\pi_\nu(n)f) = \eta(n)J_{\sigma,\nu}(f)$ by

$$J_{\sigma,\nu}(f) = \int_N \eta(n)^{-1}a(s^*n)^{\nu+\rho}f(k(s^*n))dn$$

for a differentiable function f in $L^2_\sigma(K)$ and the longest element $s \in W(A)$. Here $W(A)$ is the Weyl group defined as the quotient of $M^* = N_K(\mathfrak{a})$, the normalizer of $\mathfrak{a}$ in K, by M and s^* is an element of M^* mapping to the longest element $s \in W(A)$.

Multiplicity one theorem tells that there is at most one intertwiner (up to constant) from the space of K-finite vectors of π_ν into the subspace $A_\eta(N \setminus G)$ of moderate growth functions [16, 8.1] in $C^\infty_\eta(N \setminus G)$. If exist, then the construction is as follows: for each differentiable $f \in L^2_\sigma(K)$ it associates a function $J_f(g)$ in $C^\infty_\eta(N \setminus G)$ defined by

$$J_f(g) = J_{\sigma,\nu}(\pi_\nu(g)f), \ (g \in G).$$

These $J_f(g)$ functions are of moderate growth on G, and in particular so on the subgroup A. We want to have an explicit formula for the A-radial part of $J_f(g)$ with f belongs to a special K-type τ in π_ν.

2. Preliminaries

2.1. Classical formulas

In this section we collect some classical formulas and their combinations that is used in our evaluation. Let $K_\mu(z)$ be the Bessel function defined for $\mu, z \in \mathbb{C}$, by the integral

$$K_\mu(z) = \frac{1}{2}\int_0^\infty \exp\Big(-(t+t^{-1})\frac{z}{2}\Big)t^\mu \frac{dt}{t}. \tag{2.1}$$

Our object is to evaluate the integral $J_{f_u}(g)$, further denote it by $J_u(g)$, in terms of the modified Bessel functions of the second-order $K_\mu(z)$ when $u = 0, \pm 1, \pm 2$.

We recall the Euler integral of the second kind in the form

$$\Gamma(\nu) = c^\nu \int_0^\infty \exp(-ct)t^\nu \frac{dt}{t} \tag{2.2}$$

for $c \in \mathbb{R}_{>0}$ and $\mathrm{Re}(\nu) > 0$.

For $a, b, c, \in \mathbb{R}^*$ and $\alpha, \beta \in \mathbb{R}$ such that $\alpha^2 + \beta^2 = 1$ and $n \in \mathbb{N}$, we set

$$F^{(n)}_{(a,b)} = \left(\frac{a}{\pi}\right)^{\frac{1}{2}} \exp\left(\frac{b^2}{a}\right) \int_{\mathbb{R}} x^n \exp(-ax^2 + 2\sqrt{-1}bx)dx$$

and

$$G^{(n)}_{(a,b,c)} = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\exp(-c(x^2+y^2) - a(\alpha x + \beta y)^2 + 2\sqrt{-1}b(\alpha x + \beta y))}{(\alpha x + \beta y)^{-n}} dxdy.$$

We need the following formulas.

Proposition 2.1. *Let $a, c \in \mathbb{R}^*_+$ and $b \in \mathbb{R}$. Then*

$$(2.3) \quad F^{(0)}_{(a,b)} = 1, \quad F^{(1)}_{(a,b)} = \frac{b}{a}\sqrt{-1}, \quad F^{(2)}_{(a,b)} = \frac{a-2b^2}{2a^2},$$

$$(2.4) \quad G^{(0)}_{(a,b,c)} = \frac{\pi \exp(\frac{-b^2}{a+c})}{(c^2+ac)^{\frac{1}{2}}}, \quad \frac{G^{(1)}_{(a,b,c)}}{G^{(0)}_{(a,b,c)}} = \frac{b\sqrt{-1}}{a+c}, \quad \frac{G^{(2)}_{(a,b,c)}}{G^{(0)}_{(a,b,c)}} = \frac{a+c-2b^2}{2(a+c)^2}.$$

Proof. By formula (4.11) of [5], we have that

$$\int_{\mathbb{R}} \exp(-ax^2 + 2\sqrt{-1}bx)dx = \left(\frac{\pi}{a}\right)^{\frac{1}{2}} \exp\left(-\frac{b^2}{a}\right).$$

Then (2.3) can be verified by applying the operators $\partial/\partial a$ and $\partial/\partial b$ to both sides of the above formula. The first formula in (2.4) follows from the first one in (2.3) and using a similar argument as above, we can derive other formulas. □

2.2. The first modification of the radial part of Jacquet integrals

For our purposes, it will be enough to consider the A-radial part of the Jacquet integral because of the Iwasawa decomposition.

We put $a_i = \exp(t_i)$ for the element $a = a(t_1, t_2)$ of the $\mathbb{R}$-split torus A. For a fixed pair $\nu = (\nu_1, \nu_2) \in \mathbb{C}^2$, by definition of the character $e^{\nu+\rho}$, one has

$$\begin{aligned} e^{\nu+\rho}(a) =& (\cosh(t_1) + \sinh(t_1))^{\nu_1+3}(\cosh(t_2) + \sinh(t_2))^{\nu_2+1} \\ =& a_1^{\nu_1+3} a_2^{\nu_2+1}. \end{aligned}$$

In our case $s^* = I_{2,2}$ and hence by setting $a(s^{-1}n) = a(t'_1, t'_2)$, one can see that

$$a'_1 = 1/\sqrt{\Delta_1} \quad \text{and} \quad a'_2 = \sqrt{\Delta_1/\Delta_2},$$

where $a'_i = \exp(t'_i)(i = 1, 2)$. Here the Δ_1, Δ_2 are as follows:

$$\Delta_1 = 1 + n_1^2 + \bar{n}_2 n_2 + (\bar{n}_0 n_2 + n_0 \bar{n}_2)(n_1 + n_3) + \bar{n}_0 n_0 (1 + \bar{n}_2 n_2 + n_3^2),$$

$$\Delta_2 = 1 + n_1^2 + 2n_2\bar{n}_2 + n_3^2 + (n_1 n_3 - n_2 \bar{n}_2)^2$$

for $n = n(n_0, n_1, n_2, n_3) \in N$.

For convenience we shall rewrite Δ_1 in terms of Δ_2 and Δ_3, where Δ_3 denotes the sum $1 + n_2\bar{n}_2 + n_3^2$.

