COMMUNICATIONS IN
NUMBER THEORY AND PHYSICS
Volume 1, Number 3, 579–603, 2007

# On solutions to Walcher's extended holomorphic anomaly equation

YUKIKO KONISHI AND SATOSHI MINABE

We give a generalization of Yamaguchi–Yau's result to Walcher's extended holomorphic anomaly equation.

## 1. Introduction

Let $X$ be a non-singular quintic hypersurface in $\mathbb{CP}^4$. The case of the $X$ and its mirror is the most well-studied example of the mirror symmetry. After the construction of the mirror family of Calabi–Yau 3-folds [10], the genus 0 Gromov–Witten (GW) potential of $X$ were computed via the Yukawa coupling of the mirror family [4]. The predicted mirror formula was proved first by Givental [7].

For higher genera, Bershadsky–Cecotti–Ooguri–Vafa (BCOV) [2] has predicted that the GW potential at genus $g$ is obtained as a certain limit of the B-model closed topological string amplitude $\mathcal{F}^{(g)}$ of genus $g$.[1] They have also proposed a partial differential equation (PDE) for $\mathcal{F}^{(g)}$, called the BCOV holomorphic anomaly equation, which determines $\mathcal{F}^{(g)}$ up to a holomorphic function. The prediction of BCOV for the genus 1 GW potential was proved by Zinger [21].

Recently the open string analogue of the mirror symmetry has been developed by Walcher [18] for the pair $(X, L)$ of the quintic 3-fold $X$ defined over $\mathbb{R}$ (called a real quintic) and the set of real points $L = X(\mathbb{R})$ which is a Lagrangian submanifold of $X$. Open mirror symmetry gave the prediction for the generating function for the disk GW invariants of $X$ with boundary in $L$ and it was proved by Pandharipande–Solomon–Walcher [16]. Then, Walcher [19] further proposed the open string analogue of BCOV, the extended holomorphic anomaly equation, which is a PDE for the B-model topological string amplitude $\mathcal{F}^{(g,h)}$ for world-sheets with $g$ handles and $h$ boundaries.[2]

[1]For genus $g = 0$, the third covariant derivative of $\mathcal{F}^{(0)}$ is the Yukawa coupling, and for $g = 1$, it is recently proved that $\mathcal{F}^{(1)}$ is the Quillen's norm function [6]. For genus $g \geq 2$, the mathematical definition of $\mathcal{F}^{(g)}$ is yet to be known.

[2]There is also a proposal by Bonelli–Tanzini [3].

At present there are two ways to solve the BCOV holomorphic anomaly equation. The one is to repeatedly use the identity called the special geometry relation, or equivalently to draw Feynman diagrams associated to the perturbative expansion of a certain path integral [2]. The other is to solve the system of PDE's due to Yamaguchi–Yau [20]. They showed that $\mathcal{F}^{(g)}$ multiplied by $(g-1)$th powers of the Yukawa coupling, is a polynomial in finite number of generators and rewrite the holomorphic anomaly equation as PDE's with respect to these generators. This result were then reformulated into a more useful form by Hosono–Konishi [12, § 3.4].

It is a natural problem to generalize these methods to Walcher's extended holomorphic anomaly equation. The generalization of the Feynman rule method can be obtained from the result of Cook–Ooguri–Yang [5]. The objective of this article is to generalize Yamaguchi–Yau's and Hosono–Konishi's results to the extended holomorphic anomaly equation. It gives a more tractable method in computations than the one given by the Feynman rule.

The organization of the paper is as follows. In Section 2, we recall the special Kähler geometry of the B-model complex moduli space and Walcher's extended holomorphic anomaly equation. We also describe the Feynman rule. In Section 3, we rewrite the holomorphic anomaly equation as PDE's (Theorem 3.8). Section 4 is devoted to discussion on holomorphic ambiguities. We tried to fix them with certain *ad hoc* assumptions in several cases. In appendices, we include the Feynman diagrams and the solution of the PDE's for $(g,h)=(0,4)$.

After we finished writing this paper, we were informed that Alim–Länge [1] also obtained a generalization of Yamaguchi–Yau's result.

## 2. Walcher's extended holomorphic anomaly equation

### 2.1. Special Kähler geometry

Recall the mirror family of the quintic hypersurface $X \subset \mathbb{P}^4$ constructed in [10]. Let $W_\psi$ be the hypersurface in $\mathbb{P}^4$ defined by

$$\sum_{i=0}^{4} x_i^5 - 5\psi \prod_{i=0}^{4} x_i = 0.$$

After taking the quotient by $(\mathbb{Z}/5\mathbb{Z})^3$ and a crepant resolution $Y_\psi$ of $W_\psi/(\mathbb{Z}/5\mathbb{Z})^3$, we obtain a one-parameter family of Calabi–Yau 3-folds

$\pi : \mathcal{Y} \to \mathcal{M}_{\mathrm{cpl}} := \mathbb{P}^1 \setminus \{0, \frac{1}{5^5}, \infty\}$, where a local coordinate $z$ of $\mathcal{M}_{\mathrm{cpl}}$ is given by $z = (5\psi)^{-5}$.

Consider the variation of Hodge structure of weight 3 on the middle cohomology groups $H^3(Y_z, \mathbb{C})$. Let $0 \subset F^3 \subset F^2 \subset F^1 \subset F^0 = R^3\pi_*\mathbb{C} \otimes \mathcal{O}_{\mathcal{M}_{\mathrm{cpl}}}$ be the Hodge filtration and $\nabla$ be the Gauss–Manin connection. The holomorphic line bundle $\mathcal{L} := F^3$ over $\mathcal{M}_{\mathrm{cpl}}$ is called the vacuum line bundle (the fiber of $\mathcal{L}$ at $z$ is $H^{3,0}(Y_z)$). Let $\Omega(z)$ be a local holomorphic section trivializing $\mathcal{L}$, i.e., a nowhere vanishing $(3,0)$-form on $Y_z$. The Yukawa coupling $C_{zzz}$ is define by

$$C_{zzz} := \int_{X_z} \Omega(z) \wedge (\nabla_{\partial_z})^3 \Omega(z),$$

which is a holomorphic section of $\mathrm{Sym}^3(T^*_{\mathcal{M}_{\mathrm{cpl}}}) \otimes (\mathcal{L}^*)^2$, where $T^*_{\mathcal{M}_{\mathrm{cpl}}}$ denotes the holomorphic cotangent bundle of $\mathcal{M}_{\mathrm{cpl}}$. A suitable choice of $\Omega(z)$ gives (see [4])

$$C_{zzz} = \frac{5}{(1 - 5^5 z) z^3}.$$

It also gives the following Picard–Fuchs operator $\mathcal{D}$ which governs the periods of $\Omega(z)$:

$$\mathcal{D} = \theta_z^4 - 5z(5\theta_z + 1)(5\theta_z + 2)(5\theta_z + 3)(5\theta_z + 4),$$

where $\theta_z = z\frac{d}{dz}$.

Consider the pairing

$$(\phi, \psi) := \sqrt{-1} \int_{Y_z} \phi \wedge \psi, \quad \phi, \psi \in H^3(Y_z, \mathbb{C}).$$

Then $(\ ,\ \overline{\ })$ induces a Hermitian metric on $\mathcal{L}$. Let $K(z, \bar{z}) := -\log(\Omega(z), \overline{\Omega(z)})$. This defines a Kähler metric (the Weil–Peterson metric) $G_{z\bar{z}} := \partial_z \partial_{\bar{z}} K$ on $\mathcal{M}_{\mathrm{cpl}}$. There is a unique holomorphic Hermitian connection $D$ on $(T_{\mathcal{M}_{\mathrm{cpl}}})^m \otimes \mathcal{L}^n$ whose $(1,0)$-part $D_z$ is given by

$$D_z = \partial_z + m\Gamma^z_{zz} + n(-\partial_z K),$$

where $\Gamma^z_{zz} = G^{z\bar{z}} \partial_z G_{z\bar{z}}$. An important property of $G_{z\bar{z}}$ is the following identity called the special geometry relation [17]

$$\partial_{\bar{z}} \Gamma^z_{zz} = 2G_{z\bar{z}} - C_{zzz} C_{\bar{z}\bar{z}\bar{z}} e^{2K} G^{z\bar{z}} G^{z\bar{z}}, \tag{2.1}$$

where $C_{\bar{z}\bar{z}\bar{z}} := \overline{C_{zzz}}$.

