On a class of non-simply connected Calabi–Yau 3-folds

Vincent Bouchard and Ron Donagi

We obtain a detailed classification for a class of non-simply connected Calabi–Yau 3-folds which are of potential interest for a wide range of problems in string phenomenology. These 3-folds arise as quotients of Schoen's Calabi–Yau 3-folds, which are fiber products over $\mathbb{P}^1$ of two rational elliptic surfaces. The quotient is by a freely acting finite abelian group preserving the fibrations. Our work involves a classification of restricted finite automorphism groups of rational elliptic surfaces.

1. Introduction

The aim of this paper is to construct and study a class of non-simply connected Calabi–Yau 3-folds which are of potential interest for a wide range of problems in string phenomenology. These 3-folds arise as quotients of Schoen's Calabi–Yau 3-folds, which are fiber products over $\mathbb{P}^1$ of two rational elliptic surfaces $\beta : B \to \mathbb{P}^1, \beta' : B' \to \mathbb{P}^1$. Our aim is to classify pairs of actions $\phi_B, \phi_{B'}$ of a finite abelian group G on the two rational elliptic surfaces $\beta : B \to \mathbb{P}^1, \beta' : B' \to \mathbb{P}^1$, compatible with the same action $\phi_{\mathbb{P}^1}$ of G on $\mathbb{P}^1$, such that the resulting action $\phi_{\tilde{X}} := \phi_B \times_{\phi_{\mathbb{P}^1}} \phi_{B'}$ of G on the fiber product $\tilde{X} := B \times_{\mathbb{P}^1} B' = \{(p, p') \in B \times B' | \beta'(p') = \beta(p)\}$ is free, yielding a non-simply connected quotient Calabi–Yau 3-fold $X := \tilde{X}/G$ with fundamental group $\pi_1(X) = G$. This aim is accomplished in Theorem 7.1.

The direct motivation for this came from the problem of deriving the Standard Model of particle physics as the low-energy limit of an heterotic string compactification. We discuss some of the issues involved and review the known results in Section 1.1. We then describe the precise version of our problem in Section 1.2. The remainder of the paper is outlined in Section 1.3.

1.1. Physics motivation

The main motivation for this work comes from physics. An important open problem in string theory is to find a low-energy limit of the theory that reproduces observed physics, such as the Standard Model of particle physics. There are many approaches to this problem; one of which consists in studying 4-dimensional compactifications of the so-called $E_8 \times E_8$ heterotic string theory.

Roughly speaking, an $E_8 \times E_8$ heterotic vacuum is characterized by a triple (X, V, G), where X is a Calabi–Yau 3-fold and V is a stable vector bundle on X with structure group $G \subseteq E_8$.[1] In this model, the low-energy physics translates into topological and geometrical properties of this triple. It turns out that obtaining realistic physics out of $E_8 \times E_8$ compactifications on smooth simply connected Calabi–Yau 3-folds X runs into various difficulties.

Many problems may be circumvented by considering instead smooth non-simply connected Calabi–Yau 3-folds. For instance, an heterotic vacuum on a smooth Calabi–Yau 3-fold X with fundamental group $\pi_1(X) = \mathbb{Z}_2$ reproducing precisely the massless spectrum of the Minimal Supersymmetric Standard Model of particle physics has recently been found [2, 3]. It is plausible that many more such realistic vacua exist in the landscape of string vacua.

Hence, the holy grail of connecting string theory to the real world motivates the search for non-simply connected Calabi–Yau 3-folds. Sporadic examples of non-simply connected Calabi–Yau 3-folds already exist in the literature, but they are rather isolated compared to the wild jungle of simply connected Calabi–Yau 3-folds. In this paper we undertake the study of a large class of non-simply connected Calabi–Yau 3-folds, based on Schoen's construction of smooth simply connected Calabi–Yau 3-folds as (resolutions of) fiber products of two rational elliptic surfaces [15].

1.2. Geometric setup

Our aim is to construct smooth Calabi–Yau 3-folds X with fundamental group $\pi_1(X) = G$, where G is a finite abelian group. Every such X has a smooth simply connected universal cover $\tilde{X}$, which comes with a freely acting

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[1]For the readers familiar with this construction, here we implicitly consider the hidden bundle in the second E_8 factor to be trivial.

group of automorphisms $G_{\tilde{X}} \cong G$. Moreover, by pulling back the Calabi–Yau metric of X we see that the universal cover $\tilde{X}$ is also Calabi–Yau.

Conversely, let $\tilde{X}$ be a smooth simply connected Calabi–Yau 3-fold, with a freely acting finite abelian group of automorphisms $G_{\tilde{X}}$. It turns out that $G_{\tilde{X}}$ automatically preserves the volume form,[2] so the quotient manifold $X = \tilde{X}/G_{\tilde{X}}$ is a smooth Calabi–Yau 3-fold with fundamental group $\pi_1(X) \cong G_{\tilde{X}}$.
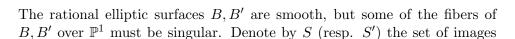
In this paper we use the second point of view to construct a large class of non-simply connected Calabi–Yau 3-folds. Namely, we classify all Calabi–Yau 3-folds $\tilde{X}$ constructed as a smooth fiber product of two rational elliptic surfaces and admitting a freely acting finite abelian group of automorphisms $G_{\tilde{X}}$. Since we will eventually also need to use the spectral construction to build Standard Model bundles on X, we require that $G_{\tilde{X}}$ preserves the elliptic fibration of $\tilde{X}$, so that the quotient 3-fold X is torus-fibered.[3]

Let us now describe in more details the family of Calabi–Yau 3-folds we are interested in. Let B and B' be rational elliptic surfaces. Consider the fiber product $\tilde{X} := B \times_{\mathbb{P}^1} B'$. That is, construct the 3-fold

(1.1) $$\tilde{X} = \{(p, p') \in B \times B' | \beta'(p') = \beta(p)\},$$

where $\beta : B \to \mathbb{P}^1$ and $\beta' : B' \to \mathbb{P}^1$ are the elliptic fibrations of the rational elliptic surfaces B and B'. $\tilde{X}$ can also be described by the commutative diagram shown next.

(1.2)

The rational elliptic surfaces B, B' are smooth, but some of the fibers of B, B' over $\mathbb{P}^1$ must be singular. Denote by S (resp. S') the set of images

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[2]The argument, due to Beauville, goes as follows. Let $\tilde{X}$ be a Calabi–Yau 3-fold, by which we mean a 3-fold with SU(3) holonomy; then $h^{1,0}(\tilde{X}) = h^{2,0}(\tilde{X}) = \chi(\mathcal{O}_{\tilde{X}}) = 0$. If $G_{\tilde{X}}$ acts freely on X, then the quotient manifold $X = \tilde{X}/G_{\tilde{X}}$ also satisfies $h^{1,0}(X) = h^{2,0}(X) = \chi(\mathcal{O}_X) = 0$. It follows that $h^{3,0}(X) = 1$, hence there is a non-zero $(3,0)$-form Ω on X. Since its pullback on $\tilde{X}$ is nowhere vanishing, Ω must also be nowhere vanishing on X; therefore X is Calabi–Yau [1].

[3]By torus-fibered we mean that X has a fibration with generic fiber an elliptic curve, but the fibration does not necessarily have a section.

of the singular fibers of B (resp. B') in $\mathbb{P}^1$. The fiber product $\tilde{X}$ will be smooth except at points (p, p') where both p and p' are singular points of their respective fibers. The image in $\mathbb{P}^1$ of the singularities is therefore the intersection $S'' = S \cap S'$. In other words, $\tilde{X}$ is smooth if and only if $S'' = \emptyset$, that is, no singular fibers of B and B' are paired in the fiber product.

It was shown by Schoen [15] that the smooth $\tilde{X}$ are Calabi–Yau. In fact, Schoen considered the more general case when $\tilde{X}$ has ordinary double point singularities. More precisely, he studied the case where for any $p \in S''$, the singular fibers $\beta^{-1}(p)$ and $(\beta')^{-1}(p)$ are semistable — of type I_k, $k \geq 0$, in Kodaira's notation. Then, if the singular fibers above the points in S'' satisfy certain additional requirements, the minimal resolution of the singular 3-fold $\tilde{X}$ is also Calabi–Yau. However, in this paper we will not consider these additional smooth Calabi–Yau 3-folds and focus on smooth fiber products.

Since the Euler characteristic of T^2 is 0, it is clear that the Euler characteristic of the fiber product is given by the sum of the Euler characteristics of the singular fibers above the points in S''. In particular, if $S'' = \emptyset$, then the fiber product $\tilde{X}$ is smooth, and $\chi(\tilde{X}) = 0$. One can show that $\tilde{X}$ is simply connected and its Hodge diamond is given by

$$
(1.3) \qquad
\begin{matrix}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 19 & & 0 & \\
1 & & 19 & & 19 & & 1 \\
 & 0 & & 19 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{matrix}
$$

The cohomology group $H^2(\tilde{X}, \mathbb{Z}) \simeq \mathrm{Pic}(\tilde{X})$ of $\tilde{X}$ is given by

$$
(1.4) \qquad H^2(\tilde{X}, \mathbb{Z}) \cong \frac{H^2(B, \mathbb{Z}) \oplus H^2(B', \mathbb{Z})}{H^2(\mathbb{P}^1, \mathbb{Z})}.
$$

So our aim is to give a complete list of pairs $(\tilde{X}, G_{\tilde{X}})$ with $\tilde{X}$ a smooth fiber product of two rational elliptic surfaces and $G_{\tilde{X}}$ a freely acting finite abelian group of automorphisms preserving the elliptic fibration. Because of the fiber product nature of $\tilde{X}$ described by the commutative diagram (1.2), our strategy will be to reduce the classification of pairs $(\tilde{X}, G_{\tilde{X}})$ to a classification of finite automorphism groups of rational elliptic surfaces B.

An automorphism $\tau_{\tilde{X}} : \tilde{X} \to \tilde{X}$ of a Calabi–Yau 3-fold $\tilde{X} = B \times_{\mathbb{P}^1} B'$ that preserves the elliptic fibration has the form $\tau_{\tilde{X}} = \tau_B \times_{\mathbb{P}^1} \tau_{B'}$, where τ_B and $\tau_{B'}$ are automorphisms of the rational elliptic surfaces B and B'.

Moreover, since there is no free automorphism of a rational elliptic surface,[4] it is clear that if $\tau_{\tilde{X}}$ acts freely on $\tilde{X}$, then both τ_B and $\tau_{B'}$ must act non-trivially on B and B', respectively. Hence, if $\tilde{X}$ is a smooth fiber product with a freely acting, fibration-preserving automorphism group $G_{\tilde{X}}$, then it must be the fiber product of two rational elliptic surfaces B and B' with automorphism groups $G_B \cong G_{B'} \cong G_{\tilde{X}}$.[5]

The direct consequence of this observation is that the complete list of pairs $(\tilde{X}, G_{\tilde{X}})$ can be easily reconstructed from the list of pairs (B, G_B). Namely, given the complete list of pairs (B, G_B), one has to consider "pairs of pairs" (B, G_B) and $(B', G_{B'})$ with $G_B \cong G_{B'}$ and consistent induced actions on the $\mathbb{P}^1$ base to construct all possible $(\tilde{X}, G_{\tilde{X}})$. Henceforth, we will be interested in finite automorphism groups G_B of rational elliptic surfaces B.

Notice however that not all automorphisms of rational elliptic surfaces B can be lifted to free automorphisms of $\tilde{X}$. For $\tau_{\tilde{X}} = \tau_B \times_{\mathbb{P}^1} \tau_{B'}$ to be free on $\tilde{X}$, it is clear that the fibers of B and B' intersecting the fixed loci of τ_B and $\tau_{B'}$ must not be paired in the fiber product. This gives restrictions on allowed automorphisms of rational elliptic surfaces, which we explore in Section 3.3. In those cases where the lifting to a free automorphism is possible, we typically get infinitely many such free autmorphisms, parametrized by a finite number of cosets of an infinite group. However, only a finite number of distinct quotients arise: automorphisms in the same coset are related via conjugation by a translation automorphism, hence they result in isomorphic quotients. This is discussed in Section 5.4.

1.3. Outline

We start by recalling important properties of rational elliptic surfaces in Section 2. We study automorphisms of rational elliptic surfaces in Section 3, and explore the properties that they must satisfy in order to lift to free automorphisms on smooth fiber products. In particular, we split the task of classifying such automorphisms into separate questions about the "linear" part, which preserves the zero-section, and the "translation" part, which has trivial linearization. Then, in Sections 4 and 5 we study what we call *automorphisms of the second kind*, which are automorphisms acting non-trivially on the $\mathbb{P}^1$ base. The list of suitable linearized actions is obtained

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[4]See Section 3.1.1 for a short proof of this fact.

[5]Note here that G_B and $G_{B'}$ are not necessarily the full automorphism groups of the rational elliptic surfaces, but may be subgroups thereof.

in table 4 in Section 4, and the corresponding translations are classified in Section 5. In Section 6, we combine our analysis of the previous sections to understand finite automorphism groups of rational elliptic surfaces that can be lifted to free automorphism groups on smooth fiber products. We produce in tables 6 to 10 a complete list of pairs (B, G_B), where G_B is a finite automorphism group of B that can be lifted to a free finite automorphism group of $\tilde{X}$. The list of 3-folds is obtained from all pairs B, B' with the same group G and the same action on $\mathbb{P}^1$; this is accomplished in our main Theorem 7.1, stated in Section 7. We conclude this section by giving some examples of non-simply connected Calabi–Yau 3-folds obtained from our results. Finally, we present in Section 8 some open questions and avenues for future research.

1.4. Hitchiker's guide to the paper

Since our calculations are sometimes rather detailed, it may help the reader to have one brief outline of the entire argument.

We want to classify free, fibration-preserving actions of a finite group G on a smooth Calabi–Yau fiber product $\tilde{X} = B \times_{\mathbb{P}^1} B'$ and the quotient Calabi–Yau $X = \tilde{X}/G$. Each group element acts as an automorphism $\tau_{\tilde{X}}$ which itself, by assumption, is a fiber product $\tau_{\tilde{X}} = \tau_B \times_{\tau_{\mathbb{P}^1}} \tau_{B'}$ where τ_B is an automorphism of B, $\tau_{B'}$ is an automorphism of B', and they both lift the same automorphism $\tau_{\mathbb{P}^1}$ of $\mathbb{P}^1$.

The automorphism τ_B is "affine," in the sense that it does not necessarily send the zero-section σ to itself; it has a *linearization* α_B which does send σ to itself. The linearization lifts the same automorphism $\tau_{\mathbb{P}^1}$ of $\mathbb{P}^1$, and in fact differs from τ_B by a translation (by a section ξ of $B \to \mathbb{P}^1$): $\tau_B = t_\xi \circ \alpha_B$. So we reduce the determination of G actions to consideration of the possible automorphisms $\tau_{\tilde{X}}$; these are reduced first to analysis of the possible τ_B, and then to a study of the possible α_B and ξ.

In Section 3.2 we introduce notation for the orders of the various finite order automorphisms: $n := \text{order}(\tau_B)$, $m := \text{order}(\alpha_B)$, $\bar{m} := \text{order}(\tau_{\mathbb{P}^1})$. There are positive integers d, k such that:

$$n = md = \bar{m}kd.$$

We say that an automorphism is of the *first kind* if $m = 1$, and of the *second kind* if $m > 1$.

A key notion is that of a *suitable pair* (B, τ_B) (Definition 3.9): (B, τ_B) is suitable if $k = 1$ (i.e., $m = \bar{m}$) and either:

- $m = 1$ (first kind), or:

- $m > 1$ (second kind), and:
    - the subgroup $\langle \tau_B \rangle$ generated by τ_B acts *freely* on the fiber f_∞ above one of the fixed points $\infty \in \mathbb{P}^1$ of $\tau_{\mathbb{P}^1}$, and:
    - if $d > 1$ we also require that f_∞ be smooth.

In Section 3.3 we show that an automorphism τ_B, in conjunction with a similar automorphism $\tau_{B'}$, can be lifted to a free $\tau_{\tilde{X}}$ if and only if the pairs (B, τ_B) and $(B', \tau_{B'})$ are suitable.

The next reduction uses the notion (Definition 4.1) of a *suitable σ-pair*, i.e., a pair (B, α_B) where α_B is an automorphism sending σ to itself and satisfying:

- $m = \bar{m}$;

- f_∞ is of Kodaira type I_{mr} for some integer $r \geq 0$;

- α_B fixes the neutral component of f_∞ pointwise.

We show in Section 3 that if (B, τ_B) is a suitable pair, then its linearization (B, α_B) is a suitable σ-pair. In particular, in the cases relevant to us, $m = \bar{m}$, so we will use m and d (or m and $n = md$) as our basic discrete parameters.

Suitable σ-pairs are classified in Proposition 4.8 and table 4. In that table we also provide basic information about the rational elliptic surface $\widehat{\mathcal{B}}$ obtained as the Kodaira model of the quotient $B/\langle \alpha_B \rangle$. The classification is done under the general-position assumption that the fiber f_∞ is smooth. As explained in Section 4.3.1, this assumption is harmless: it is satisfied at the generic point of each of the moduli spaces we are concerned with, so we do not miss any components by imposing it. The classification is based on the notion of the *deficiency* of a singular fiber of the Kodaira model $\widehat{\mathcal{B}}$ of the quotient $B/\langle \alpha_B \rangle$ (Definition 4.2). Topological considerations lead to constraints on fiber deficiencies (Lemmas 4.4 and 4.6). These imply that m must be between 1 and 6, determine the suitability of $\widehat{\mathcal{B}}$ in terms of its special fibers f_0, f_∞ over the fixed points 0 and ∞ of $\tau_{\mathbb{P}^1}$ (Lemma 4.7), and lead to the proof of Proposition 4.8.

Given α_B, for each positive integer i let $\mathcal{P}_i$ be the endomorphism of the Mordell–Weil group MW induced by the element $\sum_{j=0}^{i} \alpha_B^j$ of the group ring $\mathbb{Z}[G]$ of G. We also set $\Phi_i(\xi) := \langle \mathcal{P}_i(\xi), \mathcal{P}_i(\xi) \rangle$ where $\langle \cdot, \cdot \rangle$ is the height pairing, reviewed in Section 2. Note that $\Phi_m(\xi) = 0$ if and only if $\mathcal{P}_m(\xi)$ is a

torsion section of some finite order d. Hence, for any section $\xi \in \ker(\Phi_m)$, the automorphism $\tau_B = t_\xi \circ \alpha_B$ has order $n = md$. According to Lemma 5.1, the determination of the kernel of Φ_m amounts to calculating the perpendicular to the sublattice MW^{α_B} in MW. This rather tedious calculation is carrried out case by case in Lemma 5.2.

The possible orders d of $\mathcal{P}_m(\xi)$ must divide the order of the torsion subgroup $\mathrm{MW}_{\mathrm{tors}}$ (Lemma 3.2), and if the fiber f_0 is smooth (where $0 \in \mathbb{P}^1$ is the other fixed point of $\tau_{\mathbb{P}^1}$) then $d = 1$. This leaves a fairly short list of possibilities for $d > 1$, worked out in Lemma 5.3. We find that $d = 1$ for ξ in the subgroup $\ker(\Phi_m)_{d=1} = \mathcal{P}_m^{-1}(0)$ of $\ker(\Phi_m) = \mathcal{P}_m^{-1}(\mathrm{MW}_{\mathrm{tors}})$. In each case there is one other possible value for d, which always turns out to be 2 or 3 and which is obtained for ξ in the union of the non-trivial cosets of $\ker(\Phi_m)_{d=1}$ in $\ker(\Phi_m)$.

Given a suitable σ-pair (B, α_B) of order m, the set of *allowed sections* AS is then given by $\ker(\Phi_m)$, minus the cosets of $\mathrm{Im}(1 - \alpha_B)$ in $\ker(\Phi_m)$ containing sections which intersect the fiber at infinity f_∞ at a torsion point of order less than $n = dm$. In Lemma 3.4 and Proposition 5.4 we show that ξ is an allowed section in this sense if and only if the pair $(B, \tau_B = t_\xi \circ \alpha_B)$ is suitable.

Quite remarkably, it turns out that allowed sections always exist: in each of the relevant cases there is at least one coset that satisfies the requirements, that is, the sections belonging to it meet the fiber f_∞ at a point of order precisely n. This is noted in Remark 6.3, and the proof is outlined in Appendix. The result is that each possibility allowed by our lattice-theoretic calculations actually corresponds to a non-empty family of rational elliptic surfaces and thus produces a family of Schoen 3-folds with a free action of the appropriate finite group.

At this point, all the pieces are ready to be assembled. The two propositions in Section 6 tabulate rational elliptic surfaces with finite automorphism groups of the first and second kind, respectively. These in turn are used in the main theorem (7.1) to produce the desired list of non-simply connected Calabi–Yau 3-folds.

