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Moment zeta functions for toric Calabi–Yau hypersurfaces

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Moment zeta functions provide a diophantine formulation for the distribution of rational points on a family of algebraic varieties over finite fields. They also form algebraic approximations to Dwork's p-adic unit root zeta functions. In this paper, we use l-adic cohomology to calculate all the higher moment zeta functions for the mirror family of the Calabi-Yau family of smooth projective hypersurfaces over finite fields. Our main result is a complete determination of the purity decomposition and the trivial factors for the moment zeta functions.

1. Introduction

Let $n \geq 2$ be a positive integer. We consider the following family

$$X_\lambda : x_1 + \cdots + x_n + \frac{1}{x_1 \cdots x_n} = \lambda$$

of $(n-1)$-dimensional toric Calabi–Yau hypersurfaces in $\mathbb{G}_m^n$ parameterized by $\lambda \in \mathbb{A}^1$. Let $\mathbb{P}_\Delta$ be the projective toric variety associated to the Newton polytope of the above Laurent polynomial. The projective closure Y_λ of X_λ in $\mathbb{P}_\Delta$ is simply the quotient by $G = (\mathbb{Z}/(n+1)\mathbb{Z})^{n-1}$ of the following Dwork family of projective Calabi–Yau hypersurfaces in $\mathbb{P}^n$:

$$W_\lambda : x_0^{n+1} + \cdots + x_n^{n+1} = \lambda x_0 \cdots x_n.$$

The crepant resolution of the family Y_λ is the mirror family of W_λ.

Let $\mathbb{F}_q$ be a finite field of q elements with characteristic p. In this paper, we are interested in the moment zeta function [29] which measures the arithmetic variation of the zeta function of X_λ over $\mathbb{F}_q$ as λ varies in $\mathbb{F}_q$. The moment zeta function grew out of the second author's study [26, 27, 28] of Dwork's unit root conjecture. Its general properties were studied in Fu–Wan [9] and Wan [24, 29]. Note that the zeta function of Y_λ differs from the zeta function of X_λ by some trivial factors, see Section 7 in [32].

The zeta function of the Dwork family W_λ over finite fields had been studied extensively in the literature, first by Dwork [7] and Katz [15], and more recently in connection with arithmetic mirror symmetry by Candelas *et al.* [2, 3], and by Wan [31, 32] and Fu–Wan [12]. By the congruence mirror theorem in [31, 32], the zeta function of X_λ is the most primitive piece of the zeta function of W_λ. Thus, we shall restrict ourself to the family X_λ. The Hasse-Weil zeta function (but not its higher moment zeta function which would seem to be too hard at the moment) in a similar number field example is studied in a recent paper by Harris *et al.* [14].

More precisely, for a positive integer d, let $N_d(k)$ denote the number of points on the family X_λ such that $x_i \in \mathbb{F}_{q^{dk}}$ for all $1 \leq i \leq n$ and $\lambda \in \mathbb{F}_{q^k}$. The dth moment zeta function of the morphism $X_\lambda \to \lambda \in \mathbb{A}^1$ is defined to be

$$Z_d(\mathbb{A}^1, X_\lambda) = \exp\left(\sum_{k=1}^{\infty} \frac{N_d(k)}{k} T^k\right) \in 1 + T\mathbb{Z}[[T]].$$

This sequence $Z_d(\mathbb{A}^1, X_\lambda)$ $(d = 1, 2, \ldots)$ of power series gives a simple diophantine reformulation on the arithmetic variation of the zeta function of the family X_λ. It is a rational function in T for each d. In the special case $n = 2$, X_λ is a family of elliptic curves and the moment zeta function $Z_d(\mathbb{A}^1, X_\lambda)$ is closely related to arithmetic of modular forms [25]. In general, Dwork's unit root zeta functions [8] attached to this family are the p-adic limits of this sequence of moment zeta functions. They are thus infinite p-adic moment zeta functions in some sense. Our aim of this paper is to give a precise study of this sequence $Z_d(\mathbb{A}^1, X_\lambda)$. One main consequence of our results is a determination of the purity decomposition and the trivial factors for the moment zeta function $Z_d(\mathbb{A}^1, X_\lambda)$ for all d, all n such that $(n+1)$ divides $(q-1)$. This provides the first higher dimensional example for which all higher moment zeta functions are determined. Let

$$S_d(T) = \prod_{k=0}^{[(n-2)/2]} \frac{1 - q^{dk}T}{1 - q^{dk+1}T} \prod_{i=0}^{n-1} (1 - q^{di+1}T)^{(-1)^{i+1}\binom{n}{i+1}}.$$

Theorem 1.1. *Assume that $(n+1)$ divides $(q-1)$. Then, the dth moment zeta function has the following factorization*

$$Z_d(\mathbb{A}^1, X_\lambda)^{(-1)^{n-1}} = P_d(T) \left(\frac{Q_d(T)}{P(d,T)}\right)^{n+1} R_d(T) S_d(T).$$

We now explain each of the above factors. First, $P_d(T)$ is the non-trivial factor which has the form

$$P_d(T) = \prod_{a+b=d, 0\le b\le n} P_{a,b}(T)^{(-1)^{b-1}(b-1)},$$

and each $P_{a,b}(T)$ is a polynomial in $1 + T\mathbb{Z}[T]$, pure of weight $d(n-1)+1$, whose degree r is given explicitly in Theorem 3.11 and which satisfies the functional equation

$$P_d(T) = \pm T^r q^{(d(n-1)+1)r/2} P_d\left(\frac{1}{q^{d(n-1)+1}T}\right).$$

Second, $P(d,T) \in 1 + T\mathbb{Z}[T]$ is the dth Adams operation (see Definition 3.2) of the "non-trivial" factor in the zeta function of a singular fibre X_t, where $t = (n+1)\zeta_{n+1}$ and $\zeta_{n+1} = 1$. It is a polynomial of degree $(n-1)$ whose weights are completely determined. Third, the quasi-trivial factor $Q_d(T)$ coming from a finite singularity has the form

$$Q_d(T) = \prod_{a+b=d, 0\le b\le n} Q_{a,b}(T)^{(-1)^{b-1}(b-1)},$$

where $Q_{a,b}(T)$ is a polynomial whose degree $D_{n,a,b}$ and the weights of its roots are given in Corollaries 3.7 and 3.8. Finally, the trivial factor $R_d(T)$ is given by

$$R_d(T) = (1 - q^{d(n-1)/2}T)(1 - q^{(d(n-1)/2)+1}T)(1 - q^{(d(n-2)/2)+1}T)$$

if n and d are even,

$$R_d(T) = (1 - q^{(d(n-2)/2)+1}T)$$

if n is even and d is odd,

$$R_d(T) = (1 - q^{(d(n-1)/2)}T)$$

if n and d are odd,

$$R_d(T) = (1 - q^{(d(n-1)/2)+1}T)^{-1}$$

if n is odd and d is even.

Corollary 1.2. *Assume that p does not divide $n+1$. Let $N_d(k)$ denote the number of points on the family X_λ such that $x_i \in \mathbb{F}_{q^{dk}}$ for all $1 \le i \le n$ and $\lambda \in \mathbb{F}_{q^k}$. Then for every positive integer k, we have the estimate*

$$\left| N_d(k) - \left(\frac{(q^{kd}-1)^n}{q^{k(d-1)}} + \frac{1}{2}(1+(-1)^d) q^{k((d(n-1)/2)+1)} \right) \right| \\ \le (D+2) q^{k((d(n-1)+1)/2)},$$

where D is the total degree of $P_d(T)(Q_d(T)/P(d,T))^{n+1}$.

Since the first Hodge number $h^{0,n-1}(X_\lambda) = 1$, the zeta function of each fibre X_λ has at most one non-trivial p-adic unit root. One deduces the p-adic continuity result: If $nm+1 \le d_1 \le d_2$ are positive integers such that

$$d_1 \equiv d_2 \pmod{(p-1)p^m},$$

then

$$Z_{d_1}(\mathbb{A}^1, X_\lambda) \equiv Z_{d_2}(\mathbb{A}^1, X_\lambda) \pmod{p^{m+1}}.$$

For a p-adic integer $s \in \mathbb{Z}_p$ and a residue class $r \in \mathbb{Z}/(p-1)\mathbb{Z}$, let $\{d_i\}_{i=1}^\infty$ be a sequence of positive integers in the residue class $r \bmod (p-1)$, going to infinity as complex numbers but approaching to s as p-adic numbers, then the limit

$$\zeta_{r,s}(\mathbb{A}^1, X_\lambda) = \lim_{i\to\infty} Z_{d_i}(\mathbb{A}^1, X_\lambda) \in 1 + T\mathbb{Z}_p[[T]]$$

exists as a formal p-adic power series. This limit depends only on s and r, not on the particular chosen sequence $\{d_i\}_{i=1}^\infty$. The limit $\zeta_{r,s}(\mathbb{A}^1, X_\lambda)$ is precisely Dwork's unit root zeta function attached to the family X_λ. It is a p-adic meromorphic function in T for every $s \in \mathbb{Z}_p$ and $r \in \mathbb{Z}/(p-1)\mathbb{Z}$, as conjectured by Dwork [8] and proven by Wan [28]. It should be viewed as a two variable p-adic zeta function in (s, T). The results of the present paper can be combined with the p-adic methods in [28] to obtain some new information on these unit root zeta functions. These applications will be spelled out in another paper.

We now briefly explain the ideas in proving the above theorem. For a prime $\ell \ne p$, let $K \in \mathcal{D}_c^b(\mathbb{A}_k^1, \bar{\mathbb{Q}}_\ell)$ be the complex obtained by taking direct image with compact support with respect to the morphism $X_\lambda \to \lambda$ of the trivial ℓ-adic sheaf $\bar{\mathbb{Q}}_\ell$ on X. The cohomology sheaves $\mathcal{H}^j(K)$ are the relative ℓ-adic cohomology with compact support of the family X_λ. Then, $Z_d(\mathbb{A}^1, X_\lambda)$ can be expressed in terms of the L-function over $\mathbb{A}^1$ of the dth

Adams operation of the sheaf $\mathcal{H}^j(K)$:

$$Z_d(\mathbb{A}^1, X_\lambda) = \prod_{j=0}^{2(n-1)} L(\mathbb{A}^1, [\mathcal{H}^j(K)]^d)^{(-1)^j}.$$

The L-function of the dth power Adams operation is defined to be

$$L(\mathbb{A}^1, [\mathcal{H}^j(K)]^d) = \prod_{x\in|\mathbb{A}^1|} \frac{1}{\det(I - F_x^d T^{\deg(x)} | \mathcal{H}^j(K)^{I_x})},$$

where $|\mathbb{A}^1|$ denotes the set of closed points on $\mathbb{A}^1$, I_x denotes the inertia group at x and F_x denotes the Frobenius element at x. The dth moment zeta function $Z_d(\mathbb{A}^1, X_\lambda)$ is thus a rational function in T for each positive integer d.

Fix a prime number ℓ different from p. Let $\mathcal{F}$ be the non-trivial part of the relative ℓ-adic cohomology with compact support of the family X_λ parameterized by $\lambda \in \mathbb{A}^1$. Then $\mathcal{F}$ is the non-trivial part of the middle dimensional relative cohomology $\mathcal{H}^{n-1}(K)$. It is a geometrically irreducible smooth sheaf on the dense open set

$$U = \mathbb{A}_k^1 - \{(n+1)\zeta : \zeta^{n+1} = 1\},$$

of rank n and punctually pure of weight $n-1$. The dth moment zeta function is then given up to trivial factors, by the dth moment L-function:

$$Z_d(\mathbb{A}^1, X_\lambda) \sim L(\mathbb{A}^1, [\mathcal{F}]^d)^{(-1)^{n-1}},$$

where $[\mathcal{F}]^d$ denotes the dth Adams operation of the sheaf $\mathcal{F}$ on $\mathbb{A}^1$. As a virtual sheaf on U, one can write (Lemma 4.2 in [26]) the dth Adams operation as

$$[\mathcal{F}]^d = \sum_{a+b=d} (-1)^{b-1}(b-1)[\mathrm{Sym}^a \mathcal{F} \otimes \wedge^b \mathcal{F}],$$

where a and b are non-negative integers.

Let

$$\mathcal{G}_{a,b} := \mathrm{Sym}^a \mathcal{F} \otimes \wedge^b \mathcal{F},$$

which is an ℓ-adic sheaf on $\mathbb{A}^1$, smooth on U, vanishing if $b > n$. Thus, we shall assume that $0 \le b \le n$ from now on. The generic rank of $\mathcal{G}_{a,b}$ is $\binom{n+a-1}{a}\binom{n}{b}$, which goes to infinity as a goes to infinity. On the smooth part

U, the dth moment L-function is then given by the formula

$$L(U,[\mathcal{F}]^d)=\prod_{b=0}^{n}L(U,\mathcal{G}_{d-b,b})^{(-1)^{b-1}(b-1)}.$$

Thus, to a large extent, the moment zeta functions are reduced to the study of the L-function $L(U,\mathcal{G}_{a,b})$ of the sheaf $\mathcal{G}_{a,b}$ for all non-negative integers a and b. To understand the purity decomposition and the trivial factors of this last L-function, the key is to determine the local and global monodromy of the sheaf $\mathcal{F}$. This is accomplished in Section 2. As a consequence, we obtain

Theorem 1.3. *Assume that $(n+1)$ divides $(q-1)$. Let a and b be non-negative integers with $0\le b\le n$. Then, we have the formula*

$$L(U,\mathcal{G}_{a,b})=\frac{P_{a,b}(T)Q_{a,b}(T)^{n+1}\prod_{k=0}^{[(a+b)(n-1)/2]}(1-q^kT)^{\alpha_{a,b}(k)}}{(1-q^{(a+b)(n-1)/2}T)^{\delta_{a,b}}(1-q^{(a+b)(n-1)/2+1}T)^{\delta_{a,b}}},$$

where $P_{a,b}(T),Q_{a,b}(T)\in 1+T\mathbb{Z}[T]$ are polynomials whose degrees are explicitly given, $P_{a,b}(T)$ is pure of weight $(a+b)(n-1)+1$, $Q_{a,b}(T)$ is mixed of weights $\le(a+b)(n-1)+1$ (the precise weights of its roots are given in Corollaries 3.7 and 3.8), $\delta_{a,b}=0$ or 1 is explicitly given by Proposition 3.10, $\alpha_{a,b}(k)$ is the coefficient of x^kz^b in the power series

$$\left\{\frac{(1-x^n)\cdots(1-x^{a+n-1})}{(1-x^2)\cdots(1-x^a)}\right\}(1+z)(1+xz)\cdots(1+x^{n-1}z),$$

where the quantity in the bracket is understood to be $1-x^n$ if $a=1$, and $1-x$ if $a=0$.

