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Invariants of algebraic curves and topological expansion

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For any arbitrary algebraic curve, we define an infinite sequence of invariants. We study their properties, in particular their variation under a variation of the curve, and their modular properties. We also study their limits when the curve becomes singular. In addition, we find that they can be used to define a formal series, which satisfies formally an Hirota equation, and we thus obtain a new way of constructing a τ-function attached to an algebraic curve.

These invariants are constructed in order to coincide with the topological expansion of a matrix formal integral, when the algebraic curve is chosen as the large N limit of the matrix model's spectral curve. Surprisingly, we find that the same invariants also give the topological expansion of other models, in particular the matrix model with an external field, and the so-called double scaling limit of matrix models, i.e., the (p, q) minimal models of conformal field theory.

As an example to illustrate the efficiency of our method, we apply it to the Kontsevitch integral, and we give a new and extremely easy proof that Kontsevitch integral depends only on odd times, and that it is a KdV τ-function.

1. Introduction

Computing the topological expansion of various matrix integrals has been an interesting problem for more than 30 years. The reason for it, is that formal matrix integrals (cf [41] for a definition of formal integrals), are known to be combinatorics generating functionals. Some formal matrix integrals enumerate maps, or colored maps of given topology [14, 26, 27], some count intersection numbers (the Kontsevitch integral and its generalizations [58]), and physicists have also tried to reach the limit of continuous maps through critical limits, and thus tried to recover Liouville's field theory [28, 57].

Many methods have been invented to compute those formal matrix integrals, and the most successful is undoubtedly the "loop equations" method [24, 53], which is in fact nothing but integration by parts, or Tutte's equations

[73, 74], or Schwinger–Dyson equations, or Ward identities, or Virasoro constraints, or W-algebra [26]. Until recently, those loop equations were solved only for the first few orders (mostly planar or torus), and case by case (for each matrix model). One of the most remarkable methods was obtained in [7]. Let us also mention that other methods were invented using orthogonal polynomials [67] (only in the case were the formal integral comes from an actual convergent integral), or topological string theory methods [11, 22, 65] using the so-called holomorphic anomaly equations.

In 2004, a new method for computing the large N expansion of matrix integrals was introduced in [33], and further developed in [17, 40]. The starting point of that method was not new, it was the same as in [7], it consists in solving the loop equations recursively in powers of the expansion parameter $1/N^2$, where N is the size of the matrix. To leading order, loop equations become algebraic equations, and give rise to an algebraic curve $\mathcal{E}(x, y) = 0$ where $\mathcal{E}$ is some polynomial in two variables, which we call the "classical spectral curve".

The new feature which was introduced in [33] was to use contour integrals and functions on the curve rather than on the x-plane as in [7]. When written on the curve, the loop equations, together with the Cauchy residue formula and the Riemann bilinear identity, simplify enormously, and take a very universal structure which can be written entirely in terms of geometric properties of the curve. In other words, the solution of loop equations of many different matrix models, depends only on the properties of the spectral curve, and not on the matrix model which gives that curve. In particular, they can be written for any arbitrary algebraic curve, even for curves which do not come from matrix models. It is thus tempting to define "free energies" for any algebraic curve. This is what we do in this article.

Therefore, in this article, for any arbitrary algebraic curve $\mathcal{E}(x, y) = 0$, we define an infinite sequence of complex numbers $F^{(g)}(\mathcal{E})$, computed as residues of meromorphic forms on the curve. Out of these $F^{(g)}(\mathcal{E})$'s, we build a formal power series:

$$\ln Z_N(\mathcal{E}) = -\sum_{g=0}^{\infty} N^{2-2g} F^{(g)}(\mathcal{E}) \tag{1.1}$$

and we study its properties.

We compute the variations of $F^{(g)}$ under variations of the curve (variations of its complex structure, its moduli and modular transformations). We show that the $F^{(g)}$'s are invariant under some transformations of the curve, namely under transformations of the curve which preserve the symplectic form up to a sign $\pm dx \wedge dy$.

We also show that $Z_N(\mathcal{E})$ satisfies bilinear Hirota equations, and thus $Z_N(\mathcal{E})$ is a formal τ-function and we construct the associated formal Baker–Hakiezer function [9].

We thus have a notion of a τ-function associated to an algebraic curve. Such notion has already been encountered in the literature [9], and it is not clear whether our definition coincides with other existing definitions. What can be understood so far, is that we are defining a sort of quantum deformation of a classical τ-function whose spectral curve is $\mathcal{E}$. The classical τ-function being only the dispersionless limit $\ln Z_\infty(\mathcal{E}) = -F^{(0)}(\mathcal{E})$, while our $Z_N(\mathcal{E})$ concerns the full system.

Almost by definition, if $\mathcal{E}$ is the algebraic curve coming from the large N limit of the loop equations of a matrix model, then $Z_N(\mathcal{E})$ is the matrix integral.

What is more interesting is to see what is $Z_N(\mathcal{E})$ for curves not coming from the large N limit of the loop equations of a matrix model.

We study in details a few examples.

- The double scaling limit of a matrix model. It has been well known since [26, 51], that if we fine tune the parameters of a matrix model so that the algebraic curve $\mathcal{E}$ develops a singularity, the free energies become singular and the most singular part of the free energies form the KP-hierarchy τ-function (KdV hierarchy for the one-matrix model). We show, by looking at the double scaling limit of matrix models, that the KP τ-function (resp. KdV τ-function), coincides with our definition for the classical limit of the (p, q) systems (resp. $(p, 2)$).
- It has been well known since the works of Kontsevitch [58], that the KdV τ-function can be represented by another matrix integral called Kontsevitch integral. Kontsevitch introduced that integral as a counting function for intersection numbers, and proved that it is a KdV τ-function. One of the key features is that it depends on the eigenvalues of a diagonal matrix Λ, only through the quantities $t_k = \mathrm{Tr}\,\Lambda^{-k}$ for odd k (cf. [26, 49]). Another important known property is that, if $t_k = 0$ for $k > p$, it coincides with the $(p, 2)$ τ-function found from the double scaling limit of the one-matrix model, i.e., the $(p, 2)$ conformal minimal model.

Here, we prove that the Kontsevitch matrix integral coincides with our $Z_N(\mathcal{E})$ when $\mathcal{E}$ is the large N limit of the Schwinger–Dyson equation of the

Kontsevitch integral. The remarkable fact, is that for our $Z_N(\mathcal{E})$, the above properties (i.e., the fact that it depends only on odd t_k's and the fact that it gives the $(p,2)$ τ-function if $t_k = 0$ for $k > p$) are trivial. We thus provide a new proof of those properties, and maybe a new interpretation.

2. Main results of this article

In this section we just sketch briefly the contents of the main body of the article.

2.1. Definitions

Given a polynomial of two variables $\mathcal{E}(x,y)$, we construct an infinite sequence of multilinear meromorphic forms over the curve of equation $\mathcal{E}(x,y)=0$, which we call:

$$W_k^{(g)}(p_1,p_2,\dots,p_k), \quad k,g \in \mathbf{N}. \tag{2.1}$$

In particular $W_0^{(g)} = -F^{(g)}$ are complex numbers $F^{(g)}(\mathcal{E})$.

The $F^{(g)}$'s and the $W_k^{(g)}$'s are defined in terms of residues near the branch points of the curve only, i.e., they depend only on the local behavior of the curve near its branch points.

Then we show some properties:

- the $W_k^{(g)}$'s are *symmetric* in their k variables;
- there is a "loop insertion operator" which increases $k \to k+1$:

$$D_{B(p_{k+1},.)} W_k^{(g)}(p_1,p_2,\dots,p_k) = W_{k+1}^{(g)}(p_1,p_2,\dots,p_k,p_{k+1}); \tag{2.2}$$

- there is an inverse operator which contracts $k \to k-1$:

$$\begin{aligned}&\operatorname*{Res}_{p_k \to \text{branch points}} \Phi(p_k)\, W_k^{(g)}(p_1,p_2,\dots,p_k)\\ &\quad = (2g+k-3)\, W_{k-1}^{(g)}(p_1,p_2,\dots,p_{k-1}).\end{aligned} \tag{2.3}$$

The $F^{(g)}$'s and the $W_k^{(g)}$'s are defined in a way which mimics the solution of matrix models loop equations, and almost by definition, they coincide with matrix model's free energy and correlation functions when the polynomial $\mathcal{E}$

is chosen as the classical large N limit of the matrix model's spectral curve:

$$\ln\left(\int dM \exp -N \operatorname{Tr} V(M)\right) = -\sum_{g=0}^{\infty} N^{2-2g} F^{(g)}(\mathcal{E}_{1\mathrm{MM}}) \tag{2.4}$$

and

$$\begin{aligned}&\left\langle \operatorname{Tr}\frac{dx_1}{x_1-M}\operatorname{Tr}\frac{dx_2}{x_2-M}\cdots\operatorname{Tr}\frac{dx_k}{x_k-M}\right\rangle\\ &\quad=\sum_{g=0}^{\infty} N^{2-2g-k}\, W_k^{(g)}(p(x_1),p(x_2),\ldots,p(x_k)).\end{aligned} \tag{2.5}$$

The same construction works also for the two-matrix model and the matrix model in an external field:

$$\begin{aligned} F^{(g)}_{1\mathrm{MM}} &= F^{(g)}(\mathcal{E}_{1\mathrm{MM}}),\\ F^{(g)}_{2\mathrm{MM}} &= F^{(g)}(\mathcal{E}_{2\mathrm{MM}}),\\ F^{(g)}_{\mathrm{ext.field}} &= F^{(g)}(\mathcal{E}_{\mathrm{ext.field}}),\end{aligned} \tag{2.6}$$

in particular, it works for the Kontsevitch integral

$$F^{(g)}_{\mathrm{Kontsevitch}} = F^{(g)}(\mathcal{E}_{\mathrm{Kontsevitch}}), \tag{2.7}$$

where the LHS is the topological expansion of the corresponding matrix integral, and the RHS is the functional defined in this article, applied to the curve $\mathcal{E}(x,y)=0$ coming from the large N limit of the Schwinger–Dyson equations of the corresponding matrix model.

Let us emphasize that not every curve $\mathcal{E}$ is the large N limit of a matrix model's spectral curve, and thus our functional $F^{(g)}(\mathcal{E})$ is defined beyond matrix models, and is really an algebro-geometric object. It has many remarkable properties, and we list below some of the most important ones:

2.2. Remarkable properties

Theorem 4.8 Diagrammatic representation:

$$W^{(g)}_{k+1}(p,p_1,\ldots,p_k) = \sum_{G\in\mathcal{G}^{(g)}_{k+1}(p,p_1,\ldots,p_k)} w(G) = w\left(\sum_{G\in\mathcal{G}^{(g)}_{k+1}(p,p_1,\ldots,p_k)} G\right), \tag{2.8}$$

where $\mathcal{G}_{k+1}^{(g)}(p,p_1,\dots,p_k)$ *is a set of trivalent graphs* (*built on trees*), *and* w *is a Feynman-like weight function associating values to edges and integrals* (*residues in fact*) *to vertices of the graph.*

This theorem is important because it makes all formulae particularly easy to remember, and many theorems below can be proved in a diagrammatic way.

Theorem 8.1 Singular limits: *If the curve becomes singular, the functional* $F^{(g)}$ *commutes with the singular limit, i.e.,*

$$\lim F^{(g)}(\mathcal{E}) = F^{(g)}(\lim \mathcal{E}). \tag{2.9}$$

Theorem 9.2 Integrability: *the formal series:*

$$\ln Z_N(\mathcal{E}) = -\sum_{g=0}^{\infty} N^{2-2g}\, F^{(g)}(\mathcal{E}) \tag{2.10}$$

obeys Hirota's bilinear equations, and thus is a τ*-function.*

Theorem 5.3 Homogeneity: $F^{(g)}(\mathcal{E})$ *is homogeneous of degree* $2-2g$ *in the moduli of the curve:*

$$(2-2g)F^{(g)} = \sum_k t_k \frac{\partial F^{(g)}}{\partial t_k}. \tag{2.11}$$

Theorem 5.1 Deformations: *If the curve* $\mathcal{E}$ *is deformed into* $\mathcal{E}+\delta\mathcal{E}$, *the differential* $y\,dx$ *is deformed into* $y\,dx \to y\,dx + \delta(y\,dx)$ *where* $\delta(y\,dx)$ *is a meromorphic one-form which we denote* $\delta(y\,dx) = -\Omega$, *and which can be written as:* $\Omega = \int_{\partial\Omega} W_2^{(0)}\Lambda$ *for some appropriate contour* $\partial\Omega$ *and some appropriate function* Λ. *Then we have*

$$\delta W_k^{(g)}(p_1,p_2,\dots,p_k) = \int_{\partial\Omega} \Lambda(p_{k+1})\, W_{k+1}^{(g)}(p_1,p_2,\dots,p_k,p_{k+1}). \tag{2.12}$$

Theorem 7.1 Symplectic invariance: $F^{(g)}(\mathcal{E})$ *is unchanged under the following changes of curve* $\mathcal{E}(x,y)$*:*

$$\begin{aligned}
&y \to y + R(x), \quad R(x)\text{=rational fraction of } x,\\
&y \to cy, \quad x \to c^{-1}x, \quad c\text{=complex number},\\
&y \to -y, \quad x \to x,\\
&y \to x, \quad x \to y.
\end{aligned} \tag{2.13}$$

all those transformations conserve the symplectic form $dx \wedge dy$ *up to the sign.*

Theorem 6.2 Modular transformations: *The modular dependence of $F^{(g)}(\mathcal{E})$ is only in the Bergmann kernel (defined in Section 3.1.5), and thus the modular transformations of $F^{(g)}(\mathcal{E})$ are derived from those of the Bergmann kernel. Under a modular transformation, the Bergmann kernel is changed into $B(p,q) \to B(p,q) + 2i\pi\, du(p)\kappa\, du(q)$, and thus we introduce a new kernel for any arbitrary symmetric matrix κ:*

$$B_\kappa(p,q) \to B(p,q) + 2i\pi\, du(p)\kappa\, du(q). \tag{2.14}$$

We thus define some $F_\kappa^{(g)}(\mathcal{E})$, and we compute:

$$\frac{\partial F^{(g)}}{\partial \kappa}. \tag{2.15}$$

We also remark that when $\kappa = (\overline{\tau} - \tau)^{-1}$, $F_\kappa^{(g)}(\mathcal{E})$ is modular invariant.

2.3. Some applications, Kontsevitch's integral

2.3.1. (p,q) minimal models, KP and KdV hierarchies. It is well known [20, 26] that some rational singular limits of matrix models correspond to (p,q) minimal models, and Theorem 8.1 implies that

$$F_{(p,q)}^{(g)} = F^{(g)}(\mathcal{E}_{(p,q)}) \tag{2.16}$$

and it is well known that (p,q) minimal models are some reductions of KdV hierarchy for $q = 2$ and KP hierarchy for general (p,q).