2. Preliminaries

Let us start by recalling some important properties of rational elliptic surfaces. Note that everything in the following will be defined over the complex numbers.

2.1. General properties

Let B be a rational elliptic surface and denote by $\beta : B \to \mathbb{P}^1$ its elliptic fibration over $\mathbb{P}^1$. Let $\sigma : \mathbb{P}^1 \to B$ be the zero section.

The Euler characteristic of B is $\chi(B) = 12$. Since the Euler characteristic of a smooth elliptic fiber is $\chi(T^2) = 0$, B must have singular fibers with Euler characteristics adding up to 12. As is well known, the complete list of possible fibers has been worked out by Kodaira [10]: it includes two infinite families — I_n and I_n^* for $n \geq 0$, and six exceptional cases — II, III, IV, II^*, III^*, IV^*. Moreover, B is simply connected; therefore the single nontrivial cohomology group is given by $H^2(B, \mathbb{Z}) \cong \mathbb{Z}^{10}$.

2.2. Mordell–Weil group

The *Mordell–Weil group* MW of a rational elliptic surface B is the group of rational sections. A well-known theorem by Mordell and Weil states that MW is finitely generated.

Define the subgroup $T \subset H^2(B, \mathbb{Z})$ generated by the cohomology classes of the zero section σ, the generic fiber f and the irreducible components of the singular fibers not intersecting σ. Then the Mordell–Weil group of B can be identified with the quotient:

$$(2.1) \qquad \mathrm{MW} \cong \frac{H^2(B, \mathbb{Z})}{T}.$$

2.3. Mordell–Weil lattice

Let $\mathrm{MW}_{\mathrm{tors}}$ be the torsion subgroup of the Mordell–Weil group. Define

$$\mathrm{MW}_{\mathrm{lat}} = \frac{\mathrm{MW}}{\mathrm{MW}_{\mathrm{tors}}}.$$

Following [13, 16], we can endow the group $\mathrm{MW}_{\mathrm{lat}}$ with an additional lattice structure — hence the subscript lat. Let $\mu, \nu \in \mathrm{MW}$ be two sections. Define the height pairing of μ and ν by

$$(2.2) \qquad \langle \mu, \nu \rangle = 1 + \mu\sigma + \nu\sigma - \mu\nu - \sum_{s_i} \mathrm{contr}_{s_i}(\mu, \nu),$$

where the sum runs over the singular fibers s_i of B, multiplication denotes intersection numbers of the sections and $\mathrm{contr}_{s_i}(\mu, \nu)$ is the entry (μ_i, ν_i)

of the inverse of the intersection matrix T_{s_i} of the A-D-E lattice associated with the singular fiber s_i. Here μ_i, ν_i denote the component of singular fiber s_i intersecting the sections μ, ν, respectively. In particular, the numbers $\mathrm{contr}_{s_i}(\mu, \nu)$ are rational, and zero whenever μ or ν intersects the neutral component of s_i. This defines the structure of a positive-definite lattice on $\mathrm{MW}_{\mathrm{lat}}$.

Define the narrow Mordell–Weil lattice to be the sublattice $\mathrm{MW}_{\mathrm{lat}}^0$ consisting of sections $\mu \in \mathrm{MW}_{\mathrm{lat}}$ passing through the neutral component of every fiber. For $\mu, \nu \in \mathrm{MW}_{\mathrm{lat}}^0$, the height pairing takes the simplified form

$$\langle \mu, \nu \rangle = 1 + \mu\sigma + \nu\sigma - \mu\nu,$$

which is integer valued. It promotes $\mathrm{MW}_{\mathrm{lat}}^0$ to the status of a positive-definite even integral lattice [16]. Moreover, $\mathrm{MW}_{\mathrm{lat}}^0 \cong \mathrm{MW}_{\mathrm{lat}}^*$, where $\mathrm{MW}_{\mathrm{lat}}^*$ denotes the dual lattice of $\mathrm{MW}_{\mathrm{lat}}$.

Recall now a few properties that will be useful later.

Proposition 2.1 [13, Proposition 3.5]. *A section $\eta \in \mathrm{MW}$ is torsion if and only if $\langle \eta, \eta \rangle = 0$. Moreover, for rational elliptic surfaces defined over the complex number field, any non-zero torsion section is disjoint from the zero section. This implies (using (2.2)) that a section $\eta \in \mathrm{MW}$, $\eta \neq \sigma$, is torsion if and only if $\sum_{s_i} \mathrm{contr}_{s_i}(\eta, \eta) = 2$.*

Lemma 2.2 [16, Lemma 10.7]. *For any rational elliptic surface, the number of the sections ξ which are disjoint from the zero section σ is finite and at most 240. Every such ξ satisfies $\langle \xi, \xi \rangle \leq 2$.*

Theorem 2.3 [16, Theorem 10.8]. *The Mordell–Weil group MW of a rational elliptic surface is generated by the sections ξ which are disjoint from the zero section σ, hence by those satisfying $\langle \xi, \xi \rangle \leq 2$.*

Finally, the following result on lattices, which is used in [13] to prove the theorem above, will also be useful.

Lemma 2.4 [13, Lemma 5.1]. *Let L be a root lattice of type A_n, D_n or E_k, and let L^* be its dual lattice, namely the weight lattice of type A_n, D_n or E_k. Then L^* has a basis consisting of minimal vectors of minimal norm*

$$\mu(A_n^*) = \frac{n}{n+1}, \quad \mu(D_n^*) = 1, \quad \mu(E_k^*) = \frac{10-k}{9-k} = \frac{4}{3}, \frac{3}{2}, 2 \ (k=6,7,8),$$

except in the case $L^* = D_n^*(n > 4)$ where a vector of norm $n/4$ should be added.

2.4. Weierstrass model

Any rational elliptic surface B can be described as the minimal resolution of a Weierstrass model $W(\mathcal{O}(1), A_4, A_6)$ over $\mathbb{P}^1$, where $\mathcal{O}(1)$ is the tautological line bundle on $\mathbb{P}^1$, and the sections $A_4 \in H^0(\mathbb{P}^1, \mathcal{O}(4))$ and $A_6 \in H^0(\mathbb{P}^1, \mathcal{O}(6))$ can be described by polynomials in t of orders 4 and 6, respectively. The Weierstrass model $W(\mathcal{O}(1), A_4, A_6)$ is defined by the equation

$$(2.3) \qquad y^2 z = x^3 + A_4(t) x z^2 + A_6(t) z^3,$$

where $[x : y : z]$ are homogeneous coordinates[6] of $\mathbb{P}^2$. The zero section σ is simply $[0 : 1 : 0]$.

The map $\rho : B \to W$ from B to its Weierstrass model W is given by blowing down all the irreducible components of the singular fibers of the elliptic fibration not meeting the zero section. Thus, the fibers of W are all irreducible, but W is singular (unless $W = B$); all its singularities are rational doublepoints, and B is the minimal desingularization.

The surfaces B, W are fibered over the same base $\mathbb{P}^1$. Let t_i be the points in this $\mathbb{P}^1$ base where the fibers of β are singular. The points t_i are given by the vanishing of the discriminant of the elliptic fiber

$$(2.4) \qquad \Delta = 4 A_4(t)^3 + 27 A_6(t)^2.$$

As is well known, at each of the t_is the Kodaira type of the singular fiber in the resolved surface B can be read off directly from the order of vanishing of Δ, A_4 and A_6.

3. Automorphisms of rational elliptic surfaces

Let us now turn to the study of automorphisms of finite order on rational elliptic surfaces.

Let B be a rational elliptic surface, $\beta : B \to \mathbb{P}^1$ its elliptic fibration and $\sigma : \mathbb{P}^1 \to B$ the zero section.

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[6]More precisely (2.3) defines a divisor in the 3-fold given by the $\mathbb{P}^2$ fibration $P = \mathbb{P}(\mathcal{O}(2) \oplus \mathcal{O}(3) \oplus \mathcal{O}) \to \mathbb{P}^1$, where $\mathbb{P}()$ means projectivization. $[x : y : z]$ is then a section of the bundle P.

3.1. General properties

3.1.1. Fixed locus. Let τ_B be a non-trivial, fixed point free automorphism of a smooth rational surface B. If the order n of τ_B is prime, all non-trivial elements in the group $\langle \tau_B \rangle$ generated by τ_B are fixed point free, so the quotient $B' = B/\langle \tau_B \rangle$ is smooth and the quotient map $q : B \to B'$ is a covering map. But by Luroth's theorem B' is rational, hence simply connected, so it cannot admit a non-trivial cover. Inductively, we see that this holds for any $n > 1$: the fixed locus of any non-trivial automorphism τ_B of a rational elliptic (or just rational) surface B must be non-empty.

3.1.2. Types of automorphisms. Any automorphism preserves the canonical class. For a rational elliptic surface B, the canonical class is $K_B = -f$, where f is the class of a generic elliptic fiber. Thus, any automorphism of B preserves the fiber class, hence the elliptic fibration. In other words, any automorphism $\tau_B : B \to B$ maps fibers to fibers.

As a result, an automorphism of a rational elliptic surface can be of one of two types: either it leaves each fiber stable, or it permutes fibers and descends to a non-trivial automorphism on $\mathbb{P}^1$. We call the former *automorphisms of the first kind*, and the latter *automorphisms of the second kind*.

3.1.3. Group homomorphisms. Let us now introduce a few useful group homomorphisms. Denote by $\mathrm{Aut}(B)$ the automorphism group of B, by $\mathrm{Aut}(\mathbb{P}^1)$ the automorphism group of $\mathbb{P}^1$ and by $\mathrm{Aut}_\sigma(B)$ the group of automorphisms of B that preserve the zero section σ.

A first group homomorphism that we consider is given by $t : \mathrm{MW} \to \mathrm{Aut}(B)$, which sends a section $\xi \in \mathrm{MW}$ to the automorphism $t_\xi : B \to B$ defined by translation by the section ξ. The automorphisms t_ξ are of the first kind, since they leave each fiber stable.

There is a canonical group homomorphism $\rho : \mathrm{Aut}(B) \to \mathrm{Aut}(\mathbb{P}^1)$ which sends

$$\rho : \tau_B \longmapsto \rho(\tau_B) := \tau_{\mathbb{P}^1}.$$

Explicitly, $\tau_{\mathbb{P}^1}$ is specified by $\tau_{\mathbb{P}^1} \circ \beta = \beta \circ \tau_B$. In other words, ρ associates to any automorphism $\tau_B : B \to B$ its induced automorphism on the $\mathbb{P}^1$ base $\tau_{\mathbb{P}^1} : \mathbb{P}^1 \to \mathbb{P}^1$. An automorphism τ_B is of the first kind if and only if $\tau_{\mathbb{P}^1} = 1$, while it is of the second kind otherwise. In particular, $\rho(t_\xi) = 1$ for all $\xi \in \mathrm{MW}$.

There is another group homomorphism $\lambda : \mathrm{Aut}(B) \to \mathrm{Aut}_\sigma(B)$ acting by

$$\lambda : \tau_B \longmapsto \lambda(\tau_B) := \alpha_B,$$

where $\alpha_B \in \mathrm{Aut}_\sigma(B)$ is the *linearization of τ_B*. The precise form of the map goes as follows. Let $\xi = \tau_B(\sigma)$ be the section defined by applying τ_B to the zero section σ. Then define the linearization α_B of τ_B by $\alpha_B = t_{-\xi} \circ \tau_B$. By definition, $\alpha_B(\sigma) = \sigma$, hence $\alpha_B \in \mathrm{Aut}_\sigma(B)$. To see that λ is compatible with the group structure, recall that fiber by fiber is an isomorphism between genus 1 curves which sends the origin to the origin, hence a group homomorphism.

Remark 3.1. In the following, we will always use the symbols $\tau_{\mathbb{P}^1}$ and α_B for the images of τ_B under ρ and λ, respectively.

3.2. Automorphisms of finite order

Let $\tau_B \in \mathrm{Aut}(B)$ be an automorphism of finite order. Denote by n the order of τ_B, by m the order of α_B, and by $\bar{m}$ the order of $\tau_{\mathbb{P}^1}$. We have that

$$n = md = \bar{m}kd,$$

for some integers $d, k > 0$.

Lemma 3.2. *k divides the order of the group of complex multiplications (automorphisms fixing the origin) of the generic elliptic fiber. In particular, $k \in \{1, 2, 3, 4, 6\}$. d divides the order of the torsion subgroup $\mathrm{MW}_{\mathrm{tors}}$.*

Proof. Since $\tau_{\mathbb{P}^1} = \rho(\alpha_\beta)$ has order $\bar{m}$, we get that $(\alpha_B)^{\bar{m}}$ must leave each fiber stable. By definition, it must also fix the zero section. Hence it must act on each elliptic fiber separately and fix the zero point; that is, it must act by complex multiplication on the generic elliptic fiber. Consequently, $k = m/\bar{m}$ must divide the order of the group of complex multiplications of the generic elliptic fiber. Since the involution -1 is always there, the only possible groups of complex multiplications of elliptic curves have order $2, 4$ or 6. Hence $k \in \{1, 2, 3, 4, 6\}$.

Now $\tau_B = t_\xi \circ \alpha_B$, where $\xi := \tau_B(\sigma)$. Since $\alpha_B^m = 1$, we have that $(\tau_B)^m$ is simply given by translation by a section. Since it is of finite order, it must be given by translation by a torsion section. Hence, the order d of $(\tau_B)^m$ divides the order of the torsion subgroup $\mathrm{MW}_{\mathrm{tors}}$. $\qquad\square$

Lemma 3.3. *If $k > 1$, then $(\tau_B)^{\bar{m}}$ has a fixed curve intersecting each elliptic fiber.*

Proof. As we have seen, if $k > 1$, $(\alpha_B)^{\bar{m}}$ is given by complex multiplication on the generic elliptic fiber, which has a fixed curve (the zero section) intersecting each elliptic fiber. $(\tau_B)^{\bar{m}}$ simply composes $(\alpha_B)^{\bar{m}}$ with translation by a section; therefore, $(\tau_B)^{\bar{m}}$ also has a fixed curve intersecting each elliptic fiber. □

Let us now define the group homomorphism $\mathcal{P}_i : \mathrm{MW} \to \mathrm{MW}$ by

$$(3.1) \qquad \mathcal{P}_i : \xi \longmapsto \mathcal{P}_i(\xi) := \alpha_B^{i-1}(\xi) \boxplus \alpha_B^{i-2}(\xi) \boxplus \cdots \boxplus \xi,$$

where $\xi \in \mathrm{MW}$ and $\boxplus$ denotes addition of sections in the Mordell–Weil group.

Define also the map $\Phi_i : \mathrm{MW} \to \mathbb{Q}$ by

$$(3.2) \qquad \Phi_i : \xi \longmapsto \Phi_i(\xi) := \langle \mathcal{P}_i(\xi), \mathcal{P}_i(\xi) \rangle,$$

where $\xi \in \mathrm{MW}$ and $\langle \cdot, \cdot \rangle$ is the height pairing introduced in (2.2).

Lemma 3.4. *The following data are equivalent:*

(1) *$\tau_B \in \mathrm{Aut}(B)$, of finite order n;*

(2) *Pairs (α_B, ξ), with $\alpha_B \in \mathrm{Aut}_\sigma(B)$ of order m dividing n and $\Phi_m(\xi) = 0$ (or equivalently, $\mathcal{P}_m(\xi) \in \mathrm{MW}_{\mathrm{tors}}$, a torsion section of order $d = n/m$).*

Proof. Let $\tau_B \in \mathrm{Aut}(B)$ be of finite order n, with linearization $\alpha_B := \lambda(\tau_B) \in \mathrm{Aut}_\sigma(B)$ of order m. Let $\xi = \tau_B(\sigma)$; then $\tau_B = t_\xi \circ \alpha_B$. By definition, $\alpha_B(\sigma) = \sigma$. We need to show that $\Phi_m(\xi) = 0$.

We know that

$$
\begin{aligned}
1 = \tau_B^n \\
= (t_\xi \circ \alpha_B)^n \\
= (t_{\mathcal{P}_m(\xi)} \circ \alpha_B^m)^d \\
(3.3) \qquad = (t_{\mathcal{P}_m(\xi)})^d,
\end{aligned}
$$

where we used the fact that $\alpha_B^m = 1$. This implies that $\mathcal{P}_m(\xi)$ is a torsion section of order $d = n/m$. From Proposition 2.1, we know that a section η is torsion if and only if $\langle \eta, \eta \rangle = 0$. Hence $\Phi_m(\xi) = 0$.

Conversely, given an automorphism $\alpha_B \in \mathrm{Aut}_\sigma(B)$ of order m, and a section ξ with $\Phi_m(\xi) = 0$, we form the automorphism $\tau_B = t_\xi \circ \alpha_B$. Then τ_B is of order n, with $n = md$ where d is the order of the torsion section $\mathcal{P}_m(\xi)$. $\qquad\square$

It follows that in order to classify all automorphisms $\tau_B \in \mathrm{Aut}(B)$ of finite order, we must construct all pairs (α_B, ξ) as before.

3.2.1. Automorphisms of the first kind of finite order. An automorphism $\tau_B \in \mathrm{Aut}(B)$ is of the first kind if and only if $\bar{m} = 1$, hence the order of α_B is $m = k$. From the previous lemmas, either τ_B has a fixed curve intersecting each elliptic fiber, or α_B is the identity and $\tau_B = t_\eta$, where t_η means translation by a torsion section η of order $n = d$.

The complete list of rational elliptic surfaces with finite subgroups of the automorphism groups generated by translations by torsion sections is easy to produce. All possible configurations of singular fibers of rational elliptic surfaces are classified in [12, 14], and for each configuration, the torsion group $\mathrm{MW}_{\mathrm{tors}}$ of the Mordell–Weil group of sections is tabulated. This was also described in terms of lattices in [13]. Hence, the complete list of finite subgroups of the automorphism groups generated by translations by torsion sections can be harvested naturally from the list of torsion groups of rational elliptic surfaces.

3.2.2. Automorphisms of the second kind of finite order. Automorphisms of the second kind correspond to the cases where $\bar{m} > 1$. Then α_B is of order $m = \bar{m}k > 1$. From the previous lemmas, either $(\tau_B)^{\bar{m}}$ has a fixed curve intersecting each elliptic fiber, or $k = 1$ and $m = \bar{m} > 1$, that is, the order of α_B is equal to the order of the induced automorphism $\tau_{\mathbb{P}^1}$ on the $\mathbb{P}^1$ base.

3.3. Restrictions

In this work, we are interested in automorphisms of rational elliptic surfaces that can be lifted to free automorphisms on smooth fiber products of two rational elliptic surfaces. Accordingly, we impose two restrictions: the fiber product must be smooth, and the automorphism on the fiber product must be free. Since we focus on automorphisms of rational elliptic surfaces, we would like to understand what these restrictions on the fiber product impose on the constituent automorphisms of rational elliptic surfaces.

Take two rational elliptic surfaces B and B', and form the fiber product $\tilde{X} = B \times_{\mathbb{P}^1} B'$.

Suppose that $\tau_B \in \mathrm{Aut}(B)$ has integers $(n, m, \bar{m}, k, d)$ defined in the previous section. Similarly, suppose that $\tau_{B'} \in \mathrm{Aut}(B')$ has integers $(n', m', \bar{m}', k', d')$. One can form the automorphism $\tau_{\tilde{X}} = \tau_B \times_{\mathbb{P}^1} \tau_{B'}$ on the fiber product $\tilde{X}$ if and only if they have consistent induced action on the $\mathbb{P}^1$: $\rho(\tau_B) = \rho'(\tau_{B'})$ and in particular, $\bar{m} = \bar{m}'$. In fact, any automorphism $\tau_{\tilde{X}} \in \mathrm{Aut}(\tilde{X})$ that preserves the elliptic fibration of $\tilde{X}$ can be constructed in this way.

We want the fiber product $\tilde{X}$ to be smooth, and the group $\langle \tau_{\tilde{X}} \rangle$ generated by $\tau_{\tilde{X}}$ to act freely on $\tilde{X}$. Recall from the introduction that $\tilde{X}$ is smooth if and only if $S'' = 0$, that is, the singular fibers of B and B' are not paired in the fiber product. Furthermore, it is clear that $\tau_{\tilde{X}}$ will be free if and only if the fibers of B and B' intersecting the fixed locus of τ_B and $\tau_{B'}$ are not paired in the fiber product $\tilde{X} = B \times_{\mathbb{P}^1} B'$.

But what do these restrictions imply on the rational elliptic surfaces B and B' and the automorphisms τ_B and $\tau_{B'}$?

Lemma 3.5. *If $\langle \tau_{\tilde{X}} \rangle$ is free on $\tilde{X}$, then $n = n'$ and $k = k' = 1$.*

Proof. Suppose that $n < n'$. Then, $(\tau_B)^n$ is the identity on B; hence, $(\tau_{\tilde{X}})^n$ is not free on the fiber product $B \times_{\mathbb{P}^1} B'$. Similarly for $n' < n$; therefore if $\langle \tau_{\tilde{X}} \rangle$ is free, then $n = n'$.