Now we introduce the open disk amplitude with two insertions $\triangle_{zz}$, which is the open sector analogue of the Yukawa coupling. Let $\mathcal{T}$ be a holomorphic section of $\mathcal{L}^*$ locally given by

$$\mathcal{T} = 60\ \tau(z), \quad \tau(z) = \sum_{n=0}^{\infty} \frac{(7/2)_{5n}}{{(3/2)_n}^5} z^{n+\frac{1}{2}}. \tag{2.2}$$

Here $(\alpha)_n$ is the Pochhammer symbol: $(\alpha)_n := \alpha(\alpha+1)\cdots(\alpha+n-1)$ for $n > 0$ and $(\alpha)_0 := 1$. $\mathcal{T}$ is a solution to

$$\mathcal{D}\mathcal{T} = \frac{60}{2^4}\sqrt{z}. \tag{2.3}$$

Following [19], we define a $C^\infty$-section $\triangle_{zz}$ of $\mathrm{Sym}^2(T^*_{\mathcal{M}_{\mathrm{cpl}}}) \otimes \mathcal{L}^*$ by

$$\triangle_{zz} = D_z D_z \mathcal{T} - \frac{e^K C_{zzz}}{G_{z\bar{z}}} \overline{D}_{\bar{z}} \overline{\mathcal{T}}\ , \tag{2.4}$$

where $\overline{D}_{\bar{z}} = \partial_{\bar{z}} + \partial_{\bar{z}} K$ denotes the $(0,1)$-part of $\overline{D}$. By (2.1), it follows that $\triangle_{zz}$ satisfies the equation

$$\partial_{\bar{z}} \triangle_{zz} = -C_{zzz} e^K G^{z\bar{z}} \triangle_{\bar{z}\bar{z}}, \tag{2.5}$$

where $\triangle_{\bar{z}\bar{z}} := \overline{\triangle_{zz}}$.

**Remark 2.1.** In [19], it is argued that $\mathcal{T}$ and $\triangle_{zz}$ should be written as

$$\mathcal{T}(z) = \int_{Y_z} \Omega(z) \wedge \tilde{\nu}(z), \quad \triangle_{zz} = \int_{Y_z} \Omega(z) \wedge (\nabla_{\partial_z})^2 \tilde{\nu}(z),$$

where $\tilde{\nu}$ is a $C^\infty$-section of the Hodge bundle $F^0$ which is the "real horizontal lift" of a certain Griffiths normal function $\nu$ associated to a family of homologically trivial 2-cycles.[3] The normal function $\nu$ should be determined from the Lagrangian submanifold $L \subset X$ under the mirror symmetry with D-branes.

[3]By definition, $\nu$ is a holomorphic and horizontal section of the intermediate Jacobian fibration $\mathcal{J}^3 \to \mathcal{M}_{\mathrm{cpl}}$ of $\mathcal{Y} \to \mathcal{M}_{\mathrm{cpl}}$. See, e.g., [9, 11].

### 2.2. Extended holomorphic anomaly equation

Let $\mathcal{F}^{(g,h)}$ be the B-model topological string amplitude of genus $g$ with $h$ boundaries, and let

$$\mathcal{F}_0^{(g,h)} := \mathcal{F}^{(g,h)}, \quad \mathcal{F}_n^{(g,h)} := D_z \mathcal{F}_{n-1}^{(g,h)} \ (n \geq 1).$$

$\mathcal{F}_n^{(g,h)}$ is a $C^\infty$-section of the line bundle $(T^*_{\mathcal{M}_{\mathrm{cpl}}})^n \otimes \mathcal{L}^{2g-2+h}$. For $(g,h) = (0,0), (0,1)$,

$$\mathcal{F}_3^{(0,0)} = C_{zzz}, \quad \mathcal{F}_2^{(0,1)} = \triangle_{zz}. \tag{2.6}$$

For $(g,h) = (1,0), (0,2)$,[4]

(2.7)
$$\begin{aligned}
\mathcal{F}_1^{(1,0)} &= \frac{1}{2}\partial_z \log\Big(e^{\left(4-\frac{\chi}{12}\right)K} {G_{z\bar{z}}}^{-1}(1-5^5 z)^{-\frac{1}{6}} z^{-1-\frac{c_2.H}{12}}\Big), \\
\mathcal{F}_1^{(0,2)} &= -\triangle_{zz}\triangle^z - \frac{1}{2}C_{zzz}\triangle^z\triangle^z + \frac{N}{2}\partial_z K + f^{(0,2)}, \quad f^{(0,2)} = \frac{75}{2(1-5^5 z)},
\end{aligned}$$

where $\chi = -200$, $c_2 \cdot H = 50$, $N = 1$ and $\triangle^z = -\frac{\triangle_{zz}}{C_{zzz}}$ (cf. § 2.3).

As in [19], define

$$C_{\bar{z}}^{zz} = C_{\bar{z}\bar{z}\bar{z}} e^{2K} {G_{z\bar{z}}}^{-2}, \quad \triangle_{\bar{z}}^{z} = \triangle_{\bar{z}\bar{z}} e^{K} {G_{z\bar{z}}}^{-1}.$$

Then Walcher's extended holomorphic anomaly equation for $(g,h) \neq (0,0)$, $(1,0), (0,1), (0,2)$ is as follows.

(2.8)
$$\partial_{\bar{z}} \mathcal{F}^{(g,h)} = \frac{1}{2} C_{\bar{z}}^{zz} \left( \sum_{g_1,g_2,h_1,h_2} \mathcal{F}_1^{(g_1,h_1)} \mathcal{F}_1^{(g_2,h_2)} + \mathcal{F}_2^{(g-1,h)} \right) - \triangle_{\bar{z}}^{z} \mathcal{F}_1^{(g,h-1)} .$$

In the RHS, the summation is over $g_1, h_1, g_2, h_2 \geq 0$ satisfying $g_1 + g_2 = g$, $h_1 + h_2 = h$ and $(g_1,h_1), (g_2,h_2) \neq (0,0), (0,1)$. The second and the third terms in the RHS should be set to zero if $g = 0$ and $h = 0$, respectively.

---

[4] $\mathcal{F}_1^{(1,0)}$ and $\mathcal{F}_1^{(0,2)}$ are solutions to the following (extended) holomorphic anomaly equations [2, 19].

$$\partial_{\bar{z}} \mathcal{F}_1^{(1,0)} = \frac{1}{2} C_{zzz} C_{\bar{z}}^{zz} - \left(\frac{\chi}{24} - 1\right) G_{z\bar{z}}, \quad \partial_{\bar{z}} \mathcal{F}_1^{(0,2)} = -\triangle_{zz}\triangle_{\bar{z}}^{z} + \frac{N}{2} G_{z\bar{z}}.$$

### 2.3. Propagators and terminators

We introduce the propagators $S^{zz}, S^z, S$ and the terminators $\triangle^z, \triangle$ [2, 19]. By definition, they are solutions to

$$\begin{aligned}\partial_{\bar{z}} S^{zz} = C_{\bar{z}}^{zz}, \quad \partial_{\bar{z}} S^z = S^{zz} G_{z\bar{z}}, \quad \partial_{\bar{z}} S = S^z G_{z\bar{z}},\\ \partial_{\bar{z}} \triangle^z = \triangle_{\bar{z}}^z, \quad \partial_{\bar{z}} \triangle = \triangle^z G_{z\bar{z}}.\end{aligned} \tag{2.9}$$

These equation can be solved by using (2.1) and (2.5). The solutions of the propagators are [2, p. 391]:

$$\begin{aligned}
S^{zz} &= \frac{1}{C_{zzz}}\big(2\partial_z \log(e^K|f|^2) - \partial_z \log(vG_{z\bar{z}})\big),\\
S^z &= \frac{1}{C_{zzz}}\big((\partial_z \log(e^K|f|^2)^2 - v^{-1}\partial_z v \partial_z \log(e^K|f|^2)\big),\\
S &= \left(S^z - \frac{1}{2}D_z S^{zz} - \frac{1}{2}(S^{zz})^2 C_{zzz}\right)\partial_z \log(e^K|f|^2) + \frac{1}{2}D_z S^z\\
&\quad + \frac{1}{2}S^{zz}S^z C_{zzz}.
\end{aligned} \tag{2.10}$$

Here $f, v$ are holomorphic functions of $z$. We take $f = z^{-1/5}$ and $v = \frac{dz}{d\psi}$ ($z = \frac{1}{5^5\psi^5}$), so that $S^{zz}, S^z, S$ do not diverge at $z = \infty$.[5] Solutions of the terminators are [19, (3.12)]

$$\triangle^z = -\frac{\triangle_{zz}}{C_{zzz}}, \quad \triangle = D_z \triangle^z. \tag{2.11}$$

### 2.4. Feynman Rule

We describe the Feynman rule which gives a solution to (2.8).