Notice that the $x \leftrightarrow y$ symmetry of Theorem 7.1 (i.e., Equation (2.13)) implies the famous $(p,q) \leftrightarrow (q,p)$ duality [26, 28, 56].

2.3.2. Kontsevitch integral's properties. Kontsevitch's integral is defined as

$$\begin{aligned} Z_{\text{Kontsevitch}}(\Lambda) &= \int dM\, \mathrm{e}^{-N\,\mathrm{Tr}\,(M^3/3) - M\Lambda^2}, \\ \ln Z_{\text{Kontsevitch}} &= -\sum_{g=0}^{\infty} N^{2-2g}\, F^{(g)}_{\text{Kontsevitch}} \end{aligned} \tag{2.17}$$

and we define the Kontsevitch's times:

$$t_k = \frac{1}{N}\,\mathrm{Tr}\,\Lambda^{-k}. \tag{2.18}$$

It is straightforward to write the Schwinger–Dyson equations and find the classical spectral curve:

$$\mathcal{E}_{\text{Kontsevitch}} = \begin{cases} x(z) = z + \dfrac{1}{2N}\operatorname{Tr}\dfrac{1}{\Lambda}\dfrac{1}{z-\Lambda}, \\ y(z) = z^2 + t_1. \end{cases} \tag{2.19}$$

According to theorem 10.3, we have

$$F^{(g)}_{\text{Kontsevitch}} = F^{(g)}(\mathcal{E}_{\text{Kontsevitch}}). \tag{2.20}$$

Using the $x \leftrightarrow y$ invariance of Equation (2.13), we see that the only branch point in y is located at $z=0$, and since the $F^{(g)}$'s depend only on the local behavior near the branch point, we may perform a Taylor expansion of $x(z)$ near $z=0$:

$$\mathcal{E}_{\text{Kontsevitch}}(t_1, t_2, \ldots) = \begin{cases} x(z) = z - \dfrac{1}{2}\displaystyle\sum_{k=0}^{\infty} t_{k+2} z^k, \\ y(z) = z^2 + t_1. \end{cases} \tag{2.21}$$

From the symplectic invariance Theorem 7.1, (i.e., Equation (2.13)), we may add to x any rational function of y, i.e., of z^2, thus we may substract to x its even part, and thus the following curves are related by symplectic invariance:

$$\mathcal{E}_{\text{Kontsevitch}}(t_1, t_2, t_3, \ldots) \sim \mathcal{E}_{\text{Kontsevitch}}(t_1, 0, t_3, 0, t_5, \ldots). \tag{2.22}$$

We thus have a very easy proof that $F^{(g)}_{\text{Kontsevitch}}$ depends only on odd times.

Moreover, if $t_k = 0$ for $k > p+2$, we have

$$\mathcal{E}_{\text{Kontsevitch}}(t_1, t_2, \ldots, t_{p+2}, 0, \ldots) = \begin{cases} x(z) = z - \dfrac{1}{2}\displaystyle\sum_{k=0}^{p} t_{k+2} z^k, \\ y(z) = z^2 + t_1, \end{cases} \tag{2.23}$$

which is exactly the curve of the $(p,2)$ minimal model, i.e., it satisfies KdV hierarchy. We thus have a very easy proof that $Z_{\text{Kontsevitch}}$ is a KdV τ-function.

Those are old and classical results about the Kontsevitch integral, and we just propose a new proof, in order to illustrate the power of the tools we introduce.

3. Algebraic curves, reminder and notations

We begin by recalling some elements of algebraic geometry, which are used to fix the notations. We refer the reader to [46] or [47] for further details about algebro-geometric concepts.

	$\longrightarrow$	**Summary of notations**
$\mathcal{E}(x,y)=0$	$\longrightarrow$	classical spectral curve
$d_1+1=\deg_x \mathcal{E}$	$\longrightarrow$	x-degree of the polynomial $\mathcal{E}$
$d_2+1=\deg_y \mathcal{E}$	$\longrightarrow$	y-degree of the polynomial $\mathcal{E}$ (number of sheets)
$\mathbf{a}=\{a_i\}$	$\longrightarrow$	set of branch points $dx(a_i)=0$
$\alpha=\{\alpha_i\}$	$\longrightarrow$	poles of $y\,dx$
$\not{g}$	$\longrightarrow$	genus of the curve
$\underline{\mathcal{A}}_i \cap \underline{\mathcal{B}}_j=\delta_{ij}$	$\longrightarrow$	cannonical basis of non-contractible cycles
du_i	$\longrightarrow$	cannonical holomorphic forms $\oint_{\underline{\mathcal{A}}_j} du_i=\delta_{ij}$
$\tau_{ij}=\oint_{\underline{\mathcal{B}}_j} du_i$	$\longrightarrow$	Riemann's matrix of periods
$u_i(p)=\int_{p_0}^{p} du_i$	$\longrightarrow$	Abel map
$\mathcal{A}=\underline{\mathcal{A}}-\kappa(\underline{\mathcal{B}}-\tau\underline{\mathcal{A}})$	$\longrightarrow$	κ-modified $\mathcal{A}$-cycles
$\mathcal{B}=\underline{\mathcal{B}}-\tau\underline{\mathcal{A}}$	$\longrightarrow$	κ-modified $\mathcal{B}$-cycles, $\mathcal{A}_i\cap\mathcal{B}_j=\delta_{ij}$
$dS_{q_1,q_2}(p)$	$\longrightarrow$	third kind differential with simple poles q_1 and q_2, such that $\mathrm{Res}_{q_1} dS_{q_1,q_2}=1=-\mathrm{Res}_{q_2} dS_{q_1,q_2}$ and $\oint_{\mathcal{A}_i} dS_{q_1,q_2}=0$
$B(p,q)$	$\longrightarrow$	Bergmann kernel, i.e., second kind differential with double pole at $p=q$, no residue and vanishing $\mathcal{A}$-cycle integrals
$z=\dfrac{\vec{n}+\tau\vec{m}}{2}$	$\longrightarrow$	regular odd characteristic, i.e., $\sum_{i=1}^{g} n_i m_i=\mathrm{odd}$
$dh_z=\sum_i \left.\dfrac{\partial\theta_z(\vec{v})}{\partial v_i}\right\vert_{v=0} du_i$	$\longrightarrow$	holomorphic form with only double zeroes
$\underline{E}(p,q)=\dfrac{\theta_z(u(p)-u(q),\tau)}{\sqrt{dh_z(p)dh_z(q)}}$	$\longrightarrow$	prime form independent of z with a simple zero at $p=q$
$\Phi(p)=\int_o^p y\,dx$	$\longrightarrow$	some antiderivative of $y\,dx$ defined on the universal covering
$\overline{p}$, $x(\overline{p})=x(p)$	$\longrightarrow$	if p is near a branch points a, then $\overline{p}\neq p$ is the unique other point near a such that $x(\overline{p})=x(p)$
$p^i(p)$, $x(p^i)=x(p)$	$\longrightarrow$	the p^i's, $i=0,\dots,d_2$, are the pre-images of $x(p)$ on the curve. By convention $p^0(p)=p$
$D_\Omega=\delta_\Omega+\mathrm{Tr}\left(\kappa\,\delta_\Omega\tau\,\kappa\dfrac{\partial}{\partial\kappa}\right)$	$\longrightarrow$	covariant variation w.r.t $\Omega=\delta(y\,dx)$

Consider an (embedded) algebraic curve given by its equation:

$$\mathcal{E}(x,y)=0, \tag{3.1}$$

where $\mathcal{E}$ is an almost arbitrary polynomial of two variables. This is equivalent to considering a compact Riemann surface $\overline{\Sigma}$ and 2 meromorphic functions x and y, such that

$$\forall p \in \overline{\Sigma}, \quad \mathcal{E}(x(p),y(p))=0. \tag{3.2}$$

We only require that $\mathcal{E}(x,y)$ is not factorizable, and that all branch points (zeroes of dx) are simple, i.e., near a branch point a_i, y behaves like a square root $\sqrt{x-x(a_i)}$.

3.1. Some properties of algebraic curves

3.1.1. Sheets. For each complex x, there exist $d_2+1=\deg_y \mathcal{E}$ solutions for y of $\mathcal{E}(x,y)=0$. This means that there are exactly d_2+1 points on the Riemann surface $\overline{\Sigma}$ for which $x(p)=x$: $\overline{\Sigma}$ has a sheet structure with d_2+1 x-sheets. We call them:

$$x(p)=x \leftrightarrow p=p^i(x), \quad i=0,\dots,d_2. \tag{3.3}$$

3.1.2. Branch points and conjugated points Let a_i, $i=1,\dots,n$, $\mathbf{a}=\{a_1,\dots,a_n\}$ be the set of branch points, solutions of $dx=0$:

$$\forall a \in \mathbf{a}, \quad dx(a)=0. \tag{3.4}$$

Since we assume that the branch points are simple zeros of dx, we have the following property: if p is in the vicinity of a branch point a_i, there is a unique point $\overline{p}\neq p$, such that $x(\overline{p})=x(p)$, which is also in the vicinity of a_i. $\overline{p}$ depends on i, and in general, $\overline{p}$ is not globaly defined (see figure 1 for an example).

Notice that $\overline{p}$ is one of the p^k defined in the previous section.

3.1.3. Genus, cycles, Abel map If the curve has genus $\not{g}$, there are $2\not{g}$ homologically independent non-trivial cycles, and we may choose a (not unique) canonical basis:

$$\underline{\mathcal{A}}_i \cap \underline{\mathcal{B}}_j=\delta_{ij}, \quad \underline{\mathcal{A}}_i \cap \underline{\mathcal{A}}_j=0, \quad \underline{\mathcal{B}}_i \cap \underline{\mathcal{B}}_j=0. \tag{3.5}$$

The simply connected domain obtained by removing all $\underline{\mathcal{A}}$ and $\underline{\mathcal{B}}$-cycles from the curve is called the "fundamental domain" (see figure 2 for the example of the torus).

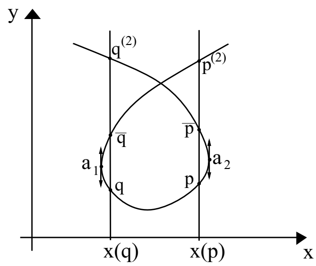


Figure 1: Example of an algebraic curve with two x-branch points a_1 and a_2 and a three sheeted structure (x has three pre-images). One can see that the map $p \to \overline{p}$ is not globally defined, for instance when $q \to p$, we have $\overline{q} \to p^{(2)}$. The notion of conjugated point depends on the branch point.

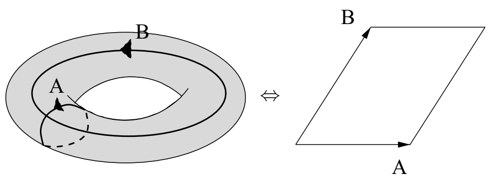


Figure 2: Example of canonical cycles and the corresponding fundamental domain in the case of the torus (genus $\mathfrak{g} = 1$).

On a genus $\mathfrak{g}$ curve, there are $\mathfrak{g}$ linearly independent holomorphic forms $du_1, \dots, du_{\mathfrak{g}}$, which we choose normalized on the $\underline{\mathcal{A}}$-cycles:

$$\oint_{\underline{\mathcal{A}}_j} du_i = \delta_{ij}. \tag{3.6}$$

The Riemann matrix of period is defined by the $\underline{\mathcal{B}}$-cycles

$$\tau_{ij} = \oint_{\underline{\mathcal{B}}_j} du_i. \tag{3.7}$$

They have the property that

$$\tau_{ij} = \tau_{ji}, \quad \mathrm{Im}\, \tau > 0. \tag{3.8}$$

Given a base point p_0 on the curve (we assume it is not on any $\underline{\mathcal{A}}$ or $\underline{\mathcal{B}}$-cycle), we define the **Abel map**

$$u_i(p) = \int_{p_0}^{p} du_i, \tag{3.9}$$

where the integration path is in the fundamental domain. The $\mathfrak{g}$ dimensional vector $u(p) = (u_1(p), \dots, u_{\mathfrak{g}}(p))$ maps the curve into its Jacobian.

3.1.4. Theta-functions and prime forms. We say that $z \in \mathbf{C}^{\mathfrak{g}}$ is a characteristic if there exist two vectors with integer coefficients $\vec{a} \in Z^{\mathfrak{g}}$ and $\vec{b} \in Z^{\mathfrak{g}}$ such that

$$z = \frac{\vec{a} + \tau.\vec{b}}{2}. \tag{3.10}$$

z is called an odd characteristic if

$$\sum_{i=1}^{\mathfrak{g}} a_i b_i = \text{odd}. \tag{3.11}$$

Given a characteristic $z = (\vec{a} + \tau\vec{b})/2$, and given a symmetric matrix $\tau_{ij} = \tau_{ji}$ such that $\operatorname{Im}\tau$ is positive definite, and a vector $v \in \mathbf{C}^{\mathfrak{g}}$, we define the θ_z function:

$$\theta_z(v,\tau) = \sum_{\vec{n} \in Z^{\mathfrak{g}}} \exp(i\pi(\vec{n} - \vec{b}/2)^t \tau(\vec{n} - \vec{b}/2)) \exp(2i\pi(v + \vec{a}/2)^t.(\vec{n} + \vec{b}/2)). \tag{3.12}$$

If z is an odd characteristic, θ_z is an odd function of v, and in particular $\theta_z(0,\tau) = 0$ and we define the following holomorphic form:

$$dh_z(p) = \sum_{i=1}^{\mathfrak{g}} du_i(p) \cdot \left.\frac{\partial \theta_z(v)}{\partial v_i}\right|_{v=0}. \tag{3.13}$$

All its $\mathfrak{g} - 1$ zeroes are double zeroes, so that it makes sense to consider its square root defined on the fundamental domain. The prime form is

$$\underline{E}(p,q) = \frac{\theta_z(u(p) - u(q))}{\sqrt{dh_z(p)\, dh_z(q)}}. \tag{3.14}$$

It is independent of z, and it vanishes only if $p = q$ (and with a simple zero), and has no pole.

3.1.5. Bergmann kernel. We define a bilinear meromorphic form called the "Bergmann kernel" [10, 48]:

$$\underline{B}(p,q) = \text{Bergmann kernel} \tag{3.15}$$

as the unique one-form in p, which has a double pole with no residue at $p = q$ and no other pole, and which is normalized such that

$$\underline{B}(p,q) \sim_{p\to q} \frac{dz(p)dz(q)}{(z(p)-z(q))^2} + \text{finite}, \qquad \oint_{\underline{\mathcal{A}}_I} \underline{B} = 0, \tag{3.16}$$

where z is any local coordinate on the curve in the vicinity of q. The Bergmann kernel depends only on the complex structure of the curve, and not on the details of $\mathcal{E}$. For instance, if $\mathcal{E}$ has genus zero, $\underline{B}$ is the Bergmann kernel of the projective complex plane (the Riemann sphere), and if the curve has genus 1, $\underline{B}$ is related to the Weierstrass function.