Suppose now that $k > 1$. Lemma 3.3 tells us that $(\tau_B)^{\bar{m}}$ has a fixed curve intersecting each elliptic fiber of B. Since the fixed locus of $(\tau_{B'})^{\bar{m}}$ cannot be empty, some of the fibers of B and B' intersecting the fixed locus of $(\tau_B)^{\bar{m}}$ and $(\tau_{B'})^{\bar{m}}$ will necessarily be paired in the fiber product. Hence, $(\tau_{\tilde{X}})^{\bar{m}}$ will not be free. Similarly for $k' > 1$. Therefore, if $\langle \tau_{\tilde{X}} \rangle$ is free, then $k = k' = 1$. $\qquad\square$

Henceforth we may set $n = n'$ and $k = k' = 1$, which implies that $d = d'$ and $m = \bar{m} = \bar{m}' = m'$, that is, the orders of α_B and $\alpha_{B'}$ are the same as the order of the induced automorphism $\tau_{\mathbb{P}^1}$ of the $\mathbb{P}^1$ base. We are left with only two independent integers, (n, m), with $n = dm$; both τ_B and $\tau_{B'}$ have order n, and α_B, $\alpha_{B'}$ and $\tau_{\mathbb{P}^1}$ all have order m.

3.3.1. $m = 1$. Consider first the case with $m = 1$ (automorphisms of the first kind). In this case, both τ_B and $\tau_{B'}$ are given by translation by torsion sections η and η' of order $n = d$, and there is no induced action on the $\mathbb{P}^1$. These automorphisms act freely on the smooth elliptic fibers of B and B', respectively. Hence, the only fibers of B (resp. B') intersecting the fixed locus of τ_B (resp. $\tau_{B'}$) must be singular. By choosing the isomorphism between the $\mathbb{P}^1$ bases of B and B' to form the fiber product, in this case we

can always make sure that the singular fibers of B and B' are not paired in the fiber product, thus ensuring simultaneously that the fiber product $\tilde{X}$ is smooth and the automorphism group $\langle \tau_{\tilde{X}} \rangle$ is free.

3.3.2. $m > 1$. Suppose now that $m > 1$ (automorphisms of the second kind). Without loss of generality, we can choose homogeneous coordinates $[u : v]$ on $\mathbb{P}^1$ such that $\tau_{\mathbb{P}^1}$ is given by

$$(3.4) \qquad \tau_{\mathbb{P}^1} : [u : v] \longmapsto [u : \omega v],$$

where ω is an mth root of unity.

Lemma 3.6. *Suppose that $m > 1$. $\langle \tau_{\tilde{X}} \rangle$ is free if and only if $\langle \tau_B \rangle$ is free on f_∞ and $\langle \tau_{B'} \rangle$ on f_0', or vice versa. Moreover, since we assume that $\tilde{X}$ is smooth, f_0 and f_0' are not both singular, and the same holds for f_∞ and f_∞'.*

Proof. Using the previous parameterization, it is clear that $0, \infty \in \mathbb{P}^1$ are the only fixed points of $\langle \tau_{\mathbb{P}^1} \rangle$. The fixed locus of τ_B (resp. $\tau_{B'}$) must then be contained in the fibers $f_0 := \beta^{-1}(0)$ (resp. $f_0' := (\beta')^{-1}(0)$) and $f_\infty := \beta^{-1}(\infty)$ (resp. $f_\infty' := (\beta')^{-1}(\infty)$). Moreover, we know that the fixed loci of τ_B and $\tau_{B'}$ cannot be empty; hence, τ_B cannot be free on f_0 and f_∞ at the same time; the same holds for $\tau_{B'}$. Therefore, the automorphism group generated by $\tau_{\tilde{X}} = \tau_B \times_{\mathbb{P}^1} \tau_{B'}$ is free on $\tilde{X}$ if and only if $\langle \tau_B \rangle$ is free on f_∞ and $\langle \tau_{B'} \rangle$ on f_0', or vice versa.

The second claim is that if $\tilde{X}$ is smooth, then f_0 and f_0' are not both singular, and the same holds for f_∞ and f_∞'. The fibers of $\tilde{X}$ above 0 and ∞ are given by $f_0 \times f_0'$ and $f_\infty \times f_\infty'$, respectively. Hence, the claim follows directly from the fact that $\tilde{X}$ is smooth if and only if $S'' = 0$. $\qquad \square$

3.3.3. $m > 1$, $d > 1$. We can extract a stronger restriction for the case $m > 1$, $d > 1$, due to the following lemma.

Lemma 3.7. *For $m > 1$ and $d > 1$, if τ_B is free on f_∞, then f_0 must be singular.*

Proof. Suppose that $m > 1$, $d > 1$, and τ_B is free on f_∞. Since $d > 1$, $(\tau_B)^m$ is given by translation by a non-zero torsion section of order d. Since τ_B is free on f_∞, it cannot be free on f_0, so $(\tau_B)^m$ is also not free on f_0. However, translation by a non-zero torsion section acts freely on a smooth elliptic fiber. Therefore f_0 must be singular. $\qquad \square$

Hence, we get the following corollary.

Corollary 3.8. *Suppose that $m > 1$, $d > 1$, $\langle \tau_{\tilde{X}} \rangle$ is free, and $\tilde{X}$ is smooth. Then, up to interchanging 0 and ∞ if necessary:*

- *$\langle \tau_B \rangle$ is free on f_∞ and $\langle \tau_{B'} \rangle$ on f'_0;*

- *f_0 and f'_∞ are singular;*

- *f_∞ and f'_0 are smooth.*

Proof. This follows immediately from Lemmas 3.6 and 3.7. $\qquad\square$

We now have quite a few restrictions on the rational elliptic surfaces B and B', and their automorphisms τ_B and $\tau_{B'}$, which we formalize in the following subsection.

3.3.4. Suitable pairs. From the previous analysis, we introduce the concept of a suitable pair (B, τ_B).

Definition 3.9. Let $\tau_B \in \mathrm{Aut}(B)$ with integers $(n, m, \bar{m}, d, k)$. We say that the pair (B, τ_B) is *suitable* if $k = 1$, which implies that $m = \bar{m}$, and one of the following conditions holds:

(1) $m = 1$ (first kind);

(2) $m > 1$ (second kind), $d = 1$ and the group generated by τ_B acts freely on f_∞;

(3) $m > 1$ (second kind), $d > 1$, f_∞ is smooth and the group generated by τ_B acts freely on f_∞. (In this case Lemma 3.7 implies that f_0 is singular.)

According to the discussion above, any free action on a smooth fiber product comes from a pair of suitable pairs (B, τ_B) and $(B', \tau_{B'})$, and conversely any suitable pair (B, τ_B) can be lifted to a free automorphism on a smooth fiber product by combining it with another appropriate suitable pair $(B', \tau_{B'})$.

Notice that when $m = 1$ (condition 1), suitable pairs are already classified by torsion groups of rational elliptic surfaces, as explained in Section 3.2.1. Our main aim is then to classify suitable pairs (B, τ_B) satisfying conditions 2 or 3.

Proposition 3.10. *Let (B, τ_B) be a suitable pair with $m > 1$, $d = 1$ (condition 2). Then f_∞ is either smooth or singular of type I_{nr}, for some integer $r > 0$.*

Proof. By definition, f_∞ is either a smooth elliptic curve or a singular fiber of one of Kodaira's type. By looking at Kodaira's list of singular fibers [10], it is easy to see that the only type of singular fiber that admits a free cyclic automorphism of order n is I_{nr} for some integer $r > 0$. □

Hence, when $m > 1$, $d = 1$ (condition 2), f_∞ must be either smooth or of type I_{nr}, while when $m > 1$, $d > 1$ (condition 3), f_∞ must be smooth.

We can also extract a requirement on the linearizations α_B.

Proposition 3.11. *Let (B, τ_B) be a suitable pair with $m > 1$ (conditions 2 and 3). Then the linearization α_B must fix the neutral component of the fiber f_∞ pointwise.*

Proof. Suppose first that f_∞ is smooth. Since (B, τ_B) is suitable, τ_B acts freely on f_∞. We know that α_B must fix the zero section; hence it cannot be free on f_∞. Therefore, if $\tau_B = t_\xi \circ \alpha_B$ is free on f_∞, then α_B must fix f_∞ pointwise.

Suppose now that f_∞ is of type I_{nr}. Let $\Theta_0, \ldots, \Theta_{nr-1}$ be the components of the singular fibers f_∞. Since α_B fixes the zero section, it must send the neutral component Θ_0 to itself. Now α_B must also send each other components Θ_i, $i = 1, \ldots, nr - 1$ to itself, otherwise $\tau_B = t_\xi \circ \alpha_B$ would have fixed points. Hence α_B must send each component to itself. Then, for τ_B to be free, on each component α_B must either fix the component pointwise or fix only the intersection points of the component with its neighbor components. But since α_B fixes the zero section, which is not an intersection point of the neutral component Θ_0 with its neighbors, we obtain that α_B must fix the neutral component of f_∞ pointwise. □

We are now in a position to classify suitable pairs (B, τ_B) satisfying conditions 2 and 3. Using the insight of Lemma 3.4, we first study the linearizations α_B, and then scrutinize the subgroup of sections $\ker(\Phi_m) \subseteq$ MW.

4. The linearizations α_B

Let us study first the linearizations α_B.

As usual, let $\alpha_B \in \mathrm{Aut}_\sigma(B)$ be of order m and $\tau_{\mathbb{P}^1} \in \mathrm{Aut}(\mathbb{P}^1)$ be of order $\bar{m}$. Recall from the previous section that we want to classify all B admitting
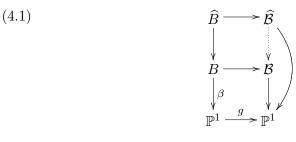
an α_B such that

(1) $m = \bar{m}$;

(2) f_∞ of B is either smooth or of type I_{mr};

(3) α_B fixes the neutral component of f_∞ pointwise.

Definition 4.1. We will call a pair (B, α_B) satisfying these three conditions a *suitable σ-pair* (not to be confused with suitable pairs (B, τ_B)).

Since α_B fixes the zero section it must act on the Weierstrass model of B. We could use this point of view to classify all such automorphisms. However, we prefer to use a different viewpoint.

Suppose that (B, α_B) is a suitable σ-pair. Let $\mathcal{B} = B/\langle \alpha_B \rangle$ be the quotient surface. Let $\widehat{\mathcal{B}}$ be the relatively minimal (or Kodaira) model of $\mathcal{B}$. Recall that this is obtained from a resolution $\widehat{\mathcal{B}}^+$ of $\mathcal{B}$ by blowing down all (-1)-curves in fibers. Although $\widehat{\mathcal{B}}$ may no longer map to $\mathcal{B}$, it does map to $\mathbb{P}^1$, in fact it is an elliptic fibration over $\mathbb{P}^1$. Since it is birational to a quotient of a rational surface, it is itself rational, so it is a smooth rational elliptic surface.

Hence, any suitable σ-pair (B, α_B) with α_B of order m fits in a commutative diagram

(4.1)

$$
\begin{array}{ccc}
\widehat{B} & \longrightarrow & \widehat{\mathcal{B}} \\
\downarrow & & \downarrow \\
B & \longrightarrow & \mathcal{B} \\
\beta \downarrow & & \downarrow \\
\mathbb{P}^1 & \xrightarrow{\ g\ } & \mathbb{P}^1
\end{array}
$$

Here $g : \mathbb{P}^1 \to \mathbb{P}^1$ is the mth power map totally ramified at $0, \infty \in \mathbb{P}^1$, and $\widehat{B}$ is the minimal resolution of the fiber product $\widehat{B}^-$.

Conversely, we can start with any rational elliptic surface $\widehat{\mathcal{B}}$ and recover (B, α_B). First we pull back via the map g to get the fiber product $\widehat{B}^-$. The $\mathbb{P}^1$ which is the source of g has an automorphism $\alpha_{\mathbb{P}^1}$ of order m which is a deck transformation with respect to g, that is $g \circ \alpha_{\mathbb{P}^1} = g$ (and generates the group of such automorphisms); this together with the identity on $\widehat{\mathcal{B}}$ induces a natural automorphism $\alpha_{\widehat{B}^-}$ on the fiber product $\widehat{B}^-$. We resolve $\widehat{B}^-$ to get $\widehat{B}$; and then blow down the (-1) curves to obtain a σ-pair (B, α_B). In general, uniqueness of the Kodaira model guarantees that starting with (B, α_B), going to the Kodaira model $\widehat{\mathcal{B}}$ of its quotient, and then going through the

previous steps brings us back to the original (B, α_B). We need to understand which smooth rational elliptic surfaces $\widehat{\mathcal{B}}$ are suitable in the sense that they yield a suitable σ-pair (B, α_B).

We illustrate the issue with two examples. First, consider a $\widehat{\mathcal{B}}$ whose fiber over ∞ is of type I_M. The inverse image in the fiber product $\widehat{B}^-$ of the original I_M fiber still looks like an I_M fiber, but the surface $\widehat{B}^-$ in which it sits is singular: it has an A_{m-1} singularity at each of the M singular points of the I_M fiber. We resolve these singularities to obtain the smooth $\widehat{B}$, containing an I_{mM} fiber over ∞. This elliptic surface $\widehat{B}$ is minimal, so is isomorphic to B. The natural automorphism of the fiber product $\widehat{B}^-$ induces an α_B on B. This α_B acts trivially on the M original components, but non-trivially (via various roots of unity) on the $(m-1)M$ new components arising from the resolution of the A_{m-1} singularities. In particular, we see that we have a suitable σ-pair (B, α_B).

On the other hand, consider the case that the fiber of $\widehat{\mathcal{B}}$ over ∞ is of type I_0^*, and $m = 2$. The fiber product $\widehat{B}^-$ is now not normal. When we normalize, the central component of the I_0^* fiber is replaced with an elliptic curve. The resulting elliptic surface $\widehat{B}$ is not minimal; B is obtained from it by blowing down the four rational components of the ∞ fiber, and has a smooth (I_0) fiber there. In this case (B, α_B) is not a suitable σ-pair even though its fiber over ∞ is smooth: α_B acts non-trivially on this smooth elliptic curve which is the neutral (and only) component.

The general pattern is obtained in Lemma 4.6 next, and depends on the notion of deficiency which we now define.

4.1. Deficiencies

The first step consists in understanding the relationship between the singular fibers of B and $\widehat{\mathcal{B}}$. If the map g is etale at $p \in \mathbb{P}^1$ — which in this case is true for any $p \in \mathbb{P}^1$ not equal to 0 or ∞ — then the fiber of $\widehat{\mathcal{B}}$ (or $\mathcal{B}$) above $g(p)$ is identical to the fiber of B above p. The crucial analysis is then for the fibers f_0 and f_∞ of B, where the map g is totally ramified of order m.

This question was addressed by Miranda and Persson in section 7 of [11] using the Weierstrass models of $\widehat{\mathcal{B}}$ and B. Table 7.1 of [11] gives the relation between the types of the fiber of $\widehat{\mathcal{B}}$ over $g(p)$ and the fiber of B over p for the points p where g is ramified of order m.[7] We reproduce this table in table 1 for convenience.

---

[7]In their notation, B is Y and $\widehat{\mathcal{B}}$ is X.

Table 1: Relation between the types of singular fibers of B and $\widehat{\mathcal{B}}$ over the ramified points p of order m ($p = 0, \infty$) [11].

Fiber of $\widehat{\mathcal{B}}$ over $g(p)$	Fiber of B over p
I_0	I_0
I_M	I_{mM}
I_M^*	I_{mM} if m even; $\qquad I_{mM}^*$ if m odd
II	I_0 if $m = 0$ mod 6; $\;II$ if $m = 1$ mod 6; $\quad IV$ if $m = 2$ mod 6 I_0^* if $m = 3$ mod 6; $\;IV^*$ if $m = 4$ mod 6; $\;II^*$ if $m = 5$ mod 6
III	I_0 if $m = 0$ mod 4; $\;III$ if $m = 1$ mod 4; I_0^* if $m = 2$ mod 4; $\;III^*$ if $m = 3$ mod 4;
IV	I_0 if $m = 0$ mod 3; $\;IV$ if $m = 1$ mod 3; $\quad IV^*$ if $m = 2$ mod 3
IV^*	I_0 if $m = 0$ mod 3; $\;IV^*$ if $m = 1$ mod 3; $\;IV$ if $m = 2$ mod 3
III^*	I_0 if $m = 0$ mod 4; $\;III^*$ if $m = 1$ mod 4; I_0^* if $m = 2$ mod 4; $\;III$ if $m = 3$ mod 4;
II^*	I_0 if $m = 0$ mod 6; $\;II^*$ if $m = 1$ mod 6; $\quad IV^*$ if $m = 2$ mod 6 I_0^* if $m = 3$ mod 6; $\;IV$ if $m = 4$ mod 6; $\quad II$ if $m = 5$ mod 6

The next step consists in understanding the allowed configurations of singular fibers for $\widehat{\mathcal{B}}$.

We can define an integer, which we call the *deficiency*, associated to singular fibers of $\widehat{\mathcal{B}}$, and depending on the order m of g. Basically, the deficiency formalizes the change in Euler characteristic of the singular fibers between B and $\widehat{\mathcal{B}}$.

Definition 4.2. Let $\mathcal{S}$ be a fiber of $\widehat{\mathcal{B}}$ over a point $p \in \mathbb{P}^1$. We define the *deficiency* $D(\mathcal{S})$ of $\mathcal{S}$ by

$$(4.2) \qquad\qquad D(\mathcal{S}) = \chi(\mathcal{S}) - \sum_{S \in W} \chi(S),$$

where W is the set of fibers over the preimages $g^{-1}(p)$ of p.

Proposition 4.3. $D(\mathcal{S}) = 0$ *if $\mathcal{S}$ is smooth.*

Proof. This is obvious from (4.2), table 1 and the fact that the fiber of $\widehat{\mathcal{B}}$ above $g(p)$ is identical to the fiber of B above p at the points p where g is etale. $\qquad\square$

Lemma 4.4. *Let $\mathcal{R}$ be the set of singular fibers of $\widehat{\mathcal{B}}$. Then*

$$\sum_{\mathcal{S} \in \mathcal{R}} D(\mathcal{S}) = 0.$$

Proof. By definition,

$$\sum_{\mathcal{S} \in \mathcal{R}} D(\mathcal{S}) = \left(\sum_{\mathcal{S} \in \mathcal{R}} \chi(\mathcal{S}) \right) - \left(\sum_{\mathcal{S} \in \mathcal{R}} \sum_{S \in S_{\mathcal{S}}} \chi(S) \right).$$

But $\sum_{\mathcal{S} \in \mathcal{R}} \chi(\mathcal{S}) = \chi(\widehat{\mathcal{B}}) = 12$, and since any singular fiber of B is pullbacked from a singular fiber of $\widehat{\mathcal{B}}$,

$$\sum_{\mathcal{S} \in \mathcal{R}} \sum_{S \in S_{\mathcal{S}}} \chi(S) = \sum_{S \in R} \chi(S)$$
$$= 12,$$

where R is the set of singular fibers of B. Hence, $\sum_{\mathcal{S} \in \mathcal{R}} D(\mathcal{S}) = 0$. $\square$

Using table 1 relating singular fibers of $\widehat{\mathcal{B}}$ and B over the ramified points $0, \infty \in \mathbb{P}^1$, we can find explicit formulae for the deficiencies of the singular fibers of $\widehat{\mathcal{B}}$.

Lemma 4.5. *Let $\mathcal{S}$ be a singular fiber of $\widehat{\mathcal{B}}$ over a point $p \in \mathbb{P}^1$. Then the deficiency of $\mathcal{S}$ is given as in table 2.*

Proof. This is a simple calculation from the definition of the deficiency (4.2) and table 1. Note that the last line cannot be expressed neatly in terms of

Table 2: Deficiencies of singular fibers $\mathcal{S}$ of $\mathcal{B}$.

Position of p	Type of $\mathcal{S}$	Deficiency $D(\mathcal{S})$
$p \neq 0, \infty$	Any type	$D(\mathcal{S}) = (1 - m)\chi(\mathcal{S})$
$p = 0$	I_M	$D(\mathcal{S}) = (1 - m)\chi(\mathcal{S})$
or $p = \infty$	I_M^*	$D(\mathcal{S}) = (1 - m)\chi(\mathcal{S}) + 6(m - \delta)$ with $\delta = m \bmod 2$
	II, III, IV	$D(\mathcal{S}) = (1 - \delta)\chi(\mathcal{S})$ with $\delta = m \bmod (12/\chi(\mathcal{S}))$
	II^*, III^*, IV^*	$D(\mathcal{S}) = \epsilon(12 - \chi(\mathcal{S}))$ with $\epsilon = (m - 1) \bmod(12/(12 - \chi(\mathcal{S})))$

δ, due to the peculiarity of clock arithmetic: for $\delta \neq 0$, we have $\epsilon = \delta - 1$, but this fails for $\delta = 0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

4.2. Suitable rational elliptic surfaces

Using Lemma 4.4 and table 2 we are in a position to generate a complete list of suitable quotient surfaces $\widehat{\mathcal{B}}$, taking into account the three requirements in the definition of a suitable σ-pair (B, α_B) in the beginning of this section.