For non-negative integers $g$,$h$,$m$ and $n$, we define $\widetilde{C}_{n:m}^{(g,h)}$ recursively as follows.

$$\widetilde{C}_{0:m}^{(0,0)} = \widetilde{C}_{1:m}^{(0,0)} = \widetilde{C}_{2:m}^{(0,0)} = 0, \tag{2.12}$$

$$\widetilde{C}_{0:m}^{(0,1)} = \widetilde{C}_{1:m}^{(0,1)} = 0, \tag{2.13}$$

$$\widetilde{C}_{0:1}^{(0,2)} = -\frac{N}{2}, \tag{2.14}$$

[5] If rewritten in the $\psi$-coordinate, (2.10) are the same as those used in [19, 3.11; 20, (2.21)].

$$\widetilde{C}_{0:0}^{(1,0)} = 0, \quad \widetilde{C}_{0:1}^{(1,0)} = \frac{\chi}{24} - 1, \tag{2.15}$$

$$\widetilde{C}_{n:0}^{(g,h)} = \mathcal{F}_n^{(g,h)} \quad \text{if} \quad 2g - 2 + h + n \geq 1, \tag{2.16}$$

$$\widetilde{C}_{n:m+1}^{(g,h)} = (2g - 2 + h + n + m)\widetilde{C}_{n:m}^{(g,h)}. \tag{2.17}$$

**Definition 2.2.** A Feynman diagram $G$ is a finite labeled graph

$$G = (V; E_0^{\text{in}}, E_1^{\text{in}}, E_2^{\text{in}}, E_0^{\text{out}}, E_1^{\text{out}}; j),$$

which consists of the following data.

(i) Each vertex $v \in V$ is labeled by a pair of non-negative integers $(g_v, h_v)$.

(ii) There are three kinds of inner edges $E^{\text{in}} = E_0^{\text{in}} \sqcup E_1^{\text{in}} \sqcup E_2^{\text{in}}$ and two kinds of outer edges $E^{\text{out}} = E_0^{\text{out}} \sqcup E_1^{\text{out}}$. The end points of the edges are specified by the collection of maps $j = (j_0^{\text{in}}, j_1^{\text{in}}, j_2^{\text{in}}, j_0^{\text{out}}, j_1^{\text{out}})$:

$$j_0^{\text{in}} : E_0^{\text{in}} \to (V \times V)/\sigma, \quad j_1^{\text{in}} : E_1^{\text{in}} \to V \times V, \quad j_2^{\text{in}} : E_2^{\text{in}} \to (V \times V)/\sigma,$$
$$j_0^{\text{out}} : E_0^{\text{out}} \to V, \quad j_1^{\text{out}} : E_0^{\text{out}} \to V,$$

where $\sigma : V \times V \to V \times V$ is the involution interchanging the first and the second factors.

In a more plain language, an edge of type $E_i^{\text{in}}$ has both endpoints attached to vertices, and an edge of type $E_i^{\text{out}}$ has only one endpoint attached to a vertex. We represent edges of types $E_0^{\text{in}}$ and $E_0^{\text{out}}$ by solid lines, edges of types $E_2^{\text{in}}$ and $E_1^{\text{out}}$ by dashed lines and an edge of type $E_1^{\text{in}}$ by a half-solid, half-dashed line. See figure 1.

For a vertex $v \in V$, we set

$$L_{i,v} = \{e \in E_i^{\text{in}} \mid j_i^{\text{in}}(e) = \{v, v\}\}, \quad L_i = \bigsqcup_{v \in V} L_{i,v}, \ \ (i = 0, 2),$$
$$L_{1,v} = \{e \in E_1^{\text{in}} \mid j_1^{\text{in}}(e) = (v, v)\}.$$

In other words, $L_{i,v}$ is the set of self-loops attached to the vertex $v$ whose edges are of the type $E_i^{\text{in}}$. Define non-negative integers $n_v^{\text{in}}$, $n_v^{\text{out}}$, $m_v^{\text{in}}$ and

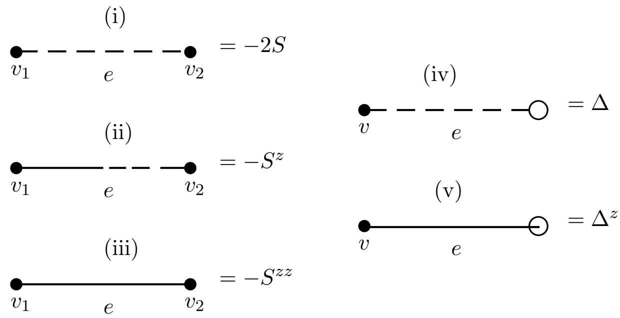


Figure 1: Three types of inner edges and propagators: (i) $e \in E_0^{\text{in}}$, $j_0^{\text{in}}(e) = \{v_1, v_2\}$, (ii) $e \in E_1^{\text{in}}$, $j_1^{\text{in}}(e) = (v_1, v_2)$, (iii) $e \in E_2^{\text{in}}$, $j_2^{\text{in}}(e) = \{v_1, v_2\}$. Two types of outer edges and terminators: (iv) $e \in E_0^{\text{out}}$, $j_0^{\text{out}}(e) = v$, (v) $e \in E_1^{\text{out}}$, $j_1^{\text{out}}(e) = v$.

$m_v^{\text{out}}$ by

$$\begin{aligned}
n_v^{\text{in}} &= \#\{e \in E_2^{\text{in}} \mid v \in j_2^{\text{in}}(e)\} + \#\{e \in E_1^{\text{in}} \mid j_1^{\text{in}}(e) = (v, *)\} \\
&\quad + \#L_{2,v}, \\
m_v^{\text{in}} &= \#\{e \in E_0^{\text{in}} \mid v \in j_0^{\text{in}}(e)\} + \#\{e \in E_1^{\text{in}} \mid j_1^{\text{in}}(e) = (*, v)\} \\
&\quad + \#L_{0,v}, \\
n_v^{\text{out}} &= \#\{e \in E_1^{\text{out}} \mid v \in j_1^{\text{out}}(e)\}, \quad m_v^{\text{out}} = \#\{e \in E_0^{\text{out}} \mid v \in j_0^{\text{out}}(e)\}.
\end{aligned}$$

The valence $\text{val}(v)$ of $v \in V$ is given by $\text{val}(v) = n_v + m_v$, where $n_v := n_v^{\text{in}} + n_v^{\text{out}}$ (the number of solid lines attached to $v$), $m_v := m_v^{\text{in}} + m_v^{\text{out}}$ (the number of dashed lines attached to $v$). See figure 2.

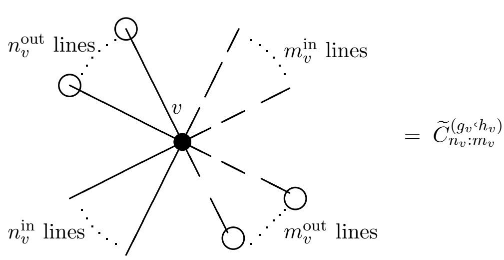


Figure 2: A vertex $v$ labeled by $(g_v, h_v)$ to which $n_v = n_v^{\text{in}} + n_v^{\text{out}}$ solid lines and $m_v = m_v^{\text{in}} + m_v^{\text{out}}$ dashed lines are attached and its value.

**Definition 2.3.** (i) For a Feynman diagram $G$, define

(2.18)
$$F_G = \prod_{v\in V} \widetilde{C}^{(g_v,h_v)}_{n_v:m_v} \cdot \prod_{e\in E_0^{\mathrm{in}}} (-2S) \cdot \prod_{e\in E_1^{\mathrm{in}}} (-S^z) \cdot \prod_{e\in E_2^{\mathrm{in}}} (-S^{zz}) \cdot \prod_{e\in E_0^{\mathrm{out}}} \Delta \cdot \prod_{e\in E_1^{\mathrm{out}}} \Delta^z.$$

(ii) Let $\mathrm{Aut}(G)$ be the automorphism group of $G$. Define the group $A_G$ by

$$A_G = \prod_{e\in L_0\sqcup L_2} \mathbb{Z}/2\mathbb{Z} \ltimes \mathrm{Aut}(G),$$

i.e., $A_G$ fits into the following exact sequence:

$$1 \to (\mathbb{Z}/2\mathbb{Z})^{\#L_0+\#L_2} \to A_G \to \mathrm{Aut}(G) \to 1.$$

This means that each self-loop of type $E_0^{\mathrm{in}}$ and $E_2^{\mathrm{in}}$ contributes the factor 2 to $\#A_G$.

**Definition 2.4.** Let $\mathbb{G}(g,h)$ be the set of (isomorphism classes of) Feynman diagrams $G$ which satisfy the following conditions.

(i) $G$ is connected.

(ii) For any $v \in V$, $\widetilde{C}^{(g_v,h_v)}_{n_v:m_v} \neq 0$.

(iii) $G$ satisfies $\sum_{v\in V} g_v + \#E^{\mathrm{in}} - \#V + 1 = g$ and $\sum_{v\in V} h_v + \#E^{\mathrm{out}} = h$.

(iv) For any $v \in V$, $\mathrm{val}(v) > 0$.

Note that the set $\mathbb{G}(g,h)$ is a finite set. Note also that the graph whose amplitude is $\mathcal{F}^{(g,h)}$, i.e., the graph with only one vertex with label $(g,h)$ and without edges is not a member of $\mathbb{G}(g,h)$ by (iv).

Define

$$\mathcal{F}^{(g,h)}_{\mathrm{FD}} := -\sum_{G\in\mathbb{G}(g,h)} \frac{1}{\#A_G} F_G. \tag{2.19}$$

The next result follows from [5].

