

Birational Calabi–Yau threefolds and BPS state counting

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This paper contains some applications of Bridgeland–Douglas stability conditions on triangulated categories, and Joyce’s work on counting invariants of semistable objects, to the study of birational geometry. We introduce the notion of motivic Gopakumar–Vafa invariants as counting invariants of D2-branes, and show that they are invariant under birational transformations between Calabi–Yau threefolds. The result is similar to the fact that birational Calabi–Yau threefolds have the same betti numbers or Hodge numbers.

1. Introduction

First of all, let us recall the following well-known fact.

Theorem 1.1 [28]. *Let $\phi: W \dashrightarrow X$ be a birational map between smooth projective Calabi–Yau threefolds. Then we have*

$$b_i(W) = b_i(X) \quad \text{for all } i \in \mathbb{Z}.$$

Here $b_i(*)$ is the i -th betti number of $*$.

This result has been generalized for all dimensions by [1, 47] using the method of *p-adic integration*. Later on the equality of Hodge numbers $h^{p,q}(X)$ (more generally stringy Hodge numbers $h_{st}^{p,q}(X)$ for varieties with log terminal singularities) under birational maps has been proved in [2, 10, 32] using the method of *motivic integration*. (Also see [19, 48] for related works.)

In terms of string theory, the numbers $b_i(X)$, $h^{p,q}(X)$ are interpreted as “counting invariants” of BPS D0-branes, which are mathematically stable zero-dimensional sheaves $\{\mathcal{O}_x\}_{x \in X}$. In this paper, we shall address the following question.

Question 1.2. *How do counting invariants of BPS D2-branes (i.e., stable one-dimensional sheaves) transform under birational transformations?*

A similar problem has been studied in [31, 33, 35, 36] for Gromov–Witten invariants and in [17, 40] for Donaldson–Thomas invariants. In this paper, we are interested in Question 1.2 for Gopakumar–Vafa invariants, which were introduced by physicists Gopakumar and Vafa [13]. In [16], Hosono et al. proposed a first mathematical formulation of them. The purpose of this paper is to introduce another mathematical formulation which we call *motivic Gopakumar–Vafa invariants* and study their behavior under birational transformations. Our method is quite different from the works cited before, and uses *Bridgeland–Douglas stability conditions on triangulated categories* [7, 11, 12], and *Joyce’s counting invariants of semistable objects* [20–23].

1.1. Gopakumar–Vafa invariants

Let X be a Calabi–Yau threefold over \mathbb{C} , $\beta \in N_1(X)$ and $g \in \mathbb{Z}_{\geq 0}$, where $N_1(X)$ is the \mathbb{R} -vector space of numerical classes of one cycles. The 0-point genus g Gromov–Witten invariant of X in numerical class β is defined as the integration over the virtual fundamental class of the moduli space of stable maps $\overline{\mathcal{M}}_{g,0}(X, \beta)$,

$$N_g^\beta := \int_{\overline{\mathcal{M}}_{g,0}^{\text{virt}}(X, \beta)} 1_{\overline{\mathcal{M}}_{g,0}(X, \beta)} \in \mathbb{Q}.$$

Although the invariants N_g^β are not integers in general, Gopakumar and Vafa [13] claimed the following integrality of the generating function involving N_g^β , based on the string duality between Type IIA and M-theory.

Conjecture 1.3. *There are integers $n_g^{\beta, \text{conj}}$ and the following equality in $\mathbb{Q}[[N_1(X), \lambda^{\pm 1}]]$,*

$$\sum_{\beta, g} N_g^\beta q^\beta \lambda^{2g-2} = \sum_{\beta, g, k} \frac{n_g^{\beta, \text{conj}}}{k} \left(2 \sin \left(\frac{k\lambda}{2} \right) \right)^{2g-2} q^{k\beta}.$$

They also claimed that the integer $n_g^{\beta, \text{conj}}$ should be defined by the “virtual counting of genus g Jacobians” in the moduli space of the BPS-branes wrapping around holomorphic curves in X , and some computations are done in [26]. Its mathematical proposal by Hosono et al. [16] uses the relative Lefschetz action on the intersection cohomology of the moduli space

of one-dimensional semistable sheaves E with

$$(1.1) \quad (\mathrm{ch}_2(E), \mathrm{ch}_3(E)) = (\beta, 1),$$

which we denote by M^β . Using an $\mathfrak{sl}_2 \times \mathfrak{sl}_2$ -action on $IH^*(\widetilde{M}^\beta)$ where $\widetilde{M}^\beta \rightarrow M^\beta$ is the normalization, they defined an invariant $\tilde{n}_g^\beta \in \mathbb{Z}$ and conjectured that the invariants \tilde{n}_g^β satisfy Conjecture 1.3.

However it seems that the invariants \tilde{n}_g^β are unlikely to be deformation invariant, hence not equal to $n_g^{\beta, \mathrm{cong}}$ exactly, since the definition of \tilde{n}_g^β does not involve virtual classes. Now there is another approach of Gopakumar–Vafa invariants using the notion of stable pairs, proposed by Pandharipande and Thomas [38].

1.2. Main result

Based on the work [16], we will construct invariants (cf. Definition 4.25)

$$n_g^\beta(X) \in \mathbb{Z}, \quad \text{for } g \geq 0, \beta \in N_1(X),$$

as a refinement of \tilde{n}_g^β . Roughly speaking $n_g^\beta(X)$ is defined using a certain motivic invariant, similar to the virtual Poincaré polynomial of M^β . Furthermore $n_g^\beta(X)$ is also defined for a non-effective one cycle class β . At least $n_g^\beta(X)$ coincides with \tilde{n}_g^β if the moduli space M^β is smooth and β is represented by an effective one cycle. The following is our main theorem.

Theorem 1.4. *Let $\phi: W \dashrightarrow X$ be a birational map between smooth projective Calabi–Yau threefolds. Then we have*

$$n_g^\beta(W) = n_g^{\phi_*\beta}(X),$$

for all $g \geq 0$ and $\beta \in N_1(W)$.

It is worth mentioning that in the proof of Theorem 1.4, putting $\beta = 0$ would result Theorem 1.1 (cf. Remark 5.7). The definition of n_g^β also does not involve virtual classes, so we do not claim that n_g^β satisfy Conjecture 1.3. However we have obtained a certain mathematical approximation of Gopakumar–Vafa invariants, which have birational invariance property.

1.3. Strategy of the proof of Theorem 1.4

We use the notion of stability conditions on triangulated categories introduced by T. Bridgeland [7], based on M. Douglas’s work on Π -stability

[11, 12]. Roughly speaking a stability condition on a triangulated category \mathcal{D} consists of data

$$Z: K(\mathcal{D}) \longrightarrow \mathbb{C}, \quad \mathcal{P}(\phi) \subset \mathcal{D},$$

where Z is a group homomorphism and $\mathcal{P}(\phi)$ is a subcategory for each $\phi \in \mathbb{R}$, which satisfy some axioms (Definition 3.1). We work over the triangulated category $\mathcal{D} = \mathcal{D}_X$ defined by

$$\mathcal{D}_X := \{E \in D^b(\text{Coh}(X)) \mid \dim \text{Supp}(E) \leq 1\}.$$

In terms of string theory, the set of objects $\{\mathcal{P}(\phi) \mid \phi \in \mathbb{R}\}$ is supposed to represent the set of BPS-branes at some point of the so-called *stringy Kähler moduli space*, the subspace of the moduli space of $\mathcal{N} = 2$ super conformal field theories. Indeed Bridgeland [7] showed that the set of good stability conditions $\text{Stab}(X)$ is a complex manifold, and expected that it describes the stringy Kähler moduli space mathematically. In this paper, we will construct a connected open subset (cf. Lemma 3.6),

$$U_X \subset \text{Stab}(X),$$

which corresponds to the *neighborhood of the large volume limit* in string theory. Then our invariant $n_g^\beta(X)$ is defined as a certain counting invariant of objects $E \in \mathcal{P}(\phi)$ which satisfy (1.1) with respect to some point $(Z, \mathcal{P}) \in U_X$.

Next let us consider a birational map $\phi: W \dashrightarrow X$ from another Calabi–Yau threefolds W . Then due to Bridgeland [8], we have the equivalence of triangulated categories,

$$\Phi: \mathcal{D}_W \longrightarrow \mathcal{D}_X,$$

which gives an isomorphism $\Phi_*: \text{Stab}(W) \rightarrow \text{Stab}(X)$. We claim that the closures of Φ_*U_W and U_X intersect, in particular they are contained in the same connected component of $\text{Stab}(X)$ (cf. Lemma 5.2). Then Question 1.2 is rephrased as follows.

Question 1.5. *How do counting invariants vary by changing stability conditions, from $\sigma \in U_X$ to $\tau \in \Phi_*U_W$?*

Now we use D. Joyce’s theory on configurations on abelian categories and counting invariants of semistable objects [20–23]. Especially in [23], he studies how counting invariants of semistable objects vary under change of stability conditions. Although his works focus on stability conditions on abelian categories, his arguments also apply in our case. The reason

is as follows: roughly speaking, a theory of stability conditions on abelian categories corresponds to a local theory on $\text{Stab}(X)$. Thus Joyce’s works enable us to study how counting invariants vary locally, and actually we will see they do not vary at all. Obviously we can answer Question 1.5, and conclude Theorem 1.4 by this argument.

The content of this paper is as follows. In Section 2 we review the mathematical proposal of Gopakumar–Vafa invariants in [16]. In Section 3 we review Bridgeland’s work on stability conditions on triangulated categories [7], and construct some stability conditions we need. In Section 4 we introduce our invariant $n_g^\beta(X)$, and prove Theorem 1.4 in Section 5. In Section 6 we prove some technical lemmas.

1.4. Terminology used in this paper

In this paper, all the varieties are defined over \mathbb{C} . We say X is Calabi–Yau if X is smooth projective with trivial canonical bundle. For a variety X , we denote by $D(X)$ the derived category of coherent sheaves on X . For a triangulated category \mathcal{D} , its K -group is denoted by $K(\mathcal{D})$. We use the following standard terminology used in birational geometry [30],

$$N^1(X) := \bigoplus_{D \subset X} \mathbb{R}D / \cong, \quad N_1(X) := \bigoplus_{C \subset X} \mathbb{R}C / \cong.$$

In the definition of $N^1(X)$, $D \subset X$ is a divisor and $D_1 \cong D_2$ if and only if $D_1 \cdot C = D_2 \cdot C$ for any curve C on X . Similarly in the definition of $N_1(X)$, C is a curve on X and $C_1 \cong C_2$ if and only if $D \cdot C_1 = D \cdot C_2$ for any divisor D . Clearly we have the perfect pairing,

$$N^1(X) \times N_1(X) \ni (D, C) \longmapsto D \cdot C \in \mathbb{R},$$

which identifies $N_1(X)$ with the dual of $N^1(X)$. We set $N^1(X)_{\mathbb{C}} = N^1(X) \otimes_{\mathbb{R}} \mathbb{C}$ and

$$\begin{aligned} \overline{NE}(X) &:= \overline{\{\text{Cone of effective curves}\}} \subset N_1(X), \\ A(X)_{\mathbb{C}} &:= \{B + i\omega \in N^1(X)_{\mathbb{C}} \mid \omega \text{ is ample}\} \subset N^1(X)_{\mathbb{C}}. \end{aligned}$$

Suppose that a birational map $\phi: W \dashrightarrow X$ is an isomorphism in codimension one. We use the following isomorphisms,

$$\phi_*: N^1(W) \longrightarrow N^1(X), \quad \phi_*: N_1(W) \longrightarrow N_1(X),$$

where the left-hand side is the strict transform and the right-hand side is the inverse of the dual of the LHS. For a non-zero $\beta \in N_1(X)$, $\text{Chow}_\beta(X)$ is the subvariety of the Chow variety $\text{Chow}(X)$, representing effective one cycles on X with numerical class β . One can refer [29, chapter 1, section 3] for the existence of the variety $\text{Chow}_\beta(X)$. We set $\text{Chow}_\beta(X) = \text{Spec } \mathbb{C}$ when $\beta = 0$. For a coherent sheaf E on X with $\dim \text{Supp}(E) \leq 1$, we set

$$(1.2) \quad s(E) := \sum_{p \in X} \text{length}_{\mathcal{O}_{X,p}}(E_p) \overline{\{p\}} \in \text{Chow}_\beta(X),$$

where $\beta = \text{ch}_2(E)$ and p runs through all the codimension two points.

Let X be a Calabi–Yau threefold. For an object $E \in \mathcal{D}_X$ and $v = (\beta, k) \in N_1(X) \oplus \mathbb{Z}$, we say E is of numerical type v if

$$(\text{ch}_2(E), \text{ch}_3(E)) = (\beta, k).$$

2. Review of work of Hosono et al.

In this section, we briefly review the work of Hosono et al. [16].

2.1. Representations of \mathfrak{sl}_2

First let us recall that the Lie algebra \mathfrak{sl}_2 is generated by three elements,

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which satisfy the relation,

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f.$$

For each $j \in \frac{1}{2}\mathbb{Z}$, there is a unique irreducible representation of \mathfrak{sl}_2 (up to isomorphism) of dimension $2j + 1$, called the *spin j representation*, and denoted by (j) . For $V = (j)$, there is an eigenvector $v \in V$ of h such that $fv = 0$ and

$$(j) = \langle v, ev, \dots, e^{2j}v \rangle, \quad e^{2j+1}v = 0,$$

with $he^k v = (-2j + 2k)v$, $0 \leq k \leq 2j$.

Let X be a normal projective variety and $IH^*(X)$ is the intersection cohomology of X introduced in [4]. Note that if X is connected and smooth,

we have

$$IH^i(X) = H^{i+\dim X}(X, \mathbb{C}),$$

for any $i \in \mathbb{Z}$. Let H be an ample divisor on X , and η be the Lefschetz operator,

$$\eta = H \wedge : IH^*(X) \longrightarrow IH^{*+2}(X).$$

It is well known that the operator $\eta^i : IH^{-i}(X) \rightarrow IH^i(X)$ is an isomorphism. (cf. [4, theorems 5.4.10 and 6.2.10].) Using this, one can construct an \mathfrak{sl}_2 -action on $IH^*(X)$ as follows (see [39, section 2] for details). First we find a homogeneous basis of $IH^*(X)$ which consist of primitive elements. Here $v \in IH^{-i}(X)$ for $i \geq 0$ is primitive if $\eta^{i+1}v = 0$. For such a basis $\{v_\alpha\}$, $IH^*(X)$ is written as a direct sum of the subspaces generated by

$$(2.1) \quad \{v_\alpha, \eta v_\alpha, \dots, \eta^{i_\alpha} v_\alpha\},$$

with $\deg v_\alpha = -i_\alpha$. Then define the representation of \mathfrak{sl}_2 on $IH^*(X)$ by letting $e \in \mathfrak{sl}_2$ act as η , $h \in \mathfrak{sl}_2$ act as multiplying by the degree and the action of $f \in \mathfrak{sl}_2$ is defined inductively from the requirement of $f v_\alpha = 0$. Hence the subspace (2.1) gives $\text{spin}(\frac{i_\alpha}{2})$ -representation. For a complex torus T of dimension g , we have the following formula (see [16, section 2]).

$$IH^*(T) = I_g := \left[\left(\frac{1}{2} \right) \oplus 2(0) \right]^{\otimes g}.$$

Lemma 2.1. *The \mathfrak{sl}_2 -representation type of $IH^*(X)$ does not depend on a choice of an ample divisor H .*

Proof. Let H' be another ample divisor, and consider the operator $\eta'(*) = H' \wedge *$. Let $T_j, T'_j \subset IH^*(X)$ be the sub \mathfrak{sl}_2 -representations with respect to the operators η, η' respectively, consisting of direct sums of $\text{spin}(j')$ -representations for $j' \geq j$. Suppose that T_j, T'_j have the same \mathfrak{sl}_2 -representation types. Then the minimal degrees of the following graded vector spaces,

$$(2.2) \quad IH^*(X)/T_j, \quad IH^*(X)/T'_j,$$

are same, say $d \in (-2j, 0]$. Also the subspaces of degree d elements in (2.2) have the same dimensions, say l . Then we see

$$T_{-d/2} \cong T_j \oplus \left(-\frac{d}{2} \right)^{\oplus l}, \quad T'_{-d/2} \cong T'_j \oplus \left(-\frac{d}{2} \right)^{\oplus l},$$

as \mathfrak{sl}_2 -representations. Therefore $T_{-d/2}$ and $T_{-d/2'}$ have the same \mathfrak{sl}_2 -representation types. By the induction we obtain the lemma. \square

2.2. Relative Lefschetz actions

Let $f: X \rightarrow A$ be a projective morphism between normal projective varieties. The idea of [16] is to define the $(\mathfrak{sl}_2)_L \times (\mathfrak{sl}_2)_R$ -action on $IH^*(X)$ using the Lefschetz operators in fiber directions and base directions. Let $H_A, H_{X/A}$ be an ample divisor on A , a relative ample divisor on X over A respectively. We denote by $D(\mathbb{C}_X)$, $\text{Perv}(\mathbb{C}_X)$ the derived category of constructible sheaves on X (with its classical topology), the heart of the middle perverse t -structure on $D(\mathbb{C}_X)$ respectively. We have the perverse Leray spectral sequence,

$$E_2^{r,s} = H^r(A, {}^p R^s f_* \mathcal{IC}_X) \implies IH^{r+s}(X),$$

where $\mathcal{IC}_X \in \text{Perv}(\mathbb{C}_X)$ is the intersection complex on X , and ${}^p R^s f_* \mathcal{IC}_X \in \text{Perv}(\mathbb{C}_A)$ is the s -th cohomology of $\mathbf{R}f_* \mathcal{IC}_X$ with respect to the middle perverse t -structure on $D(\mathbb{C}_A)$. It is known that this spectral sequence degenerates at E_2 -terms (cf. [4, theorem 6.2.5]), and we have two operators,

$$\eta_L = H_{X/A} \wedge: E_2^{r,s} \longrightarrow E_2^{r,s+2}, \quad \eta_R = H_A \wedge: E_2^{r,s} \longrightarrow E_2^{r+2,s},$$

such that $\eta_L^s: E_2^{r,-s} \xrightarrow{\cong} E_2^{r,s}$ and $\eta_R^r: E_2^{-r,s} \xrightarrow{\cong} E_2^{r,s}$. As in Paragraph 2.1, these two actions define an $(\mathfrak{sl}_2)_L \times (\mathfrak{sl}_2)_R$ -action on $IH^*(X)$ (cf. [16, corollary 2.1]). Also by the same argument of Lemma 2.1, the $(\mathfrak{sl}_2)_L \times (\mathfrak{sl}_2)_R$ -representation type of $IH^*(X)$ does not depend on $H_{X/A}, H_A$.

