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Frobenius polynomials for Calabi–Yau equations

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We describe a variation of Dwork' s unit-root method to determine the degree 4 Frobenius polynomial for members of a 1-modulus Calabi–Yau family over $\mathbb{P}^1$ in terms of the holomorphic period near a point of maximal unipotent monodromy. The method is illustrated on a couple of examples from the list [3]. For singular points, we find that the Frobenius polynomial splits in a product of two linear factors and a quadratic part $1 - a_pT + p^3T^2$. We identify weight 4 modular forms which reproduce the a_p as Fourier coefficients.

1. Introduction

Given a projective morphism $f : X \longrightarrow \mathbb{P}^1$ with smooth generic $n-1$-dimensional fibre, the sheaf $R^{n-1}f_*(\mathbb{Q}_X)$ restricts to a $\mathbb{Q}$-local system $\mathbb{H}$ over the smooth locus $S \subset \mathbb{P}^1$ of f and hence determines, after the choice of a base-point $s_0 \in S$, a monodromy representation $\pi_1(S, s_0) \longrightarrow \mathrm{Aut}(\mathbb{H}_{s_0})$. The local system $\mathbb{H}$ carries a non-degenerate $(-1)^{n-1}$-symmetric pairing

$$< -, - >: \mathbb{H} \otimes \mathbb{H} \longrightarrow \mathbb{Q}_S$$

induced by the intersection form in the fibres. Hence we can identify $\mathbb{H}$ with its dual and the monodromy representation lands in a symplectic ($n-1$ odd) or orthogonal group ($n-1$ even). The primitive part of $\mathbb{H}$ underlies a variation of Hodge structures (VHS), polarized by $< -, - >$, see [18].

We call a sub-VHS $\mathbb{L} \subset \mathbb{H}$ a CY(n)-*local system* if the local monodromy around $0 \in \mathbb{P}^1 \setminus S$ is unipotent and consists of a single Jordan block of size n. Hence, $\mathbb{L}$ is irreducible of rank n and the non-vanishing subquotients $Gr^W_{2k}(k = 0, 1, \ldots, n-1)$ of the monodromy weight filtration all have dimension equal to one. The Hodge filtration $F^{\cdot}$ of the limiting mixed Hodge structure at 0 is opposite to the weight filtration [9, 17]. If ω is a section of the smallest Hodge space F^{n-1} and γ a local section of $\mathbb{L}$ near 0, then the *period function*

$$f_0 :=< \gamma, \omega >$$

is holomorphic near 0 and satisfies a linear differential equation of order n, called the associated *Picard–Fuchs* equation.

We call a linear differential operator of order n

$$P = \frac{d^n}{dx} + a_{n-1}(x)\frac{d^{n-1}}{dx} + \cdots + a_0(x) \in \mathbb{Q}(x)\left[\frac{d}{dx}\right]$$

a CY(n)-*operator* if

1. P has maximal unipotent monodromy at 0 (MUM).
2. P is self-dual in the sense that

$$P = (-1)^n \exp\left(-\frac{2}{n}\int a_{n-1}\right) \circ P^* \circ \exp\left(\frac{2}{n}\int a_{n-1}\right),$$

   where $\circ$ means the composition of differential operators.
3. P has a convergent power series solution $f_0(x) \in \mathbb{Z}[[x]]$ with $f_0(0) = 1$.

The first condition implies that the operator P is irreducible and can (after left multiplication by x^n) be written in the form

$$\theta^n + xP_1(\theta) + x^2 P_2(\theta) + \cdots + x^d P_d(\theta),$$

where $\theta := x\frac{d}{dx}$ and $P_i(\theta) \in \mathbb{Q}[\theta]$ is a polynomial in θ. We remark that $\exp(\int a_{n-1}) \in \mathbb{Q}(x)$ precisely if the differential Galois group of $\mathcal{P}$ belongs to $\mathrm{SL}(n)$. In the second condition P^* is the formal adjoint of P. The condition is equivalent to the condition that the transformed operator

$$\tilde{P} = \exp\left(\frac{1}{n}\int a_{n-1}\right) \circ P \circ \exp\left(-\frac{1}{n}\int a_{n-1}\right) = \frac{d^n}{dx} + 0\frac{d^{n-1}}{dx} + \cdots$$

satisfies

$$\tilde{P} = (-1)^n \tilde{P}^*$$

which translates into $\lfloor (n-1)/2 \rfloor$ differential-polynomial conditions on the coefficients a_i. These express the conditions that the differential Galois group of P is in the symplectic or orthogonal group. For $n = 4$, one finds the condition of [2]:

$$a_1 = \frac{1}{2}a_2a_3 - \frac{1}{8}a_3^3 + a_2' - \frac{3}{4}a_3a_3 - \frac{1}{2}a_3''$$

If the Yukawa coupling is non-constant, then the differential Galois group is $\mathrm{Gal}(P)^0 = \mathrm{Sp}(4)$ in general, see [6]. In [3], one finds a list with more than 350 examples of such fourth-order operators.

Because of the MUM-condition, the solution $f_0(x)$ from the third condition is unique and conversely determines the operator P. As f_0 is a G-function, the operator P is a G-operator and hence by a theorem of Katz is of fuchsian type with rational exponents, see [4].

A Picard–Fuchs operator that arises from a geometrical situation as sketched above will satisfy the first two conditions and the period function f_0 will be a so-called G-function, see [4]. It would therefore perhaps seem more natural to require f_0 to be a G-function. However, requiring integrality of the solution covers all interesting examples and helps fixing the coordinate x. In [2], for $n = 4$, further integrality properties for the mirror map and Yukawa coupling were required.

CY(2)-operators arise from families of elliptic curves, CY(3)-operators arise from families of $K3$ with Picard-number 19 with a point of maximal degeneration (type III in the terminology of [12]). CY(4)-operators arise from families of Calabi–Yau 3-folds with $h^{12} = 1$ that are studied in mirror symmetry, [7].

Dwork and Bombieri have conjectured conversely that all G-operators come from geometry. So one may ask: is the local system of solutions $\mathrm{Sol}(P)$ of a CY(n)-operator of the form $\mathbb{C} \otimes \mathbb{L}$, where $\mathbb{L}$ is a CY(n)-local system in the above sense? When can one achieve $\mathbb{L} = \mathbb{H}$? If $\mathbb{L} = \mathbb{H}$, can one find a family $f : X \longrightarrow \mathbb{P}^1$ with generic fibre a *Calabi–Yau* $n - 1$-fold?

We refer to [11] for a conjectural approach via mirror symmetry for CY(4)-operators.

Now suppose the whole situation is defined over $\mathbb{Z}$ and consider the reduction of $X \longrightarrow \mathbb{P}^1$ modulo some prime number p. The object $\mathcal{L} \subset R^{n-1}f_*(\mathbb{Q}_l)$ ($l \neq p$ now defines an l-adic sheaf on $\mathbb{P}^1$, lisse (that is, smooth) in some subset S. In particular, for each point $s : \mathrm{Spec}(k) \longrightarrow S$, one has an action of $\mathrm{Gal}(\bar{k}/k)$ on the stalk $\mathcal{L}_s$. Hence one obtains a Frobenius element $\mathrm{Frob}_s \in \mathrm{Aut}(\mathcal{L}_s)$ and

$$P_s(T) := \det(1 - T \cdot \mathrm{Frob}_s) \in \mathbb{Z}[T]$$

determines a factor of the zeta function of the reduction $X_s \mod p$.