The paper is organized as follows. In Section 2, we determine both the local and the global monodromy of the sheaf $\mathcal{F}$. These results are then used in Section 3 to calculate the L-function of the sheaf $\mathcal{G}_{a,b}$ and its local factors at bad points. In Section 4, we treat the degenerate case when p divides $n+1$.

2. The monodromy via Fourier transform

Let $k=\mathbb{F}_q$ be a finite field of characteristic p, $n\ge 2$ an integer, $X\subset\mathbb{A}_k^{n+1}$ the hypersurface defined by $x_1\cdots x_{n+1}=1$, and $\sigma:X\to\mathbb{A}_k^1$ the restriction of the sum map $(x_1,\dots,x_{n+1})\to x_1+\cdots+x_{n+1}$ to X. Fix a prime $\ell\ne p$. We want to study the local monodromy of the non-trivial part of the object $K:=\mathrm{R}\sigma_!\bar{\mathbb{Q}}_\ell\in\mathcal{D}_c^b(\mathbb{A}_k^1,\bar{\mathbb{Q}}_\ell)$, which parameterizes the cohomology of the family described in Section 1. Let η be a generic point of $\mathbb{A}_k^1$. The main results are summarized in the following theorem:

Theorem 2.1. *The cohomology sheaves $\mathcal{H}^j(K) = \mathrm{R}^j\sigma_!\bar{\mathbb{Q}}_\ell$ vanish for $j < n-1$ and $j > 2n-2$. We have isomorphisms*

$$\mathcal{H}^j(K) \cong \bar{\mathbb{Q}}_\ell^{\binom{n}{j-n+2}}(n-1-j),$$

for $n \le j \le 2n-2$, and an exact sequence

$$0 \to \bar{\mathbb{Q}}_\ell^n \to \mathcal{H}^{n-1}(K) \to \mathcal{F} \to 0,$$

where $\mathcal{F}$ is the extension by direct image of a geometrically irreducible smooth sheaf on the dense open set $U = \mathbb{A}^1_k - \{(n+1)\zeta : \zeta^{n+1} = 1\}$, of rank n and punctually pure of weight $n-1$. It is endowed with a non-degenerate pairing $\Phi : \mathcal{F} \times \mathcal{F} \to \bar{\mathbb{Q}}_\ell(1-n)$, which is symmetric if n is odd and skew-symmetric if n is even. As a representation of the inertia group at infinity, $\mathcal{F}$ is unipotent with a single Jordan block.

If p does not divide $n+1$, $\mathcal{F}$ is everywhere tamely ramified. The inertia group at each of the $n+1$ singular points $x = (n+1)\zeta$ acts on $\mathcal{F}_{\bar{\eta}}$ with invariant subspace of codimension 1. On the quotient $\mathcal{F}_{\bar{\eta}}/\mathcal{F}_{\bar{\eta}}^{I_x}$, I_x acts trivially if n is even, and through its unique character of order 2 if n is odd.

If p divides $n+1$, let $n+1 = p^a m$, with m prime to p. Then $\mathcal{F}$ is smooth on $\mathbb{G}_m$, and the inertia group at 0 acts with invariant subspace of dimension $m-1$. The action of I_0 on the quotient $\mathcal{F}_{\bar{\eta}}/\mathcal{F}_{\bar{\eta}}^{I_0}$ is totally wild, with a single break $1/(p^a-1)$ with multiplicity $m(p^a-1) = n-m+1$. In particular, the Swan conductor at 0 is m.

The determinant of $\mathcal{F}$ is the geometrically constant sheaf $\bar{\mathbb{Q}}_\ell(-n(n-1)/2)$ if n is even or p divides $n+1$, and the pulled back Kummer sheaf

$$\mathcal{L}_{\chi(\lambda^{n+1}-(n+1)^{n+1})}\left(\frac{-n(n-1)}{2}\right),$$

if n is odd and $(p, n+1) = 1$, where χ is the unique character of order 2 of the inertia group I_0.

The geometric monodromy group of $\mathcal{F}$ is given by

$Sp(n,\Phi)$	*if n is even*
$O(n,\Phi)$	*if n is odd and $(p,n+1)=1$*
$SO(n,\Phi)$	*if n is odd, $p \mid n+1$ and $(p,n) \neq (2,5)$ or $(2,7)$*
G_2 in its standard 7-dimensional representation	*if $p=2$, $n=7$*
$SL(2)$ in sym^4 of its standard representation	*if $p=2$, $n=5$*

.

We will deduce most of the properties of the object K from the properties of its Fourier transform $L \in \mathcal{D}^b_c(\mathbb{A}^1_k, \bar{\mathbb{Q}}_\ell)$ with respect to a fixed non-trivial additive character $\psi : k \to \mathbb{C}^\star \xrightarrow{\sim} \bar{\mathbb{Q}}^\star_\ell$. The Fourier transform L is closely related to the Kloosterman sheaf. This connection of the Dwork family with Kloosterman sums was first discovered by Katz [16] (Section 5.5) who uses the properties of the family to get information on certain Kloosterman sums. We will use this connection the other way around and apply Katz's fundamental results for the Kloosterman sheaf.

Recall (cf. [20]) that the Fourier transform is defined by

$$FT_\psi(K) = \mathrm{R}\pi_{2!}(\pi_1^\star(K) \otimes \mu^\star \mathcal{L}_\psi)[1],$$

where $\pi_1, \pi_2 : \mathbb{A}^2_k \to \mathbb{A}^1_k$ are the projections, $\mu : \mathbb{A}^2_k \to \mathbb{A}^1_k$ is the product map and $\mathcal{L}_\psi$ is the Artin–Schreier sheaf on $\mathbb{A}^1_k$ associated to the character ψ. It is an auto-equivalence of the triangulated category $\mathcal{D}^b_c(\mathbb{A}^1_k, \bar{\mathbb{Q}}_\ell)$, and has the following involution property:

$$FT_{\bar\psi} FT_\psi(K) = K(-1).$$

One of the main advantages of this equivalence is that, following Laumon (cf. [22]), the local properties of the object K can be read from those of its Fourier transform. This is the method that we will use to deduce most of the results about K.

Let us first determine what the Fourier transform of K is explicitly. Using proper base change on the cartesian diagram

$$\begin{array}{ccc} X & \xleftarrow{\tilde\pi_1} & X \times \mathbb{A}^1_k \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\tilde\sigma} \\ \mathbb{A}^1_k & \xleftarrow{\pi_1} & \mathbb{A}^2_k \end{array},$$

we get

$$\pi_1^\star(K) = \pi_1^\star(\mathrm{R}\sigma_! \bar{\mathbb{Q}}_\ell) = \mathrm{R}\tilde\sigma_! \tilde\pi_1^\star \bar{\mathbb{Q}}_\ell = \mathrm{R}\tilde\sigma_! \bar{\mathbb{Q}}_\ell.$$

By the projection formula, we have then

$$L = \mathrm{R}\pi_{2!}((\mathrm{R}\tilde\sigma_! \bar{\mathbb{Q}}_\ell) \otimes \mu^\star \mathcal{L}_\psi)[1] = \mathrm{R}\pi_{2!}(\mathrm{R}\tilde\sigma_!(\tilde\sigma^\star \mu^\star \mathcal{L}_\psi))[1] = \mathrm{R}\tilde\pi_{2!}(\tilde\mu^\star \mathcal{L}_\psi)[1]$$

where $\tilde\pi_1$ and $\tilde\pi_2$ are the projections of $X \times \mathbb{A}^1_k$ onto its factors and $\tilde\mu : X \times \mathbb{A}^1_k \to \mathbb{A}^1_k$ is the map $((x_1, \ldots, x_{n+1}), t) \mapsto t(x_1 + \cdots + x_{n+1})$.

Extend the canonical map $L \to j_\star j^\star L$ to a distinguished triangle

$$M \to L \to j_\star j^\star L \to \tag{2.1}$$

in $\mathcal{D}_c^b(\mathbb{A}_k^1, \bar{\mathbb{Q}}_\ell)$, where $j : \mathbb{A}_k^1 - \{0\} \hookrightarrow \mathbb{A}_k^1$ is the open immersion. The object M is punctual supported at 0, since $L \to j_\star j^\star L$ is an isomorphism away from 0.

At 0, the object L is just $\mathrm{R}\Gamma_c(X \otimes \bar{k}, \bar{\mathbb{Q}}_\ell)[1]$ by proper base change. Since X is just the product of n copies of $\mathbb{G}_m$, we have

$$L_0 = \bigotimes_{i=1}^{n} \mathrm{R}\Gamma_c(\mathbb{G}_{m,\bar{k}}, \bar{\mathbb{Q}}_\ell)[1].$$

From $\mathrm{H}_c^1(\mathbb{G}_{m,\bar{k}}, \bar{\mathbb{Q}}_\ell) = \bar{\mathbb{Q}}_\ell$, $\mathrm{H}_c^2(\mathbb{G}_{m,\bar{k}}, \bar{\mathbb{Q}}_\ell) = \bar{\mathbb{Q}}_\ell(-1)$ and $\mathrm{H}_c^i(\mathbb{G}_{m,\bar{k}}, \bar{\mathbb{Q}}_\ell) = 0$ for $i \neq 1, 2$, we conclude

$$\mathcal{H}^{i-1}(L)_0 = \bar{\mathbb{Q}}_\ell^{\binom{n}{i-n}}(n-i),$$

for $n \leq i \leq 2n$, and 0 otherwise. Thus, we get a quasi-isomorphism

$$L_0 \cong \bigoplus_{i=n}^{2n} \bar{\mathbb{Q}}_\ell^{\binom{n}{i-n}}(n-i)[1-i].$$

Away from 0, we have $L = \mathrm{R}\tilde{\pi}_{2!}(\tilde{\mu}^\star \mathcal{L}_\psi)[1]$, where we now regard $\tilde{\pi}_{2!}$ as the projection $X \times \mathbb{G}_m \to \mathbb{G}_m$. Consider the automorphism ϕ of $\mathbb{A}_k^{n+1} \times \mathbb{G}_m$ given by $\phi((x_1, \ldots, x_{n+1}), t) = ((tx_1, \ldots, tx_{n+1}), t)$. The image of $X \times \mathbb{G}_m$ under ϕ is the variety Y defined by the equation $x_1 \cdots x_{n+1} = t^{n+1}$, and $\tilde{\mu} = \tilde{\sigma} \circ \phi$. Since ϕ is an automorphism, $\phi^\star = \mathrm{R}\phi_\star = \mathrm{R}\phi_!$, and we get

$$\begin{aligned} j^\star L &= \mathrm{R}\tilde{\pi}_{2!}(\tilde{\mu}^\star \mathcal{L}_\psi)[1] = \mathrm{R}\tilde{\pi}_{2!}(\phi^\star \tilde{\sigma}^\star \mathcal{L}_\psi)[1] \\ &= \mathrm{R}(\tilde{\pi}_2 \phi)_!(\tilde{\sigma}^\star \mathcal{L}_\psi)[1] = \mathrm{R}\tilde{\pi}_{2!}(\tilde{\sigma}^\star \mathcal{L}_\psi)[1]. \end{aligned}$$

The stalk of $j^\star L$ at a geometric point $t \in \mathbb{G}_{m,\bar{k}}$ is then

$$\mathrm{R}\Gamma_c(\{x_1 \cdots x_{n+1} = t^{n+1}\}, \mathcal{L}_{\psi(\sum x_i)})[1].$$

By [5, Théorème 7.4], we deduce that $\mathcal{H}^i(j^\star L) = 0$ for $i \neq n-1$, and $\mathcal{H}^{n-1}$ $(j^\star L)$ is the pull-back by the $(n+1)$th power map of the Kloosterman sheaf given in [5, Théorème 7.8], and, more generally, in [17, 4.1.1]. Therefore, we have a quasi-isomorphism

$$j^{\star}L \cong [n+1]^{\star}\mathrm{Kl}_{n+1}(\psi)[1-n].$$

Denote by $\mathcal{L}$ the sheaf $[n+1]^{\star}\mathrm{Kl}_{n+1}(\psi)$ on $\mathbb{G}_m$. It is geometrically irreducible, because it is already irreducible as a representation of the inertia group at 0: by [5, Théorème 7.8], the action of a topological generator is unipotent with a single Jordan block. In particular, the invariant subspace for the inertia action at 0 has dimension 1, so the stalk of $j_{\star}j^{\star}L$ at 0 is quasi-isomorphic to $\bar{\mathbb{Q}}_\ell[1-n]$.

Taking stalks at 0 in the distinguished triangle (2.1), we get

$$M_0 \to \bigoplus_{i=n}^{2n} \bar{\mathbb{Q}}_\ell^{\binom{n}{i-n}}(n-i)[1-i] \to \bar{\mathbb{Q}}_\ell[1-n] \to$$

and consequently a quasi-isomorphism

$$M_0 \simeq \bigoplus_{i=n+1}^{2n} \bar{\mathbb{Q}}_\ell^{\binom{n}{i-n}}(n-i)[1-i] = \bigoplus_{i=1}^{n} \bar{\mathbb{Q}}_\ell^{\binom{n}{i}}(-i)[1-n-i].$$

Then since M is punctual supported at 0, the distinguished triangle (1) reads

$$\bigoplus_{i=1}^{n} \bar{\mathbb{Q}}_\ell^{\binom{n}{i}}(-i)[1-n-i]_0 \to L \to j_{\star}\mathcal{L}[1-n] \to .$$

Taking Fourier transform with respect to the complex conjugate character $\bar{\psi}$ and using the facts that $FT_{\bar{\psi}}FT_{\psi}(K) = K(-1)$ and that the Fourier transform of the punctual sheaf $(\bar{\mathbb{Q}}_\ell)_0$ is the shifted constant sheaf $\bar{\mathbb{Q}}_\ell[1]$, we get the distinguished triangle

$$\bigoplus_{i=1}^{n} \bar{\mathbb{Q}}_\ell^{\binom{n}{i}}(-i)[2-n-i] \to K(-1) \to FT_{\bar{\psi}}(j_{\star}\mathcal{L})[1-n] \to .$$

Since $\mathcal{L}$ is a geometrically irreducible sheaf of rank ≥ 2, its direct image $j_{\star}\mathcal{L}$ is a Fourier sheaf in the sense of [17, 8.2] (cf. [17, Lemma 8.3.1]). Then its Fourier transform is a sheaf of the same kind, by [17, Theorem 8.2.5]. Namely, it is the extension by direct image to $\mathbb{A}^1$ of a geometrically irreducible sheaf on a dense open set $U \subset \mathbb{A}^1$, and we get a distinguished triangle

$$\bigoplus_{i=1}^{n} \bar{\mathbb{Q}}_\ell^{\binom{n}{i}}(1-i)[2-n-i] \to K \to \mathcal{F}[1-n] \to,$$

where $\mathcal{F} = FT_{\bar{\psi}}(j_\star\mathcal{L})$ (1). Taking the associated long exact sequence of cohomology sheaves and using the fact that $\mathcal{F}$ has no punctual sections, we get an exact sequence

$$0 \to \bar{\mathbb{Q}}_\ell^n \to \mathcal{H}^{n-1}(K) \to \mathcal{F} \to 0 \tag{2.2}$$

and isomorphisms

$$\mathcal{H}^j(K) \tilde{\to} \bar{\mathbb{Q}}_\ell^{\binom{n}{j-n+2}}(n-1-j) \quad \text{for } n \leq j \leq 2n-2 \tag{2.3}$$

and

$$\mathcal{H}^j(K) = 0 \quad \text{for } j \notin \{n-1, \ldots, 2n-2\}.$$

Thus the cohomology of our family has a “constant part”, which has dimension $\binom{n}{j-n+2}$ and is pure of weight $2(j-n+1)$ on degree j for every $j = n-1, \ldots, 2n-2$, and a non-constant geometrically irreducible part on degree $n-1$ given by the sheaf $\mathcal{F}$. If $n+1$ is prime to p, this sheaf is the pull-back by the $(n+1)$th power map of a hypergeometric sheaf, as defined by Katz in [18] (Section 8). Namely, using the same notation as in the reference, it is $[n+1]^\star\mathrm{Hyp}_{(n+1)^{n+1}}(!, \psi,$all non-trivial characters χ of order dividing $n+1$; n times the trivial character) (cf. [18, Theorem 9.3.2]). We will not make use of this fact in what follows.