**Proposition 2.5.** *$\partial_{\bar{z}}\mathcal{F}^{(g,h)}_{\mathrm{FD}} =$ the RHS of* (2.8).

Therefore, the general solution $\mathcal{F}^{(g,h)}$ of Walcher's holomorphic anomaly equation is of the form

$$\mathcal{F}^{(g,h)} = \mathcal{F}^{(g,h)}_{\mathrm{FD}} + f^{(g,h)}, \tag{2.20}$$

where $f^{(g,h)}$ is the holomorphic ambiguity which cannot be determined from the equation (2.8).

### 2.5. Holomorphic ambiguity

Recall that the holomorphic ambiguity $f^{(g,0)}$ $(g \geq 2)$ for the closed sector $h = 0$ is of the form [2; 20, (2.30)]

$$f^{(g,0)} = \frac{a_0 + a_1 z + \cdots + a_{2g-1} z^{2g-1}}{(1-5^5 z)^{2g-2}} + \sum_{i=0}^{\lfloor \frac{2g-2}{5} \rfloor} z^j.$$

Huang–Klemm–Quackenbush [13] determined $f^{(g,0)}$ up to $g \leq 51$ by using the vanishing of the BPS numbers $n^g_d$ (cf. footnote 7), the gap condition at the conifold point $z = \frac{1}{5^5}$ and the regularity condition at the orbifold point $z = \infty$.

For $h > 0$, we assume that $\mathcal{F}^{(g,h)}$ has poles of order at most $2g-2+h$ at $z = \frac{1}{5^5}$ and also that the asymptotic behavior at $z = \infty$ is $F^{(g,h)} \sim z^{\frac{2g-2+h}{2}}$ [19, § 3.3]. Therefore we put the following ansatz for $f^{(g,h)}$:

$$\begin{aligned} f^{(g,h)} &= \frac{a_0 + a_1 z + \cdots + a_{3g-3+\frac{3h}{2}} z^{3g-3+\frac{3h}{2}}}{(1-5^5 z)^{2g-2+h}} \quad (h \text{ even}), \\ f^{(g,h)} &= \frac{\sqrt{z}\left(a_0 + a_1 z + \cdots + a_{3g-3+\frac{3h-1}{2}} z^{3g-3+\frac{3h-1}{2}}\right)}{(1-5^5 z)^{2g-2+h}} \quad (h \text{ odd}). \end{aligned} \tag{2.21}$$

## 3. Polynomiality and PDE's for $\mathcal{F}^{(g,h)}$

In this section, we consider extending Yamaguchi–Yau's and Hosono–Konishi's results [20, 12] to $\mathcal{F}^{(g,h)}$.

### 3.1. The generators of polynomial ring

Let $\theta_z = z\frac{\partial}{\partial z}$. We define

$$
\begin{aligned}
& A_p = \frac{(\theta_z)^p G_{z\bar{z}}}{G_{z\bar{z}}}, \quad B_p = \frac{(\theta_z)^p e^{-K}}{e^{-K}} \quad (p = 1, 2, \ldots), \\
& Q_p = z^{1/2}(\theta_z)^p \mathcal{T} \quad (p = 0, 1, 2, \ldots), \\
& R_1 = z^{5/2}\frac{e^K C_{zzz}}{G_{z\bar{z}}}\overline{D}_{\bar{z}}\overline{\mathcal{T}}, \quad R_2 = z^{7/2}e^K C_{zzz}\overline{\mathcal{T}}.
\end{aligned} \tag{3.1}
$$

The generators $A_p$'s and $B_p$'s were defined in [20]. The new ingredients are $Q_p$'s, $R_1$ and $R_2$ which are necessary for incorporating $\triangle_{zz}$.

Consider the polynomial ring

$$
I = \mathbb{C}(z)[A_1, B_1, B_2, B_3, Q_0, Q_1, Q_2, Q_3, R_1, R_2] \tag{3.2}
$$

with coefficients in the field of rational functions $\mathbb{C}(z)$.

**Lemma 3.1.** 1. *$A_p \in I$ $(p \geq 2)$, $B_p \in I$ $(p \geq 4)$, $Q_p \in I$ $(p \geq 4)$.*
*2. $\theta_z I \subseteq I$*

*Proof.* First, notice that the logarithmic derivation $\theta_z$ acts as follows:

$$
\begin{aligned}
& \theta_z A_p = A_{p+1} - A_p A_1, \quad \theta_z B_p = B_{p+1} - B_p B_1, \\
& \theta_z Q_p = \frac{1}{2}Q_p + Q_{p+1}, \\
& \theta_z R_1 = \Big(\frac{5}{2} - A_1 - B_1 + \frac{\theta_z C_{zzz}}{C_{zzz}}\Big)R_1 + R_2, \\
& \theta_z R_2 = \Big(\frac{7}{2} - B_1 + \frac{\theta_z C_{zzz}}{C_{zzz}}\Big)R_2.
\end{aligned} \tag{3.3}
$$

Next we show $A_2, B_4, Q_4 \in I$. By the special geometry relation (2.1), we have

$$
A_2 = -2A_1B_1 + 2B_1^2 + 2B_1 - 4B_2 + \frac{\theta_z(zC_{zzz})}{zC_{zzz}}(1 + A_1 + 2B_1) + h(z). \tag{3.4}
$$

Here $h(z)$ is determined by comparing the behavior of the RHS and the LHS at $z = 0$:

$$
h(z) = \frac{1 - 3 \cdot 5^4 z}{1 - 5^5 z}.
$$

Let us write the Picard–Fuchs operator as $\mathcal{D} = \sum_{p=0}^{4} H_p(z)\theta_z{}^p$, where $H_p(z) \in \mathbb{C}[z]$. Since $\mathcal{D}e^{-K} = 0$, $B_4$ satisfies

$$(3.5) \qquad B_4 = -\sum_{p=1}^{3} \frac{H_p(z)}{H_4(z)} B_p - \frac{H_0(z)}{H_4(z)} = 0.$$

Moreover, since $\mathcal{T}$ satisfies (2.3),

$$(3.6) \qquad Q_4 = -\sum_{p=0}^{3} \frac{H_p(z)}{H_4(z)} Q_p + \frac{60}{2^4} z.$$

These together with (3.3) implies that $I$ is closed with respect to the logarithmic derivation $\theta_z$. Moreover, by applying $\theta_z$ recursively, we can show that $A_p \in I$ $(p \geq 3)$, $B_p \in I$ $(p \geq 5)$, $Q_p \in I$ $(p \geq 5)$. $\square$

### 3.2. Polynomiality

For simplicity, we will use the notation

$$(3.7) \qquad V_1 = A_1 + 2B_1 + 1, \quad V_2 = B_2 - B_1 V_1$$

from here on.

Since $D_z$ acts on $(T_{\mathcal{M}_{\mathrm{cpl}}})^m \otimes \mathcal{L}^n$ as

$$D_z = \frac{1}{z}(\theta_z + mA_1 + nB_1),$$

we have the following

**Lemma 3.2.** *Let $f$ be a section of $(T_{\mathcal{M}_{\mathrm{cpl}}})^m \otimes \mathcal{L}^n$. Then $D_z f \in I$, if $f \in I$ and $D_z f \in \sqrt{z} I$ if $f \in \sqrt{z} I$.*

**Lemma 3.3.** $\mathcal{F}_n^{(g,h)} \in z^{h/2}\ I$.

*Proof.* We prove the lemma by induction on $(g,h)$.

For $(g,h) = (0,0),(1,0),(0,1),(0,2)$, the lemma is true since

$$
\begin{aligned}
\mathcal{F}_3^{(0,0)} &= C_{zzz} \in I,\\
\mathcal{F}_1^{(1,0)} &= \frac{1}{2z}\Big[-A_1 - \frac{62}{3}B_1 - \frac{31}{6} - \frac{1}{6}\frac{\theta_z(1-5^5 z)}{(1-5^5 z)}\Big] \in I,\\
\mathcal{F}_2^{(0,1)} &= \triangle_{zz} = z^{-5/2}[Q_2 - V_1 Q_1 - V_2 Q_0 - R_1] \in \sqrt{z}\, I,\\
\mathcal{F}_1^{(0,2)} &= \frac{1\triangle_{zz}}{2C_{zzz}} - \frac{B_1}{2z} + f^{(0,2)} \in I.
\end{aligned} \tag{3.8}
$$

For $(g,h) \neq (0,0),(1,0),(0,1),(0,2)$, assume that $\mathcal{F}_n^{(g',h')} \in z^{h'/2} I$ holds for every $(g',h') \neq (g,h)$ such that $g' \leq g$ and $h' \leq h$. Consider the contribution $F_G$ from a Feynman diagram $G \in \mathbb{G}(g,h)$ to $\mathcal{F}_{\mathrm{FD}}^{(g,h)}$ (2.18). The assumption of the induction implies that a vertex factor satisfies $\widetilde{C}_{n_v;m_v}^{(g_v,h_v)} \in z^{h_v/2} I$. As for edge factors, the followings hold. From (2.10),

$$
S^{zz} = \frac{1}{zC_{zzz}}\left(-A_1 - 2B_1 - \frac{8}{5}\right) \in I, \quad S^{z} = \frac{1}{z^2 C_{zzz}}\left(B_2 + 3B_1 + \frac{2}{25}\right) \in I. \tag{3.9}
$$