Properties

$$\underline{B}(p,q) = \underline{B}(q,p), \qquad \oint_{q\in\underline{\mathcal{B}}_i} \underline{B}(p,q) = 2i\pi\, du_i(p). \tag{3.17}$$

For any odd characteristic z, we have

$$\underline{B}(p,q) = d_p\, d_q \ln\left(\theta_z(u(p)-u(q))\right). \tag{3.18}$$

If $f(p)$ is any meromorphic function, its differential is given by

$$df(p) = \operatorname*{Res}_{q\to p} \underline{B}(p,q) f(q). \tag{3.19}$$

3.1.6. Third type differentials. Given two points q_1 and q_2 on the curve, we define the one-form $\underline{dS}_{q_1,q_2}$ by

$$\underline{dS}_{q_1,q_2}(p) = \int_{q_2}^{q_1} \underline{B}(p,q), \tag{3.20}$$

where the integration path is in the fundamental domain.

$\underline{dS}_{q_1,q_2}$ is the unique meromorphic form with only simple poles at q_1 and q_2, such that:

$$\operatorname*{Res}_{q_1} \underline{dS}_{q_1,q_2} = 1 = -\operatorname*{Res}_{q_2} \underline{dS}_{q_1,q_2}, \qquad \oint_{\underline{\mathcal{A}}_i} \underline{dS}_{q_1,q_2} = 0. \tag{3.21}$$

3.1.7. Modified set of cycles. In order to easily deal with modular properties of the objects we are going to introduce, it is convenient to define some modified cycles and kernels with an arbitrary symmetric matrix κ.

When $\kappa = 0$, all those quantities reduce to the unmodified ones. The modular transformations of the modified objects, merely amount to a change of κ.

We thus choose an arbitrary $\mathfrak{g} \times \mathfrak{g}$ symmetric matrix κ with complex coefficients, and we define another set of cycles:

$$\begin{aligned} \mathcal{A}_i &= \underline{\mathcal{A}}_i - \sum_j \kappa_{ij} \left(\underline{\mathcal{B}}_j - \sum_l \tau_{jl} \underline{\mathcal{A}}_l \right), \\ \mathcal{B}_i &= \underline{\mathcal{B}}_i - \sum_j \tau_{ij} \underline{\mathcal{A}}_j. \end{aligned} \tag{3.22}$$

They satisfy

$$\mathcal{A}_i \cap \mathcal{B}_j = \delta_{ij}, \quad \mathcal{A}_i \cap \mathcal{A}_j = 0, \quad \mathcal{B}_i \cap \mathcal{B}_j = 0 \tag{3.23}$$

and we straightforwardly have

$$\oint_{\mathcal{A}_i} du_j = \delta_{ij}, \quad \oint_{\mathcal{B}_i} du_j = 0. \tag{3.24}$$

3.1.8. Modified Bergmann kernel. We also define the modified Bergmann kernel, normalized on $\mathcal{A}$ instead of $\underline{\mathcal{A}}$:

$$B(p,q) = \underline{B}(p,q) + 2\mathrm{i}\pi \sum_{i,j} du_i(p)\, \kappa_{ij}\, du_j(q). \tag{3.25}$$

It is such that

$$B(p,q) = B(q,p), \quad \oint_{\mathcal{A}_I} B = 0, \quad \oint_{q \in \mathcal{B}_i} B(p,q) = 2i\pi\, du_i(p) \tag{3.26}$$

and if $f(p)$ is any meromorphic function, its differential is given by:

$$df(p) = \operatorname*{Res}_{q \to p} B(p,q) f(q). \tag{3.27}$$

- For $\kappa = 0$ we have $\underline{B} = B$.
- For $\kappa = (\overline{\tau} - \tau)^{-1}$, B is the Schiffer kernel [10, 48], and it is modular invariant.

3.1.9. Modified prime form. Similarly we define a modified prime form:

$$E(p,q) = \underline{E}(p,q)\, \mathrm{e}^{2i\pi u^t(p) \kappa u(q)}. \tag{3.28}$$

It vanishes only if $p = q$ (with a simple zero), and has no pole.

3.1.10. Modified third type differentials. In the same fashion, we define the modified third type differentials dS_{q_1,q_2} by

$$dS_{q_1,q_2}(p) = \int_{q_2}^{q_1} B(p,q), \tag{3.29}$$

where the integration path is in the fundamental domain.

dS_{q_1,q_2} is the unique meromorphic form with only simple poles at q_1 and q_2, such that

$$\operatorname*{Res}_{q_1} dS_{q_1,q_2} = 1 = -\operatorname*{Res}_{q_2} dS_{q_1,q_2}, \qquad \oint_{\mathcal{A}_i} dS_{q_1,q_2} = 0. \tag{3.30}$$

Properties

$$dS_{q_1,q_2} = -dS_{q_2,q_1}, \tag{3.31}$$

$$\begin{aligned} dS_{q_1,q_2}(p) = {} & d_p \ln\left(\frac{\theta_z(u(p)-u(q_1))}{\theta_z(u(p)-u(q_2))}\right) \\ & + 2i\pi \sum_{i,j} du_i(p)\kappa_{ij}(u_j(q_1)-u_j(q_2)), \end{aligned} \tag{3.32}$$

$$\oint_{\mathcal{B}_i} dS_{q_1,q_2} = 2i\pi(u_i(q_1)-u_i(q_2)), \tag{3.33}$$

$$d_{q_1}(dS_{q_1,q_2}(p)) = B(q_1,p), \tag{3.34}$$

$$\int_{p_1}^{p_2} dS_{q_1,q_2} = \int_{q_1}^{q_2} dS_{p_1,p_2}. \tag{3.35}$$

Cauchy residue formula: for any meromorphic function $f(p)$ we have

$$f(p) = -\operatorname*{Res}_{q_1\to p} dS_{q_1,q_2}(p) f(q_1). \tag{3.36}$$

3.1.11. Bergmann τ-function. The Bergmann τ-function τ_{Bx} was introduced and studied in [37, 61, 62], it is such that

$$\frac{\partial \ln(\tau_{Bx})}{\partial x(a_i)} = \operatorname*{Res}_{p\to a_i} \frac{B(p,\overline{p})}{dx(p)}. \tag{3.37}$$

It is well defined because the Rauch variational formula [68] implies that the RHS is a closed form. Notice that τ_{Bx} is defined only up to a multiplicative constant which will play no role in all the sequel.

3.2. Examples: genus 0 and 1

3.2.1. Genus 0. If the curve $\mathcal{E}$ has a genus $\overline{g} = 0$, it is conformally equivalent to the Riemann sphere, i.e., the complex plane with a point at ∞, and there exists a rationnal parametrization of the curve. It means that there exists two rational functions $X(p)$ and $Y(p)$ such that:

$$\mathcal{E}(x,y) = 0 \leftrightarrow \exists p \in \mathbf{C}, \quad x = X(p),\ \ y = Y(p). \tag{3.38}$$

In this case, the Bergmann kernel is the Bergmann kernel of the Riemann sphere:

$$B(p,q) = \underline{B}(p,q) = \frac{dp\, dq}{(p-q)^2} = d_p\, d_q\, \ln{(p-q)}. \tag{3.39}$$

The prime form is

$$E(p,q) = \underline{E}(p,q) = \frac{p-q}{\sqrt{dp\, dq}}. \tag{3.40}$$

3.2.2. Genus 1. If the curve has genus $\overline{g} = 1$, then it can be parametrized on a rhombus corresonding to the fundamental domain of a torus (see figure 2). It means that there exists two elliptical functions $X(p)$ and $Y(p)$ such that (see [75] for elliptical functions):

$$\begin{gathered} \mathcal{E}(x,y) = 0 \leftrightarrow \exists p \in \mathbf{C}, \quad x = X(p),\ \ y = Y(p) \\ X(p+1) = X(p+\tau) = X(p), \quad Y(p+1) = Y(p+\tau) = Y(p) \end{gathered} \tag{3.41}$$

Then, the Bergmann kernel is the corresponding Weierstrass function [75]:

$$\underline{B}(p,q) = \left(\wp(p-q,\tau) + \frac{\pi}{\operatorname{Im}\tau}\right) dp\, dq. \tag{3.42}$$

The prime form is

$$\underline{E}(p,q) = \frac{\theta_1(p-q,\tau)}{\theta_1'(0,\tau)\sqrt{dp\, dq}}. \tag{3.43}$$

When $\kappa = (-1/2i\text{Im}\tau)$, the modified Bergmann kernel is the Schiffer kernel, and if $\overline{g} = 1$ it is the Weierstrass function:

$$B(p,q) = \wp(p-q,\tau)\, dp\, dq. \tag{3.44}$$

3.3. Riemann bilinear identity

If ω_1 and ω_2 are two meromorphic forms on the curve. Let p_0 be an arbitrary base point, we consider the function Φ_1 defined on the fundamental domain by

$$\Phi_1(p) = \int_{p_0}^{p} \omega_1. \tag{3.45}$$

We have

$$\operatorname*{Res}_{p\to \text{all poles}} \Phi_1(p)\omega_2(p) = \frac{1}{2i\pi} \sum_{i=1}^{\mathfrak{g}} \oint_{\underline{\mathcal{A}}_i} \omega_1 \oint_{\underline{\mathcal{B}}_i} \omega_2 - \oint_{\underline{\mathcal{B}}_i} \omega_1 \oint_{\underline{\mathcal{A}}_i} \omega_2. \tag{3.46}$$

Note that this identity holds also for the modified cycles with any κ:

$$\operatorname*{Res}_{p\to \text{all poles}} \Phi_1(p)\omega_2(p) = \frac{1}{2i\pi} \sum_{i=1}^{\mathfrak{g}} \oint_{\mathcal{A}_i} \omega_1 \oint_{\mathcal{B}_i} \omega_2 - \oint_{\mathcal{B}_i} \omega_1 \oint_{\mathcal{A}_i} \omega_2. \tag{3.47}$$

In particular with $\omega_1(p) = B(p,q)$, we have

$$\operatorname*{Res}_{p\to \text{all poles}} dS_{p,p_0}(q)\,\omega(p) = -\sum_{i=1}^{\mathfrak{g}} du_i(q) \oint_{\mathcal{A}_i} \omega \tag{3.48}$$

and

$$\omega(q) = \operatorname*{Res}_{p\to \text{poles of } \omega} dS_{p,p_0}(q)\,\omega(p) + \sum_{i=1}^{\mathfrak{g}} du_i(q) \oint_{\mathcal{A}_i} \omega. \tag{3.49}$$

3.4. Moduli of the curve

The curve $\mathcal{E}(x,y) = 0$ is parameterized by

- A genus $\mathfrak{g}$ compact Riemann surface $\bar{\Sigma}$, with periods τ_{ij}.
- Punctures α_i at the poles of x and y, whose moduli are given by the negative coefficients of the Laurent series of $y\,dx$ near the poles.
- The $\mathcal{A}$-cycle integrals of $y\,dx$, called filling fractions.

3.4.1. Filling fractions. We define

$$\epsilon_i = \frac{1}{2i\pi} \oint_{\mathcal{A}_i} y\,dx \tag{3.50}$$

which are called "filling fractions" by analogy with matrix models [35].

3.4.2. Moduli of the poles. Consider a pole α of $y\,dx$, define the "*temperatures*"

$$t_\alpha = \operatorname*{Res}_{\alpha} \; y\,dx. \tag{3.51}$$

Notice that

$$\sum_\alpha t_\alpha = 0. \tag{3.52}$$

Then consider the three cases:

- Either α is a pole of x of degree d_α, then we define the local parameter near α as

$$z_\alpha(p) = x(p)^{(1/d_\alpha)}; \tag{3.53}$$

- or α is not a pole of x neither a branch point (thus it is a pole of y), then we define the local parameter near α as

$$z_\alpha(p) = \frac{1}{x(p) - x(\alpha)}. \tag{3.54}$$

- or α is not a pole of x, and it is a branch point (thus it is a pole of y), then we define the local parameter near α as

$$z_\alpha(p) = \frac{1}{\sqrt{x(p) - x(\alpha)}}. \tag{3.55}$$

In all cases, in the vicinity of α, we define the "*potential*"

$$V_\alpha(p) = \operatorname*{Res}_{q\to\alpha} \; y(q)\,dx(q) \ln\left(1 - \frac{z_\alpha(p)}{z_\alpha(q)}\right), \tag{3.56}$$

which is a polynomial in $z_\alpha(p)$:

$$V_\alpha(p) = \sum_{k=1}^{\deg V_\alpha} t_{\alpha,k}\, z_\alpha^k(p). \tag{3.57}$$

The $t_{\alpha,k}$ are the moduli of the pole α.

We have the following properties:

$$dV_\alpha(p) = \mathop{\mathrm{Res}}_{q\to\alpha} y(q)\,dx(q)\,\frac{dz_\alpha(p)}{z_\alpha(p)-z_\alpha(q)}, \tag{3.58}$$

$$\mathop{\mathrm{Res}}_{\alpha}\, dV_\alpha = 0 \tag{3.59}$$

and

$$y(p)\,dx(p) \mathop{\sim}_{p\to\alpha} dV_\alpha(p) - t_\alpha\,\frac{dz_\alpha(p)}{z_\alpha(p)} + O\left(\frac{dz_\alpha(p)}{z_\alpha(p)^2}\right). \tag{3.60}$$

We have from Equation (3.49)

$$y(p)\,dx(p) = -\sum_\alpha \mathop{\mathrm{Res}}_{q\to\alpha} B(p,q)V_\alpha(q) + \sum_\alpha t_\alpha dS_{\alpha,o}(p) + 2i\pi\sum_i \epsilon_i\,du_i(p). \tag{3.61}$$

If we introduce

$$B_{\alpha,k}(p) = -\mathop{\mathrm{Res}}_{q\to\alpha} B(p,q)\,z_\alpha(q)^k, \tag{3.62}$$

we can turn this expression to

$$y\,dx = \sum_{\alpha,k} t_{\alpha,k}B_{\alpha,k} + \sum_\alpha t_\alpha\,dS_{\alpha,o} + 2i\pi\sum_i \epsilon_i\,du_i(p) \tag{3.63}$$

in order to exhibit the moduli of the curve.

4. Definition of correlation functions and free energies

In all this section, the curve $\mathcal{E}(x,y)=0$ and a symmetric matrix κ are given and fixed. The unfamiliar reader may choose $\kappa=0$ since most usual applications (matrix models) correspond to that case.