Proposition 2.2. *The sheaf $\mathcal{F}$ is smooth of rank n and punctually pure of weight $n-1$ on $U = \mathbb{A}^1_{\bar{k}} - \{(n+1)\zeta : \zeta^{n+1} = 1\}$.*

Proof. If $n+1$ is prime to p, by [10, lemma 1.4], the wild inertia group of $\mathbb{A}^1_{\bar{k}}$ at infinity acts on $\mathcal{L}$ as $\bigoplus_{\zeta^{n+1}=1} \mathcal{L}_{\psi_{(n+1)\zeta}}$, where $\psi_{(n+1)\zeta}(t) = \psi((n+1)\zeta t)$. By [18, Lemma 7.3.9], $\mathcal{F}$ is smooth at $t \in \mathbb{A}^1_{\bar{k}}$ if and only if all breaks of $\mathcal{L} \otimes \mathcal{L}_{\bar{\psi}_t}$ at infinity are ≥ 1. But, as a representation of P_∞,

$$\mathcal{L} \otimes \mathcal{L}_{\bar{\psi}_t} = \bigoplus_{\zeta^{n+1}=1} \mathcal{L}_{\psi_{(n+1)\zeta - t}}$$

has all its breaks equal to 1 unless $t = (n+1)\zeta$ for some $\zeta \in \mu_{n+1}(\bar{k})$. This proves that $\mathcal{F}$ is smooth on U. If p divides $n+1$, all breaks of $\mathcal{L}$ at infinity are <1, so $\mathcal{F}$ is smooth on $U = \mathbb{G}_{m,\bar{k}}$ by [17, 8.5.8].

Since $\mathcal{L}$ is pure of weight n, so is its direct image $j_\star\mathcal{L}[0]$ as a derived category object. The Fourier transform preserves purity and shifts weights by 1, so $\mathcal{F}(-1)[0]$ is pure of weight $n+1$ as a derived category object. In particular, on the open set where $\mathcal{F}$ is smooth, it is punctually pure of weight $(n+1)-2=n-1$. To compute the rank, we use the formula in [18, 7.3.9], which gives

$$\mathrm{rank}(\mathcal{F}) = \mathrm{drop}_0(\mathcal{L}) = (n+1)-1 = n. \qquad \square$$

Proposition 2.3. *There is a non-degenerate pairing $\Phi : \mathcal{F}_U \times \mathcal{F}_U \to \bar{\mathbb{Q}}_\ell$ $(1-n)$ which is symmetric for n odd and skew-symmetric for n even.*

Proof. According to [17, 4.1.3], the dual of the sheaf $\mathrm{Kl}_{n+1}(\psi)$ on $\mathbb{G}_{m,k}$ is $\mathrm{Kl}_{n+1}(\bar\psi)(n-1)$. Therefore, the dual of the object $j_\star\mathcal{L}[0] \in \mathcal{D}^b_c(\mathbb{A}^1_k, \bar{\mathbb{Q}}_\ell)$ is $j_\star\bar{\mathcal{L}}[0](n-1)$, where $\bar{\mathcal{L}} = [n+1]^\star \mathrm{Kl}_{n+1}(\bar\psi)$.

By [20, Théorème 2.1.5], the dual of the Fourier transform with respect to ψ of an object is the Fourier transform with respect to $\bar\psi$ of the dual object. Therefore, the dual of $FT_{\bar\psi}(j_\star\mathcal{L}[0]) = \mathcal{F}[0]$ is $FT_\psi(j_\star\bar{\mathcal{L}}[0](n-1)) = \mathcal{F}[0](n-1)$. In particular, we have a non-degenerate pairing on the open set U, where $\mathcal{F}$ is smooth: $\mathcal{F}_U \times \mathcal{F}_U \to \bar{\mathbb{Q}}_\ell(1-n)$. Since $\mathcal{F}_U$ is irreducible, the pairing is unique up to a scalar and either symmetric of skew-symmetric. The actual sign is given by the usual cup product sign, since $\mathcal{F}$ is a subsheaf of $\mathrm{R}^{n-1}\sigma_!(\bar{\mathbb{Q}}_\ell)$. $\square$

Proposition 2.4. *The sheaf $\mathcal{F}$ is tamely ramified at infinity. The tame inertia group at infinity I_∞^{tame} acts unipotently on $\mathcal{F}_{\bar\eta}$ with a single Jordan block.*

Proof. Since $\mathcal{L}$ is tamely ramified at 0 and the inertia group acts unipotently with a single Jordan block, the same is true for $\mathcal{F}$ at ∞ by [18, Theorem 7.5.4]. $\square$

Proposition 2.5. *Suppose that $n+1$ is prime to p. Then $\mathcal{F}$ is everywhere tamely ramified, and for every $(n+1)$th root of unity ζ in $\bar k$, the action of the inertia group at $(n+1)\zeta$ on $\mathcal{F}_{\bar\eta}$ has invariant subspace of codimension* 1.

Proof. Let ζ be a $(n+1)$th root of unity in $\bar k$. Then

$$\zeta : (x_1, \ldots, x_{n+1}) \to (\zeta x_1, \ldots, \zeta x_{n+1})$$

is an automorphism of X. Therefore,

$$K := \mathrm{R}\sigma_!\bar{\mathbb{Q}}_\ell = \mathrm{R}(\sigma\circ\zeta)_!\bar{\mathbb{Q}}_\ell = \mathrm{R}(\tilde\zeta\circ\sigma)_!\bar{\mathbb{Q}}_\ell = [\tilde\zeta]_\star \mathrm{R}\sigma_!\bar{\mathbb{Q}}_\ell = [\tilde\zeta]_\star K,$$

where $[\tilde{\zeta}] : \mathbb{A}^1_k \to \mathbb{A}^1_k$ is multiplied by ζ. So the sheaf $\mathcal{F}$ is invariant under multiplication by $(n+1)$th roots of unity on $\mathbb{A}^1_k$. In particular, the local monodromies at $(n+1)\zeta$ are isomorphic for all $\zeta \in \mu_{n+1}(\bar{k})$.

By the Euler–Poincaré formula,

$$\chi_c(\mathcal{F}) = \operatorname{rank}(\mathcal{F}) - \sum_{t \in (n+1)\mu_{n+1}(\bar{k})} (\operatorname{drop}_t \mathcal{F} + \operatorname{swan}_t \mathcal{F}),$$

since $\mathcal{F}$ is tamely ramified at infinity and smooth on U. We can compute this Euler characteristic directly:

$$\chi_c(K) = \chi_c(\mathrm{R}\sigma_! \bar{\mathbb{Q}}_\ell) = \chi_c(X, \bar{\mathbb{Q}}_\ell) = 0,$$

since X is a product of copies of $\mathbb{G}_m$. Therefore,

$$\begin{aligned} 0 = \chi_c(K) &= \sum_{j=n-1}^{2n-2} (-1)^j \chi_c(\mathcal{H}^j(K)) \\ &= (-1)^{n-1}\chi_c(\mathcal{F}) + (-1)^{n-1} n + \sum_{j=n}^{2n-2} (-1)^j \binom{n}{j-n+2} \\ &= (-1)^{n-1}\chi_c(\mathcal{F}) + \sum_{j=1}^{n} (-1)^{j+n} \binom{n}{j} = (-1)^{n-1}\chi_c(\mathcal{F}) - (-1)^n, \end{aligned}$$

so $\chi_c(\mathcal{F}) = -1$. We conclude that

$$\sum_{t \in (n+1)\mu_{n+1}(\bar{k})} (\operatorname{drop}_t \mathcal{F} + \operatorname{swan}_t \mathcal{F}) = n+1$$

and therefore the only possibility is $\operatorname{drop}_t \mathcal{F} = 1$ and $\operatorname{swan}_t \mathcal{F} = 0$ for every $t \in (n+1)\mu_{n+1}(\bar{k})$. In particular, $\mathcal{F}$ is everywhere tamely ramified. □

Proposition 2.6. *Suppose that $n+1$ is prime to p, and let $t \in (n+1)\mu_{n+1}(\bar{k})$. If n is even, the inertia group I_t acts trivially on the 1-dimensional space $\mathcal{F}_{\bar{\eta}}/\mathcal{F}_{\bar{\eta}}^{I_t}$. That is, the action of I_t on $\mathcal{F}_{\bar{\eta}}$ is unipotent with a Jordan block of size 2 and all other blocks of size 1. If $t \in \mathbb{F}_q$, the action of a geometric Frobenius element at t on $\mathcal{F}_{\bar{\eta}}^{I_t}$ has one of $\pm q^{(n-2)/2}$ as an eigenvalue, and all other eigenvalues of absolute value $q^{(n-1)/2}$.*

If n is odd, I_t acts on the 1-dimensional space $\mathcal{F}_{\bar{\eta}}/\mathcal{F}_{\bar{\eta}}^{I_t}$ via its unique character of order 2. In particular, the action of I_t on $\mathcal{F}_{\bar{\eta}}$ is semisimple. If $t \in \mathbb{F}_q$, the action of a geometric Frobenius element at t on $\mathcal{F}_{\bar{\eta}}^{I_t}$ has all eigenvalues of absolute value $q^{(n-1)/2}$.

Proof. This can be proven using the Picard–Lefschetz formulas (cf. [6, exposé XV]), since the fibres of $\sigma : X \to \mathbb{A}^1$ have only isolated ordinary quadratic singularities. Alternatively, one may use the explicit description of the monodromy at infinity of the Kloosterman sheaf and Laumon's local Fourier transform theory.

According to [10, Theorem 1.1], the action of the inertia group at infinity on $\mathrm{Kl}_{n+1}(\psi)$ is given by $[n+1]_\star \mathcal{L}_{\psi_{n+1}}$ if n is even and $[n+1]_\star(\mathcal{L}_{\psi_{n+1}} \otimes \mathcal{L}_{\chi_2})$ if n is odd. Therefore, the action on $[n+1]^\star \mathrm{Kl}_{n+1}(\psi)$ is given by $\oplus_{\zeta^{n+1}=1} \mathcal{L}_{\psi_{(n+1)\zeta}}$ if n is even and $\oplus_{\zeta^{n+1}=1} \mathcal{L}_{\psi_{(n+1)\zeta}} \otimes \mathcal{L}_{\chi_2}$ if n is odd (cf. [10, Lemma 1.4]). We conclude by [18, 7.4.1 and 7.5.4].

In particular, the Frobenius eigenvalues of $\mathcal{F}_{\bar\eta}^{I_t}$ all have weight $n-1$ if n is odd by [17, 7.0.8]. If n is even, there are $n-2$ eigenvalues of weight $n-1$ and one of weight $n-2$. Since the local L-function has integral coefficients, the non-real eigenvalues must appear in complex conjugate pairs, and therefore the one with weight $n-2$ must be real, necessarily $\pm q^{(n-2)/2}$. □

Proposition 2.7. *Suppose that p divides $n+1$, and write $n+1 = p^a m$ with $(p, m) = 1$. Then the inertia group at 0 acts with invariant subspace of dimension $m-1$, and its action on the quotient $\mathcal{F}_{\bar\eta}/\mathcal{F}_{\bar\eta}^{I_0}$ is totally wild, with a single break $1/(p^a-1)$ with multiplicity $m(p^a-1) = n-m+1$.*

Proof. In this case,

$$\mathcal{L} = j_\star[n+1]^\star \mathrm{Kl}_{n+1}(\psi) = j_\star[m]^\star[p^a]^\star \mathrm{Kl}_{n+1}(\psi) = j_\star[m]^\star \mathrm{Kl}_{n+1}(\psi'),$$

where ψ' is the additive character given by $\psi'(t) = \psi(t^{p^a})$. We deduce by [17, 1.13.1] that $\mathcal{L}$ is totally wild at ∞ with a single break $m/(n+1) < 1$ with multiplicity $n+1$. Therefore, by [18, 7.5.4], we conclude that $\mathcal{F}$ has break $m/(n-m+1)$ at 0 with multiplicity $n-m+1$. In particular, the Swan conductor at 0 is m.

It remains to compute the tame part of the monodromy at 0. By the Euler–Poincaré formula,

$$-1 = \chi_c(\mathcal{F}) = \dim \mathcal{F}_{\bar\eta}^{I_0} - \mathrm{swan}_0 \mathcal{F} = \dim \mathcal{F}_{\bar\eta}^{I_0} - m.$$

Thus, $\dim \mathcal{F}_{\bar\eta}^{I_0} = m-1$, which is precisely the codimension of the wild part. Therefore, the inertia group at 0 has dimension $m-1$ invariant subspace, and the action in $\mathcal{F}_{\bar\eta}/\mathcal{F}_{\bar\eta}^{I_0}$ is totally wild, with a single break $m/(n-m+1) = 1/(p^a-1)$ with multiplicity $n-m+1$. □

Proposition 2.8. *The L-function of $\mathcal{F}$ on $\mathbb{A}^1_k$ is given by*

$$L(\mathbb{A}^1, \mathcal{F}, T) = 1 - T.$$

The eigenvalues of a geometric Frobenius element F_∞ at infinity acting on $\mathcal{F}$ are $1, q, \ldots, q^{n-1}$.