Lemma 4.6. *Let $\alpha_B \in \mathrm{Aut}_\sigma(B)$ be of order m. If (B, α_B) is a suitable σ-pair, then the fiber f_∞ of $\widehat{\mathcal{B}}$ is either smooth or of type I_r for some $r \geq 1$, and*

$$D(f_0) = (m - 1)\,(12 - \chi(f_0)),$$

where f_0 is the fiber above 0 of $\widehat{\mathcal{B}}$. In this case we say that $\widehat{\mathcal{B}}$ is suitable of order m.

Proof. If (B, α_B) is a suitable σ-pair, then f_∞ of B is either smooth or of type I_{mr}, and α_B fixes the neutral component of f_∞ pointwise.

Suppose that f_∞ is smooth. Since α_B fixes f_∞ pointwise, it is clear that the fiber f_∞ of $\widehat{\mathcal{B}}$ will also be smooth.[8]

If f_∞ of B is of type I_{mr}, then table 1 says that f_∞ of $\widehat{\mathcal{B}}$ is either of type I_r or of type I_r^* and m is even. But the latter case occurs only if α_B does not fix the neutral component of f_∞ of B pointwise. Hence f_∞ of $\widehat{\mathcal{B}}$ must be of type I_r.

Now, let $\mathcal{R}$ be the set of singular fibers of $\widehat{\mathcal{B}}$. We know that $\sum_{\mathcal{S} \in \mathcal{R}} D(\mathcal{S}) = 0$. Using table 2, and the fact that f_∞ of $\widehat{\mathcal{B}}$ is either smooth or of type I_r, we get

$$0 = \sum_{\mathcal{S} \in \mathcal{R}} D(\mathcal{S}) = D(f_0) + (1 - m)\left(\chi(f_\infty) + \sum_{\mathcal{S} \in \mathcal{R} \setminus \{f_0, f_\infty\}} \chi(\mathcal{S}) \right).$$

---

[8]In general, the fixed locus of any automorphism α of a manifold is a submanifold [6]. The differential $d\alpha$ acting on the tangent space at a fixed point has the tangent space to the fixed locus as its $+1$ eigenspace. The quotient is smooth if and only if there is a single normal direction, that is, codim $= 1$. In our case, we have a curve of fixed points on a smooth surface, so the quotient is smooth there, hence the fiber (in the resolved $\widehat{\mathcal{B}}$) is the naive quotient, which is a smooth elliptic curve.

Table 3: List of allowed f_0 for $\widehat{\mathcal{B}}$.

m	Type of f_0
6	II^*
5	II^*
4	II^*, III^*
3	II^*, III^*, IV^*
2	$II^*, III^*, IV^*, I_4^*, I_3^*, I_2^*, I_1^*, I_0^*$

But for a rational elliptic surface

$$\chi(f_\infty) + \sum_{\mathcal{S}\in\mathcal{R}\setminus\{f_0,f_\infty\}} \chi(\mathcal{S}) = 12 - \chi(f_0),$$

and the lemma is proved.   □

We can now produce a list of all suitable quotient surfaces $\widehat{\mathcal{B}}$.

Lemma 4.7. *A rational elliptic surface $\widehat{\mathcal{B}}$ is suitable of order m if and only if f_∞ is either smooth or of type I_r and f_0 is of one of the types listed in table 3. All other ms are not allowed.*

Proof. This follows from Lemma 4.6 and table 2 after a straightforward calculation.   □

This is enough to produce a complete list of suitable quotient surfaces $\widehat{\mathcal{B}}$. We scan through Persson's list of configurations of singular fibers [14], and keep all configurations with f_0 as in Lemma 4.7 and f_∞ either smooth or of type I_k. Each of these configurations is suitable for one or more ms.

4.3. List of suitable σ-pairs (B, α_B)

To produce the list of suitable σ-pairs (B, α_B), we pullback each rational elliptic surface $\widehat{\mathcal{B}}$ suitable of order m that we found in the previous section, via the map $g : \mathbb{P}^1 \to \mathbb{P}^1$ of order m as in (4.1). We use table 1 to extract the types of the fibers f_0 and f_∞ of B. This generates all suitable σ-pairs (B, α_B).

4.3.1. A "simplification". The choice of the m-sheeted covering map g (as in (4.1)) is equivalent to the choice of its two branch points $0, \infty \in \mathbb{P}^1$. Now the point 0 is rigid, since by Lemma 4.7 it must correspond to a singular

fiber. But we are free to move the point ∞ over the Zariski open subset of $\mathbb{P}^1$ parametrizing fibers of type I_r (including the smooth ones, $r = 0$). So any $\widehat{\mathcal{B}}$ and m as in Lemma 4.7 produce a one-parameter family of suitable σ-pairs (B, α_B). At most finitely many of these involve a singular fiber f_∞. It follows that any suitable σ-pair (B, α_B) coming from a $\widehat{\mathcal{B}}$ with singular fiber f_∞ is the smooth specialization of a one-parameter family of suitable σ-pairs (B, α_B) coming from quotient surfaces $\widehat{\mathcal{B}}$ with smooth fibers f_∞.

From now on we will therefore focus on surfaces B and $\widehat{\mathcal{B}}$ with f_∞ smooth.

4.3.2. The list. Keeping this in mind, we get the following result.

Proposition 4.8. *The list of suitable $\widehat{\mathcal{B}}$ and their associated suitable σ-pairs (B, α_B) with smooth f_∞ is given in table 4. We indicate the Mordell–Weil lattices of the rational elliptic surfaces B and $\widehat{\mathcal{B}}$: for each Mordell–Weil lattice the allowed configurations of singular fibers can be found using Oguiso and Shioda's and Persson's list [12–14]. We also write down the root lattice T associated to the singular fibers of B, as in [13]. In the table I_0 means a smooth fiber.*

Note that while writing down this table we found a small mistake in Oguiso and Shioda's list [13] for case 30 in our list, the quotient surface of which corresponds to case 33 in their list. There, it is stated that MW $= A_1^* \oplus \langle 1/6 \rangle$ for the quotient surface $\widehat{\mathcal{B}}$, but it should be MW $= \frac{1}{6} \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix} \right)$.

5. The sections ξ

In the previous section we produced a list of suitable σ-pairs (B, α_B). We now want to complete the pairs (B, α_B) to suitable pairs (B, τ_B) as defined in Definition 3.9. More precisely, we want to complete the automorphisms α_B to automorphisms $\tau_B = t_\xi \circ \alpha_B$ of the second kind satisfying conditions 2 or 3 in Definition 3.9, where $\xi \in$ MW is a section.

Throughout this section we let $m, n = md$ denote the orders of α_B, τ_B, respectively. We recall from Lemma 3.4 that ξ must satisfy $\Phi_m(\xi) = 0$. Equivalently, $\mathcal{P}_m(\xi)$ is a torsion section; its order is precisely the integer $d = n/m$. This is also the order of the point where $\mathcal{P}_m(\xi)$ meets the fiber f_∞, which by the "simplification" of Subsection 4.3.1 can and will be taken to be smooth.

We will determine the group $\ker(\Phi_m)$ in Subsection 5.1. In 5.2 we will determine its subgroup $\ker(\mathcal{P}_m) = \ker(\Phi_m)_{d=1}$; the possibility that $d > 1$

Table 4: List of suitable $\hat{\mathcal{B}}$ and the associated suitable σ-pairs (B, α_B) with smooth f_∞.

m	$\hat{\mathcal{B}}$			B				num
	f_0	MW$_{\text{lat}}$	MW$_{\text{tors}}$	f_0	MW$_{\text{lat}}$	MW$_{\text{tors}}$	T	
6	II^*	0	0	I_0	E_8	0	0	1
5	II^*	0	0	II	E_8	0	0	2
4	II^*	0	0	IV	E_6^*	0	A_2	3
	III^*	A_1^*	0	I_0	E_8	0	0	4
	II^*	0	$\mathbb{Z}_2$	I_0^*	D_4^*	$\mathbb{Z}_2$	$A_1^{\oplus4}$	5
3		0	0		D_4^*	0	D_4	6
	III^*	A_1^*	0	III	E_7^*	0	A_1	7
		0	$\mathbb{Z}_2$		D_4^*	$\mathbb{Z}_2$	$A_1^{\oplus4}$	8
	IV^*	A_2^*	0	I_0	E_8	0	0	9
		$\langle 1/6 \rangle$	0		$D_4^* \oplus A_1^*$	0	$A_1^{\oplus3}$	10
		0	$\mathbb{Z}_3$		A_2^*	$\mathbb{Z}_3$	$A_2^{\oplus3}$	11
2	II^*	0	0	IV^*	A_2^*	0	E_6	12
	III^*	A_1^*	0	I_0^*	D_4^*	0	D_4	13
		0	$\mathbb{Z}_2$		$A_1^{*\oplus2}$	$\mathbb{Z}_2$	$D_4 \oplus A_1^{\oplus2}$	14
	IV^*	A_2^*	0	IV	E_6^*	0	A_2	15

continued

Table 4: *Continued*

		$\mathbb{Z}$			$\mathbb{Z}$		#
	$\langle 1/6 \rangle$	0	$\frac{1}{6}\begin{pmatrix} 2 & 1 & 0 & -1 \\ 1 & 5 & 3 & 1 \\ 0 & 3 & 6 & 3 \\ -1 & 1 & 3 & 5 \end{pmatrix}$		0	$A_2 \oplus A_1^{\oplus 2}$	16
I_4^*	0	$\mathbb{Z}_3$	A_2^*	I_8	$\mathbb{Z}_3$	$A_2^{\oplus 3}$	17
I_3^*	0	$\mathbb{Z}_2$	A_1^*	I_6	$\mathbb{Z}_2$	A_7	18
I_2^*	$\langle 1/4 \rangle$	0	$A_2^* \oplus A_1^*$	I_4	0	A_5	19
	$A_1^{*\oplus 2}$	0	D_5^*		0	A_3	20
	A_1^*	$\mathbb{Z}_2$	A_3^*		$\mathbb{Z}_2$	$A_3 \oplus A_1^{\oplus 2}$	21
	0	$(\mathbb{Z}_2)^2$	$\langle 1/4 \rangle$		$(\mathbb{Z}_2)^2$	$A_3 \oplus A_1^{\oplus 4}$	22
I_1^*	A_3^*	0	E_7^*	I_2	0	A_1	23
	$A_1^* \oplus \langle 1/4 \rangle$	0	$D_4^* \oplus A_1^*$		0	$A_1^{\oplus 3}$	24
	$\langle 1/12 \rangle$	0	$A_2^* \oplus \langle 1/6 \rangle$		0	$A_2^{\oplus 2} \oplus A_1$	25
	$\langle 1/4 \rangle$	$\mathbb{Z}_2$	$A_1^{*\oplus 3}$		$\mathbb{Z}_2$	$A_1^{\oplus 5}$	26
	D_4^*	$\mathbb{Z}_4$	A_1^*		$\mathbb{Z}_4$	$A_3^{\oplus 2} \oplus A_1$	27
I_0^*	$A_1^{*\oplus 3}$	0	E_8	I_0	0	0	28
	$\frac{1}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$	0	D_6^*		0	$A_1^{\oplus 2}$	29
	$A_1^{*\oplus 2}$	0	$A_2^{*\oplus 2}$		0	$A_2^{\oplus 2}$	30
	A_1^*	$\mathbb{Z}_2$	D_4^*		$\mathbb{Z}_2$	$A_1^{\oplus 4}$	31
	$\langle 1/4 \rangle$	$(\mathbb{Z}_2)^2$	$A_1^{*\oplus 2}$		$(\mathbb{Z}_2)^2$	$A_1^{\oplus 6}$	32
	0	$\mathbb{Z}_2$	$A_1^{*\oplus 2}$		$\mathbb{Z}_2$	$A_3^{\oplus 2}$	33
		$(\mathbb{Z}_2)^2$	0		$(\mathbb{Z}_2)^2$	$D_4^{\oplus 2}$	34

arises when these two groups are not equal. The remaining condition for suitability is that the action on f_∞ should be free; this is discussed in 5.3. An overview of the entire analysis is given in 5.4.

5.1. Calculation of $\ker(\Phi_m)$

Lemma 5.1. $\ker(\Phi_m) = [(\mathrm{MW}^{\alpha_B})^\perp \ in \ \mathrm{MW}].$[9]

Proof. Consider the map $\mathrm{MW} \to \mathrm{MW} \otimes \mathbb{C}$. Under this map, α_B becomes an automorphism of order m of a vector space, and so its roots are mth roots of 1. But Φ_m kills all roots other than 1, so $(\Phi_m)/m$ is projection onto the invariants, and its kernel is the subspace perpendicular to the space of α_B-invariants. But both $\ker(\Phi_m)$ and $[(\mathrm{MW}^{\alpha_B})^\perp \ in \ \mathrm{MW}]$ are the inverse images of their images under $\mathrm{MW} \to \mathrm{MW} \otimes \mathbb{C}$; therefore $\ker(\Phi_m) = [(\mathrm{MW}^{\alpha_B})^\perp \ in \ \mathrm{MW}]$. $\square$

We now know how to find $\ker(\Phi_m)$; we only have to calculate, for each case, the orthogonal complement of MW^{α_B} in MW, and we know that $\mathrm{MW}^{\alpha_B} = m\widehat{\mathrm{MW}}$,[10] where $\widehat{\mathrm{MW}}$ is the Mordell–Weil group of the quotient surface $\widehat{B}$ and m is the order of α_B [16].

This can be done for each case, using the following result.

Lemma 5.2. *For each of the following cases, the embedding is unique up to isomorphisms, and the orthogonal complement is as follows. We write in brackets the case number of table 4 that these embeddings correspond to. In all the other cases of table 4, $\mathrm{MW}_{\mathrm{lat}}(\widehat{\mathcal{B}}) = 0$.*

$$\left[A_1^\perp \ in \ E_8\right] = E_7 \ (4), \quad \left[\langle 3/2 \rangle^\perp \ in \ E_7^*\right] = E_6^* \ (7),$$

$$\left[A_2^\perp \ in \ E_8\right] = E_6 \ (9), \quad \left[(A_1^*)^\perp \ in \ D_4^* \oplus A_1^*\right] = D_4^* \ (10),$$

$$\left[(\langle 1 \rangle)^\perp \ in \ D_4^*\right] = \langle 1 \rangle^{\oplus 3} \ (13), \quad \left[(2A_2^*)^\perp \ in \ E_6^*\right] = D_4 \ (15),$$

$$\left[\left(\left\langle \frac{1}{3} \right\rangle\right)^\perp \ in \ \frac{1}{6} \begin{pmatrix} 2 & 1 & 0 & -1 \\ 1 & 5 & 3 & 1 \\ 0 & 3 & 6 & 3 \\ -1 & 1 & 3 & 5 \end{pmatrix}\right] = \langle 1 \rangle^{\oplus 3} \ (16),$$

$$\left[(A_1^*)^\perp \ in \ A_2^* \oplus A_1^*\right] = A_2^* \ (19), \quad \left[(\langle 1 \rangle^{\oplus 2})^\perp \ in \ D_5^*\right] = \langle 1 \rangle^{\oplus 3} \ (20),$$

---

[9]$(\mathrm{MW}_B^\alpha)^\perp$ in MW denotes the orthogonal complement of MW_B^α in MW with respect to the height pairing.

[10]By mA we mean that the intersection matrix of A is multiplied by m.

$$\left[(\langle 1 \rangle \oplus \mathbb{Z}_2)^{\perp} \ in \ A_3^* \oplus \mathbb{Z}_2\right] = \langle 1 \rangle^{\oplus 2} \oplus \mathbb{Z}_2 \ (21),$$

$$\left[(2A_3^*)^{\perp} \ in \ E_7^*\right] = D_4 \ (23),$$

$$\left[(\langle 1 \rangle \oplus A_1^*)^{\perp} \ in \ D_4^* \oplus A_1^*\right] = \langle 1 \rangle^{\oplus 3} \ (24),$$

$$\left[\left(\left\langle \frac{1}{6} \right\rangle\right)^{\perp} \ in \ A_2^* \oplus \left\langle \frac{1}{6} \right\rangle\right] = A_2^* \ (25),$$

$$\left[(A_1^* \oplus \mathbb{Z}_2)^{\perp} \ in \ A_1^{*\oplus 3} \oplus \mathbb{Z}_2\right] = A_1^{*\oplus 2} \oplus \mathbb{Z}_2 \ (26),$$

$$\left[(D_4)^{\perp} \ in \ E_8\right] = D_4 \ (28),$$

$$\left[(\langle 1 \rangle^{\oplus 3})^{\perp} \ in \ D_6^*\right] = \langle 1 \rangle^{\oplus 3} \ (29), \quad \left[(A_2^*)^{\perp} \ in \ A_2^{*\oplus 2}\right] = A_2^* \ (30),$$

$$\left[(\langle 1 \rangle^{\oplus 2} \oplus \mathbb{Z}_2)^{\perp} \ in \ D_4^* \oplus \mathbb{Z}_2\right] = \langle 1 \rangle^{\oplus 2} \oplus \mathbb{Z}_2 \ (31),$$

$$\left[(\langle 1 \rangle \oplus (\mathbb{Z}_2)^2)^{\perp} \ in \ A_1^{*\oplus 2} \oplus (\mathbb{Z}_2)^2\right] = \langle 1 \rangle \oplus (\mathbb{Z}_2)^2 \ (32),$$

$$\left[(A_1^* \oplus \mathbb{Z}_2)^{\perp} \ in \ A_1^{*\oplus 2} \oplus \mathbb{Z}_2\right] = A_1^* \oplus \mathbb{Z}_2 \ (33).$$

Proof. For most of the cases this is just a simple exercise in linear algebra. Here we only prove a few cases; the other cases are either straighforward or can be proved similarly.

Case 32. We want $[(\langle 1 \rangle \oplus (\mathbb{Z}_2)^2)^{\perp}$ in $A_1^{*\oplus 2} \oplus (\mathbb{Z}_2)^2]$. There are only four elements of $A_1^{*\oplus 2}$ of squared length 1: the sums of one generator from each A_1^*. In $A_1^{*\oplus 2} \oplus (\mathbb{Z}_2)^2$, we can use the same four elements plus the torsion elements. The embedding of $\langle 1 \rangle \oplus (\mathbb{Z}_2)^2$ is determined by the choice of one of these four elements of $A_1^{*\oplus 2}$ plus the torsion elements. The orthogonal complement is the sum of the entire torsion $(\mathbb{Z}_2)^2$ plus the orthogonal complement in $A_1^{*\oplus 2}$, which is the complementary copy of $\langle 1 \rangle$.

Case 31. Here we want $[(\langle 1 \rangle^{\oplus 2} \oplus \mathbb{Z}_2)^{\perp}$ in $D_4^* \oplus \mathbb{Z}_2]$. Let us use the explicit description of D_4^* as the square lattice $\mathbb{Z}^4$ plus the single non-integral element $\frac{1}{2}(1,1,1,1)$. The 24 elements of length 1 are plus/minus the four unit vectors in $\mathbb{Z}^4$, and $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$. Triality interchanges all of these, so we can take the image of the generator of the first $\langle 1 \rangle$ to be, say, $(1,0,0,0)$. The image of the generator of the second $\langle 1 \rangle$ must then be $(0,1,0,0)$, up to permutation. The orthogonal complement is then $(0,0,a,b) = \langle 1 \rangle^2$, and $\left[(\langle 1 \rangle^{\oplus 2} \oplus \mathbb{Z}_2)^{\perp} \text{ in } D_4^* \oplus \mathbb{Z}_2\right] = \langle 1 \rangle^{\oplus 2} \oplus \mathbb{Z}_2$.

Case 15. We want $\left[(2A_2^*)^{\perp} \text{ in } E_6^*\right]$. That is, we need two vectors in E_6^* with length 4/3 and intersection $-2/3$. The minimal points in E_6^* have length 4/3, and there are 54 of them: 30 of the form $\pm\frac{1}{3}(0,-2,-2,1,1,$

$1, 1, 0$), and another 24 of the form $\pm\frac{1}{6}(\pm 3, 5, -1, -1, -1, -1, -1, \mp 3)$ [7]. We find three possible embeddings, given by sending the two generators of A_2^* to one of the three following possibilities, up to permutations:

$$\left\{ \frac{1}{3}(0, -2, -2, 1, 1, 1, 1, 0), \frac{1}{3}(0, 1, 1, -2, -2, 1, 1, 0) \right\}$$

$$\left\{ \frac{1}{3}(0, -2, -2, 1, 1, 1, 1, 0), \frac{1}{6}(3, 5, -1, -1, -1, -1, -1, -3) \right\}$$

$$\left\{ \frac{1}{6}(-3, -1, 5, -1, -1, -1, -1, 3), \frac{1}{6}(3, 5, -1, -1, -1, -1, -1, -3) \right\}$$

These three embeddings are presumably equivalent, but we did not check this. Instead, we perform the calculations separately in the three cases. In each case, we calculate the intersection numbers of these vectors with the basis of E_6^* given in [7, p. 127, Equation (125)]. From this we work out the orthogonal complement for each embedding in this basis of E_6^*, and we get in each case the intersection matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -2 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}$$

for the orthogonal complement, which corresponds to D_4.