2.3. Definition of HST (Hosono, Saito, Takahashi) invariants

Let X be a projective variety. For $\beta \in N_1(X)$ and an ample divisor H on X , let M^β be the moduli space of H -Gieseker semistable sheaves E on X (cf. [18]), pure of dimension one, with numerical type $(\beta, 1)$. Let $\widetilde{M}^\beta \rightarrow M^\beta$ be the normalization. By the same argument as in [37, chapter 5, section 4], there is a natural map,

$$\pi_\beta: M^\beta \ni E \longmapsto s(E) \in \text{Chow}_\beta(X),$$

and let S^β be the normalization of the image of π_β . The induced morphism $\pi_\beta: \widetilde{M}^\beta \rightarrow S^\beta$ is projective, hence defines an $(\mathfrak{sl}_2)_L \times (\mathfrak{sl}_2)_R$ -action on

$IH^*(\widetilde{M}^\beta)$. One can rearrange its action in the following formula (cf. [16, theorem 2.4]),

$$IH^*(\widetilde{M}^\beta) = \bigoplus_{g \geq 0} I_g \otimes R_g(\beta),$$

where $R_g(\beta)$ is a virtual $(\mathfrak{sl}_2)_R$ -representation.

Definition 2.2 [16, definition 3.6]. We define \tilde{n}_g^β to be

$$(2.3) \quad \tilde{n}_g^\beta = \sum_j (-1)^{2j} (2j+1) \cdot N_j \in \mathbb{Z},$$

after writing $R_g(\beta) = \sum_j N_j \cdot (j)_R$ as a virtual representation.

Remark 2.3. In [16, definition 3.6], the invariant \tilde{n}_g^β is defined as $\mathrm{Tr}_{R_g(\beta)} (-1)^{h_R}$, which coincides with formula (2.3).

Remark 2.4. As pointed out in [38], the invariants \tilde{n}_g^β are unlikely to be BPS-invariants discussed in [13]. We need to involve virtual classes to define appropriate BPS-counting which are deformation invariant.

There is an alternative way of defining \tilde{n}_g^β pointed out by [39], and it is much more useful for our purposes. Let V be the space of an $(\mathfrak{sl}_2)_L \times (\mathfrak{sl}_2)_R$ -representation. Then the operator $h_L + h_R$ defines the grading $V = \bigoplus V_n$ with $e_R V_n \subset V_{n+2}$. We can decompose V into the direct sum of the vector subspaces spanned by

$$v, e_R v, \dots, e_R^{l-1} v,$$

where $v \in V_\alpha$ for some α , $e_R^l v = 0$, and there is no $v' \in V$ with $e_R v' = v$. Such a subspace is called a *Jordan cell of size l and minimal degree α* . Let $\nu_l^\alpha \in \mathbb{Z}_{\geq 0}$ be

$$(2.4) \quad \nu_l^\alpha = \#\{\text{Jordan cells of size } l \text{ and minimal degree } \alpha \text{ in } V\}.$$

Note that ν_l^α depends only on the e_R -action and the grading induced by $h_L + h_R$. We have the following.

Proposition 2.5 [39]. For $V = IH^*(\widetilde{M}^\beta)$, we have

$$\tilde{n}_g^\beta = \sum_{\alpha+l \geq 1} (-1)^{\alpha+g} l \nu_l^\alpha \left\{ \binom{\alpha+l+g}{2g+1} - \binom{\alpha+l+g-2}{2g+1} \right\}.$$

3. Stability conditions on triangulated categories

In this section we briefly recall the notion of stability conditions on triangulated categories [7], fix some notation and prove some fundamental properties.

3.1. Generalities

The notion of stability conditions on a triangulated category \mathcal{D} was introduced by T. Bridgeland [7] motivated by M. Douglas's work on Π -stability [11, 12]. Here we only introduce its definition and some terminology used in this paper, and do not explain its review too much. For the readers who are not familiar with [7], we recommend consulting the original paper [7].

Definition 3.1. A stability condition on \mathcal{D} consists of data $\sigma = (Z, \mathcal{P})$,

$$Z: K(\mathcal{D}) \longrightarrow \mathbb{C}, \quad \mathcal{P}(\phi) \subset \mathcal{D},$$

where Z is a group homomorphism, $\mathcal{P}(\phi)$ is a subcategory for each $\phi \in \mathbb{R}$ which satisfies,

- $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$.
- If $\phi_1 > \phi_2$ and $A_i \in \mathcal{P}(\phi_i)$, then $\text{Hom}(A_1, A_2) = 0$.
- If $E \in \mathcal{P}(\phi)$ is non-zero, then $Z(E) = m(E) \exp(i\pi\phi)$ for some $m(E) \in \mathbb{R}_{>0}$.
- For a non-zero object $E \in \mathcal{T}$, we have the following collection of triangles:

$$\begin{array}{ccccccc}
 0 = E_0 & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow & \cdots & \longrightarrow & E_n = E \\
 & & \swarrow & & \swarrow & & \swarrow & & \swarrow \\
 & & A_1 & & A_2 & & & & A_n \\
 & & \nwarrow & & \nwarrow & & \nwarrow & & \nwarrow \\
 & & [1] & & [1] & & & & [1]
 \end{array}$$

such that $A_j \in \mathcal{P}(\phi_j)$ with $\phi_1 > \phi_2 > \cdots > \phi_n$.

A non-zero object in $\mathcal{P}(\phi)$ is called *semistable of phase ϕ* . The mass of E is defined to be

$$m_\sigma(E) = \sum_j |Z(A_j)|.$$

For an interval $I \subset \mathbb{R}$, we denote by $\mathcal{P}(I)$ the smallest extension closed subcategory of \mathcal{D} which contains $\mathcal{P}(\phi)$ with $\phi \in I$. It is easy to see that $\mathcal{P}((0, 1])$ is the heart of a bounded t -structure on \mathcal{D} . This gives an alternative way of constructing a stability condition.

Proposition 3.2 [7, proposition 4.2]. *Giving a stability condition is equivalent to giving the heart of a bounded t -structure $\mathcal{A} \subset \mathcal{D}$ together with a group homomorphism $Z: K(\mathcal{A}) \rightarrow \mathbb{C}$ such that for a non-zero object $E \in \mathcal{A}$ one has*

$$(3.1) \quad Z(E) \in \mathbb{H} := \{r \exp(i\pi\phi) \mid r > 0, 0 < \phi \leq 1\},$$

and the pair (Z, \mathcal{A}) satisfies the Harder–Narasimhan property.

The set of stability conditions which satisfy the local finiteness (cf. [7, definition 5.7]) is denoted by $\text{Stab}(\mathcal{D})$. It is shown in [7, section 6] that $\text{Stab}(\mathcal{D})$ has a natural topology. Furthermore for each connected component $\Sigma \subset \text{Stab}(\mathcal{D})$, there exists a linear subspace $V(\Sigma) \subset \text{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C})$ with a norm such that we have the local homeomorphism (cf. [7, theorem 1.2]),

$$\mathcal{Z}: \Sigma \ni (Z, \mathcal{P}) \mapsto Z \in V(\Sigma).$$

3.2. Stability conditions on Calabi–Yau threefolds

For a Calabi–Yau threefold X , we consider the following triangulated category as in the introduction,

$$\mathcal{D}_X := \{E \in D(X) \mid \dim \text{Supp}(E) \leq 1\}.$$

Here we introduce the subspace of $\text{Stab}(\mathcal{D}_X)$, coming from the points corresponding to the neighborhood of the large volume limit at X . For $B + i\omega \in N^1(X)_{\mathbb{C}}$, we set $Z_{(B,\omega)}: K(\mathcal{D}_X) \rightarrow \mathbb{C}$ as

$$Z_{(B,\omega)}(E) = -\text{ch}_3(E) + (B + i\omega) \cdot \text{ch}_2(E).$$

Remark 3.3. Note that $Z_{(B,\omega)}$ factors as follows,

$$Z_{(B,\omega)}: K(\mathcal{D}_X) \xrightarrow{(\text{ch}_2, \text{ch}_3)} N_1(X) \oplus \mathbb{Z} \longrightarrow \mathbb{C}.$$

Here the right arrow takes $v = (\beta, k)$ to $-k + (B + i\omega)\beta$, which we write as $Z_{(B,\omega)}(v)$ by abuse of notation.

We also set $\text{Coh}_{\leq 1}(X)$ as $\text{Coh}_{\leq 1}(X) := \text{Coh}(X) \cap \mathcal{D}_X$. Note that $\text{Coh}_{\leq 1}(X)$ is the heart of a bounded t-structure on \mathcal{D}_X . We have the following lemma.

Lemma 3.4. *For $B + i\omega \in A(X)_{\mathbb{C}}$, the pair*

$$\sigma_{(B,\omega)} := (Z_{(B,\omega)}, \text{Coh}_{\leq 1}(X))$$

determines a point in $\text{Stab}(\mathcal{D}_X)$ in the sense of Proposition 3.2.

Proof. The proof of this lemma is exactly same as in [42, lemma 4.1]. In fact for a non-zero $E \in \text{Coh}_{\leq 1}(X)$, we have $\text{Im } Z_{(B,\omega)}(E) > 0$ when $\dim \text{Supp}(E) = 1$ and $Z_{(B,\omega)}(E) \in \mathbb{R}_{<0}$ when $\dim \text{Supp}(E) = 0$. Thus (3.1) holds. One can also check the Harder–Narasimhan property as in [42, lemma 4.1]. \square

Remark 3.5. For an object $E \in \text{Coh}_{\leq 1}(X)$, it is $\sigma_{(0,\omega)}$ -semistable if and only if for any non-trivial subobject $F \subset E$ in $\text{Coh}_{\leq 1}(X)$, one has

$$\frac{\text{ch}_3(F)}{\omega \cdot \text{ch}_2(F)} \leq \frac{\text{ch}_3(E)}{\omega \cdot \text{ch}_2(E)},$$

i.e., E is ω -Gieseker semistable sheaf.

We define $\text{Stab}(X)$ to be the following fiber product,

$$\begin{array}{ccc} \text{Stab}(X) & \longrightarrow & \text{Stab}(\mathcal{D}_X) \\ \downarrow & & \downarrow \\ N^1(X)_{\mathbb{C}} & \longrightarrow & \text{Hom}(K(\mathcal{D}_X), \mathbb{C}), \end{array}$$

where the right arrow takes (Z, \mathcal{P}) to Z and the bottom arrow takes $B + i\omega$ to $Z_{(B,\omega)}$. Note that the stability conditions constructed in Lemma 3.4 are contained in $\text{Stab}(X)$. Let $U_X \subset \text{Stab}(X)$ be

$$U_X := \{\sigma_{(B,\omega)} \in \text{Stab}(X) \mid B + i\omega \in A(X)_{\mathbb{C}}\}.$$

We have to check the following, whose proof will be postponed in Section 6.

Lemma 3.6. *The subset U_X is an open connected subset of $\text{Stab}(X)$.*

By [7, theorem 1.1] and Lemma 3.6, the map

$$\mathcal{Z}_X: \text{Stab}(X) \ni (Z, \mathcal{P}) \mapsto Z \in N^1(X)_{\mathbb{C}},$$

restricts to a homeomorphism between U_X and $A(X)_{\mathbb{C}}$.

3.3. Wall and chamber structures

In this paragraph, we recall the notion of wall and chamber structures on the space of stability conditions. The wall and chamber structure is introduced in [6] on the space of stability conditions on K3 surfaces, and we show that our space $\text{Stab}(X)$ also possesses such a structure. Let

$$\text{Stab}^*(X) \subset \text{Stab}(X),$$

be the connected component which contains U_X . Let $\mathcal{S} \subset \mathcal{D}_X$ be a set of objects. For $\sigma \in \text{Stab}^*(X)$, we call \mathcal{S} has *bounded mass* if there is $m > 0$ such that

$$m_{\sigma}(E) < m,$$

for any $E \in \mathcal{S}$. It is easy to show that if this condition holds for $\sigma \in \text{Stab}^*(X)$, then it also holds for any $\sigma' \in \text{Stab}^*(X)$. We have the following proposition (cf. [6, proposition 9.3]).

Proposition 3.7. *For a fixed compact subset $\bar{U} \subset \text{Stab}^*(X)$, there is a finite number of real codimension one submanifolds $\{\mathcal{W}_{\gamma} \mid \gamma \in \Gamma\}$ such that each connected component*

$$\mathcal{C} \subset \bar{U} \setminus \bigcup_{\gamma \in \Gamma} \mathcal{W}_{\gamma}$$

has the following property. If $E \in \mathcal{S}$ is σ -semistable for some $\sigma \in \mathcal{C}$, then E is semistable in σ' for all $\sigma' \in \mathcal{C}$.

Proof. The same proof of [6, proposition 9.3] applies once we show the analogue of [6, lemma 9.2] in our case, i.e., the set of numerical classes,

$$(3.2) \quad \{(\text{ch}_2(E), \text{ch}_3(E)) \mid E \in \mathcal{S}\} \subset N_1(X) \oplus \mathbb{Z},$$

is a finite set. For an ample divisor ω , let us take $\sigma = \sigma_{(0, \omega)}$. Also for $E \in \mathcal{S}$, let A_1, \dots, A_n be the σ -semistable factors. Then the bounded mass

condition for \mathcal{S} implies that the values

$$|\mathrm{ch}_3(A_i)|, \quad |\mathrm{ch}_2(A_i) \cdot \omega|,$$

are bounded. On the other hand, the following space,

$$\{c \in \overline{NE}(X) \mid c \cdot \omega \leq 1\} \subset N_1(X),$$

is compact. In fact openness of the ample cone immediately implies the compactness of the earlier space. Since $\mathrm{ch}_2(A_i)$ or $-\mathrm{ch}_2(A_i)$ is contained in $\overline{NE}(X)$, we see that the pair $(\mathrm{ch}_2(A_i), \mathrm{ch}_3(A_i))$ has finite number of possibilities. Hence (3.2) is also finite. \square

We call a connected component $\mathcal{C} \subset \overline{U} \setminus \bigcup_{\gamma \in \Gamma} \mathcal{W}_\gamma$ *chamber*.