Proof. By (2.2) and (2.3), we have

$$\begin{aligned} L(\mathbb{A}^1, K, T) &= \prod_{j=n-1}^{2n-2} L(\mathbb{A}^1, \mathcal{H}^j(K), T)^{(-1)^j} \\ &= \prod_{j=n-1}^{2n-2} (1 - q^{j+2-n}T)^{(-1)^{j+1}\binom{n}{j+2-n}} \cdot L(\mathbb{A}^1, \mathcal{F}, T)^{(-1)^{n-1}} \\ &= \prod_{j=1}^{n} (1 - q^j T)^{(-1)^{j+n-1}\binom{n}{j}} \cdot L(\mathbb{A}^1, \mathcal{F}, T)^{(-1)^{n-1}}. \end{aligned}$$

On the other hand, we have

$$L(\mathbb{A}^1, K, T) = L(\mathbb{A}^1, R\sigma_! \bar{\mathbb{Q}}_\ell, T) = Z(X, T).$$

Since X is a product of n copies of the torus $\mathbb{G}_m$, we get

$$L(\mathbb{A}^1, K, T) = \prod_{j=0}^{n} (1 - q^j T)^{(-1)^{j+n-1}\binom{n}{j}}.$$

Comparing both expressions, we conclude that $L(\mathbb{A}^1, \mathcal{F}, T) = 1 - T$.

Let $j : U \to \mathbb{P}^1$ be the inclusion. Since $\mathcal{F}$ is irreducible and not geometrically constant, $H^0(\mathbb{P}^1, j_\star\mathcal{F}) = H^2(\mathbb{P}^1, j_\star\mathcal{F}) = 0$. On the other hand, the Euler–Poincaré formula gives

$$\chi(\mathbb{P}^1, j_\star\mathcal{F}) = n + 1 - \sum_{\zeta^{n+1}=1} 1 = 0,$$

if $n+1$ is prime to p, and

$$\chi(\mathbb{P}^1, j_\star\mathcal{F}) = 1 + \dim \mathcal{F}^{I_0} - \mathrm{Sw}_0\mathcal{F} = 1 + (m-1) - m = 0,$$

if p divides $n+1$. In either case, $H^1(\mathbb{P}^1, j_\star\mathcal{F}) = 0$. Therefore, the L-function of $j_\star\mathcal{F}$ on $\mathbb{P}^1$ is trivial, and we deduce

$$L(\mathbb{A}^1, \mathcal{F}, T) = L(\mathbb{P}^1, j_\star\mathcal{F}, T)\det(1 - TF_\infty|\mathcal{F}^{I_\infty}) = \det(1 - TF_\infty|\mathcal{F}^{I_\infty}).$$

In particular, the action of D_∞/I_∞ on the one-dimensional space $\mathcal{F}^{I_\infty}$ is trivial, and the eigenvalues of a geometric Frobenius element acting on $\mathcal{F}$ are $1, q, \dots, q^{n-1}$ by [17, 7.0.7]. $\square$

Proposition 2.9. *If n is even or p divides $n+1$, the determinant of $\mathcal{F}$ is the geometrically constant sheaf $\bar{\mathbb{Q}}_\ell(-n(n-1)/2)$. If n is odd and $(p, n+1) = 1$,*

$$\det(\mathcal{F}) = \mathcal{L}_{\chi(t^{n+1}-(n+1)^{n+1})}(-n(n-1)/2),$$

where χ is the unique character of order 2 of the inertia group of $\mathbb{A}^1$ at 0 and $\mathcal{L}_{\chi(t^{n+1}-(n+1)^{n+1})}$ is the pull-back of the extension by zero to $\mathbb{A}^1$ of the corresponding Kummer sheaf on $\mathbb{G}_m$ under the map $t \mapsto t^{n+1} - (n+1)^{n+1}$.

Proof. If n is even, there is a non-degenerate symplectic pairing $\mathcal{F} \times \mathcal{F} \to \bar{\mathbb{Q}}_\ell(1-n)$. In particular, $\mathcal{F}$ is geometrically symplectically self-dual, and therefore its determinant is geometrically trivial.

If p divides $n+1$, let $n+1 = p^a m$ as in Proposition 2.7. If ζ is a primitive mth root of unity, exactly as in the proof of Proposition 2.5 we get an isomorphism $\mathcal{F} \cong [t \to \zeta t]^\star \cdot \mathcal{F}$. In particular, there is a sheaf $\mathcal{G}$ on $\mathbb{G}_m$ such that $\mathcal{F}_{|\mathbb{G}_m} = [m]^\star \mathcal{G}$, where $[m] : \mathbb{G}_m \to \mathbb{G}_m$ is the mth power map. By [17, 1.13.1] and Proposition 2.7, as a representation of the wild inertia group at 0, the sheaf $\mathcal{G}$ has a single positive break $1/m(p^a - 1) = 1/(n+1-m)$ with multiplicity $n+1-m = m(p^a-1) > 1$, and Swan conductor 1. At infinity, the inertia group acts quasi-unipotently with a single Jordan block, and after tensoring with a suitable Kummer sheaf we can assume that the action is unipotent. Then $\det \mathcal{G}$ is smooth of rank 1 on $\mathbb{G}_m$, unramified at infinity and its break at $0 \leq 1/(n+1-m) < 1$. Since this break (which is the Swan conductor of $\det \mathcal{G}$ at 0) is an integer, it has to be zero. Thus, $\det \mathcal{G}$ is tamely ramified at zero, and therefore geometrically trivial, and the same is true for $\det \mathcal{F} = [m]^\star \det \mathcal{G}$.

So in both cases, there is some ℓ-adic unit α such that $\det \mathcal{F} \cong \alpha^{\deg}$, where $\alpha^{\deg}$ is the pull-back to $\pi_1(\mathbb{G}_{m,k})$ of the character of

$$\pi_1(\mathbb{G}_{m,k})/\pi_1(\mathbb{G}_{m,\bar{k}}) \cong \mathrm{Gal}(\bar{k}/k)$$

that maps the canonical generator F to α. To find the value of α, we need to compute the determinant of the action of an element of degree 1 of

$\pi_1(\mathbb{G}_{m,k})$ on $\det \mathcal{F}$. But from Proposition 2.8, we know that the action of the geometric Frobenius element at infinity (which has degree 1) on $\mathcal{F}$ has eigenvalues $1, q, \ldots, q^{n-1}$. Therefore, $\alpha = q^{1+2+\cdots+(n-1)} = q^{n(n-1)/2}$, and

$$\det \mathcal{F} \cong (q^{n(n-1)/2})^{\deg} = \bar{\mathbb{Q}}_\ell(-n(n-1)/2).$$

If n is odd and $(p, n+1) = 1$, from Propositions 2.4 and 2.6, we know that $\det \mathcal{F}$ is smooth on $U = \mathbb{A}^1_{\bar{k}} - \{(n+1)\zeta : \zeta^{n+1} = 1\}$, unramified at infinity and tamely ramified at the $n+1$ singular points $(n+1)\zeta$, with the inertia groups acting via their character χ of order two. Therefore, $(\det \mathcal{F}) \otimes \check{\mathcal{L}}_{\chi(t^{n+1}-(n+1)^{n+1})}$ is everywhere unramified, and thus geometrically trivial. So there is some ℓ-adic unit α such that $\det \mathcal{F} \cong \alpha^{\deg} \otimes \mathcal{L}_{\chi(t^{n+1}-(n+1)^{n+1})}$. To find the exact value of α, we again evaluate the determinant at $t = \infty$ to be $q^{n(n-1)/2}$ using Proposition 2.8. On the other hand, using

$$\mathcal{L}_{\chi(t^{n+1}-(n+1)^{n+1})} = \mathcal{L}_{\chi(t^{n+1})} \otimes \mathcal{L}_{\chi(1+((n+1)^{n+1}/t^{n+1}))} = \mathcal{L}_{\chi(1+((n+1)^{n+1}/t^{n+1}))},$$

since χ has order 2 and $n+1$ is even, we conclude that the Frobenius element at infinity acts trivially on $\mathcal{L}_{\chi(t^{n+1}-(n+1)^{n+1})}$, and therefore $\alpha = q^{n(n-1)/2}$ and

$$\det(\mathcal{F}) = \mathcal{L}_{\chi(t^{n+1}-(n+1)^{n+1})}\left(\frac{-n(n-1)}{2}\right).$$

□

Corollary 2.10. *Suppose that n is odd and $(p, n+1) = 1$, and let $t \in \mathbb{F}_q$. Then the action of a geometric Frobenius element F_t at t on $\mathcal{F}$ has $\chi(t^{n+1} - (n+1)^{n+1})q^{(n-1)/2}$ as an eigenvalue (where $\chi : \mathbb{F}_q^\star \to \mathbb{C}^\star$ is the unique character of order 2) and the remaining eigenvalues appear in complex conjugate pairs.*

Proof. From the previous theorem, we know that the product of the eigenvalues is $\chi(t^{n+1} - (n+1)^{n+1})q^{n(n-1)/2}$. They all have absolute value $q^{(n-1)/2}$ and, given that $\mathcal{F}((n-1)/2)$ is self-dual, they are permuted by the map $z \mapsto q^{n-1}/z$. So the non-real eigenvalues show up in complex conjugate pairs. There are an odd number of real eigenvalues, all of them necessarily equal to $q^{(n-1)/2}$ or $-q^{(n-1)/2}$. Grouping them in pairs of identical eigenvalues, we are left with just one, whose sign must be $\chi(t^{n+1} - (n+1)^{n+1})$ (since the product of the other ones is positive). □

Proposition 2.11. *The geometric monodromy group G of $\mathcal{F}$ is given by*

$Sp(n,\Phi)$	*if n is even*
$O(n,\Phi)$	*if n is odd and $(p,n+1)=1$*
$SO(n,\Phi)$	*if n is odd, $p\mid n+1$ and $(p,n)\neq(2,5)$ or $(2,7)$*
G_2 in its standard 7-dimensional representation	*if $p=2$, $n=7$*
$SL(2)$ in sym^4 of its standard representation	*if $p=2$, $n=5$.*

Proof. The connected component G_0 of G containing the identity is semisimple by [4, 1.3.9]. Since G contains a unipotent element with a single Jordan block, its Lie algebra $\mathfrak{g}$ is simple and contains a nilpotent element with a single Jordan block and the representation $\mathfrak{g}\to\mathrm{End}(\mathcal{F}_{\bar\eta})$ is faithful and irreducible, by [17, 11.5.2.3]. By Proposition 2.3, we have an a priori inclusion $G\subset\mathrm{Sp}(n,\Phi)$ for n even and $G\subset O(n,\Phi)$ for n odd.

Suppose that $n+1$ is prime to p. Then G contains pseudo-reflections (i.e., elements with invariant subspace of codimension 1). Since any element in G normalizes $\mathfrak{g}$, from [18, Theorem 1.5], we conclude that $\mathfrak{g}=\mathfrak{sp}_n$ if n is even and $\mathfrak{g}=\mathfrak{so}_n$ if n is odd. Consequently, $G=\mathrm{Sp}(n,\Phi)$ if n is even and $G=\mathrm{SO}(n,\Phi)$ or $O(n,\Phi)$ if n is odd. But the local monodromies at the points $t\in(n+1)\mu_{n+1}(\bar k)$ contain elements of determinant -1, so G must be the full orthogonal group.

When p divides $n+1$, we will make use of the classification theorem in [17, 11.6]. According to it, the possibilities for $\mathfrak{g}$ are: $\mathfrak{sl}_2$ in the $(n-1)$th symmetric power of its standard representation, $\mathfrak{sp}_n$ if n is even, $\mathfrak{so}_n$ if n is odd and $\mathfrak{g}_2$ in its standard 7-dimensional representation if $n=7$.

Suppose that $\mathfrak{g}=\mathfrak{sl}_2$, and let $n+1=p^am$ with m prime to p. As in the proof of Proposition 2.9, we find a smooth sheaf $\mathcal{G}$ on $\mathbb{G}_m$ such that $\mathcal{F}_{|\mathbb{G}_m}=[m]^\star\mathcal{G}$. Since the geometric monodromy group of $\mathcal{F}$ has finite index in that of $\mathcal{G}$, their Lie algebras are the same.

Let G' be the monodromy group of $\mathcal{G}$. The proof of [17, 11.5.2.4] shows that we have a faithful representation $G'\hookrightarrow GL(2)$ if n is even and $G'\hookrightarrow\mathrm{SO}(3)\times\mu_n\subset GL(3)$ if n is odd. Let $\mathcal{H}$ be the corresponding sheaf. As a representation of the wild inertia group P_0 at 0, the breaks of $\mathcal{G}$ are 0 and $1/(n+1-m)$, so the breaks of $\mathcal{H}$ are at most $1/(n+1-m)$. In particular, the Swan conductor of $\mathcal{H}$ as a representation of P_0 is $\leq 2/(n+1-m)$ if n is even ($\leq 3/(n+1-m)$ if n is odd). If $n+1-m>3$ (or >2 if n is even), this automatically implies that $\mathcal{H}$ is tame at zero as a representation of $\pi_1(\mathbb{G}_{m,\bar k})$ (since the Swan conductor is an integer) and therefore if factors through the abelian tame fundamental group of $\mathbb{G}_m$. In particular, the monodromy group would be finite, which contradicts the assumption that $\mathfrak{g} = \mathfrak{sl}_2$. This rules out the possibility $\mathfrak{g} = \mathfrak{sl}_2$ for all cases except $(p, n) = (2, 3)$, $(2, 5)$ or $(3, 2)$.

Therefore, the classification theorem forces $\mathfrak{g} = \mathfrak{sp}_n$ if n is even and $\mathfrak{g} = \mathfrak{so}_n$ if n is odd as long as $(p, n) \neq (2, 3)$, $(2, 5)$, $(2, 7)$ or $(3, 2)$. So in that case $G = \mathrm{Sp}(n, \Phi)$ if n is even, and $G = \mathrm{SO}(n, \Phi)$ if n is odd (since the determinant of $\mathcal{F}$ is geometrically trivial by Proposition 2.9).

If $(p, n) = (2, 3)$, $(2, 7)$ or $(3, 2)$, $n + 1$ is a power of p, so $\mathcal{F}$ is totally wild at 0 with Swan conductor 1. By [17, Theorem 8.7.1], applied to the sheaf $\iota^\star\mathcal{F}$ (where $\iota : \mathbb{G}_m \to \mathbb{G}_m$ is the inversion map), $\iota^\star\mathcal{F}$ is just a translation of a Kloosterman sheaf on $\mathbb{G}_m$, so it has the same geometric monodromy group. Using [17, Theorem 11.1], we conclude that $G = \mathrm{Sp}(n, \Phi)$ if $(p, n) = (3, 2)$, $G = \mathrm{SO}(n, \Phi)$ if $(p, n) = (2, 3)$ and $G = G_2$ if $(p, n) = (2, 7)$.