Lemma 2.2. *Put $n_i = x_i + \sqrt{-1}y_i$ with $x_i, y_i \in \mathbb{R}$ $(i = 0, 2)$. Then we have the following identities for Δ_1 and Δ_2:*

$$\Delta_1\Delta_3 = (X_0^2 + Y_0^2)\Delta_3^2 + \Delta_2$$

with $(X_0, Y_0) = \Big(x_0 + \dfrac{n_1 + n_3}{\Delta_3} x_2, y_0 + \dfrac{n_1 + n_3}{\Delta_3} y_2\Big).$

$$(1 + n_3^2)\Delta_2 = (1 + N_1^2)\Delta_3^2, \text{ with } N_1 = \frac{(1 + n_3^2)n_1 - n_2\bar{n}_2 n_3}{\Delta_3}.$$

Proof. (a) To prove this part, by direct computation, one can see that

$$\Delta_2 = (1 + n_1^2 + n_2\bar{n}_2)\Delta_3 - (n_1 + n_3)^2 n_2\bar{n}_2$$

and hence (a) is immediate.
(b) It is straightforward to check that $\sqrt{\Delta_2}$ is the complex norm of

$$(1 - n_1 n_3 + n_2\bar{n}_2) + \sqrt{-1}(n_1 + n_3).$$

The lemma follows. □

For an integer u, define a function $f_u(k)$ on K by

$$f_u(k) := \det(k_2)^u, \quad k = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \in K.$$

Lemma 2.3. *The function $f_u(k)$ belongs to $L^2_{\sigma_{(0,\varepsilon)}}(K)$ if $\varepsilon(-1)=(-1)^u$. In particular, we have*

$$f_u(k(I_{2,2}n)) = \left(\frac{1-n_1n_3+n_2\bar{n}_2+\sqrt{-1}(n_1+n_3)}{1-n_1n_3+n_2\bar{n}_2-\sqrt{-1}(n_1+n_3)}\right)^{\frac{u}{2}}$$

for $n=(n_0,n_1,n_2,n_3)\in N$.

Proof. For the factor $k(I_{2,2}n)$ of the Iwasawa decomposition of $I_{2,2}n$ with $n\in N$, there are $k_1,k_2\in U(2)$ such that $k(I_{2,2}n)=\begin{pmatrix}k_1 & 0\\ 0 & k_2\end{pmatrix}\in K$. Put $N_1=\begin{pmatrix}n_1 & n_2\\ \bar{n}_2 & n_3\end{pmatrix}$ for $n=n(n_0,n_1,n_2,n_3)\in N$. One can see that

$$\frac{\det(k_1)}{\det(k_2)}=\frac{\det(1-\sqrt{-1}N_1)}{\det(1+\sqrt{-1}\mathrm{N}_1)}.$$

Since $\det(k_1)\det(k_2)=1$, the function f_u has the required expression. □

Note that the K-submodule in $L^2_{\sigma_{(0,\varepsilon)}}(K)$ generated by $f_u(k)$ is isomorphic to V_u when u satisfying the condition in Lemma 1.3. By setting $J_u:=J_{f_u}$ for Jacquet function J_{f_u}, the function $J_u(a)$ on A is given by the integral expression

$$a^{\rho-\nu}\int_N a(I_{2,2}n)^{\nu+\rho}\exp(-2\sqrt{-1}\Big(\frac{a_1}{a_2}\mathrm{Re}(\bar{c}n_0)+c_2a_2^2n_3\Big))f_u(k(I_{2,2}n))dn$$

for a character η depending on $c\in\mathbb{C}$ and $c_2\in\mathbb{R}$. For future convenience, we choose a new coordinate

$$y=(y_1,y_2)=\Big(\frac{a_1}{a_2},a_2^2\Big).$$

Since $f\to J_f(g)$ is the Whittaker realization of π_ν, $J_{f_u}(a)$ is the radial part of a Whittaker function on G belonging to π_ν. Thus, in the new coordinate system, we can summarize the following lemma.

Lemma 2.4. *The radial part of the moderate growth Whittaker function $W_{(\nu_1,\nu_2)}(y_1,y_2;u)=y_1^3y_2^2\tilde{W}_{(\nu_1,\nu_2)}(y_1,y_2;u)$ (up to constant) associated with the K-type τ_u can be written in the form*

$$\begin{aligned}\tilde{W}_{(\nu_1,\nu_2)}(y_1,y_2;u)=&y_1^{-\nu_1}y_2^{-\frac{\nu_1+\nu_2}{2}}\int_N\Delta_1^{-\frac{\nu_1-\nu_2}{2}-1}\Delta_2^{-\frac{\nu_2+1}{2}}\\&\times\exp(-2\sqrt{-1}\Big(y_1\mathrm{Re}(\bar{c}n_0)+c_2y_2n_3\Big)f_u(k(I_{2,2}n))dn,\end{aligned}$$

where dn is a multiplicative Haar measure on N.

Note here that the results in [3] led us to the determination of the Whittaker function associated to certain K-types in π_ν, because the intertwiner corresponding to the functional $J_{\sigma,\nu}$ is an intertwiner of $\mathfrak{g}$-equivariant. The assumptions for $\nu = (\nu_1, \nu_2)$ in Subsection 1.2 imply that $L^2_{\sigma_{(0,\varepsilon)}}(K)$ is infinitesimally irreducible. Hence, in fact, it suffices to consider the cases $u = 0$ and 1 for our purpose.

3. Explicit formulas

In this section we consider the integral J_u when $u = 0, \pm 1, \pm 2$. Actually the results corresponding to $u = 0, \pm 1$ are quite similar to that integrals on $\mathrm{Sp}(2, \mathbb{R})$ in [6], which could be explained by the coincidence of the restricted root system of type C_2. Throughout this paper we denote by I the interval $[0, \infty)$.

Now we shall give a normalization of Haar measure of N. In Section 1, the subalgebra $\mathfrak{n}$ is regarded as $\mathbb{R}^6$ with coordinates $(\phi_i)_{1\le i\le 6}$. Let $d\phi$ be the corresponding Lebesgue measure on $\mathfrak{n}$. Since the exponential map of $\mathfrak{n}$ onto N is an analytic isomorphism, there exists a unique Haar measure dn on N that corresponds to $d\phi$.

Set $n_i = x_i + \sqrt{-1}y_i$ $(i = 0, 2)$. For $\mu_1, \mu_2 \in \mathbb{C}$ and non-degenerated unitary character η such that $c_0^2 + c_1^2 = 1$ and $c_2 = \pm 1$, let us evaluate

$$J = \int_{\mathbb{R}^6} \Delta_1^{\mu_1} \Delta_2^{\mu_2} \exp(-2\sqrt{-1}(c_0 x_0 A_1 + c_1 y_0 A_1 - n_3 A_2)) dn$$

where $dn = dx_0 dy_0 dn_1 dx_2 dy_2 dn_3$ and A_1, A_2 are positive real parameters.

Lemma 3.1. *We have that the integral J defined above is equal to*

$$\frac{\pi}{\Gamma(-\mu_1)\Gamma(-\mu_2)} \int_{\mathbb{R}^4} \int_0^\infty \int_0^\infty \frac{t_1^{-\mu_1-1} t_2^{-\mu_2}}{1+n_3^2} \exp\left(-\frac{A_1^2}{t_1} - \frac{\Delta_2}{\Delta_3^2}(t_1+t_2)\right)$$
$$\exp\Big(2\sqrt{-1}\{-n_3 A_2 + \frac{N_1+n_3}{1+n_3^2}(c_0 x_2 + c_1 y_2) A_1\}\Big) \Delta_3^{\mu_1+2\mu_2+1} dn$$

with $dn = dN_1 dx_2 dy_2 dn_3 \dfrac{dt_1}{t_1}\dfrac{dt_2}{t_2}$

Proof. Firstly we change the system variable from

$$(x_0, y_0, x_2, y_2, n_1, n_3) \text{ to } (X_0, Y_0, x_2, y_2, N_1, n_3).$$

Here X_0, Y_0 and N_1 are defined in Lemma 2.2. Then

$$dx_0dy_0dx_2dy_2dn_1dn_3 = \frac{\Delta_3}{1+n_3^2}dX_0dY_0dx_2dy_2dN_1dn_3.$$

Moreover,

$$(c_0x_0 + c_1y_0)A_1 = (c_0X_0 + c_1Y_0)A_1 - \frac{N_1 + n_3}{1+n_3^2}(c_0x_2 + c_1y_2)A_1.$$