3.4. Moduli theory of semistable objects

In this paragraph, we consider the moduli problem of σ -semistable objects $E \in \mathcal{D}_X$ for $\sigma \in \mathrm{Stab}(X)$. Such a moduli theory is studied in [41] for stability conditions on K3 surfaces and abelian surfaces, and we use some basic arguments developed there. Let \mathcal{M} be the moduli stack of objects $E \in \mathcal{D}_X$, satisfying the following condition,

$$(3.3) \quad \mathrm{Ext}^i(E, E) = 0, \quad \text{for } i < 0.$$

Then by the result of Lieblich [34], the stack \mathcal{M} is an Artin stack of locally finite type over \mathbb{C} . More precisely, Lieblich showed that the stack of $E \in D(X)$ satisfying the previous condition is an Artin stack of locally finite type. However for a family of objects $\mathcal{E} \in D(X \times S)$, the condition $\mathcal{E}_s \in \mathcal{D}_X$ is obviously an open condition, thus \mathcal{M} is also an Artin stack of locally finite type.

For $\sigma = (Z, \mathcal{P}) \in \overline{U}_X$, $v \in N_1(X) \oplus \mathbb{Z}$ and $\phi \in \mathbb{R}$, we can consider the substack,

$$i: \mathcal{M}^{(v, \phi)}(\sigma) \hookrightarrow \mathcal{M},$$

which is the moduli stack of $E \in \mathcal{P}(\phi)$ of numerical type v . The purpose here is to show that $\mathcal{M}^{(v, \phi)}(\sigma)$ is algebraic. Recall that a set of objects $\mathcal{S} \subset \mathcal{D}_X$ is called *bounded* if there is a finite type \mathbb{C} -scheme S together with an object $\mathcal{E} \in D(X \times S)$ such that any object $E \in \mathcal{S}$ is isomorphic to \mathcal{E}_s for some $s \in S$. We use the following lemma, whose proof will be postponed until Section 6.

Lemma 3.8. *Let us take $\sigma = (Z, \mathcal{P}) \in \text{Stab}^*(X)$ such that Z is defined over \mathbb{Q} . Then the set of objects,*

$$\mathcal{M}^{(v,\phi)}(\sigma)(\text{Spec } \mathbb{C}) = \{E \in \mathcal{P}(\phi) \mid E \text{ is of numerical type } v\},$$

is bounded.

Using this lemma and the argument in [41], we show the following proposition.

Proposition 3.9. *The stack $\mathcal{M}^{(v,\phi)}(\sigma)$ is an Artin stack of finite type over \mathbb{C} , and $i: \mathcal{M}^{(v,\phi)}(\sigma) \hookrightarrow \mathcal{M}$ is an open immersion.*

Proof. For B and ω are rational, the set of \mathbb{C} -valued points of $\mathcal{M}^{(v,\phi)}(\sigma_{(B,\omega)})$ is bounded by Lemma 3.8. Hence by [41, lemma 3.13 (ii), proposition 3.18] the result is true for such $\sigma_{(B,\omega)}$. Then one can apply [41, theorem 3.20, step 1], and conclude the result for any $\sigma \in \overline{U}_X$. \square

4. Motivic Gopakumar–Vafa invariants

The purpose of this section is to introduce the invariants $n_g^\beta(X) \in \mathbb{Z}$, from a certain motivic invariant of varieties over a Chow variety.

4.1. Motivic invariants of varieties

Let A be a projective variety over \mathbb{C} . First let us recall the Grothendieck group of varieties over A .

Definition 4.1. We define the \mathbb{Z} -module $K_0(\text{Var}/A)$ to be

$$K_0(\text{Var}/A) = \bigoplus \mathbb{Z}[(X, \pi)] / \sim,$$

where $[(X, \pi)]$ is an isomorphism class of a pair of a quasi-projective variety X together with a morphism $\pi: X \rightarrow A$. The equivalence relation \sim is generated by

$$[(X, \pi)] = [(Z, \pi|_Z)] + [(X \setminus Z, \pi|_{X \setminus Z})],$$

for closed subvarieties $Z \subset X$.

Let $\pi: X \rightarrow A$ be a projective morphism with X smooth and connected. There is an induced morphism $X \rightarrow \tilde{A}$, where \tilde{A} is the normalization of

$\pi(A)$. Then as in Paragraph 2.2, $I\tilde{H}^*(X)$ carries an $(\mathfrak{sl}_2)_L \times (\mathfrak{sl}_2)_R$ -action with respect to the morphism $X \rightarrow \tilde{A}$. Let $\nu_l^\alpha \in \mathbb{Z}$ be the number of Jordan cells in $I\tilde{H}^*(X)$ for this action, defined in (2.4). We set $P(X, \pi) \in \mathbb{Z}[t, s]$ as follows:

$$P(X, \pi) = t^{\dim X} \sum_{\alpha, l} \nu_l^\alpha t^\alpha s^{l-1}.$$

We show the following proposition.

Proposition 4.2. *There exists a map,*

$$\Upsilon_A: K_0(\text{Var}/A) \longrightarrow \mathbb{Z}[t, s],$$

such that for any projective morphism $\pi: X \rightarrow A$ with X smooth and connected, we have

$$\Upsilon_A([(X, \pi)]) = P(X, \pi).$$

Proof. Let X be a connected smooth projective variety with a morphism $\pi: X \rightarrow A$. Let $Z \subset X$ be a smooth closed subvariety and take the blow-up $p: X^\dagger \rightarrow X$ along Z ,

$$(4.1) \quad \begin{array}{ccccc} E & \xrightarrow{j} & X^\dagger & & \\ q \downarrow & & p \downarrow & \searrow \pi^\dagger & \\ Z & \xrightarrow{i} & X & \xrightarrow{\pi} & A. \end{array}$$

By the result of [5, theorem 5.1], $K_0(\text{Var}/A)$ is generated by such $[(X, \pi)]$ with relation

$$[(X^\dagger, \pi^\dagger)] - [(E, \pi^\dagger \circ j)] = [(X, \pi)] - [(Z, \pi \circ i)].$$

Hence it is enough to show that

$$(4.2) \quad P(X^\dagger, \pi^\dagger) - P(E, \pi^\dagger \circ j) = P(X, \pi) - P(Z, \pi \circ i).$$

Let $d = \dim X$ and r be the codimension of Z in X . Note that $q: E \rightarrow Z$ is a \mathbb{P}^{r-1} -bundle. By the isomorphism (4.3) in Lemma 4.3 next, we have

$$\frac{P(X^\dagger, \pi^\dagger)}{t^d} = \frac{P(X, \pi)}{t^d} + \frac{t^{-r+2}P(Z, \pi \circ i)}{t^{d-r}} + \cdots + \frac{t^{r-2}P(Z, \pi \circ i)}{t^{d-r}},$$

since μ_l^α is determined by the grading and e_R -actions. Thus we have

$$P(X^\dagger, \pi^\dagger) = P(X, \pi) + t^2 P(Z, \pi \circ i) + \cdots + t^{2(r-1)} P(Z, \pi \circ i).$$

Similarly the isomorphism (4.4) in Lemma 4.3 yields,

$$P(E, \pi^\dagger \circ j) = P(Z, \pi \circ i) + t^2 P(Z, \pi \circ i) + \cdots + t^{2(r-1)} P(Z, \pi \circ i).$$

Hence the Equation (4.2) holds. \square

We have used the following lemma.

Lemma 4.3. *In the diagram (4.1), we have the following isomorphisms,*

$$(4.3) \quad IH^*(X^\dagger) \cong IH^*(X) \oplus \bigoplus_{i=0}^{r-2} IH^*(Z)[-r+2+2i],$$

$$(4.4) \quad IH^*(E) \cong \bigoplus_{i=0}^{r-1} IH^*(Z)[-r+1+2i],$$

which preserve e_R -actions.

Proof. Let $\mathcal{O}_E(1)$ be the tautological line bundle on E and set $\xi = c_1(\mathcal{O}_E(1)) \in H^2(E, \mathbb{C})$. We have the following isomorphism,

$$(4.5) \quad H^*(Z, \mathbb{C}) \oplus H^*(Z, \mathbb{C})[-2] \oplus \cdots \oplus H^*(Z, \mathbb{C})[-2(r-1)] \xrightarrow{\cong} H^*(E, \mathbb{C}),$$

which sends $(v_0, v_1, \dots, v_{r-1})$ to $\sum \xi^i q^* v_i$. It is obvious that (4.5) preserves e_R -actions, hence (4.4) follows.

Next in [14, p. 605], it is shown that we have the isomorphism,

$$(4.6) \quad H^*(X^\dagger, \mathbb{C}) \xrightarrow{\cong} H^*(X, \mathbb{C}) \oplus \frac{H^*(E, \mathbb{C})}{q^* H^*(Z, \mathbb{C})},$$

which sends $v \in H^*(X, \mathbb{C})$ to $(p_* v, j^* v)$. For an ample divisor H_A on A , we have

$$\pi^* H_A \wedge p_* v = p_*(p^* \pi^* H_A \wedge v), \quad j^* \pi^\dagger H_A \wedge j^* v = j^*(\pi^\dagger H_A \wedge v).$$

Hence the isomorphism (4.6) preserves e_R -actions. Using (4.4) and (4.6), we obtain the isomorphism (4.3). \square

In the following, we compute some examples of our invariant $P(X, \pi)$.

Example 4.4. (i) Let A be a d -dimensional smooth projective variety and consider the element $[(A, \text{id})] \in K_0(\text{Var}/A)$. Let $b_i(A) \in \mathbb{Z}$ be the i -th betti number of A . Then one can easily see

$$\nu_l^\alpha = \begin{cases} b_{d+\alpha}(A) - b_{d+\alpha-2}(A), & \text{if } l = -\alpha + 1, \\ 0, & \text{otherwise.} \end{cases}$$

Hence we have

$$P(A, \text{id}) = \sum_{\alpha=0}^d (b_\alpha(A) - b_{\alpha-2}(A)) t^\alpha s^{-\alpha+d}.$$

(ii) Let $\pi: X \rightarrow A$ be a projective bundle with fiber \mathbb{P}^{r-1} . Then applying (i) and (4.4), we have

$$\begin{aligned} P(X, \pi) &= P(A, \text{id})(1 + t^2 + \cdots + t^{2(r-1)}), \\ &= \sum_{\alpha=0}^d \sum_{k=0}^{r-1} (b_\alpha(A) - b_{\alpha-2}(A)) t^{\alpha+2k} s^{-\alpha+d}. \end{aligned}$$

(iii) Let $i: Z \hookrightarrow A$ be a smooth subvariety of codimension r . Let $\pi: X \rightarrow A$ be a blow-up along Z . Using (4.3), we have

$$\begin{aligned} P(X, \pi) &= P(A, \text{id}) + P(Z, i)(t^2 + t^4 + \cdots + t^{2(r-1)}), \\ &= \sum_{\alpha=0}^d (b_\alpha(A) - b_{\alpha-2}(A)) t^\alpha s^{-\alpha+d} \\ &\quad + \sum_{\alpha=0}^{d-r} \sum_{k=1}^{r-1} (b_\alpha(Z) - b_{\alpha-2}(Z)) t^{\alpha+2k} s^{-\alpha+d-r}. \end{aligned}$$

Here we have used Lemma 4.5 next, which shows $P(Z, i) = P(Z, \text{id})$. Lemma 4.5 will also be used in Lemma 5.5

Lemma 4.5. *Let $u: A \rightarrow A'$ be a finite morphism between projective varieties. Then the following diagram commutes,*

$$\begin{array}{ccc} K_0(\text{Var}/A) & \xrightarrow{\Upsilon_A} & \mathbb{Z}[t, s] \\ u_* \downarrow & \nearrow \Upsilon_{A'} & \\ K_0(\text{Var}/A') & & \end{array}$$

Here u_* takes $\pi: X \rightarrow A$ to $u \circ \pi: X \rightarrow A \rightarrow A'$.

Proof. It is enough to check the commutativity for $[(X, \pi)] \in K_0(\text{Var}/A)$, where X is a smooth projective variety and $\pi: X \rightarrow A$ is a morphism. Let $H_{A'}$ be an ample divisor on A' , and set $H_A = u^*H_{A'}$. Note that the divisor H_A is also ample because u is finite. Since the definition of Υ_A does not depend on a choice of an ample divisor on A , one may compute $\Upsilon_A(X, \pi)$ using the e_R -action on $IH^*(X)$ given by $\pi^*H_A \wedge *$. Also one may compute $\Upsilon_{A'}(X, u \circ \pi)$ by the e_R -action given by $(u \circ \pi)^*H_{A'} \wedge *$. Since $H_A = u^*H_{A'}$, both e_R -actions are same, hence $\Upsilon_A(X, \pi) = \Upsilon_{A'}(X, u \circ \pi)$. \square

Remark 4.6. For $Q = \sum q^{a,b}t^a s^b \in \mathbb{Z}[t, s]$, let $m(Q)$ be the integer

$$m(Q) = \max\{a + 2b \mid q^{a,b} \neq 0\}.$$

Let $\pi: X \rightarrow A$ be a projective morphism with X smooth and connected. Then it is clear that $m(Q) = 2 \dim X$ for $Q = \Upsilon_A([(X, \pi)])$. In particular one can recover ν_t^α from $t^{-m(Q)/2}Q$. Using this and the motivic property of Υ_A , one can easily show that for any quasi-projective variety X with a morphism $\pi: X \rightarrow A$ and $Q = \Upsilon_A([(X, \pi)])$, the integer $m(Q)$ is even.

Based on Remark 4.6 we introduce the following notation, which will be used in Paragraph 4.4.

Definition 4.7. We define the subset $\mathbb{Z}[t, s]^\dagger \subset \mathbb{Z}[t, s]$ to be the set of polynomials Q such that $m(Q)$ is even. For $Q \in \mathbb{Z}[t, s]^\dagger$, we define the operation Q^b by

$$Q^b := t^{-m(Q)/2}Q \in \mathbb{Z}[t^{-1}, t, s].$$

When $A = \text{Spec } \mathbb{C}$, we write $K_0(\text{Var}/A)$ as $K_0(\text{Var}/\mathbb{C})$, and just write its elements as $[X] \in K_0(\text{Var}/\mathbb{C})$ omitting the structure morphism $X \rightarrow \text{Spec } \mathbb{C}$. Also we write $\Upsilon_{\text{Spec } \mathbb{C}}$ as Υ for simplicity.

Remark 4.8. If $A = \text{Spec } \mathbb{C}$, all the Jordan cells have length one. Hence $\Upsilon(X) \in \mathbb{Z}[t]$ is nothing but the virtual Poincaré polynomial of X .

There is a ring structure on $K_0(\text{Var}/\mathbb{C})$ defined by

$$([X_1], [X_2]) \mapsto [X_1 \times X_2],$$

with unit $[\text{Spec } \mathbb{C}]$. Also we have the following natural map,

$$\Pi: K_0(\text{Var}/\mathbb{C}) \times K_0(\text{Var}/A) \longrightarrow K_0(\text{Var}/A),$$

which takes the pair $([T], [(X, \pi)])$ to

$$T \times X \xrightarrow{\text{pr}} X \xrightarrow{\pi} A,$$

where pr is the projection to X . The operation Π makes $K_0(\text{Var}/A)$ a $K_0(\text{Var}/\mathbb{C})$ -algebra. We have the following lemma.

Lemma 4.9. *The following diagram is commutative.*

$$\begin{array}{ccc} \Pi: K_0(\text{Var}/\mathbb{C}) \times K_0(\text{Var}/A) & \longrightarrow & K_0(\text{Var}/A) \\ \Upsilon \times \Upsilon_A \downarrow & & \Upsilon_A \downarrow \\ \mathbb{Z}[t] \times \mathbb{Z}[t, s] & \longrightarrow & \mathbb{Z}[t, s], \end{array}$$

where the bottom arrow takes $(Q_1(t), Q_2(t, s))$ to $Q_1(t)Q_2(t, s)$.

