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On Landau–Ginzburg models for Fano varieties

Victor Przyjalkowski

We observe a method for finding weak Landau–Ginzburg models for Fano varieties and find them for smooth Fano threefolds of genera 9, 10, and 12.

In the late 1980s, physicists discovered a phenomenon of mirror symmetry. They found that given a Calabi–Yau variety one can construct the so-called superconformal field theory. This can be done in two ways: "algebro-geometric" and "symplectic". Based on this, they suggested that for each Calabi–Yau X there is another one Y (not necessary uniquely determined), whose algebro-geometric properties "correspond" to symplectic ones of X and symplectic ones "corresponds" to algebro-geometric ones of X. In particular, the Hodge diamond of Y is a reflection (a mirror image) of the Hodge diamond of X about a 45° line. That is why X and Y are called a mirror pair.

Later, in order to formalize this empiric approach, mathematicians formulated a series of mirror symmetry conjectures. They generalize the correspondence to Fano varieties (Batyrev, Givental, Hori, Vafa, etc.). The pair for a Fano variety X is conjecturally a Landau–Ginzburg model, that is, a (non-compact) manifold M with complex-valued function f on it.

The dual model M has a series of properties that correspond to the properties of X. Homological mirror symmetry conjecture (Kontsevich [18]), for instance, states that the derived category of coherent sheaves on X is isomorphic to the category of Lagrangian vanishing cycles on M. This gives the correspondence between singular fibers of f and the exceptional collection of X. One of the main problems in the mirror symmetry is to find a Landau–Ginzburg model for a Fano variety. We use the mirror symmetry conjecture of Hodge structure variations to find a candidate for it. This conjecture states the following. Given genus 0 two-pointed Gromov–Witten invariants of X (the expected numbers of rational curves of given degree that intersect two general representatives of homological classes on X), one may define a (small) quantum cohomology ring, i.e., the deformation of the cohomology of X with Gromov–Witten invariants as structural constants. The quantum multiplication in this ring gives the quantum $\mathcal{D}$-module. The conjecture is that the regularization of this $\mathcal{D}$-module is isomorphic to a Picard–Fuchs $\mathcal{D}$-module of M. The advantage of this conjecture is that we can effectively check it in some cases. In the paper, we check it in the cases of some threefolds and discuss some approaches to finding candidates to Landau–Ginzburg models. More about homological mirror symmetry and mirror symmetry of Hodge structure variations for the varieties we consider in the paper and for the other examples see in [15].

Consider a smooth Fano threefold V with Picard group $\mathbb{Z}$. We may associate a differential operator L_V of type D3 with it (see [12, Definition 2.10]). The differential equation associated with this operator has a unique analytical solution of type $1 + a_1 t + a_2 t^2 + \cdots$ (the fundamental term of the regularized I-series; see [22] for the detailed description of the solutions of equations of type DN). It is easy to calculate the coefficients of the fundamental term using the recursive procedure.

Consider a Laurent polynomial $f \in \mathbb{C}[x, x^{-1}, y, y^{-1}, z, z^{-1}]$. Let b_i be the constant term of f^i.

Definition. The polynomial f is called *a very weak Landau–Ginzburg model* for V, if $a_i = b_i$ for all $i \geq 0$. The polynomial f is called *a weak Landau–Ginzburg model* for V, if f is a very weak Landau–Ginzburg model for V and a general element of the pencil $\mathbb{T} = \{1 - tf = 0, t \in \mathbb{C}\}$ is birational to a K3 surface.

The similar definition may be formulated for the varieties of an arbitrary dimension (see Definition 2.2).

The motivation of this definition is the following. The series $1 + b_1 t + b_2 t^2 + \cdots$ is the solution of the Picard–Fuchs equation for $\mathbb{T}$ (see Proposition 2.3). The mirror symmetry conjecture of Hodge structure variations states that the regularized quantum operator a variety (which is of type D3 in the 3-fold case) coincides with the Picard–Fuchs operator of its mirror dual Landau–Ginzburg model. The general mirror symmetry philosophy says that the general fiber of the Landau–Ginzburg model is a smooth Calabi–Yau variety.

Finding weak Landau–Ginzburg models reduces to a computational problem. That is, given a_is, one should find f with free terms b_is coinciding with a_is. The K3-condition usually follows from the degree reasons and the Bertini Theorem (see below). The difficulty is computational. In this paper, we discuss some approaches to solving this problem. In particular, we discuss conjectures related with toric degenerations of Fano varieties. Using these approaches we find weak Landau–Ginzburg models of Fano threefolds V_{16} (of genus 9), V_{18} (of genus 10), and V_{22} (of genus 12).