We apply all these replacement for the integration of J together with the insertion of

$$(\Delta_1/\Delta_3)^{\mu_1} = \frac{1}{\Gamma(-\mu_1)}\int_0^\infty \exp(-\Delta_1 t_1/\Delta_3)t_1^{-\mu_1}\frac{dt_1}{t_1},$$

which is the Euler integral of the second kind (2.2). Then J is equal to

$$\begin{aligned}&\frac{1}{\Gamma(-\mu_1)}\int_I\int_{\mathbb{R}^6}\exp\left(-(X_0^2+Y_0^2)t_1 - \frac{\Delta_2}{\Delta_3^2}t_1\right)\frac{\Delta_2^{\mu_2}\Delta_3^{\mu_1+1}}{t_1^{\mu_1}(1+n_3^2)}\\ &\times\exp(-2\sqrt{-1}(c_0X_0+c_1Y_0)A_1)\\ &\times\exp\left(2\sqrt{-1}(\pm n_3A_2 + \frac{N_1+n_3}{1+n_3^2}(c_0x_2+c_1y_2)A_1)\right)\end{aligned}$$

with respect to $dX_0dY_0dx_2dy_2dN_1dn_3\dfrac{dt_1}{t_1}$.

Note here that we use the equation

$$\Delta_1/\Delta_3 = X_0^2 + Y_0^2 + \Delta_2/\Delta_3^2$$

in Lemma 2.2. Now we can execute the integrations with respect to the variables X_0, Y_0 applying formula (2.3) with $n = 0$ to obtain

$$\begin{aligned}J = \frac{\pi}{\Gamma(-\mu_1)}&\int_{\mathbb{R}^4}\int_I \exp\left(2\sqrt{-1}\{\pm n_3A_2 + \frac{N_1+n_3}{1+n_3^2}(c_0x_2+c_1y_2)A_1\}\right)\\ &\exp\left(-\frac{A_1^2}{t_1} - \frac{\Delta_2}{\Delta_3^2}t_1\right)\Delta_2^{\mu_2}\Delta_3^{\mu_1+1}\frac{t_1^{-\mu_1-1}}{1+n_3^2}\frac{dt_1}{t_1}dN_1dx_2dy_2dn_3\frac{dt_1}{t_1}.\end{aligned}$$

To complete the proof we remove the factor $\Delta_2^{\mu_2}$ by applying formula (2.2) again

$$\Delta_2^{\mu_2} = \frac{\Delta_3^{2\mu_2}}{\Gamma(-\mu_2)}\int_0^\infty \exp(-\Delta_2t_2/\Delta_3^2)t_2^{-\mu_2}\frac{dt_2}{t_2}.$$

This completes the proof of our Lemma. □

3.1. The standard cases $|\, u \,| \leq 1$

In this subsection we discuss the main results of this paper. These standard cases seem to be very useful for the the Jacquet vectors corresponding to the minimal K-types of other principal series representations. Let

$$\Gamma(s_1, s_2) = \frac{\Gamma_{\pm}(s_1, \nu_1)\Gamma_{\pm}(s_1, \nu_2)\Gamma_{\pm}(s_2, (\nu_1+\nu_2)/2)\Gamma_{\pm}(s_2, (\nu_1-\nu_2)/2)}{\Gamma_{\pm}(s_1+s_2, \nu_1+\nu_2)\Gamma_{\pm}(s_1+s_2, \nu_1-\nu_2)}$$

with

$$\Gamma_{\pm}(s,t) := \Gamma\left(\frac{s+t}{2}\right)\Gamma\left(\frac{s-t}{2}\right)$$

for suitable $s_i, \nu_i \in \mathbb{C}, (i = 1, 2)$.

Set $\nu = (\nu_1, \nu_2) \in \mathbb{C}^2$. Let us begin with the case $u = 0$, i.e., the class one case.

Theorem 3.2. *Let $\pi_\nu = \mathrm{Ind}_P^G(1_M \otimes e^{\nu+\rho} \otimes 1_N)$ be an irreducible representation. For a non-degenerated unitary character η of N we have the following assertions on the A-radial part of the primary Whittaker function $W_{(\nu_1,\nu_2)}(y_1, y_2; 0) = y_1^3 y_2^2 \tilde{W}_{(\nu_1,\nu_2)}(y_1, y_2; 0)$. The function $\tilde{W}_{(\nu_1,\nu_2)}(y_1, y_2; 0)$ has the following integral expressions:*

1. *We have*

$$\begin{aligned}
&\tilde{W}_{(\nu_1,\nu_2)}(y_1, y_2; 0) \\
&\quad = \int_0^\infty \int_0^\infty K_{\frac{\nu_1+\nu_2}{2}}(2\sqrt{t_2/t_1}) K_{\frac{\nu_2-\nu_1}{2}}(2\sqrt{t_1 t_2}) \\
&\qquad \exp\left(-|c_2|y_2 t_1 - \frac{|c_2|y_2}{t_1} - \frac{t_2}{|c_2|y_2} - (c_0^2+c_1^2)|c_2|\frac{y_1^2 y_2}{t_2}\right) \frac{dt_1}{t_1}\frac{dt_2}{t_2}.
\end{aligned}$$

2. *The function $\tilde{W}_{(\nu_1,\nu_2)}(y_1, y_2; 0)$ is identified with*

$$\left(\frac{y_1}{y_2}\right)^{\frac{\nu_2}{2}} \int_0^\infty \int_0^\infty K_{\frac{\nu_1}{2}}(X) K_{\frac{\nu_2}{2}}(Y) \left(\frac{x(1+x)}{y(1+y)}\right)^{\frac{\nu_1}{4}} \left(\frac{x^2y^2}{1+x+y}\right)^{\frac{\nu_2}{4}} \frac{dx}{x}\frac{dy}{y}$$

with $X = 2|c_2|y_2\left(\frac{(1+x)(1+y)}{xy}\right)^{\frac{1}{2}}$ *and* $Y = 2(c_0^2+c_1^2)^{\frac{1}{2}} y_1 (1+x+y)^{\frac{1}{2}}$,

3. *The Mellin–Barnes's integral expression of* $\tilde{W}_{(\nu_1,\nu_2)}(y_1, y_2; 0)$ *is*

$$\tilde{W}_{(\nu_1,\nu_2)}(y_1, y_2; 0) = \frac{1}{(2\sqrt{-1})^2} \int_{s_1} \int_{s_2} V_{(\nu_1,\nu_2)}(s_1, s_2) y_1^{-s_1} y_2^{-s_2} ds_1 ds_2.$$

Here the paths of integrations are the vertical lines from $\alpha_i - \sqrt{-1}\infty$ *to* $\alpha_i + \sqrt{-1}\infty$ *with real number* α_i *such that*

$$\alpha_1 > |\mathrm{Re}(\nu_1)|, |\mathrm{Re}(\nu_1)|, \ \alpha_2 > |\mathrm{Re}(\nu_1 + \nu_2)|/2, |\mathrm{Re}(\nu_1 - \nu_2)|/2$$

and the integrand $V_{(\nu_1,\nu_2)}(s_1, s_2)$ *is equal to*

$$\Gamma(s_1, s_2) \times {}_3F_2 \left(\begin{array}{c} \frac{s_1}{2}, \frac{s_2}{2} + \frac{\nu_2 - \nu_1}{4}, \frac{s_2}{2} - \frac{\nu_2 - \nu_2}{4} \\ \frac{s_2 + s_1}{2} + \frac{\nu_2 + \nu_1}{4}, \frac{s_2 + s_1}{2} - \frac{\nu_1 + \nu_2}{4} \end{array} \middle| 1 \right).$$

Proof. In order to get the desired result we shall evaluate the integration J in Lemma 3.1 with the assumption for η, because of $f_0 = 1$ and Lemma 2.4.