Proof. It is enough to check

$$(4.7) \quad \Upsilon_A([T \times X, \pi \circ \text{pr}]) = \Upsilon(T) \cdot \Upsilon_A([(X, \pi)]),$$

for smooth projective T and X with a morphism $\pi: X \rightarrow A$. We have the isomorphism as graded vector spaces,

$$IH^*(T \times X) \cong IH^*(T) \otimes IH^*(X).$$

Let us introduce the e_R -action on the right-hand side, by letting e_R act on $IH^*(T)$ trivially. Then this isomorphism preserves the e_R -actions, which shows the equality (4.7) immediately. \square

Remark 4.10. More generally a motivic invariant is defined to be a ring homomorphism $\Upsilon: K(\text{Var}/\mathbb{C}) \rightarrow \Lambda$ for some ring Λ . Then a motivic invariant relative to A is defined to be a map,

$$K_0(\text{Var}/A) \otimes_{K_0(\text{Var}/\mathbb{C})} \Lambda \longrightarrow M,$$

for some Λ -module M . Lemma 4.9 implies that Υ_A is obtained in this way.

Remark 4.11. Let $\pi: X \rightarrow A$ be a morphism and $p: Z \rightarrow X$ be a Zariski locally trivial fibration with fiber F . Then we have $[(Z, \pi \circ p)] = \Pi([F], [(X, \pi)])$, which yields the equality $\Upsilon_A([(Z, \pi \circ p)]) = \Upsilon(F) \cdot \Upsilon_A([(X, \pi)])$ by Lemma 4.9 (cf. [24, lemma 4.2]).

4.2. Motivic invariants of Artin stacks

Here we extend the invariant constructed in the previous paragraph to the invariant of Artin stacks over a projective variety A . The material of this paragraph is a slight generalization of Joyce’s work [24]. In *loc.cite*, he works on the motivic invariants over $K_0(\text{Var}/\mathbb{C})$ such as virtual Poincaré polynomials. For our purpose we have to extend the results in [24] to invariants over $K_0(\text{Var}/A)$ such as Υ_A . However the proofs are straightforward generalizations and we will leave some details to readers. Let \mathcal{R} be an Artin stack of locally finite type over \mathbb{C} . Following Joyce [24], we introduce the Grothendieck group of Artin stacks over \mathcal{R} , denoted by $K_0(\text{St}/\mathcal{R})$ in this paper. It was denoted by $\text{SF}(\mathcal{R})$ in Joyce’s papers [21–24].

Definition 4.12 [24, definition 3.1]. We define $K_0(\text{St}/\mathcal{R})$ to be the \mathbb{Q} -vector space generated by equivalence classes of pairs $[(\mathcal{X}, \rho)]$, where $\rho: \mathcal{X} \rightarrow \mathcal{R}$ is a 1-morphism of Artin stacks, \mathcal{X} is of finite type over \mathbb{C} with affine geometric stabilizers, such that for each closed substack $\mathcal{Y} \subset \mathcal{X}$, one has

$$[(\mathcal{X}, \rho)] = [(\mathcal{Y}, \rho|_{\mathcal{Y}})] + [(\mathcal{X} \setminus \mathcal{Y}, \rho|_{\mathcal{X} \setminus \mathcal{Y}})].$$

The following lemma is a generalization of [24, theorem 4.10]. Next we set $\Lambda = \mathbb{Q}(t, s)$.

Lemma 4.13. *Let A be a projective variety. Then $\Upsilon_A: K_0(\text{Var}/A) \rightarrow \mathbb{Z}[t, s]$ extends to the map,*

$$\Upsilon'_A: K_0(\text{St}/A) \longrightarrow \Lambda = \mathbb{Q}(t, s),$$

such that for a 1-morphism $\rho: \mathcal{X} \rightarrow A$ with $\mathcal{X} \cong [X/G]$, X is a quasi-projective variety and G a special algebraic \mathbb{C} -group (cf. [24, definition 2.1]), we have

$$(4.8) \quad \Upsilon'_A([(X, \pi)]) = \frac{\Upsilon_A([(X, \pi)])}{\Upsilon([G])}.$$

Here π is the composition, $\pi: X \rightarrow [X/G] \cong \mathcal{X} \xrightarrow{\rho} A$.

Proof. When $A = \text{Spec } \mathbb{C}$, Lemma is proved in [24, theorem 4.10]. Also $\Upsilon([G])$ is non-zero in $\mathbb{Z}[t]$ by [24, lemma 4.7], hence the right-hand side of (4.8) makes sense. We have to check that for $\rho: \mathcal{X} \rightarrow A$ with \mathcal{X} 1-isomorphic to $[X/G]$, the value $\Upsilon_A([(X, \pi)])/\Upsilon([G])$ does not depend on a choice of X , G , and an isomorphism $\mathcal{X} \cong [X/G]$. This follows from Remark 4.11 and

exactly the same proof of [24, proposition 4.8]. Finally as in the proof of [24, theorem 4.10], any Artin stack with affine geometric stabilizers \mathcal{X} is stratified by global quotient stacks, and can define $\Upsilon'_A([\mathcal{X}, \rho])$ by the formula (4.8) and the linearity. Then the same proof of [24, theorem 4.10] shows that $\Upsilon'_A([\mathcal{X}, \rho])$ does not depend on a choice of such a stratification. \square

We will use a Λ -module with more relations than $K_0(\text{St}/\mathcal{R})$, denoted by $K_0(\text{St}/\mathcal{R})$. It was denoted by $\text{SF}(\mathcal{R}, \Upsilon, \Lambda)$ in [24].

Definition 4.14 [24, definition 4.11]. We define $K_0(\text{St}/\mathcal{R})_\Upsilon$ to be the Λ -module generated by equivalence classes of pairs $[(\mathcal{X}, \rho)]$, where $\rho: \mathcal{X} \rightarrow \mathcal{R}$ is a 1-morphism of Artin stacks, \mathcal{X} is of finite type over \mathbb{C} with affine geometric stabilizers, such that

- (i) For each closed substack $\mathcal{Y} \subset \mathcal{X}$, one has

$$[(\mathcal{X}, \rho)] = [(\mathcal{Y}, \rho|_{\mathcal{Y}})] + [(\mathcal{X} \setminus \mathcal{Y}, \rho|_{\mathcal{X} \setminus \mathcal{Y}})].$$

- (ii) Let \mathcal{X} be a finite type Artin \mathbb{C} -stack with affine geometric stabilizers together with a 1-morphism $\rho: \mathcal{X} \rightarrow \mathcal{R}$. Let T be a quasi-projective variety, and $\text{pr}: T \times \mathcal{X} \rightarrow \mathcal{X}$ the projection. Then

$$[(T \times \mathcal{X}, \rho \circ \text{pr})] = \Upsilon([T])[(\mathcal{X}, \rho)].$$

- (iii) Let $\rho: \mathcal{X} \rightarrow \mathcal{R}$ be as before and $\mathcal{X} \cong [X/G]$ with X quasi-projective, G a special algebraic group acting on X . Then we have

$$[(\mathcal{X}, \rho)] = \Upsilon([G])^{-1}[(X, \pi)],$$

where π is the composition $\pi: X \rightarrow [X/G] \cong \mathcal{X} \xrightarrow{\rho} \mathcal{R}$.

Remark 4.15. As in the proof of [24, theorem 4.10], any Artin stack of finite type is stratified by global quotient stacks. Then using (i), (iii) in Definition 4.14, one can show that Λ -module $K_0(\text{St}/\mathcal{R})$ is spanned over Λ by $[(X, \pi)]$, where X is a variety and $\pi: X \rightarrow \mathcal{R}$ is a 1-morphism.

Now we descend the map Υ'_A to $K_0(\text{St}/A)_\Upsilon$.

Lemma 4.16. *There is a Λ -module homomorphism $\tilde{\Upsilon}_A: K_0(\text{St}/A)_\Upsilon \rightarrow \Lambda$ such that the following diagram commutes,*

$$\begin{array}{ccc} K_0(\text{St}/A) & \xrightarrow{\Upsilon'_A} & \Lambda \\ \downarrow & \nearrow \tilde{\Upsilon}_A & \\ K_0(\text{St}/A)_\Upsilon & & \end{array}$$

Here the left arrow is the natural quotient map.

Proof. For the relation (ii) of Definition 4.14, we have

$$\Upsilon'_A([(T \times \mathcal{X}, \rho \circ \text{pr})]) = \Upsilon(T) \cdot \Upsilon'_A([\mathcal{X}, \rho]),$$

by Lemma 4.9 and the construction of Υ'_A in Lemma 4.13. The compatibility with Definition 4.14 (iii) follows from (4.8). \square

Let $p: \mathcal{R}' \rightarrow \mathcal{R}$ be a 1-morphism between Artin stacks of locally finite type. Then there is the notion of its push-forward,

$$(4.9) \quad p_*: K_0(\text{St}/\mathcal{R}') \longrightarrow K_0(\text{St}/\mathcal{R}),$$

by taking a 1-morphism $\rho: \mathcal{X} \rightarrow \mathcal{R}'$ to $p \circ \rho: \mathcal{X} \rightarrow \mathcal{R}$. Moreover if p is of finite type, there is the notion of its pull-back,

$$(4.10) \quad p^*: K_0(\text{St}/\mathcal{R}) \longrightarrow K_0(\text{St}/\mathcal{R}'),$$

by taking a 1-morphism $\rho: \mathcal{X} \rightarrow \mathcal{R}$ to the fiber product $\mathcal{X} \times_{\mathcal{R}} \mathcal{R}' \rightarrow \mathcal{R}'$ (cf. [24, definition 3.4]). These operations descend to Λ -module homomorphisms between $K_0(\text{St}/\mathcal{R})_\Upsilon$ and $K_0(\text{St}/\mathcal{R}')_\Upsilon$ (cf. [24, theorem 4.13]).

4.3. Ringel–Hall product

Let \mathcal{M} be the moduli stack of $E \in \mathcal{D}_X$ satisfying (3.3), which we discussed in Section 3. Let $\mathcal{A} \subset \mathcal{D}_X$ be the heart of a bounded t -structure, and take

$v \in N_1(X) \oplus \mathbb{Z}$. We consider the substacks

$$\mathfrak{D}\mathfrak{b}\mathfrak{j}(\mathcal{A}) \subset \mathcal{M}, \quad \mathfrak{D}\mathfrak{b}\mathfrak{j}^v(\mathcal{A}) \subset \mathcal{M}$$

the stack of objects in \mathcal{A} , the stack of objects in \mathcal{A} of numerical type v , respectively. Suppose that $\mathfrak{D}\mathfrak{b}\mathfrak{j}(\mathcal{A})$ is an open substack of \mathcal{M} , hence in particular $\mathfrak{D}\mathfrak{b}\mathfrak{j}(\mathcal{A}), \mathfrak{D}\mathfrak{b}\mathfrak{j}^v(\mathcal{A})$ are Artin stacks of locally finite type. (This condition holds when $\mathcal{A} = \text{Coh}_{\leq 1}(X)$.) Then there is an associative product $*$ on $K_0(\text{St}/\mathfrak{D}\mathfrak{b}\mathfrak{j}(\mathcal{A}))$ and $K_0(\text{St}/\mathfrak{D}\mathfrak{b}\mathfrak{j}(\mathcal{A}))_{\Upsilon}$, based on *Ringel–Hall algebras* [21]. Let $\mathfrak{E}\mathfrak{r}(\mathcal{A})$ be the stack of the exact sequences, $0 \rightarrow E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow 0$ in \mathcal{A} . We have the diagram,

$$(4.11) \quad \begin{array}{ccc} \mathfrak{E}\mathfrak{r}(\mathcal{A}) & \xrightarrow{p_2} & \mathfrak{D}\mathfrak{b}\mathfrak{j}(\mathcal{A}) \\ & \downarrow (p_1, p_3) & \\ & \mathfrak{D}\mathfrak{b}\mathfrak{j}(\mathcal{A}) \times \mathfrak{D}\mathfrak{b}\mathfrak{j}(\mathcal{A}), & \end{array}$$

where p_i takes $0 \rightarrow E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow 0$ to E_i . For elements $f_i \in K_0(\text{St}/\mathfrak{D}\mathfrak{b}\mathfrak{j}(\mathcal{A}))$ with $i = 1, 2$, one can define $f_1 * f_2$ as

$$(4.12) \quad f_1 * f_2 = p_{2*}(p_1 \times p_3)^*(f_1 \times f_2) \in K_0(\text{St}/\mathfrak{D}\mathfrak{b}\mathfrak{j}(\mathcal{A})),$$

where $p_{2*}, (p_1 \times p_3)^*$ are defined in (4.9), (4.10) respectively. Note that if $f_i \in K_0(\text{St}/\mathfrak{D}\mathfrak{b}\mathfrak{j}^{v_i}(\mathcal{A}))$, the element $f_1 * f_2$ is contained in $K_0(\text{St}/\mathfrak{D}\mathfrak{b}\mathfrak{j}^{v_1+v_2}(\mathcal{A}))$. See [21, section 5] for the details.

Theorem 4.17 [21, theorem 5.2]. *The operation $*$ makes $K_0(\text{St}/\mathfrak{D}\mathfrak{b}\mathfrak{j}(\mathcal{A}))$ and $K_0(\text{St}/\mathfrak{D}\mathfrak{b}\mathfrak{j}(\mathcal{A}))_{\Upsilon}$ associative algebras with unit $[(\text{Spec } \mathbb{C}, \rho_0)]$, where $\rho_0: \text{Spec } \mathbb{C} \rightarrow \mathfrak{D}\mathfrak{b}\mathfrak{j}(\mathcal{A})$ corresponds to the zero object.*

In order to simplify the expositions in the following sections, we introduce *Hall algebras of derived categories* $(\mathcal{H}(X), *)$, introduced by Toën [43]. We emphasize that in our proof, we will not use the algebra $(\mathcal{H}(X), *)$ essentially. The algebra $(\mathcal{H}(X), *)$ contains $(K_0(\text{St}/\mathfrak{D}\mathfrak{b}\mathfrak{j}(\mathcal{A})), *)$ as a subalgebra, and all the computations in the proof will be made in the latter algebra for suitable hearts of bounded t -structures $\mathcal{A} \subset \mathcal{D}_X$. As we do not need its actual definition, we only give its rough explanation and properties. Let $\tilde{\mathcal{D}}$ be a dg-category of finite type (cf. [45, definition 2.4]) whose homotopy category is $D(X)$. Then Toën and Vaquié [45, theorem 0.1] showed that the stack of objects in $\tilde{\mathcal{D}}$, which they denote $\underline{\text{Perf}}(X)$ in [45, definition 3.28], is an ∞ -stack of locally geometric and locally of finite presentation. Then

$\mathcal{H}(X)$ is defined by the ∞ -stack version of our notion of stack functions over $\underline{Perf}(X)$ (in [43, paragraph 3.3], it is denoted by $\mathcal{H}_{abs}(\widetilde{\mathcal{D}})$), and the $*$ -product is defined in the similar way of (4.12). Furthermore by [45, corollary 3.21], the stack \mathcal{M} is realized as an open substack of $\underline{Perf}(X)$. Thus if there is a 1-morphism,

$$\rho: \mathcal{X} \longrightarrow \mathcal{M},$$

with \mathcal{X} an Artin stack of finite type, it defines an element $[(\mathcal{X}, \rho)] \in \mathcal{H}(X)$. In particular for the heart of a bounded t -structure $\mathcal{A} \subset \mathcal{D}_X$ with $\mathfrak{D}\mathfrak{b}\mathfrak{j}(\mathcal{A}) \subset \mathcal{M}$ open, the algebra $(K_0(\mathrm{St}/\mathfrak{D}\mathfrak{b}\mathfrak{j}(\mathcal{A})), *)$ is realized as a subalgebra of $\mathcal{H}(X)$ (see [45, 46 – paragraph 3.3] for details).

4.4. Counting invariants of moduli stacks

Next we assume X is a Calabi–Yau threefold, and use the space of stability conditions $\mathrm{Stab}(X)$ and the open subset U_X in Section 3. Let us take $v \in N_1(X) \oplus \mathbb{Z}$, $\sigma \in \overline{U}_X$ and $\phi \in \mathbb{R}$. We consider the substack

$$i: \mathcal{M}^{(v, \phi)}(\sigma) \hookrightarrow \mathcal{M},$$

which is the moduli stack of $E \in \mathcal{P}(\phi)$ of numerical type v . By Proposition 3.9, $\mathcal{M}^{(v, \phi)}(\sigma)$ is an Artin stack of finite type. Also for an interval $I \subset \mathbb{R}$, let $C_\sigma(I) \subset N_1(X) \oplus \mathbb{Z}$ be the image of the map,

$$(\mathrm{ch}_2, \mathrm{ch}_3): \mathcal{P}(I) \setminus \{0\} \longrightarrow N_1(X) \oplus \mathbb{Z}.$$

Definition 4.18. For $\sigma \in \overline{U}_X$, we define $\delta^{(v, \phi)}(\sigma) \in \mathcal{H}(X)$ to be

$$\delta^{(v, \phi)}(\sigma) = [(\mathcal{M}^{(v, \phi)}(\sigma), i)] \in \mathcal{H}(X).$$

Also define $\epsilon^{(v, \phi)}(\sigma) \in \mathcal{H}(X)$ as follows,

$$(4.13) \quad \epsilon^{(v, \phi)}(\sigma) = \sum_{v_1 + \dots + v_n = v} \frac{(-1)^{n-1}}{n} \delta^{(v_1, \phi)}(\sigma) * \dots * \delta^{(v_n, \phi)}(\sigma),$$

where $v_i \in C_\sigma(\phi)$.