1. Definitions and conventions

The variety is a reduced irreducible scheme of finite type. The variety X is called $\mathbb{Q}$-*factorial* if $\mathrm{Cl}(X) \otimes \mathbb{Q} \cong \mathrm{Pic}(X) \otimes \mathbb{Q}$ (where $\mathrm{Cl}(X)$ is the group of Weil divisor classes on X). It is said to be $\mathbb{Q}$-*Gorenstein* if $mK_X \in \mathrm{Pic}(X)$ for some $m \in \mathbb{N}$. The $\mathbb{Q}$-Gorenstein variety X is said to have *canonical singularities* if for each resolution $f : X' \to X$ the relative canonical $\mathbb{Q}$-divisor $K_{X'} - f^*(K_X)$ is effective.

The (*local*) *deformation* is a flat morphism $\mathcal{X} \to S$, where $S = (S, s_0)$ is a germ of a smooth variety (usually the germ of a curve). The fiber over the central point X_{s_0} is called *the special fiber*. The fiber over other point is called *the general fiber*. We say that the general fiber *degenerates* to the special fiber and $\mathcal{X}$ is *a degeneration* to X_{s_0}.

Let X be a smooth algebraic variety with Picard group $\mathbb{Z}$. Let $\gamma_1, \ldots, \gamma_m \in H^*(X, \mathbb{Z})$, $k_1, \ldots, k_m \in \mathbb{Z}_{\geq 0}$, and $\beta \in H_2(X, \mathbb{Z})$ be the class of algebraic curves of anticanonical degree $d \geq 0$. We denote genus 0 Gromov–Witten invariant with descendants that correspond to this data (see [19, VI–2.1]) by

$$\langle \tau_{k_1}\gamma_1, \ldots, \tau_{k_m}\gamma_m \rangle_\beta = \langle \tau_{k_1}\gamma_1, \ldots, \tau_{k_m}\gamma_m \rangle_d.$$

Everything is over $\mathbb{C}$.

2. Weak Landau–Ginzburg models

2.1. Quantum side

Let X be a smooth Fano variety of dimension N with Picard group $\mathbb{Z}$. Consider a torus $\mathbb{T}_{\mathrm{NS}^\vee} \cong \mathrm{Spec}\, B$, $B = \mathbb{C}[t_0, t_0^{-1}]$, twice dual to the Neron–Severi lattice. Let $H_H^*(X) \subset H^*(X, \mathbb{Q})$ be the subspace generated by the anticanonical class $H = -K_X$. This subspace is tautologically closed with respect to the multiplication, i.e., for any $\gamma_1, \gamma_2 \in H_H^*(X)$ the product $\gamma_1 \cdot \gamma_2$ lies in $H_H^*(X)$. The multiplication structure on the cohomology ring may be deformed. That is, one can consider *a quantum cohomology ring* $QH^*(X) = H^*(X) \otimes \mathbb{C}[t_0]$ (see [19, Definition 0.0.2]) with quantum multiplication $\star : QH^*(X) \times QH^*(X) \to QH^*(X)$, i.e. the bilinear map given by

$$\gamma_1 \star \gamma_2 = \sum_{\gamma, d} t_0^d \langle \gamma_1, \gamma_2, \gamma^\vee \rangle_d \gamma$$

for all $\gamma_1, \gamma_2, \gamma \in H^*(X)$, where $\gamma^\vee$ is the Poincaré dual class to γ (we identify elements of $\gamma \in H^*(X)$ and $\gamma \otimes 1 \in QH^*(X)$). The constant term of $\gamma_1 \star \gamma_2$ (with respect to t_0) is $\gamma_1 \cdot \gamma_2$. The subspace $QH^*_H(X) = H^*_H(X) \otimes \mathbb{C}[t_0]$ is not closed with respect to $\star$ in general. The examples of varieties V with non-closed subspaces $H^*_H(X)$ are Grassmannians $G(k,n)$, k, $n-k>1$ of dimension >4 (for instance, $G(2,5)$) or their hyperplane sections of dimension ≥ 4.

The variety is called *quantum minimal* if $H^*_H(X)$ is closed with respect to the quantum multiplication (see [22, Definition 1.2.1]). The examples of such varieties are Fano complete intersections or Fano threefolds.

Let HQ be a trivial vector bundle over $\mathbb{T}_{\mathrm{NS}^\vee}$ with fiber $H^*_H(X)$. Let $S = H^0(HQ)$ and $\star : S \times S \to S$ be the quantum multiplication (we may consider the quantum multiplication as an operation on $S \cong QH^*_H(X) \otimes \mathbb{C}[t_0, t_0^{-1}]$). Let $\mathcal{D} = B[t_0(\partial/\partial t_0)]$ and $D = t_0(\partial/\partial t_0)$. Consider a (flat) connection ∇ on HQ defined on the sections H^i as

$$\left(\nabla(H^i), t_0\frac{\partial}{\partial t_0}\right) = K_V \star H^i$$

(the pairing is the natural pairing between differential forms and vector fields). This connection provides the structure of $\mathcal{D}$-module for S by $D(H^i) = (\nabla(H^i), D)$.