4.1. Notations

Consider an arbitrary point $p\in\bar\Sigma$, and a point q of $\bar\Sigma$ which is in the vicinity of a branch point a_i (so that $\overline{q}$ is well defined). We define

Definition 4.1. Diagrammatic rules:

$$\text{vertex} : \omega(q) = (y(q) - y(\overline{q}))\, dx(q), \tag{4.1}$$

$$\text{line-propagator: } B(p, q), \tag{4.2}$$

$$\text{arrow-propagator: } dE_q(p) = \frac{1}{2} \int_q^{\overline{q}} B(\xi, p), \tag{4.3}$$

where the integration path is a path which lies entirely in a vincinity of a_i (thus it is uniquely defined)[1].

$$\text{root} : \Phi(q) = \int_o^q y\, dx, \tag{4.4}$$

where o is an arbitrary base point on the curve, i.e., Φ is an arbitrary antiderivative of $y\, dx$, i.e., $d\Phi = y\, dx$.

The reason why we call these objects diagramatic rules and vertices, propagator or root, is explained in Section 4.5 below.

Notice that dE depends on i, i.e., on which branch point we are considering, but we omit to mention the index i in order to make the notations easier to read. In all what follows it is always clear which i is being considered.

Notation for subset of indices

Given a set of points of the curve $\{p_1, p_2, \ldots, p_n\}$, if $K = \{i_1, i_2, \ldots, i_k\}$ is any subset of $\{1, 2, \ldots, n\}$, we denote:

$$\mathbf{p}_K = \{p_{i_1}, p_{i_2}, \ldots, p_{i_k}\}. \tag{4.5}$$

4.2. Correlation functions and free energies

4.2.1. Correlation functions. The k-point correlation functions to order g, $W_k^{(g)}$, are meromorphic multilinear forms, defined by the following recursive triangular system:

---

[1] This definition is the opposite of the notation used in [19, 40] since the integral goes from q to $\overline{q}$ instead of going from $\overline{q}$ to q.

Definition 4.2. Correlation functions

$$W_k^{(g)} = 0 \quad \text{if } g < 0. \tag{4.6}$$

$$W_1^{(0)}(p) = 0. \tag{4.7}$$

$$\boxed{W_2^{(0)}(p_1, p_2) = B(p_1, p_2)} \tag{4.8}$$

and define recursively (remember that $\mathbf{p_K}$ is a k-uplet of points cf Equation 4.5):

$$\boxed{\begin{array}{l} W_{k+1}^{(g)}(p, \mathbf{p_K}) = \operatorname{Res}_{q\to\mathbf{a}} dE_q(p)\omega(q) \Big(\sum_{m=0}^{g} \sum_{J\subset K} \\ \qquad\qquad W_{|J|+1}^{(m)}(q, \mathbf{p_J}) W_{k-|J|+1}^{(g-m)}(\overline{q}, \mathbf{p_{K/J}}) + W_{k+2}^{(g-1)}(q, \overline{q}, \mathbf{p_K}) \Big) \end{array}} \tag{4.9}$$

This system is triangular because all terms in the RHS have lower $2g + k$ than in the LHS and given $W_1^{(0)}$ and $W_2^{(0)}$, it has a unique solution.

Notice that $W_{k+1}^{(g)}(p, p_1, \ldots, p_k)$ is a multilinear meromorphic form in each of its arguments, it is clearly symmetric in the last k-ones, and we prove below (Theorem 4.6) that it is in fact symmetric in all its arguments.

More properties of $W_{k+1}^{(g)}$ are studied below in Section 4.4.

4.2.2. Free energies. We define the *free energies* which are complex numbers:

Definition 4.3. Free energies.

For $g > 1$

$$\boxed{F^{(g)} = \frac{1}{2-2g} \sum_i \operatorname*{Res}_{q\to a_i} \Phi(q) W_1^{(g)}(q)} \tag{4.10}$$

and

$$\boxed{F^{(1)} = -\frac{1}{2}\ln(\tau_{Bx}) - \frac{1}{24}\ln\left(\prod_i y'(a_i)\right) - \ln(\det\kappa)} \tag{4.11}$$

where

$$y'(a_i) = \frac{dy(a_i)}{dz_i(a_i)}, \quad z_i(p) = \sqrt{x(p) - x(a_i)} \tag{4.12}$$

and τ_{Bx} is the Bergmann τ-function defined in Equation (3.39).

$F^{(0)}$ is defined in the next section.

4.2.3. Leading order free energy $F^{(0)}$. Let us define $F^{(0)}$ as follows:

$$\boxed{F^{(0)} = \frac{1}{2}\sum_\alpha \mathop{\mathrm{Res}}_\alpha V_\alpha y\, dx + \frac{1}{2}\sum_\alpha t_\alpha \mu_\alpha - \frac{1}{4i\pi}\sum_i \oint_{\mathcal{A}_i} y\, dx \oint_{\mathcal{B}_i} y\, dx} \tag{4.13}$$

where μ_α is given by

$$\mu_\alpha = \int_\alpha^o \left(y\, dx - dV_\alpha + t_\alpha \frac{dz_\alpha}{z_\alpha}\right) + V_\alpha(o) - t_\alpha \ln\left(z_\alpha(o)\right). \tag{4.14}$$

Notice that μ_α depends on some base point o, but the sum $\sum_\alpha t_\alpha \mu_\alpha$ does not.

4.2.4. Special free energies and correlation functions. All the quantities defined so far, were defined with the κ-modified cycles and modified Bergmann kernel. Let us also define them for $\kappa = 0$ (for instance $F^{(1)}$ obviously needs another definition).

Therefore we also define the unmodified quantities corresponding to $\kappa = 0$, as

$$\forall k, g, \quad \underline{W}_k^{(g)}(p_1, \dots, p_k) := W_k^{(g)}(p_1, \dots, p_k)\Big|_{\kappa=0}, \tag{4.15}$$

$$\text{for } g > 1, \quad \underline{F}^{(g)} := \frac{1}{2-2g}\sum_i \mathop{\mathrm{Res}}_{q\to a_i} \Phi(q)\underline{W}_1^{(g)}(q) = F^{(g)}|_{\kappa=0}, \tag{4.16}$$

$$\underline{F}^{(1)} = -\frac{1}{2}\ln\left(\tau_{\underline{B}x}\right) - \frac{1}{24}\ln\left(\prod_i y'(a_i)\right), \tag{4.17}$$

and

$$\underline{F}^{(0)} = \frac{1}{2}\sum_\alpha \mathop{\mathrm{Res}}_\alpha V_\alpha y\, dx + \frac{1}{2}\sum_\alpha t_\alpha \mu_\alpha - \frac{1}{4i\pi}\sum_i \oint_{\underline{\mathcal{A}}_i} y\, dx \oint_{\underline{\mathcal{B}}_i} y\, dx. \tag{4.18}$$

Remark 4.1. *The special functions, except $\underline{F}^{(1)}$ and $\underline{F}^{(0)}$, are obtained by changing B and dS by $\underline{B}$ and $\underline{dS}$ in the diagrammatic rules defined in Section 4.5.*

4.3. τ-Function

Definition 4.4. We define the τ-function as the formal power series in N^{-2}:

$$\boxed{\ln\left(Z_N(\mathcal{E})\right) = -\sum_{g=0}^{\infty} N^{2-2g} F^{(g)}.} \tag{4.19}$$

We show in Section 9 that $Z_N(\mathcal{E})$ is indeed a τ-function because it obeys Hirota equations, order by order in N^{-2}.

4.4. Properties of correlation functions

The loop functions defined in Definition 4.2 satisfy the following theorems, whose proofs can be found in Appendix A:

Theorem 4.1. *The correlation function $W_3^{(0)}$ is worth:*

$$W_3^{(0)}(p, p_1, p_2) = \operatorname*{Res}_{q \to \mathbf{a}} \frac{B(q,p)B(q,p_1)B(q,p_2)}{dx(q)\, dy(q)} \tag{4.20}$$

In particular, $W_3^{(0)}$ is symmetric in its three variables.

Theorem 4.2. *For $(k, g) \neq (1, 0)$, the loop function $W_{k+1}^{(g)}(p, p_1, \ldots, p_k)$ has poles (in any of its variables $p, p_1, \ldots, p_k$) only at the branch points.*

Theorem 4.3. *For every $\mathcal{A}$ cycle we have*

$$\forall i = 1, \ldots, \not{g}, \quad \oint_{p \in \mathcal{A}_i} W_{k+1}^{(g)}(p, p_1, \ldots, p_k) = 0, \tag{4.21}$$

$$\forall i = 1, \ldots, \not{g}, \ \forall m = 1, \ldots, k, \quad \oint_{p_m \in \mathcal{A}_i} W_{k+1}^{(g)}(p, p_1, \ldots, p_k) = 0. \tag{4.22}$$

Theorem 4.4. *For every k and g, we have*

$$\sum_i \frac{W_{k+1}^{(g)}(p^i, p_1, \ldots, p_k)}{dx(p^i)} = \delta_{k,1}\delta_{g,0} \frac{dx(p_1)}{(x(p)-x(p_1))^2} \tag{4.23}$$

and if $k \geq 1$:

$$\sum_i \frac{W_{k+1}^{(g)}(p_1, p^i, p_2, \ldots, p_k)}{dx(p^i)} = \delta_{k,1}\delta_{g,0} \frac{dx(p_1)}{(x(p)-x(p_1))^2}, \tag{4.24}$$

where we recall that p^i are all the points such that $x(p^i) = x(p)$ (see Section 3.1).

Theorem 4.5. *For $(k, g) \neq (0, 1)$,*

$$\begin{aligned} P_k^{(g)}(x(p), \mathbf{p_K}) = \frac{1}{dx(p)^2} \sum_i \Big[& -2y(p^i)\, dx(p) W_{k+1}^{(g)}(p^i, \mathbf{p_K}) \\ & + W_{k+2}^{(g-1)}(p^i, p^i, \mathbf{p_K}) + \sum_{m=0}^{g} \\ & \sum_{J \subset K} W_{j+1}^{(m)}(p^i, \mathbf{p_J}) W_{k-j+1}^{(g-m)}(p^i, \mathbf{p_{K/J}}) \Big] \end{aligned} \tag{4.25}$$

is a rational function of $x(p)$, with no poles at branch points.

Theorem 4.6. *$W_k^{(g)}$ is a symmetric function of its k variables.*

Corollary 4.1.

$$\forall i, \quad \mathop{\mathrm{Res}}_{p \to a_i} W_{k+1}^{(g)}(p, p_1, \ldots, p_k) = 0, \tag{4.26}$$

$$\forall i, \quad \mathop{\mathrm{Res}}_{p \to a_i} x(p) W_{k+1}^{(g)}(p, p_1, \ldots, p_k) = 0, \tag{4.27}$$

$$\sum_i \mathop{\mathrm{Res}}_{p \to a_i} y(p) W_{k+1}^{(g)}(p, p_1, \ldots, p_k) = 0, \tag{4.28}$$

$$\sum_i \mathop{\mathrm{Res}}_{p \to a_i} x(p) y(p) W_{k+1}^{(g)}(p, p_1, \ldots, p_k) = 0. \tag{4.29}$$

Theorem 4.7. *For $k \geq 1$ we have*

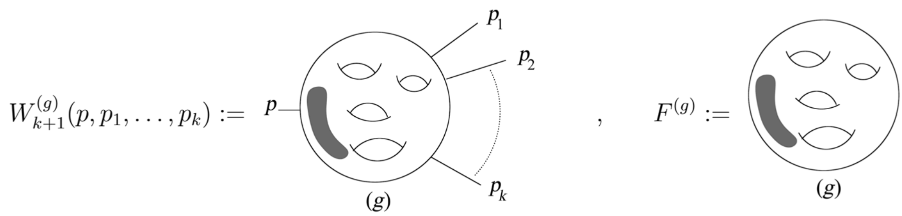
$$\begin{aligned} \mathop{\mathrm{Res}}_{p_{k+1} \to \mathbf{a}, p_1, \ldots, p_k} \Phi(p_{k+1}) W_{k+1}^{(g)}(\mathbf{p_K}, p_{k+1}) = {} & (2g + k - 2) W_k^{(g)}(\mathbf{p_K}) \\ & + \delta_{g,0}\delta_{k,1} y(p_1)\, dx(p_1). \end{aligned} \tag{4.30}$$

Notice that for $k = 0$ and $g \geq 2$, it holds by definition if we define $W_0^{(g)} = -F^{(g)}$.

4.5. Diagrammatic representation

The recursive definitions of $W_k^{(g)}$ and $F^{(g)}$ can be represented *graphically.*

We represent the multilinear form $W_k^{(g)}(p_1, \ldots, p_k)$ as a blob-like "surface" with g holes and k legs (or punctures) labeled with the variables $p_1, \ldots, p_k$, and $F^{(g)}$ with 0 legs and g holes.

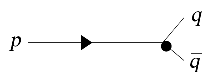


(4.31)

We represent the Bergmann kernel $B(p, q)$ (which is also $W_2^{(0)}$, i.e., a blob with 2 legs and no hole) as a straight non-oriented line between p and q

$$B(p,q) := p \text{———} q. \tag{4.32}$$

We represent $(dE_q(p)/\omega(q))$ as a straight arrowed line with the arrow from p toward q, and with a tri-valent vertex whose legs are q and $\overline{q}$

(4.33) $\dfrac{dE_q(p)}{\omega(q)} :=$ 

Graphs

Definition 4.5. For any $k \geq 0$ and $g \geq 0$ such that $k + 2g \geq 3$, we define:

Let $\mathcal{G}_{k+1}^{(g)}(p, p_1, \ldots, p_k)$ be the set of connected trivalent graphs defined as follows:

1. There are $2g + k - 1$ tri-valent vertices called vertices.
2. There is one 1-valent vertex labeled by p, called the root.
3. There are k 1-valent vertices labeled with $p_1, \ldots, p_k$ called the leaves.
4. There are $3g + 2k - 1$ edges.

5. Edges can be arrowed or non-arrowed. There are $k+g$ non-arrowed edges and $2g+k-1$ arrowed edges.
6. The edge starting at p has an arrow leaving from the root p.
7. The k edges ending at the leaves $p_1, \dots, p_k$ are non-arrowed.
8. The arrowed edges form a "spanning[2] planar[3] binary skeleton[4] tree" with root p. The arrows are oriented from root towards leaves. In particular, this induces a partial ordering of all vertices.
9. There are k non-arrowed edges going from a vertex to a leaf, and g non-arrowed edges joining two inner vertices. Two inner vertices can be connected by a non-arrowed edge only if one is the parent of the other along the tree.
10. If an arrowed edge and a non-arrowed inner edge come out of a vertex, then the arrowed edge is the left child. This rule only applies when the non-arrowed edge links this vertex to one of its descendants (not one of its parents).

We have the following useful lemma:

Lemma 4.1. *There is a* 1 *to* $3g+2k-1$ *map from* $\mathcal{G}_{k+1}^{(g)}(p, \mathbf{p_K})$ *to* $\mathcal{G}_{k+2}^{(g)}(p, \mathbf{p_K}, p_{k+1})$.