Case 23. We want to compute $[(2A_3^*)^\perp \text{ in } E_7^*]$. The intersection matrix of $2A_3^*$ is

$$\frac{1}{2} \begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}.$$

So we are looking for three elements a_1, a_2 and a_3 of E_7^* with these intersection numbers. The minimal norm in E_7^* is $3/2$, and there are 56 minimal vectors of the form $\pm\frac{1}{4}(-3, -3, 1, 1, 1, 1, 1, 1)$ [7]. So without loss of generality we can take $a_1 = \frac{1}{4}(-3, -3, 1, 1, 1, 1, 1, 1)$. Now a_3 must also be a vector of minimal length, and it must have intersection number $\frac{1}{2}$ with a_1. There are two possibilities, up to permutations: $a_3 = \frac{1}{4}(-3, 1, -3, 1, 1, 1, 1, 1)$ or $a_3' = \frac{1}{4}(-1, -1, -1, -1, -1, -1, 3, 3)$. Now a_2 must have norm 2. We note that vectors in E_7 have integer norm, while vectors in the non-trivial coset $E_7^* - E_7$ have half-integer (but not integer) norm. Hence, a_2 must be in E_7; that is, it must be a minimal vector of E_7. It is a permutation of either

$(-1, 1, 0, 0, 0, 0, 0, 0)$ or $\frac{1}{2}(1, 1, 1, 1, -1, -1, -1, -1)$. Imposing the required intersection numbers with a_3 and a_1, we get two possibilities for a_2, regardless of whether we consider a_3 or a_3': $a_2 = (-1, 0, 0, 0, 0, 0, 0, 1)$ or $a_2' = \frac{1}{2}(-1, -1, -1, -1, 1, 1, 1, 1)$, up to permutations. Hence in total we have four choices of embeddings for $2A_3^*$ in E_7^*. Then, we compute the intersection numbers of our as with the generators of E_7^* given in [7, p. 125, Equation (115)]. It is simple linear algebra to find the orthogonal complement of our four embeddings in E_7^* in this basis, and it is straightforward to show that all four orthogonal complements have intersection matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -2 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix},$$

which again corresponds to D_4.

   Case 16. In this case we want $\left[\langle 1/3 \rangle^\perp \text{ in } L^*\right]$, where $L = \left[A_1^\perp \text{ in } A_5\right]$. The intersection matrix for L^* was computed in [13]; this is the matrix in the statement of the lemma. Now we want to embed a vector of length $\frac{1}{3}$ in L^*. Note that the quotient L^*/L is $(\mathbb{Z}_2)^2 \times \mathbb{Z}_3$. We write down representatives for the 12 non-trivial cosets of L in L^*. The square of each element of L^* is a rational number which modulo 2 depends only on its image in L^*/L. The possible values are $0, \frac{1}{3}, \frac{5}{6}, 1, \frac{3}{2}, \frac{4}{3}$ mod 2. It follows that the shortest possible length squared is $\frac{1}{3}$, and any minimal vector is, up to equivalence, of the form $\pm\frac{1}{6}(1, 1, 1, 1, -2, -2)$. Taking an explicit basis for L^* we compute the intersection number of the generators with our minimal vector, and find the orthogonal complement in L^* in this basis. We find that the intersection matrix is simply the 3×3 identity matrix, hence the orthogonal complement is the cubic lattice $\mathbb{Z}^3$, or in the earlier notation $\langle 1 \rangle^{\oplus 3}$. $\qquad\square$

Combining Lemmas 5.1 and 5.2 we have an explicit lattice description of $\ker(\Phi_m)$ for each case in table 4.

5.2. Cases where $d > 1$ is allowed

The next step consists in finding the cases where $d > 1$ is allowed. Given a section $\xi \in \ker(\Phi_m)$, we know that the automorphism $\tau_B = t_\xi \circ \alpha_B$ is of order n, $n = dm$ for an integer $d \geq 1$. However, the integer d depends on the section ξ. For some sections $\xi \in \ker(\Phi_m)$, d may equal 1, while for others it may be greater than 1. We need to find out when this happens.

Table 5: List of cases where $d > 1$ is allowed.

Case	d	$\ker(\Phi_m)$	$\ker(\Phi_m)_{d=1}$
8	2	$D_4^* \oplus \mathbb{Z}_2$	D_4^*
14	2	$A_1^{*\oplus 2} \oplus \mathbb{Z}_2$	$\langle 1 \rangle^{\oplus 2} \oplus \mathbb{Z}_2$
17	3	$A_2^* \oplus \mathbb{Z}_3$	A_2^*
22	2	$\langle 1/4 \rangle \oplus (\mathbb{Z}_2)^2$	$\langle 1 \rangle \oplus (\mathbb{Z}_2)^2$
26	2	$A_1^{*\oplus 2} \oplus \mathbb{Z}_2$	$\langle 1 \rangle^{\oplus 2} \oplus \mathbb{Z}_2$
27	2	$A_1^* \oplus \mathbb{Z}_4$	$A_1^* \oplus \mathbb{Z}_2$

We showed in Lemma 3.2 that d divides the order of $\mathrm{MW}_{\mathrm{tors}}$. Hence, if $\mathrm{MW}_{\mathrm{tors}} = 0$, then $d = 1$. We also showed in Lemma 3.7 that if f_0 is smooth, then $d = 1$. Thus, the only cases in table 4 that may have $d > 1$ are cases 8, 14, 17, 18, 21, 22, 26 and 27.

For the cases where $d > 1$ is possible, the group $\ker(\Phi_m)$ will split into the disjoint union of a subgroup $\ker(\Phi_m)_{d=1}$ of sections yielding $d = 1$, and its complement $\ker(\Phi_m)_{d>1}$ of sections with $d > 1$.

Lemma 5.3. *The only cases where $d > 1$ is allowed are listed in table 5. We write down the case number of table 4, the allowed integer d, the group $\ker(\Phi_m)$ and its subgroup $\ker(\Phi_m)_{d=1}$. For each case with $d = 2$, $\ker(\Phi_m)_{d=2}$ is the non-trivial coset of $\ker(\Phi_m)_{d=1}$ in $\ker(\Phi_m)$, while when $d = 3$, $\ker(\Phi_m)_{d=3}$ is the union of the two non-trivial cosets of $\ker(\Phi_m)_{d=1}$ in $\ker(\Phi_m)$.*

Proof. We will prove this lemma case by case. But let us first state the general philosophy of the proof. For the eight possible cases listed before the lemma, given a section $\xi \in \ker(\Phi_m)$, we want to know whether $\mathcal{P}_m(\xi)$ is the zero section or a non-zero torsion section. Our strategy goes as follows. We first find the action of α_B on the components of the singular fibers. Then we use Proposition 2.1 and the fact that the torsion sections are α_B-invariant to find the components that the torsion sections intersect. Finally, using this result and the fact that the height pairing of any section with a torsion section is zero, we are able to decide whether $d > 1$ is allowed, and find the decomposition of $\ker(\Phi_m)$.

Case 8. $m = 3$. $\ker(\Phi_3) = \mathrm{MW} = D_4^* \oplus \mathbb{Z}_2$. f_0 is of type III, and $T = A_1^{\oplus 4}$. Let R_i, $i = 0, 1$ denote the components of f_0, and $S_{j,k}$, with $j = 1, 2, 3$ and $k = 0, 1$ the components of the three other singular fibers with associated

root lattice A_1. The zero components are the neutral components as will always be the case in the following.

α_B acts on the components as follows. On f_0, it sends each component to itself: $\alpha_B : R_i \mapsto R_i$ for $i = 0, 1$. α_B permutes the other singular fibers, hence if we denote them appropriately we obtain that $\alpha_B : S_{j,k} \mapsto S_{j+1,k}$ for $j = 1, 2, 3$ and $k = 0, 1$, where $S_{4,k} \equiv S_{1,k}$ is understood.

Let η be the non-zero torsion section of order 2. We know from Proposition 2.1 that $\sum_{s_i} \text{contr}_{s_i}(\eta, \eta) = 2$, where s_i denotes the reducible fiber of B. For a singular fiber s with associated root lattice A_1, we have that $\text{contr}_s(\eta, \eta) = 0$ if η intersects the neutral component, and $\text{contr}_s(\eta, \eta) = 1/2$ otherwise. Hence, we directly obtain that η must intersect R_1 and $S_{j,1}$ for $j = 1, 2, 3$.

Take a section $\xi \in \ker(\Phi_3)$. Suppose that it intersects R_0. Then, from the action of α_B on f_0 we get that $\mathcal{P}_3(\xi)$ also intersects R_0, hence $\mathcal{P}_3(\xi) = \sigma$ and $d = 1$. Now suppose that ξ intersects R_1; we get that $\mathcal{P}_3(\xi)$ also intersects R_1, that is $\mathcal{P}_3(\xi) = \eta$ and $d = 2$. Therefore, any section $\xi \in \ker(\Phi_3)$ intersecting R_0 satisfies $\mathcal{P}_3(\xi) = \sigma$, while any section $\xi \in \ker(\Phi_3)$ intersecting R_1 satisfies $\mathcal{P}_3(\xi) = \eta$.

Now take any section $\xi \in \ker(\Phi_3)$. If it intersects R_0, then $\xi \boxplus \eta^{11}$ intersects R_1; and vice versa if ξ intersects R_1. Hence, we obtain that $\ker(\Phi_3)$ breaks into the disjoint union of its index 2 sublattice $\ker(\Phi_3)_{d=1} = D_4^*$, and the non-trivial coset $\ker(\Phi_3)_{d=2}$.

Case 14. $m = 2$. $\ker(\Phi_2) = \text{MW} = A_1^{*\oplus 2} \oplus \mathbb{Z}_2$. f_0 is of type I_0^*, and $T = D_4 \oplus A_1^{\oplus 2}$. We denote the five irreducible components of f_0 by $R_{0,0}$, $R_{1,0}$, $R_{0,1}$, $R_{1,1}$ and R'; $R_{0,0}$ is the neutral component, and R' is the component of I_0^* with multiplicity 2. We write $S_{j,k}$, with $j = 1, 2$ and $k = 0, 1$ for the components of the two other singular fibers with root lattices A_1.

α_B acts on the two A_1 fibers by $\alpha_B : S_{j,k} \mapsto S_{j+1,k}$, with $S_{3,k} \equiv S_{1,k}$. Using the fact that the fiber f_0 of the quotient surface $\widehat{\mathcal{B}}$ is of type III^*, we obtain the action of α_B on the components of f_0:

$$\alpha_B : \{R_{0,0}, R_{1,0}, R_{0,1}, R_{1,1}, R'\} \longmapsto \{R_{0,0}, R_{0,1}, R_{1,0}, R_{1,1}, R'\}.$$

The contributions $\text{contr}(\eta, \eta)$ associated to fibers with root lattices A_1 are either 0 or 1/2. For a fiber of type I_0^* (with root lattice D_4), the contributions can be either 0 for the neutral component or 1 for the four components with subscripts — recall that no sections intersect the component R' with multiplicity 2. Hence, from Proposition 2.1, and the fact that the non-zero

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[11]As usual, $\boxplus$ denotes addition in the Mordell–Weil group.

torsion section η of order 2 must be α_B-invariant, we get that η intersects the non-neutral components $R_{1,1}$, $S_{1,1}$ and $S_{2,1}$.

From the action of α_B, we get that any section $\xi \in \ker(\Phi_2)$ intersecting either $R_{0,0}$ or $R_{1,1}$ must satisfy $\mathcal{P}_2(\xi) = \sigma$ and $d = 1$, while any section $\xi \in \ker(\Phi_2)$ intersecting either $R_{0,1}$ or $R_{1,0}$ must satisfy $\mathcal{P}_2(\xi) = \eta$ and $d = 2$. However, we need more information to determine the intersection numbers of the sections $\xi \in \ker(\Phi_2)$.

Now let $\xi \in \ker(\Phi_2)$ be a section corresponding to a minimal point of the Mordell–Weil lattice $A_1^{*\oplus 2}$, which has length $1/2$. Since for this configuration of singular fibers $\sum_{s_i} \mathrm{contr}_{s_i}(x) \leq 2$, we know that ξ must be disjoint from the zero section. Hence

$$\langle \xi, \xi \rangle = \frac{1}{2} = 2 - \sum_{s_i} \mathrm{contr}_{s_i}(\xi).$$

Using this constraint, the fact that $\langle \xi, \eta \rangle = 0$ and the intersection numbers of η, we can list the possible intersection numbers of ξ. We obtain four possibilities: $(R_{1,0}, S_{1,1}, S_{2,0})$, $(R_{1,0}, S_{1,0}, S_{2,1})$, $(R_{0,1}, S_{1,0}, S_{2,1})$ and $(R_{0,1}, S_{1,1}, S_{2,0})$. In all of these cases, ξ intersects either $R_{0,1}$ or $R_{1,0}$. Hence, the sections ξ corresponding to the minimal points of the Mordell–Weil lattice must satisfy $\mathcal{P}_2(\xi) = \eta$.

Therefore, we get that $\ker(\Phi_2)$ splits into the disjoint union of the 45 degree rotated sublattice $\ker(\Phi_2)_{d=1} = \langle 1 \rangle^{\oplus 2} \oplus \mathbb{Z}_2$ and its non-trivial coset $\ker(\Phi_2)_{d=2}$.

Case 17. $m = 2$. $\ker(\Phi_2) = MW = A_2^* \oplus \mathbb{Z}_3$. f_0 is of type IV, and $T = A_2^{\oplus 3}$. As usual, let R_i, $i = 0, 1, 2$ be the components of f_0, and $S_{j,k}$, $j = 1, 2$ and $k = 0, 1, 2$ be the components of the two other singular fibers.

The contributions associated to these three reducible fibers with root lattices A_2 are either 0 for the neutral component, or $2/3$ for the two other irreducible components.

α_B sends each component of f_0 to itself, that is, $\alpha_B : R_i \mapsto R_i$ for $i = 0, 1, 2$. For the two other fibers, as usual $\alpha_B : S_{j,k} \mapsto S_{j+1,k}$ for $j = 1, 2$ and $k = 0, 1, 2$, with $S_{3,k} \equiv S_{1,k}$.

There are two non-zero torsion sections of order 3; denote them by η and $\eta \boxplus \eta$. Using Proposition 2.1, we find that η intersects R_1, $S_{1,1}$ and $S_{2,1}$, and accordingly $\eta \boxplus \eta$ intersects R_2, $S_{1,2}$ and $S_{2,2}$.

Since α_B sends each component of f_0 to itself, we get that any section $\xi \in \ker(\Phi_2)$ intersecting the neutral component R_0 must satisfy $\mathcal{P}_2(\xi) = \sigma$, any section $\xi \in \ker(\Phi_2)$ intersecting R_1 must satisfy $\mathcal{P}_2(\xi) = \eta \boxplus \eta$, while any section $\xi \in \ker(\Phi_2)$ intersecting R_2 must satisfy $\mathcal{P}_2(\xi) = \eta$. In the two last cases, we obtain that $d = 3$, since η and $\eta \boxplus \eta$ are of order 3.

Now take any section $\xi \in \ker(\Phi_2)$. Suppose that ξ intersects R_l for some l. Then, from the intersection form of the torsion sections, we get that $\xi \boxplus \eta$ will intersect R_{l+1}, and $\xi \boxplus \eta \boxplus \eta$ will intersect R_{l+2}, where $R_3 \equiv R_0$.

Hence, we get that $\ker(\Phi_2)$ breaks into the disjoint union of the index 3 sublattice $\ker(\Phi_2)_{d=1} = A_2^*$, and $\ker(\Phi_2)_{d=3}$ which is the union of its two non-trivial cosets.

Case 18. $m = 2$. $\ker(\Phi_2) = \mathrm{MW} = A_1^* \oplus \mathbb{Z}_2$. f_0 is of type I_8, and $T = A_7$. Let R_i, $i = 0, \ldots, 7$ denote the components of f_0.

The contributions for f_0 are 0 for the neutral component, $7/8$ for R_1 and R_7, $3/2$ for R_2 and R_6, $15/8$ for R_3 and R_5 and 2 for R_4. Hence, from Proposition 2.1 the non-zero torsion section η of order 2 must intersect R_4.

The action of α_B on the components of f_0 is given by $\alpha_B : R_i \mapsto R_{8-i}$, for $i = 0, \ldots, 7$, with $R_8 \equiv R_0$. Hence, we directly get that for any section $\xi \ker(\Phi_2)$, $\mathcal{P}_2(\xi)$ must intersect R_0, that is, $\mathcal{P}_2(\xi) = \sigma$ and $d = 1$. Therefore, it turns out that in this case $d > 1$ is not possible.

Case 21. $m = 2$. $\ker(\Phi_2) = \langle 1 \rangle^{\oplus 2} \oplus \mathbb{Z}_2$. f_0 is of type I_4, and $T = A_3 \oplus A_1^{\oplus 2}$. Denote by R_i, $i = 0, 1, 2, 3$ the components of f_0, and by $S_{j,k}$, $j = 1, 2$ and $k = 0, 1$ the components of the two other singular fibers with root lattices A_1.

The contributions for f_0 are either 0 for the neutral component, $3/4$ for R_1 and R_3, and 1 for R_2. For the two other reducible fibers, it is either 0 for the neutral components, or $1/2$ otherwise.

From Proposition 2.1, the non-zero torsion section η of order 2 must intersect R_2, $S_{1,1}$ and $S_{2,1}$. Since the action of α_B on f_0 is given by $\alpha_B : R_i \mapsto R_{4-i}$ for $i = 0, 1, 2, 3$, with $R_4 \equiv R_0$, we get that for any section $\xi \in \ker(\Phi_2)$, $\mathcal{P}_2(\xi)$ must intersect R_0, that is, $\mathcal{P}_2(\xi) = \sigma$. Therefore, in this case as well it turns out that $d > 1$ is not possible.

Case 22. $m = 2$. $\ker(\Phi_2) = \mathrm{MW} = \langle 1/4 \rangle \oplus (\mathbb{Z}_2)^2$. f_0 is of type I_4, and $T = A_3 \oplus A_1^{\oplus 4}$. Denote by R_i, $i = 0, 1, 2, 3$ the components of f_0, and by $S_{j,k}$, $j = 1, 2, 3, 4$ and $k = 0, 1$ the components of the four other singular fibers with root lattices A_1.

The contributions for f_0 are either 0 for the neutral component, $3/4$ for R_1 and R_3 and 1 for R_2. For the four other reducible fibers, it is either 0 for the neutral components, or $1/2$ otherwise.

The action of α_B on f_0 is given by $\alpha_B : R_i \mapsto R_{4-i}$ for $i = 0, 1, 2, 3$, with $R_4 \equiv R_0$. On the four other fibers, it is given by

$$\alpha_B : \{S_{1,k}, S_{2,k}, S_{3,k}, S_{4,k}\} \longmapsto \{S_{2,k}, S_{1,k}, S_{4,k}, S_{3,k}\},$$

for $k = 0, 1$.

There are three non-zero torsion sections of order 2, which we denote by η_1, η_2 and $\eta_1 \boxplus \eta_2$. From Proposition 2.1 and the fact that they are all

α_B-invariant, we get that η_1 intersects the non-neutral components R_2, $S_{1,1}$ and $S_{2,1}$, η_2 intersects R_2, $S_{3,1}$ and $S_{4,1}$ and $\eta_1 \boxplus \eta_2$ intersects $S_{1,1}$, $S_{2,1}$, $S_{3,1}$ and $S_{4,1}$.

Take any section $\xi \in \ker(\Phi_2)$. From the action of α_B on f_0, we know that $\mathcal{P}_2(\xi)$ must intersect R_0. Hence, the only possibilities are that $\mathcal{P}_2(\xi) = \sigma$ or $\mathcal{P}_2(\xi) = \eta_1 \boxplus \eta_2$. To distinguish between these two cases we must look at the intersection numbers of the sections ξ with the other singular fibers.