Let Q be this $\mathcal{D}$-module. It is not regular in general. To obtain the regular $\mathcal{D}$-module, we need "to regularize" it. Let $\mathbb{G}_{\mathrm{m}} = \mathrm{Spec}\,[t, t^{-1}]$. Let $E = \mathcal{D}_{\mathbb{G}_{\mathrm{m}}}/\mathcal{D}_{\mathbb{G}_{\mathrm{m}}}(t(\partial/\partial t) - t)$ be the exponential $\mathcal{D}_{\mathbb{G}_{\mathrm{m}}}$-module. Consider the inclusion $\mathbb{Z}(-K_X) \hookrightarrow \mathrm{Pic}\, X$. The natural isomorphism $\mathrm{Pic}(X) \cong \mathrm{NS}(X)$ (X is Fano) and double dualization provide the morphism $j : \mathbb{G}_{\mathrm{m}} \to \mathbb{T}_{\mathrm{NS}^\vee}$. Define *the regularization* of Q as $Q^{\mathrm{reg}} = j^*(\mu_*(Q \boxtimes j_*(E)))$, where $\mu : \mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}$ is the multiplication, and $\boxtimes$ is the external tensor product (i.e., Q^{reg} is a convolution with the anticanonical exponential $\mathcal{D}$-module). It may be represented as $\mathcal{D}_{\mathbb{G}_{\mathrm{m}}}/\mathcal{D}_{\mathbb{G}_{\mathrm{m}}}(t(\partial/\partial t)L_X)$. We denote $t(d/dt)$ also by D. The differential operator L_X is called *of type* DN (see [12, 2.10]). This operator is explicitly written in [12, Example 2.11] for $N=3$ in terms of structural constants of quantum multiplication by the anticanonical class (that is, two-pointed Gromov–Witten invariants). Thus, there is an operator of type D3 associated with every smooth Fano threefold with Picard group $\mathbb{Z}$. There are 17 families of such Fanos (the Iskovskikh list). For all of them the operators of type D3 are known, see, for instance, [21, 4.4].

Definition 2.1. (A unique) analytic solution of $L_X I = 0$ of type

$$I^X_{H_0} = 1 + a_1 t + a_2 t^2 + \cdots a_i \in \mathbb{C}[[t]], \quad a_i \in \mathbb{C},$$

is called *the fundamental term of the regularized I-series* of X.

Let $\mathbf{1}$ be the class in $H^0(X, \mathbb{Z})$ dual to the fundamental class of X. Then this series is of the form

$$I^X_{H^0} = 1 + \sum_{d \geq 2} \langle \tau_{d-2} \mathbf{1} \rangle_d \cdot t^d$$

(see [22, Corollary 2.2.6]).

2.2. Picard–Fuchs side

Consider a torus $\mathbb{T} = \mathbb{G}^n_{\mathrm{m}} = \prod_{i=1}^n \operatorname{Spec} \mathbb{C}[x_i, x_i^{-1}]$ and a function f on it. This function is represented by Laurent polynomial: $f = f(x_1, x_1^{-1}, \ldots, x_n, x_n^{-1})$. Let $\phi_f(i)$ be the constant term (i.e. the coefficient at $x_1^0 \cdot \ldots \cdot x_n^0$) of f^i. Put

$$\Phi_f = \sum_{i=0}^{\infty} \phi_f(i) \cdot t^i \in \mathbb{C}[[t]].$$

Definition 2.2. The series $\Phi_f = \sum_{i=0}^{\infty} \phi_f(i) \cdot t^i$ is called *the constant terms series* of f.

Definition 2.3. Let X be a smooth n-dimensional quantum minimal Fano variety and $I^X_{H^0} \in \mathbb{C}[[t]]$ be its fundamental term of regularized I-series. The Laurent polynomial $f \in \mathbb{C}[\mathbb{Z}^n]$ is called *a very weak Landau–Ginzburg model* for X if

$$\Phi_f(t) = I^X_{H^0}(t).$$

The Laurent polynomial $f \in \mathbb{C}[\mathbb{Z}^n]$ is called *a weak Landau–Ginzburg model* for X if it is a very weak Landau–Ginzburg model for X and for almost all $t \in \mathbb{C}$ the hypersurface $(1 - tf = 0)$ is birational to a Calabi–Yau variety.