Proof. If G is a graph in $\mathcal{G}_{k+2}^{(g)}(p, p_1, \dots, p_k, p_{k+1})$, remove the non-arrowed edge attached to the leaf p_{k+1} and remove the corresponding vertex, and merge the incoming and the other outgoing edges of that vertex. You clearly get a graph $G' \in \mathcal{G}_{k+1}^{(g)}(p, p_1, \dots, p_k)$. It is clear that the same graph is obtained $3g+2k-1$ times (the number of edges of G'). And it is clear that from any $G' \in \mathcal{G}_{k+1}^{(g)}(p, p_1, \dots, p_k)$, you can obtain $3g+2k-1$ graphs $G \in \mathcal{G}_{k+2}^{(g)}(p, p_1, \dots, p_k, p_{k+1})$ by adding a new vertex on any edge, and linking this new vertex to the leaf p_{k+1}. □

Example of $\mathcal{G}_1^{(2)}(p)$ As an example, let us build step by step all the graphs of $\mathcal{G}_1^{(2)}(p)$, i.e., $g=2$ and $k=0$.

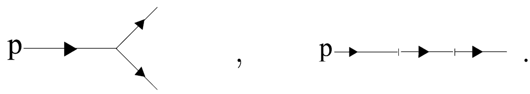
---

[2]It goes through all vertices.

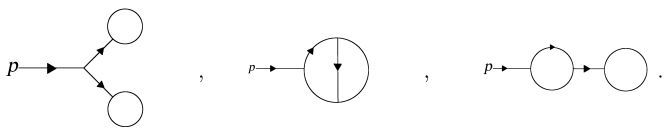
[3]planar tree means that the left child and right child are not equivalent. The right child is marked by a black disk on the outgoing edge.

[4]a binary skeleton tree is a binary tree from which we have removed the leaves, i.e. a tree with vertices of valence 1, 2 or 3.

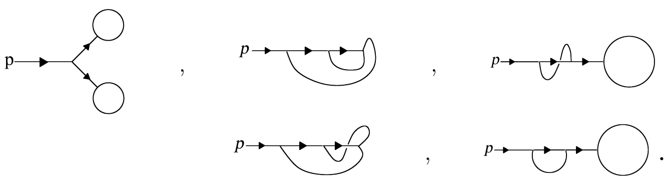
We first draw all planar binary skeleton trees with one root p and $2g+k-1=3$ arrowed edges:

(4.34) 

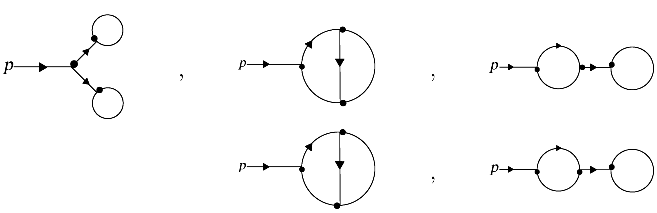
Then, we draw $g+k=2$ non-arrowed edges in all possible ways such that every vertex is tri-valent, also satisfying rule 9 of Definition.4.5. There is only one possibility for the first graph and two for the second one:

(4.35) 

It just remains to specify the left and right children for each vertex. The only possibilities in accordance with rule 10 of Definition 4.5 are[5]:

(4.36) 

In order to simplify the drawing, we can draw a black dot to specify the right child. This way one gets only planar graphs.

(4.37) 

Remark that without the prescriptions 9 and 10, one would get 13 different graphs whereas we only have 5.

Weight of a graph Consider a graph $G \in \mathcal{G}_{k+1}^{(g)}(p, p_1, \ldots, p_k)$. Then, to each vertex $i = 1, \ldots, 2g+k-1$ of G, we associate a label q_i, and we associate q_i to the beginning of the left child edge, and $\overline{q}_i$ to the right child edge.

[5]Note that the graphs are not necessarily planar.

Thus, each edge (arrowed or not), links two labels which are points on the Riemann surface $\bar{\Sigma}$.

- To an arrowed edge going from q' toward q, we associate a factor $(dE_q(q')/(y(q) - y(\overline{q}))\, dx(q))$.
- To a non-arrowed edge going between q' and q we associate a factor $B(q', q)$.
- Following the arrows backwards (i.e., from leaves to root), for each vertex q, we take a residue at $q \to \mathbf{a}$, i.e., we sum over all branch points.

After taking all the residues, we get the weight of the graph:

$$w(G), \tag{4.38}$$

which is a multilinear form in $p, p_1, \ldots, p_k$.

Similarly, we define weights of linear combinations of graphs by

$$w(\alpha G_1 + \beta G_2) = \alpha w(G_1) + \beta w(G_2) \tag{4.39}$$

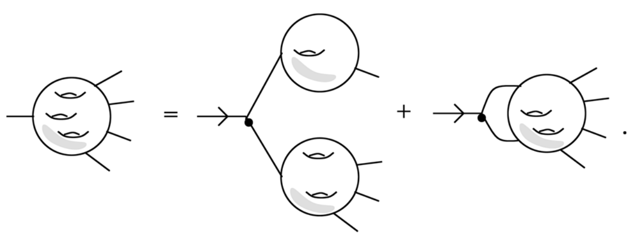
and for a disconnected graph, i.e., a product of two graphs:

$$w(G_1 G_2) = w(G_1) w(G_2). \tag{4.40}$$

Theorem 4.8. *We have*

$$W_{k+1}^{(g)}(p, p_1, \ldots, p_k) = \sum_{G \in \mathcal{G}_{k+1}^{(g)}(p,p_1,\ldots,p_k)} w(G) = w\left(\sum_{G \in \mathcal{G}_{k+1}^{(g)}(p,p_1,\ldots,p_k)} G \right). \tag{4.41}$$

Proof. This is precisely what the recursion Equations 4.9 of Definition 4.2 are doing. Indeed, one can represent them diagrammatically by



(4.42)

□

Such graphical notations are very convenient, and are a good support for intuition and even help proving some relationships. It was immediately noticed after [33] that those diagrams look very much like Feynman graphs, and there was a hope that they could be the Feynman's graphs for the Kodaira–Spencer theory. But they ARE NOT Feynman graphs, because Feynman graphs cannot have non-local restrictions like the fact that non-oriented lines can join only a vertex and one of its descendent.

Those graphs are merely a notation for the recursive Definition 4.2.

Lemma 4.2. (**Symmetry factor**) *The weight of two graphs differing by the exchange of the right and left children of a vertex are the same. Indeed, the distinction between right and left child is just a way of encoding symmetry factors.*

Proof. This property follows directly from Theorem 4.4 and the definition (Equation (4.9)). Consider one term contributing to the first part of RHS of Equation (4.9):

$$\begin{aligned}
&\operatorname*{Res}_{q\to\mathbf{a}} \frac{dE_q(p)}{\omega(q)} W^{(m)}_{|J|+1}(q,\mathbf{P_J}) W^{(g-m)}_{k-|J|+1}(\overline{q},\mathbf{P_{K/J}}) \\
&\quad = -\operatorname*{Res}_{q\to\mathbf{a}} \frac{dE_q(p)}{\omega(q)} W^{(m)}_{|J|+1}(q,\mathbf{P_J}) W^{(g-m)}_{k-|J|+1}(q,\mathbf{P_{K/J}}) \\
&\quad = \operatorname*{Res}_{q\to\mathbf{a}} \frac{dE_q(p)}{\omega(q)} W^{(m)}_{|J|+1}(\overline{q},\mathbf{P_J}) W^{(g-m)}_{k-|J|+1}(q,\mathbf{P_{K/J}}),
\end{aligned} \tag{4.43}$$

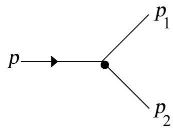
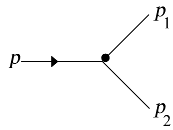
where the equalities are obtained by adding terms without residues at the branch points to the integrand and using Theorem 4.4. One can perform the same trick for the second term in Equation (4.9) and this proves the lemma. □

4.6. Examples

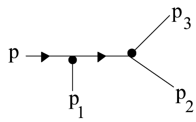
Let us present some examples of correlation functions and free energy for low order.

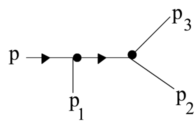
Correlation functions To leading order, one has the first correlation functions given by

$$W^{(0)}_2(p,q) = B(p,q). \tag{4.44}$$

$W_3^{(0)}(p,p_1,p_2) =$  $+$ 

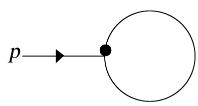
$$= \mathop{\mathrm{Res}}_{q\to\mathbf{a}} \frac{dE_q(p)}{\omega(q)} \left[B(q,p_1)B(\overline{q},p_2) + B(\overline{q},p_1)B(q,p_2)\right] \tag{4.45}$$

$W_4^{(0)}(p,p_1,p_2,p_3) =$  $+$ perm. (1,2,3)

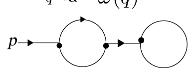
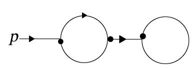
$+$  $+$ perm. (1,2,3)

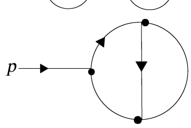
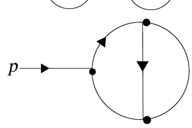
$$\begin{aligned}
&= \mathop{\mathrm{Res}}_{q\to\mathbf{a}} \mathop{\mathrm{Res}}_{r\to\mathbf{a}} \frac{dE_q(p)}{\omega(q)} \frac{dE_r(q)}{\omega(r)} \left[B(\overline{q},p_1)B(r,p_2)B(\overline{r},p_3)\right.\\
&+B(\overline{q},p_1)B(\overline{r},p_2)B(r,p_3) + B(\overline{q},p_2)B(r,p_1)B(\overline{r},p_3)\\
&+B(\overline{q},p_2)B(\overline{r},p_1)B(r,p_3) + B(\overline{q},p_3)B(r,p_2)B(\overline{r},p_1)\\
&\left.+B(\overline{q},p_3)B(\overline{r},p_2)B(r,p_1)\right]\\
&+ \mathop{\mathrm{Res}}_{q\to\mathbf{a}} \mathop{\mathrm{Res}}_{r\to\mathbf{a}} \frac{dE_q(p)}{\omega(q)} \frac{dE_r(\overline{q})}{\omega(r)} \left[B(q,p_1)B(r,p_2)B(\overline{r},p_3)\right.\\
&+B(q,p_1)B(\overline{r},p_2)B(r,p_3) + B(q,p_2)B(r,p_1)B(\overline{r},p_3)\\
&+B(q,p_2)B(\overline{r},p_1)B(r,p_3) + B(q,p_3)B(r,p_2)B(\overline{r},p_1)\\
&\left.+B(q,p_3)B(\overline{r},p_2)B(r,p_1)\right]
\end{aligned} \tag{4.46}$$

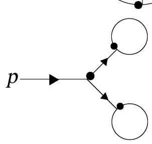
First orders for the one point correlation function read:

$W_1^{(1)}(p) =$ 

$$= \mathop{\mathrm{Res}}_{q\to\mathbf{a}} \frac{dE_q(p)}{\omega(q)} B(q,\overline{q}) \tag{4.47}$$

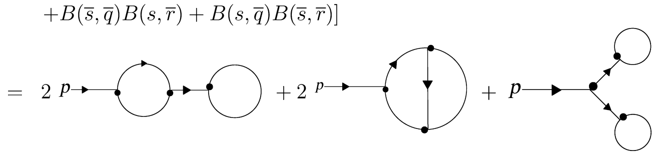
$W_1^{(2)}(p) =$  $+$ 

$+$  $+$ 

$+$ 

$$\begin{aligned}
&= \mathop{\mathrm{Res}}_{q\to\mathbf{a}} \mathop{\mathrm{Res}}_{r\to\mathbf{a}} \mathop{\mathrm{Res}}_{s\to\mathbf{a}} \frac{dE_q(p)}{\omega(q)} \frac{dE_r(q)}{\omega(r)} \frac{dE_s(\overline{q})}{\omega(s)} B(r,\overline{r})B(s,\overline{s})\\
&+ \mathop{\mathrm{Res}}_{q\to\mathbf{a}} \mathop{\mathrm{Res}}_{r\to\mathbf{a}} \mathop{\mathrm{Res}}_{s\to\mathbf{a}} \frac{dE_q(p)}{\omega(q)} \frac{dE_r(q)}{\omega(r)} \frac{dE_s(\overline{r})}{\omega(s)} B(r,\overline{q})B(s,\overline{s})
\end{aligned}$$

$$+\mathop{\mathrm{Res}}_{q\to\mathbf{a}}\mathop{\mathrm{Res}}_{r\to\mathbf{a}}\mathop{\mathrm{Res}}_{s\to\mathbf{a}}\frac{dE_q(p)}{\omega(q)}\frac{dE_r(q)}{\omega(r)}\frac{dE_s(r)}{\omega(s)}[B(\overline{q},\overline{r})B(s,\overline{s})$$
$$+B(\overline{s},\overline{q})B(s,\overline{r})+B(s,\overline{q})B(\overline{s},\overline{r})]$$



(4.48)

where the last expression is obtained using Lemma 4.2.

Free energy The second-order free energy reads

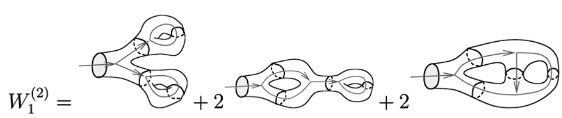
$$\begin{aligned}-2F^{(2)} = &\mathop{\mathrm{Res}}_{p\to\mathbf{a}}\mathop{\mathrm{Res}}_{q\to\mathbf{a}}\mathop{\mathrm{Res}}_{r\to\mathbf{a}}\mathop{\mathrm{Res}}_{s\to\mathbf{a}}\frac{\Phi(p)\,dE_q(p)}{\omega(q)}\frac{dE_r(q)}{\omega(r)}\frac{dE_s(\overline{q})}{\omega(s)}B(r,\overline{r})B(s,\overline{s})\\
&+\mathop{\mathrm{Res}}_{p\to\mathbf{a}}\mathop{\mathrm{Res}}_{q\to\mathbf{a}}\mathop{\mathrm{Res}}_{r\to\mathbf{a}}\mathop{\mathrm{Res}}_{s\to\mathbf{a}}\frac{\Phi(p)\,dE_q(p)}{\omega(q)}\frac{dE_r(q)}{\omega(r)}\frac{\Phi(p)\,dE_s(\overline{r})}{\omega(s)}B(r,\overline{q})B(s,\overline{s})\\
&+\mathop{\mathrm{Res}}_{p\to\mathbf{a}}\mathop{\mathrm{Res}}_{q\to\mathbf{a}}\mathop{\mathrm{Res}}_{r\to\mathbf{a}}\mathop{\mathrm{Res}}_{s\to\mathbf{a}}\frac{\Phi(p)\,dE_q(p)}{\omega(q)}\frac{dE_r(q)}{\omega(r)}\frac{dE_s(r)}{\omega(s)}[B(\overline{q},\overline{r})B(s,\overline{s})\\
&+B(\overline{s},\overline{q})B(s,\overline{r})+B(s,\overline{q})B(\overline{s},\overline{r})]\end{aligned}$$

(4.49)

4.7. Remark: Teichmuller pant gluings

Every Riemann surface of genus g with k punctures can be decomposed into $2g+k$ pants whose boundaries are $3g+k$ closed geodesics (in the metric with constant negative curvature). The number of ways (in the combinatorial sense) of gluing $2g+k$ pants by their boundaries is clearly the same as the number of diagrams of $\mathcal{G}_k^{(g)}$, and each diagram corresponds to one pant decomposition.