Let $\xi \in \ker(\Phi_2)$ correspond to a minimal point in the Mordell–Weil lattice $\langle 1/4 \rangle$, which has length $1/4$. Since for this configuration of singular fibers $\sum_{s_i} \mathrm{contr}_{s_i}(\xi) \le 3$, we know that ξ must be disjoint from σ. Hence

$$\langle \xi, \xi \rangle = \frac{1}{4} = 2 - \sum_{s_i} \mathrm{contr}_{s_i}(\xi).$$

We can use this constraint, the fact that $\langle \xi, \eta_1 \rangle = \langle \xi, \eta_2 \rangle = \langle \xi, \eta_1 \boxplus \eta_2 \rangle = 0$ and the intersection numbers of the torsion sections before to list the possible intersection numbers of ξ. We get the eight different combinations (listing only the non-neutral components that ξ intersects) ($R_{1 \text{ or } 3}$, $S_{1,1 \text{ or } 2,1}$, $S_{3,1 \text{ or } 4,1}$). Using the action of α_B before, for each of these cases it is easy to see that the section $\mathcal{P}_2(\xi)$ will intersect the components $R_0, S_{1,1}, S_{2,1}, S_{3,1}$ and $S_{4,1}$. That is, for any section ξ corresponding to a minimal point of the Mordell–Weil lattice, we have that $\mathcal{P}_2(\xi) = \eta_1 \boxplus \eta_2$ and $d = 2$.

Therefore, we obtain that $\ker(\Phi_2)$ splits into the disjoint union of the sublattice $\ker(\Phi_2)_{d=1} = \langle 1 \rangle \oplus (\mathbb{Z}_2)^2$ and its non-trivial coset $\ker(\Phi_2)_{d=2}$.
Case 26. $m = 2$. $\ker(\Phi_2) = A_1^{*\oplus 2} \oplus \mathbb{Z}_2$. f_0 is of type I_2, and $T = A_1^{\oplus 5}$. Let R_i, $i = 0, 1$ be the components of f_0, and $S_{j,k}$, $j = 1, 2, 3, 4$, $k = 0, 1$ be the components of the four other singular fibers with root lattices A_1.

The contributions for all the singular fibers are either 0 for the neutral component, or $1/2$ otherwise. α_B sends each component of f_0 to itself, and permutes two by two the other components. That is, if we name the components appropriately,

$$\alpha_B : \ \{R_i, S_{1,k}, S_{2,k}, S_{3,k}, S_{4,k}\} \longmapsto \{R_i, S_{2,k}, S_{1,k}, S_{4,k}, S_{3,k}\},$$

for $i = 0, 1$ and $k = 0, 1$.

There is one non-zero torsion section of order 2, which we denote by η. From Proposition 2.1 and the fact that η is α_B-invariant, we get that it must intersect the components $R_0, S_{1,1}, S_{2,1}, S_{3,1}$ and $S_{4,1}$.

Take any section $\xi \in \ker(\Phi_2)$. From the action of α_B on f_0 we know that $\mathcal{P}_2(\xi)$ must intersect R_0. Hence, both possibilities, $\mathcal{P}_2(\xi) = \sigma$ or $\mathcal{P}_2(\xi) = \eta$, may occur.

Suppose now that $\xi \in \mathrm{MW}$ corresponds to a minimal point in the Mordell–Weil lattice $\mathrm{MW}_{\mathrm{lat}} = A_1^{*\oplus 3}$, hence it has length $1/2$. Since for this configuration $\sum_{s_i} \mathrm{contr}_{s_i}(\xi, \xi) \leq 5/2$, it follows that ξ is disjoint from the zero section. Hence we get

$$\langle \xi, \xi \rangle = \frac{1}{2} = 2 - \sum_{s_i} \mathrm{contr}_{s_i}(\xi, \xi).$$

Using also the fact that $\langle \xi, \eta \rangle = 0$ and the intersection numbers of η, we can list the possible intersection numbers of the minimal ξ. We obtain six possibilities: listing only the non-neutral components that ξ intersects, $(R_1, S_{1,1}, S_{2,1}), (R_1, S_{3,1}, S_{4,1}), (R_1, S_{1,1}, S_{3,1}), (R_1, S_{1,1}, S_{4,1}), (R_1, S_{2,1}, S_{3,1})$ and $(R_1, S_{2,1}, S_{4,1})$.

From the action of α_B, we see that the two first possibilities correspond to the two generators of the α_B-invariant sublattice A_1^* of $\mathrm{MW}_{\mathrm{lat}}$. Hence, the orthogonal complement $A_1^{*\oplus 2}$ is generated by the minimal sections with the four last possible intersection numbers. For each of these case, $\mathcal{P}_2(\xi)$ intersects $(R_0, S_{1,1}, S_{2,1}, S_{3,1}, S_{4,1})$; hence $\mathcal{P}_2(\xi) = \eta$. That is, for any section corresponding to a minimal point of $\ker(\Phi_2)$, we have that $\mathcal{P}_2(\xi) = \eta$ and $d = 2$.

Therefore, we obtain that $\ker(\Phi_2)$ splits into the disjoint union of the 45 degree rotated sublattice $\ker(\Phi_2)_{d=1} = \langle 1 \rangle^{\oplus 2} \oplus \mathbb{Z}_2$ and its non-trivial coset $\ker(\Phi_2)_{d=2}$.

Case 27. $m = 2$. $\ker(\Phi_2) = \mathrm{MW} = A_1^* \oplus \mathbb{Z}_4$. f_0 is of type I_2, and $T = A_3^{\oplus 2} \oplus A_1$. Let R_i, $i = 0, 1$ be the components of f_0, and $S_{j,k}$, $j = 1, 2$ and $k = 0, 1, 2, 3$ be the components of the two other singular fibers with root lattices A_3.

The contributions for f_0 are either 0 for the neutral component, or $1/2$ otherwise. For the two other fibers, the contributions are either 0 for the neutral component, $3/4$ for the components 1 and 3 and 1 for the middle component 2.

There are three non-zero torsion sections of order 4, which we denote by η, $\eta \boxplus \eta$ and $\eta \boxplus \eta \boxplus \eta$. From Proposition 2.1, we get that η intersects R_1, $S_{1,1}$ and $S_{2,1}$, $\eta \boxplus \eta$ intersects R_0, $S_{1,2}$ and $S_{2,2}$ and $\eta \boxplus \eta \boxplus \eta$ intersects R_1, $S_{1,3}$ and $S_{2,3}$.

α_B sends each component of f_0 to itself, and permutes the other components. That is,

$$\alpha_B : \{R_i, S_{1,k}, S_{2,k}\} \longmapsto \{R_i, S_{2,k}, S_{1,k}\},$$

for $i = 0, 1$ and $k = 0, 1, 2, 3$. Hence, for any section $\xi \in \ker(\Phi_2)$, we know that $\mathcal{P}_2(\xi)$ must intersect the neutral component R_0. Thus, either $\mathcal{P}_2(\xi) = \sigma$, or $\mathcal{P}_2(\xi) = \eta \boxplus \eta$. In the latter case, $d = 2$, since $\eta \boxplus \eta$ generates the subgroup $\mathbb{Z}_2 \subset \mathbb{Z}_4$.

Now, take a section $\xi \in \ker(\Phi_2)$. It is easy to show that if ξ satisfies $\mathcal{P}_2(\xi) = \sigma$, then $\xi \boxplus \eta$ satisfies $\mathcal{P}_2(\xi \boxplus \eta) = \eta \boxplus \eta$, $\xi \boxplus \eta \boxplus \eta$ satisfies $\mathcal{P}_2(\xi \boxplus \eta \boxplus \eta) = \sigma$ and $\mathcal{P}_2(\xi \boxplus \eta \boxplus \eta \boxplus \eta) = \eta \boxplus \eta$. Hence, we get that $\mathrm{AS} = A_1^* \oplus \mathbb{Z}_4$ splits into the disjoint union of the index 2 sublattice $\ker(\Phi_2)_{d=1} = A_1^* \oplus \mathbb{Z}_2$, and its non-trivial coset $\ker(\Phi_2)_{d=2}$. $\qquad\square$

5.3. Free action on f_∞

We have now computed explicitly $\ker(\Phi_m)$ for each case in table 4, and found out when $d > 1$ is possible. However, this is not the end of the story; we also want $\langle \tau_B \rangle = \langle t_\xi \circ \alpha_B \rangle$ to act freely on f_∞.

Proposition 5.4. *Let (B, α_B) be a suitable σ-pair with a smooth f_∞. Take a section $\xi \in \ker(\Phi_m)$, and define the automorphism $\tau_B = t_\xi \circ \alpha_B$, of order n, with $n = dm$. Then the following statements are equivalent:*

(1) $\langle \tau_B \rangle$ *acts freely on f_∞;*

(2) $\mathcal{P}_i(\xi)\big|_\infty \neq 0$, *for $i = 1, \ldots, n - 1$;*

(3) $\xi\big|_\infty$ *is a torsion point of the smooth elliptic curve of order precisely n.*

Proof. We know that f_∞ is smooth, and that α_B fixes f_∞ pointwise. Since translation by a non-zero section acts freely on a smooth elliptic curve, $\langle \tau_B \rangle$ will be free on f_∞ if and only if $\mathcal{P}_i(\xi)$ is non-zero on f_∞ for $i = 1, \ldots, n - 1$. Now since α_B fixes f_∞ pointwise, this can happen if and only if ξ intersects f_∞ at a torsion point of order precisely n. $\qquad\square$

5.4. Summary

Hence, taking into account Lemma 5.2 and Proposition 5.4, we now know how to construct the set of allowed sections, which we call AS, for each case in table 4. Let us summarize how we proceed.

For each suitable σ-pair (B, α_B) listed in table 4, we first extract $\ker(\Phi_m)$ using Lemmas 5.1 and 5.2. Then, for the cases listed in table 5 where $d > 1$ is possible, we split $\ker(\Phi_m)$ into the disjoint union of $\ker(\Phi_m)_{d=1}$ and

$\ker(\Phi_m)_{d>1}$ according to the result of Lemma 5.3. This gives us the set of sections which yield automorphisms τ_B of order n for each case.

Then we must remove from these sets the sections which do not intersect f_∞ at a torsion point of order precisely n, according to Proposition 5.4. This can be done as follows. Consider the map $(1 - \alpha_B) : \mathrm{MW} \to \mathrm{MW}$. Its image $\mathrm{Im}(1 - \alpha_B)$ is a finite index subgroup of $\ker(\Phi_m)$. Since α_B fixes f_∞ pointwise, any section in $\mathrm{Im}(1 - \alpha_B)$ will necessarily intersect f_∞ at zero.[12]

As a consequence, all sections in a given coset of $\mathrm{Im}(1 - \alpha_B)$ in $\ker(\Phi_m)$ will intersect f_∞ at the same torsion point (or 0). This tells us that to obtain the set of allowed sections AS from $\ker(\Phi_m)$ for a given n (hence a given d), we first need to compute the subgroup $\mathrm{Im}(1 - \alpha_B) \subset \ker(\Phi_m)$, and then keep only the cosets containing sections intersecting f_∞ at a torsion point of order precisely n. We thus obtain a complete list of pairs (B, τ_B), with smooth f_∞ and τ_B acting freely on f_∞.

Remark 5.5. Note that one can show that all sections in a given coset of $\mathrm{Im}(1 - \alpha_B) \subset \ker(\Phi_m)$ lead to isomorphic quotients $B/\langle \tau_B \rangle$, hence the number of non-isomorphic quotients, for a given α_B, is at most the cardinality of the quotient group $\ker(\Phi_m)/\mathrm{Im}(1 - \alpha_B)$. To see that, consider two automorphisms $\tau_B = t_\xi \circ \alpha_B$ and $\tau_B' = t_{\xi'} \circ \alpha_B$, with $\xi, \xi' \in \ker(\Phi_m)$. A sufficient condition for the quotients to be isomorphic is that there is an isomorphism $a : B \to B$ intertwining the τ's: $a \circ \tau_B = \tau_B' \circ a$. Take $a = t_\zeta$ for some section ζ. Explicitly, the condition then says:

$$(5.1) \qquad t_\zeta \circ t_\xi \circ \alpha_B = t_{\xi'} \circ \alpha_B \circ t_\zeta.$$

Since these two automorphisms have the same linearization, they agree if the two of them take the zero section to the same section. We thus get the condition

$$(5.2) \qquad \xi' - \xi = (1 - \alpha_B)\zeta,$$

which implies that all sections in a given coset of $\mathrm{Im}(1 - \alpha_B) \subset \ker(\Phi_m)$ produce isomorphic quotients.

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[12]Note also that when $d > 1$ is allowed, $\mathrm{Im}(1 - \alpha_B)$ is always a subgroup of $\ker(\Phi_m)_{d=1}$, since for any $\zeta \in \mathrm{MW}$, $\mathcal{P}_m((1 - \alpha_B)\zeta) = \sigma$.

6. The list(s)

Combining the results of the previous sections, we now have two lists:

(1) A list of rational elliptic surfaces B with automorphism groups generated by translation by torsion sections — this was extracted directly from the torsion groups of the rational elliptic surfaces in Persson's list [12, 14];

(2) A list of pairs (B, τ_B), where τ_B is an automorphism of the second kind, f_∞ is smooth and τ_B acts freely on f_∞. In other words, we have a list of rational elliptic surfaces B, with smooth f_∞, and cyclic automorphism groups generated by automorphisms of the second kind acting freely on f_∞.

By combining appropriately these two lists, we are now in a position to produce a complete list of rational elliptic surfaces with finite automorphism groups (not necessarily cyclic) that can be lifted to free automorphism groups on smooth fiber products. The list naturally splits into two sublists, depending on whether the automorphism group acts trivially on the $\mathbb{P}^1$ base or not.

6.1. Trivial action on $\mathbb{P}^1$

In this case we obtain the following result.

Proposition 6.1. *The list of rational elliptic surfaces with finite automorphism groups acting trivially on $\mathbb{P}^1$ is given in table 6. In this table we write down the automorphism group G_B (which consists in translation by torsion sections), the dimension $\dim$ of the moduli space, and the configuration of singular fibers at a generic point in the moduli space. We also write down the root lattice T associated to the singular fibers and the Mordell–Weil group MW, for the generic configuration of singular fibers.*

It is understood that for cases 5 to 8 there are various specializations in the moduli space where some of the singular fibers collide to produce different configurations of singular fibers, and that these specializations do not necessarily have the same root lattice T and Mordell–Weil group MW as the generic configuration. The specializations that have the same torsion group MW_{tors} as the generic configuration are shown in table 7. Additional degenerations are possible in which the torsion group MW_{tors} becomes larger — for example, both the $(\mathbb{Z}_2)^2$ locus and the $\mathbb{Z}_4$ locus can degenerate to the

Table 6: List of rational elliptic surfaces with finite automorphism groups acting trivially on $\mathbb{P}^1$.

#	G_B	dim	Singular fibers	T	MW
1	$\mathbb{Z}_3 \times \mathbb{Z}_3$	0	$\{4I_3\}$	$A_2^{\oplus 4}$	$(\mathbb{Z}_3)^2$
2	$\mathbb{Z}_4 \times \mathbb{Z}_2$	0	$\{2I_4, 2I_2\}$	$(A_3 \oplus A_1)^{\oplus 2}$	$\mathbb{Z}_4 \times \mathbb{Z}_2$
3	$\mathbb{Z}_6$	0	$\{I_6, I_3, I_2, I_1\}$	$A_5 \oplus A_2 \oplus A_1$	$\mathbb{Z}_6$
4	$\mathbb{Z}_5$	0	$\{2I_5, 2I_1\}$	$A_4^{\oplus 2}$	$\mathbb{Z}_5$
5	$\mathbb{Z}_4$	1	$\{2I_4, I_2, 2I_1\}$	$A_3^{\oplus 2} \oplus A_1$	$A_1^* \oplus \mathbb{Z}_4$
6	$\mathbb{Z}_2 \times \mathbb{Z}_2$	2	$\{6I_2\}$	$A_1^{\oplus 6}$	$A_1^{*\oplus 2} \oplus (\mathbb{Z}_2)^2$
7	$\mathbb{Z}_3$	2	$\{3I_3, 3I_1\}$	$A_2^{\oplus 3}$	$A_2^* \oplus \mathbb{Z}_3$
8	$\mathbb{Z}_2$	4	$\{4I_2, 4I_1\}$	$A_1^{\oplus 4}$	$D_4^* \oplus \mathbb{Z}_2$

Table 7: Specializations for table 6.

#	G_B	Specializations
5	$\mathbb{Z}_4$	$\{2I_4, I_2, 2I_1\}$, $\{I_8, I_2, 2I_1\}$, $\{I_1^*, I_4, I_1\}$
6	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\{6I_2\}$, $\{I_4, 4I_2\}$, $\{I_0^*, 3I_2\}$, $\{I_2^*, 2I_2\}$, $\{2I_0^*\}$
7	$\mathbb{Z}_3$	$\{3I_3, 3I_1\}$, $\{3I_3, I_2, I_1\}$, $\{IV, 2I_3, 2I_1\}$, $\{I_6, I_3, 3I_1\}$, $\{IV, 2I_3, I_2\}$, $\{I_6, IV, 2I_1\}$, $\{I_9, 3I_1\}$, $\{3IV\}$, $\{IV^*, I_3, I_1\}$, $\{IV^*, IV\}$
8	$\mathbb{Z}_4$	$\{4I_2, 4I_1\}$, $\{5I_2, 2I_1\}$, $\{III, 3I_2, 3I_1\}$, $\{I_4, 2I_2, 4I_1\}$, $\{III, 4I_2, I_1\}$, $\{2III, 2I_2, 2I_1\}$, $\{I_3, 4I_2, I_1\}$, $\{I_4, 3I_2, 2I_1\}$, $\{I_4, III, I_2, 3I_1\}$, $\{2I_4, 4I_1\}$, $\{I_6, I_2, 4I_1\}$, $\{2III, 3I_2\}$, $\{I_3, III, 3I_2\}$, $\{I_4, III, 2I_2, I_1\}$, $\{I_4, 2III, 2I_1\}$, $\{I_4, I_3, 2I_2, I_1\}$, $\{I_0^*, 2I_2, 2I_1\}$, $\{I_6, 2I_2, 2I_1\}$, $\{I_6, III, 3I_1\}$, $\{I_8, 4I_1\}$, $\{4III\}$, $\{I_4, 2III, I_2\}$, $\{I_4, I_3, III, I_2\}$, $\{I_0^*, III, I_2, I_1\}$, $\{I_0^*, I_4, 2I_1\}$, $\{I_6, III, I_2, I_1\}$, $\{I_1^*, 2I_2, I_1\}$, $\{I_2^*, I_2, 2I_1\}$, $\{I_0^*, 2III\}$, $\{I_1^*, III, I_2\}$, $\{I_2^*, III, I_1\}$, $\{III^*, I_2, I_1\}$, $\{I_4^*, 2I_1\}$, $\{III^*, III\}$

$\mathbb{Z}_4 \times \mathbb{Z}_2$ locus. We did not list such degenerations under the subgroup, as they can readily be found listed under the larger group.

Proof. As mentioned earlier, this list can be obtained directly from Persson's list of rational elliptic surfaces [12, 14] by keeping all the surfaces with non-trivial torsion groups MW_{tors}. (Alternatively, one could work with the

main table in [13]; but note that the torsion group in their case 70 should be $\mathbb{Z}_4$, rather than $(\mathbb{Z}_2)^2$ as they claim. This is responsible for the first specialization, $\{I_8, I_2, 2I_1\}$, of the $\mathbb{Z}_4$ surface, as listed in the first line of our table 7.)

What remains is the calculation of the dimension of the moduli space, that is, the number of complex deformations. For this we present first a sketch, for which we are grateful to the referee, and then a detailed and explicit calculation.

It is well–known that semi-stable rational elliptic surfaces with fixed configuration and Mordell–Weil group of rank r have moduli space of dimension r. In cases 5 to 7, the configuration uniquely determines the torsion group $\mathrm{MW}_{\mathrm{tors}}$ (see [13] — this follows from the height pairing in [16]); so only case 8 remains where both $\mathbb{Z}_2$-torsion and torsion free are possible.

To deal with case 8, start with case 5 in table 4; this surface has configuration of singular fibers $\{III^*, I_2, I_1\}$, a two-torsion section and no moduli (it is an extremal rational elliptic surface). We want to get rid of the III^* fiber. According to table 1, this requires a base change of degree 4 with ramification of order 4 at the III^* fiber, to obtain the desired configuration $\{4I_2, 4I_1\}$. However, in this case generally we do not know whether the pullbacked surface has $\mathbb{Z}_2$-torsion or not. But here the situation is special; if the ramification index at the III^* fiber is $(2, 2)$, then we obtain two I_0^* fibers in the pullback, which can then be eliminated by a quadratic twist (which transforms "star"-fibers into "unstarred"-fibers, see for instance [12]). We thus obtain the desired configuration of singular fibers with $\mathbb{Z}_2$-torsion in MW, since quadratic twists preserve two-torsion sections. We can write down this base change explicitly as follows. We normalize the fiber III^* to sit at ∞. We can choose three pre-images for the base change — let us choose 0 and ∞ as pre-images for ∞, and 1 as one pre-image for 0. Such base changes take the form

$$t \longmapsto \frac{(at^3 + bt^2 + ct + d)(t - 1)}{t^2}.$$

We deduce that the moduli space is at least 4-dimensional. On the other hand, by the previous argument, the dimension of the moduli space can at most equal the rank of MW, which is 4 here. Hence the moduli space in this case is 4-dimensional as claimed. (This last argument actually works for all cases 5 to 8, by considering appropriate base changes.)