The meaning of the definition is the following (see [4, pp. 50–52] or [23, 10]). Consider functions $F_t = 1 - t \cdot f \in \mathbb{C}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}][t]$. They provide a pencil $\mathbb{T} \to \mathbb{B} = \mathbb{P}[u : v] \setminus (0 : 1)$ with fibers $Y_t = (F_t = 0)$, $t \in \mathbb{B}$.

The following proposition is a sort of a mathematical folklore.

Proposition 2.3. *Let the Newton polytope of $f \in \mathbb{C}[\mathbb{Z}^n]$ contains* 0 *in the interior. Let $t \in \mathbb{B}$ be the local coordinate around* $(0:1)$*. Then there is a fiberwise $n-1$-form $\omega_t \in \Omega^{n-1}_{\mathbb{T}/\mathbb{B}}$ and* (*locally defined*) *fiberwise $n-1$-cycle Δ_t, such that*

$$\Phi_f(t) = \int_{\Delta_t} \omega_t.$$

This means that $\Phi_f(t)$ is a solution of the Picard–Fuchs equation for the pencil $\{Y_t\}$.

Proof. The following argument is based on [13, § 3].

Let

$$T_s = \{(x_1, \ldots, x_n) \in \mathbb{T} : |x_1| = \cdots = |x_n| = s\},$$

$T = T_1$, and $R_\delta = \cup T_s$, $\delta \le s \le 1$ (we consider the natural metric on the torus given by $\mathbb{T} \hookrightarrow \mathbb{C}^n$). Let t be small enough such that $Y_t \cap T = \emptyset$, that is, $|f(T)| < |1/t|$. Let δ be small enough, such that $Y_t \cap T_\delta = \emptyset$ (f has terms of negative degree). We may assume that R_δ and Y_t intersect transversally. Let $\Delta_t = Y_t \cap R_\delta \in Y_t$. Let $Y_t^\varepsilon = \{p \in \mathbb{T} | \, \exists v \in Y_t : |p - v| < \varepsilon\}$, where ε is small enough. Then $\Delta_t^\varepsilon = R_\delta \cap Y_t^\varepsilon$ is a "tube" over Δ_t. Clearly $\partial(R_\delta \setminus Y_t^\varepsilon) = T + T_\delta - \Delta_t^\varepsilon$, so $T + T_\delta$ and Δ_t^ε are homologically equivalent.

Let

$$\Omega_t = \frac{1}{(2\pi i)^n} \frac{1}{F_t} \prod_{i=1}^{n} \frac{dx_i}{x_i}.$$

Consider the integral

$$\Phi(t) = \int_{T+T_\delta} \Omega_t.$$

It is easy to see that $\int_{T_\delta} \Omega_t$ tends to zero under $\delta \to 0$. Since it is constant, it equals zero. Therefore, integrating step by step, we have $\Phi(t) = \int_T \Omega_t = \Phi_f(t)$.

On the other hand, by Poincaré residue theorem

$$\Phi(t) = \int_{\Delta_t^\varepsilon} \Omega_t = \int_{\Delta_t} \operatorname{Res}_{Y_t} \Omega_t = \int_{\Delta_t} \omega_t.$$

□

Let $PF_f = PF_f(t, \partial/\partial t)$ be a Picard–Fuchs operator of $\{Y_t\}$. Let m be the order of PF_f and r be the degree with respect to t. Let Y be a semistable compactification of $\{Y_t\}$ (so we have the map $\widetilde{f} : Y \to \mathbb{P}^1$; denote it for the simplicity by f). Let m_f be the dimension of transcendental part of $R^{n-1} f_! \mathbb{Z}_Y$ (the algorithm for computing it see in [9]). Let r_f be the number of singularities of f (counted with multiplicities). Then, $m \leq m_f$ and $r \leq r_f$. Thus, the first few coefficients of the expansion of the solution of the Picard–Fuchs equation determine the other ones. This means that if the first few coefficients of the expansion the solution of $L\Phi = 0$, $L \in \mathbb{C}[t, \partial/\partial t]$, coincide with the first few coefficients of the expansion of Φ_f, then $L = PF_f$.

3. Main theorem

Theorem 3.1. *Weak Landau–Ginzburg models*

(1) *The Laurent polynomial*

$$f_{16} = \frac{1}{xyz} + 2\left(\frac{1}{xy} + \frac{1}{xz} + \frac{1}{yz}\right) + 3\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) + \left(\frac{x}{y} + \frac{x}{z} + \frac{y}{x} + \frac{y}{z} + \frac{z}{x} + \frac{z}{y}\right) + 4 + (x + y + z)$$

is a weak Landau–Ginzburg model for Fano variety V_{16}.