Example with $k=1$ and $g=2$:



5. Variations of the curve

The goal of this section is to study how the various $F^{(g)}$ and correlation functions change under the variations of moduli of the curve.

Consider an infinitesimal variation of the curve $\mathcal{E} \to \mathcal{E} + \delta\mathcal{E}$. It induces a variation of the function $y(x)$ at fixed x:

$$\delta_\Omega \, y|_x \, dx = -\Omega. \tag{5.1}$$

If we use a local coordinate z, we may prefer to work at fixed z instead of fixed x, we have a Poisson structure:

$$\delta_\Omega \, y|_z \, dx - \delta_\Omega \, x|_z \, dy = -\Omega. \tag{5.2}$$

The possible Ω's can be classified as first type (holomorphic), second type (residueless, and vanishing $\mathcal{A}$-cycles) and third type (only simple poles and vanishing $\mathcal{A}$-cycles), see [12] for this classification.

5.1. Rauch variational formula

Equation 5.2 implies that the variation of position of a branch point a_i is given by:

$$\delta_\Omega \, x(a_i) = \frac{\Omega(a_i)}{dy(a_i)}. \tag{5.3}$$

We assume here that (Ω/dy) has no pole at branch points. Then, Rauch variational formula [47, 68] implies that the change of the Bergmann kernel is

$$\begin{aligned}\delta_\Omega \underline{B}(p,q)|_{x(p),x(q)} &= \sum_i \frac{\Omega(a_i)}{dy(a_i)} \operatorname*{Res}_{r\to a_i} \frac{\underline{B}(r,p)\underline{B}(r,q)}{dx(r)} \\ &= \sum_i \operatorname*{Res}_{r\to a_i} \frac{\Omega(r)\underline{B}(r,p)\underline{B}(r,q)}{dx(r)\, dy(r)}.\end{aligned} \tag{5.4}$$

In particular after integrating over a $\underline{\mathcal{B}}$-cycle we have

$$\delta_\Omega du(p)|_{x(p)} = \sum_i \operatorname*{Res}_{r\to a_i} \frac{\Omega(r)\underline{B}(r,p)du(r)}{dx(r)\, dy(r)}, \tag{5.5}$$

and integrating again over a $\underline{\mathcal{B}}$-cycle:

$$\delta_\Omega \tau = 2i\pi \sum_i \operatorname*{Res}_{r\to a_i} \frac{\Omega(r)\, du(r)\, du^t(r)}{dx(r)\, dy(r)}. \tag{5.6}$$

Let us compute the variations of the κ-modified Bergmann kernel:

$$
\begin{aligned}
\delta_\Omega B(p,q)|_{x(p),x(q)} &= \delta_\Omega \underline{B}(p,q)|_{x(p),x(q)} + 2i\pi \delta_\Omega \, du^t(p)|_{x(p)} \, \kappa \, du(q) \\
&\quad + 2i\pi \, du^t(p) \, \kappa \, \delta_\Omega \, du(q)|_{x(q)} \\
&= \operatorname*{Res}_{r\to \mathbf{a}} \frac{\Omega(r)\underline{B}(r,p)\underline{B}(r,q)}{dx(r)\, dy(r)} \\
&\quad + 2i\pi \operatorname*{Res}_{r\to \mathbf{a}} \frac{\Omega(r)\underline{B}(r,p)\, du^t(r)\kappa\, du(q)}{dx(r)\, dy(r)} \\
&\quad + 2i\pi \operatorname*{Res}_{r\to \mathbf{a}} \frac{\Omega(r)\underline{B}(r,q)\, du^t(p)\, \kappa\, du(r)}{dx(r)\, dy(r)} \\
&= \operatorname*{Res}_{r\to \mathbf{a}} \frac{\Omega(r)\underline{B}(r,p)\underline{B}(r,q)}{dx(r)\, dy(r)} \\
&\quad + \operatorname*{Res}_{r\to \mathbf{a}} \frac{\Omega(r)\underline{B}(r,p)(B(r,q) - \underline{B}(r,q))}{dx(r)\, dy(r)} \\
&\quad + \operatorname*{Res}_{r\to \mathbf{a}} \frac{\Omega(r)(B(r,p) - \underline{B}(r,p))\underline{B}(r,q)}{dx(r)\, dy(r)} \\
&= \operatorname*{Res}_{r\to \mathbf{a}} \frac{\Omega(r)B(r,p)B(r,q)}{dx(r)\, dy(r)} \\
&\quad + 4\pi^2 \operatorname*{Res}_{r\to \mathbf{a}} \frac{\Omega(r)\, du^t(p)\kappa\, du(r)\, du^t(r)\kappa\, du(q)}{dx(r)\, dy(r)} \\
&= \operatorname*{Res}_{r\to \mathbf{a}} \frac{\Omega(r)B(r,p)B(r,q)}{dx(r)\, dy(r)} - 2i\pi\, du^t(p)\kappa\, \delta_\Omega \tau\, \kappa\, du(q), \qquad (5.7)
\end{aligned}
$$

i.e.

$$
\begin{aligned}
\left(\delta_\Omega + \operatorname{tr}\left(\kappa\, \delta_\Omega \tau\, \kappa \frac{\partial}{\partial \kappa}\right)\right)_{x(p),x(q)} B(p,q) &= \operatorname*{Res}_{r\to \mathbf{a}} \frac{\Omega(r)B(r,p)B(r,q)}{dx(r)\, dy(r)} \\
&= -2 \operatorname*{Res}_{r\to \mathbf{a}} \frac{\Omega(r)\, dE_r(p) B(r,q)}{\omega(r)}. \qquad (5.8)
\end{aligned}
$$

We thus define the *covariant variation*:

$$
\boxed{D_\Omega = \delta_\Omega + \operatorname{Tr}\left(\kappa\, \delta_\Omega \tau\, \kappa \frac{\partial}{\partial \kappa}\right).} \qquad (5.9)
$$

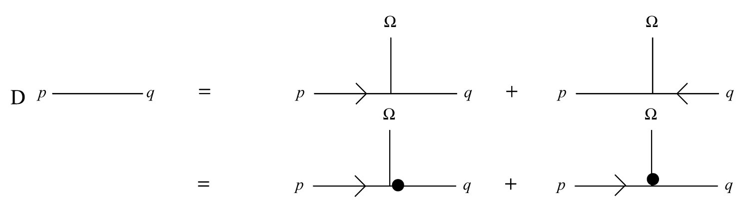
It is more convenient to rewrite Equation (5.8) as follows:

$$
\begin{aligned}
D_\Omega B(p,q) &= -2 \operatorname*{Res}_{r\to\mathbf{a}} \frac{\Omega(r)\, dE_r(p) B(r,q)}{\omega(r)} \\
&= -2 \operatorname*{Res}_{r\to\mathbf{a}} \operatorname*{Res}_{s\to r} \frac{\Omega(r)\, dE_r(p) B(s,q)}{(y(r)-y(\overline{r}))(x(s)-x(r))} \\
&= 2 \operatorname*{Res}_{r\to\mathbf{a}} \operatorname*{Res}_{s\to \overline{r}} \frac{\Omega(r)\, dE_r(p) B(s,q)}{(y(r)-y(\overline{r}))(x(s)-x(r))} \\
&= 2 \operatorname*{Res}_{r\to\mathbf{a}} \frac{\Omega(r)\, dE_r(p) B(\overline{r},q)}{\omega(r)} \\
&= \operatorname*{Res}_{r\to\mathbf{a}} \frac{dE_r(p)}{\omega(r)} \left[\Omega(r) B(\overline{r},q) + \Omega(\overline{r}) B(r,q)\right]
\end{aligned}
\tag{5.10}
$$

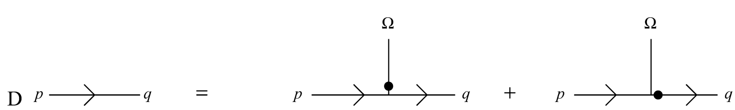
because now we recognize the propagator and vertex of Definition 4.1. Similarly, by integrating once with respect to q, near a branch point a_j we get

$$
\begin{aligned}
D_\Omega\, dE_q(p)|_{x(p),x(q)} &= -2 \operatorname*{Res}_{r\to\mathbf{a}} \frac{dE_r(p)}{\omega(r)}\, \Omega(r)\, dE_q(r) \\
&= \operatorname*{Res}_{r\to\mathbf{a}} \frac{dE_r(p)}{\omega(r)} [\Omega(r)\, dE_q(\overline{r}) + \Omega(\overline{r})\, dE_q(r)]
\end{aligned}
\tag{5.11}
$$

Those two relations can be depicted



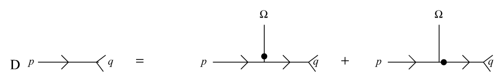
and



From this last variation, one can compute the covariant variations of the correlation functions and free energies through the following lemma:

Lemma 5.1. *For any symmetric bilinear form $f(q,p) = f(p,q)$:*

$$D_\Omega \left(\operatorname*{Res}_{q \to \mathbf{a}} \frac{dE_q(p)}{\omega(q)} f(q,\overline{q}) \right)_{x(p)} = 2 \sum_{i,j} \operatorname*{Res}_{r \to a_i} \operatorname*{Res}_{q \to a_j} \frac{dE_r(p)}{\omega(r)} \Omega(r) \frac{dE_q(r)}{\omega(q)} f(q,\overline{q})$$
$$+ \sum_j \operatorname*{Res}_{q \to a_j} \frac{dE_q(p)}{\omega(q)} D_\Omega(f(q,\overline{q}))_{x(q)}.$$

(5.12)


Graphically, this means that taking the variation of a diagram just consists in adding a leg Ω in all possible edges of the graph. In particular if Ω can be written as

$$\Omega(p) = \int_{\partial\Omega} B(p,q)\Lambda(q), \tag{5.13}$$

where the path $\partial\Omega$ does not intersect small circles around branch points[6], then we have

Theorem 5.1. *Variations of correlation functions and free energies: For $g + k > 1$ we have*

$$\boxed{D_\Omega W_k^{(g)}(p_1,\ldots,p_k)\Big|_{x(p_i)} = \int_{\partial\Omega} W_{k+1}^{(g)}(p_1,\ldots,p_k,q)\Lambda(q)} \tag{5.14}$$

and, for $g \geq 1$,

$$\boxed{D_\Omega F^{(g)} = -\int_{\partial\Omega} W_1^{(g)}(p)\Lambda(p).} \tag{5.15}$$

This theorem is proved in Appendix B. It follows directly from Lemmas 4.1 and 5.1.

[6]This excludes the case where Ω corresponds to the variation of an hard edge, cf. [12, 16, 36].

5.2. Loop insertion operator

In particular for any point q lying away from the branch points, if we choose

$$\Omega(p) = B(p,q) \tag{5.16}$$

we call $D_{B(.,q)}$ the loop insertion operator, by analogy with matrix models [7,39].

It acts on the correlation functions and free energies as follows:

Theorem 5.2.

$$D_B\, W_k^{(g)}(p_1,\dots,p_k) = W_{k+1}^{(g)}(p_1,\dots,p_k,q), \tag{5.17}$$

$$D_B\, F^{(g)} = -W_1^{(g)}(q) \tag{5.18}$$

and

$$D_B\, F^{(0)} = y(q)\,dx(q) + \frac{1}{4i\pi}\left(\kappa \oint_{\mathcal{B}} y\,dx\right)^t \oint_{\mathcal{B}}\oint_{\mathcal{B}} W_{3,0}\kappa \oint_{\mathcal{B}} y\,dx. \tag{5.19}$$

Thus, the loop insertion operator, adds one leg to correlation functions.

5.3. Variations with respect to the moduli

Let us consider canonical variations of the curve corresponding to each moduli of the curve defined in Section 3.4. We use Equation (3.63)

$$y\,dx = \sum_{\alpha,k} t_{\alpha,k} B_{\alpha,k} + \sum_{\alpha} t_\alpha\, dS_{\alpha,o} + 2i\pi \sum_i \epsilon_i\, du_i(p) \tag{5.20}$$

to identify the Ω's corresponding to varying only one modulus.

Variation of filling fractions Consider the variation of the curve

$$\Omega(p) = -2i\pi\, du_i(p) = -\oint_{\mathcal{B}_i} B(p,q). \tag{5.21}$$

i.e., $\partial\Omega = \mathcal{B}_i$ and $\Lambda = -1$. It is such that

$$\delta_\Omega \epsilon_j = \delta_{ij}, \quad \delta_\Omega t_\alpha = 0, \quad \delta_\Omega V_\alpha = 0. \tag{5.22}$$

Therefore it is equivalent to varying only the filling fraction $\epsilon_i = (1/2i\pi) \oint_{\mathcal{A}_i} y\,dx$:

$$D_{-2i\pi du_i} = \frac{\partial}{\partial \epsilon_i}. \tag{5.23}$$

Theorem 5.1 gives

$$\frac{\partial}{\partial \epsilon_i} W_k^{(g)}(p_1, \dots, p_k) = -\oint_{\mathcal{B}_i} W_{k+1}^{(g)}(p_1, \dots, p_k, q), \tag{5.24}$$

and

$$\frac{\partial}{\partial \epsilon_i} F^{(g)} = \oint_{\mathcal{B}_i} W_1^{(g)}(q), \tag{5.25}$$

and

$$\frac{\partial}{\partial \epsilon_i} F^{(0)} = -\oint_{\mathcal{B}_j} y\,dx + \frac{1}{4i\pi} \left(\kappa \oint_{\mathcal{B}} y\,dx\right)^t \delta_{-2i\pi\,du_i}(\tau)\,\kappa \oint_{\mathcal{B}} y\,dx. \tag{5.26}$$

Variation of temperatures

Let α and α' be two distinct poles of $y\,dx$. Consider the variation of the curve

$$\Omega(p) = -dS_{\alpha,\alpha'}(p) = \int_{\alpha}^{\alpha'} B(p,q), \quad \text{i.e., } \partial\Omega = [\alpha, \alpha'], \quad \Lambda = 1. \tag{5.27}$$

It is such that

$$\delta_\Omega \epsilon_j = 0, \quad \delta_\Omega t_\beta = \delta_{\alpha,\beta} - \delta_{\alpha',\beta}, \quad \delta_\Omega V_\beta = 0. \tag{5.28}$$

Therefore it is equivalent to varying only the temperatures t_α and $t_{\alpha'}$:

$$D_{-dS_{\alpha,\alpha'}} = \frac{\partial}{\partial t_\alpha} - \frac{\partial}{\partial t'_\alpha}. \tag{5.29}$$

Notice that it is impossible to vary only one temperature t_α since we have $\sum_\beta t_\beta = 0$.