Step 1. Integration for N_1.
To integrate J in the statement of Lemma 3.1 with respect to N_1, we use the expression of Δ_2 in Lemma 2.2 and apply (2.3) with

$$(n, a, b) = \Big(0, \frac{t_1 + t_2}{1 + n_3^2}, \frac{c_0 x_2 + c_1 y_2}{1 + n_3^2} A_1\Big).$$

Then we find that

$$\begin{aligned} J =& \frac{\pi}{\Gamma(-\mu_1)\Gamma(-\mu_2)} \int_{\mathbb{R}^4} \int_{I^2} \exp\Big(2\sqrt{-1}\{-n_3 A_2 + \frac{n_3}{1 + n_3^2}(c_0 x_2 + c_1 y_2) A_1\}\Big) \\ & \exp\Big(-\frac{A_1^2}{t_1} - \frac{P}{1 + n_3^2} - \frac{(c_0 x_0 + c_1 x_2)^2}{P(1 + n_3^2)} A_1^2\Big) \Big(\frac{\pi}{P(1 + n_3^2)}\Big)^{\frac{1}{2}} \frac{\Delta_3^{\mu_1 + 2\mu_2 + 1}}{t_1^{\mu_1 + 1} t_2^{\mu_2}} dn \end{aligned}$$

with $P = t_1 + t_2$.

Step 2. Integration for n_2.
We apply (2.2) with $(c, \nu) = \Big(\frac{A_1^2 \Delta_3}{1 + n_3^2}, -\mu_1 - 2\mu_2 - 1\Big)$ to rewrite $\Delta_3^{\mu_1 + 2\mu_2 + 1}$

as

$$\frac{(A_1^{-2}(1+n_3^2))^{\mu_1+2\mu_2+1}}{\Gamma(-\mu_1-2\mu_2-1)}\int_0^{\infty}\exp\Big(-A_1^2t_3-\frac{A_1^2t_3(x_2^2+y_2^2)}{1+n_3^2}\Big)t_3^{-\mu_1-2\mu_2-1}\frac{dt_3}{t_3}.$$

Substitute this into the last expression of J and using (2.4) for the variables x_2 and y_2 by choosing

$$(c,a,b)=\Big(\frac{A_1^2t_3}{1+n_3^2},\frac{A_1^2}{P(1+n_3^2)},\frac{n_3A_1}{1+n_3^2}\Big).$$

Thus we can rewrite J as

$$\frac{\pi^{\frac{5}{2}}A_1^{-2\mu_1-4\mu_2-4}}{\Gamma(-\mu_1)\Gamma(-\mu_2)\Gamma(-\mu_1-2\mu_2-1)}\int_{\mathbb{R}}\int_{I^3}\exp(-2\sqrt{-1}n_3A_2)\frac{(1+n_3^2)^{\mu_1+2\mu_2+\frac{3}{2}}}{(Pt_3+1)^{\frac{1}{2}}}$$
$$\exp\left(-\frac{A_1^2}{t_1}-\frac{P}{1+n_3^2}-A_1^2t_3-\frac{n_3^2P}{(1+n_3^2)(Pt_3+1)}\right)$$
$$\times\frac{t_1^{-\mu_1-1}t_2^{-\mu_2}}{t_3^{\mu_1+2\mu_2+\frac{3}{2}}}\frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dt_3}{t_3}dn_3.$$

Changing the variables (u_1,u_2,u_3) from (t_1,t_2,t_3) defined through

$$u_1=t_3+\frac{1}{P},\quad u_2=\frac{t_3P}{(1+n_3^2)},\quad u_3=\frac{t_2}{t_1},$$

the integration J has the following expression:

$$J=\frac{\pi^{\frac{3}{2}}A_1^{-2\mu_1-4\mu_2-4}}{\Gamma(-\mu_1)\Gamma(-\mu_2)\Gamma(-\mu_1-2\mu_2-1)}\int_{\mathbb{R}}\int_{I^3}\frac{(1+u_3)^{\mu_1+\mu_2+1}Q^{\mu_2}}{u_1^{\mu_2+\frac{1}{2}}u_2^{\mu_1+2\mu_2+\frac{3}{2}}u_3^{\mu_2}}$$
$$\exp(-2\sqrt{-1}n_3A_2)\exp\Big(-u_1A_1^2\Big(1+\frac{u_3}{Q}\Big)-\frac{1+u_2}{u_1}\Big)dn_3\frac{du_1}{u_1}\frac{du_2}{u_2}\frac{du_3}{u_3}$$

with $Q=1+(1+n_3^2)u_2$.

Step 3. Integration for n_3.
Before performing this step, we again change the variables by the rule

$$u_1=\frac{1+x}{A_1^2A_2}t_2,\quad u_2=x,\quad u_3=y+\frac{xy}{1+x}n_3^2$$

to get

$$J = \frac{\pi^{\frac{3}{2}} A_1^{-2\mu_1-2\mu_2-3} A_2^{\mu_2+\frac{1}{2}}}{\Gamma(-\mu_1)\Gamma(-\mu_2)\Gamma(-\mu_1-2\mu_2-1)} \int_{\mathbb{R}}\int_{I^3} \exp\Big(-\frac{1+x+y}{A_2}t_2 - \frac{A_1^2A_2}{t_2}\Big)$$
$$\exp(-2\sqrt{-1}n_3A_2)\frac{((1+x)(1+y)+xyn_3^2)^{\mu_1+\mu_2+1}}{(x(1+x))^{\mu_1+\mu_2+\frac{3}{2}}(xy)^{\mu_2}t_2^{\mu_2+\frac{1}{2}}} dn_3 \frac{dx}{x}\frac{dy}{y}\frac{dt_2}{t_2}.$$

By suitable substitution, from (2.3) one can derive that

(3.1)
$$\int_{\mathbb{R}} (ax^2+b)^{\nu} \exp(2\sqrt{-1}cx)dx = \mathrm{sgn}(c)\frac{\sqrt{\pi/a}}{\Gamma(-\nu)} \int_0^{\infty} \exp\Big(-bt - \frac{c^2}{at}\Big)t^{-\nu-\frac{1}{2}}\frac{dt}{t}.$$

for $a, b \in \mathbb{R}_+$ and $c \in \mathbb{R}$. Apply this formula to the above expression of J by choosing

$$(x,a,b,c,t) = \Big(n_3, \frac{x}{1+x}, \frac{1+y}{y}, A_2, t_1A_2\Big)$$

and put

$$(\mu_1,\mu_2,A_1,A_2) = \Big(\frac{-\nu_1+\nu_2-2}{2}, \frac{-\nu_2-1}{2}, y_1\sqrt{c_0^2+c_1^2}, y_2|c_2|\Big).$$

We then arrive at an evaluation of $\tilde{W}_{(\nu_1,\nu_2)}(y_1,y_2;0)$, that is,

$$\frac{\pi^3 sgn(c_2)(c_0^2+c_1^2)^{\frac{\nu_1}{2}}|c_2|^{\frac{\nu_1-\nu_2}{2}}y_2^{-\nu_2}}{\Gamma(\frac{\nu_1+1}{2})\Gamma(\frac{\nu_2+1}{2})\Gamma(\frac{\nu_1-\nu_2}{2}+1)\Gamma(\frac{\nu_1+\nu_2}{2}+1)} \int_{I^4} x^{\frac{\nu_1+\nu_2}{2}} y^{\frac{\nu_2-\nu_1}{2}} t_1^{\frac{\nu_1}{2}} t_2^{\frac{\nu_2}{2}}$$
$$\exp\Big(-|c_2|y_2\Big(\frac{1+x+y}{c_2^2y_2^2}t_2 + (c_0^2+c_1^2)\frac{y_1^2}{t_2} + \frac{1+y}{y}t_1 + \frac{1+x}{xt_1}\Big)\Big)dn.$$

with $dn = \frac{dx}{x}\frac{dy}{y}\frac{dt_1}{t_1}\frac{dt_2}{t_2}$.