By Lemma 3.2, $S$ also satisfies $S \in I$. Similarly by (3.8) the terminators (2.11) satisfy

$$
\triangle^z, \triangle \in \sqrt{z} I.
$$

Therefore, by the condition (iii) in Definition 2.4, we have $F_G \in z^{h/2} I$ and thus $\mathcal{F}_{\mathrm{FD}}^{(g,h)} \in z^{h/2} I$. As to the holomorphic ambiguity $f^{(g,h)}$, it satisfies $f^{(g,h)} \in z^{h/2}\mathbb{C}(z) \subset z^{h/2}\, I$ by assumption (2.21). Therefore $\mathcal{F}^{(g,h)} \in z^{h/2}\, I$. For $n \geq 1$, $\mathcal{F}_n^{(g,h)} \in z^{h/2}\, I$ by Lemma 3.2. □

**Definition 3.4.** Let $g,h,n \geq 0$ be integers satisfying $2g-2+h+n > 0$. We define

$$
P_n^{(g,h)} = (z^3 C_{zzz})^{g+h-1} z^{h/2} \mathcal{F}_n^{(g,h)}, \quad P^{(g,h)} := P_0^{(g,n)}. \tag{3.10}
$$

For other values of $(g,h,n)$, we set $P_n^{(g,h)} = 0$.

Lemma 3.3 implies that

$$
P_n^{(g,h)} \in I.
$$

**Remark 3.5.** Let $x = z^3 C_{zzz} = \frac{5}{1-5^5 z}$. Consider the graded ring

$$\mathbb{C}[x, A_1, B_1, B_2, B_3, Q_0, \ldots, Q_3, R_1, R_2],$$

where the grading is given by $\deg x = 1$, $\deg A_1 = 1$, $\deg B_p = p$ $(p = 1, 2, 3)$, $\deg Q_p = p$ $(p = 0, 1, 2, 3)$, $\deg R_1 = 2$ and $\deg R_2 = 3$. Then $P^{(g,h)}$ belongs to this ring and its degree is at most $3(g + h - 1)$.

### 3.3. Rewriting the extended holomorphic anomaly equation (2.8)

There are relations among the $\partial_{\bar{z}}$-derivatives of the generators (3.1).

**Lemma 3.6.**

$$\begin{aligned} \partial_{\bar{z}} B_2 &= V_1 \partial_{\bar{z}} B_1, \\ \partial_{\bar{z}} B_3 &= (A_2 + 2A_1 + 3B_1 + 3B_2 + 3A_1 B_1 + 1) \partial_{\bar{z}} B_1 \\ &= \Big( - V_2 + \frac{\theta_z(z^3 C_{zzz})}{z^3 C_{zzz}} V_1 + h(z) - 1 \Big) \partial_{\bar{z}} B_1 \\ \partial_{\bar{z}} Q_p &= 0 \ \ (p = 0, 1, 2, \ldots), \\ \partial_{\bar{z}} R_2 &= -R_1 \partial_{\bar{z}} B_1. \end{aligned} \tag{3.11}$$

*Proof.* The first and the second equations were obtained from (2.1) in [20]. The third is trivial, since $Q_p$'s do not depend on $\bar{z}$. The calculation of $\partial_{\bar{z}} R_2$ is as follows.

$$\partial_{\bar{z}} R_2 = z^{\alpha+1} C_{zzz} (\partial_{\bar{z}} \bar{T} + \partial_{\bar{z}} K \cdot \bar{T}) = z G_{z\bar{z}} R_1 = -R_1 \partial_{\bar{z}} B_1,$$

where we have used the identity $G_{z\bar{z}} = \partial_z \partial_{\bar{z}} K(z, \bar{z}) = -\partial_{\bar{z}} B_1 / z$. □

If one assumes that $\partial_{\bar{z}} A_1$, $\partial_{\bar{z}} B_1$, $\partial_{\bar{z}} R_1$ are independent, then Walcher's extended holomorphic equation (2.8) is rewritten as follows.

**Lemma 3.7.** *The equation* (2.8) *is equivalent to the system of* PDE*'s:*

$$\begin{aligned} &\Big[ -R_1 \frac{\partial}{\partial R_2} - 2 \frac{\partial}{\partial A_1} + \frac{\partial}{\partial B_1} + V_1 \frac{\partial}{\partial B_2} \\ &\quad + \left( -V_2 + \frac{\theta_z(z^3 C_{zzz})}{z^3 C_{zzz}} V_1 + h(z) - 1 \right) \frac{\partial}{\partial B_3} \Big] P^{(g,h)} = 0, \end{aligned} \tag{3.12}$$

$$\frac{\partial P^{(g,h)}}{\partial A_1} = -\frac{1}{2}\left(\sum_{\substack{g_1+g_2=g,\\ h_1+h_2=h}} P_1^{(g_1,h_1)}P_1^{(g_2,h_2)} + P_2^{(g-1,h)}\right) + (B_1Q_0 - Q_1)P_1^{(g,h-1)}, \tag{3.13}$$

$$\frac{\partial P^{(g,h)}}{\partial R_1} = -P_1^{(g,h-1)}. \tag{3.14}$$

*Here the summation in* (3.13) *runs over* $(g_1,h_1),(g_2,h_2)$ *such that* $(g_i,h_i)\neq (0,0),(0,1)$.

*Proof.* By (2.8), we have

$$\partial_{\bar z}P^{(g,h)} = \frac{1}{2}\partial_{\bar z}(zC_{zzz}S^{zz})\cdot\left(\sum_{\substack{g_1+g_2=g,\\ h_1+h_2=h}} P_1^{(g_1,h_1)}P_1^{(g_2,h_2)} + P_2^{(g-1,h)}\right) - \partial_{\bar z}(z^{5/2}C_{zzz}\triangle^z)\cdot P_1^{(g,h-1)}.$$

Note that, by (3.9) and (3.11),

$$\begin{aligned}\partial_{\bar z}(zC_{zzz}S^{zz}) &= -(\partial_{\bar z}A_1 + 2\partial_{\bar z}B_1),\\ \partial_{\bar z}(z^{\frac{5}{2}}C_{zzz}\triangle^z) &= -(\partial_{\bar z}A_1 + 2\partial_{\bar z}B_1)(-Q_1 + B_1Q_0) + \partial_{\bar z}R_1.\end{aligned}$$

On the other hand, by (3.11), $\partial_{\bar z}$ in the LHS is as follows:

$$\partial_{\bar z} = \partial_{\bar z}R_1\frac{\partial}{\partial R_1} + \partial_{\bar z}A_1\frac{\partial}{\partial A_1} + \partial_{\bar z}B_1\left[-R_1\frac{\partial}{\partial R_2} + \frac{\partial}{\partial B_1} + V_1\frac{\partial}{\partial B_2} + \left(-V_2 + \frac{\theta_z(z^3C_{zzz})}{z^3C_{zzz}}V_1 + h(z) - 1\right)\frac{\partial}{\partial B_3}\right].$$

Inserting these and comparing the coefficients of $\partial_{\bar z}A_1, \partial_{\bar z}B_1, \partial_{\bar z}R_1$, one obtains Lemma 3.7. □

To write the equations in a more useful form, we change the generators. We define

$$\begin{aligned} &u = B_1, \quad v_1 = V_1 + \frac{3}{5}, \quad v_2 = V_2 + \frac{2}{25},\\ &v_3 = B_3 - B_1\left(-V_2 + \frac{\theta_z(z^3C_{zzz})}{z^3C_{zzz}}V_1 + h(z) - 1\right) + s(z),\end{aligned}$$

$$
\begin{aligned}
(3.15)\quad m_1 &= \frac{2}{25}Q_0 + \frac{3}{5}Q_1 + Q_2 - R_1,\\
m_2 &= Q_0\Big(s(z) - \frac{2}{25}\frac{\theta_z(z^3C_{zzz})}{z^3C_{zzz}}\Big) + Q_1\Big(\frac{23}{25} - h(z)\Big) - Q_2\frac{\theta_z(z^3C_{zzz})}{z^3C_{zzz}}\\
&\quad + Q_3 - R_2 - B_1R_1,
\end{aligned}
$$

where

$$
s(z) = \frac{12}{25} - \frac{1}{5}h(z) + \frac{3}{25}\frac{\theta_z(z^3C_{zzz})}{z^3C_{zzz}}. \tag{3.16}
$$

Define the ring

$$
J := \mathbb{C}(z)[u, v_1, v_2, v_3, Q_0, Q_1, Q_2, Q_3, m_1, m_2].
$$

It is isomorphic to $I$ since (3.15) is invertible. Notice that $\theta_z : J \to J$ increases the degree in $u$ at most by 1.