Theorem 5.1 gives

$$\left(\frac{\partial}{\partial t_\alpha} - \frac{\partial}{\partial t'_\alpha}\right) W_k^{(g)}(p_1,\dots,p_k) = \int_\alpha^{\alpha'} W_{k+1}^{(g)}(p_1,\dots,p_k,q), \tag{5.30}$$

$$\left(\frac{\partial}{\partial t_\alpha} - \frac{\partial}{\partial t'_\alpha}\right) F^{(g)} = \int_{\alpha'}^{\alpha} W_1^{(g)}(q) \tag{5.31}$$

and

$$\left(\frac{\partial}{\partial t_\alpha} - \frac{\partial}{\partial t'_\alpha}\right) F^{(o)} = \mu_\alpha - \mu_{\alpha'} + \frac{1}{4i\pi}\left(\kappa \oint_{\mathcal{B}} y\,dx\right)^t \delta_{-dS_{\alpha,\alpha'}}(\tau)\ \kappa \oint_{\mathcal{B}} y\,dx. \tag{5.32}$$

Variation of the moduli of the poles Let α be a pole of $y\,dx$. Consider the variation of the curve

$$\Omega(p) = -B_{\alpha,k} = \operatorname*{Res}_{\alpha} B(p,q) z_\alpha^k(q), \tag{5.33}$$

i.e., $\partial\Omega$ is a small circle around α and $\Lambda = (1/2i\pi) z_\alpha^k$. It is such that

$$\delta_\Omega \epsilon_j = 0, \quad \delta_\Omega t_\beta = 0, \quad \delta_\Omega t_{\beta,k'} = \delta_{\alpha,\beta}\delta_{k,k'}. \tag{5.34}$$

Therefore it is equivalent to varying only the coefficient $t_{\alpha,k}$:

$$D_{-B_{\alpha,k}} = \frac{\partial}{\partial t_{\alpha,k}}. \tag{5.35}$$

Theorem 5.1 gives

$$\frac{\partial}{\partial t_{\alpha,k}} W_k^{(g)}(p_1,\dots,p_k) = \operatorname*{Res}_{\alpha} z_\alpha^k(q) W_{k+1}^{(g)}(p_1,\dots,p_k,q), \tag{5.36}$$

$$\frac{\partial}{\partial t_{\alpha,k}} F^{(g)} = -\operatorname*{Res}_{\alpha} z_\alpha^k(q) W_1^{(g)}(q) \tag{5.37}$$

and

$$\frac{\partial}{\partial t_{\alpha,k}} F^{(o)} = \operatorname*{Res}_{\alpha} y\,dx z_\alpha^k + \frac{1}{4i\pi}\left(\kappa \oint_{\mathcal{B}} y\,dx\right)^t \delta_{B_{\alpha,k}}(\tau)\kappa \oint_{\mathcal{B}} y\,dx. \tag{5.38}$$

5.4. Homogeneity

Theorem 5.3. *For $g > 1$, we have the homogeneity property:*

$$(2-2g)F^{(g)} = \sum_{\alpha,k} t_{\alpha,k}\frac{\partial}{\partial t_{\alpha,k}}F^{(g)} + \sum_{\alpha} t_\alpha \frac{\partial}{t_\alpha}F^{(g)} + \sum_i \epsilon_i \frac{\partial}{\partial \epsilon_i}F^{(g)} \tag{5.39}$$

i.e., $F^{(g)}$ is homogeneous of degree $2-2g$.

The proof is given in Appendix B.

5.5. Variations of $F^{(0)}$ with respect to the moduli

In this section we compute the first and second deriatives of $F^{(0)}$ with respect to the moduli of the curve. This paragraph is only for bookkeeping since those expressions have been known for some time [12, 13, 63]. Here we set $\kappa = 0$.

First derivatives of $F^{(0)}$

$$\frac{\partial F^{(0)}}{\partial t_{\alpha,k}} = \operatorname*{Res}_{\alpha} z_\alpha^k\, y\, dx, \tag{5.40}$$

$$\frac{\partial F^{(0)}}{\partial t_{\alpha,\beta}} = \left(\frac{\partial}{\partial t_\alpha} - \frac{\partial}{\partial t_\beta}\right) F^{(0)} = \mu_\alpha - \mu_\beta, \tag{5.41}$$

$$\frac{\partial F^{(0)}}{\partial \epsilon_i} = -\oint_{\mathcal{B}_i} y\, dx. \tag{5.42}$$

Second derivatives of $F^{(0)}$

$$\frac{\partial^2 F^{(0)}}{\partial t_{\alpha,k}\,\partial t_{\beta,l}} = (\delta_{\alpha,\beta}-1)\ \operatorname*{Res}_{p\to\alpha}\ \operatorname*{Res}_{q\to\beta} z_\alpha(p)^k B(p,q) z_\beta(q)^l, \tag{5.43}$$

$$\frac{\partial^2 F^{(0)}}{\partial t_{\alpha,k}\,\partial t_{\gamma,\beta}} = \operatorname*{Res}_{\alpha} z_\alpha^k\, dS_{\gamma,\beta}, \tag{5.44}$$

$$\frac{\partial^2 F^{(0)}}{\partial t_{\alpha,k}\,\partial \epsilon_i} = 2i\pi \operatorname*{Res}_{\alpha} z_\alpha^k\, du_i = -\oint_{\mathcal{B}_i} B_{\alpha,k}, \tag{5.45}$$

$$\frac{\partial^2 F^{(0)}}{\partial \epsilon_i\, \partial t_{\alpha,\beta}} = 2i\pi(u_i(\beta) - u_i(\alpha)), \tag{5.46}$$

$$\frac{\partial^2 F^{(0)}}{\partial \epsilon_i\, \partial \epsilon_j} = 0(-2i\pi\tau_{ij} \quad \text{for } \underline{F}^{(0)}), \tag{5.47}$$

$$\frac{\partial^2 F^{(0)}}{\partial t_{\alpha,\beta}^2} = \ln\left(d\zeta_\alpha(\alpha)\, d\zeta_\beta(\beta) E(\alpha,\beta)^2\right), \tag{5.48}$$

$$\frac{\partial^2 F^{(0)}}{\partial t_{\alpha,\beta}\, \partial t_{\alpha,\gamma}} = \ln\left(\frac{d\zeta_\alpha(\alpha) E(\alpha,\beta) E(\alpha,\gamma)}{E(\beta,\gamma)}\right), \tag{5.49}$$

$$\frac{\partial^2 F^{(0)}}{\partial t_{\alpha,\beta}\, \partial t_{\delta,\gamma}} = \ln\left(\frac{E(\delta,\beta) E(\alpha,\gamma)}{E(\alpha,\delta) E(\beta,\gamma)}\right), \tag{5.50}$$

where $\zeta_\alpha = (1/z_\alpha)$ is a local coordinate around the pole α.

Remark 5.1. *The definition of $F^{(0)}$ given in Equation (4.13) is nothing but the homogeneity property since it is written in terms of the first derivatives. One can also write a formula focussing more on the second-order derivatives of $F^{(0)}$:*

$$\begin{aligned} F^{(0)} = &-\frac{1}{2}\sum_{\alpha,\beta} \operatorname*{Res}_{p\to\alpha} \operatorname*{Res}_{q\to\beta} V_\alpha(p) B(p,q) V_\beta(q) + \sum_{\alpha,\beta} t_\beta \operatorname*{Res}_{\alpha} V_\alpha\, dS_{\beta,o} \\ &-\frac{1}{2}\sum_{\alpha,\beta} t_\alpha t_\beta \ln(\gamma_{\alpha,\beta}) - \frac{1}{2}\epsilon^t \oint_{\mathcal{B}} y\, dx, \end{aligned} \tag{5.51}$$

where

$$\ln \gamma_{\alpha,\alpha} = -\int_\alpha^o \left(dS_{\alpha,o'} + \frac{dz_\alpha}{z_\alpha}\right) + \ln(z_\alpha(o)) \tag{5.52}$$

and

$$\ln \gamma_{\alpha,\beta} = \ln\left(\frac{E(\alpha,\beta) E(o,o')}{E(\alpha,o') E(\beta,o)}\right). \tag{5.53}$$

One can notice that, in these terms,

$$\frac{\partial^2 F^{(0)}}{\partial t_{\alpha,\beta}^2} = \ln(\gamma_{\alpha,\alpha}\gamma_{\beta,\beta}). \tag{5.54}$$

6. Variations with respect to κ and modular transformations

6.1. Variations with respect to κ

We have introduced the matrix κ in order to easily compute modular transformations of our functions. Somehow variations of κ play the role of infinitesimal modular transformations. Therefore it is important to compute $\partial/\partial\kappa$, and we will use this result in Section 6.2.

First, notice that $W_k^{(g)}(p_1,\dots,p_k)$ is a polynomial in κ of degree $3g+2k-3$, and $F^{(g)}$ is a polynomial in κ of degree $3g-3$ for $g>1$ (number of propagators in a graph of $\mathcal{G}_k^{(g)}$).

Theorem 6.1.

$$\boxed{\begin{aligned} 2i\pi\frac{\partial}{\partial\kappa_{ij}}\,W_k^{(g)}(\mathbf{p_K}) &= \frac{1}{2}\oint_{r\in\mathcal{B}_j}\oint_{s\in\mathcal{B}_i} W_{k+2}^{(g-1)}(\mathbf{p_K},r,s)\\ &\quad+\frac{1}{2}\sum_h\sum_{L\subset K}\oint_{r\in\mathcal{B}_i} W_{|L|+1}^{(h)}(\mathbf{p_L},r)\\ &\quad\oint_{s\in\mathcal{B}_j} W_{k-|L|+1}^{(g-h)}(\mathbf{p_{K/L}},s)\end{aligned}} \tag{6.1}$$

and in particular for $g\geq 2$:

$$\boxed{\begin{aligned} -2i\pi\frac{\partial}{\partial\kappa_{ij}}\,F^{(g)} &= \frac{1}{2}\oint_{r\in\mathcal{B}_j}\oint_{s\in\mathcal{B}_i} W_2^{(g-1)}(r,s)\\ &\quad+\frac{1}{2}\sum_{h=1}^{g-1}\oint_{r\in\mathcal{B}_i} W_1^{(h)}(r)\oint_{s\in\mathcal{B}_j} W_1^{(g-h)}(s)\end{aligned}} \tag{6.2}$$

and

$$\frac{\partial}{\partial\kappa_{ij}}\,F^{(1)} = \frac{1}{\kappa_{ji}}. \tag{6.3}$$

This theorem is proved in Appendix B

Notice that these equations are to be compared with the Kodaira–Spencer theory [1, 2, 11, 42].

6.2. Modular transformations

Consider a modular change of cycles:

$$\begin{pmatrix} \underline{\mathcal{A}} \\ \underline{\mathcal{B}} \end{pmatrix} = \begin{pmatrix} \delta_{\mathcal{A}\mathcal{A}'} & \delta_{\mathcal{A}\mathcal{B}'} \\ \delta_{\mathcal{B}\mathcal{A}'} & \delta_{\mathcal{B}\mathcal{B}'} \end{pmatrix} \begin{pmatrix} \underline{\mathcal{A}'} \\ \underline{\mathcal{B}'} \end{pmatrix}, \quad \begin{pmatrix} \underline{\mathcal{A}'} \\ \underline{\mathcal{B}'} \end{pmatrix} = \begin{pmatrix} \delta_{\mathcal{A}'\mathcal{A}} & \delta_{\mathcal{A}'\mathcal{B}} \\ \delta_{\mathcal{B}'\mathcal{A}} & \delta_{\mathcal{B}'\mathcal{B}} \end{pmatrix} \begin{pmatrix} \underline{\mathcal{A}} \\ \underline{\mathcal{B}} \end{pmatrix}, \tag{6.4}$$

where $\delta_{\mathcal{A}'\mathcal{A}} = \delta^t_{\mathcal{B}\mathcal{B}'}$, $\delta_{\mathcal{A}'\mathcal{B}} = -\delta^t_{\mathcal{A}\mathcal{B}'}$, $\delta_{\mathcal{B}'\mathcal{B}} = \delta^t_{\mathcal{A}\mathcal{A}'}$, $\delta_{\mathcal{B}'\mathcal{A}} = -\delta^t_{\mathcal{B}\mathcal{A}'}$ and the matrices $\delta_{\mathcal{A}\mathcal{A}'}, \delta_{\mathcal{A}\mathcal{B}'}, \delta_{\mathcal{B}\mathcal{A}'}, \delta_{\mathcal{B}\mathcal{B}'}$ have integer coefficients and satisfy $\delta_{\mathcal{A}\mathcal{A}'}\delta^t_{\mathcal{B}\mathcal{B}'} - \delta_{\mathcal{B}\mathcal{A}'}\delta^t_{\mathcal{A}\mathcal{B}'} = Id$.

Under this transformation of the cycle homology basis, the Abel map and the matrix of period change like

$$du' = \mathcal{J}\, du, \quad du = \mathcal{J}^{-1}\, du' \tag{6.5}$$

with $\mathcal{J} = (\delta^t_{\mathcal{A}\mathcal{A}'} + \tau'\delta^t_{\mathcal{A}\mathcal{B}'}) = (\delta_{\mathcal{B}\mathcal{B}'} - \tau\delta_{\mathcal{A}\mathcal{B}'})^{-1}$ and

$$\begin{aligned} \tau' &= (\delta_{\mathcal{B}\mathcal{B}'} - \tau\delta_{\mathcal{A}\mathcal{B}'})^{-1}(-\delta_{\mathcal{B}\mathcal{A}'} + \tau\delta_{\mathcal{A}\mathcal{A}'}), \\ \tau &= (\delta^t_{\mathcal{A}\mathcal{A}'} + \tau'\delta^t_{\mathcal{A}\mathcal{B}'})^{-1}\,(\delta^t_{\mathcal{B}\mathcal{A}'} + \tau'\delta^t_{\mathcal{B}\mathcal{B}'}). \end{aligned} \tag{6.6}$$

Let us define the following symmetric matrix:

$$\widehat{\kappa} = \widehat{\kappa}^t = (\delta_{\mathcal{B}\mathcal{B}'}\delta^{-1}_{\mathcal{A}\mathcal{B}'} - \tau)^{-1} = \delta_{\mathcal{A}\mathcal{B}'}\mathcal{J} \tag{6.7}$$

it is such that the Bergmann kernel changes like:

$$\underline{B}' = \underline{B} + 2i\pi\, du^t\widehat{\kappa}\, du. \tag{6.8}$$

The κ-modified Bergmann kernel changes like

$$\begin{aligned} B'(p,q) &= \underline{B}(p,q) + 2i\pi\left[du'^t(p)\kappa\, du'(q) + du^t(p)\widehat{\kappa}\, du(q)\right] \\ &= \underline{B}(p,q) + 2i\pi\, du^t(p)\left(\widehat{\kappa} + \mathcal{J}^t\kappa\mathcal{J}\right)\, du(q). \end{aligned} \tag{6.9}$$

In other words, the effect of a modular change of cycles is equivalent to a change $\kappa \to \widehat{\kappa} + \mathcal{J}^t\kappa\mathcal{J}$ in the definition of the kernel $B(p,q)$.