We have to check the following, whose proof will be given in Section 6.

Lemma 4.19. *The sum (4.13) is a finite sum.*

Remark 4.20. Suppose that for $\phi \in \mathbb{R}$, there is the heart of a t -structure $\mathcal{A} \subset \mathcal{D}_X$ such that $\mathfrak{D}\mathfrak{b}\mathfrak{j}(\mathcal{A}) \subset \mathcal{M}$ is open and $\mathcal{P}(\phi) \subset \mathcal{A}$. Then $\delta^{(v,\phi)}(\sigma)$ is contained in the subalgebra $K_0(\text{St}/\mathfrak{D}\mathfrak{b}\mathfrak{j}(\mathcal{A}))$ and $\epsilon^{(v,\phi)}(\sigma)$ coincides with the one defined in the algebra $K_0(\text{St}/\mathfrak{D}\mathfrak{b}\mathfrak{j}(\mathcal{A}))$ as in [22, definition 8.1].

Remark 4.21. Suppose any σ -semistable object of numerical type v is stable. Then we must have $\epsilon^{(v,\phi)}(\sigma) = \delta^{(v,\phi)}(\sigma)$.

For $v = (\beta, k) \in N_1(X) \oplus \mathbb{Z}$, let us consider the following open substacks of \mathcal{M} ,

$$\mathfrak{C}\mathfrak{o}\mathfrak{h}(X) := \mathfrak{D}\mathfrak{b}\mathfrak{j}(\text{Coh}_{\leq 1}(X)), \quad \mathfrak{C}\mathfrak{o}\mathfrak{h}^v(X) := \mathfrak{D}\mathfrak{b}\mathfrak{j}^v(\text{Coh}_{\leq 1}(X)).$$

By Remark 4.20, for $0 < \phi \leq 1$ and $\sigma \in U_X$, we have

$$(4.14) \quad \epsilon^{(v,\phi)}(\sigma) \in K_0(\text{St}/\mathfrak{C}\mathfrak{o}\mathfrak{h}^v(X)).$$

Using the same argument as in [37, chapter 5, section 4], we have the following 1-morphism,

$$\pi: \mathfrak{C}\mathfrak{o}\mathfrak{h}^v(X) \ni E \longmapsto s(E) \in \text{Chow}_\beta(X).$$

Now we define the element $P(v, \sigma) \in \Lambda$.

Definition 4.22. For $v \in N_1(X) \oplus \mathbb{Z}$ and $\sigma \in U_X$ as before, we define $P(v, \sigma) \in \Lambda$ as follows.

- If $v \in C_\sigma(\phi)$ with $0 < \phi \leq 1$, we define

$$P(v, \sigma) := (\mathbb{L} - 1)\Upsilon'_A(\pi_*\epsilon^{(v,\phi)}(\sigma)),$$

where $\mathbb{L} = \Upsilon(\mathbb{A}^1)$ and $A = \text{Chow}_\beta(X)$. Note that $\pi_*\epsilon^{(v,\phi)}(\sigma) \in K_0(\text{St}/A)$ makes sense by (4.14).

- If $v \in C_\sigma(\phi)$ with $1 < \phi \leq 2$, we define

$$P(v, \sigma) := P(-v, \sigma).$$

Note that in this case $-v \in C_\sigma(\phi - 1)$, hence the right-hand side makes sense.

- Otherwise we define $P(v, \sigma) = 0$.

Remark 4.23. The definition of $P(v, \sigma)$ for $v \in C_\sigma(\phi)$ with $1 < \phi \leq 2$ is motivated by the following observation. Let $\mathfrak{Coh}^v(X)[1]$ be the stack of objects $E \in \text{Coh}_{\leq 1}(X)[1]$ of numerical type v . Then we have

$$\epsilon^{(v, \phi)}(\sigma) \in K_0(\text{St}/\mathfrak{Coh}^v(X)[1]).$$

Also we have the 1-morphism,

$$\pi' : \mathfrak{Coh}^v(X)[1] \ni E \mapsto -s(E) \in A' := \text{Chow}_{-\beta}(X).$$

Hence it is reasonable to define

$$(4.15) \quad P(v, \sigma) = (\mathbb{L} - 1)\Upsilon'_{A'}(\pi'_*\epsilon^{(v, \phi)}(\sigma)).$$

However we can easily see that the right-hand side of (4.15) is equal to $P(-v, \sigma)$, hence we define $P(v, \sigma)$ as in Definition 4.22 to reduce the exposition.

We have the following proposition. (Recall that we have defined $\mathbb{Z}[t, s]^\dagger$ and Q^b for $Q \in \mathbb{Z}[t, s]^\dagger$ in Definition 4.7.)

Proposition 4.24. *If $\sigma = \sigma_{(0, \omega)}$ with ω ample and $v = (\beta, 1)$, we have $P(v, \sigma) \in \mathbb{Z}[t, s]^\dagger$.*

Proof. We may assume $v \in C_\sigma(\phi)$ for $0 < \phi \leq 1$. If $\sigma = \sigma_{(0, \omega)}$ and $v = (\beta, 1)$, any semistable object in σ of type v is stable. In fact suppose there is a σ -semistable object $E \in \text{Coh}_{\leq 1}(X)$ which is not stable. Then there is an exact sequence $0 \rightarrow E_1 \rightarrow E \rightarrow E_2 \rightarrow 0$ in $\text{Coh}_{\leq 1}(X)$ such that

$$(4.16) \quad \arg Z_{(0, \omega)}(E) = \arg Z_{(0, \omega)}(E_1) = \arg Z_{(0, \omega)}(E_2).$$

Because $\text{ch}_3(E) = 1$, we have $\text{Re } Z_{(0, \omega)}(E) < 0$, hence (4.16) implies $\text{Re } Z_{(0, \omega)}(E_i) < 0$ for $i = 1, 2$. Since $-\text{ch}_3(E_i) = \text{Re } Z_{(0, \omega)}(E_i)$, this contradicts to $1 = \text{ch}_3(E) = \text{ch}_3(E_1) + \text{ch}_3(E_2)$.

By Remark 4.21 we have

$$\epsilon^{(v, \phi)}(\sigma) = \delta^{(v, \phi)}(\sigma) = [([M^v/\mathbb{G}_m], \rho)],$$

for some projective variety M^v , \mathbb{G}_m acting on M^v trivially and ρ is a 1-morphism $[M^v/\mathbb{G}_m] \rightarrow \mathfrak{Coh}(X)$. (The factor \mathbb{G}_m comes from the stabilizers $\text{Aut}(E) \cong \mathbb{G}_m$ for stable objects E .) In fact M^v is the moduli space of ω -Gieseker stable sheaves of Chern character v . Thus by Remark 4.6 and noting $\mathbb{L} - 1 = \Upsilon(\mathbb{G}_m)$, we have $P(v, \sigma) \in \mathbb{Z}[t, s]^\dagger$. \square

Now we define the notion of motivic Gopakumar–Vafa invariant.

Definition 4.25. For $\sigma = \sigma_{(0,\omega)} \in U_X$ and $v = (\beta, 1)$, write $P(v, \sigma)^b$ as

$$P(v, \sigma)^b = \sum_{\alpha, l} \nu_l^\alpha(\beta) t^\alpha s^{l-1}.$$

By Proposition 4.24, it is possible to define $P(v, \sigma)^b$. Then define the *motivic Gopakumar–Vafa invariant* $n_g^\beta(X)$ as follows:

$$(4.17) \quad n_g^\beta(X) = \sum_{\alpha+l \geq 1} (-1)^{\alpha+g} l \nu_l^\alpha(\beta) \times \left\{ \binom{\alpha+l+g}{2g+1} - \binom{\alpha+l+g-2}{2g+1} \right\}.$$

Remark 4.26. The motivation of (4.17) in Definition 4.25 comes from Proposition 2.5. Obviously the invariant $n_g^\beta(X)$ coincides with \tilde{n}_g^β if β is represented by an effective one cycle, and the moduli space M^β in Paragraph 2.3 is smooth.

Remark 4.27. Still the definition of n_g^β does not involve virtual classes, so n_g^β is unlikely to be deformation invariant. On the other hand, our construction of BPS-count has a possibility to involve virtual classes using Behrend’s constructible functions [3]. As in the proof of Proposition 4.24, $P(v, \sigma)$ in Definition 4.25 is equal to $\Upsilon_A([(M^v, \rho)])$ for some projective variety over A , $\rho: M^v \rightarrow A$. Then using Behrend’s constructible function $\nu: M^v \rightarrow \mathbb{Z}$, one might try to define $P'(v, \sigma)$ something like

$$P'(v, \sigma) = \sum_n \pm n \Upsilon_A([\nu^{-1}(n), \rho|_{\nu^{-1}(n)}]),$$

and construct $n_g^{\beta'}$ by the same way as in Definition 4.25. In this paper we stick to Definition 4.25 in order to show birational invariance, though it seems interesting to pursue this construction.

In the next paragraph, we will show that $n_g^\beta(X)$ does not depend on a choice of ω .

4.5. Local transformation formula of the counting invariants

The aim of this paragraph is to give the transformation formula of $\epsilon^{(v,\phi)}(\sigma)$ under small deformations of σ . Again we assume X is a Calabi–Yau threefold. Let us fix the following data:

$$v \in N_1(X) \oplus \mathbb{Z}, \quad \phi \in \mathbb{R}, \quad \sigma = (Z, \mathcal{P}) \in \overline{\mathcal{U}}_X.$$

Furthermore we fix an open neighborhood $\sigma \in \mathcal{U}$ in $\text{Stab}(X)$ such that $\overline{\mathcal{U}}$ is compact. We set $\mathcal{S} \subset \mathcal{D}_X$ to be the set of objects,

$$\mathcal{S} := \{E \in \mathcal{D}_X \mid E \text{ is semistable in some } \sigma' = (Z', \mathcal{P}') \in \overline{\mathcal{U}} \text{ with } |Z'(E)| \leq |Z'(v)|\}.$$

Then \mathcal{S} has bounded mass, hence there is a wall and chamber structure on $\overline{\mathcal{U}}$ as in Proposition 3.7. Let $\mathcal{C} \subset \overline{\mathcal{U}}$ be a chamber with $\sigma \in \overline{\mathcal{C}}$. By the definition of the topology on $\text{Stab}(X)$ (cf. [7, Section 6]), we can take $\tau = (W, \mathcal{Q}) \in \mathcal{U}_X \cap \mathcal{C}$ and $0 < \varepsilon < 1/6$ such that

$$\mathcal{P}(\phi) \subset \mathcal{Q}((\phi - \varepsilon, \phi + \varepsilon)), \quad \mathcal{Q}(\phi) \subset \mathcal{P}((\phi - \varepsilon, \phi + \varepsilon)),$$

for any $\phi \in \mathbb{R}$. Furthermore we can take W to be defined over \mathbb{Q} . Note that for any $v \in C_\tau((\phi - \varepsilon, \phi + \varepsilon))$, there are uniquely determined $\phi(v), \phi'(v) \in (\phi - \frac{1}{2}, \phi + \frac{1}{2})$ such that

$$(4.18) \quad W(v) \in \mathbb{R}_{>0} \exp(\pi i \phi(v)), \quad Z(v) \in \mathbb{R}_{>0} \exp(\pi i \phi'(v)),$$

by our choice of ε . Recall the definition of $\mathbb{H} \subset \mathbb{C}$ in (3.1).

Proposition 4.28. *Let σ , τ and $\varepsilon > 0$ be as before. There is a unique sequence of functions $u_n: \mathbb{H}^{2n} \rightarrow \mathbb{Q}$ such that we have the following in $\mathcal{H}(X)$,*

$$(4.19) \quad \begin{aligned} \epsilon^{(v,\phi)}(\sigma) &= \sum_{v_1 + \dots + v_n = v} u_n(z_1, \dots, z_n, w_1, \dots, w_n) \\ &\quad \times \epsilon^{(v_1, \phi_1)}(\tau) * \dots * \epsilon^{(v_n, \phi_n)}(\tau), \end{aligned}$$

$$(4.20) \quad = \epsilon^{(v,\phi)}(\tau) + [\text{multiple commutators of } \epsilon^{(v_i, \phi_i)}(\tau)],$$

where $v_i \in C_\tau((\phi - \varepsilon, \phi + \varepsilon))$, $\phi_i = \phi(v_i)$ and

$$z_i = \exp\left(-\pi i \left(\phi - \frac{1}{2}\right)\right) Z(v_i), \quad w_i = \exp\left(-\pi i \left(\phi - \frac{1}{2}\right)\right) W(v_i).$$

Here (4.19) is a finite sum, and $[\dots]$ in (4.20) is a finite \mathbb{Q} -linear combination of multiple commutators of $\epsilon^{(v_i, \phi_i)}(\tau)$.

Proof. This is an application of the arguments in [23, theorem 5.2] to Bridgeland's stability conditions, and the proof is same as in [41, equation (68)]. Note that (4.18) implies $z_i, w_i \in \mathbb{H}$, thus (4.19) makes sense. By [41, proposition 3.18], there is $\psi \in \mathbb{R}$ such that

$$\psi - 1 < \phi - \varepsilon < \phi + \varepsilon < \psi,$$

and $\mathfrak{D}\mathbf{b}j(\mathcal{A}_\psi) \subset \mathcal{M}$ is open for $\mathcal{A}_\psi = \mathcal{Q}((\psi - 1, \psi])$. Since all the terms in (4.19) are contained in $K_0(\mathrm{St}/\mathfrak{D}\mathbf{b}j(\mathcal{A}_\psi))$, it is enough to show (4.19) in $K_0(\mathrm{St}/\mathfrak{D}\mathbf{b}j(\mathcal{A}_\psi))$. Then the straightforward adaptation of the arguments in [41, proposition 5.23] which deduces [41, equation (68)] gives the desired equality. (Note that in *loc.cite*, we worked over an algebra $A(\mathcal{A}_\psi, \Lambda, \chi)$, not over $K_0(\mathrm{St}/\mathfrak{D}\mathbf{b}j(\mathcal{A}_\psi))$.) However readers can find that the same argument is applied. Moreover in *loc.cite*, the coefficients are given in the form $U(\{v_i\}_{1 \leq i \leq n}, \tau, \sigma)$. These are obviously rephrased in terms of the functions $u_n: \mathbb{H}^{2n} \rightarrow \mathbb{Q}$, as in [25, paragraph 3.1]. Finally the formula (4.20) follows from [23, theorem 5.2]. \square

Remark 4.29. Suppose that there is the heart of a t -structure $\mathcal{A} \subset \mathcal{D}_X$ with $\mathfrak{D}\mathbf{b}j(\mathcal{A}) \subset \mathcal{M}$ open, and all the terms $\epsilon^{(v, \phi)}(\sigma)$, $\epsilon^{(v_i, \phi_i)}(\tau)$ in (4.19) are contained in $K_0(\mathrm{St}/\mathfrak{D}\mathbf{b}j(\mathcal{A}))$. Then (4.19), (4.20) hold in the algebra $K_0(\mathrm{St}/\mathfrak{D}\mathbf{b}j(\mathcal{A}))$.

The explicit formula of u_n (cf. [23, definition 4.4]) is complicated and we do not need this. Now we show the following proposition.