Now we regard $P^{(g,h)} \in J$. Then (3.12) implies $P^{(g,h)}$ is independent of $u$. In turn, $P_n^{(g,h)} \in J$ has degree at most $n$ in $u$. Following [12, (3-4.c)], let us define $u$-independent polynomials $Y_0, Y_1, W_0, W_1, W_2 \in J$ by

$$
\begin{gathered}
Y_0 + u\,Y_1 = P_1^{(g,h-1)},\\
W_0 + uW_1 + u^2W_2 = (\text{the RHS of (3.13)}).
\end{gathered} \tag{3.17}
$$

Then applying the change of generators (3.15) to the equations (3.12)–(3.14), we obtain

**Theorem 3.8.** *The equation* (2.8) *is equivalent to the following system of* PDE*'s for* $P^{(g,h)} \in J$*:*

$$
\begin{aligned}
&\frac{\partial}{\partial u}P^{(g,h)} = 0,\\
&\frac{\partial}{\partial m_1}P^{(g,h)} = Y_0, \quad \frac{\partial}{\partial m_2}P^{(g,h)} = Y_1,\\
&\frac{\partial}{\partial v_1}P^{(g,h)} = W_0, \quad \frac{\partial}{\partial v_2}P^{(g,h)} = -W_1 + \frac{\theta_z(z^3C_{zzz})}{z^3C_{zzz}}W_2,\\
&\frac{\partial}{\partial v_3}P^{(g,h)} = -W_2.
\end{aligned} \tag{3.18}
$$

Let us comment on the constant of integration. Decompose $P^{(g,h)}$ as

$$P^{(g,h)} = \hat{P}^{(g,h)} + P^{(g,h)}|_{v_1,v_2,v_3,m_1,m_2=0},$$

where $\hat{P}^{(g,h)}$ consists of terms of degree $\geq 1$ with respect to at least one of $v_1, v_2, v_3, m_1, m_2$. The equations (3.18) can determine $\hat{P}^{(g,h)}$, but not the second term. The latter is a priori a polynomial in $Q_0, Q_1, Q_2, Q_3$ with $\mathbb{C}(z)$ coefficients. However, the choice of the new generators (3.15) is "good" (cf. [12, (3-4.d)]) so that we have the following

**Proposition 3.9.**

$$P^{(g,h)}|_{v_1,v_2,v_3,m_1,m_2=0} = (z^3 C_{zzz})^{g+h-1} z^{h/2} f^{(g,h)}.$$

*Proof.* We have

$$S^{zz} = -\frac{v_1}{zC_{zzz}}, \quad S^z = \frac{uv_1 + v_2}{z^2 C_{zzz}},$$

$$S = \frac{1}{z^3 C}\left[-\frac{1}{2}u^2 v_1 - \left(u + \frac{5^5 z}{2(1-5^5 z)}\right) v_2 + \frac{v_3}{2}\right],$$

$$\triangle^z = \frac{1}{z^{5/2} C_{zzz}}(-m_1 + Q_1 v_1 + Q_0 v_2),$$

$$\triangle = \frac{1}{z^{7/2} C_{zzz}}\Bigg[um_1 - m_2 - uQ_0 v_1 - v_2\left(uQ_0 + \frac{5^5 z}{1-5^5 z}Q_0 + Q_1\right) + Q_0 v_3\Bigg].$$

Notice that every monomial term in the propagators $S^{zz}, S^z, S$ and the terminators $\triangle^z, \triangle$ contains at least one of $v_1, v_2, v_3, m_1, m_2$. Therefore the Feynman diagram part $\mathcal{F}^{(g,h)}_{\mathrm{FD}}$ of $\mathcal{F}^{(g,h)}$ has degree at least 1 with respect to one of $v_1, v_2, v_3, m_1, m_2$ by (2.18) and (2.19). This implies that the first term in the RHS of

$$P^{(g,h)} = (z^3 C_{zzz})^{g+h-1} z^{h/2} \mathcal{F}^{(g,h)}_{\mathrm{FD}} + (z^3 C_{zzz})^{g+h-1} z^{h/2} f^{(g,h)}$$

vanishes as $v_1, v_2, v_3, m_1, m_2$ tends to zero. This proves the proposition. $\square$

## 4. Discussion

We discuss how to fix the holomorphic ambiguity $f^{(g,h)}$.

Let $\omega_0(z), \omega_1(z), \omega_2(z), \omega_3(z)$ be the following solutions to the Picard–Fuchs equation $\mathcal{D}\omega = 0$ about $z = 0$.

$$\omega_i(z) = \partial_\rho^i \left( \sum_{n \geq 0} \frac{(5\rho+1)_{5n}}{{(\rho+1)_n}^5} z^{n+\rho} \right) \Bigg|_{\rho=0}.$$

Let $t = \omega_1(z)/\omega_0(z)$ be the mirror map and consider the inverse $z = z(q)$, where $q = e^t$. Explicitly, these are

$$\begin{aligned}
\omega_0(z) &= 1 + 120z + 113400z^2 + \cdots, \\
\omega_1(z) &= \omega_0(z) \log z + 770z + 810225z^2 + \cdots, \\
t &= \log z + 770z + 717825z^2 + \frac{3225308000}{3} z^3 + \cdots, \\
z &= q - 770q^2 + 171525q^3 + \cdots.
\end{aligned}$$

Let

$$F_A^{(g,h)} = \lim_{\bar{z} \to 0} \mathcal{F}^{(g,h)} \omega_0(z)^{2g+h-2}, \tag{4.1}$$

for $(g,h)$ satisfying $2g + h - 2 > 0$.[6] The limit $\bar{z} \to 0$ in the RHS means

$$G_{z\bar{z}} \longrightarrow \frac{dt}{dz}, \quad e^{-K} \longrightarrow \omega_0(z), \quad \triangle_{zz} \longrightarrow D_z D_z \mathcal{T}.$$

[6]For $(g,h) = (0,0), (1,0), (0,1), (0,2)$, one should consider

$$\partial_t^n F_A^{(g,h)} = \left( \frac{dz}{dt} \right)^n \lim_{\bar{z} \to 0} \mathcal{F}_n^{(g,h)} \omega_0^{2g+h-2},$$

where $n = 3, 1, 2, 1$, respectively.

Define $n_d^{(g,h)}$ for $h > 0$[7] by the formula [14; 15; 19, (3.22)]:

$$\begin{aligned}&\text{the terms in positive powers in } q \text{ of } \sum_{g=0}^{\infty} {g_s}^{2g+h-2} F_A^{(g,h)} \\ &= \sum_{g=0}^{\infty} \sum_d \sum_k n_d^{(g,h)} \frac{1}{k} \left(2 \sin \frac{k g_s}{2}\right)^{2g+h-2} q^{kd/2}.\end{aligned} \tag{4.2}$$

Here the summation of $k$ is over positive odd integers and that of $d$ is over positive even (resp. odd) integers when $h$ is even (resp. odd).

**Remark 4.1.** It is expected that $F_A^{(g,h)}$ is the $A$-model topological string amplitude of genus $g$ with $h$ boundaries for the real quintic 3-fold $(X, L)$, and that $n_d^{(g,h)}$ be the BPS numbers in the class $d \in H_2(X, L; \mathbb{Z})$. See [7, 21] for $(g, h) = (0, 0), (1, 0)$ and [16, 18] for $(g, h) = (0, 1)$. It is expected that $n_d^{(g,h)}$ are even[8] integers for $h > 0$.

The holomorphic ambiguity could be fixed by assuming some conditions on $F_A^{(g,h)}$ and $n_d^{(g,h)}$. In [19], the following boundary conditions were proposed.

(i) $n_d^{(g,h)} = 0$ if $n_d^{2g+h-1} = 0$.

(ii) If $h$ is even, the $q$-constant term in $F_A^{(g,h)}$ vanishes except for $(g, h) = (0, 2)$.

However, these do not give enough equations to fix unknown parameters in $f^{(g,h)}$ unless $(g, h) = (0, 1), (0, 2), (0, 3), (1, 1)$ (these cases were done in [19]; see also commens in [1, Appendix B]). Without any other guiding principle, we might assume the following *ad hoc* condition together with (ii):

(i') $n_d^{(g,h)} = 0$ for $d \leq d_0$, where $d_0$ is the smallest number necessary to completely determine unknown parameters in $f^{(g,h)}$.