The integrand in the above integral expression of $\tilde{W}_{(\nu_1,\nu_2)}(y_1,y_2;0)$ is rapidly decreasing at both zero and infinity for each variable of x, y, t_1 and t_2. Hence the integral converges and $\tilde{W}_{(\nu_1,\nu_2)}(y_1,y_2;0)$ is well defined to the whole plane $\mathbb{C}^2$ if $\frac{\nu_1+1}{2}, \frac{\nu_2+1}{2}, \frac{\nu_1-\nu_2+2}{2}, \frac{\nu_1+\nu_2+2}{2}$ are not negative integers simultaneously.

By a simple substitution, (2.1) can be transformed in the form

$$K_\nu(2\sqrt{ab}) = \frac{1}{2}(a/b)^{\frac{\nu}{2}} \int_0^\infty \exp(-ax - b/x)x^\nu \frac{dx}{x}, \quad a, b \in \mathbb{R}_{>0}. \tag{3.2}$$

To get the first expression in our theorem we apply (3.2), for the variables x and y, with

$$(a, b, \nu) = \Big(\frac{t_2}{|c_2|y_2}, \frac{|c_2|y_2}{t_1}, \frac{\nu_1+\nu_2}{2}\Big) \text{ and } \Big(\frac{t_2}{|c_2|y_2}, |c_2|t_1y_2, \frac{\nu_2-\nu_1}{2}\Big).$$

2. In the above expression of $\tilde{W}_{(\nu_1,\nu_2)}(y_1, y_2; 0)$ we again utilize (3.2) for the variables t_1 and t_2 by choosing (a, b, ν) as

$$\left(|c_2|y_2\frac{1+y}{y}, |c_2|y_2\frac{1+x}{x}, \frac{\nu_1}{2}\right) \quad \text{and} \quad \left(\frac{1+x+y}{|c_2|y_2}, (c_0^2+c_1^2)|c_2|y_1^2y_2, \frac{\nu_2}{2}\right),$$

respectively. Then we obtain the second expression.
3. For this one, the method of proof is similar to that of [6]. □

We now turn to the discussion of non-class one case i.e., $u = \pm 1$.

Theorem 3.3. *Let $\pi_\nu = \mathrm{Ind}_P^G(\sigma_{(0,-1)} \otimes e^{\nu+\rho} \otimes 1_N)$ be an irreducible representation with $\nu = (\nu_1, \nu_2) \in \mathbb{C}^2$. For a normalized character η of N we have the following assertions on the A-radial part of the primary Whittaker function $W_{(\nu_1,\nu_2)}(y_1, y_2; u) = y_1^3y_2^2\tilde{W}_{(\nu_1,\nu_2)}(y_1, y_2; u)$. The function $\tilde{W}_{(\nu_1,\nu_2)}(y_1, y_2; u)$ has the following integral expressions:*

1. *For $u = \pm 1$, we have that $\tilde{W}_{(\nu_1,\nu_2)}(y_1, y_2; u)$ is equal to*

$$\frac{y_1y_2}{2^2}\int_0^\infty\int_0^\infty K_{\frac{\nu_1+\nu_2}{2}}\left(2\sqrt{\frac{t_2}{t_1}}\right)K_{\frac{\nu_2-\nu_1}{2}}(2\sqrt{t_1t_2})\left(\sqrt{\frac{t_1}{t_2}} - \frac{u}{\sqrt{t_1t_2}}\right)$$
$$\times \exp\left(-|c_2|y_2t_1 - \frac{|c_2|y_2}{t_1} - \frac{t_2}{|c_2|y_2} - (c_0^2+c_1^2)|c_2|\frac{y_1^2y_2}{t_2}\right)\frac{dt_1}{t_1}\frac{dt_2}{t_2}.$$

2. *The function $\tilde{W}_{(\nu_1,\nu_2)}(y_1, y_2; u)$ is identified with*

$$y_1^{\frac{\nu_2+1}{2}}y_2^{-\frac{\nu_2-1}{2}}\int_0^\infty\int_0^\infty K_{\frac{\nu_2-1}{2}}(Y)\left(\frac{x(1+x)}{y(1+y)}\right)^{\frac{\nu_1}{4}}\left(\frac{x^2y^2}{1+x+y}\right)^{\frac{\nu_2-1}{4}}$$
$$\times\left(y\Big(\frac{x(1+x)}{y(1+y)}\Big)^{1/4}K_{\frac{\nu_1+1}{2}}(X) + ux\Big(\frac{y(1+y)}{x(1+x)}\Big)^{1/4}K_{\frac{\nu_1-1}{2}}(X)\right)\frac{dx}{x}\frac{dy}{y}$$

with $X = 2|c_2|y_2\left(\frac{(1+x)(1+y)}{xy}\right)^{\frac{1}{2}}$ and $Y = 2(c_0^2+c_1^2)^{\frac{1}{2}}y_1(1+x+y)^{\frac{1}{2}}$,

3. *The Mellin–Barnes's integral expression of* $\tilde{W}_{(\nu_1,\nu_2)}(y_1, y_2; u)$ *is*

$$\frac{1}{(2\sqrt{-1})^2}\int_{s_1}\int_{s_2}(V^1_{(\nu_1,\nu_2)}(s_1,s_2) - uV^2_{(\nu_1,\nu_2)}(s_1,s_2))y_1^{-s_1}y_2^{-s_2}ds_1ds_2.$$