For the remaining case $p = 2$, $n = 5$, we have two possibilities, $\mathfrak{g} = \mathfrak{so}_5$ or $\mathfrak{g} = \mathfrak{sl}_2$ in the fourth symmetric power of its standard representation. In the first case, G would be SO(5), since the determinant is trivial. We will rule out this possibility by computing the third moment of $\mathcal{F}$ over $\mathbb{F}_{2^{16}}$. Suppose that $G = \mathrm{SO}(5)$, and let V be the stalk of $\mathcal{F}$ at the generic point of $\mathbb{A}^1$, viewed as a representation of SO(5). The alternating square of $\wedge^2 V$ of V is irreducible, and the symmetric square $\mathrm{sym}^2 V$ contains the trivial representation and another irreducible factor W. So $V \otimes V$ decomposes as $\wedge^2 V \oplus \mathbf{1} \oplus W$. None of these irreducible factors is isomorphic to V, so $V \otimes V \otimes V \cong \mathrm{Hom}_G(V \otimes V, V)$ (since V is self-dual) does not contain the trivial representation. Therefore $\mathrm{H}^2_c(\mathbb{G}_{m,\bar{k}}, \mathcal{F}^{\otimes 3})$ vanishes, being the dual of $(V \otimes V \otimes V)^G = 0$. Since $\mathcal{F}^{\otimes 3}$ does not have punctual sections, its H^0_c vanishes too, and then the trace formula gives

$$\left|\sum_{t\in k^\star} \mathrm{Tr}(F_t|\mathcal{F}_t)^3\right| = |\mathrm{Tr}(F|\mathrm{H}^1_c(\mathbb{G}_{m,\bar{k}}, \mathcal{F}^{\otimes 3}))| \leq \dim \mathrm{H}^1_c(\mathbb{G}_{m,\bar{k}}, \mathcal{F}^{\otimes 3}) q^{6+(1/2)}$$

since $\mathcal{F}^{\otimes 3}$ is pure of weight 12. Now $\mathcal{F}^{\otimes 3}$ has rank 125, it is smooth on $\mathbb{G}_m$, tamely ramified at infinity and all its breaks at 0 are $\leq$1 (since the only breaks of $\mathcal{F}$ at 0 are 0 and 1). Therefore, its Swan conductor at 0 is at most 125, and then the Euler–Poincaré formula gives

$$\dim\ H^1_c(\mathbb{G}_{m,\bar{k}}, \mathcal{F}^{\otimes 3}) = -\chi(\mathbb{G}_{m,\bar{k}}, \mathcal{F}^{\otimes 3}) = \mathrm{Sw}_0(\mathcal{F}^{\otimes 3}) \leq 125$$

so

$$\left|\sum_{t\in k^\star} \mathrm{Tr}(F_t|\mathcal{F}_t)^3\right| \leq 125 \cdot q^{6+(1/2)}.$$

Now using the explicit formula given in Proposition 4.1, we find for $k = \mathbb{F}_{2^{16}}$ that

$$\sum_{t \in k^\star} \mathrm{Tr}(F_t|\mathcal{F}_t)^3 \simeq 5.48857 \cdot 10^{33} > 2.5353 \cdot 10^{33} \simeq 125 \cdot 2^{16(6+(1/2))}$$

in contradiction with the inequality above. So $\mathfrak{g} = \mathfrak{sl}_2$ in sym^4 of its standard representation, and therefore $G_0 = \mathrm{SL}(2)$ in sym^4 of its standard representation. G_0 is normal in G, being its identity component. For every $g \in G$, conjugation by g gives an automorphism of G_0. But every automorphism of $S = L(2)$ is inner, so there is an element $g_0 \in G_0$ such that gg_0^{-1} is in the centralizer of G_0. Now the centralizer of G_0 in $\mathrm{GL}(5)$ is the set of scalar matrices (a matrix commuting with all matrices of the form

$$\mathrm{sym}^4 \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \text{ and } \mathrm{sym}^4 \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$$

must already be a scalar). But $G \subset \mathrm{SO}(5)$, and the only scalar matrix in $\mathrm{SO}(5)$ is the identity. Therefore, $g = g_0 \in G_0$, and $G = G_0 = \mathrm{SL}(2)$ in sym^4 of its standard representation. □

3. *L*-functions of symmetric and alternating powers of $\mathcal{F}$

Throughout this section, we will assume that $n+1$ is prime to p. We will describe the L-function of the smooth sheaf $\mathrm{Sym}^a\mathcal{F} \otimes \wedge^b\mathcal{F}$ on the set $U = \mathbb{A}^1_k - \{(n+1)\zeta : \zeta^{n+1} = 1\}$.

Proposition 3.1. *The L-function of $\mathcal{F}$ on U is given by*

$$L(U, \mathcal{F}, T) = (1-T)P(T)^{n+1},$$

where $P(T) \in 1 + T\mathbb{Z}[T]$ is a polynomial of degree $n-1$. If n is odd, all reciprocal roots of $P(T)$ have absolute value $q^{(n-1)/2}$. If n is even, $P(T) = (1 \pm q^{(n-2)/2}T)P_1(T)$, where all reciprocal roots of $P_1(T)$ have absolute value $q^{(n-1)/2}$.

Proof. Since $\mathcal{F}$ is smooth, geometrically irreducible and not geometrically constant on U,

$$L(U, \mathcal{F}, T) = \det(1 - F \cdot T|\mathrm{H}^1_c(U \otimes \bar{k}, \mathcal{F})).$$

If $j : U \to \mathbb{P}^1$ is the inclusion, the Euler–Poincaré formula gives

$$\chi(\mathbb{P}^1_{\bar{k}}, j_\star \mathcal{F}) = 1 + n - (n+1) = 0.$$

Therefore, $\mathrm{H}^i(\mathbb{P}^1_{\bar{k}}, j_\star \mathcal{F}) = 0$ for all i, and we get an isomorphism

$$\mathrm{H}^1_c\left(U \otimes \bar{k}, \mathcal{F}\right) \cong \left(\bigoplus_{\zeta^{n+1}=1} \mathcal{F}^{I_{(n+1)\zeta}} \right) \oplus \mathcal{F}^{I_\infty}.$$

A similar argument gives

$$\mathcal{F}^{I_\infty} \cong \mathrm{H}^1_c(\mathbb{A}^1_{\bar{k}}, \mathcal{F}).$$

By Proposition 2.8, we have then

$$L(U, \mathcal{F}, T) = (1 - T) \prod_{\zeta^{n+1}=1} \det(1 - F \cdot T | \mathcal{F}^{I_{(n+1)\zeta}}).$$

But the isomorphism $\mathcal{F} \cong [\zeta]^\star \mathcal{F}$ implies that

$$P(T) = \det(1 - F \cdot T | \mathcal{F}^{I_{(n+1)\zeta}})$$

is independent of ζ. The absolute values of the reciprocal roots of the polynomial $P(T)$ are given by Proposition 2.6. □

Definition 3.2. Let $P(T)$ be the polynomial of degree $n-1$ in the above proposition. Write

$$P(T) = \prod_{i=1}^{n-1} (1 - \alpha_i T).$$

For a positive integer d, the dth Adams operation of $P(T)$ is defined to be

$$P(d, T) = \prod_{i=1}^{n-1} (1 - \alpha_i^d T).$$

We now turn to the study of the L-function of the sheaf $\mathcal{G}_{a,b} := \mathrm{sym}^a \mathcal{F} \otimes \wedge^b \mathcal{F}$, which is smooth of rank $\binom{n+a-1}{a}\binom{n}{b}$ and pure of weight $(a+b)(n-1)$ on U. Let us find the bad factor of the L-function at infinity first. The local monodromy of $\mathcal{G}_{a,b}$ it infinity is clearly unipotent, since that of $\mathcal{F}$ is. By Proposition 3.1, the eigenvalues of the geometric Frobenius element at infinity acting on $\mathcal{G}_{a,b}$ are $q^{i_1+\cdots+i_a+j_1+\cdots+j_b}$ for all possible choices of integers $0 \leq i_1 \leq i_2 \leq \cdots \leq i_a \leq n-1$ and $0 \leq j_1 < j_2 < \cdots < j_b \leq n-1$. Let $N_{n,a,b,k}$ be the number of such possible choices with $i_1 + \cdots + i_a + j_1 + \cdots + j_b = k$, that is,

$$N_{n,a,b,k} = \#\{(i_1, \ldots, i_a, j_1, \ldots, j_b) : 0 \leq i_1 \leq i_2 \leq \cdots \leq i_a \leq n-1, \\ 0 \leq j_1 < j_2 < \cdots < j_b \leq n-1, i_1 + \cdots + i_a + j_1 + \cdots + j_b = k\}.$$

It is clear that $N_{n,a,b,k} = N_{n,a,b,(a+b)(n-1)-k}$ (just change $i_l \mapsto n-1-i_{a+1-l}$ and $j_l \mapsto n-1-j_{b+1-l}$) and $N_{n,a,b,k} = 0$ for $k < b(b-1)/2$ and $k > (a+b)(n-1) - b(b-1)/2$.

Proposition 3.3. *The dimension of the invariant subspace $\mathcal{G}_{a,b}^{I_\infty}$ is $N_{n,a,b,c}$ where $c = \lfloor (a+b)(n-1)/2 \rfloor$. If $(a+b)(n+1)$ is even, all Jordan blocks for the action of I_∞ on $\mathcal{G}_{a,b}$ have odd size, and the number of blocks of size $2k+1$ is $N_{n,a,b,c-k} - N_{n,a,b,c-k-1}$ for all $k \geq 0$. If $(a+b)(n+1)$ is odd, all Jordan blocks for the action of I_∞ on $\mathcal{G}_{a,b}$ have even size, and the number of blocks of size $2k+2$ is $N_{n,a,b,c-k} - N_{n,a,b,c-k-1}$ for all $k \geq 0$.*

Proof. This is just a translation of [4, 1.8.4] and [17, 7.0.7] to this particular situation, considering that $\mathcal{G}_{a,b}$ is pure of weight $(a+b)(n-1)$ and all Frobenius eigenvalues of $\mathcal{G}_{a,b}$ at infinity are integral powers of q (that is, they have even weight). In fact, the multiplicity $N_{n,a,b,0}$ of the minimun Froebnius eigenvalue q^0 is equal to the number of Jordan blocks with length $(a+b)(n-1)+1$. Removing these blocks, then the multiplicity $N_{n,a,b,1} - N_{n,a,b,0}$ of the minimun remaining Frobenius eigenvalue q is equal to the number of blocks with length $(a+b)(n-1)-1$. By induction, for $0 < k \leq c$, one deduces that $N_{n,a,b,k} - N_{n,a,b,k-1}$ is equal to the number of blocks with length $(a+b)(n-1)-2k+1$ and with minimun Frobenius eigenvalue q^k. The dimension of the invariant subspace $\mathcal{G}_{a,b}^{I_\infty}$ is simply the total number of Jordan blocks:

$$\sum_{k=0}^{c} (N_{n,a,b,k} - N_{n,a,b,k-1}) = N_{n,a,b,c}.$$

□

Corollary 3.4. *The local L-function of $j_\star \mathcal{G}_{a,b}$ at infinity has degree $N_{n,a,b,c}$ and is given by*

$$\det(1 - F_\infty \cdot T | \mathcal{G}_{a,b}^{I_\infty}) = \prod_{k=0}^{c} (1 - q^k T)^{\alpha_{a,b}(k)},$$

where $\alpha_{a,b}(k) = N_{n,a,b,k} - N_{n,a,b,k-1}$ *is the coefficient of* $x^k z^b$ *in the expansion of*

$$\left\{ \frac{(1-x^n)\cdots(1-x^{a+n-1})}{(1-x^2)\cdots(1-x^a)} \right\} (1+z)(1+xz)\cdots(1+x^{n-1}z).$$

Proof. We construct a generating function for $\alpha(k)$ in the following way. Let $C_{n,a,k} = \#\{(i_1,\ldots,i_a) : 0 \le i_1 \le i_2 \le \cdots \le i_a \le n-1, i_1+\cdots+i_a = k\} = \#\{(h_0,\ldots,h_{n-1}) : 0 \le h_i, h_0 + \cdots + h_{n-1} = a, h_1 + 2h_2 + \cdots + (n-1) h_{n-1} = k\}$ (to check that both numbers agree, just let h_j be the number of $l = 1,\ldots,a$ such that $i_l = j$). By [11, Section 3], we have

$$\sum_{k\ge 0}(C_{n,a,k} - C_{n,a,k-1})x^k = \left\{ \frac{(1-x^n)\cdots(1-x^{n+a-1})}{(1-x^2)\cdots(1-x^a)} \right\},$$

where the quantity in the bracket is understood to be $1-x^n$ if $a=1$, and $1-x$ if $a=0$. Let

$$B_{n,b,j} = \#\{(j_1,\ldots,j_b) : 0 \le j_1 < \cdots < j_b \le n-1, j_1+\cdots+j_b = j\}.$$

It is the coefficient of $x^j z^b$ in the expansion of $(1+z)(1+xz)\cdots(1+x^{n-1}z)$. Then

$$N_{n,a,b,k} = \sum_{j=0}^{k} C_{n,a,k-j}B_{n,b,j},$$

and thus

$$\alpha_{a,b}(k) = N_{n,a,b,k} - N_{n,a,b,k-1} = \sum_{j=0}^{k-1}(C_{n,a,k-j} - C_{n,a,k-j-1})B_{n,b,j} + B_{n,b,k}.$$

Therefore $\alpha_{a,b}(k)$ is the coefficient of $x^k z^b$ in the expansion of

$$\left\{ \frac{(1-x^n)\cdots(1-x^{a+n-1})}{(1-x^2)\cdots(1-x^a)} \right\} (1+z)(1+xz)\cdots(1+x^{n-1}z).$$

In particular, the number $N_{n,a,b,c}$ is the coefficient of $x^c z^b$ in the expansion of the power series

$$\frac{(1-x^n)\cdots(1-x^{a+n-1})}{(1-x)\cdots(1-x^a)}(1+z)(1+xz)\cdots(1+x^{n-1}z).$$