For example, $d_0 = 3$ for $(g, h) = (0, 3), (1, 1)$, $d_0 = 6$ for $(g, h) = (1, 2), (0, 4)$ and $d_0 = 9$ for $(g, h) = (1, 3), (0, 5)$. When we fix holomorphic ambiguities

[7]For $h = 0$, the BPS number $n_d^g$ is defined by [2.8]

$$\sum_{g=0}^{\infty} {g_s}^{2g-2} F_A^{(g,0)} = \sum_{g=0}^{\infty} \sum_{d>0} \sum_{k>0} n_d^g \frac{1}{k} \left(2 \sin \frac{k g_s}{2}\right)^{2g-2} q^{kd} + \text{ polynomial in } \log q.$$

[8]The authors thank a referee for pointing out this property.

with these assumption, for $(g,h)=(0,4)$ we obtain

$$f^{(0,4)}=\frac{2-20125\,z+70618750\,z^2-86493078125\,z^3}{10000\,(1-3125\,z)^2},$$

and $n_8^{(0,4)}=-307669500$, $n_{10}^{(0,4)}=-1290543544800$ and so on. In the cases of $(g,h)=(0,4),(0,5),(0,6),(1,1),(1,2),(1,3),(1,4)$, we obtain integral $n_d^{(g,h)}$ for small $d$. For $(g,h)=(0,4),(0,5),(0,6)$, they are even, but for $(g,h)=(1,1),(1,2)$, some of them are odd. For $(g,h)=(0,7),(1,5),(2,1)$, the holomorphic ambiguities determined by our assumptions do not give integral $n_d^{(g,h)}$'s. It will be interesting to investigate systematic ways to fix holomorphic ambiguities so as to get the BPS numbers $n_d^{(g,h)}$ with the desired properties, as was done in the closed case [13].

As a final remark, let us comment on the expansion about the conifold point $z=\frac{1}{5^5}$. If one imposes the gap condition to $\mathcal{F}^{(0,4)}$ such as the one found in [13, (1.2)] instead of $n_6^{(0,4)}=0$, then the integrality of $n_d^{(0,4)}$'s does not hold.

## Appendix A. Examples of Feynman diagrams

Feynman diagrams for $\mathcal{F}_{\mathrm{FD}}^{(0,3)}$ and $\mathcal{F}_{\mathrm{FD}}^{(1,1)}$ have been given in equations (2.109) and (2.108) of [19] respectively ($\#\mathbb{G}(0,3)=\#\mathbb{G}(1,1)=4$). For $(g,h)=(0,4)$, we have $\#\mathbb{G}(0,4)=19$. See figure 3. It is clear that the number of Feynman diagrams grows rapidly as $g$ and $h$ increase. For example, one can check that $\#\mathbb{G}(0,5)=83$, $\#\mathbb{G}(1,2)=29$, $\#\mathbb{G}(2,1)=97$.

## Appendix B. $P^{(0,4)}$

$$\begin{aligned}
P^{(0,4)} &= z^2(z^3C_{zzz})^3 f^{(0,4)} - \frac{z(2-9500z+16015625z^2){m_1}^2}{20(-1+3125z)^3} \\
&+ \frac{(-9+12500z){m_1}^4}{120(-1+3125z)} + \frac{75z^2(-1+3145z)m_2}{4(-1+3125z)^3} + \frac{{m_1}^3 m_2}{6} \\
&+ \frac{5z{m_2}^2}{4(-1+3125z)} + m_1\left(\frac{375z^3(-3+3125z)}{2(-1+3125z)^4} + \frac{375z^2 m_2}{2(-1+3125z)^2}\right) \\
&- \frac{{Q_1}^4{v_1}^5}{8} + \left(\frac{(-3+25000z){Q_0}^4}{40(-1+3125z)} - \frac{{Q_0}^3 Q_1}{6}\right){v_2}^4 \\
&+ {v_1}^4\left(\frac{m_1{Q_1}^3}{2} + \frac{(-9+12500z){Q_1}^4}{120(-1+3125z)} - \frac{Q_0{Q_1}^3 v_2}{2}\right)
\end{aligned}$$

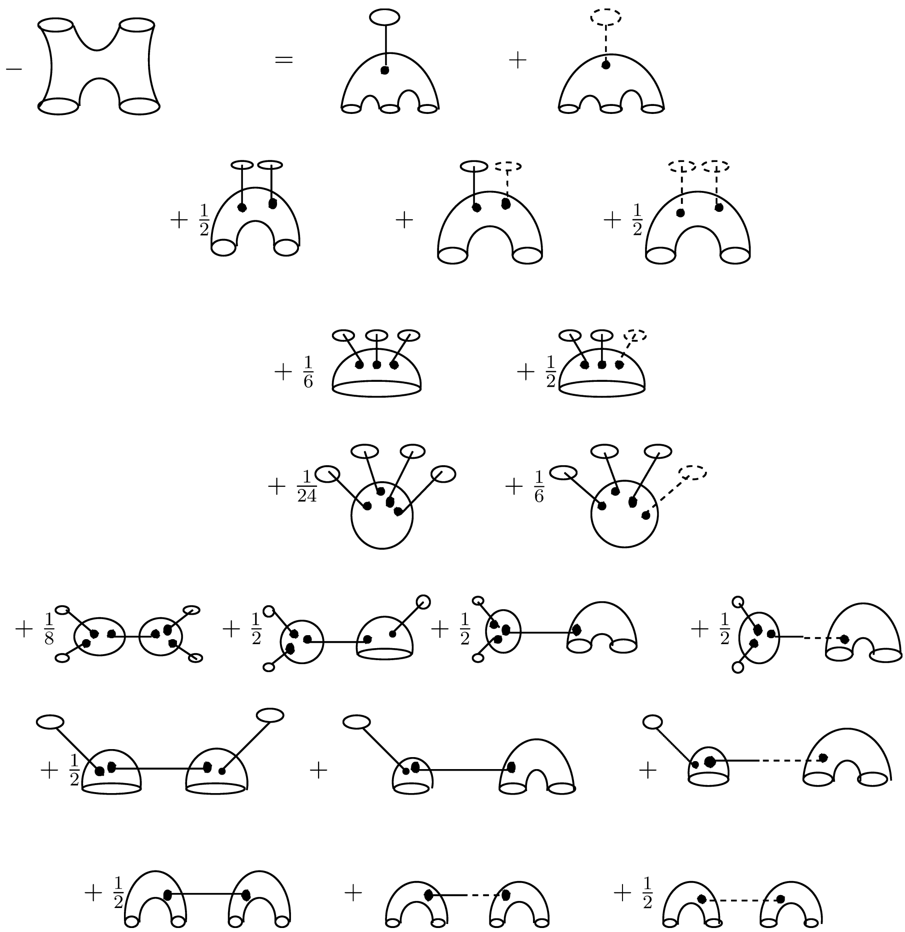


Figure 3: The elements $G$ in $\mathbb{G}(0,4)$ and the orders of $A_G$. The vertices are expressed as bordered Riemann surfaces to visualize the labeling.

$$
\begin{aligned}
&+\left(\frac{25z^2}{8(-1+3125z)^2}-\frac{75z^2(-1+3145z)Q_0}{4(-1+3125z)^3}-\frac{375z^2m_1Q_0}{2(-1+3125z)^2}\right.\\
&\left.-\frac{{m_1}^3Q_0}{6}-\frac{5zm_2Q_0}{2(-1+3125z)}\right)v_3+\frac{5z{Q_0}^2{v_3}^2}{4(-1+3125z)}\\
&+{v_2}^2\left(\frac{z(-1+4750z+119921875z^2){Q_0}^2}{10(-1+3125z)^3}\right.\\
&+m_1\left(\frac{-5zQ_0}{2(-1+3125z)}+\frac{m_2{Q_0}^2}{2}\right)-\frac{8000z^2Q_0Q_1}{(-1+3125z)^2}+\frac{5z{Q_1}^2}{4(-1+3125z)}
\end{aligned}
$$