To get a computational handle on these Frobenius polynomials, it turns out to be useful to change to a de Rham-type description of the cohomology [14]. It was Dwork who realized early that there is a tight interaction between the Gauss–Manin connection and the Frobenius operator. This leads in general to a relation between periods and the zeta function and in 1958 he gave his famous p-adic analytic formula for the Frobenius polynomial in terms of a solution of the Picard–Fuchs differential equation for the Legendre family of elliptic curves, which we will now review.

The affine part of the Legendre family is given by

$$X_s : y^2 = x(x-1)(x-s),$$

where $s \neq 0, 1$. Over $\mathbb{C}$, the relative de Rham cohomology H^1_{dR} of the family is free of rank 2, and the Hodge filtration $\mathrm{Fil}^1 H^1_{\mathrm{dR}}$ is generated by the differential

$$\omega := \frac{dx}{y}.$$

Let ∇ be the Gauss–Manin connection on H^1_{dR}. Then, ω satisfies the differential equation

$$s(s-1)\omega'' + (1-2s)\omega' - \frac{1}{4}\omega = 0,$$

where $\omega' = \nabla(\omega)$. Let f_0 be the unique solution in $\mathbb{C}[[s]]$ to the above differential equation satisfying $f_0(s) = 1$. f_0 is then given by the hypergeometric series

$$f_0(s) = F\left(\frac{1}{2}, \frac{1}{2}, 1, s\right) = \sum_{j=0}^{\infty} \left(\frac{(1/2)_j}{j!}\right)^2 s^j.$$

Now let $s_0 \in \mathbb{F}_{p^a}$ such that $f_0^{(p-1)/2}(s_0) \neq 0$, where $f_0^{(p-1)/2}$ is the truncation of f_0 up to degree $(p-1)/2$. Let $\hat{s}$ be the Teichmüller lifting of s_0 to $W(\mathbb{F}_{p^a})$. The formal power series

$$h(s) := \frac{f_0(s)}{f_0(s^p)}$$

converges at $\hat{s}$ and can be evaluated there. If $\epsilon = (-1)^{(p-1)/2}$, the element

$$\pi := \epsilon^a f_0(\hat{s}) f_0(\hat{s}^p) \cdots f_0(\hat{s}^{p^{a-1}})$$

is a reciprocal zero of the Frobenius polynomial, and the zeta function of X_{s_0} is given by

$$\zeta(X_{s_0}, T) = \frac{(1-\pi T)(1-p^a/\pi T)}{(1-T)(1-p^a T)}.$$

Thus, Dwork found a way to derive a formula for the Frobenius polynomial, which does only depend (up to ϵ) on the solution of the Picard–Fuchs differential equation. The geometrical origin of ϵ lies in the geometry of the singular fibre X_0, which has a node with tangent cone $x^2 + y^2 = 0$, which splits over $\mathbb{F}_p$ precisely when $\epsilon = 1$.

In this paper, we will consider the following

Question. Given a CY(n)-operator P of $f : X \longrightarrow \mathbb{P}^1$ defined over $\mathbb{Z}$, is there a way to calculate the Frobenius polynomials $P_s(T)$?

We describe a method to solve this problem for $n = 4$ (modulo "ϵ") and illustrate the procedure on some non-trivial examples.

2. Dworks unit-root crystals

We give a short introduction to the theory of Hodge F-crystals, which provides a framework to formalize the interaction between the Gauss–Manin connection and the Frobenius operator. (see [8, 13, 20, 22]).

Let k be a perfect field of characteristic $p > 0$, and let $W(k)$ be the ring of Witt vectors of k. Let A be the ring $W(k)[z][g(z)^{-1}]$, where g is a polynomial in z (which will be specified later according to the actual situation), and let A_n be the ring $A/p^{n+1}A$. By $A_\infty := \varprojlim A/p^{n+1}A$, we denote the p-adic completion of A.

Let σ be the absolute Frobenius on k, given by $\sigma(x) = x^p$. Following [22], we define

Definition 2.1. 1. An F-crystal over $W(k)$ is a free $W(k)$-module H of finite rank with a σ-linear endomorphism

$$F : H \to H$$

such that $F \otimes \mathbb{Q}_p : H \otimes \mathbb{Q}_p \to H \otimes \mathbb{Q}_p$ is an isomorphism. If F itself is an isomorphism, we call H a unit-root F-crystal.

2. A Hodge F-crystal over $W(k)$ is an F-crystal H equipped with a filtration by free $W(k)$-submodules

$$H = \mathrm{Fil}^0 H \supset \mathrm{Fil}^1 H \supset \cdots \supset \mathrm{Fil}^{N-1} H \supset \mathrm{Fil}^N H = 0$$

(called the Hodge filtration on H) which satisfies $F(\mathrm{Fil}^i H) \subset p^i H$ for all i.

The Frobenius automorphism σ on k lifts canonically to an automorphism σ on $W(k)$.

There are different lifts of the Frobenius σ on A_∞, which restrict to σ on $W(k)$ and reduce to the p-th power map modulo p. Let $\phi : A_\infty \to A_\infty$ be such a lift of Frobenius.

Definition 2.2. An F-crystal over A_∞ is a finitely generated free A_∞-module H with an integrable and p-adically nilpotent connection

$$\nabla : H \to \Omega_{A_\infty/W(k)} \otimes_A H$$

such that for every lift $\phi : A_\infty \to A_\infty$ of Frobenius, there exists a homomorphism of A_∞-modules

$$F(\phi) : \phi^* H \to H$$

such that the square

$$\begin{tikzcd} H \arrow[r, "\nabla"] \arrow[d, "F(\phi)\phi^*"'] & \Omega^1_{A_\infty/W(k)} \otimes H \arrow[d, "\phi\otimes F(\phi)\phi^*"] \\ H \arrow[r, "\nabla"] & \Omega^1_{A_\infty/W(k)} \otimes H \end{tikzcd}$$

is commutative, and such that $F(\phi) \otimes \mathbb{Q}_p : \phi^* H \otimes \mathbb{Q}_p \to H \otimes \mathbb{Q}_p$ is an isomorphism. If $F(\phi)$ itself is an isomorphism, we call H a unit-root crystal.

From now on, to simplify the notation, we set $F := F(\phi)\phi^*$.

Definition 2.3. A divisible Hodge F-crystal H is an F-crystal H equipped with a filtration by free A_∞-submodules

$$H = \mathrm{Fil}^0 H \supset \mathrm{Fil}^1 H \supset \cdots \supset \mathrm{Fil}^{N-1} H \supset \mathrm{Fil}^N H$$

(called the Hodge filtration on H) which satisfies

1. $\nabla \mathrm{Fil}^i H \subset \Omega^1_{A_\infty/W(k)} \otimes_{A_\infty} \mathrm{Fil}^{i-1} H$;
2. $F(\mathrm{Fil}^i H) \subset p^i H$.

Proposition 2.1. *Let H be a divisible Hodge F-crystal where $H/\mathrm{Fil}^1 H$ is free of rank 1. Then $\wedge^2 H$ is a divisible Hodge F-crystal, with homomorphism* *of A_∞-modules*

$$\frac{1}{p}\wedge^2 F : \wedge^2 H \to \wedge^2 H$$

and with Hodge filtration given by

$$\mathrm{Fil}^{i-1}(\wedge^2 H) = \sum_{k=0}^{i} \mathrm{Fil}^k H \wedge \mathrm{Fil}^{i-k} H$$

for $i \geq 1$.