□

We now look for the bad factors of the L-function at the finite singular points $t = (n+1)\zeta$ with $\zeta^{n+1} = 1$ and $\zeta \in \mathbb{F}_q$. Suppose that n is even. Then the local monodromy at t is unipotent, with a Jordan block of size 2 and all other blocks of size 1. The Frobenius eigenvalues on $\mathcal{F}^{I_t}$ are $\epsilon q^{(n-2)/2}$, with $\epsilon = 1$ or -1, and $(n-2)/2$ pairs of conjugate complex numbers $\alpha_1, \ldots, \alpha_{(n-1)/2}, \bar{\alpha}_1, \ldots, \bar{\alpha}_{(n-1)/2}$ of absolute value $q^{(n-1)/2}$. That is, as a representation of I_t, $\mathcal{F} \cong U_2 \oplus \mathbf{1}^{n-2}$, where U_m denotes the unique (up to isomorphism) non-trivial unipotent tame representation of I_t of dimension m with a single Jordan block. Therefore, we get isomorphisms

$$\mathrm{Sym}^a \mathcal{F} \cong \bigoplus_{i=0}^{a} \mathrm{Sym}^i U_2 \otimes \mathrm{Sym}^{a-i} \mathbf{1}^{n-2} = \bigoplus_{i=0}^{a} U_{i+1}^{\binom{n-3+a-i}{n-3}}$$
$$\wedge^b \mathcal{F} \cong \wedge^b \mathbf{1}^{n-2} \oplus (U_2 \otimes \wedge^{b-1}\mathbf{1}^{n-2}) \oplus \wedge^{b-2}\mathbf{1}^{n-2} \cong \mathbf{1}^{\binom{n-2}{b-2}+\binom{n-2}{b}} \oplus U_2^{\binom{n-2}{b-1}}.$$

Lemma 3.5. *Let V and W be vector spaces of dimensions $n \geq 2$ and 2, respectively, over an algebraically closed field k of characteristic 0, and let $T : V \to V$ and $U : W \to W$ be unipotent endomorphisms with a single Jordan block. Then $T \otimes U : V \otimes W \to V \otimes W$ is unipotent with two Jordan blocks of sizes $n+1$ and $n-1$.*

Proof. Let $\{\mathbf{x}, \mathbf{y}\}$ be a basis for W such that $U(\mathbf{x}) = \mathbf{x}$ and $U(\mathbf{y}) = \mathbf{x} + \mathbf{y}$. We claim that the invariant subspace of $T \otimes U$ is the subspace of elements that can be written as $\mathbf{v} \otimes \mathbf{x} + (\mathbf{v} - T(\mathbf{v})) \otimes \mathbf{y}$ for $\mathbf{v} \in \mathrm{Ker}((T - I_V)^2)$, which has dimension 2 by hypothesis:

$$(T \otimes U)(\mathbf{v} \otimes \mathbf{x} + (\mathbf{v} - T(\mathbf{v})) \otimes \mathbf{y}) = T(\mathbf{v}) \otimes \mathbf{x} + T(\mathbf{v} - T(\mathbf{v})) \otimes (\mathbf{x} + \mathbf{y})$$
$$= T(\mathbf{v}) \otimes \mathbf{x} + (\mathbf{v} - T(\mathbf{v})) \otimes (\mathbf{x} + \mathbf{y}) = \mathbf{v} \otimes \mathbf{x} + (\mathbf{v} - T(\mathbf{v})) \otimes \mathbf{y}.$$

Conversely, if $(T \otimes U)(\mathbf{v} \otimes \mathbf{x} + \mathbf{w} \otimes \mathbf{y}) = \mathbf{v} \otimes \mathbf{x} + \mathbf{w} \otimes \mathbf{y}$, we get $T(\mathbf{w}) = \mathbf{w}$ and $T(\mathbf{v}) + T(\mathbf{w}) = \mathbf{v}$, so $\mathbf{w} = T(\mathbf{w}) = \mathbf{v} - T(\mathbf{v}) =$ and $(T - I_V)^2(\mathbf{v}) = 0$. This shows that $T \otimes U$ has precisely two Jordan blocks. From

$$T \otimes U - I \otimes I = (T - I) \otimes (U - I) + I \otimes (U - I) + (T - I) \otimes I$$

we get that $(T \otimes U - I \otimes I)^{n+1}$ is a sum of terms $(T - I)^\alpha \otimes (U - I)^\beta$ with $\alpha + \beta \geq n + 1$ and therefore equal to 0, since $(T - I)^n = (U - I)^2 = 0$. So the Jordan blocks of $T \otimes U$ have size $\leq n + 1$. Finally, if $\mathbf{v} \in V$ is a vector such that $\mathbf{w} := (T - I)^{n-1}(\mathbf{v}) \neq 0$ and $\mathbf{x}, \mathbf{y} \in W$ are as above, the same expression shows that $(T \otimes U - I \otimes I)^n(\mathbf{v} \otimes \mathbf{y}) = (n-1)(T - I)^{n-1}(\mathbf{v}) \otimes$

$(U-I)(\mathbf{y}) = (n-1)\mathbf{w}\otimes\mathbf{x} \neq 0$, so $\mathbf{v}\otimes\mathbf{y}$ generates a Jordan block of size $n+1$, and the other block must have size $2n-(n+1) = n-1$. □

Corollary 3.6. *Suppose that n is even. As a representation of I_t, $\mathcal{G}_{a,b} = \operatorname{Sym}^a \mathcal{F} \otimes \wedge^b \mathcal{F}$ is isomorphic to*

$$\bigoplus_{i=0}^{a} U_{i+1}^{\binom{n-3+a-i}{n-3}\left[\binom{n-2}{b-2}+\binom{n-2}{b}\right]} \oplus U_i^{\binom{n-3+a-i}{n-3}\binom{n-2}{b-1}} \oplus U_{i+2}^{\binom{n-3+a-i}{n-3}\binom{n-2}{b-1}} = \bigoplus_{i=0}^{a+2} U_i^{d(i)},$$

where

$$d(0) = \binom{n-3+a}{n-3}\binom{n-2}{b-1},$$

$$d(1) = \binom{n-4+a}{n-3}\binom{n-2}{b-1} + \binom{n-3+a}{n-3}\left[\binom{n-2}{b-2}+\binom{n-2}{b}\right],$$

and for $2 \leq i \leq a+2$,

$$d(i) = \left[\binom{n-3+a-i}{n-3} + \binom{n-1+a-i}{n-3}\right]\binom{n-2}{b-1} + \binom{n-2+a-i}{n-3}\left[\binom{n-2}{b-2}+\binom{n-2}{b}\right].$$

Corollary 3.7. *Suppose that n is even. The local L-function of $j_\star \mathcal{G}_{a,b}$ at t, $\det(1-F_t\cdot T|\mathcal{G}_{a,b}^{I_t})$ has degree*

$$D_{n,a,b} := \sum_{i=0}^{a+2} d(i) = \binom{n-2+a}{n-2}\binom{n}{b}.$$

For every $i = 1,\ldots,a+2$, it has $d(i)$ roots which are pure of weight $(a+b)(n-1)-(i-1)$.

For n odd, the situation is much simpler. In that case, as a representation of I_t, $\mathcal{F} \cong \chi_2 \oplus \mathbf{1}^{n-1}$, where $\chi_2 : I_t \to \bar{\mathbb{Q}}_\ell^\star$ is the unique character of order 2. Therefore, we get isomorphisms

$$\operatorname{Sym}^a \mathcal{F} \cong \bigoplus_{i=0}^{a} \operatorname{Sym}^i \chi_2 \otimes \operatorname{Sym}^{a-i} \mathbf{1}^{n-1} \cong \bigoplus_{\substack{i=0\\ i,\,\text{even}}}^{a} \mathbf{1}^{\binom{n-2+a-i}{n-2}} \oplus \bigoplus_{\substack{i=0\\ i,\,\text{odd}}}^{a} \chi_2^{\binom{n-2+a-i}{n-2}}$$

$$\wedge^b \mathcal{F} \cong \wedge^b \mathbf{1}^{n-1} \oplus (\chi_2 \otimes \wedge^{b-1}\mathbf{1}^{n-1}) \cong \mathbf{1}^{\binom{n-1}{b}} \oplus \chi_2^{\binom{n-1}{b-1}}.$$

$$\mathrm{Sym}^a \mathcal{F} \otimes \wedge^b \mathcal{F} \cong \mathbf{1}^{\alpha\binom{n-1}{b}+\beta\binom{n-1}{b-1}} \oplus \chi_2^{\alpha\binom{n-1}{b-1}+\beta\binom{n-1}{b}}$$

where

$$\alpha = \sum_{\substack{i=0\\ i,\,\mathrm{even}}}^{a} \binom{n-2+a-i}{n-2}, \beta = \sum_{\substack{i=0\\ i,\,\mathrm{odd}}}^{a} \binom{n-2+a-i}{n-2}$$

Corollary 3.8. *Suppose that n is odd. The local L-function of $j_\star\mathcal{G}_{a,b}$ at t, $\det(1 - F_t \cdot T|\mathcal{G}_{a,b}^{I_t})$ has degree*

$$D_{n,a,b} := \binom{n-1}{b} \sum_{\substack{i=0\\ i,\,\mathrm{even}}}^{a} \binom{n-2+a-i}{n-2} + \binom{n-1}{b-1} \sum_{\substack{i=0\\ i,\,\mathrm{odd}}}^{a} \binom{n-2+a-i}{n-2}.$$

All its roots are pure of weight $(a+b)(n-1)$.

Consider the sheaf $j_\star\mathcal{G}_{a,b}$ on $\mathbb{P}^1$. Since $\check{\mathcal{F}} \cong \mathcal{F}(n-1)$, we get an isomorphism $\check{\mathcal{G}}_{a,b} \cong \mathcal{G}_{a,b}((n-1)(a+b))$. By Poincaré duality, we conclude that there is a perfect pairing of $\mathrm{Gal}(\bar{k}/k)$-modules

$$H^i(\mathbb{P}^1_{\bar{k}}, j_\star\mathcal{G}_{a,b}) \times H^{2-i}(\mathbb{P}^1_{\bar{k}}, j_\star\mathcal{G}_{a,b}) \to \bar{\mathbb{Q}}_\ell((a+b)(1-n)-1)$$

for $i = 0, 1, 2$. Since $\mathcal{G}_{a,b}$ is smooth on U, the zeroth cohomology group $H^0(\mathbb{P}^1_{\bar{k}}, j_\star\mathcal{G}_{a,b})$ corresponds to the maximal geometrically constant subsheaf of $\mathcal{G}_{a,b}$. Since $\mathcal{G}_{a,b}$ is pure of weight $(n-1)(a+b)$ and all Frobenius eigenvalues of $j_\star\mathcal{G}_{a,b}$ at infinity are integral powers of q, such a subsheaf must be a direct sum of copies of $\bar{\mathbb{Q}}_\ell((1-n)(a+b)/2)$. Incidentally, this shows that $H^0(\mathbb{P}^1_{\bar{k}}, j_\star\mathcal{G}_{a,b}) = 0$ if $(n-1)(a+b)$ is odd. Therefore, we have:

Proposition 3.9. *The L-function of $j_\star\mathcal{G}_{a,b}$ on $\mathbb{P}^1$ has the form*

$$L(\mathbb{P}^1, j_\star\mathcal{G}_{a,b}) = \frac{P_{a,b}(T)}{(1-q^{(a+b)(n-1)/2}T)^{\delta_{a,b}}(1-q^{(a+b)(n-1)/2+1}T)^{\delta_{a,b}}},$$

where $\delta_{a,b} = \dim H^0(\mathbb{P}^1_{\bar{k}}, j_\star\mathcal{G}_{a,b})$, and $P_{a,b}(T)$ is a polynomial that satisfies the functional equation

$$P_{a,b}(T) = \pm T^r q^{((a+b)(n-1)+1)r/2} P_{a,b}\left(\frac{1}{q^{(a+b)(n-1)+1}T}\right),$$

where $r = \deg(P_{a,b})$.

Proof. We have just seen that

$$H^0(\mathbb{P}^1_{\bar{k}}, j_\star \mathcal{G}_{a,b}) = \bar{\mathbb{Q}}_\ell \left(\frac{(1-n)(a+b)}{2} \right)^{\delta_{a,b}},$$

and Poincaré duality implies that

$$H^2(\mathbb{P}^1_{\bar{k}}, j_\star \mathcal{G}_{a,b}) = \bar{\mathbb{Q}}_\ell \left(\frac{(1-n)(a+b)}{2} - 1 \right)^{\delta_{a,b}}.$$

This gives the denominator.

The numerator is $P_{a,b}(T) = (1-\alpha_1 T)\cdots(1-\alpha_r T)$, where $\alpha_1, \dots, \alpha_r$ are the Frobenius eigenvalues of $H^1(\mathbb{P}^1_{\bar{k}}, j_\star \mathcal{G}_{a,b})$. By Poincaré duality, these eigenvalues are permuted by $\alpha \mapsto q^{(a+b)(n-1)+1}/\alpha$. In particular, $(\prod \alpha_i)^2 = q^{((a+b)(n-1)+1)r}$. We have

$$\begin{aligned} P_{a,b}\left(\frac{1}{q^{(a+b)(n-1)+1} T} \right) &= \left(1 - \frac{1}{\alpha_1 T}\right) \cdots \left(1 - \frac{1}{\alpha_r T}\right) \\ = \frac{1}{\alpha_1 \cdots \alpha_r T^r} (\alpha_1 T - 1) \cdots (\alpha_r T - 1) &= \frac{(-1)^r}{\pm T^r q^{((a+b)(n-1)+1)r/2}} P_{a,b}(T) \end{aligned}$$

and the functional equation follows. □

To find the dimension of $H^0(\mathbb{P}^1_{\bar{k}}, j_\star \mathcal{G}_{a,b})$, we will use the knowledge of the global monodromy of $\mathcal{F}$, as in [19]. Let V be the geometric generic fibre of $\mathcal{F}$, regarded as a representation of $\pi_1(U \otimes \bar{k})$. We know that the Zariski closure G of the image of $\pi_1(U \otimes \bar{k})$ in $\mathrm{GL}(V)$ is $\mathrm{Sp}(n)$ if n is even and $O(n)$ if n is odd. The dimension we are looking for is

$$\dim(\mathrm{Sym}^a(V) \otimes \wedge^b(V))^G = \dim \mathrm{Hom}_G(\mathrm{Sym}^a(V), \wedge^b(V)),$$

since V is self-dual as a representation of G.