(2) *The Laurent polynomial*

$$f_{18} = 2\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) + \left(\frac{x}{yz} + \frac{y}{xz} + \frac{z}{xy}\right) + \left(\frac{x}{y} + \frac{x}{z} + \frac{y}{x} + \frac{y}{z} + \frac{z}{x} + \frac{z}{y}\right) + 3 + (x + y + z)$$

is a weak Landau–Ginzburg model for Fano variety V_{18}.

(3) *The Laurent polynomial*

$$f_{22} = \frac{xy}{z} + \frac{y}{z} + \frac{x}{z} + x + y + \frac{1}{z} + 4 + \frac{1}{x} + \frac{1}{y} + z + \frac{1}{xy} + \frac{z}{x} + \frac{z}{y} + \frac{z}{xy}$$

is a weak Landau–Ginzburg model for Fano variety V_{22}.

Proof. The operators of type $D3$ are

$$D^3 - 4t(2D + 1)(3D^2 + 3D + 1) + 16t^2(D + 1)^3$$

for V_{16},

$$D^3 - 3t(2D + 1)(3D^2 + 3D + 1) - 27t^2(D + 1)^3$$

for V_{18}, and

$$D^3 - \frac{2}{5}t(2D+1)(17D^2+17D+16) - \frac{56}{25}t^2(D+1)(11D^2+22D+12) - \frac{126}{125}t^3(D+1)(D+2)(2D+3) - \frac{1504}{625}t^4(D+1)(D+2)(D+3)$$

for V_{22} (see [12]). The degrees of Picard–Fuchs operators for pencils that are given by f_{16}, f_{18}, and f_{22} are bounded by 3 with respect to D and 4 with respect to t. One may check that the first few coefficients of the expansion of the constant terms series and the fundamental term of regularized I-series coincide (see last paragraph in Section 2.2). Thus, these polynomials are very weak Landau–Ginzburg models.

Compactify the torus to $\mathbb{P}^3$ in the standard way. Then the elements of the pencils that are given by f_{16}, f_{18}, and f_{22} are quartics in $\mathbb{P}^3$, and so have trivial canonical class. By Bertini's theorem, the general element of the pencil may have only base points as singularities. It is easy tho check that all base points of any pencil are Du Val, so the canonical class of minimal model of the general fiber of it is trivial. By [5, Theorem 5.1], the general fiber is birational to a K3 surface. □

Remark 3.2. The definition of weak Landau–Ginzburg model is numerical. So, it is natural that the polynomials from Theorem 3.1 are not unique. For instance, polynomials obtained from them by some coordinate changes or resizing $x \to \alpha x$, $y \to \beta y$, $z \to \gamma z$ are also weak Landau–Ginzburg models.

3.1. Singularities

The philosophy of mirror symmetry says that for every smooth variety there is a dual Landau–Ginzburg model, that is, the pencil of projective varieties, whose symplectic properties correspond to algebro-geometric properties of the variety and vice-versa. In particular, the sheaf of vanishing cycles on the element of the pencil "lying over 0" corresponds to the horizontal Hodge cohomologies of the variety and the elements with isolated singularities correspond to the bounded derived category of coherent sheaves on the variety. The fibers of weak Landau–Ginzburg models are non-compact. Consider a toric variety that is given by a Newton polytope of a weak Landau–Ginzburg model. Compactify these models (as lying in $\mathbb{T} \hookrightarrow \mathbb{P}^3$) and resolve singularities of the compactifications. Then the singularities of the fibers are[1]:

[1]Remind that the fiber of weak Landau–Ginzburg model f over infinity, which is the fiber over 0 in the standard coordinates, is given by $f = 0$.

f_{16}: the (reducible) curve of genus 3 over the infinity and two conjugate points defined over the quadratic extension of $\mathbb{Q}$.

f_{18}: the (reducible) curve of genus 2 over the infinity and two conjugate points defined over the quadratic extension of $\mathbb{Q}$.

f_{22}: three conjugate points defined over the extension of $\mathbb{Q}$ of degree 3.

The fibers of the pencils over the infinity in these cases coincide with the expectation. Namely, the genera of the schemes of singularities equal the dimensions of the intermediate Jacobians of the corresponding varieties. More about these models see in [15].

The images of singular points are the singular points of the differential operators of type $D3$ for these varieties.

Remark 3.3. The aim of this paper is to describe Landau–Ginzburg models for smooth Fano threefolds with Picard group $\mathbb{Z}$. For most of them (for complete intersections in projective or weighted projective varieties) they are known (see [14, 7.2]). The last cases that we have not found yet are V_{10}, V_{12}, and V_{14}.

4. Finding weak Landau–Ginzburg models

Unfortunately, finding weak Landau–Ginzburg models is a very complicated computational problem. In this section we discuss some (empiric) ways of simplifying it.