Proof. Since $H/\mathrm{Fil}^1 H$ is of rank 1, $\mathrm{Fil}^0 \wedge \mathrm{Fil}^0 = \mathrm{Fil}^0 \wedge \mathrm{Fil}^1$.

Let $a \in \mathrm{Fil}^k H$ and $b \in \mathrm{Fil}^{i-k} H$.Then, $a \wedge b \in \mathrm{Fil}^{i-1}(\wedge^2 H)$ and

$$\frac{1}{p}\wedge^2 F(a \wedge b) = \frac{1}{p} Fa \wedge Fb \in \frac{1}{p} p^k H \wedge p^{i-k} H = p^{i-1} \wedge^2 H.$$

For $i \geq 2$,

$$\nabla(a \wedge b) = \nabla(a) \wedge b + a \wedge \nabla(b) \in \Omega_{A_\infty/W(k)} \otimes_{A_\infty} \mathrm{Fil}^{i-2}(\wedge^2 H).$$

Let k' be a perfect field extension of k and let $e_0 : A_0 \to k'$ be a k-morphism. Let $e_0(z) = \alpha_0$, let α be the Teichmüller lifting of α_0 in $W(k')$ and let $e : A_\infty \to W(k')$ be the $W(k)$-morphism with $e(z) = \alpha$. By H_α, we denote the Teichmüller representative $H_\alpha := H \otimes_{(A_\infty, e)} W(k')$ of the crystal H at the point e_0, which is an F-crystal with corresponding map $F_\alpha := e^* F$. If H is a Hodge F-crystal, then so is H_α.

On $W(k')[[z-\alpha]]$, we put the natural connection ∇ and choose the lift of Frobenius given by $\phi(z) = z^p$. □

Theorem 2.1 ([22, *Theorem* 2.1] *or* [13, *Theorem* 4.1])**.** *Let $\bar{k}$ be the algebraic closure of k, and let H be a divisible Hodge F-crystal over A_∞.*

If $H/\mathrm{Fil}^1 H$ is of rank 1 and if for every k-morphism $e_0 : A_0 \to \bar{k}$ with $e_0(z) = \alpha_0$ and $\alpha \in W(\bar{k})$ a Teichmüller lifting of α_0, H_α contains a direct factor of rank 1, transversal to $\mathrm{Fil}^1 H_\alpha$, which is fixed by the map induced by F on H_α, then there exists a unique unit-root F-subcrystal U of H such that $H = U \oplus \mathrm{Fil}^1 H$ as A_∞-modules.

Suppose that over A_∞, U is locally generated by u. Write $F(u) = r(z)u$ for $r(z) \in A_\infty^*$. Then we have

1. *Let $e_0 : A_0 \to k'$ be a k-morphism to a perfect field extension k' of k with $e_0(z) = \alpha_0$ where u is defined. Let α be the Teichmüller lifting*

of α_0. *Then there exists an* $f_0 \in W(k')[[z-\alpha]]$ *such that* $v := f_0 \cdot u \in W(k')[[z-\alpha]] \otimes_{A_\infty} H$ *is horizontal with regard to* ∇ *and the quotient* f_0/f_0^ϕ *is in fact the expansion of an element in* A_∞.

2. *There exists* $c \in W(\bar{k})$ *such that* $c \cdot v \in W(\bar{k}) \otimes_{W(k)} H$ *is fixed by* F *and* $r(z) = (cf_0)/(cf_0)^\phi$.

The fact that $f_0/f_0^\phi \in A_\infty$ although $f_0 \in W(k)[[z-\alpha]]$ means that f_0/f_0^ϕ is a local expression of a "global" function. Although f_0 itself does only converge in a neighbourhood of α, the global function expressed by the ratio f_0/f_0^ϕ converges at any Teichmüller point in $\mathrm{Spec}(A_\infty)$.

2.1. CY(4)-operators and the corresponding crystals

Now let $\mathcal{P}$ be a CY(4)-operator. We assume that $\mathcal{P}$ is the Picard–Fuchs operator on a rank 4 submodule $H \subset H^3_{\mathrm{dR}}(X/S_\infty)$ for some family $f : X \to S_\infty$ of smooth CY 3-folds.

Let k be the finite field with p^r elements. From now on, we have $S_\infty = \mathrm{Spec}(A_\infty)$, where $A = W(k)[z][(zs(z)g(z))^{-1}]$ for some polynomials $g(z)$ and $s(z)$. We assume that over the roots of $s(z)$, the family becomes singular. We will specify the polynomial $g(z)$ later (see Section 2.4); it will be chosen in a way such that over each Teichmüller point $\alpha \in S_\infty$, the Frobenius polynomial on H_α is of the form

$$P := 1 + aT + bpT^2 + ap^3T^3 + p^6T^4.$$

with four different reciprocal roots

$$r_1, \quad pr_2, \quad \frac{p^2}{r_2}, \quad \frac{p^3}{r_1},$$

where r_1 and r_2 are p-adic units. Hence, giving a formula for the polynomial P is equivalent to giving formulas for the p-adic units r_1 and r_2.

In general, if $f : V \to V$ is a homomorphism of vector spaces, then the eigenvalues of $\wedge^2 f : \wedge^2 V \to \wedge^2 V$ are given by products ab, where a and b are eigenvalues of f corresponding to linearly independent eigenvectors.

Let $\alpha_0 \in S_0$, and let $\alpha \in S_\infty$ be the Teichmüller lifting of α_0. By Proposition 2.1, the Frobenius automorphism on each fibre $\wedge^2 H_\alpha$ of the crystal $\wedge^2 H$ is given by $\frac{1}{p} \wedge^2 F_\alpha$, where F_α is the Frobenius on $H_\alpha \subset H^3_{\mathrm{dR}}(X_\alpha)$. The eigenvalues of the relative Frobenius $(\wedge^2 F_\alpha)^r$ on the fibres $\wedge^2 H_\alpha$ are of the form $a_\alpha b_\alpha / p$, where a_α and b_α are eigenvalues of the relative Frobenius F_α^r

on the corresponding fibre H_α. Thus, if r_1 is the unit root on a fibre H_α, and $\hat{r}_1$ is the unit root on the corresponding fibre $\wedge^2 H_\alpha$, then the roots of the Frobenius polynomial $\det(1 - TF_\alpha^r)$ on H_α are given by

$$r_1, \quad \frac{p\hat{r}_1}{r_1}, \quad \frac{p^2 r_1}{\hat{r}_1}, \quad \frac{p^3}{r_1}. \tag{2.1}$$

We will give p-adic analytic formulas for the unit roots r_1 and $\hat{r}_1$.