Suppose $n = 2m$ is even. The representations of $G = \mathrm{Sp}(n)$ are in one-to-one correspondence with the representations of the Lie algebra $\mathfrak{g} = \mathfrak{sp}_n$. If $L_1, \dots, L_m$ are generators of the weight lattice for $\mathfrak{g}$, then $\mathrm{Sym}^d V$ is the irreducible representation with maximal weight dL_1, and the kernel of the natural contraction map $\wedge^d V \to \wedge^{d-2} V$ is the irreducible representation of maximal weight $L_1 + \cdots + L_d$ for $1 \le d \le m$ [13, ch.17]. Therefore, we have

$$\wedge^b V \cong W(L_1 + \cdots + L_b) \oplus W(L_1 + \cdots + L_{b-2}) \oplus \cdots \oplus V,$$

if $b \le m$ is odd and

$$\wedge^b V \cong W(L_1 + \cdots + L_b) \oplus W(L_1 + \cdots + L_{b-2}) \oplus \cdots \oplus \mathbf{1},$$

if $b \leq m$ is even and $\wedge^b V \cong \wedge^{n-b} V$ for $m \leq b \leq n$. So $\mathrm{Sym}^a V \otimes \wedge^b V$ contains exactly one copy of the trivial representation if $a = 0$ and $b \leq n$ is even or if $a = 1$ and $b \leq n$ is odd, and does not contain the trivial representation otherwise.

Suppose $n = 2m + 1$ is odd. The representations of $\mathrm{SO}(n)$, the connected component of G containing the identity, are in one-to-one correspondence with the representations of the Lie algebra $\mathfrak{g} = \mathfrak{so}_n$ contained in the tensor algebra of the standard representation. Each of them gives rise to two different representations of $O(n)$ (given one of them, the other one is obtained by tensoring with the determinant). If $L_1, \ldots, L_m$ are generators of the weight lattice for $\mathfrak{g}$, then $\wedge^d V$ is the irreducible representation with maximal weight $L_1 + \cdots + L_d$ for $d \leq m$, $\wedge^d V \cong \wedge^{n-d} V$ for $m + 1 \leq d \leq n$, and the kernel of the natural contraction map $\mathrm{Sym}^d V \to \mathrm{Sym}^{d-2} V$ is the irreducible representation of maximal weight dL_1 (cf. [13, ch.19]). Therefore we have

$$\mathrm{Sym}^a V \cong W(aL_1) \oplus W((a-2)L_1) \oplus \cdots \oplus V,$$

if a is odd and

$$\mathrm{Sym}^a V \cong W(aL_1) \oplus W((a-2)L_1) \oplus \cdots \oplus \mathbf{1},$$

if a is even. So $\mathrm{Sym}^a V \otimes \wedge^b V$ (as a representation of $\mathfrak{g}$) contains exactly one copy of the trivial representation if a is even and $b = 0$ or n, or if a is odd and $b = 1$ or $n - 1$, and does not contain the trivial representation otherwise.

For G itself, since the determinant becomes trivial only in even tensor powers of the standard representation, we get that $\mathrm{Sym}^a V \otimes \wedge^b V$ contains exactly one copy of the trivial representation and no copies of the determinant representation if a is even and $b = 0$, or if a is odd and $b = 1$. It contains exactly one copy of the determinant representation and no copies of the trivial representation if a is even and $b = n$ or if a is odd and $b = n - 1$. It does not contain the trivial or the determinant representations otherwise. Therefore we get

Proposition 3.10. *The dimension $\delta_{a,b} = \dim H^0(\mathbb{P}^1_{\bar{k}}, j_\star \mathcal{G}_{a,b})$ is*

$$\text{if } n \text{ is even} \begin{cases} 1 & \text{if } a = 0 \text{ and } b \leq n \text{ is even or } a = 1 \text{ and } b \leq n \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{if } n \text{ is odd} \begin{cases} 1 & \text{if } a \text{ is even and } b = 0 \text{ or } a \text{ is odd and } b = 1 \\ 0 & \text{otherwise} \end{cases}$$

Putting everything together, we get the following expression for the L-function of $\mathcal{G}_{a,b}$, in the case where $n+1$ divides $q-1$ (which is always true after a finite extension of the base field):

Theorem 3.11. *Assume that $(n+1)$ divides $(q-1)$. The L-function of $\mathcal{G}_{a,b}$ on U has degree $n\binom{n+a-1}{a}\binom{n}{b}$ and is given by*

$$L(U,\mathcal{G}_{a,b}) = \frac{P_{a,b}(T)Q_{a,b}(T)^{n+1}\prod_{k=0}^{[(a+b)(n-1)/2]}(1-q^kT)^{\alpha_{a,b}(k)}}{(1-q^{(a+b)(n-1)/2}T)^{\delta_{a,b}}(1-q^{(a+b)(n-1)/2+1}T)^{\delta_{a,b}}}$$

where $\delta_{a,b}=0$ or 1 is given by Proposition 3.10, $\alpha_{a,b}(k)=N_{n,a,b,k}-N_{n,a,b,k-1}$, $Q_{a,b}(T)$ is a polynomial whose degree $D_{n,a,b}$ and the weights of its roots are given in Corollaries 3.7 and 3.8 and $P_{a,b}(T)$ is a polynomial in $1+T\mathbb{Z}[T]$ of degree

$$n\binom{n+a-1}{a}\binom{n}{b}+2\delta_{a,b}-N_{n,a,b,c}-(n+1)D_{n,a,b},$$

where $c=[(a+b)(n-1)/2]$. Furthermore, $P_{a,b}(T)$ is pure of weight $(a+b)(n-1)+1$ and it satisfies the functional equation

$$P_{a,b}(T)=\pm T^r q^{((a+b)(n-1)+1)r/2}P_{a,b}(1/q^{(a+b)(n-1)+1}T).$$

Proof. For $t\in(n+1)\mu_{n+1}$, the factor

$$Q_{a,b}(T)=\det(1-F_t\cdot T|\mathcal{G}_{a,b}^{I_t})$$

is independent of t. Its degree $D_{n,a,b}$ and the weights of its roots are given in Corollaries 3.7 and 3.8.

The degree of the L-function is the negative Euler characteristic $-\chi(U,\mathcal{G}_{a,b})$. Since $\mathcal{G}_{a,b}$ is everywhere tamely ramified, this Euler characteristic is $\chi(U)\mathrm{rank}(\mathcal{G}_{a,b})=-n\binom{n+a-1}{a}\binom{n}{b}$. The stated formula is just the decomposition

$$\begin{aligned}L(U,\mathcal{G}_{a,b}) = {} & L(\mathbb{P}^1,j_\star\mathcal{G}_{a,b})\det(1-F_\infty\cdot T|\mathcal{G}_{a,b}^{I_\infty})\\ & \times\prod_{t\in(n+1)\mu_{n+1}}\det(1-F_t\cdot T|\mathcal{G}_{a,b}^{I_t}).\end{aligned}$$

The shape of each of the factors has already been determined. □

Corollary 3.12. *The L-function of $\mathcal{G}_{a,b}$ on $\mathbb{A}^1$ is given, with the same notation as in the previous theorem, by*

$$L(\mathbb{A}^1,\mathcal{G}_{a,b}) = \frac{P_{a,b}(T)E_{a,b}(T)^{n+1}\prod_{k=0}^{[(a+b)(n-1)/2]}(1-q^kT)^{\alpha_{a,b}(k)}}{(1-q^{(a+b)(n-1)/2}T)^{\delta_{a,b}}(1-q^{(a+b)(n-1)/2+1}T)^{\delta_{a,b}}}$$

where the polynomial $E_{a,b}(T)$ is a factor of $Q_{a,b}(T)$, whose degree is given by

$$\begin{array}{ll} \binom{n-2+a}{n-2}\binom{n-1}{b-1} & \textit{if } n \textit{ is even} \\ \binom{n-1}{b}\sum\limits_{\substack{i=2\\ i,\,\text{even}}}^{a}\binom{n-2+a-i}{n-2}+\binom{n-1}{b-1}\sum\limits_{\substack{i=0\\ i,\,\text{odd}}}^{a}\binom{n-2+a-i}{n-2} & \textit{if } n \textit{ is odd} \end{array}$$

Its reciprocal roots are pure of weight $(a+b)(n-1)$ if n is odd, and mixed of weights $\leq(a+b)(n-1)$ if n is even.

Proof. Let $j : U \hookrightarrow \mathbb{A}^1$ be the inclusion. The isomorphism $\mathcal{F} \cong j_\star j^\star\mathcal{F}$ gives an injection of sheaves $\mathcal{G}_{a,b} = \mathrm{Sym}^a\,\mathcal{F}\otimes\wedge^b\mathcal{F} = \mathrm{Sym}^a\, j_\star j^\star\mathcal{F}\otimes\wedge^b j_\star j^\star\mathcal{F}\hookrightarrow j_\star j^\star(\mathrm{Sym}^a\,\mathcal{F}\otimes\wedge^b\mathcal{F}) = j_\star j^\star\mathcal{G}_{a,b}$.

For $t\in(n+1)\mu_{n+1}$, let

$$Q'_{a,b}(T) = \det(1-F_t\cdot T|\mathrm{Sym}^a\,\mathcal{F}_t\otimes\wedge^b\,\mathcal{F}_t),$$

which does not depend on the choice of t. Since $\mathcal{F}_t$ has dimension $n-1$, $Q'_{a,b}(T)$ is a polynomial of degree $\binom{n-2+a}{n-2}\binom{n-1}{b}$ and, by the previous injection, it divides the polynomial $Q_{a,b}(T)$. The L-function of $\mathcal{G}_{a,b}$ on $\mathbb{A}^1$ is then given by

$$\begin{aligned} L(\mathbb{A}^1,\mathcal{G}_{a,b}) &= \frac{L(U,\mathcal{G}_{a,b})}{Q'_{a,b}(T)^{n+1}} \\ &= \frac{P_{a,b}(T)E_{a,b}(T)^{n+1}\prod_{k=0}^{[(a+b)(n-1)/2]}(1-q^kT)^{\alpha_{a,b}(k)}}{(1-q^{(a+b)(n-1)/2}T)^{\delta_{a,b}}(1-q^{(a+b)(n-1)/2+1}T)^{\delta_{a,b}}}, \end{aligned}$$

where $E_{a,b}(T) = Q_{a,b}(T)/Q'_{a,b}(T)$ is a polynomial of degree $D_{n,a,b}-\binom{n-2+a}{n-2}\binom{n-1}{b}$. Replacing $D_{n,a,b}$ by its explicit value gives the formulae stated. □

Theorem 3.13. *Assume that $(n+1)$ divides $(q-1)$. Let*

$$P_d(T) = \prod_{b=0}^{n}P_{d-b,b}(T)^{(-1)^{b-1}(b-1)},\quad Q_d(T) = \prod_{b=0}^{n}Q_{d-b,b}(T)^{(-1)^{b-1}(b-1)}.$$

Then, the L-function of $[\mathcal{F}]^d$ on U is given by

$$L(U,[\mathcal{F}]^d) = P_d(T)Q_d(T)^{n+1}R_d(T)\prod_{k=0}^{[(n-2)/2]}\frac{1-q^{dk}T}{1-q^{dk+1}T},$$

where $R_d(T)$ is given by

$$R_d(T) = (1-q^{d(n-1)/2}T)(1-q^{(d(n-1)/2)+1}T)(1-q^{(d(n-2)/2)+1}T)$$

if n and d are even,

$$R_d(T) = (1-q^{(d(n-2)/2)+1}T)$$

if n is even and d is odd,

$$R_d(T) = (1-q^{(d(n-1)/2)}T)$$

if n and d are odd,

$$R_d(T) = (1-q^{(d(n-1)/2)+1}T)^{-1}$$

if n is odd and d is even.

Proof. From the L-function decomposition

$$L(U,[\mathcal{F}]^d) = \prod_{b=0}^{n} L(U,\mathcal{G}_{d-b,b})^{(-1)^{b-1}(b-1)}$$

and Theorem 3.11, we get

$$L(U,[\mathcal{F}]^d) = P_d(T)Q_d(T)^{n+1}\frac{\prod_{k=0}^{[d(n-1)/2]}(1-q^kT)^{\sum_{b=0}^{n}(-1)^{b-1}(b-1)\alpha_{d-b,b}(k)}}{(1-q^{d(n-1)/2}T)^{\delta_d}(1-q^{d(n-1)/2+1}T)^{\delta_d}}$$

where

$$\delta_d = \sum_{b=0}^{n}(-1)^{b-1}(b-1)\delta_{d-b,b}.$$

Using Proposition 3.10, we find $\delta_d = -d+(d-1) = -1$ if n and d are even, $\delta_d = 1$ if n is odd and d is even and $\delta_d = 0$ if d is odd. Let $N(T)$ denote the numerator of the previous expression. It remains to compute $N(T)$.