4.1. Canonical degenerations and numerical invariants

Theorem 4.1 (Kawamata [16])**.** *Let $\mathcal{X} \to S$ be a deformation (S is a germ of a curve). Suppose that X_{s_0} has canonical singularities. Then $\mathcal{X}$ has canonical singularities. In particular, for any $s \in S$ the fiber X_s is canonical.*

Corollary 4.2. *The total space $\mathcal{X}$ is $\mathbb{Q}$-Gorenstein. Thus, by adjunction, for any $s \in S$ we have $-K_{X_s} = -K_{\mathcal{X}}|_{X_s}$. In particular, the anticanonical degree $(-K_s)^{\dim X_s}$ does not depend on $s \in S$.*

Let $\mathcal{F}$ be a coherent sheaf on $\mathcal{X}$ which is flat over S. Then the Euler characteristic $\chi(X_s, \mathcal{F}_s)$ does not depend on $s \in S$ (see, for instance, [8, Proposition 3.8]). Let X_s be a canonical Fano variety for every $s \in S$, $s \neq s_0$ and X_{s_0} be a canonical almost Fano variety (that is, its anticanonical divisor is nef and big). By Kawamata–Viehweg vanishing theorem ([17, Theorem

2.17]), $\chi(X_s, -K_s) = h^0(X_s, -K_s)$. Thus, $h^0(X_s, -K_s)$ does not depend on $s \in S$.

Proposition 4.3. *Let $\pi : \mathcal{X} \to S$ be a deformation such that the general fiber is a Fano variety of Picard rank k and the special fiber X_{s_0} is irreducible, projective, and normal almost Fano variety. Let all fibers have canonical singularities. Let $\mathrm{Pic}(X_{s_0}) = \mathbb{Z}^m$. Then $m \leq k$.*

Proof (the idea is due to Ivo Radloff). We use here the Picard groups and cohomology groups with coefficients in $\mathbb{Q}$. Let $\Delta = \{t : |t - s_0| < \varepsilon\} \subset S$ be a small enough neighborhood of s_0 and $X = \pi^{-1}(\Delta)$. Then $H^2(X) = H^2(X_{s_0})$, as X_{s_0} is a deformation retract of X. By Kawamata–Viehweg vanishing theorem and an exponential exact sequence $H^2(X) \cong \mathrm{Pic}(X)$ and $H^2(X_{s_0}) \cong \mathrm{Pic}(X_{s_0})$ (X is a relative Fano variety). Thus, we need to show that there is no linear sheaf $\mathcal{L}$ such that the restriction $\mathcal{L}|_{X_s} \cong \mathcal{O}_{X_s}$ for $s \neq s_0$ and positive for $s = s_0$ (the numerical equivalence over $\mathbb{Q}$ is the same as the linear equivalence).

Suppose it is. By semicontinuity (see, for example, [8, Theorem 3.6]), there is a section of $\mathcal{L}|_{s_0}$. It is non-zero by assumption, so it is an effective divisor. Denote the dimension of the fibers by n. (We apply an intersection theory to sheaves as to the linear systems associated to them.) The special fiber is projective, so there is a divisor $\mathcal{D}$ on X whose restriction on the special fiber is ample. So, $\mathcal{D}^{n-1} \cdot \mathcal{L} \cdot X_{s_0} = (\mathcal{D}|_{X_{s_0}})^{n-1} \cdot \mathcal{L}|_{s_0} > 0$. The intersection number does not depend on the fiber as all fibers are numerically equivalent. The sheave $\mathcal{L}$ restricted to the general fiber is numerically trivial by assumption. Contradiction. $\square$

Corollary 4.4. *Let $\mathcal{X} \to S$ be a degeneration of Fano variety of Picard rank* 1 *to the toric canonical variety X_{s_0}. Then the anticanonical degree $(-K_{X_s})^{\dim X_s}$, $h^0(-K_{X_s})$, and the Picard rank of X_s do not depend on $s \in S$.*

Proof. It follows from Corollary 4.2, and discussion after it and Proposition 4.3. $\square$

4.2. Toric varieties and Laurent polynomials

Consider a torus $\mathbb{T} = \mathrm{Spec}\,\mathbb{C}[M] \cong \mathrm{Spec}\,(\mathbb{C}^*)^n$, where $M \cong \mathbb{Z}^n$. Let $x_1, \ldots, x_n$ be the coordinates on the one-dimensional tori. Put $x^m = x_1^{m_1} \cdot \ldots \cdot x_n^{m_n}$ for $m = (m_1, \ldots, m_n)$. Then any function f on $\mathbb{T}$ can be uniquely represented as $f = \sum_{m \in M} a_m x^m$. Let $\mathrm{Supp}(f) = \{m \in M, a_m \neq 0\}$. The convex hull of $\mathrm{Supp}(f)$ in $M_{\mathbb{R}} = M \otimes \mathbb{R}$ is called *the Newton polyhedra* of f.