Proposition 4.30. *For any $v = (\beta, k) \in N_1(X) \oplus \mathbb{Z}$, the element $P(v, \sigma) \in \Lambda$ does not depend on a choice of $\sigma \in U_X$.*

Proof. For $\sigma \in U_X$, let us take $\tau \in U_X$ and $\varepsilon > 0$ as in Proposition 4.28. It is enough to show $P(v, \sigma) = P(v, \tau)$ in this situation. First assume $v \notin C_\sigma(\phi)$ for any $\phi \in \mathbb{R}$, thus $P(v, \sigma) = 0$. If $P(v, \tau) \neq 0$, there is some τ -semistable object $E \in \mathcal{D}_X$ of numerical type v . Because τ is contained in a chamber, E must be also semistable in σ , which is a contradiction. (See the comment in [7] after [7, proposition 8.1].) Hence $P(v, \tau) = 0$ follows.

Next suppose $v \in C_\sigma(\phi)$ for some ϕ . We may assume $0 < \phi \leq 1$. If $\phi = 1$, then $\beta = 0$ and $v \in C_\tau(1)$. Since we have

$$\mathcal{P}(1) = \mathcal{Q}(1) = \{\text{zero-dimensional sheaves}\},$$

it follows that $\delta^{(v_i, 1)}(\sigma) = \delta^{(v_i, 1)}(\tau)$ for any $v_i \in C_\sigma(1) = C_\tau(1)$. Hence $\epsilon^{(v, 1)}(\sigma) = \epsilon^{(v, 1)}(\tau)$ and $P(v, \sigma) = P(v, \tau)$ follows.

Finally suppose $0 < \phi < 1$. We can take $\varepsilon > 0$ sufficiently small such that $0 < \phi - \varepsilon < \phi + \varepsilon < 1$. Then all the terms $\epsilon^{(v_i, \phi_i)}(\tau)$ in (4.19) are contained in $K_0(\text{St}/\mathfrak{Coh}(X))$, and (4.20) holds in $K_0(\text{St}/\mathfrak{Coh}(X))$. Then applying Lemma 4.16 and Remark 4.15, it is enough to show the following: for two varieties U_1, U_2 with 1-morphisms $\rho_i: U_i \rightarrow \mathfrak{Coh}^{v_i}(X)$, where $v_i \in C_\tau((\phi - \varepsilon, \phi + \varepsilon))$ with $v_1 + v_2 = v$, we have

$$(4.21) \quad \Upsilon'_A(\pi_*[f_1, f_2]) = 0, \quad f_i = [(U_i, \rho_i)] \in K_0(\text{St}/\mathfrak{Coh}(X)),$$

where $A = \text{Chow}_\beta(X)$. Note that if there is an exact sequence $0 \rightarrow E_1 \rightarrow E \rightarrow E_2 \rightarrow 0$ in $\text{Coh}_{\leq 1}(X)$, we have $s(E) = s(E_1 \oplus E_2)$. Using this and Lemma 4.31 next, we can conclude (4.21) holds. \square

Lemma 4.31. *Let $\mathcal{A} \subset \mathcal{D}_X$ be the heart of a t -structure with $\mathfrak{D}\mathfrak{b}\mathfrak{j}(\mathcal{A}) \subset \mathcal{M}$ open. For $v \in N_1(X) \oplus \mathbb{Z}$, assume that there is a 1-morphism $\pi: \mathfrak{D}\mathfrak{b}\mathfrak{j}^v(\mathcal{A}) \rightarrow A$, where A is a projective variety, which satisfies,*

$$(4.22) \quad \begin{aligned} \pi([E]) &= \pi([E_1 \oplus E_2]), \quad \text{for any exact sequence} \\ 0 &\rightarrow E_1 \rightarrow E \rightarrow E_2 \rightarrow 0 \text{ in } \mathcal{A}, \end{aligned}$$

where $E \in \mathcal{A}$ is of numerical type v . Then we have

$$(4.23) \quad \Upsilon'_A(\pi_*[f_1, f_2]) = 0, \quad f_i = [(U_i, \rho_i)] \in K_0(\text{St}/\mathfrak{D}\mathfrak{b}\mathfrak{j}(\mathcal{A})),$$

where U_1, U_2 are quasi-projective varieties.

Proof. For \mathbb{C} -valued points $p_i \in U_i$, let $E(p_i) \in \mathcal{A}$ be the objects corresponding to $\rho_i(p_i)$. Let us decompose $U_1 \times U_2$ into finite locally closed pieces,

$$U_1 \times U_2 = \coprod_k W_k,$$

such that the dimensions of $\text{Ext}^j(E(p_1), E(p_2)), \text{Ext}^j(E(p_2), E(p_1))$ are constant on each W_k for $j = 0, 1$. Furthermore we may assume that the

bundles

$$\bigcup_{(p_1, p_1) \in W_k} \text{Ext}^j(E(p_2), E(p_1)) \rightarrow W_k, \quad \bigcup_{(p_1, p_2) \in W_k} \text{Ext}^j(E(p_1), E(p_2)) \rightarrow W_k,$$

are trivial bundles with fibers V_k^j, \bar{V}_k^j for $j = 0, 1$ respectively. Let us consider the diagram (4.11) which defines $*$ -product on $K_0(\text{St}/\mathfrak{D}\mathfrak{b}\mathfrak{j}(\mathcal{A}))$. Then the set of \mathbb{C} -valued points of the following stack,

$$(4.24) \quad W_k \times_{\mathfrak{D}\mathfrak{b}\mathfrak{j}(\mathcal{A}) \times \mathfrak{D}\mathfrak{b}\mathfrak{j}(\mathcal{A})} \mathfrak{E}\mathfrak{r}(\mathcal{A}),$$

is identified with the \mathbb{C} -valued points of $W_k \times V_k^1$. Let $0 \rightarrow E(p_1) \rightarrow E \rightarrow E(p_2) \rightarrow 0$ be an exact sequence in \mathcal{A} which represents a \mathbb{C} -valued point of (4.24). Then the stabilizers at this point in (4.24) is identified with the fiber at (id, id) of the following morphism,

$$\text{Aut}(0 \rightarrow E(p_1) \rightarrow E \rightarrow E(p_2) \rightarrow 0) \rightarrow \text{Aut}(E(p_1)) \times \text{Aut}(E(p_2)),$$

which is isomorphic to $V_k^0 = \text{Hom}(E(p_2), E(p_1))$. Hence we have

$$(p_1, p_3)^* \{(f_1, f_2)|_{W_k}\} = [W_k \times V_k^1 / V_k^0],$$

where V_k^0 acts on $W_k \times V_k^1$ trivially. Therefore we can write $f_1 * f_2$ in the following form,

$$f_1 * f_2 = \sum_k [([W_k \times V_k^1 / V_k^0], \rho'_k)],$$

for some 1-morphism $\rho'_k: [W_k \times V_k^1 / V_k^0] \rightarrow \mathfrak{D}\mathfrak{b}\mathfrak{j}^v(\mathcal{A})$. (Also see [23, theorem 5.18].) Let us consider the composition,

$$(4.25) \quad W_k \times V_k^1 \rightarrow [W_k \times V_k^1 / V_k^0] \xrightarrow{\rho'_k} \mathfrak{D}\mathfrak{b}\mathfrak{j}^v(\mathcal{A}) \xrightarrow{\pi} A.$$

By the assumption (4.22), the earlier morphism is nothing but the following map,

$$W_k \times V_k^1 \ni (p_1, p_2, v) \mapsto \pi([E(p_1) \oplus E(p_2)]).$$

Hence the morphism (4.25) descends to the morphism, $\rho_k^\dagger: W_k \rightarrow A$, which takes (p_1, p_2) to $\pi([E(p_1) \oplus E(p_2)])$. Therefore by Lemma 4.9 and

using (4.8), we have

$$\Upsilon'_A(\pi_*(f_1 * f_2)) = \sum_k \mathbb{L}^{\dim V_k^1 - \dim V_k^0} \Upsilon_A([(W_k, \rho_k^\dagger)]).$$

(Recall that we have defined $\mathbb{L} = \Upsilon(\mathbb{A}^1)$ in Definition 4.22.) Arguing as in the same way for $\Upsilon'_A(\pi_*(f_2 * f_1))$ and taking their difference, we obtain

$$\Upsilon'_A(\pi_*[f_1, f_2]) = \sum_k (\mathbb{L}^{\dim V_k^1 - \dim V_k^0} - \mathbb{L}^{\dim \bar{V}_k^1 - \dim \bar{V}_k^0}) \Upsilon_A([(W_k, \rho_k^\dagger)]).$$

Then Sublemma 4.32 next shows $\dim V_k^1 - \dim V_k^0 = \dim \bar{V}_k^1 - \dim \bar{V}_k^0$, hence (4.21) follows. \square

We have used the following sublemma.

Sublemma 4.32. *Let $\mathcal{A} \subset \mathcal{D}_X$ be the heart of a t -structure and take $E, F \in \mathcal{A}$. Then one has*

$$\dim \operatorname{Ext}^1(E, F) - \dim \operatorname{Hom}(E, F) = \dim \operatorname{Ext}^1(F, E) - \dim \operatorname{Hom}(F, E).$$

Proof. Since X is a Calabi–Yau threefold, we have

$$\begin{aligned} \chi(E, F) &:= \sum_k (-1)^k \dim \operatorname{Ext}^k(E, F) \\ &= -\dim \operatorname{Ext}^1(E, F) + \dim \operatorname{Hom}(E, F) \\ &\quad + \dim \operatorname{Ext}^1(F, E) - \dim \operatorname{Hom}(F, E), \end{aligned}$$

by Serre duality. On the other hand Riemann–Roch implies $\chi(E, F) = 0$ because $\operatorname{ch}_i(E) = \operatorname{ch}_i(F) = 0$ for $i = 0, 1$. \square

Combined with Proposition 4.24, we have the following.

Corollary 4.33. *For any $\sigma \in U_X$ and $v = (\beta, 1)$, we have $P(v, \sigma) = P(v, \sigma_{(0, \omega)}) \in \mathbb{Z}[t, s]^\dagger$, and $n_g^\beta(X)$ does not depend on a choice of ω .*

5. Birational invariance of the counting invariants

Now we state our main theorem.

Theorem 5.1. *Let $\phi: W \dashrightarrow X$ be a birational map between smooth projective Calabi–Yau threefolds. Then for $\beta \in N_1(W)$, one has*

$$n_g^\beta(W) = n_g^{\phi_*\beta}(X).$$

The strategy is as follows. First we enlarge the definition of $P(v, \sigma) \in \Lambda$ for some boundary points $\sigma \in \bar{U}_X$, and show $P(v, \sigma) = P(v, \tau)$ for $\tau \in U_X$. Next we compare $P(v, \sigma)$ with $P(v', \sigma')$ defined for $\sigma' \in \bar{U}_W$, using the derived equivalence [8], $\Phi: \mathcal{D}_W \rightarrow \mathcal{D}_X$.

5.1. Perverse t -structures on \mathcal{D}_X

Before giving the proof of Theorem 5.1, we investigate some boundary points in \bar{U}_X . We assume there is a diagram of birational maps,

$$(5.1) \quad \begin{array}{ccc} (C^\dagger \subset W) & & (X \supset C) \\ & \searrow g & \swarrow f \\ & (0 \in Y), & \end{array}$$

where C and C^\dagger are tree of rational curves. Furthermore we assume that relative Picard numbers of f and g are one, and $\phi: W \dashrightarrow X$ is not an isomorphism. In this case the diagram (5.1) is called a *flop* (cf. [28]). The main technical tool we use here is the notion of *perverse t -structures* associated to $f: X \rightarrow Y$. It was introduced by T. Bridgeland [8] to construct the derived equivalence between W and X . Next we collect some results we need.

Proposition 5.2. *There are hearts of bounded t -structures ${}^p\text{Per}(\mathcal{D}_X) \subset \mathcal{D}_X$ for $p = -1, 0$ which satisfy the following.*

- (i) *For any $E \in {}^p\text{Per}(\mathcal{D}_X)$, we have $\mathbf{R}f_*E \in \text{Coh}_{\leq 1}(Y)$.*
- (ii) *There is an equivalence $\Phi: \mathcal{D}_W \rightarrow \mathcal{D}_X$ which restrict to the equivalence,*

$$\Phi: {}^{-1}\text{Per}(\mathcal{D}_W) \longrightarrow {}^0\text{Per}(\mathcal{D}_X).$$

Furthermore Φ induce the following commutative diagrams,

$$(5.2) \quad \begin{array}{ccc} \mathcal{D}_W & \xrightarrow{\Phi} & \mathcal{D}_X \\ (\text{ch}_2, \text{ch}_3) \downarrow & & \downarrow (\text{ch}_2, \text{ch}_3) \\ N_1(W) \oplus \mathbb{Z} & \xrightarrow{\phi_*} & N_1(X) \oplus \mathbb{Z}, \end{array} \quad \begin{array}{ccc} \text{Stab}(W) & \xrightarrow{\Phi_*} & \text{Stab}(X) \\ \mathcal{Z}_W \downarrow & & \downarrow \mathcal{Z}_X \\ N^1(W)_{\mathbb{C}} & \xrightarrow{\phi_*} & N^1(X)_{\mathbb{C}}. \end{array}$$

Here ϕ_* in the left diagram takes (β, k) to $(\phi_*\beta, k)$, and Φ_* is the natural isomorphism induced by the equivalence Φ .

(iii) Let H be a relatively ample divisor on X over Y . Then for a sufficiently small $0 < \delta \ll 1$ and an ample divisor ω' on Y , the pairs

$$(5.3) \quad \begin{aligned} \sigma_{(-\delta H, f^* \omega')} &= (Z_{(-\delta H, f^* \omega')}, {}^0\text{Per}(\mathcal{D}_X)), \\ \sigma_{(\delta H, f^* \omega')} &= (Z_{(\delta H, f^* \omega')}, {}^{-1}\text{Per}(\mathcal{D}_X)), \end{aligned}$$

determine stability conditions contained in $\overline{\mathcal{U}}_X$.

(iv) For $p = -1, 0$, the stack of objects $E \in {}^p\text{Per}(\mathcal{D}_X)$,

$${}^p\mathfrak{P}\text{er}(X) \subset \mathcal{M},$$

is an open substack of \mathcal{M} .

Proof. In [8, section 3], Bridgeland constructed the hearts of some bounded t -structures ${}^p\text{Per}(X/Y)$ on $D(X)$ for $p = -1, 0$. For simplicity we discuss the case of $p = 0$. According to [46, lemma 3.1], the abelian category ${}^0\text{Per}(X/Y)$ is obtained from $\text{Coh}(X)$ as a tilting of the torsion pair,

$$\begin{aligned} \mathcal{T}_0 &= \{T \in \text{Coh}(X) \mid R^1 f_* T = 0\}, \\ \mathcal{F}_0 &= \{F \in \text{Coh}(X) \mid f_* F = 0, \text{Hom}(\mathfrak{C}, F) = 0\}, \end{aligned}$$

where $\mathfrak{C} := \{E \in \text{Coh}(X) \mid \mathbf{R}f_* E = 0\}$, i.e., ${}^0\text{Per}(X/Y)$ is generated by $\mathcal{F}_0[1]$ and \mathcal{T}_0 . Let us define ${}^p\text{Per}(\mathcal{D}_X)$ to be

$${}^p\text{Per}(\mathcal{D}_X) := \mathcal{D}_X \cap {}^p\text{Per}(X/Y).$$

We have to check that ${}^p\text{Per}(\mathcal{D}_X)$ is the heart of a bounded t -structure on \mathcal{D}_X . Since $\mathcal{F}_0 \subset \text{Coh}_{\leq 1}(X)$, the pair $(\mathcal{F}_0, \mathcal{T}_0 \cap \text{Coh}_{\leq 1}(X))$ also determines a torsion pair on $\text{Coh}_{\leq 1}(X)$, and the corresponding tilting is ${}^0\text{Per}(\mathcal{D}_X)$. Thus ${}^0\text{Per}(\mathcal{D}_X)$ is the heart of a bounded t -structure on \mathcal{D}_X (cf. [15]).

(i) For $E \in {}^p\text{Per}(\mathcal{D}_X)$, the object $\mathbf{R}f_* E$ must be a sheaf by the definition of ${}^p\text{Per}(X/Y)$ in [8, section 3].

(ii) In [8], Bridgeland constructed the equivalence,

$$\Phi: D(W) \longrightarrow D(X),$$

which restricts to an equivalence between ${}^{-1}\text{Per}(W/Y)$ and ${}^0\text{Per}(X/Y)$. Furthermore Chen [9] showed that Φ is given by a Fourier–Mukai functor with kernel $\mathcal{O}_{W \times_Y X}$. Because $\phi: W \dashrightarrow X$ is an isomorphism in codimension one, the equivalence Φ takes \mathcal{D}_W to \mathcal{D}_X .