These dimensions can also be obtained by explicit calculations. We start with an argument which occurs in the proof of Theorem 7.1 next. This says that the number of deformations of a 3-fold X which is a quotient of a

Schoen's 3-fold is given by

$$h = h_B + h_{B'} + e,$$

where h_B and $h_{B'}$ are the number of deformations of the $G = G_B = G_{B'}$ action on the two rational elliptic surfaces B and B', and $e = 3$ when $m = 1$, while $e = 1$ otherwise. Now, since

$$H^2(X, \mathbb{Q}) = \frac{H^2(B, \mathbb{Q})^G \oplus H^2(B', \mathbb{Q})^G}{H^2(\mathbb{P}^1, \mathbb{Q})},$$

where the G superscript denotes the G-invariant part, we obtain that the number of G deformations of B is given by

$$h_B = \dim H^2(B, \mathbb{Q})^G - \frac{e+1}{2}.$$

To compute the dimension of the G-invariant part of $H^2(B, \mathbb{Q})$, we use the Lefschetz fixed-point theorem. We obtain the general formula:

$$\dim H^2(B, \mathbb{Q})^G = \frac{12 + \sum_{i=1}^{n-1} f_i}{n} - 2,$$

where n is the order of G and f_i, $i = 1, \ldots, n-1$ are the numbers of fixed points of the non-zero elements of G. Thus, by counting the number of fixed points of the action on B of each element in G, we obtain $\dim H^2(B, \mathbb{Q})^G$, hence the number of deformations h_B. This gives the numbers dim in the table.

Finally, we need to check that all the cases listed in table 7 are indeed specializations of the generic configuration. For most cases this is straightforward. For the surfaces involving star fibers, a simple way to do so is to count the number of fixed points as before for these subfamilies to obtain their number of deformations h_B. Recall that whenever the j-invariant of an elliptic fibration is non-constant, the complex structure is determined locally by the positions in the base of the singular fibers. So if h_B is bigger than the number of the singular fibers minus $3 = \dim PGL(2)$, this means that the surface is in a subspace of a bigger moduli space, hence it is a specialization of a more generic configuration. Using this simple argument we can prove that all the configurations of table 7 are specializations of the generic configuration, for each G_B. Let us do a few cases explicitly.

For case #5 with $G_B = \mathbb{Z}_4$, we want to show that the configuration $\{I_1^*, I_4, I_1\}$ is a degeneration of the generic configuration $\{2I_4, I_2, 2I_1\}$. The

order 4 elements of $\mathbb{Z}_4$ have one fixed point on the I_1^* fiber and one fixed point on the I_1 fiber, while the order 2 element has three fixed points on I_1^* and one fixed point on I_1. Hence we get that $h_B = 1$; but this is an extremal surface, which, according to [11], is unique. Hence it must be a specialization of the generic configuration; that is, we showed that the fibers I_4, I_2 and I_1 collide to give an I_1^* fiber.

For case #6 with $G_B = \mathbb{Z}_2 \times \mathbb{Z}_2$, we need to show that $3I_2$ collide to give a I_0^*. For the configuration $\{I_0^*, 3I_2\}$, each element of G_B has two fixed points on I_0^* and two fixed points on one of the I_2 fibers. Hence $h_B = 2$, which means that it is indeed a degeneration of the generic configuration $\{6I_2\}$. For this case we also need to show that $4I_2$s collide to give an I_2^*. For the configuration $\{I_2^*, 2I_2\}$, two elements have two fixed points on I_2^* and two fixed points on one of the two I_2 fibers, and one element has four fixed points on I_2^* and acts freely on the I_2s. Hence $h_B = 2$ again, and the surface is a specialization of the generic configuration.

For case #7 with $G_B = \mathbb{Z}_3$, we need to check that $2I_3$ and $2I_1$ collide to give a IV^* fiber. For the configuration $\{IV^*, I_3, I_1\}$, each non-zero element of G_B has two fixed points on IV^* and one fixed point on I_1. Hence $h_B = 2$, that is, it is a specialization of the generic configuration with $\{3I_3, 3I_1\}$.

Finally, for case #8 with $G_B = \mathbb{Z}_2$ it remains to be checked that $3I_2$ and $3I_1$ collide to give a III^* fiber, and that $4I_2$ and $2I_1$ collide to give a I_4^* fiber. For the configuration $\{III^*, I_2, I_1\}$, the non-zero element has three fixed points on III^* and one fixed point on I_1, hence $h_B = 4$ and it is a specialization of the generic configuration. For the configuarion $\{I_4^*, 2I_1\}$, the non-zero element has two fixed points on I_4^* and one fixed point on each I_1s, hence $h_B = 4$ and it is also a specialization of the generic configuration. $\qquad\square$

6.2. Non-trivial action on $\mathbb{P}^1$

In this case a little bit more work is needed. We obtained in the previous section a list of suitable pairs (B, τ_B). However, some of these surfaces also have non-trivial torsion group $\mathrm{MW}_{\mathrm{tors}}$. In this case, we can form "mixed" finite automorphism groups, with one factor generated by an automorphism of the second kind and the other factors generated by translation by torsion sections. Taking this into account, we obtain the following lemma.

Proposition 6.2. *The list of rational elliptic surfaces with f_∞ smooth and finite automorphism groups G_B acting non-trivially on $\mathbb{P}^1$ and freely on f_∞*

is given in tables 8 and 9. For each entry in the tables, we write down, following the notation of the previous sections:

- *the order m of the linearization α_B;*

- *the order d of the torsion section $\mathcal{P}_m(\xi)$;*

- *the dimension dim of the moduli space of complex deformations preserving the automorphism group;*

- *the generic configuration of singular fibers in the moduli space (the fiber in bold typeface corresponds to the special fiber f_0; hence if there is no bold singular fiber it means that f_0 is smooth);*

- *the root lattice T associated to the singular fibers of B;*

- *the Mordell–Weil group MW of B;*

- *the α_B-invariant subgroup MW^{α_B} of MW;*

- *the subgroup of sections $\ker(\Phi_m) \subset \mathrm{MW}$;*

- *when $d > 1$ is possible, the subgroup $\ker(\Phi_m)_{d=1} \subset \ker(\Phi_m)$ — the set $\ker(\Phi_m)_{d>1}$ is then the union of the non-trivial cosets of $\ker(\Phi_m)_{d=1} \subset \ker(\Phi_m)$;*

- *the corresponding case number num of table 4.*

For each entry in the table it is understood that there may be various specializations in the moduli space where some of the singular fibers collide to produce different configurations of singular fibers, with the same associated root lattice T and Mordell–Weil group MW. These specializations are shown in table 10; we use semicolons to segregate strata of a given dimension, which are then separated by commas. In table 10 we only list the cases where there are specializations.

Remark 6.3. Recall that to get the set of allowed sections AS in each case, we must keep only the cosets of $\mathrm{Im}(1 - \alpha_B) \subset \ker(\Phi_m)$ which contain sections intersecting f_∞ at a torsion point of order precisely n, with $n = dm$. One can show that AS is non-empty for all lines in tables 8 and 9; we sketch a case-by-case proof in the Appendix.

Remark 6.4. Note that, as explained in Section 4.3.1, there are codimension one subspaces in the moduli space where some of the singular fibers collide to give a singular fiber at ∞. However the full automorphism group

Table 8: List of rational elliptic surfaces with finite automorphism groups satisfying the criteria of Proposition 6.2.

#	m	d	dim	Sing. fibers	T	$\mathbb{Z}_3 \times \mathbb{Z}_3$ MW	MW^{α_B}	$\ker(\Phi_m)$	$\ker(\Phi_m)_{d=1}$	num
9	3	1	1	$\{3I_3, 3I_1\}$	$A_2^{\oplus 3}$	$A_2^* \oplus \mathbb{Z}_3$ $\mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_3$	MW	–	11
10	4	1	1	$\{4I_2, 4I_1\}$	$A_1^{\oplus 4}$	$D_4^* \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	MW	–	5
11	2	2/1	1	$\{\mathbf{I_2}, 2I_4, 2I_1\}$	$A_3^{\oplus 2} \oplus A_1$	$A_1^* \oplus \mathbb{Z}_4$	$\mathbb{Z}_4$	MW	$A_1^* \oplus \mathbb{Z}_2$	27
12	2	2	1	$\{\mathbf{I_4}, 4I_2\}$	$A_3 \oplus A_1^{\oplus 4}$	$\langle 1/4 \rangle \oplus (\mathbb{Z}_2)^2$ $\mathbb{Z}_6$	$(\mathbb{Z}_2)^2$	MW	$\langle 1 \rangle \oplus (\mathbb{Z}_2)^2$	22
13	6	1	1	$\{12I_1\}$	0	E_8	0	MW	–	1
14	3	2/1	1	$\{\mathbf{III}, 3I_2, 3I_1\}$	$A_1^{\oplus 4}$	$D_4^* \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	MW	D_4^*	8
15	2	3/1	1	$\{\mathbf{IV}, 2I_3, 2I_1\}$	$A_2^{\oplus 3}$	$A_2^* \oplus \mathbb{Z}_3$ $\mathbb{Z}_5$	$\mathbb{Z}_3$	MW	A_2^*	17
16	5	1	1	$\{\mathbf{II}, 10I_1\}$	0	E_8 $\mathbb{Z}_4$	0	MW	–	2
17	4	1	2	$\{12I_1\}$	0	E_8	A_1	E_7	–	4
18	4	1	1	$\{\mathbf{IV}, 8I_1\}$	A_2	E_6^*	0	MW	–	3
19	2	2	2	$\{\mathbf{I_2}, 4I_2, 2I_1\}$	$A_1^{\oplus 5}$	$A_1^{*\oplus 3} \oplus \mathbb{Z}_2$	$A_1^* \oplus \mathbb{Z}_2$	$A_1^{\oplus 2} \oplus \mathbb{Z}_2$	$\langle 1 \rangle^{\oplus 2} \oplus \mathbb{Z}_2$	26
20	2	2	1	$\{\mathbf{I_0^*}, 2I_2, 2I_1\}$	$D_4 \oplus A_1^{\oplus 2}$	$A_1^{*\oplus 2} \oplus \mathbb{Z}_2$ $\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2$	MW	$\langle 1 \rangle^{\oplus 2} \oplus \mathbb{Z}_2$	14

continued

Table 8: *Continued.*

$$\mathbb{Z}_3 \times \mathbb{Z}_3$$

#	m	d	dim	Sing. fibers	T	MW	MW$^{\alpha_B}$	ker(Φ_m)	ker$(\Phi_m)_{d=1}$	num
21	2	1	3	$\{4I_2, 4I_1\}$	$A_1^{\oplus 4}$	$D_4^* \oplus \mathbb{Z}_2$	$\langle 1 \rangle^{\oplus 2} \oplus \mathbb{Z}_2$	$\langle 1 \rangle^{\oplus 2} \oplus \mathbb{Z}_2$	–	31
22	2	1	2	$\{6I_2\}$	$A_1^{\oplus 6}$	$A_1^{*\oplus 2} \oplus (\mathbb{Z}_2)^2$	$\langle 1 \rangle \oplus (\mathbb{Z}_2)^2$	$\langle 1 \rangle \oplus (\mathbb{Z}_2)^2$	–	32
23	2	1	2	$\{2I_4, 4I_1\}$	$A_3^{\oplus 2}$	$A_1^{*\oplus 2} \oplus \mathbb{Z}_2$	$A_1^* \oplus \mathbb{Z}_2$	$A_1^* \oplus \mathbb{Z}_2$	–	33
24	2	1	2	$\{\mathbf{I_2}, 4I_2, 2I_1\}$	$A_1^{\oplus 5}$	$A_1^{*\oplus 3} \oplus \mathbb{Z}_2$	$A_1^* \oplus \mathbb{Z}_2$	$A_1^{*\oplus 2} \oplus \mathbb{Z}_2$	$\langle 1 \rangle^{\oplus 2} \oplus \mathbb{Z}_2$	26
25	2	1	2	$\{\mathbf{I_4}, 2I_2, 4I_1\}$	$A_3 \oplus A_1^{\oplus 2}$	$A_3^* \oplus \mathbb{Z}_2$	$\langle 1 \rangle \oplus \mathbb{Z}_2$	$\langle 1 \rangle^{\oplus 2} \oplus \mathbb{Z}_2$	–	21
26	2	1	1	$\{\mathbf{I_4}, 4I_2\}$	$A_3 \oplus A_1^{\oplus 4}$	$\langle 1/4 \rangle \oplus (\mathbb{Z}_2)^2$	$(\mathbb{Z}_2)^2$	MW	$\langle 1 \rangle \oplus (\mathbb{Z}_2)^2$	22
27	2	1	1	$\{\mathbf{I_8}, 4I_1\}$	A_7	$A_1^* \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	MW	–	18
28	2	1	1	$\{\mathbf{I_0^*}, 2I_2, 2I_1\}$	$D_4 \oplus A_1^{\oplus 2}$	$A_1^{*\oplus 2} \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	MW	$\langle 1 \rangle^{\oplus 2} \oplus \mathbb{Z}_2$	14
29	2	1	0	$\{2I_0^*\}$	$D_4^{\oplus 2}$	$(\mathbb{Z}_2)^2$ $\mathbb{Z}_3$	$(\mathbb{Z}_2)^2$	MW	–	34
30	3	1	3	$\{12I_1\}$	0	E_8	A_2	E_6	–	9
31	3	1	2	$\{3I_2, 6I_1\}$	$A_1^{\oplus 3}$	$D_4^* \oplus A_1^*$	A_1^*	D_4^*	–	10
32	3	1	2	$\{\mathbf{III}, 9I_1\}$	A_1	E_7^*	$\langle 3/2 \rangle$	E_6^*	–	7
33	3	1	1	$\{\mathbf{I_0^*}, 6I_1\}$	D_4	D_4^*	0	MW	–	6

Table 9: List of rational elliptic surfaces with finite automorphism groups satisfying the criteria of Proposition 6.2.

#	m	d	dim	Sing. fibers	T	$\mathbb{Z}_2$				num
						MW	MW^{α_B}	$\ker(\Phi_m)$	$\ker(\Phi_m)_{d=1}$	
34	2	1	5	$\{12I_1\}$	0	E_8	D_4	D_4	—	28
35	2	1	4	$\{2I_2, 8I_1\}$	$A_1^{\oplus 2}$	D_6^*	$\langle 1 \rangle^{\oplus 3}$	$\langle 1 \rangle^{\oplus 3}$	—	29
36	2	1	4	$\{\mathbf{I_2}, 10I_1\}$	A_1	E_7^*	$2A_3^*$	D_4	—	23
37	2	1	3	$\{2I_3, 6I_1\}$	$A_2^{\oplus 2}$	$A_2^{*\oplus 2}$	A_2^*	A_2^*	—	30
38	2	1	3	$\{\mathbf{I_2}, 2I_2, 6I_1\}$	$A_1^{\oplus 3}$	$D_4^* \oplus A_1^*$	$A_1^* \oplus \langle 1 \rangle$	$\langle 1 \rangle^{\oplus 3}$	—	24
39	2	1	3	$\{\mathbf{I_4}, 8I_1\}$	A_3	D_5^*	$\langle 1 \rangle^{\oplus 2}$	$\langle 1 \rangle^{\oplus 3}$	—	20
40	2	1	3	$\{\mathbf{IV}, 8I_1\}$	A_2	E_6^*	$2A_2^*$	D_4	—	15
41	2	1	2	$\{\mathbf{I_2}, 2I_3, 4I_1\}$	$A_2^{\oplus 2} \oplus A_1$	$A_2^* \oplus \langle 1/6 \rangle$	$\langle 1/6 \rangle$	A_2^*	—	25
42	2	1	2	$\{\mathbf{I_6}, 6I_1\}$	A_5	$A_2^* \oplus A_1^*$	A_1^*	A_2^*	—	19
43	2	1	2	$\{\mathbf{IV}, 2I_2, 4I_1\}$	$A_2 \oplus A_1^{\oplus 2}$	$\frac{1}{6}\begin{pmatrix} 2 & 1 & 0 & -1 \\ 1 & 5 & 3 & 1 \\ 0 & 3 & 6 & 3 \\ -1 & 1 & 3 & 5 \end{pmatrix}$	$\langle 1/3 \rangle$	$\langle 1 \rangle^{\oplus 3}$	—	16
44	2	1	2	$\{\mathbf{I_0^*}, 6I_1\}$	D_4	D_4^*	$\langle 1 \rangle$	$\langle 1 \rangle^{\oplus 3}$	—	13
45	2	1	1	$\{\mathbf{IV^*}, 4I_1\}$	E_6	A_2^*	0	MW	—	12

Table 10: Specializations for tables 8 and 9.

#	G_B	Specializations
9	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$\{3I_3, 3I_1\}; \{3IV\}$
10	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\{4I_2, 4I_1\}; \{4III\}$
13	$\mathbb{Z}_6$	$\{12I_1\}; \{6II\}$
14		$\{III, 3I_2, 3I_1\}; \{4III\}$
15		$\{IV, 2I_3, 2I_1\}; \{3IV\}$
16	$\mathbb{Z}_5$	$\{II, 10I_1\}; \{6II\}$
17	$\mathbb{Z}_4$	$\{12I_1\}; \{4II, 4I_1\}$
18		$\{IV, 8I_1\}; \{IV, 4II\}$
19		$\{5I_2, 2I_1\}; \{3I_2, 2III\}$
20		$\{I_0^*, 2I_2, 2I_1\}; \{I_0^*, 2III\}$
21	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\{4I_2, 4I_1\}; \{2III, 2I_2, 2I_1\}; \{4III\}$
24		$\{5I_2, 2I_1\}; \{3I_2, 2III\}$
25		$\{I_4, 2I_2, 4I_1\}; \{I_4, 2III, 2I_1\}$
28		$\{I_0^*, 2I_2, 2I_1\}; \{I_0^*, 2III\}$
30	$\mathbb{Z}_3$	$\{12I_1\}; \{3II, 6I_1\}; \{6II\}$
31		$\{3I_2, 6I_1\}; \{3I_2, 3II\}, \{3III, 3I_1\}$
32		$\{III, 9I_1\}; \{III, 3II, 3I_1\}$
33		$\{I_0^*, 6I_1\}; \{I_0^*, 3II\}$
34	$\mathbb{Z}_2$	$\{12I_1\}; \{2II, 8I_1\}; \{4II, 4I_1\}; \{6II\}$
35		$\{2I_2, 8I_1\}; \{2I_2, 2II, 4I_1\}, \{2III, 6I_1\}; \{2I_2, 4II\},$ $\{2III, 2II, 2I_1\}$
36		$\{I_2, 10I_1\}; \{I_2, 2II, 6I_1\}; \{I_2, 4II, 2I_1\}$
37		$\{2I_3, 6I_1\}; \{2I_3, 2II, 2I_1\}, \{2IV, 4I_1\}; \{2IV, 2II\}$
38		$\{3I_2, 6I_1\}; \{3I_2, 2II, 2I_1\}, \{I_2, 2III, 4I_1\}; \{I_2, 2III, 2II\}$
39		$\{I_4, 8I_1\}; \{I_4, 2II, 4I_1\}; \{I_4, 4II\}$
40		$\{IV, 8I_1\}; \{IV, 2II, 4I_1\}; \{IV, 4II\}$
41		$\{2I_3, I_2, 4I_1\}; \{2I_3, I_2, 2II\}, \{2IV, I_2, 2I_1\}$
42		$\{I_6, 6I_1\}; \{I_6, 2II, 2I_1\}$
43		$\{IV, 2I_2, 4I_1\}; \{IV, 2I_2, 2II\}, \{IV, 2III, 2I_1\}$
44		$\{I_0^*, 6I_1\}; \{I_0^*, 2II, 2I_1\}$
45		$\{IV^*, 4I_1\}; \{IV^*, 2II\}$

may not act freely on f_∞ at these points, depending on the singular type of f_∞. In any case, we excluded these cases from our analysis for the reasons

pointed out in Section 4.3.1, and considered only generic configurations of singular fibers in a given moduli space, where f_∞ is smooth.

Remark 6.5. Note that various entries in the tables correspond to specializations of other entries with the same automorphism group. We decided to write separate entries in tables 8 and 9 whenever the surfaces have different Mordell–Weil groups, since the set of allowed sections AS in the construction of the automorphism τ_B is then different, resulting in different possibilities for spectral covers and eventually vector bundles, cf. [4]. We will come back to this in Remark 6.6.