2.2. Horizontal sections for CY differential operators of order 4 and 5

Let $\mathcal{P}$ be a CY(4)-operator. The differential equation $\mathcal{P}y = 0$ can be written in the form

$$y^{(4)} + a_3 y^{(3)} + a_2 y^{(2)} + a_1 y^{(1)} + a_0 y = 0,$$

where the coefficients a_i satisfy the following relation:

$$a_1 = \frac{1}{2}a_2 a_3 - \frac{1}{8}a_3^3 + a_2' - \frac{3}{4}a_3 a_3' - \frac{1}{2}a_3''. \tag{2.2}$$

Proposition 2.2 [22]. *Let $\mathcal{P}$ be a* CY(4) *differential operator and let (H, ∇) be a $\mathbb{Q}(z)/\mathbb{Q}$ differential module. Let $\omega \in H$ such that*

$$\nabla^4\omega + a_3\nabla^3\omega + a_2\nabla^2\omega + a_1\nabla\omega + a_0\omega = 0$$

and let $f_0 \in \mathbb{Q}[[z]]$ be a formal solution to the differential equation $\mathcal{P}y = 0$. If $Y := \exp\left(1/2\int a_3\right) \in \mathbb{Q}[[z]]$, then the following element $u_4 \in H \otimes_{\mathbb{Q}[z]} \mathbb{Q}[[z]]$ is horizontal with regard to ∇:

$$\begin{aligned} u_4 = {} & Y[f_0\nabla^3(\omega) - f_0'\nabla^2(\omega) + f_0''\nabla(\omega) - f_0'''\omega] \\ & +(Ya_3 - Y')[f_0\nabla^2(\omega) - f_0''\omega] \\ & +(Ya_2 - (Ya_3)' + Y'')[f_0\nabla(\omega) - f_0'\omega]. \end{aligned} \tag{2.3}$$

Proof. See [22]. The proof is by direct computation, using (2.2). $\square$

Now let $\mathcal{Q}$ be a CY(5)-operator. The differential equation $\mathcal{Q}y = 0$ can be written in the form

$$y^{(5)} + b_4 y^{(4)} + b_3 y^{(3)} + b_2 y^{(2)} + b_1 y^{(1)} + b_0 y = 0.$$

Proposition 2.3. *The operator $\mathcal{Q}$ satisfies the second condition for $CY(5)$ of the introduction if and only if the coefficients $b_i(z)$ satisfy the relations*

$$b_2 = \frac{3}{5}b_3b_4 - \frac{4}{25}b_4^3 + \frac{3}{2}b_3' - \frac{6}{5}b_4b_4' - b_4'' \tag{2.4}$$

and

$$\begin{aligned} b_0 = {} & \frac{1}{2}b_1' - \frac{2}{125}b_3b_4^3 + \frac{1}{5}b_1b_4 - \frac{1}{10}b_3b_4'' + \frac{2}{5}b_4'''b_4 + \frac{4}{5}b_4''b_4' + \frac{16}{125}b_4'b_4^3 \\ & + \frac{12}{25}(b_4')^2b_4 - \frac{3}{10}b_3''b_4 + \frac{8}{25}b_4^2b_4'' - \frac{3}{10}b_3'b_4' - \frac{3}{25}b_4^2b_3' - \frac{1}{4}b_3''' + \frac{16}{3125}b_4^5 \\ & + \frac{1}{5}b_4'''' - \frac{3}{25}b_3b_4'b_4. \end{aligned} \tag{2.5}$$

Proof. By direct calculation, for details we refer to [6]. □

Proposition 2.4. *Let $\mathcal{Q}$ be a $CY(5)$ differential operator and let (H, ∇) be a $\mathbb{Q}(z)/\mathbb{Q}$ differential module. Let $\eta \in H$ such that*

$$\nabla^5\eta + b_4\nabla^4\eta + b_3\nabla^3\eta + b_2\nabla^2\eta + b_1\nabla\eta + b_0\eta = 0$$

and let $F_0 \in \mathbb{Q}[[z]]$ be a formal solution to the differential equation $\mathcal{Q}y = 0$. If $Y := \exp\left(2/5 \int b_4\right) \in \mathbb{Q}[[z]]$, then the following element $u_5 \in H \otimes_{\mathbb{Q}[z]} \mathbb{Q}[[z]]$ is horizontal with regard to ∇ :

$$\begin{aligned} u_5 = {} & Y[F_0\nabla^4(\eta) - F_0'\nabla^3(\eta) + F_0''\nabla^2(\eta) - F_0'''\nabla(\eta) + F_0''''\eta] \\ & + (Yb_4 - Y')\left[F_0\nabla^3(\eta) - \frac{1}{3}F_0'\nabla^2(\eta) - \frac{1}{3}F_0''\nabla(\eta) + F_0'''\eta\right] \\ & + (Yb_3 - (Yb_4)' + Y'')[F_0\nabla^2(\eta) + F_0''\eta] \\ & + \left(\frac{4}{3}((Yb_4)' - Y'') - \alpha b_3\right)F_0'\nabla(\eta) \\ & + \left(\frac{1}{2}((Yb_3)' - \frac{4}{3}((Yb_4)'' - Y'''))\right)[F_0'\eta + F_0\nabla(\eta)] \\ & + \left(Yb_1 - \frac{1}{2}((Yb_3)' - \frac{4}{3}((Yb_4)' - Y''''))\right)F_0\eta, \end{aligned} \tag{2.6}$$

Proof. Applying the identities (2.4) and (2.6), one directly verifies that u_5 satisfies $\nabla(u_5) = 0$. □

2.3. Dwork's congruences

Let p be a prime number. We say that a sequence $(c_n)_{n\in\mathbb{N}}$ satisfies the *Dwork-congruences for* p, if the associated sequence $C(n) := c(n)/c(\lfloor \frac{n}{p} \rfloor) \in \mathbb{Z}_p$ satisfies

$$C(n) \equiv C(n+mp^s) \bmod p^s$$

for all $n, s \in N$ and $m = \{0, 1, \ldots, p-1\}$ and if $c(0) = 1$. We say that the Dwork-congruences hold for a CY(n) differential operator $\mathcal{P}$ if the Dwork-congruences hold for the sequence $(c_n)_{n\in\mathbb{N}}$ of coefficients of the holomorphic solution

$$f_0(z) = \sum_{n=0}^{\infty} c_n z^n$$

to the differential equation $\mathcal{P}y = 0$ around $z = 0$. Dwork shows (see [10, Corollaries 1 and 2]) that hypergeometric-type numbers satisfy these Dwork congruences for all p.

Theorem 2.2. (*see* [10, *Lemma* 3.4]) *Let* $y(z) = \sum_n c_n z^n$ *such that* (c_n) *satisfies the Dwork congruences. Let* $D := \{x \in \mathbb{Z}_p, |y^{(p-1)}(x)| = 1\}$. *Then, for all* $x \in D$,

$$\frac{y(z)}{y(z^p)}|_{z=x} \equiv \frac{y^{(p^s-1)}(x)}{y^{(p^{s-1}-1)}(x^p)} \bmod p^s.$$

This leads to an efficient evaluation of the left hand side at Teichmüller points. (Here $y^{(p^s-1)}(z)$ is the polynomial obtained from $y(z)$ by truncation at z^{p^s}.) This crucial fact was used in all of our computations.

2.4. A formula for the roots of the Frobenius polynomial

Let $\mathcal{P} := \mathcal{P}(\theta, z)$ be a CY(4)-operator, where θ denotes the logarithmic derivative $z\partial/\partial z$.

As before, we assume that $\mathcal{P}$ is the Picard–Fuchs operator on a rank 4 submodule $H \subset H^3_{\mathrm{dR}}(X/S_\infty)$ for a family $f : X \to S_\infty$ of smooth CY 3-folds.