$$
\begin{aligned}
&+m_1{}^2\left(\frac{(-9+43750z)Q_0{}^2}{20(-1+3125z)}-\frac{Q_0Q_1}{2}\right)-\frac{m_1Q_0{}^3v_3}{2}\Bigg)\\
&+v_2{}^3\left(\frac{5zQ_0{}^2}{4(-1+3125z)}-\frac{m_2Q_0{}^3}{6}+m_1\left(\frac{-((-9+59375z)Q_0{}^3)}{30(-1+3125z)}+\frac{Q_0{}^2Q_1}{2}\right)\right.\\
&\left.+\frac{Q_0{}^4v_3}{6}\right)+v_1{}^3\left(\frac{-375z^2Q_1{}^2}{4(-1+3125z)^2}-\frac{3m_1{}^2Q_1{}^2}{4}-\frac{(-9+12500z)m_1Q_1{}^3}{30(-1+3125z)}\right.\\
&-\frac{m_2Q_1{}^3}{6}+\left(\frac{3m_1Q_0Q_1{}^2}{2}+\frac{3Q_0Q_1{}^3}{10}-\frac{Q_1{}^4}{6}\right)v_2\\
&\left.-\frac{3Q_0{}^2Q_1{}^2v_2{}^2}{4}+\frac{Q_0Q_1{}^3v_3}{6}\right)+v_2\left(\frac{81875z^3}{8(-1+3125z)^3}-\frac{236625z^3Q_0}{4(-1+3125z)^3}\right.\\
&+m_1{}^2\left(\frac{5z}{4(-1+3125z)}-\frac{m_2Q_0}{2}\right)+m_1{}^3\left(\frac{-3Q_0}{10}+\frac{Q_1}{6}\right)\\
&+\frac{75z^2(-1+3145z)Q_1}{4(-1+3125z)^3}+m_1\left(\frac{z(-1+1625z)Q_0}{5(-1+3125z)^2}+\frac{375z^2Q_1}{2(-1+3125z)^2}\right)\\
&+m_2\left(\frac{-8000z^2Q_0}{(-1+3125z)^2}+\frac{5zQ_1}{2(-1+3125z)}\right)\\
&\left.+\left(\frac{8000z^2Q_0{}^2}{(-1+3125z)^2}+\frac{m_1{}^2Q_0{}^2}{2}-\frac{5zQ_0Q_1}{2(-1+3125z)}\right)v_3\right)\\
&+v_1\left(\frac{-140625z^4}{8(-1+3125z)^4}-\frac{m_1{}^4}{8}-\frac{375z^3(-3+3125z)Q_1}{2(-1+3125z)^4}\right.\\
&+\frac{z(2-9500z+16015625z^2)m_1Q_1}{10(-1+3125z)^3}-\frac{(-9+12500z)m_1{}^3Q_1}{30(-1+3125z)}\\
&-\frac{375z^2m_2Q_1}{2(-1+3125z)^2}+m_1{}^2\left(\frac{-375z^2}{4(-1+3125z)^2}-\frac{m_2Q_1}{2}\right)\\
&+\left(\frac{m_1Q_0{}^3}{2}+\frac{(-9+59375z)Q_0{}^3Q_1}{30(-1+3125z)}-\frac{Q_0{}^2Q_1{}^2}{2}\right)v_2{}^3-\frac{Q_0{}^4v_2{}^4}{8}\\
&+\left(\frac{375z^2Q_0Q_1}{2(-1+3125z)^2}+\frac{m_1{}^2Q_0Q_1}{2}\right)v_3\\
&+v_2\left(\frac{m_1{}^3Q_0}{2}-\frac{z(-1+1625z)Q_0Q_1}{5(-1+3125z)^2}-\frac{375z^2Q_1{}^2}{2(-1+3125z)^2}\right.\\
&+m_1\left(\frac{375z^2Q_0}{2(-1+3125z)^2}-\frac{5zQ_1}{2(-1+3125z)}+m_2Q_0Q_1\right)\\
&\left.\left.+m_1{}^2\left(\frac{9Q_0Q_1}{10}-\frac{Q_1{}^2}{2}\right)-m_1Q_0{}^2Q_1v_3\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ {v_2}^2 \left( \frac{-375z^2{Q_0}^2}{4(-1+3125z)^2} - \frac{3{m_1}^2{Q_0}^2}{4} + \frac{5zQ_0Q_1}{2(-1+3125z)} - \frac{m_2{Q_0}^2Q_1}{2} \right. \\
&\left. +m_1 \left( \frac{-((-9+43750z){Q_0}^2Q_1)}{10(-1+3125z)} + Q_0{Q_1}^2 \right) + \frac{{Q_0}^3Q_1v_3}{2} \right) \Bigg) \\
&+ {v_1}^2 \left( \frac{{m_1}^3Q_1}{2} - \frac{z(2-9500z+16015625z^2){Q_1}^2}{20(-1+3125z)^3} \right. \\
&+ \frac{(-9+12500z){m_1}^2{Q_1}^2}{20(-1+3125z)} + m_1 \left( \frac{375z^2Q_1}{2(-1+3125z)^2} + \frac{m_2{Q_1}^2}{2} \right) \\
&\quad + \left( \frac{3m_1{Q_0}^2Q_1}{2} + \frac{(-9+43750z){Q_0}^2{Q_1}^2}{20(-1+3125z)} - \frac{Q_0{Q_1}^3}{2} \right) {v_2}^2 \\
&- \frac{{Q_0}^3Q_1{v_2}^3}{2} - \frac{m_1Q_0{Q_1}^2v_3}{2} + v_2 \left( \frac{-375z^2Q_0Q_1}{2(-1+3125z)^2} - \frac{3{m_1}^2Q_0Q_1}{2} \right. \\
&\left.\left. + \frac{5z{Q_1}^2}{4(-1+3125z)} - \frac{m_2Q_0{Q_1}^2}{2} + m_1 \left( \frac{-9Q_0{Q_1}^2}{10} + \frac{{Q_1}^3}{2} \right) + \frac{{Q_0}^2{Q_1}^2v_3}{2} \right) \right).
\end{aligned}
$$

## Acknowledgments

Y.K. thanks Shinobu Hosono for valuable discussions and helpful comments. This work was initiated when S.M. was staying at Institut Mittag-Leffler (Djursholm, Sweden). He would like to thank the institute for support. The authors are also grateful to J.D. Länge and T. Okuda for informing them of the work of [1] at the 5th Simons Workshop on Mathematics and Physics held at Stony Brook. Finally they thank the referees for suggestions for improvement of the manuscript.

The work of Y.K. is partly supported by JSPS Research Fellowships for Young Scientists. Research of S.M. is supported in part by 21st Century COE Program at Department of Mathematics, Hokkaido University.

## References

[1] M. Alim and J.D. Länge, *Polynomial structure of the (open) topological string partition function*, J. High Energy Phys. **2007**, no. 10, 045.

[2] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, *Kodaira–Spencer theory of gravity and exact results for quantum string amplitudes*, Comm. Math. Phys. **165** (1994), 311–428.

[3] G. Bonelli and A. Tanzini, *The holomorphic anomaly for open string moduli*, J. High Energy Phys. **2007**, no. 10, 060.

[4] P. Candelas, X. C. de la Ossa, P.S. Green, and L. Parkes, *A pair of Calabi–Yau manifolds as an exactly soluble superconformal theory*, Nucl. Phys. B **359** (1991), 21–74.

[5] P.L.H. Cook, H. Ooguri, and J. Yang, *Comments on the holomorphic anomaly in open topological string theory*, Phys. Lett. **B653** (2007), 335–337.

[6] H. Fang, Z. Lu, and K.-I. Yoshikawa, *Analytic torsion for Calabi–Yau threefolds*, Preprint, arXiv:math/0601411.

[7] A.B. Givental, *Equivariant Gromov–Witten invariants*, Int. Math. Res. Not. **1996**, no. 13, 613–663.

[8] R. Gopakumar and C. Vafa, *M-theory and topological strings. I, II*, Preprint, arXiv:hep-th/9809187, arXiv:hep-th/9812127.

[9] M.L. Green, *Infinitesimal methods in Hodge theory*, in Algebraic cycles and Hodge theory (Torino, 1993), Lecture Notes in Mathematics **1594**, Springer, Berlin, 1994, 1–92.

[10] B.R. Greene and M.R. Plesser, *Duality in Calabi–Yau moduli space*, Nucl. Phys. B **338** (1990), no. 1, 15–37.

[11] P. Griffiths, ed., *Topics in transcendental algebraic geometry*, in Proceedings of a seminar held at the Institute for Advanced Study, Princeton, NJ, during the academic year 1981/1982, Annals of Mathematics Studies **106**, Princeton University Press, Princeton, NJ, 1984.

[12] S. Hosono and Y. Konishi, *Higher genus Gromov–Witten invariants of the Grassmannian, and the Pfaffian Calabi–Yau threefolds*, Preprint, arXiv:0704.2928 [math.AG].

[13] M. Huang, A. Klemm, and S. Quackenbush, *Topological string theory on compact Calabi–Yau: modularity and boundary conditions*, Preprint, arXiv:hep-th/0612125.

[14] J.M.F. Labastida, M. Marino, and C. Vafa, *Knots, links and branes at large N*, J. High Energy Phys. **2000**, no. 11, 007 (electronic).

[15] H. Ooguri and C. Vafa, *Knot invariants and topological strings*, Nucl. Phys. B **577** (2000), 419–438.

[16] R. Pandharipande, J. Solomon, and J. Walcher, *Disk enumeration on the quintic* 3*-fold*, Preprint, arXiv:math/0610901.

[17] A. Strominger, *Special geometry*, Comm. Math. Phys. **133** (1990), 163–180.

[18] J. Walcher, *Opening mirror symmetry on the quintic*, Comm. Math. Phys. **276** (2007), 671–689.

[19] J. Walcher, *Extended holomorphic anomaly and loop amplitudes in open topological string*, Preprint, arXiv:0705.4098 [hep-th].

[20] S. Yamaguchi and S.-T. Yau, *Topological String Partition Functions as Polynomials*, J. High Energy Phys. **2004**, no. 7, 047 (electronic).

[21] A. Zinger, *The reduced genus*-1 *Gromov–Witten invariants of Calabi–Yau hypersurfaces*, Preprint, arXiv:0705.2397 [math.AG].

Graduate School of Mathematical Sciences
The University of Tokyo
3-8-1 Komaba
Meguro
Tokyo 153-8914
Japan
*E-mail address*: konishi@ms.u-tokyo.ac.jp

Department of Mathematics
Hokkaido University
Kita 10, Nishi 8
Kita-Ku
Sapporo 060-0810
Japan
*E-mail address*: minabe@math.sci.hokudai.ac.jp

Received August 24, 2007