Here the paths of integrations are the vertical lines from $\alpha_i - \sqrt{-1}\infty$ *to* $\alpha_i + \sqrt{-1}\infty$ *with real number* α_i *such that*

$$\alpha_1 > |\mathrm{Re}(\nu_1)|, |\mathrm{Re}(\nu_1)|,\ \ \alpha_2 > |\mathrm{Re}(\nu_1+\nu_2)|/2, |\mathrm{Re}(\nu_1-\nu_2)|/2$$

and the integrand $V^1_{(\nu_1,\nu_2)}(s_1,s_2)$ *is equal to* $\dfrac{(\frac{s_1-1}{2})\Gamma_\pm(s_2+1,\frac{\nu_1-\nu_2}{2})}{\Gamma_\pm(s_2,\frac{\nu_1-\nu_2}{2})}$ *times*

$$\Gamma(s_1,s_2)\times {}_3F_2\left(\begin{array}{c} \frac{s_1}{2},\ \frac{s_2+1}{2}+\frac{\nu_2-\nu_1}{4},\ \frac{s_2+1}{2}-\frac{\nu_2-\nu_2}{4} \\ \frac{s_1+s_2+1}{2}+\frac{\nu_2+\nu_1}{4},\frac{s_1+s_2+1}{2}-\frac{\nu_1+\nu_2}{4} \end{array}\middle| 1\right)$$

and $V^2_{(\nu_1,\nu_2)}(s_1,s_2)$ *is equal to* $\Gamma_\pm(s_2+1,\dfrac{\nu_1+\nu_2}{2})/\Gamma_\pm(s_2,\dfrac{\nu_1+\nu_2}{2})$ *times*

$$\Gamma(s_1,s_2)\times {}_3F_2\left(\begin{array}{c} \frac{s_1-1}{2},\ \frac{s_2}{2}+\frac{\nu_2-\nu_1}{4},\ \frac{s_2}{2}-\frac{\nu_2-\nu_2}{4} \\ \frac{s_2+s_1}{2}+\frac{\nu_2+\nu_1}{4},\frac{s_2+s_1}{2}-\frac{\nu_1+\nu_2}{4} \end{array}\middle| 1\right).$$

Proof. 1. In this case, the integrand f_u in Lemma 2.4 is

$$\begin{aligned} f_u(n) &= (1+n_2\bar{n_2}-n_1n_3+u\sqrt{-1}(n_1+n_3))/\Delta_2^{\frac{1}{2}} \\ &= (1-N_1n_3+u\sqrt{-1}(N_1+n_3))\Delta_3/((1+n_3^2)\Delta_2^{\frac{1}{2}}) \end{aligned}$$

with $u=\pm1$, and it does not depend on X_0 and Y_0. By Lemma 3.1, we evaluate J_u to get the first part of this theorem. Here

$$\begin{aligned} J_u = \frac{\pi}{\Gamma(-\mu_1)\Gamma(-\mu_2)}&\int_{\mathbb{R}^4}\int_{I^2}\frac{t_1^{-\mu_1-1}t_2^{-\mu_2}}{1+n_3^2}\exp\Big(-\frac{A_1^2}{t_1}-\frac{\Delta_2}{\Delta_3^2}(t_1+t_2)\Big) \\ &\exp\Big(2\sqrt{-1}(-n_3A_2+\frac{N_1+n_3}{1+n_3^2}(c_0x_2+c_1y_2)A_1)\Big)\Delta_3^{\mu_1+2\mu_2+2}f_udn. \end{aligned}$$

As we have seen in the previous theorem, the integrations for N_1 and n_2 can be done as well by applying formulas (2.3) and (2.4). For the integration for

n_3, we use

$$\int_{\mathbb{R}} x(ax^2+b)^{\nu} \exp(2\sqrt{-1}cx)dx = \frac{\sqrt{-1}c(\pi/a)^{\frac{1}{2}}}{a\Gamma(-\nu)} \int_I \exp\Big(-bt - \frac{c^2}{at}\Big) t^{-\nu-\frac{3}{2}} \frac{dt}{t}$$

for $a, b \in \mathbb{R}_+$ and $c \in \mathbb{R}$. Then we get

$$\begin{aligned} &\tilde{W}_{\nu}(y_1, y_2; u) \\ &= \frac{2^{-4}(c_0^2+c_1^2)^{\nu_1}|c_2|^{\nu_2} y_1^4 y_2^{-\nu_2+3}}{\Gamma(\frac{\nu_1}{2}+1)\Gamma(\frac{\nu_2}{2}+1)\Gamma(\frac{\nu_1-\nu_2}{2}+1)\Gamma(\frac{\nu_1+\nu_2}{2}+1)} \int_{I^4} x^{\frac{\nu_1+\nu_2}{2}} y^{\frac{\nu_2-\nu_1}{2}} \\ &\quad \exp\bigg(-|c_2|y_2\left(\frac{1+y}{y}t_1 + \frac{1+x}{xt_1} + \frac{1+x+y}{c_2^2 y_2^2} t_2 + (c_0^2+c_1^2)\frac{y_1^2}{t_2}\right)\bigg) \\ &\quad t_1^{\frac{\nu_1}{2}} t_2^{\frac{\nu_2}{2}} \left(\sqrt{\frac{t_1}{t_2}} + \frac{u}{\sqrt{t_1 t_2}}\right) \frac{dt_1}{t_1}\frac{dt_2}{t_2}\frac{dx}{x}\frac{dy}{y}. \end{aligned}$$

Note here that the above integral converges and therefore the function $\tilde{W}_{(\nu_1,\nu_2)}(y_1, y_2; u)$ is well defined, because of the assumption for the pair (ν_1, ν_2). Hence our theorem follows. □

Remark 3.4. In Theorem 3.3 above, we write the constants c_i, $(i = 0, 1, 2)$. It looks like superfluous, because replacing $|c_2|y_2$ by y_2 we can erase this constant. However if one try to discuss other K-types that are not handled in this paper, sometime the derivatives with respect to these parameters are crucial.

4. Explicit formula, the case $u = \pm 2$

The feature of the case $u = \pm 2$ is that the K-types corresponding to $u = \pm 2$ and $u = 0$ belong to the same principal series representation $\pi_\nu = \mathrm{Ind}_P^G(1_M \otimes e^{\nu+\rho} \otimes 1_N)$. These cases do not seem to appear in the literature. In this subsection we handle this case. Note that one can do this by using $(\mathfrak{g}, K)$-module structure of the principal series representation of $SU(2,2)$ computed in [3]. We may normalize the non-degenerated unitary character η of N so that $c_0^2 + c_1^2 = c_2 = 1$ without loss of generality and call it *normalized* character.

4.1. Evaluation of Jacquet integrals

First of all we consider an evaluation of integrals with a certain integrand that is closely related to the case $u = \pm 2$.

For $\nu_1, \nu_2 \in \mathbb{C}$ and normalized character η of N let us evaluate

$$J_\lambda = A_1^{-\nu_1} A_2^{-\frac{\nu_1+\nu_2}{2}} \int_{\mathbb{R}^6} \exp\Big(-2\sqrt{-1}(c_0 x_0 A_1 + c_1 y_0 A_1 \pm n_3 A_2)\Big) \Delta_1^{-\frac{\nu_1-\nu_2}{2}-1} \Delta_2^{\frac{\nu_2+1}{2}} F_\lambda(n) dx_0 dy_0 dn_1 dx_2 dy_2 dn_3,$$

where A_1, A_2 are positive real parameters and the integrand $F_\lambda(n)$ is

$$F_\lambda(n) = \frac{1}{\Delta_2}\Big((1 + n_2\bar{n}_2 - n_1 n_3)^2 + \lambda\sqrt{-1}(n_1 + n_3)(1 + n_2\bar{n}_2 - n_1 n_3)\Big),$$

where $n_2 = x_2 + \sqrt{-1}y_2$ and $\lambda = \pm 1$.