The rank $6 = \binom{4}{2} A_\infty$-module $\wedge^2 H$ is a direct sum of an A_∞-module G of rank 5 and a rank 1 module. The rank 1 module is generated by a section that corresponds to the pairing $< -, - >$ and is horizontal with respect to ∇.

We construct a fifth order differential operator $\mathcal{Q}$ on the submodule G by choosing $\mathcal{Q}$ to be the differential operator of minimal order such that for any two linearly independent solutions $y_1(z), y_2(z)$ of the differential equation $\mathcal{P}y = 0$,

$$w := z \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

is a solution of $\mathcal{Q}w = 0$.

Proposition 2.5. *The operator $\mathcal{Q}$ satisfies the first and the second condition of* CY(5).

Proof. The statement that $\mathcal{Q}$ satisfies the first condition of CY(5) is the content of [2, Proposition 4]. A direct computation shows that since $\mathcal{P}$ is a CY(4)-operator, the coefficients of $\mathcal{Q}$ satisfy Equations (2.6) and (2.4), so the second condition of CY(5) holds. □

In all examples, it was found that the operator $\mathcal{Q}$ also has an integral power series solution, and thus satisfies the third condition of CY(5). For the moment, however, we are unable to prove this is general so we

Conjecture 2.1. *The differential operator $\mathcal{Q}$, constructed from a* CY(4)*-operator $\mathcal{P}$ as above, satisfies the third condition of* CY(5).

So if Conjecture 2.1 holds true, the differential operator $\mathcal{Q}$ is a CY(5)-operator.

$\mathcal{Q}$ can be expressed in terms of $\wedge^2\mathcal{P}(\theta, z)$ as

$$\mathcal{Q}(\theta, z) = \wedge^2\mathcal{P}(\theta - 1, z).$$

For the differential operators $\mathcal{P}$ and $\mathcal{Q}$, we use the same notation with coefficients a_i and b_i as in Section 2.2.

Proposition 2.6. *Let $\mathcal{Q}$ be the* CY(5)*-operator constructed above, and let $\omega \in H$ such that*

$$\nabla^4\omega + a_3\nabla^3\omega + a_2\nabla^2\omega + a_1\nabla\omega + a_0\omega = 0.$$

Then, the element $\eta := z\omega \wedge \nabla\omega \in G$ satisfies

$$\nabla^5\eta + b_4\nabla^4\eta + b_3\nabla^3\eta + b_2\nabla^2\eta + b_1\nabla\eta + b_0\eta = 0.$$

Proof. The proposition follows by a straightforward calculation, applying the relations between the coefficients a_i of the CY(4)-operator $\mathcal{P}$ and the coefficients b_i of the CY(5)-operator $\mathcal{Q}$ listed in [1].

It still remains to point out how to choose the polynomial $g(z)$ in the definition of the ring $W(k)[z][(zs(z)g(z))^{-1}]$ to obtain divisible Hodge F-crystals $H \subset H^3_{\mathrm{dR}}(X/S_\infty)$ and $G \subset \wedge^2 H$ which satisfy the conditions of Theorem 2.1. $\square$

The following conjecture was crucial for the choice of the polynomial $g(z)$:

Conjecture 2.2.

1. *Let f_0 be the solution of the differential equation $\mathcal{P}y = 0$ around $z = 0$ with $f_0(0) = 1$. If the coefficients c_n in the expansion*

$$f_0(z) = \sum_{n=0}^{\infty} c_n z^n$$

satisfy the Dwork congruences, then H satisfies the conditions of Theorem (2.1) if the polynomial $g(z)$ in the definition of A_∞ is chosen as $g(z) := f_0^{(p-1)}(z)$.

2. *Let $F_0(z)$ be the solution of the differential equation $\mathcal{Q}y = 0$ around $z = 0$ with $F_0(0) = 1$. If the coefficients d_n in the expansion*

$$F_0(z) = \sum_{n=0}^{\infty} d_n z^n$$

satisfy the Dwork congruences, then the sub-F-crystal $G \subset \wedge^2 H$ satisfies the conditions of Theorem (2.1) if the polynomial $g(z)$ in the definition of A_∞ is chosen as $g(z) := F_0^{(p-1)}(z)$.

According to the conjecture, it seems to be the right thing to choose $g(z) = f_0^{(p-1)}(z) F_0^{(p-1)}(z)$. So from now on, we fix the ring A_∞ by

$$A := W(k)[z][(zs(z) f_0^{(p-1)}(z) F_0^{(p-1)}(z))^{-1}].$$

This choice was confirmed by our numerous computations; for each parameter value $z = \alpha$ with $f_0^{(p-1)}(\alpha_0) \neq 0 \bmod p$ and $F_0^{(p-1)}(\alpha_0) \neq 0 \bmod p$, in the examples we considered, we were able to compute the Frobenius polynomial explicitly.

For each pair of CY(4) and CY(5) operators we treat in this paper, the functions

$$Y_4 = \exp\left(\frac{1}{2}\int a_3\right) \quad \text{and} \quad Y_5 = \exp\left(\frac{2}{5}\int b_4\right)$$

satisfy $Y_4 \in \mathbb{Q}(z)$ and $Y_5 \in \mathbb{Q}(z)$. Thus, in each of the examples we considered, the following proposition holds:

Proposition 2.7. *Let*

$$r(z) = \frac{f_0(z)}{f_0(z^p)} \quad \text{and} \quad \hat{r}(z) = \frac{F_0(z)}{F_0(z^p)}.$$

Assuming that Conjecture 2.2 holds, if $|\alpha s(\alpha) f_0^{(p-1)}(\alpha) F_0^{(p-1)}(\alpha)| = 1$, there exist constants ϵ_4 and $\epsilon_5 \in W(\bar{k})$ such that the p-adic units $r_1(\alpha)$ and $\hat{r}_1(\alpha)$ determining the Frobenius polynomial on $H_\alpha \subset H^3_{\mathrm{dR}}(X_\alpha)$ are given by

$$r_1(\alpha) = (\epsilon_4^{(1-\sigma)})^{1+\cdots+\sigma^{r-1}} r(\alpha) r(\alpha^p) \cdots r(\alpha^{p^{r-1}})$$

and

$$\hat{r_1}(\alpha) = (\epsilon_5^{(1-\sigma)})^{1+\sigma+\cdots+\sigma^{r-1}} \hat{r}(\alpha) \hat{r}(\alpha^p) \cdots \hat{r}(\alpha^{p^{r-1}}).$$

If we assume furthermore that

$$(\epsilon_4^{1-\sigma})^{1+\sigma+\cdots+\sigma^{r-1}} = (\epsilon_5^{1-\sigma})^{1+\sigma+\cdots+\sigma^{r-1}} = 1,$$

the p-adic units are given by

$$r_1(\alpha) = r(\alpha) r(\alpha^p) \cdots r(\alpha^{p^{r-1}})$$

and

$$\hat{r_1}(\alpha) = \hat{r}(\alpha) \hat{r}(\alpha^p) \cdots \hat{r}(\alpha^{p^{r-1}}).$$