For the left diagram of (5.2), let us take a divisor D on X and $E \in \mathcal{D}_W$. By Riemann–Roch theorem, we have

$$(5.4) \quad \chi(\mathcal{O}_X(D), \Phi(E)) = -D \cdot \text{ch}_2 \Phi(E) + \text{ch}_3 \Phi(E),$$

$$(5.5) \quad \chi(\Phi^{-1}\mathcal{O}_X(D), E) = -\phi_*^{-1}D \cdot \text{ch}_2(E) + \text{ch}_3(E).$$

Here we have used the fact that $\text{ch}_1 \Phi^{-1}\mathcal{O}_X(D) = \phi_*^{-1}D$. This follows because $\Phi^{-1}(\mathcal{O}_X(D))$ and $\mathcal{O}_X(\phi_*^{-1}D)$ are isomorphic over $W \setminus C^\dagger$, and C^\dagger has codimension two in W (cf. [42, lemma 3.15]). By adjunction we must have (5.4) = (5.5), and this holds for any divisor D . Thus we have

$$(\text{ch}_2 \Phi(E), \text{ch}_3 \Phi(E)) = (\phi_* \text{ch}_2(E), \text{ch}_3(E)),$$

by the definition of $\phi_*: N_1(W) \rightarrow N_1(X)$.

For the commutativity of the right diagram of (5.2), the same proof of [42, lemma 4.8] is applied, and we leave the readers to check the detail.

(iii) The same proof of [42, lemma 4.3] shows that the pairs (5.3) give stability conditions. In fact arguing as in [42, lemma 3.8 (iii)], any object in ${}^0\text{Per}(\mathcal{D}_X)$ is given by a successive extension of the following objects,

$$(5.6) \quad S_0 = \omega_{f^{-1}(0)}[1], \quad S_i = \mathcal{O}_{C_i}(-1) \quad (1 \leq i \leq m),$$

$$(5.7) \quad \widetilde{\text{Coh}}_{\leq 1}(X) := \{F \in \text{Coh}_{\leq 1}(X) \mid C_i \not\subseteq \text{Supp}(F) \text{ for all } i\}.$$

Here C_i for $1 \leq i \leq m$ are the irreducible components of C and $f^{-1}(0)$ is the scheme theoretic fiber of f at $0 \in Y$. In order to show (3.1) in Proposition 3.2, it is enough to check this for the generators (5.6) and (5.7). For $Z = Z_{(-\delta H, f^*\omega')}$, we have

$$Z(S_0) = -1 + \delta H \cdot f^{-1}(0) < 0, \quad Z(S_i) = -\delta H \cdot C_i < 0 \quad (1 \leq i \leq m),$$

$$\text{Im } Z(F) > 0 \quad \text{for } F \in \widetilde{\text{Coh}}_{\leq 1}(X) \setminus \{0\},$$

thus (3.1) holds. We leave the readers to check the Harder–Narasimhan property, applying the proof of [42, lemma 4.3]. Also the case of $p = -1$ is similarly proved. Finally we have to check that the stability conditions determined by (5.3) are contained in \overline{U}_X . Since it requires some more technical arguments, we postpone it until Section 6.

(iv) According to [46], there are vector bundles ${}^p\mathcal{E}$ on X for $p = -1, 0$ such that an object $E \in \mathcal{D}_X$ is contained in ${}^p\text{Per}(\mathcal{D}_X)$ if and only if

$$(5.8) \quad \mathbf{R}f_*\mathbf{R}\mathcal{H}om({}^p\mathcal{E}, E) \in \text{Coh}(Y).$$

Since (5.8) is an open condition, the stack ${}^p\mathfrak{Per}(X)$ is an open substack of \mathcal{M} . \square

For $v = (\beta, k) \in N_1(X) \oplus \mathbb{Z}$, let

$${}^p\mathfrak{Per}^v(X) \subset {}^p\mathfrak{Per}(X),$$

the substack of objects $E \in {}^p\text{Per}(\mathcal{D}_X)$ of numerical type v . By Proposition 5.2 (i), we have the 1-morphism,

$$\bar{\pi}: {}^p\mathfrak{Per}^v(X) \ni E \longmapsto s(\mathbf{R}f_*E) \in \text{Chow}_{f_*\beta}(Y).$$

Let $\sigma \in \bar{U}_X$ be one of (5.3), corresponding to ${}^p\text{Per}(\mathcal{D}_X)$ for $p = -1$ or 0 . As in (4.14), for $0 < \phi \leq 1$ we have

$$(5.9) \quad \epsilon^{(v,\phi)}(\sigma) \in K_0(\text{St}/{}^p\mathfrak{Per}(X)).$$

Definition 5.3. Let $\sigma \in \bar{U}_X$ be one of (5.3). We define $P(v, \sigma) \in \Lambda$ as follows.

- If $v \in C_\sigma(\phi)$ with $0 < \phi \leq 1$, we define

$$P(v, \sigma) := (\mathbb{L} - 1)\Upsilon'_{\bar{A}}(\bar{\pi}_*\epsilon^{(v,\phi)}(\sigma)),$$

where $\bar{A} = \text{Chow}_{f_*\beta}(Y)$. By (5.9), $\bar{\pi}_*\epsilon^{(v,\phi)}(\sigma) \in K_0(\text{St}/\bar{A})$ makes sense.

- If $v \in C_\sigma(\phi)$ with $1 < \phi \leq 2$, we define

$$P(v, \sigma) := P(-v, \sigma).$$

- Otherwise we define $P(v, \sigma) = 0$.

Remark 5.4. In [46], it is shown that there are vector bundles ${}^p\mathcal{E}$ on X for $p = -1, 0$ such that there are equivalences,

$${}^p\mathrm{Per}(X/Y) \xrightarrow{\sim} \mathrm{Coh}(f_*\mathcal{H}\mathrm{om}({}^p\mathcal{E}, {}^p\mathcal{E})).$$

Since the right-hand side is the module category over a non-commutative sheaf of algebras on Y , BPS counting constructed from Definition 5.3 and the formula (4.17) is interpreted as (approximation of) non-commutative Gopakumar–Vafa invariant. It seems interesting to pursue its relationship to non-commutative Donaldson–Thomas theory on conifold studied by B. Szendrői [40].

For $\sigma \in \overline{U}_X$ as in Proposition 5.2 (iii), let us write it $\sigma = (Z, \mathcal{P})$ as in Definition 3.1. We also use the abelian category

$$\mathcal{A}_\sigma = \mathcal{P} \left(\left(-\frac{1}{2}, \frac{1}{2} \right] \right),$$

and the stack of objects in \mathcal{A}_σ , $\mathfrak{Obj}(\mathcal{A}_\sigma) \subset \mathcal{M}$. By Proposition 5.2(iv) and [41, proposition 3.18], $\mathfrak{Obj}(\mathcal{A}_\sigma)$ is an open substack of \mathcal{M} . We have the following lemma.

Lemma 5.5. *Under the previous situation, the following diagram is commutative,*

$$(5.10) \quad \begin{array}{ccccc} K_0(\mathrm{St}/\mathfrak{Coh}^v(X)) \cap K_0(\mathrm{St}/{}^p\mathfrak{Per}^v(X)) & \longrightarrow & K_0(\mathrm{St}/\mathfrak{Coh}^v(X)) & \xrightarrow{\pi_*} & K_0(\mathrm{St}/A) \\ & \searrow & & & \downarrow f_* \\ & & K_0(\mathrm{St}/{}^p\mathfrak{Per}^v(X)) & \xrightarrow{\tilde{\pi}_*} & K_0(\mathrm{St}/\bar{A}) \xrightarrow{\Upsilon'_A} \Lambda. \end{array}$$

Here $A = \mathrm{Chow}_\beta(X)$ and $\bar{A} = \mathrm{Chow}_{f_*\beta}(Y)$. Furthermore if $f^*\omega' \cdot \beta = 0$, the following diagram commutes,

$$(5.11) \quad \begin{array}{ccccc} K_0(\mathrm{St}/\mathfrak{Coh}^v(X)) \cap K_0(\mathrm{St}/\mathfrak{Obj}^v(\mathcal{A}_\sigma)) & \longrightarrow & K_0(\mathrm{St}/\mathfrak{Coh}^v(X)) & \xrightarrow{\pi_*} & K_0(\mathrm{St}/A) \\ & \searrow & & & \downarrow \pi_{1*} \\ & & K_0(\mathrm{St}/\mathfrak{Obj}^v(\mathcal{A}_\sigma)) & \xrightarrow{\pi_{0*}} & K_0(\mathrm{St}/\mathbb{C}) \xrightarrow{\Upsilon'} \Lambda. \end{array}$$

Here $\pi_0: \mathcal{D}\mathfrak{b}j^v(\mathcal{A}_\sigma) \rightarrow \text{Spec } \mathbb{C}$, $\pi_1: A \rightarrow \text{Spec } \mathbb{C}$ are the structure morphisms. The diagram also commutes after the following replacements.

$$\begin{aligned} \mathfrak{Coh}^v(X) &\longmapsto \mathfrak{Coh}^v(X)[1], & \pi &\longmapsto \pi', & A &\longmapsto A', \\ \mathfrak{Coh}^v(X) &\longmapsto \mathfrak{P}\mathfrak{e}\mathfrak{r}^v(X), & \pi &\longmapsto \bar{\pi}, & A &\longmapsto \bar{A}. \end{aligned}$$

Here we have used the notation of Remark 4.23.

Proof. In both diagrams, the commutativity of the left-hand side follows from the functorial property of the push-forwards. Let us show the commutativity of the right-hand side. Since $f: X \rightarrow Y$ contracts only finite number of rational curves, the map

$$f_*: A \ni Z \longmapsto f_*Z \in \bar{A},$$

which sends an algebraic cycle Z on X to the cycle f_*Z on Y , is a finite morphism (see [29, theorem 6.8] for the existence of the above morphism). Hence we can apply Lemma 4.5, which shows the commutativity of the right-hand side of (5.10). Finally if $f^*\omega' \cdot \beta = 0$, any effective one cycle of homology class β is contracted by f . Thus $\bar{A} = \text{Spec } \mathbb{C}$, and $f_*: A \rightarrow \bar{A}$ is identified with π_1 . Therefore the commutativity of the right-hand side of (5.11) also follows from Lemma 4.5. \square

Now we show the following proposition.

Proposition 5.6. *Let $\sigma \in \bar{U}_X$ be as before and $\tau = (W, \mathcal{Q}) \in U_X$. Then for any $v \in N_1(X) \oplus \mathbb{Z}$, one has $P(v, \sigma) = P(v, \tau)$.*

Proof. It is enough to show $P(v, \sigma) = P(v, \tau)$ under the situation of Proposition 4.28. Furthermore the same proof of Proposition 4.30 shows $P(v, \sigma) = P(v, \tau) = 0$ if $v \notin C_\sigma(\phi)$ for any $\phi \in \mathbb{R}$. Thus we may assume $v \in C_\sigma(\phi)$ for some $0 < \phi \leq 1$. First we assume $0 < \phi < 1$. Let us take $\varepsilon > 0$ as in Proposition 4.28. We can choose $\varepsilon > 0$ sufficiently small so that $0 < \phi - 2\varepsilon < \phi + 2\varepsilon < 1$. Then we have

$$\mathcal{Q}((\phi - \varepsilon, \phi + \varepsilon)) \subset {}^p\text{Per}(\mathcal{D}_X) \cap \text{Coh}_{\leq 1}(X).$$

Thus all the terms in (4.19) are contained in both $K_0(\text{St}/{}^p\mathfrak{P}\mathfrak{e}\mathfrak{r}(X))$ and $K_0(\text{St}/\mathfrak{C}\mathfrak{oh}(X))$, and (4.20) holds in both algebras. Applying Lemma 4.31

for $\mathcal{A} = \text{Coh}_{\leq 1}(X)$, we have

$$(5.10) \quad (\mathbb{L} - 1)\Upsilon'_A(\pi_*\epsilon^{(v,\phi)}(\sigma)) = (\mathbb{L} - 1)\Upsilon'_A(\pi_*\epsilon^{(v,\psi)}(\tau)),$$

for some $\psi \in (\phi - \varepsilon, \phi + \varepsilon)$. (Here we have used the same notation as in Lemma 5.5.) By Definition 4.22, the right-hand side of (5.10) is $P(v, \tau)$. On the other hand we have

$$(5.11) \quad (\mathbb{L} - 1)\Upsilon'_A(\pi_*\epsilon^{(v,\phi)}(\sigma)) = (\mathbb{L} - 1)\Upsilon'_{\bar{A}}(\bar{\pi}_*\epsilon^{(v,\phi)}(\sigma)),$$

by the diagram (5.10). Then the right-hand side of (5.11) is $P(v, \sigma)$ by Definition 5.3. Hence (5.10) and (5.11) show $P(v, \sigma) = P(v, \tau)$.

Next suppose $\phi = 1$. Note that in this case $f^*\omega' \cdot \beta = 0$ for $v = (\beta, k)$. For a sufficiently small $\varepsilon > 0$, one has

$$\mathcal{Q}((1 - \varepsilon, 1 + \varepsilon)) \subset \mathcal{A}_\sigma.$$

Thus the formulae (4.19) and (4.20) hold in $K_0(\text{St}/\mathfrak{D}\text{bj}(\mathcal{A}_\sigma))$. Applying Lemma 4.31 for $\mathcal{A} = \mathcal{A}_\sigma$, we have

$$(5.12) \quad (\mathbb{L} - 1)\Upsilon'(\pi_{0*}\epsilon^{(v,1)}(\sigma)) = (\mathbb{L} - 1)\Upsilon'(\pi_{0*}\epsilon^{(v,\psi)}(\tau)),$$

for some $\psi \in (1 - \varepsilon, 1 + \varepsilon)$. If $\psi \leq 1$, then $\epsilon^{(v,\psi)}(\tau)$ is contained in both $K_0(\text{St}/\mathfrak{Coh}(X))$ and $K_0(\text{St}/\mathfrak{D}\text{bj}(\mathcal{A}_\sigma))$, hence the right-hand side of (5.12) is equal to $P(v, \tau)$ by the diagram (5.11). Also the diagram (5.11) after the following replacement,

$$\mathfrak{Coh}^v(X) \longmapsto {}^p\mathfrak{Fet}^v(X), \quad \pi \longmapsto \bar{\pi} \quad A \longmapsto \bar{A},$$

shows that the left-hand side of (5.12) is equal to $P(v, \sigma)$. Hence in this case we obtain $P(v, \sigma) = P(v, \tau)$. When $\psi > 1$, we can use the diagram (5.12) after the replacement,

$$\mathfrak{Coh}^v(X) \longmapsto \mathfrak{Coh}^v(X)[1], \quad \pi \longmapsto \pi' \quad A \longmapsto A',$$

and conclude that $P(v, \sigma) = P(v, \tau)$. In fact by Remark 4.23, we see that the right-hand side of (5.12) is equal to $P(v, \tau)$ also in this case. \square

5.2. Proof of Theorem 5.1

Proof. We may assume that there is a diagram (5.1), since any birational map $\phi: W \dashrightarrow X$ is connected by a sequence of flops [27]. Let H be a

relatively ample divisor on W over Y and

$$\sigma = (Z_{(-\delta_H, g^*\omega')}, \quad {}^{-1}\text{Per}(\mathcal{D}_W)) \in \overline{U}_W,$$

the stability condition in (5.3), applied for $g: W \rightarrow Y$. By Proposition 5.2 (ii), we have

$$\Phi_*\sigma = (Z_{(-\delta_{\phi_*H}, f^*\omega')}, \quad {}^0\text{Per}(\mathcal{D}_X)).$$

(Here we have used the right diagram of (5.2).) Since $-\phi_*H$ is relatively ample over Y , $\Phi_*\sigma$ is one of the stability conditions constructed in (5.3). Hence by Proposition 5.6, it is enough to show

$$(5.13) \quad P(W, v, \sigma) = P(X, \phi_*v, \Phi_*\sigma).$$

We may assume $v \in C_\sigma(\phi)$ for $0 < \phi \leq 1$. By Proposition 5.2 (ii), the equivalence Φ induces the isomorphism,

$$(5.14) \quad \Phi_{\text{St}}: K_0(\text{St}/{}^{-1}\mathfrak{Per}(W)) \longrightarrow K_0(\text{St}/{}^0\mathfrak{Per}(X)).$$

It is easy to see that (5.14) preserves $*$ -product. Furthermore by the left diagram of (5.2), the isomorphism (5.14) takes $\delta^{(v, \phi)}(\sigma)$ to $\delta^{(\phi_*v, \phi)}(\Phi_*\sigma)$. Thus we have

$$(5.15) \quad \Phi_{\text{St}}\epsilon^{(v, \phi)}(\sigma) = \epsilon^{(\phi_*v, \phi)}(\Phi_*\sigma).$$

On the other hand, we have the commutative diagram,

$$(5.16) \quad \begin{array}{ccc} K_0(\text{St}/{}^{-1}\mathfrak{Per}^v(W)) & \xrightarrow{\Phi_{\text{St}}} & K_0(\text{St}/{}^0\mathfrak{Per}^{\phi_*v}(X)) \\ & \searrow \pi_{W*} & \downarrow \pi_{X*} \\ & & K_0(\text{St}/\bar{A}) \xrightarrow{\Upsilon_{\bar{A}}} \Lambda. \end{array}$$

Hence the diagram (5.16) together with (5.15) imply (5.13). □

Remark 5.7. If $v = (0, 1)$ and $\sigma \in U_X$, then $P(v, \sigma)$ is equal to $\sum_i b_i(X)t^i$. Hence Proposition 5.6 and (5.13) imply $b_i(W) = b_i(X)$ for all $i \in \mathbb{Z}$.