Let X be a toric variety with an open subset $\mathbb{T}$. Let $N = \mathrm{Hom}\,(M, \mathbb{Z})$ be the dual to M lattice and $\langle \cdot, \cdot \rangle$ be a natural pairing. It is well known (see, for instance, [7]) that X is associated with *the fan* $\Sigma \subset N_{\mathbb{R}} = N \otimes \mathbb{R}$. The variety $X = X_\Sigma$ is covered by affine toric varieties (maps) $X_\sigma = \mathrm{Spec}\,\mathbb{C}[\sigma^\vee \cap M]$, where $\sigma^\vee = \{m \in M_{\mathbb{R}} | \, \langle m, v \rangle \geq 0, v \in \sigma\}$ are dual cones for one-dimensional cones $\sigma \in \Sigma$.

In the following we assume that all fans are *projective*, that is, they correspond to the projective varieties (this is equivalent to the existence of strictly convex function on the fan which is linear on all cones from Σ). All polytopes are supposed to be convex and to have the origin in the interior.

Let $L \cong \mathbb{Z}^n$ be any lattice, $L_{\mathbb{Q}} = L \otimes \mathbb{Q}$, $L_{\mathbb{R}} = L \otimes \mathbb{R}$, and $L^\vee$ be its dual. For any fan $\Sigma \in L$ we associate the polytope $P_\Sigma \in L_{\mathbb{R}}$ which is defined to be a convex hull of primitive vectors of its rays (that is, primitive vectors $v_i \in L$ that generate one-dimensional cones of Σ). Otherwise, given any polytope $P \in L_{\mathbb{Q}}$, we can construct a *normal fan* taking cones over its faces. Given a polytope in $L_{\mathbb{R}}$, define its dual as

$$P^\vee = \{m \in L_{\mathbb{R}}^\vee | \langle m, n \rangle \geq -1 \text{ for all } n \in P\}.$$

Obviously, if P have vertices in $L_{\mathbb{Q}}$, then $P^\vee$ have vertices in $L_{\mathbb{Q}}^\vee$. Therefore, given P we may construct a toric variety X_P that is given by a normal fan for $P^\vee$.[2] The polytope $P \subset L_{\mathbb{R}}$ with vertices in L is called *reflexive* if $P^\vee$ have vertices in $L^\vee$. The toric variety that corresponds to a reflexive polytope is a Gorenstein Fano with at most canonical singularities.

Let $\Sigma \subset N$ be a fan, $P = P_\Sigma \subset N_{\mathbb{R}}$ be the corresponding polytope and $P^\vee \subset M_{\mathbb{Q}}$ be its dual. The anticanonical divisor of X_Σ is the sum of boundary divisors $D_1, \ldots, D_r$ corresponding to rays given by primitive vectors $n_1, \ldots, n_r$. The point $m \in M$ is a rational function on X_Σ. Its divisor is $\sum \langle m, n_i \rangle D_i$. Let us consider $\mathbb{Q}$-divisors in the following. The element of $M_{\mathbb{Q}}$ determines a $\mathbb{Q}$-divisor by linearity. In particular, the Newton polyhedra $\Delta \in M$ of the function f lies in $P^\vee$ if and only if $\mathrm{div}(f) - K_{X_\Sigma} \in \mathrm{Pic}(X_\Sigma) \otimes \mathbb{Q}$ is effective (where $\mathrm{div}\,(f)$ is the divisor of f). Thus, functions whose Newton polytope lie in $P^\vee$ are the sections of the anticanonical sheaf, so $L(P^\vee)$ is naturally isomorphic to $|-K_{X_\Sigma}|$, where $L(P^\vee)$ is the space of Laurent polynomials with support in $P^\vee$.

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[2]In fact, the datum (P, φ), where $P \in L$ is an integral polytope containing origin and φ is a strictly convex integral piecewise linear function, is equivalent to the fan in $L^\vee$. This function we consider is the function φ such that $\varphi(n) = -1$ for all vertices of P (it is represented by the anticanonical class).

Let $P \subset N_{\mathbb{R}} \cong \mathbb{Z}^n \otimes \mathbb{R}$ be a polytope and X be the toric variety associated with the normal fan of P. Then X has canonical singularities if and only if P has the origin as the only integral point in the interior. The the anticanonical degree of X (that is, $(-K_X)^n$) is a volume of the dual polytope $P^\vee$ divided by $n!$. The dimension of the anticanonical linear system equals the number of integral points inside $P^\vee$ and on the boundary. The Picard number of X equals the dimension of the space of functions on $N_{\mathbb{R}}$ that are linear on every cone over the face of P modulo linear functions.