Proof. There exists an $\omega \in H$ such that the horizontal section w.r.t. ∇ is given by formula (2.3), while on G, it is given by formula (2.6), where $\eta = z\omega \wedge \nabla\omega$ by Proposition 2.6. These sections u_4 and u_5 play the role of the section $v = f \cdot u$ in Theorem 2.1. Hence, the section u in the theorem is given by $(f_0 Y_4)^{-1} u_4$ and $(F_0 Y_5)^{-1} u_5$, respectively, where $Y_4 = \exp\left(1/2 \int a_3\right) \in \mathbb{Q}(z)$ and $Y_5 = \exp\left(2/5 \int b_4\right) \in \mathbb{Q}(z)$. Since

$$\left(\frac{Y_4(z)}{Y_4(z^p)}|_{z=\alpha}\right)^{1+\sigma+\cdots+\sigma^{r-1}} = 1 \quad \text{and} \quad \left(\frac{Y_5(z)}{Y_5(z^p)}|_{z=\alpha}\right)^{1+\sigma+\cdots+\sigma^{r-1}} = 1,$$

by Theorem 2.1 there exist constants ϵ_4 and $\epsilon_5 \in W(\bar{k})$ (where $\bar{k}$ denotes the algebraic closure of $\mathbb{F}_p$) such that

$$r_1(\alpha) = (\epsilon_4^{(1-\sigma)})^{1+\cdots+\sigma^{r-1}} r(\alpha) r(\alpha^p) \cdots r(\alpha^{p^{r-1}})$$

and

$$\hat{r_1}(\alpha) = (\epsilon_5^{(1-\sigma)})^{1+\sigma+\cdots+\sigma^{r-1}} \hat{r}(\alpha)\hat{r}(\alpha^p)\cdots\hat{r}(\alpha^{p^{r-1}}).$$

Now we assume that the constants satisfy

$$(\epsilon_4^{1-\sigma})^{1+\sigma+\cdots+\sigma^{r-1}} = (\epsilon_5^{1-\sigma})^{1+\sigma+\cdots+\sigma^{r-1}} = 1. \tag{2.7}$$

□

Then, the proposition follows.

3. Some special Picard–Fuchs equations

We will apply the method explained in the previous section to compute Frobenius polynomials for some special fourth-order operators. These operators belong to the list [3]. A typical example is operator 45 from that list:

$$\theta^4 - 4\,x\,(2\,\theta+1)^2\left(7\,\theta^2+7\,\theta+2\right) - 128\,x^2\,(2\,\theta+1)^2\,(2\,\theta+3)^2\,.$$

This operator is a so-called *Hadamard product* of two second-order operators.

3.1. Hadamard products

The *Hadamard product* of two power series $f(x) := \sum_n a_n x^n$ and $g(x) = \sum_n b_n x^n$ is the power series defined by the coefficient-wise product:

$$f * g(x) := \sum_n a_n b_n x^n.$$

It is a classical theorem, due to Hurwitz, that if f and g satisfy linear differential equations $\mathcal{P}$ and $\mathcal{Q}$, resp., then $f * g$ satisfies a linear differential equation $\mathcal{P} * \mathcal{Q}$. Only in very special cases, the Hadamard product of two CY-operators will again be CY, but it is a general fact that if f and g satisfy differential equations of *geometrical origin*, then so does $f * g$. For a proof, we refer to [4]. Here we sketch the idea. The multiplication map

$$m : \mathbb{C}^* \times \mathbb{C}^* \longrightarrow \mathbb{C}^*, (s,t) \mapsto s \cdot t$$

can be compactified to a map

$$\mu : \widetilde{\mathbb{P}^1 \times \mathbb{P}^1} \longrightarrow \mathbb{P}^1$$

by blowing up the two points $(0,\infty)$ and $(\infty,0)$ of $\mathbb{P}^1\times\mathbb{P}^1$. Given two families $X\to\mathbb{P}^1$ and $Y\to\mathbb{P}^1$ over $\mathbb{P}^1$, we define a new family $X*Y\to\mathbb{P}^1$, as follows. The cartesian product $X\times Y$ maps to $\mathbb{P}^1\times\mathbb{P}^1$ and can be pulled back to $X*Y$ over $\mathbb{P}^1\times\mathbb{P}^1$. Via the map μ, we obtain a family over $\mathbb{P}^1$. If n resp. m is the fibre dimension of $X\to\mathbb{P}^1$ resp. $Y\to\mathbb{P}^1$, then $X*Y\to\mathbb{P}^1$ has fibre dimension $n+m+1$. The local system H^{n+m+1} of $X*Y\to\mathbb{P}^1$ contains the *convolution* of the local systems of $X\to\mathbb{P}^1$ and $Y\to\mathbb{P}^1$. Note that the critical points of $X*Y\to\mathbb{P}^1$ are, apart from 0 and ∞, the products of the critical values of the factors. In down-to-earth terms, if $X\to\mathbb{P}^1$ and $Y\to\mathbb{P}^1$ are defined by say Laurent polynomials $F(x)$ and $G(y)$ resp., then the fibre of $X*Y\to\mathbb{P}^1$ over u is defined by the equations

$$F(x)=s,\quad G(y)=t,\quad s\cdot t=u.$$

If the period functions for $X\longrightarrow\mathbb{P}^1$ and $Y\longrightarrow\mathbb{P}^1$ are represented as

$$\begin{aligned} f(s)&=\int_\gamma \operatorname{Res}\left(\frac{\omega}{F(x)-s}\right)=\sum_n a_n s^n,\\ g(t)&=\int_\delta \operatorname{Res}\left(\frac{\eta}{G(y)-t}\right)=\sum_m b_m t^m,\end{aligned}$$

then

$$\begin{aligned}\int_{\gamma\times\delta\times S^1}\frac{\omega\wedge\eta\wedge ds\wedge dt}{(F(x)-s)(G(y)-t)(st-u)}&=\int_{S^1}\sum a_n s^n b_m t^m\frac{du}{u}\\ &=\sum a_n b_n u^n=f(u)*g(u)\end{aligned}$$

is a period of $X*Y\longrightarrow\mathbb{P}^1$.

For example, if we apply this construction to the rational elliptic surfaces $X=Y$ with singular fibres of Kodaira type I_9 over 0 and I_1 over ∞ and two further fibres of type I_1, we obtain a family $X*Y\longrightarrow\mathbb{P}^1$, with generic fibre a Calabi–Yau 3-fold with $h^{12}=1$ and $\chi=164$.

3.2. Some special CY(2)-operators

We will use Hadamard products of some very special CY(2)-operators appearing in [2] from which we also take the names. These operators all are associated to *extremal rational elliptic surfaces* $X\longrightarrow\mathbb{P}^1$ with non-constant j-function. Such a surface has three or four singular fibres [16]. The six cases with three singular fibres fall into four isogeny classes and each of these gives rise to a Picard–Fuchs operator of hypergeometric type (named A, B, C, D) and one obtained by performing a Möbius transformation interchanging ∞ with the singular point $\neq 0$ (named e, h, i, j).