Lemma 4.1. *Let J_λ be as above. Then the function $\tilde{J}_\lambda = A_1^{-3} A_2^{-2} J_\lambda$ is proportional to*

$$A_2^{-\nu_2} \int_{I^4} \exp\Big(-\frac{1+x+y}{A_2} t_2 - \frac{1+y}{y} A_2 t_1 - A_2 \frac{1+x}{x t_1} - \frac{A_1^2 A_2}{t_2}\Big) \frac{x^{\frac{\nu_1+\nu_2}{2}} y^{\frac{\nu_2-\nu_1}{2}}}{t_1^{\frac{\nu_1}{2}} t_2^{\frac{\nu_2}{2}}} \Big(A_1^2 A_2^2 \frac{t_1}{t_2} + \frac{\nu_1+1}{4} - \frac{1+y}{2y} t_1 A_2 - \lambda\Big(\frac{A_1^2 A_2^2}{t_2} - \frac{A_2}{2}\Big)\Big) dX$$

where $dX = dx/x dy/y dt_1/t_1 dt_2/t_2$.

Proof. Recalling Lemma 3.1, we have

$$J_\lambda = \frac{\pi A_1^{-\nu_1} A_2^{-\frac{\nu_1+\nu_2}{2}}}{\Gamma(-\mu_1)\Gamma(-\mu_2)} \int_{\mathbb{R}^4} \int_0^\infty \int_0^\infty \frac{t_1^{-\mu_1-1} t_2^{-\mu_2}}{1+n_3^2} \exp\left(-\frac{A_1^2}{t_1} - \frac{\Delta_2}{\Delta_3^2}(t_1 + t_2)\right) \exp\Big(2\sqrt{-1}(n_3 A_2 + \frac{N_1 + n_3}{1+n_3^2}(c_0 x_2 + c_1 y_2) A_1)\Big) \Delta_3^{\mu_1+2\mu_2+1} F_\lambda(n) dn,$$

because the function $F_\lambda(n)$ does not depend on the variable n_0.

In terms of variables N_1, n_3 and Δ_3, the function $F_\lambda(n)$ is expressed by

$$F_\lambda(n) = \frac{\Delta_3^2}{(1+n_3^2)^2}(1 - n_3 N_1)(1 - n_3 N_1 + \sqrt{-1}\lambda(N_1 + n_3)).$$

Thus we are now in a position to perform the transformations with respect to the variables N_1, n_2 and n_3 as we have seen in the previous cases. In this manner, the integral J_λ can be identified with

$$\frac{\pi^3 A_2^{-\nu_2}}{\Gamma(\frac{\nu_1+1}{2}+1)\Gamma(\frac{\nu_2+1}{2}+1)\Gamma(\frac{\nu_1-\nu_2}{2}+1)\Gamma(\frac{\nu_1+\nu_2}{2}+1)} \int_I\int_I\int_I\int_I t_1^{\frac{\nu_1}{2}} t_2^{\frac{\nu_2}{2}}$$
$$\exp\left(-\frac{1+x+y}{A_2}t_2 - \frac{1+y}{y}A_2 t_1 - A_2\frac{1+x}{xt_1} - \frac{A_1^2A_2}{t_2}\right) x^{\frac{\nu_1+\nu_2}{2}} y^{\frac{\nu_2-\nu_1}{2}}$$
$$\left(\frac{t_1}{t_2}A_1^2A_2^2 + \frac{\nu_1+1}{4} - \frac{1+y}{2y}t_1A_2 - \lambda\left(\frac{A_1^2A_2^2}{t_2} - \frac{A_2}{2}\right)\right)\frac{dx}{x}\frac{dy}{y}\frac{dt_1}{t_1}\frac{dt_2}{t_2}.$$

□

Let us consider the following theorem to get the Mellin–Barnes integral expression for the case $u = \pm 2$.

Theorem 4.2. *For a normalized character η of the unipotent group N, $\nu = (\nu_1, \nu_2) \in \mathbb{C}^2$ and $u = \pm 2$, on the A-radial part of the primary Whittaker function $W_{(\nu_1,\nu_2)}(y_1, y_2; u) = y_1^3 y_2^2 \tilde{W}_{(\nu_1,\nu_2)}(y_1, y_2; u)$*

(1) We have

$$\tilde{W}_{(\nu_1,\nu_2)}(y_1, y_2; u)$$
$$= y_1^{\frac{\nu_2}{2}} y_2^{-\frac{\nu_2}{2}} \int_0^\infty\int_0^\infty \left(\frac{x(1+x)}{y(1+y)}\right)^{\frac{\nu_1}{4}} \left(\frac{x^2y^2}{1+x+y}\right)^{\frac{\nu_2}{4}}$$
$$\times \Big\{K_{\frac{\nu_1}{2}}(X)\Big(\{2uy_2 - (\nu_1+1)(\nu_2-1)\}K_{\frac{\nu_2}{2}}(Y) - 2uy_2 Y K_{\frac{\nu_2}{2}-1}(Y)\Big)$$
$$+ \frac{2y}{1+y}XYK_{\frac{\nu_1}{2}+1}(X)K_{\frac{\nu_2}{2}-1}(Y) - 2XK_{\frac{\nu_1}{2}+1}(X)K_{\frac{\nu_2}{2}}(Y)\Big\}\frac{dx}{x}\frac{dy}{y},$$

with $X = 2y_2((1+1/x)(1+1/y))^{\frac{1}{2}}$ and and $Y = 2y_1(1+x+y)^{\frac{1}{2}}$.

(2) the function $\tilde{W}_{(\nu_1,\nu_2)}(y_1, y_2; u)$ is equal to

$$\int_{I^2} \exp\Big(-y_2t_1 - \frac{y_2}{t_1} - \frac{t_2}{y_2} - \frac{y_1^2y_2}{t_2}\Big) K_{\frac{\nu_1+\nu_2}{2}}\left(2\sqrt{\frac{t_2}{t_1}}\right)\Big\{-4t_1y_2K_{\frac{\nu_2-\nu_1-2}{2}}(2\sqrt{t_1t_2})$$
$$+ K_{\frac{\nu_2-\nu_1}{2}}(2\sqrt{t_1t_2})\Big((\nu_1+1)(1-\nu_2) + \Big(\frac{2y_1^2y_2^2}{t_2} - y_2\Big)(4t_1 - u)\Big)\Big\}\frac{dt_1}{t_1}\frac{dt_2}{t_2}.$$

Proof. (1). By putting $u = 2\lambda$, one has that

$$f_u(n) = 2F_\lambda(n) - 1.$$

Change A_i by y_i in the expression of J_λ for $i = 1, 2$, then we may write

$$\tilde{W}_{(\nu_1,\nu_2)}(y_1, y_2; u) = 2\tilde{J}_\lambda(y_1, y_2) - \tilde{W}_{(\nu_1,\nu_2)}(y_1, y_2; 0).$$

By making a similar computation as in Theorem 3.2, we also assume that the corresponding integral with respect to second in the above expression is equal to $(\nu_1+1)(\nu_2+1)$ times

$$y_1^{\frac{\nu_2}{2}+3} y_2^{-\frac{\nu_2}{2}+2} \int_{I^2} \Big(\frac{x(1+x)}{y(1+y)}\Big)^{\frac{\nu_1}{4}} \Big(\frac{x^2y^2}{1+x+y}\Big)^{\frac{\nu_2}{4}} K_{\frac{\nu_1}{2}}(X) K_{\frac{\nu_2}{2}}(Y) \frac{dx}{x}\frac{dy}{y}.$$