Proof. We obtained in the previous section a list of suitable (B, τ_B), where τ_B is an automorphism of the second kind, that is, with $m > 1$, f_∞ is smooth, and τ_B acts freely on f_∞. For the cases where $\mathrm{MW}_{\mathrm{tors}}$ is non-trivial, one can combine this cyclic automorphism group with the automorphism group generated by translation by torsion sections to obtain a (perhaps non-cyclic) automorphism group.

For the group to be abelian, the generators of the cyclic factors must commute, that is, the torsion sections must be α_B-invariant (since translation by sections always commute). However, by looking at table 4 it is easy to see that for any pair (B, α_B) in our list, the torsion sections are always α_B-invariant; hence the automorphism groups that we obtain are always abelian.

We also want the full group to act freely on f_∞. We know that the cyclic automorphism group generated by an automorphism of the second kind acts freely on f_∞, by construction. Moreover, any translation by a non-zero torsion section also acts freely on the smooth fiber f_∞. But we must make sure that no element of G_B acts trivially on f_∞. The only subtle cases in that respect are cases 22 and 26. Here it may seem that one could obtain a $(\mathbb{Z}_2)^3$ automorphism group by combining the $\mathbb{Z}_2$ automorpism group acting on $\mathbb{P}^1$ with a torsion $(\mathbb{Z}_2)^2$. However, there is no free $(\mathbb{Z}_2)^3$ action on a smooth elliptic curve, hence at least one element of this group must fix the smooth fiber f_∞ pointwise. That is, only a subgroup $(\mathbb{Z}_2)^2 \subset (\mathbb{Z}_2)^3$ acts freely on f_∞.

To compute the dimension of the moduli space of complex deformations, we first follow the same argument as in the proof of Proposition 6.1, but with $e = 1$ since now $m > 1$. This gives the dimension of the moduli space for the generic configuration of singular fibers. For the entries in the table that correspond to specializations of other entries, we proceed as follows.

Assume we are given a family of elliptic surfaces, determined by a specific configuration of singular fibers, and that the positions of those singular fibers are subject to a number c of constraints. Consider another such family which is in the boundary of the former, that is, it is given by a singular configuration which is a degeneration of the first. Then the number of constraints on the new singular fibers is $\leq c$. In particular, if $c = 0$ for the big family, then $c = 0$ for any stratum in it. This is the case for all the families in table 8, in the sense that the position of the singular fibers of the quotient surfaces are unrestricted. Hence we get the dimension of the moduli space of these subfamilies by counting the number of singular fibers of the quotient surface minus $3 = \dim \mathrm{PGL}(2)$. This also accounts for the stratification of the configurations in table 9.

Taking these remarks into account the list follows from the results of the previous sections. Let us simply end the proof by commenting on a few cases.

Case 11. In this case one may think that by choosing a section $\xi \in \ker$ $(\Phi_2)_{d=2}$ to build τ_B, and combining with translation by a torsion section of order 4, one would get an automorphism group $\mathbb{Z}_4 \times \mathbb{Z}_4$. However, this is not true since $\tau_B^2 = t_{\eta \boxplus \eta}$, where $\eta \boxplus \eta$ is the order 2 torsion section; thus only a $\mathbb{Z}_4 \times \mathbb{Z}_2$ group acts faithfully. This kind of analysis applies each time $d > 1$ is allowed.

Cases 11, 14 *and* 15. In these cases we can realize the automorphism group either by taking a section $\xi \in \ker(\Phi_m)_{d=1}$ or by taking a section $\xi \in \ker(\Phi_m)_{d>1}$, and combining with translation by a torsion section. But these two realizations of the automorphism group are in fact equivalent, since they only differ by a linear combination of the generators.

Cases 12, 19, 20 *and, respectively,* 26, 24, 28. These are the same rational elliptic surfaces. The three first cases correspond to the automorphism groups obtained by choosing $\xi \in \ker(\Phi_2)_{d=2}$, while the three last cases correspond to the choice $\xi \in \ker(\Phi_2)_{d=1}$.

Cases 22, 26 *and* 29. Notice that in these cases one can also realize the $(\mathbb{Z}_2)^2$ automorphism group only by translation by torsion sections ($m = 1$). $\qquad \square$

Remark 6.6. By looking at the configurations of singular fibers in tables 8 and 9, it is straightforward to see that:

- case 18 is a specialization of case 17;

- case 20 is a specialization of case 19;

- cases 22–29 are specializations of case 21;

- cases 31–33 are specializations of case 30;

- cases 35–45 are specializations of case 34.

Tables 6–10 provide a complete list of rational elliptic surfaces with finite automorphism groups that can be lifted to free automorphism groups on smooth fiber products. Then, given any rational elliptic surface B in the tables above, we can quotient by the finite automorphism group (or a subgroup thereof) to obtain a new rational elliptic surface.

Remark 6.7. For the cases with trivial action on $\mathbb{P}^1$ given in table 6, it is interesting to note that, for a surface with the generic configuration of singular fibers, the quotient surface — by the whole torsion group — also has the same configuration of singular fibers.

7. Non-simply connected Calabi–Yau 3-folds

7.1. The main theorem

Finally, we apply our results on automorphisms of rational elliptic surfaces to the classification of the corresponding non-simply connected Calabi–Yau 3-folds.

Theorem 7.1. *The moduli space of quotients of smooth fiber products $\tilde{X} := B \times_{\mathbb{P}^1} B'$ (Schoen 3-folds) by free, fiber-preserving finite group actions has components as indicated in table 11.*

In the table, G is the group acting freely on $\tilde{X}$, m is the order of the image of G in $\mathrm{Aut}(\mathbb{P}^1)$ and h is the Hodge number $h := h^{1,1} = h^{2,1}$ of the quotient 3-fold $X := \tilde{X}/G$. The cases refer to tables 6, 8 and 9. When we list a case as $n_1 \times n_2$ we refer to the fiber product of B of case n_1 (in tables 6 to 9) with B' of case n_2. When we list a case as n, it means $n \times n$. When several cases are possible, we use semicolons to segregate strata of a given dimension, which are then separated by commas.

Proof. By assumption, $X = \tilde{X}/G$ is the quotient by some finite group G of a smooth Schoen 3-fold $\tilde{X} = B \times_{\mathbb{P}^1} B'$, where B and B' must be listed in tables 6 to 9, with the same G and the same m. What remains is to determine when two such Xs live in the same moduli space, and to calculate the Hodge numbers.

Table 11: Classification of non-simply connected Calabi–Yau 3-folds constructed as free, fiber-preserving quotients of smooth fiber products $B \times_{\mathbb{P}^1} B'$.

G	m	h	Cases
$\mathbb{Z}_3 \times \mathbb{Z}_3$	3	3	9
	1	3	1
$\mathbb{Z}_4 \times \mathbb{Z}_2$	4	3	10
	2	3	11
	2	3	12
	2	3	11×12
	1	3	2
$\mathbb{Z}_6$	6	3	13
	3	3	14
	2	3	15
	1	3	3
$\mathbb{Z}_5$	5	3	16
	1	3	4
$\mathbb{Z}_4$	4	5	$17; 10, 18$
	2	5	$19; 11, 12, 20$
	1	5	5
$\mathbb{Z}_2 \times \mathbb{Z}_2$	2	7	$21; 22-25; 11, 15, 26-28; 29$
	1	7	6
$\mathbb{Z}_3$	3	7	$30; 31, 32; 9, 14, 33$
	1	7	7
$\mathbb{Z}_2$	2	11	$34; 35, 36; 21, 37-40; 22-25, 41-44; 11, 15, 26-28, 45; 29$
	1	11	8

For the first task, we simply need to determine which configurations in tables 8 and 9 can specialize to which other configurations. This was presented in Remark 6.6.

As to the Hodge numbers, the smooth Schoen 3-folds $\tilde{X}$ have $h^{1,1}(\tilde{X}) = h^{2,1}(\tilde{X}) = 19$, so their Euler characteristic vanishes. This latter property descends to X, so $h^{1,1}(X) = h^{2,1}(X) := h$. We determine this by counting complex moduli. Any deformation of X lifts to a deformation of its universal cover $\tilde{X}$, which in turn is given by a deformation of the surfaces B and B'. Now the equivalence relation on $\tilde{X}$ which gives the quotient X is etale, in fact it is isomorphic to the product $\tilde{X} \times G$. This property is clearly invariant under deformations, so the deformed X must be the quotient of a deformed

$\tilde{X}$ by a free action of the same group G. It must therefore be included in tables 6–9. It follows that h, the number of complex moduli of our Calabi–Yau X, is given as:

$$h = h_B + h_{B'} + e$$

where h_B, $h_{B'}$ are the numbers of deformations of the G action on B, B', respectively, which are indicated in tables 6 to 9, and e gives the number of extra parameters coming from an identification of the $\mathbb{P}^1$ bases of B and B':

- $e = \dim(\mathrm{PGL}(2)) = 3$ in case $m = 1$;

- $e = 1$ if $m > 1$ (in this case we lose two of the three parameters since the points $0, \infty$ must go to each other). $\qquad\square$

7.2. Examples

In this subsection we give a few examples of non-simply connected Calabi–Yau 3-folds that we constructed in this paper.

7.2.1. Calabi–Yau 3-folds with fundamental group $\pi_1(X) \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.
According to Theorem 7.1, we constructed two Calabi–Yau 3-folds with $\pi_1(X) \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

The first one has Hodge numbers $h = 3$, and corresponds to the fiber product of two rational elliptic surfaces in the one-parameter family given by case #9 of table 8. These rational elliptic surfaces have $T = A_2^{\oplus 3}$, with smooth f_0; one $\mathbb{Z}_3$ is generated by translation by a torsion section, and the other $\mathbb{Z}_3$ acts faithfully on the $\mathbb{P}^1$ base. The generic configuration of singular fibers is $\{3I_3, 3I_1\}$. This is precisely the Calabi–Yau 3-fold studied in great detail in [5].

The second Calabi–Yau 3-fold also has Hodge numbers $h = 3$ and consists of the fiber product of two rational elliptic surfaces corresponding to case #1 of table 6. Here, the full automorphism group acts trivially on $\mathbb{P}^1$. It would be interesting to investigate the construction of standard model bundles on this 3-fold.

7.2.2. Calabi–Yau 3-folds with fundamental group $\pi_1(X) \cong \mathbb{Z}_6$.
According to Theorem 7.1, we get four Calabi–Yau 3-folds with $\pi_1(X) \cong \mathbb{Z}_6$; all have Hodge numbers $h = 3$. We are presently studying the construction of standard model bundles on these manifolds and will report on it in further publication [4].

7.2.3. Calabi–Yau 3-folds with fundamental group $\pi_1(X) \cong \mathbb{Z}_2$.
According to Theorem 7.1 we get two Calabi–Yau 3-folds with $\pi_1(X) \cong \mathbb{Z}_2$,
with Hodge numbers $h = 11$. Here we simply note that the first one is the
Calabi–Yau 3-fold that was used in $[2, 3, 8, 9]$ to construct standard model
bundles. More precisely, in $[2, 3, 8, 9]$ the covering Calabi–Yau 3-fold $\tilde{X}$ was
constructed as a smooth fiber product of two rational elliptic surfaces with
configuration of singular fibers $2I_2, 8I_1$, with smooth f_0 and $\alpha_B \simeq \mathbb{Z}_2$; this
is the generic configuration in the four-parameter family given by case #35
in table 9.

7.2.4. Schoen's constructions.   In Section 9 of [15], Schoen constructs
four different smooth fiber products with free $\mathbb{Z}_n$ action, with $n = 2, 3, 4, 6$,
respectively. He considers four types of rational elliptic surfaces, and then
fiber products of two rational elliptic surfaces of the same type. The four
types of rational elliptic surfaces he considers have smooth fibers at 0 and ∞,
and admit $\mathbb{Z}_n$ actions with fixed points only on the smooth fiber f_0. The $\mathbb{Z}_n$
actions act faithfully on the $\mathbb{P}^1$ base, hence in our notation $m = n$, that is, in
each case the order of α_B is equal to the order of τ_B. Therefore we see that
the rational elliptic surfaces of the types he considers, with $n = 2, 3, 4, 6$,
respectively, correspond to the cases in our list with $n = 2, 3, 4, 6$, $m = n$
and $f_0 = I_0$. That is, respectively, cases #34, 35, 37; #30, 31; #17 and #13.
The resulting Calabi–Yau 3-folds correspond to the first line for each of these
cyclic groups in table 11.

8. Outlook

In this paper we produced a complete classification of finite automorphism
groups of rational elliptic surfaces such that they lift to free automorphism
groups on the smooth fiber product of two rational elliptic surfaces. This
work opens up various avenues of research.

- We classified and studied in detail a large class of finite automorphisms
  of rational elliptic surfaces. However, since our aim was ultimately to
  construct non-simply connected Calabi–Yau 3-folds, we restricted our-
  selves to automorphisms acting freely on the fiber at infinity. An obvi-
  ous direction of research would be to continue our study from a purely
  2-dimensional point of view and classify all finite automorphisms of
  rational elliptic surfaces.

- As mentioned in the Introduction, one of our simplifying assumptions was to restrict ourselves to the study of Calabi–Yau 3-folds constructed as smooth fiber products of two rational elliptic surfaces. However, an important aspect of Schoen's work [15] was precisely to study the more complicated situation when the fiber product has ordinary double point singularities. He showed that in many cases, the minimal resolution of the singular Calabi–Yau 3-fold is also Calabi–Yau. It would be interesting, both from a mathematical and a physics point of view, to construct free automorphism groups acting on these smooth resolutions, in order to construct an even larger family of non-simply connected Calabi–Yau 3-folds.

- From a physics point of view, the motivation behind this work is to construct a large class of non-simply connected Calabi–Yau 3-folds suitable for string compactifications. In particular, we are interested in $E_8 \times E_8$ heterotic string vacua, which roughly speaking are given by triples (X, V, G), where X is a Calabi–Yau 3-fold and V is a stable vector bundle on X with structure group $G \subseteq E_8$. One way to obtain realistic 4-dimensional standard model physics out of these compactifications is to consider non-simply connected Calabi–Yau 3-folds X, with $G = \mathrm{SU}(4)$ or $\mathrm{SU}(5)$. In this work we produced a large class of non-simply connected X; the next step in this program is to construct and study "standard model bundles" on these 3-folds (possibly by constructing invariant stable bundles on the simply connected Calabi–Yau covers $\tilde{X}$), by which we mean $\mathrm{SU}(4)$ or $\mathrm{SU}(5)$ bundles satisfying all the requirements needed to yield realistic 4-dimensional physics. The only bundles known so far which give a consistent compactification with the correct spectrum are those constructed in [2, 3] on the Calabi–Yaus built from the 4-dimensional family of rational elliptic surfaces in case #35 of table 9.

Appendix: Sketch of the proof of remark 6.3

In this appendix we sketch a case-by-case proof of Remark 6.3, where we claimed that the set of allowed sections AS is non-empty for all lines in tables 8 and 9.

When the group is cyclic, the automorphism group is generated by an automorphism of the form

$$(A.1) \qquad \tau_B = t_\xi \circ \alpha_B, \quad \xi \in \ker(\Phi_m),$$

and AS will be non-empty if we can find a section ξ which intersects f_∞ at a torsion point of order $n = dm$. When the automorphism group is non-cyclic, one of the generators is of the form above, and the other is translation by a torsion section η. In this case we must find a section ξ as above, and also make sure that all the sections $\mathcal{P}_i(\xi)$, $i = 1, \ldots, n - 1$ intersect f_∞ at torsion points other than those where η and its multiples do, so that the full automorphism group acts freely on f_∞.

The easiest cases are when the group is cyclic and $m = n$ is prime ($\mathbb{Z}_5$, $\mathbb{Z}_3$ and $\mathbb{Z}_2$). In each of these cases we can take a section ξ corresponding to a point of $\ker(\Phi_m)$ which has minimal length in the Mordell–Weil group. Using the height pairing, it is easy to show that ξ is disjoint from the zero section, hence must intersect f_∞ at a non-zero torsion point of order n. Therefore AS is non-empty.

For the non-cyclic cases, we take a section ξ as above, and check using the height pairing that it is also disjoint from the torsion sections used to construct the second cyclic subgroup. This works in all non-cyclic cases, except case 10.

The $\mathbb{Z}_6$ cases 14 and 15 and the $\mathbb{Z}_4$ cases 19 and 20 can be treated similarly. For the $\mathbb{Z}_6$ case 13, we consider a minimal point $\xi_2 \in \ker(\Phi_2) \subset E_8$, and a minimal point $\xi_3 \in \ker(\Phi_3) \subset E_8$. Both sections are disjoint from the zero section, hence ξ_2 must intersect f_∞ at a torsion point of order 2, while ξ_3 intersects f_∞ at a torsion point of order 3. So the sum $\xi_2 \boxplus \xi_3$ must intersect f_∞ at a torsion point of order 6.

The remaining cases are 10, 17 and 18. For cases 17 and 18, we have $m = 4$, and we want to find a section ξ intersecting f_∞ at a torsion point of order 4. If we can find at least four sections $\xi_i \in \ker(\Phi_4)$, $i = 1, \ldots, 4$, which are all disjoint from each other and from the zero section, that is, $\xi_i \cdot \xi_j = -\delta_{ij}$, and $\xi_i \cdot \sigma = 0$, for all i, j, then one of them must intersect f_∞ at a 4-torsion point, since there are only three non-zero torsion points of order 2 on the smooth elliptic fiber f_∞. Now for both cases 17 and 18, using the height pairing and an explicit basis for E_7 and E_6^*, we can find four minimal points satisfying these conditions.

Finally case 10. Here we have $m = 4$ and want to construct a $\mathbb{Z}_4 \times \mathbb{Z}_2$ automorphism group. Hence we want a section ξ that intersects f_∞ at a 4-torsion point, and we also want to make sure that $\alpha_B \xi \boxplus \xi$ does not intersect f_∞ at the same point as the 2-torsion section η. If we can find at least eight sections which are all disjoint from each other and from the zero section, then one of them, call it ξ, will necessarily intersect f_∞ at a 4-torsion point but $\alpha_B \xi \boxplus \xi$ will intersect f_∞ at a 2-torsion point different

than the one where the 2-torsion section η intersects f_∞. So let us try to find eight such sections.

Describe D_4^* as the square lattice $\mathbb{Z}^4$ plus the non-integral element $\frac{1}{2}(1, 1, 1, 1)$. The 24 minimal points of norm 1 are $\pm$ the four unit vectors, and $\frac{1}{2}(\pm1, \pm1, \pm1, \pm1)$. Note that any minimal point is disjoint from σ and the torsion section η, so we need to find seven mutually disjoint sections projecting to minimal points of the lattice. Note also that for case 10 there are four I_2 fibers, and that any minimal section intersects two non-neutral components.

Consider first the three sections ξ_1, ξ_2, ξ_3 corresponding to $(1, 0, 0, 0)$, $\frac{1}{2}(1, 1, 1, 1)$ and $\frac{1}{2}(1, 1, 1, -1)$, and the two additional sections $\xi_4 = \xi_2 \boxplus \eta$, $\xi_5 = \xi_3 \boxplus \eta$. It is easy to show that these five sections are all disjoint, since their mutual height pairing is always greater than 0. Now consider the section ξ_6' corresponding to $(0, 1, 0, 0)$. It is clearly disjoint from ξ_i for $i = 2, 3, 4, 5$, and it is either disjoint from ξ_1 or $\xi_6' \cdot \xi_1 = 1$. But then $\xi_6' \boxplus \eta$ is disjoint from ξ_1. We can therefore choose ξ_6 to be either ξ_6' or $\xi_6' \boxplus \eta$, whichever is disjoint from ξ_1. We then know that ξ_1 and ξ_6 must intersect the non-neutral components of the same two I_2 fibers. Finally, consider the section ξ_7' corresponding to $(0, 0, 1, 0)$. This and $\xi_7' \boxplus \eta$ are again clearly disjoint from ξ_i for $i = 2, 3, 4, 5$, so by taking ξ_7 to be either ξ_7' or $\xi_7' \boxplus \eta$, it can be assumed to be disjoint from ξ_1. But then ξ_7, ξ_1, ξ_6 all intersect the non-neutral components of the same two I_2 fibers. It follows that ξ_7 is also disjoint from ξ_6. Hence, we have constructed a set of seven mutually disjoint sections which are also disjoint from η and σ, demonstrating that AS is non-empty in this case as well.

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HARVARD UNIVERSITY
JEFFERSON PHYSICAL LABORATORY
17 OXFORD ST.
CAMBRIDGE
MA 02138
USA
E-mail address: bouchard@physics.harvard.edu

UNIVERSITY OF PENNSYLVANIA
DEPARTMENT OF MATHEMATICS
DAVID RITTENHOUSE LAB.
209 SOUTH 33RD STREET
PHILADELPHIA
PA 19104-6395
USA
E-mail address: donagi@math.upenn.edu