Name	Operator	a_n
A	$\theta^2 - 4z(2\theta+1)^2$	$(2n)!^2/n!^4$
B	$\theta^2 - 3z(3\theta+1)(3\theta+2)$	$(3n)!/n!^3$
C	$\theta^2 - 4z(4\theta+1)(4\theta+3)$	$(4n)!/(2n)!n!^2$
D	$\theta^2 - 12z(6\theta+1)(6\theta+5)$	$(6n)!/(3n)!(2n)!n!$

Name	Operator	a_n
e	$\theta^2 - z(32\theta^2+32\theta+12)$ $+256z^2(\theta+1)^2$	$16^n \sum_k (-1)^k \binom{-1/2}{k}\binom{-1/2}{n-k}^2$
h	$\theta^2 - z(54\theta^2+54\theta+21)$ $+729z^2(\theta+1)^2$	$27^n \sum_k (-1)^k \binom{-2/3}{k}\binom{-1/3}{n-k}^2$
i	$\theta^2 - z(128\theta^2+128\theta+52)$ $+4096z^2(\theta+1)^2$	$64^n \sum_k (-1)^k \binom{-3/4}{k}\binom{-1/4}{n-k}^2$
j	$\theta^2 - z(864\theta^2+864\theta+372)$ $+18664z^2(\theta+1)^2$	$432^n \sum_k (-1)^k \binom{-5/6}{k}\binom{-1/6}{n-k}^2$

The six cases with four singular fibres are the Beauville surfaces [5] and also form four isogeny classes and lead to the six Zagier operators, called (a, b, c, d, f, g).

These are also of the form

$$\theta^2 - z(a\theta^2 + a\theta + b) - cz^2(\theta+1)^2$$

but now the discriminant $1 - az - cz^2$ is not a square, so the operator has four singular points.

Name	Operator	a_n
a	$\theta^2 - z(7\theta^2+7\theta+2) - 8z^2(\theta+1)^2$	$\sum_k \binom{n}{k}^3$
c	$\theta^2 - z(10\theta^2+10\theta+3) + 9z^2(\theta+1)^2$	$\sum_k \binom{n}{k}^2\binom{2k}{k}$
g	$\theta^2 - z(17\theta^2+17\theta+6) + 72z^2(\theta+1)^2$	$\sum_{i,j} 8^{n-i}(-1)^i\binom{n}{i}\binom{i}{j}^3$
d	$\theta^2 - z(12\theta^2+12\theta+4) + 32z^2(\theta+1)^2$	$\sum_k \binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k}$
f	$\theta^2 - z(9\theta^2+9\theta+3) + 27z^2(\theta+1)^2$	$\sum_k (-1)^k 3^{n-3k}$ $\binom{n}{3k}((3k)!/k!^3)$
b	$\theta^2 - z(11\theta^2+11\theta+3) - z^2(\theta+1)^2$	$\sum_k \binom{n}{k}^2\binom{n+k}{n}$

The ten products $A * A$, etc. form 10 of the 14 hypergeometric families from [3]. The 16 products $A * e$, etc. are not hypergeometric, but also have three singular fibres. The 24 operators $A * a$, etc. have, apart from 0 and ∞, two further singular fibres. The operators $a * a$, etc. have four singular fibres apart from 0 and ∞.

Observations

1) The Dwork congruences hold for the operators $a, b, \ldots, j$. For the Apery sequence (case b) this was also conjectured in [22]. (It follows from [10] that A, B, C, D satisfy the Dwork congruences). It follows that the Dwork congruences hold for all fourth-order Hadamard products within this group.

2) For the hypergeometric cases $A * A$, etc. and the cases $A * a$, etc. the Dwork congruences also hold for the associated fifth-order operator, although even for the simplest examples like the quintic 3-fold, this is not at all obvious. In the case of the quintic, the holomorphic solution around $z = 0$ to the fifth-order differential equation is given by the formula $F_0(z) = \sum_{n=0}^{\infty} A_n z^n$, where

$$A_n := \sum_{k=0}^{n} \frac{(5k)!}{k!^5} \frac{5(n-k)!}{(n-k)!^5} (1 + k(-5H_k + 5H_{n-k} + 5H_{5k} - 5H_{5(n-k)})),$$

   and H_k is the harmonic number $H_k = \sum_{j=1}^{k} \frac{1}{j}$. Thus, by the formula, it is not even obvious that the coefficients A_n are integers.

3) In fact, the Dwork congruences hold for *almost all* fourth-order operators from the list [3]. It is an interesting problem to try to prove these experimental facts. On the other hand, it is clear that they cannot hold in general for differential operators of geometrical origin: if we multiply f_0 with a rational function of x, we obtain a (much more complicated) CY-operator for which the congruences in general will not hold.

3.3. Computations

In the hypergeometric cases, we reproduced results obtained in [21]. In the appendix of [19], the results of our calculations on the 24 operators which are Hadamard products like $A * a$, etc. are collected. We computed coefficients (a, b) of the Frobenius polynomial

$$P(T) = 1 + aT + bpT^2 + ap^3T^3 + p^6T^4$$

for all primes p between 3 and 17 and for all possible values of $z \in \mathbb{F}_p^*$. In our computations, we assumed that Conjecture 2.2 holds true and took the constants (2.7) appearing in the formula for the unit root to be one. To generate the tables of coefficients in [19], we used the programming language MAGMA. We computed with an overall p-adic accuracy of 500 digits. This was necessary, since in the computation of the power series solutions to the differential equations $\mathcal{P}y = 0$ and $\mathcal{Q}y = 0$, denominators divisible by large powers of p occurred during the calculations (although the solutions themselves have integral coefficients). The occurrence of large denominators reduces the p-adic accuracy in MAGMA, and thus we had to compute with such a high overall accuracy to obtain correct results in the end. For the unit roots themselves, we computed the ratio

$$\frac{f_0(z)^{(p^3-1)}}{f_0(z^p)^{(p^2-1)}}|_{z=\alpha} \bmod p^3$$

with p-adic accuracy modulo p^3. We checked our results for the tuples (a, b) determined the absolute values of the complex roots of the Frobenius polynomial, which by the Weil conjectures should have absolute value $p^{-3/2}$. Needless to say, this was always fulfilled.

3.4. Example

In this section, we describe the computational steps we performed in MAGMA for one specific example. We consider the operator $A * a$, which is nr. 45 from the list [3].

We compute the Frobenius polynomial for $p = 7$ and $\alpha_0 = 2 \in \mathbb{F}_7$ with 4 digits of 7-adic precision, i.e., modulo 7^4. Since $2 \neq -\frac{1}{16}$ and $2 \neq \frac{1}{128}$ in $\mathbb{F}_7, \alpha_0$ is not a singular point of the differential equation.

First of all, we computed the truncated power series solution $f_0^{(p^{s+1}-1)}(z)$ to the differential equation

$$\mathcal{P}y = 0,$$

and obtained

$$f_0^{(7^4-1)}(z) = 1 + 8z + 360z^2 + 22400z^3 + 1695400z^4 + 143011008z^5 + \cdots .$$

Thus, $f_0^{(7-1)}(\alpha_0) = 1 \in \mathbb{F}_7$ is non-zero. Let $\alpha^{(4)}$ be the Teichmüller lifting of α_0 with 7-adic accuracy of four digits. Evaluating f_0 in this point, we obtain

$$f_0^{(7^4-1)}(\alpha^{(4)}) \equiv 1709 \bmod 7^4$$

and

$$f_0^{(7^3-1)}((\alpha^{(4)})^7) \equiv 1814 \bmod 7^4.$$

Thus, the unit root of the Frobenius polynomial is

$$r^4 := \frac{f_0^{(7^4-1)}(\alpha^{(4)})}{f_0^{(7^3-1)}((\alpha^{(4)})^7)} \equiv 582 \bmod 7^4.$$