Thus the desired result follows from the evaluation of Lemma 4.1 and above integral representation.
(2). Using (3.2) with

$$(a,b,\nu) = \Big(\frac{1+y}{y}y_2, \frac{1+x}{x}y_2, \frac{\nu_1}{2}\Big) \quad \text{and} \quad (a,b,\nu) = \Big(\frac{1+x+y}{y_2}, y_1^2y_2, \frac{\nu_2}{2}\Big),$$

we obtain that $\tilde{W}_{(\nu_1,\nu_2)}(y_1,y_2;u)y_2^{\nu_2}$ is equal to

$$\frac{1}{4}\int_{I^4} x^{\frac{\nu_1+\nu_2}{2}} y^{\frac{\nu_2-\nu_1}{2}} \exp\Big(-y_2(-\frac{1+y}{y}t_1 - \frac{1+x}{xt_1} - (1+x+y)\frac{t_2}{y_2^2} - \frac{y_1^2}{t_2})\Big)$$
$$\Big(-(\nu_1+1)(\nu_2-1) + 8\frac{y_1^2y_2^2t_1}{t_2} - 4t_1y_2\frac{1+y}{y} + uy_2 - \frac{2uy_1^2y_2^2}{t_2}\Big) t_1^{\frac{\nu_1}{2}} t_2^{\frac{\nu_2}{2}}$$

with respect to $= dt_1/t_1 dt_2/t_2 dx/x dy/y$. To complete the proof we again apply (3.2) for the variables x and y. Then we get the desired result. □

4.2. Mellin–Barnes integral representation

Let us compute the double Mellin transformation

$$V(s_1,s_2) = \int_0^\infty \int_0^\infty \tilde{W}_{(\nu_1,\nu_2)}(y_1,y_2;\pm 2) y_1^{s_1} y_2^{s_2} \frac{dy_1}{y_1}\frac{dy_2}{y_2}$$

of $\tilde{W}_{(\nu_1,\nu_2)}(y_1,y_2;\pm 2)$ from the previous theorem. Then, by applying Mellin inversion to this, we get the desired result as in the following theorem. We use the following notations:

$$b = \frac{s_2}{2} + \frac{\nu_1-\nu_2}{4}, \quad c = \frac{s_2}{2} + \frac{\nu_2-\nu_1}{4}, \quad d = \frac{s_2}{2} + \frac{\nu_1+\nu_2}{4},$$
$$e = \frac{s_2}{2} - \frac{\nu_1+\nu_2}{4} \quad \text{and} \quad a = s_1/2.$$

Lemma 4.3. *We have*

1.

$$\frac{y}{1+y}XYK_{\frac{\nu_1}{2}+1}(X)K_{\frac{\nu_2}{2}-1}(Y) \\ = \frac{1}{(2\sqrt{-1})^2}\int_{s_1}\int_{s_2} V^1_{(\nu_1,\nu_2)}(s_1,s_2)y_1^{-s_1}y_2^{-s_2}ds_1ds_2,$$

where

$$V^1_{(\nu_1,\nu_2)}(s_1,s_2) = \frac{4abc\Gamma(s_1,s_2)}{(a+d)(a+e)} \times {}_3F_2\left(\begin{array}{c} a+1,b+1,c+1 \\ d+1,e+1 \end{array}\middle| 1\right).$$

2.

$$-(\nu_1+1)(\nu_2-1)K_{\frac{\nu_1}{2}}(X)K_{\frac{\nu_2}{2}}(Y) - 2XK_{\frac{\nu_1}{2}+1}(X)K_{\frac{\nu_2}{2}}(Y) \\ = \frac{1}{(2\sqrt{-1})^2}\int_{s_1}\int_{s_2} V^2_{(\nu_1,\nu_2)}(s_1,s_2)y_1^{-s_1}y_2^{-s_2}ds_1ds_2$$

where

$$V^2_{(\nu_1,\nu_2)}(s_1,s_2) = \Gamma(s_1,s_2)(-(\nu_1+1)(\nu_2-1)-2e)\cdot {}_3F_2\left(\begin{array}{c} a,b,c \\ d,e \end{array}\middle| 1\right).$$

3.

$$K_{\frac{\nu_1}{2}}(X)\Big(K_{\frac{\nu_2}{2}}(Y) - YK_{\frac{\nu_2}{2}-1}(Y)\Big)\frac{1}{(2\sqrt{-1})^2} \\ \times \int_{s_1}\int_{s_2} V^3_{(\nu_1,\nu_2)}(s_1,s_2)y_1^{-s_1}y_2^{-s_2}ds_1ds_2$$

where

$$V^3_{(\nu_1,\nu_2)}(s_1,s_2) = \Gamma(s_1,s_2+1)\cdot(1-a)\cdot {}_3F_2\left(\begin{array}{c} a+\frac{1}{2},b,c+\frac{1}{2} \\ d,e+\frac{1}{2} \end{array}\middle| 1\right).$$

Here paths s_1, s_2 are the same with those defined in Theorem 3.1.

Proof. 1. Utilizing the formulas

$$\int_I K_\nu(ax)x^s\frac{dx}{x} = 2^{s-2}a^{-s}\Gamma_\pm(s,\nu), \quad \text{for } a>0, \quad \mathrm{Re}(s) > |\mathrm{Re}(\nu)|$$

and

$$\int_{I^2} \frac{x^a y^b (1+x+y)^{-e}}{(1+x)^c (1+y)^d} \frac{dx}{x} \frac{dy}{y}$$
$$= \frac{\Gamma(a)\Gamma(b)\Gamma(c+e-a)\Gamma(d+e-b)}{\Gamma(c+e)\Gamma(d+e)} {}_3F_2 \left(\begin{matrix} a, b, e \\ c+e, d+e \end{matrix} \middle| 1 \right)$$

for $\mathrm{Re}(c+e) > \mathrm{Re}(a) > 0, \mathrm{Re}(d+e) > \mathrm{Re}(b) > 0$ and $\mathrm{Re}(c+d+e-a-b) > 0$, which can be derived from formula (2.2.2) of [1], we get

$$V(s_1, s_2) = 2^{-4}\Gamma(d)\Gamma(c+1)\Gamma(e)\Gamma(b+1)\Gamma(a+\frac{\nu_2}{2})\Gamma(a+1)\Gamma(a-\frac{\nu_1}{2})$$
$$\Gamma(a+\frac{\nu_1}{2})[\Gamma(a+c)\Gamma(a+d+1)]^{-1} {}_3F_2 \left(\begin{matrix} d, c+1, a+\frac{\nu_2}{2} \\ a+c, a+d+1 \end{matrix} \middle| 1 \right).$$

Apply it with this form to Thomae's transformation for the hypergeometric series ${}_3F_2$ (see formula (3.3.6) of [1]), then we get the first expression. Similarly we obtain the other cases. □

By collecting the partial representations of $\tilde{W}_{(\nu_1,\nu_2)}(y_1, y_2; \pm 2)$ in the above lemma, we have

Theorem 4.4. *Let* $V^i_{(\nu_1,\nu_2)}(s_1, s_2)$ *be the function defined above for each* $i \in \{1, 2, 3\}$. *Then we have that* $\tilde{W}_{(\nu_1,\nu_2)}(y_1, y_2; \pm 2)$ *is equal to* 2^{-4} *times*

$$\int_{s_1}\int_{s_2} \Big(V^1_{(\nu_1,\nu_2)}(s_1, s_2) - V^2_{(\nu_1,\nu_2)}(s_1, s_2) \pm 2y_2 V^3_{(\nu_1,\nu_2)}(s_1, s_2)\Big) y_1^{-s_1} y_2^{-s_2} ds_1 ds_2.$$

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