Example 5.8. Let $\mathbb{P}^1 \cong C \subset X$ be a $(-1, -1)$ curve, i.e., the normal bundle $N_{C/X}$ is isomorphic to $\mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$. For $m \neq 0$, the same

computation in [16, proposition 4.5] shows,

$$n_g^{m[C]}(X) = \begin{cases} 1 & \text{if } g = 0, m = \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\phi: W \dashrightarrow X$ be a flop at C , and $C^\dagger \subset W$ the flopped curve. Since $\phi_*[C^\dagger] = -[C]$, one can see

$$n_g^{[C^\dagger]}(W) = n_g^{-[C]}(X) = 1,$$

by the computation shown. Note that since $-[C]$ is not effective, the invariant $n_g^\beta(X)$ should also be defined for non-effective one cycle classes β .

6. Some technical lemmas

In this section, we prove some postponed technical lemmas.

6.1. Proof of Lemma 3.6

Let us take $\sigma = \sigma_{(B,\omega)} \in U_X$ and $B' + i\omega' \in N^1(X)_\mathbb{C}$. First we show $\|Z_{(B,\omega)} - Z_{(B',\omega')}\|_\sigma < \infty$ (see [7, section 6] for $\|\cdot\|_\sigma$). By the definition of $\|\cdot\|_\sigma$, it is equal to

$$(6.1) \quad \sup \left\{ \frac{|\{(B - B') + i(\omega - \omega')\} \text{ch}_2(E)|}{|Z_{(B,\omega)}(E)|} : E \text{ is semistable in } \sigma \right\}.$$

Let us put $m = |Z_{(B,\omega)}(E)|$ for a σ -semistable object E . Then we have $|\omega \cdot \text{ch}_2(E)/m| \leq 1$. We set $K \subset N_1(X)$ as

$$(6.2) \quad K := \{c \in \overline{NE}(X) \mid \omega \cdot c \leq 1\} \subset N_1(X).$$

As in the proof of Proposition 3.7, the space K is compact. Therefore the function

$$K \ni c \longmapsto |\{(B - B') + i(\omega - \omega')\}c| \in \mathbb{R}$$

has a maximum value, say M . Since $\text{ch}_2(E)/m \in K$ or $-\text{ch}_2(E)/m \in K$, we have (6.1) $\leq M < \infty$.

Then by [7, proposition 6.3], the map $\text{Stab}(X) \rightarrow N^1(X)_{\mathbb{C}}$ is a local homeomorphism. Suppose that $B' + i\omega'$ satisfies

$$\|Z_{(B,\omega)} - Z_{(B',\omega')}\|_{\sigma} < \sin \pi\varepsilon,$$

for a sufficiently small ε . Then [7, theorem 7.1] guarantees the existence of a stability condition $\tau = (Z_{(B',\omega')}, \mathcal{Q})$ which satisfies $d(\mathcal{P}, \mathcal{Q}) < \varepsilon$ (see [7, section 6] for $d(*, *)$). If we know $\tau \in U_X$, we can conclude U_X is open.

To conclude $\tau \in U_X$, it is enough to check $\mathcal{Q}((0, 1]) \subset \text{Coh}_{\leq 1}(X)$. According to the proof of [7, theorem 7.1], the set of objects $\mathcal{Q}(\phi)$ for $0 < \phi \leq 1$ is obtained as follows: an object $E \in \mathcal{D}_X$ is contained in $\mathcal{Q}(\phi)$ if and only if there is a thin and enveloping subcategory $E \in \mathcal{P}((a, b))$ such that E is $Z_{(B',\omega')}$ -semistable with phase ϕ (see [7, definitions 7.2 and 7.4] for the notion of thin enveloping subcategory). Take $E \in \mathcal{Q}(\phi)$ with $E \in \mathcal{P}((a, b))$ as before. If $0 < a < b \leq 1$, one has $E \in \text{Coh}_{\leq 1}(X)$. Suppose $b > 1$. Then there is a distinguished triangle

$$(6.3) \quad H^{-1}(E)[1] \longrightarrow E \longrightarrow H^0(E),$$

with $H^{-1}(E)[1] \in \mathcal{P}((1, b))$ and $H^0(E) \in \mathcal{P}((a, 1])$. The semistability of E in $Z_{(B',\omega')}$ implies

$$(6.4) \quad \arg Z_{(B',\omega')}(H^{-1}(E)[1]) \leq \arg Z_{(B',\omega')}(H^0(E)).$$

Here \arg is taken in the interval $(\pi i(a - \varepsilon), \pi i(b + \varepsilon))$. However since $H^i(E) \in \text{Coh}_{\leq 1}(X)$, (6.4) implies $E \cong H^0(E)$ or $E \cong H^{-1}(E)[1]$. Since E has phase $0 < \phi \leq 1$ with respect to $Z_{(B',\omega')}$, one must have $E \cong H^0(E) \in \text{Coh}_{\leq 1}(X)$. The similar argument shows $E \in \text{Coh}_{\leq 1}(X)$ when $a \leq 0$.

6.2. Proof of Lemma 3.8

In fact we show the following stronger claim. Let $\Sigma \subset \text{Stab}(\mathcal{D}_X)$ be the connected component which contains U_X , and $\Sigma' \subset \Sigma$ the subset of $\sigma = (Z, \mathcal{P})$ such that $\text{Im } Z \subset \mathbb{C}$ is discrete. We show that for any $\sigma \in \Sigma'$, $\phi \in \mathbb{R}$ and $z \in \mathbb{C}$, the following set of objects,

$$M^{(z,\phi)}(\sigma) := \{E \in \mathcal{P}(\phi) \mid Z(E) = z\},$$

is bounded. For an ample divisor ω on X , let us consider the point $\sigma_{(0,\omega)} \in U_X$. By Remark 3.5, any $\sigma_{(0,\omega)}$ -semistable object is nothing but ω -Gieseker semistable sheaf up to shift. As is well known, the set of ω -Gieseker

semistable sheaves with a fixed Hilbert polynomial forms a bounded family, hence the claim is true for $\sigma = \sigma_{(0,\omega)}$.

Next suppose that the previous claim is true for some $\sigma = (Z, \mathcal{P}) \in \Sigma'$. We show that if $\tau = (W, \mathcal{Q}) \in \Sigma'$ is sufficiently close to σ , then the claim also holds for τ . Obviously once we show this, then the claim is true for any $\sigma \in \Sigma'$. In order to show the boundedness of $M^{(z,\phi)}(\tau)$, we may assume $\phi = 1/2$ by applying some element $g \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ (cf. [7, lemma 8.2]) to σ , τ . If τ is sufficiently close to σ , we have

$$\mathcal{Q}\left(\frac{1}{2}\right) \subset \mathcal{P}\left(\left(\frac{1}{4}, \frac{3}{4}\right)\right) \subset \mathcal{Q}((0, 1)).$$

For $E \in M^{(z, 1/2)}(\tau)$, let $F_i \in \mathcal{P}(\phi_i)$ for $\phi_i \in (1/4, 3/4)$, $1 \leq i \leq n$ be the semistable factors of E . Since $\mathrm{Im} W(F_i) \leq \mathrm{Im} W(E)$, there is $\delta > 0$ which does not depend on E such that $\mathrm{Im} Z(F_i) \leq \mathrm{Im} z + \delta$, if τ is enough close to σ . Because $\mathrm{Im} Z \subset \mathbb{C}$ is discrete, we see that numbers of semistable factors n , and the values $z_i = Z(F_i) \in \mathbb{C}$ have finite number of possibilities. Since $F_i \in M^{(z_i, \phi_i)}(\sigma)$, the boundedness of $M^{(z_i, \phi_i)}(\sigma)$ for each i implies the boundedness of $M^{(z, \phi)}(\tau)$.

6.3. Proof of Lemma 4.19

We may assume $0 < \phi \leq 1$ and let us take $0 < \varepsilon < 1/6$. Since $\sigma \in \overline{U}_X$, there is $\tau = (Z_{(B,\omega)}, \mathrm{Coh}_{\leq 1}(X)) \in U_X$ with B, ω rational such that $C_\sigma(\phi) \subset C_\tau((\phi - \varepsilon, \phi + \varepsilon))$. From this it is clear that there is a finite number of possibilities for n in (4.13). Hence it is enough to check the finiteness of the set,

$$\{(v_1, v_2) \mid v_1 + v_2 = v, v_i \in C_\sigma(\phi)\}.$$

We write $v_i = (\beta_i, k_i) \in N_1(X) \oplus \mathbb{Z}$. It is enough to check that the possible pairs (β_1, β_2) are finite. First we assume $0 < \phi < 1$. We may assume that $0 < \phi - \varepsilon < \phi + \varepsilon < 1$. Then $\beta_i \in \overline{NE}(X)$ and we have

$$(6.5) \quad \mathrm{Im} Z_{(B,\omega)}(v_i) \leq \mathrm{Im} Z_{(B,\omega)}(v).$$

Since (6.5) implies $\beta_i \cdot \omega \leq \beta \cdot \omega$ and (6.2) is compact, the possible pairs (β_1, β_2) must be finite.

Next we treat the case of $\phi = 1$. Then v_i is decomposed as follows,

$$v_i = \sum_j v_{ij}, \quad v_{ij} \in C_\tau(\phi_{ij}) \text{ with } \phi_{ij} \in (1 - \varepsilon, 1 + \varepsilon).$$

If we write $v_{ij} = (\beta_{ij}, k_{ij})$, then $\beta_{ij} \in \overline{NE}(X)$ or $-\beta_{ij} \in \overline{NE}(X)$. We can easily see,

$$|\beta_{ij} \cdot \omega| = |\operatorname{Im} Z_{(B,\omega)}(v_{ij})| \leq |\operatorname{Re} Z_{(B,\omega)}(v)| \cdot \tan \pi \varepsilon.$$

Again since (6.2) is compact, the possible $\{\beta_{ij}\}_{i,j}$ are finite. Thus the pair (β_1, β_2) also has a finite number of possibilities.

6.4. Proof of Proposition 5.2 (iii)

We have to show stability conditions in (5.3) are contained in \overline{U}_X . In [42, proposition 4.4], the author put the assumption that there exists a hyperplane $Y_0 \subset Y$ such that $f^{-1}(Y_0)$ is smooth. In our purpose, we have to improve the proof and show that actually stability conditions in (5.3) are contained in \overline{U}_X without such assumption.

The proof goes on as in Lemma 3.6, and we show the case of $p = 0$ for simplicity. Let $\sigma = \sigma_{(-\delta H, f^* \omega')}$ be as in (5.3), and take $B + i\omega \in N^1(X)_{\mathbb{C}}$. We also set $Z = Z_{(-\delta H, f^* \omega')}$. The value $\|Z - Z_{(B,\omega)}\|_{\sigma}$ is given by

$$(6.6) \quad \sup \left\{ \frac{|\{(-\delta H - B) + i(f^* \omega' - \omega)\} \operatorname{ch}_2(E)|}{|Z(E)|} : E \text{ is semistable in } \sigma \right\}.$$

In order to show (6.6) is finite, it is enough to give the upper bound of (6.6) for $E \in {}^0\operatorname{Per}(\mathcal{D}_X)$. Let us take $F \in \widetilde{\operatorname{Coh}}_{\leq 1}(X)$ where $\widetilde{\operatorname{Coh}}_{\leq 1}(X)$ is given by (5.7), and put $m = |Z(F)|$. Then $f^* \omega' \cdot \operatorname{ch}_2(F)/m \leq 1$. By the openness of the big cone, there is a sufficiently small rational polyhedral cone $f^* \omega' \in \Delta \subset \phi_* \overline{A}(W) \cup \overline{A}(X)$. Let K' be

$$K' := \{c \in \check{\Delta} \mid f^* \omega' \cdot c \leq 1\} \subset N_1(X),$$

where $\check{\Delta}$ is the dual cone. Then K' is compact, hence the function

$$K' \ni c \longmapsto |\{(-\delta H - B) + i(f^* \omega' - \omega)\} \cdot c| \in \mathbb{R}$$

has a maximum value, say M' . Since $F \in \widetilde{\operatorname{Coh}}_{\leq 1}(X)$, we have $\operatorname{ch}_2(F) \cdot \phi_* H' \geq 0$, where H' is an ample divisor on W . Hence $\operatorname{ch}_2(F)/m \in K'$, which implies

$$(6.7) \quad \frac{|\{(-\delta H - B) + i(f^* \omega' - \omega)\} \operatorname{ch}_2(F)|}{|Z(F)|} \leq M',$$

for all $F \in \widetilde{\operatorname{Coh}}_{\leq 1}(X)$.

Next let us take a non-zero $G \in {}^0\text{Per}(\mathcal{D}_X)$ supported on C . Since G is generated by (5.6), we can write $[G] = \sum_{i=1}^n a_i [S_i]$ in $K(\mathcal{D}_X)$ for $a_i \geq 0$. Let us set $c_i := \delta H \cdot C_i$ and $c'_i := |((-\delta H - B) - i\omega) \cdot C_i|$. We have

$$(6.8) \quad \frac{|\{(-\delta H - B) + i(f^*\omega' - \omega)\} \text{ch}_2(G)|}{|Z(G)|} \leq \frac{\sum_{i=1}^n a_i c'_i}{a_0 + \sum_{i=1}^n a_i c_i}.$$

Since $c_i > 0$ and $a_i > 0$ for some i , we have $\text{RHS} \leq M''$ for some $M'' > 0$ independent of a_i . We may take $M'' = M'$.

Finally since ${}^0\text{Per}(\mathcal{D}_X)$ is generated by (5.6) and (5.7), any $E \in {}^0\text{Per}(\mathcal{D}_X)$ is written as $[E] = [F] + [G]$ in $K(\mathcal{D}_X)$, where $F \in \widehat{\text{Coh}}_{\leq 1}(X)$ and $[G] = \sum_{i=1}^n a_i [S_i]$ for $a_i \geq 0$. We have

$$(6.9) \quad (6.6) \leq \sup \left\{ M' \cdot \frac{|Z(F)| + |Z(G)|}{|Z(F) + Z(G)|} : E \in {}^0\text{Per}(\mathcal{D}_X) \right\}$$

$$(6.10) \quad \leq M' \cdot \sup \left\{ \frac{|z| + 1}{|z + 1|} : \text{Im } z \geq 1 \right\} < \infty.$$

Now we have proved $\|Z - Z_{(B,\omega)}\|_\sigma < \infty$. As in the proof of Lemma 3.6, for any $\varepsilon > 0$ there is $B + i\omega \in A(X)_\mathbb{C}$ and a stability condition $\tau = (Z_{(B,\omega)}, \mathcal{Q})$ such that

$$d(\sigma, \tau) < \varepsilon, \quad \|Z - Z_{(B,\omega)}\|_\sigma < \sin \pi \varepsilon.$$

If we show $\tau \in U_X$, we can conclude $\sigma \in \overline{U}_X$. The same proof of the last part of Lemma 3.6 shows $\tau \in U_X$ (it is enough to notice that in the sequence (6.3), one has $H^{-1}(E)[1] \in \mathcal{P}([1, b])$ and the rest is the same), and we leave the detail to the reader.

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