To compute the second root of the Frobenius polynomial, we compute the truncated power series solution $F_0^{(7^4-1)}(z)$ of the fifth-order differential equation

$$\mathcal{Q}y = 0,$$

where $\mathcal{Q}$ is the second exterior power of the differential operator $\mathcal{P}$, given by

$$\begin{aligned}\mathcal{Q} = \theta^5 &- z(44 + 260\theta + 628\theta^2 + 792\theta^3 + 560\theta^4 + 224\theta^5)\\ &+ z^2(-6512 + 400\theta + 44160\theta^2 + 71040\theta^3 + 42240\theta^4 + 8448\theta^5)\\ &+ z^3(4177920 + 131809280\theta + 165888000\theta^2 + 105676800\theta^3\\ &+ 34406400\theta^4 + 4587520\theta^5)\\ &+ z^4 2^{22}(\theta+1)(\theta+2)^3(\theta+3).\end{aligned}$$

The solution is given by

$$F_0^{(7^4-1)} = 1 + 44z + 3652z^2 + 337712z^3 + 33909700z^4 + 3567877424z^5 + \cdots,$$

$F_0^{(7-1)}(\alpha_0) = 2 \in \mathbb{F}_7$ is non-zero and we compute

$$F_0^{(7^4-1)}(\alpha^{(4)}) \equiv 51 \bmod 7^4$$

and

$$F_0^{(7^3-1)}((\alpha^{(4)})^7) \equiv 1387 \bmod 7^4.$$

Thus,

$$\hat{r}^4 := \frac{F_0^{(7^4-1)}(\alpha^{(4)})}{F_0^{(7^3-1)}((\alpha^{(4)})^7)} \equiv 1101 \bmod 7^4.$$

Since the Frobenius polynomial (with 7-adic accuracy 4) is given by

$$P(T) = (1 - r^4 T)\left(1 - \frac{7\hat{r}^4}{r^4 T}\right)\left(1 - \frac{7^2 r^4}{\hat{r}^4 T}\right)\left(1 - \frac{7^3}{r^4 T}\right),$$

we finally obtain

$$P(T) = 7^6 T^4 - 7^3 \cdot 8T^3 + 7 \cdot 2T^2 - 8T + 1.$$

As expected, the complex roots of P do have complex absolute value $7^{-3/2}$.

Exemplarily, we now list all values (a, b) we computed for the differential operator $A * a$. If there occurs a "–" in the table instead of the tuple (a, b), then the corresponding $z \in \mathbb{F}_p$ is either a zero of $f_0^{(p-1)}$ or $F_0^{(p-1)}$ or of both, where f_0 was the power series solution of the fourth-order differential equation and F_0 was the solution of the fifth-order equation. The appearance of $(a, b)'$ means that the polynomial is *reducible*. The appearance of $(a, b)^*$ means that the corresponding z is a singular point of the differential equation.

$p = 3$

z	1	2
	–	–

$p = 5$

z	1	2	3	4
	$(6, -6)'$	$(28, 38)^*$	–	$(32, 62)^*$

$p = 7$

z	1	2	3	4	5	6
	$(2, -46)$	$(-8, 2)$	$(32, -94)^*$	$(80, 290)^*$	$(10, 50)'$	–

$p = 11$

z	1	2	3	4	5	6	7	8
	$(56, 290)'$	–	$(-16, 2)'$	$(6, 26)$	$(16, 98)$	$(12, 114)'$	$(26, 106)$	–

z	9	10
	$(-8, 2)$	$(-36, 210)'$

$p = 13$

z	1	2	3	4	5
	$(-8, 270)'$	$(20, -106)$	$(-4, 86)$	$(-204, 646)^*$	$(22, -30)$

z	6	7	8	9	10
	$(-160, 30)^*$	$(-34, 50)$	$(-16, 302)$	$(58, 146)$	$(18, 34)$

z	11	12
	$(84, 406)$	$(56, 206)'$

$p = 17$

z	1	2	3	4	5
	$(256, -322)^*$	$(256, -322)^*$	$(-24, 542)$	(44, 166)	(210, 1218)

z	6	7	8	9	10
	$(24, -178)'$	$(-100, 278)$	(22, 50)	$(-4, 70)$	(52,470)

z	11	12	13	14	15	16
	$(-84, 342)'$	–	$(22, -334)'$	(18, 258)	(184, 974)	$(-56, 302)'$

3.5. Modular forms of weight 4

In some cases, the so chosen accuracy was too low, and we had to compute mod p^4. This happened in case the parameter $\alpha_0 \in \mathbb{F}_p$ was a critical point of the differential equation. But it is somewhat of a miracle that our calculation made sense at the critical points at all. In order to understand what is supposed to happen at a singular point, recall that if the fibre X_s of a family $X \longrightarrow \mathbb{P}^1$ over $s \in \mathbb{P}^1(\mathbb{Q})$ acquires an ordinary double point, then the Frobenius polynomial should factor as

$$P(T) = (1 - \chi(p)T)(1 - p\chi(p)T)(1 - a_pT + p^3T^2)$$

for some character χ. The factor $(1 - a_pT + p^3T^2)$ is the Frobenius polynomial on the 2-dimensional pure part of H^3. This part can be identified with the H^3 of a small resolution $\tilde{X_s}$, which then is a rigid Calabi–Yau 3-fold. According to the modularity conjecture for such Calabi–Yau 3-folds, the coefficients a_p are Fourier coefficients of a weight 4 modular form for some congruence subgroup $\Gamma_0(N)$ [15].

This is exactly the phenomenon that occurs at the singular points of our differential equations. For the hypergeometric cases, we refind the results of [21]. For 16 of the 24 operators $A * a$, etc., we have two rational critical values. In 31 of the cases, we are able to identify the modular form.

We use the notation of modular forms as in [15]: the notation a/b means the bth Hecke eigenform of level a. “Twist of” means the modular forms differ by character. We remark that the critical points of the operators are reciprocal integers and the level of the corresponding modular form divides that integer. For the cases involving the operator c, one usually has equality and so the modular form for $D * c$ presumably has level 3888, which was outside the range of our table. Remark that all levels appearing only involve primes 2 and 3.

Case	Point	Form	Twist of	Point	Form	Twist of
A ∗ a	−1/16	8/1	–	1/128	64/5	8/1
B ∗ a	−1/274	27/2	27/1	1/126	54/2	–
C ∗ a	−1/64	32/3	32/2	1/512	256/3	–
D ∗ a	−1/432	216/4	216/2	1/3456	1728/16	216/1
A ∗ c	1/144	48/1	24/1	1/16	16/1	8/1
B ∗ c	1/243	243/1	–	1/27	27/1	–
C ∗ c	1/576	576/3	94/4	1/64	64/3	32/2
D ∗ c	1/3888		1944/5	1/432	432/9	216/2
A ∗ d	1/128	64/4	32/1	1/64	32/2	–
B ∗ d	1/216	9/1	–	1/108	108/4	108/2
C ∗ d	1/512	256/1	–	1/256	128/4	128/1
D ∗ d	1/3456	576/8	288/1	1/1728	864/3	864/1
A ∗ g	1/144	24/1	–	1/128	64/1	8/1
B ∗ g	1/243	243/2	243/1	1/216	54/4	54/2
C ∗ g	1/576	288/10	96/4	1/512	256/4	256/3
D ∗ g	1/3888	1944/6	1944/5	1/3456	1728/15	–

The simplest modular forms appearing are the well-known η-products $8/1 = \eta(q^2)^4\eta(q^4)^4$, $9/1 = \eta(q^3)^8$.

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