Let

$$\mathcal{H}_+ = \oplus_{b=0, b \text{ odd}}^n \mathcal{G}_{d-b,b}^{b-1}, \quad \mathcal{H}_- = \oplus_{b=0, b \text{ even}}^n \mathcal{G}_{d-b,b}^{b-1}.$$

These are "real" sheaves. Then,

$$N(T) = \frac{\det(I - F_\infty T | \mathcal{H}_+^{I_\infty})}{\det(I - F_\infty T | \mathcal{H}_-^{I_\infty})}.$$

We know that $\mathcal{H}_+$ and $\mathcal{H}_-$ are pure of weight $n-1$, the inertia group I_∞ acts unipotently on them and all their Frobenius eigenvalues at infinity are integral powers of q. If $\mu(k)$ (resp. $\nu(k)$) is the number of Frobenius eigenvalues of $\mathcal{H}_+$ (resp. of $\mathcal{H}_-$) at infinity which are equal to q^k, then by [4, 1.8.4] and [17, 7.0.7], we deduce

$$\det(I - F_\infty T | \mathcal{H}_+^{I_\infty}) = \prod_{k=0}^{[d(n-1)/2]} (1 - q^k T)^{(\mu(k) - \mu(k-1))}.$$

Similarly,

$$\det(I - F_\infty T | \mathcal{H}_-^{I_\infty}) = \prod_{k=0}^{[d(n-1)/2]} (1 - q^k T)^{(\nu(k) - \nu(k-1))}.$$

Thus,

$$N(T) = \prod_{k=0}^{[d(n-1)/2]} (1 - q^k T)^{(\mu(k) - \nu(k)) - (\mu(k-1) - \nu(k-1))}.$$

On the other hand, for every $r \geq 1$ the trace of the action of the drth power of the local Frobenius at infinity on $[\mathcal{F}]^d$ is

$$\text{Trace}(F_\infty^{dr} | \mathcal{F}) = 1 + q^{dr} + \cdots + q^{dr(n-1)}.$$

But

$$\text{Trace}(F_\infty^{dr} | \mathcal{F}) = \text{Trace}(F_\infty^{r} | [\mathcal{F}]^d) = \sum_{k \geq 0} (\mu(k) - \nu(k)) q^{kr}.$$

Since this holds for every $r \geq 1$, we conclude that $\mu(k) - \nu(k) = 1$ if $k = 0, d, \ldots, (n-1)d$ and 0 otherwise. Therefore,

$$\begin{aligned} N(T) &= \prod_{k=0}^{[d(n-1)/2]} (1 - q^k T)^{(\mu(k)-\nu(k))-(\mu(k-1)-\nu(k-1))} \\ &= (1 - q^{d(n-1)/2} T) \prod_{k=0}^{n-3/2} \frac{1 - q^{dk} T}{1 - q^{dk+1} T} \end{aligned}$$

if n is odd, and

$$\begin{aligned} N(T) &= \prod_{k=0}^{[d(n-1)/2]} (1 - q^k T)^{(\mu(k)-\nu(k))-(\mu(k-1)-\nu(k-1))} \\ &= (1 - q^{(d(n-2)/2)+1} T) \prod_{k=0}^{(n/2)-1} \frac{1 - q^{dk} T}{1 - q^{dk+1} T} \end{aligned}$$

if n is even. This combined with the explicit description of δ_d shows that

$$\frac{N(T)}{(1 - q^{d(n-1)/2} T)^{\delta_d} (1 - q^{d(n-1)/2+1} T)^{\delta_d}} = R_d(T) \prod_{k=0}^{[(n-2)/2]} \frac{1 - q^{dk} T}{1 - q^{dk+1} T}.$$

The theorem is proved. □

We can now finish the proof of Theorem 1.1. By Theorem 2.1, we deduce

$$\begin{aligned} L(\mathbb{A}^1, [\mathcal{H}^{n-1}(K)]^d) &= L(\mathbb{A}^1, [\mathcal{F}]^d) L(\mathbb{A}^1, \bar{\mathbb{Q}}_\ell^n) = L(\mathbb{A}^1, [\mathcal{F}]^d)(1 - qT)^{-n} \\ &= L(U, [\mathcal{F}]^d) P(d, T)^{-(n+1)} (1 - qT)^{-n}, \end{aligned}$$

where $P(d, T)$ is the dth Adams operation of the polynomial $P(T)$ in Definition 3.2. On the other hand, for $n \leq j \leq 2(n-1)$,

$$\begin{aligned} L(\mathbb{A}^1, [\mathcal{H}^j(K)]^d) &= L(\mathbb{A}^1, \bar{\mathbb{Q}}_\ell(d(n-1-j)))^{\binom{n}{j-n+2}} \\ &= (1 - q^{d(j-(n-1))+1} T)^{-\binom{n}{j-n+2}}. \end{aligned}$$

Also, by Theorem 2.1 and the Grothendieck trace formula,

$$Z_d(\mathbb{A}^1, X_\lambda) = \prod_{j=n-1}^{2(n-1)} L(\mathbb{A}^1, [\mathcal{H}^j(K)]^d)^{(-1)^j}.$$

Substituting the above calculation, we obtain

$$Z_d(\mathbb{A}^1, X_\lambda)^{(-1)^{n-1}} = L(U, [\mathcal{F}]^d) P(d,T)^{-(n+1)} \prod_{i=0}^{n-1} (1 - q^{di+1}T)^{(-1)^{i+1}\binom{n}{i+1}}.$$

This together with Corollary 3.11 gives Theorem 1.1. The proof is complete.

4. Zeta function in terms of Gauss sums

In this section, we give an elementary formula for the number $N_q(\lambda)$ of $\mathbb{F}_q$-rational points in the fibre X_λ in terms of Gauss sums for every $\lambda \in \mathbb{F}_q$. This type of elementary formulas for a general equation can be found in Koblitz [23]. We derive a more explicit formula in the special case of X_λ and in particular deduce an explicit formula for the zeta function of X_0. This allows us to determine the rank of the sheaf $\mathcal{F}$ when p divides $n+1$ and the local factor at 0 of the sheaf $\mathcal{F}$.

Let $\omega : \mathbb{F}_q^\star \to \mathbb{C}^*$ be a primitive character of order $q-1$. For every $k \in \mathbb{Z}$, define the Gauss sum $G_q(k)$ by

$$G_q(k) = -\sum_{a \in \mathbb{F}_q^\star} \omega(a)^{-k} \zeta_p^{\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(a)},$$

where $\zeta_p = \exp(2\pi \mathrm{i}/p)$. It is clear that $G_q(k) = 1$ if $(q-1)|k$, and $|G_k(q)| = \sqrt{q}$ otherwise. We have the inversion formula

$$\zeta_p^{\mathrm{Tr}(a)} = \sum_{k=0}^{q-2} \frac{G_q(k)}{1-q} \omega(a)^k, \tag{4.1}$$

for every $a \in \mathbb{F}_q^\star$. We find that

$$\begin{aligned} N_q(\lambda) &= \frac{1}{q} \sum_{x_0 \in \mathbb{F}_q} \sum_{x_1, \dots, x_n \in \mathbb{F}_q^\star} \zeta_p^{\mathrm{Tr}(x_0x_1 + \cdots + x_0x_n + (x_0/x_1\cdots x_n) - x_0\lambda)} \\ &= \frac{(q-1)^n}{q} + \frac{1}{q} S_q(\lambda), \end{aligned}$$

where

$$S_q(\lambda) = \sum_{x_0, x_1, \dots, x_n \in \mathbb{F}_q^\star} \zeta_p^{\mathrm{Tr}(x_0x_1 + \cdots + x_0x_n + (x_0/x_1\cdots x_n) - x_0\lambda)}.$$

Using Equation 4.1, we deduce for $\lambda \neq 0$:

$$
\begin{aligned}
S_q(\lambda) &= \sum_{x_0, x_1, \ldots, x_n \in \mathbb{F}_q^\star} \zeta_p^{\mathrm{Tr}(x_0 x_1)} \cdots \zeta_p^{\mathrm{Tr}(x_0 x_n)} \zeta_p^{\mathrm{Tr}(x_0/x_1 \cdots x_n)} \zeta_p^{\mathrm{Tr}(-x_0 \lambda)} \\
&= \sum_{k_1, \ldots, k_{n+2}=0}^{q-2} \frac{G_q(k_1) \cdots G_q(k_{n+2})}{(1-q)^{n+2}} \\
&\quad \sum_{y_i^{q-1}=1} (y_0 y_1)^{k_1} \cdots (y_0 y_n)^{k_n} \left(\frac{y_0}{y_1 \cdots y_n} \right)^{k_{n+1}} (y_0 \omega(-\lambda))^{k_{n+2}} \\
&= \sum_{k_1, \ldots, k_{n+2}=0}^{q-2} \frac{G_q(k_1) \cdots G_q(k_{n+2})}{(1-q)^{n+2}} \omega(-\lambda)^{k_{n+2}} \\
&\quad \times \sum_{y_i^{q-1}=1} y_0^{k_1 + \cdots + k_{n+2}} y_1^{k_1 - k_{n+1}} \cdots y_n^{k_n - k_{n+1}} \\
&= (-1)^n \sum_{\substack{a,b=0 \\ (n+1)a+b \equiv 0 (q-1)}}^{q-2} \frac{G_q(a)^{n+1} G_q(b)}{q-1} \omega(-\lambda)^b \\
&= (-1)^n \left(\frac{1}{q-1} + \sum_{\substack{(n+1)a+b \equiv 0 (q-1) \\ (a,b) \neq (0,0)}} \frac{G_q(a)^{n+1} G_q(b)}{q-1} \omega(-\lambda)^b \right).
\end{aligned}
$$

Thus, we obtain

Proposition 4.1. *If $\lambda \neq 0$, the number of $\mathbb{F}_q$-rational points in X_λ is given by*

$$
\begin{aligned}
N_q(\lambda) &= \frac{(q-1)^n - (-1)^n}{q} + \frac{(-1)^n}{q-1} + \frac{(-1)^n}{q(q-1)} \\
&\qquad \times \sum_{\substack{(n+1)a+b \equiv 0 (q-1) \\ (a,b) \neq (0,0)}} G_q(a)^{n+1} G_q(b) \omega(-\lambda)^b \\
&= \frac{(q-1)^n - (-1)^n}{q} + \frac{(-1)^n}{q-1} + \frac{(-1)^n}{q(q-1)} \\
&\qquad \times \sum_{k=1}^{q-2} G_q(k)^{n+1} G_q(-(n+1)k) \omega(-\lambda)^{-(n+1)k}.
\end{aligned}
$$

If $\lambda = 0$, then Equation 4.1 gives

$$\begin{aligned}
S_q(0) &= \sum_{x_0,x_1,\dots,x_n\in\mathbb{F}_q^\star} \zeta_p^{\mathrm{Tr}(x_0x_1+\cdots+x_0x_n+(x_0/x_1\cdots x_n))} \\
&= \sum_{x_0,x_1,\dots,x_n\in\mathbb{F}_q^\star} \zeta_p^{\mathrm{Tr}(x_0x_1)}\cdots\zeta_p^{\mathrm{Tr}(x_0x_n)}\zeta_p^{\mathrm{Tr}(x_0/x_1\cdots x_n)} \\
&= \sum_{k_1,\dots,k_{n+1}=0}^{q-2} \frac{G_q(k_1)\cdots G_q(k_{n+1})}{(1-q)^{n+1}} \\
&\quad \times \sum_{y_i^{q-1}=1} (y_0y_1)^{k_1}\cdots(y_0y_n)^{k_n}\left(\frac{y_0}{y_1\cdots y_n}\right)^{k_{n+1}} \\
&= \sum_{k_1,\dots,k_{n+1}=0}^{q-2} \frac{G_q(k_1)\cdots G_q(k_{n+1})}{(1-q)^{n+1}} \\
&\quad \times \sum_{y_i^{q-1}=1} y_0^{k_1+\cdots+k_{n+1}} y_1^{k_1-k_{n+1}}\cdots y_n^{k_n-k_{n+1}} \\
&= (-1)^{n+1} \sum_{\substack{k=0 \\ (n+1)k\equiv 0(q-1)}}^{q-2} G_q(k)^{n+1} \\
&= (-1)^{n+1}\left(1+\sum_{\substack{k=1 \\ (n+1)k\equiv 0(q-1)}}^{q-2} G_q(k)^{n+1}\right).
\end{aligned}$$

And therefore

$$N_q(0) = \frac{(q-1)^n-(-1)^n}{q} + \frac{(-1)^{n+1}}{q} \sum_{\substack{k=1 \\ (n+1)k\equiv 0(q-1)}}^{q-2} G_q(k)^{n+1}.$$

Writing $(n+1) = p^a m$ with $(p,m)=1$, this is

$$N_q(0) = \frac{(q-1)^n-(-1)^n}{q} + \frac{(-1)^{n+1}}{q} \sum_{\substack{k=1 \\ mk\equiv 0(q-1)}}^{q-2} G_q(k)^{n+1}. \tag{4.2}$$

Let $S_m = \{\frac{1}{m}, \frac{2}{m}, \dots, \frac{m-1}{m}\}$. It is clear that multiplication by p induces an action on S_m, called p-action:

$$r \mapsto \{pr\},$$

where $\{pr\}$ denotes the fractional part of pr. For a given $r \in S_m$, let $d(r)$ denote the length of the p-orbit containing r, that is, the smallest positive integer d such that $(p^d - 1)r \in \mathbb{Z}$. Let $S_{m,d}$ denote the set of p-orbits of length d in S_m.

Since $G_{p^d}(k) = G_{p^d}(pk)$, it is clear that if r_1 and r_2 are in the same p-orbit σ in $S_{m,d}$, $G_{p^d}(r_1(p^d-1)) = G_{p^d}(r_2(p^d-1))$. Let us denote this common value by $G_{p^d}(\sigma(p^d-1))$. Since the set of p-orbits of S_m is the union of $S_{m,d}$ for all $d \geq 1$, we have (see [30] for more general such formula)

Theorem 4.2. *The zeta function of X_0 over $\mathbb{F}_p$ is given by*

$$\begin{aligned} Z(X_0,T)^{(-1)^n} &= \prod_{i=0}^{n-1} (1-p^iT)^{\binom{n}{i+1}(-1)^i} \\ &\quad \times \prod_{d\geq 1} \prod_{\sigma \in S_{m,d}} \left(1 - T^d \frac{G_{p^d}^{n+1}(\sigma(p^d-1))}{p^d}\right). \end{aligned}$$

Proof. By Equation 4.2,

$$\begin{aligned} \log Z(X_0,T) &= \sum_{k\geq 1} \frac{T^k}{k} \frac{(p^k-1)^n - (-1)^n}{p^k} \\ &\quad + \sum_{k\geq 1} \frac{T^k}{k} \frac{(-1)^{n+1}}{p^k} \sum_{\substack{h=1 \\ mh\equiv 0(p^k-1)}}^{p^k-2} G_{p^k}(h)^{n+1}. \end{aligned}$$

The second sum is

$$\begin{aligned} &\sum_{k\geq 1} \frac{T^k}{k} \frac{(-1)^{n+1}}{p^k} \sum_{\substack{r\in S_m \\ r(p^k-1)\in\mathbb{Z}}} G_{p^k}(r(p^k-1))^{n+1} \\ &= (-1)^{n+1} \sum_{d\geq 1} \sum_{\sigma\in S_{m,d}} \sum_{r\in\sigma} \left(\sum_{k\geq 1} \frac{T^{dk}}{dk} \frac{G_{p^{dk}}(r(p^{dk}-1))^{n+1}}{p^{dk}} \right) \\ &= (-1)^{n+1} \sum_{d\geq 1} \sum_{\sigma\in S_{m,d}} \left(\sum_{k\geq 1} \frac{T^{dk}}{k} \frac{G_{p^{dk}}(\sigma(p^{dk}-1))^{n+1}}{p^{dk}} \right). \end{aligned}$$

By the Hasse–Davenport relation, this sum becomes

$$= (-1)^{n+1} \sum_{d\geq 1} \sum_{\sigma\in S_{m,d}} \left(\sum_{k\geq 1} \frac{T^{dk}}{k} \frac{G_{p^d}(\sigma(p^d-1))^{k(n+1)}}{p^{dk}} \right),$$

which gives the stated formula for the zeta function. □

Corollary 4.3. 1. *The rank of the sheaf $\mathcal{F}$ at 0 is $m-1$.*
2. *The local L-function of the sheaf $\mathcal{F}$ at 0 is given by*

$$\prod_{d\geq 1}\prod_{\sigma\in S_{m,d}}\left(1-T^d\frac{G_{p^d}^{n+1}(\sigma(p^d-1))}{p^d}\right).$$

Proof. From the given formula for the L-function, we see that the degree of the non-trivial part is given by $|S_m| = m-1$. □

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