4.3. Strategy

So, the natural strategy for finding weak Landau–Ginzburg models of Fano threefold is the following. Consider a smooth Fano threefold X with Picard group $\mathbb{Z}$. Find the fundamental term $I^X_{H_0} = \sum a_r t^r$, $a_i \in \mathbb{Q}$, of its I-series (these series are known for all 17 families of such Fanos; see, for instance, [21, 4.4]). Suppose that X degenerates to a canonical Fano X_Σ and suppose that there exists a weak Landau–Ginzburg model f for X whose Newton polyhedra lies in P_Σ. Find it. For this find all integral polytopes with the origin as a unique integral point in the interior, whose numerical data is the same as the data of X. Consider any such polytope and the Laurent polynomial $f = \sum b_{ijk} x^i y^j z^k \in \mathbb{C}[x, x^{-1}, y, y^{-1}, z, z^{-1}][b_{ijk}]$, whose Newton polyhedra is our polytope. Let $\Phi_f = \sum b_r(b_{ijk}) t^r$ be its constant terms series. To specify the coefficients b_{ijk}, solve the system of equations $\{b_r(b_{ijk}) = a_r\}$, $r = 1, \ldots, N$, where $N \in \mathbb{N}$ is big enough; to avoid resizing, normalize x, y, and z such that b_{100}, b_{010} and b_{001} are 0 or 1. Prove that $\Phi_f = I^X_{H^0}$ for f we found. For this check that for all coefficients of Φ_f holds the same recurrence that holds for the coefficients of $I^X_{H^0}$. Finally, to prove that the general element of our pencil is birational to a Calabi–Yau variety it usually suffices to compactify the torus to the projective space, compactify the fibers therein, check the degree condition, use the Bertini theorem and check that the general hypersurface admits a crepant resolution.

To "legalize" this empiric strategy, one should solve two following problems.

Problem 4.5. Prove that any smooth Fano threefold with Picard group $\mathbb{Z}$ admits a degeneration to a canonical toric variety. Find all such degenerations. Characterize them. Generalize this to more general class of Fanos or to toric varieties with worse singularities.

This problem may be solved if the singularities of toric variety are terminal Gorenstein (that is, ordinary double points) (see [11]). Unfortunately, there are only five such varieties, that is, $\mathbb{P}^3$, three-dimensional quadric, complete intersection of two quadrics, and the varieties V_5 and V_{22}. It is remarkable that we do not need the particular form of degeneration (but in some cases, as for quadric or complete intersection of two quadrics we can find them; see [1]). Unfortunately, we cannot expect that any smooth Fano variety degenerates to a *Gorenstein* canonical toric Fano variety (that is, associated with a reflexive polytope). The example is V_2, the Fano variety of degree 2: there is no reflexive polytopes of volume 1/3.

Problem 4.6. Let the smooth Fano variety X degenerate to the canonical toric variety T. Prove that there is a weak Landau–Ginzburg model f for X with Newton polytope Δ and there is a fan Σ of T such that $P_\Sigma = \Delta^\vee$.

Good references for this problem are [1, 2]. In the paper [10] the first examples of weak Landau–Ginzburg mirrors for nontoric Fano varieties with Picard number 1 (Grassmannians) were suggested. In the paper [3] their relation to toric degenerations of Grassmannians was explained.

Unfortunately, this straightforward way is too complicated for computational reasons. Firstly, there are too many such polytopes. Secondly, there are too many integral points in any three-dimensional polytope, so there are too many variables in the system of equations. Thirdly, it is complicated to solve the system of polynomial equations.

To fix these problems we put some restrictions.

To fix the first problem we consider not all such polytopes, but some natural class of them (such as reflexive polytopes, polytopes with many symmetries, or polytopes that are contained in the cube $[-1,1]\times[-1,1]\times[-1,1]$) and hope that X degenerates to the toric variety associated with such polytope. As the degree of Fano variety increases, the polytope of its degeneration tends to become simpler.

To fix the second one we consider not all functions but functions of some type. Namely, we may consider only Laurent polynomials $f = f(x, y, z)$ which are symmetric under all permutations of x, y, z. We may also consider polynomials with coefficients 1 at the vertices of their Newton polytopes. Polynomials we found are of these types.

Finally, we hope that the coefficients of polynomials are integral. So, to solve the system of equations we solve it modulo some prime numbers, lift the solutions to $\mathbb{Z}$, and check if we did this correctly. Actually, we consider all possibilities for $b_{ijk} \bmod p$ and check if the equations hold for them.

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Steklov Institute of Mathematics
8 Gubkin Street
Moscow 119991
Russia
E-mail address: victorprz@gmail.com, victorprz@mi.ras.ru

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