Thus, the modular variations of the free energies satisfy the following theorem:

Theorem 6.2. *For $g \geq 2$ the modular transformation of the free energies $F^{(g)}$ consists in changing κ to $\widehat{\kappa} + \mathcal{J}^t\kappa\mathcal{J}$ in the definitions of the modified Bergmann kernel and Abelian differential.*

The first correction $F^{(1)}$ changes like

$$F^{(1)'} = F^{(1)} - \frac{1}{2}\ln\left(\delta_{\mathcal{BB}'} - \tau\delta_{\mathcal{AB}'}\right). \tag{6.10}$$

An equivalent way of saying the same thing, is that: if we change the basis of cycles and change the matrix $\kappa \to \mathcal{J}^{t\,-1}(\kappa - \widehat{\kappa})\mathcal{J}^{-1}$, then the $F^{(g)}$'s are unchanged for $g > 1$.

Proof. The result for $g \geq 2$ comes directly from the variation of the Bergmann kernel.

$F^{(1)}$ depends on the cycles only through the Bergmann τ-function. Since,

$$\begin{aligned}
\frac{\partial \ln\left(\tau_{B'x}\right)}{\partial x(a_i)} - \frac{\partial \ln\left(\tau_{Bx}\right)}{\partial x(a_i)} &= \operatorname*{Res}_{p\to a_i} \frac{\underline{B}'(p,\overline{p}) - \underline{B}(p,\overline{p})}{dx(p)} \\
&= 2i\pi \operatorname*{Res}_{p\to a_i} \frac{du^t(p)\widehat{\kappa}\,du(\overline{p})}{dx(p)} \\
&= -2i\pi \operatorname*{Res}_{p\to a_i} \frac{du^t(p)\widehat{\kappa}\,du(p)}{dx(p)} \\
&= -\operatorname{Tr}\kappa\frac{\partial\tau}{\partial x(a_i)} \\
&= -\operatorname{Tr}\left(\delta_{\mathcal{BB}'} - \tau\delta_{\mathcal{AB}'}\right)^{-1}\frac{\partial\tau\delta_{\mathcal{AB}'}}{\partial x(a_i)} \\
&= \frac{\partial \ln\det(\delta_{\mathcal{BB}'} - \tau\delta_{\mathcal{AB}'})}{\partial x(a_i)}
\end{aligned} \tag{6.11}$$

and this characterizes the Bergmann τ-function totally (up to a general multiplicative factor), one obtains the second result of the theorem. □

Remark 6.1. *The transformation of leading order $F^{(0)}$ is more complicated and its computation is more involved as the final result depends explicitely on the position of the poles α in the fundamental domain. Let us just mention that it depends on all the parameters of the modular transformation explicitely.*

Theorem 6.3. *If one chooses $\kappa = (i/2\,\mathrm{Im}\tau)$, then $F^{(g)}(\kappa)$ is modular invariant.*

Proof. It is well known that for that value of κ, the modified Bergmann kernel is the Schiffer kernel and is modular invariant. Indeed, it is easy to check that if $\kappa = (i/2\,\mathrm{Im}\tau)$ one has $\hat{\kappa} + \mathcal{J}^t\kappa\mathcal{J} = (i/2\,\mathrm{Im}\tau')$.

Since the only modular dependence of $F^{(g)}$ for $g \geq 1$ is in the Bergmann kernel, this proves the modular invariance of the $F^{(g)}$'s. □

7. Symplectic invariance

The following theorem is mostly the reason why we call $F^{(g)}$'s invariants of the curve. This theorem seems to be rather important and it has beautiful applications as we will see in Section 10.4.1.

Theorem 7.1. *The following transformations of $\mathcal{E}$ leave the $F^{(g)}$'s unchanged:*

- $x \to (ax+b)/(cx+d)$ *and* $y \to ((cx+d)^2)/(ad-bc)y$.
- $y \to y + R(x)$ *where R is any rational function.*
- $y \to y$ *and* $x \to -x$.
- $y \to x$ *and* $x \to y$.

Notice that these are transformations which conserve the symplectic form

$$|dx \wedge dy|. \tag{7.1}$$

In particular, we have the $PSL_2(\mathbf{C})$ invariance:

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad (ad-bc)^2 = 1. \tag{7.2}$$

That symplectic invariance seems to be a very powerful tool to see if different matrix models have the same topological expansion. For example, in Section 10.4.1 we show how symplectic invariance can be used to provide a new and very easy proof of some properties of the Kontsevitch integral.

Proof. Invariance under the first two transformations is obvious from the definitions, because the only x and y dependance of the $W_k^{(g)}$'s is in $\omega(q) = (y(q) - y(\overline{q}))\, dx(q)$ which is clearly unchanged under the first two transformations (notice that the transformation $x \to (ax+b)/(cx+d)$ conserves the branch points). In fact, the first two transformations leave $W_k^{(g)}$ unchanged.

In the third transformation the only thing which changes is the sign of ω, and it is easy to see that $W_k^{(g)}$ is multiplied by $(-1)^{2g-2+k} = (-1)^k$, and $F^{(g)}$ is multiplied by $(-1)^{2g-2} = 1$.

The fourth transformation is the difficult one. The proof consists in building some "mixed correlation functions", and seeing that their definition by inverting the roles of x and y lead to the same objects. Since it is long and involves new results in the framework of matrix model, it is written in a separate paper [43].

The case of $F^{(0)}$ and $F^{(1)}$ is done separately in Appendix C. $\square$

8. Singular limits

Consider a one-parameter family of algebraic curves:

$$\mathcal{E}(x, y, t) \tag{8.1}$$

such that the curve at $t = 0$ has a singular branch point a with a p/q rational singularity, i.e., in some local coordinate z near a we have

$$\begin{cases} t = 0, \\ x(z) \sim x(a) + (z-a)^q, \\ y(z) \sim y(a) + (z-a)^p. \end{cases} \tag{8.2}$$

At $t \neq 0$, the singularity is smoothed, and we have (the local parameter is now $\zeta = z\,t^{-\nu}$):

$$\begin{cases} x(z,t) \sim x(a) + t^{q\nu}\, Q(\zeta) + O(t^{q\nu}), \\ y(z,t) \sim y(a) + t^{p\nu}\, P(\zeta) + O(t^{p\nu}), \end{cases} \tag{8.3}$$

where Q is a polynomial of degree q and P is a polynomial of degree p, and where ν is some exponent which depends on the choice of the parameter t.

The curve

$$\mathcal{E}_{\text{sing}}(\xi, \eta) = \begin{cases} \xi(\zeta) = Q(\zeta) \\ \eta(\zeta) = P(\zeta) \end{cases} = \text{Resultant}(Q - \xi, P - \eta) \tag{8.4}$$

is called the singular spectral curve.

One observes that $F^{(g)}(\mathcal{E})$ is singular in the small t limit, and it behaves like

$$F^{(g)}(\mathcal{E}(t)) \sim t^{\gamma_g} F^{(g)}_{\text{sing}} + O(t^{\gamma_g}), \quad \text{for } g \geq 2, \tag{8.5}$$

$$F^{(1)}(\mathcal{E}(t)) \sim -\frac{1}{24}\,(p-1)(q-1)\nu\, \ln(t) + O(1), \quad \text{for } g = 1. \tag{8.6}$$

$F^{(g)}_{\text{sing}}$ is called the double scaling limit of $F^{(g)}$. The exponent γ_g and $F^{(g)}_{\text{sing}}$ are given by the following theorem:

Theorem 8.1. *Singular limits:*

$$\boxed{F^{(g)}_{\text{sing}}(\mathcal{E}) = F^{(g)}(\mathcal{E}_{\text{sing}}), \quad \text{for } g \geq 2} \tag{8.7}$$

and

$$\gamma_g = (2-2g)(p+q)\nu. \tag{8.8}$$

In other words, our construction of $F^{(g)}$ commutes with the singular limit.

Proof. It is easy to see that the most singular term in the limit of the Bergmann kernel in that regime, behaves like t^0, and thus $dE_{z'}(z)$ as well. The denominator $((y(z)-y(\overline{z}))\,dx(z)$ in the recursion behaves like $t^{\nu(p+q)}$ $(P(\zeta)-P(\overline{\zeta}))Q'(\zeta)d\zeta$, and by recursion on k and g, we easily see that

$$W_k^{(g)}(z_1,\dots,z_k) \sim t^{(2-2g-k)(p+q)\nu}\,\omega_k^{(g)}(\zeta_1,\dots,\zeta_k) \tag{8.9}$$

if all z_i's are close to a, and is subdominant if some z_i's are not in the vicinity of a. The leading contribution to $W_k^{(g)}$ is thus obtained by taking z' and $\overline{z}'$ in the vicinity of a only in Equation 4.9, i.e., $\omega_k^{(g)}(\zeta_1,\dots,\zeta_k)$ obey the same recursion formula as Equation 4.9, with the curve $\mathcal{E}_{\text{sing}}$. The same holds for the free energy. □

9. Integrability

Here, we prove that Z is a τ-function, because it satisfies some Hirota equation.

9.1. Baker–Akhiezer function

Given two points ξ and η in the fundamental domain, we define the following kernel as a formal series in $1/N$:

$$K_N(\xi,\eta) = \frac{\mathrm{e}^{-N\int_\eta^\xi y\,dx}}{E(\xi,\eta)\sqrt{dx(\xi)\,dx(\eta)}}\exp\Bigg(-\sum_{g=0}^{\infty}\sum_{l=1,2-2g-l<0}^{\infty}\frac{1}{l!}N^{2-2g-l}$$
$$\int_\eta^\xi\int_\eta^\xi\cdots\int_\eta^\xi W_l^{(g)}(p_1,\dots,p_l)\Bigg), \tag{9.1}$$

where the integration path lies in the fundamental domain.

This kernel has the following properties:

- Notice that $(x(\xi) - x(\eta))K_N(\xi,\eta) \to 1$ when $\eta \to \xi$.
- We have

$$\lim_{\eta\to\xi}\left(K_N(\xi,\eta) - \frac{1}{(x(\xi)-x(\eta))}\right) = -Ny(\xi) + \frac{W_1(\xi)}{dx(\xi)}, \tag{9.2}$$

  where $W_1 = \sum_{g=1}^{\infty} N^{1-2g} W_1^{(g)}$.
- We have:

$$K_N(\xi,\eta) = K_{-N}(\eta,\xi). \tag{9.3}$$

- One may think that K_N is singular at branch points because $\ln K_N$ has poles, however, using the singular limit Theorem 8.1, we see that the leading behavior of all $W_l^{(g)}$'s is given by the $W_l^{(g)}$'s of the Airy curve $y = \sqrt{x}$ described in Section 10.5. Therefore, near a branch point a, when $\xi,\eta \to a$, to leading order K_N is the Traicy–Widom kernel [72]:

$$\begin{gathered} K_N(\xi,\eta) \sim \frac{Ai(\hat\xi)Ai'(\hat\eta) - Ai'(\hat\xi)Ai(\hat\eta)}{\hat\xi - \hat\eta}, \\ \hat\xi = N^{2/3}(x(\xi) - x(a)), \quad \hat\eta = N^{2/3}(x(\eta) - x(a)). \end{gathered} \tag{9.4}$$

  In other words, K_N is not singular near branch points.
- The only singularities of $K_N(\xi,\eta)$ are essential singularities at all the poles of $y\,dx$, with a singular part equal to $\exp\left(-N\int_\eta^\xi y\,dx\right)$, as well as a simple pole at $\xi = \eta$.

Then, given a pole α of $y\,dx$, we define for ξ in the vicinity of α:

$$\begin{aligned} \psi_{\alpha,N}(\xi) &= \frac{e^{-N V_\alpha(\xi)} e^{-N\int_\alpha^\xi (y\,dx - dV_\alpha + t_\alpha dz_\alpha/z_\alpha)}}{E(\xi,\alpha)\sqrt{dx(\xi)d\zeta_\alpha(\alpha)}} (z_\alpha(\xi))^{Nt_\alpha} \exp \\ &\quad \left(-\sum_{g=0}^{\infty}\sum_{l=1, 2-2g-l<0}^{\infty} \frac{1}{l!} N^{2-2g-l} \int_\alpha^\xi \int_\alpha^\xi \cdots \int_\alpha^\xi W_l^{(g)}(p_1,\dots,p_l)\right) \\ &= \lim_{\eta\to\alpha}\left(K(\xi,\eta)\sqrt{\frac{dx(\eta)}{d\zeta_\alpha(\eta)}}\, e^{-NV_\alpha(\eta)}(z_\alpha(\eta))^{Nt_\alpha}\right), \end{aligned} \tag{9.5}$$

where ζ_α is the local parameter near α, $\zeta_\alpha = 1/z_\alpha$, and $\phi_{\alpha,N}(\xi) = \psi_{\alpha,-N}(\xi)$.

They have the following properties:

- $\psi_{\alpha,N}$ was defined only in the vicinity of α, but it can be easily analytically continued to the whole curve, by choosing an arbitrary point o in the vicinity of α and writing:

$$\begin{aligned}&\int_\alpha^\xi \left(y\,dx - dV_\alpha + t_\alpha \frac{dz_\alpha}{z_\alpha}\right) + V_\alpha(\xi) - t_\alpha \ln\left(z_\alpha(\xi)\right)\\ &\quad = \int_o^\xi y\,dx + \int_\alpha^o \left(y\,dx - dV_\alpha + t_\alpha \frac{dz_\alpha}{z_\alpha}\right) + V_\alpha(o) - t_\alpha \ln\left(z_\alpha(o)\right).\end{aligned} \tag{9.6}$$

- Using the singular limit Theorem 8.1 near branch points, we see that the leading behavior of all $W_l^{(g)}$'s is given by the $W_l^{(g)}$'s of the Airy curve $y = \sqrt{x}$ described in Section 10.5. Therefore, near a branch point a, when $\xi, \eta \to a$ we have

$$\psi_{\alpha,N}(\xi) \sim C\, Ai(\hat{\xi}), \quad \hat{\xi} = N^{2/3}(x(\xi) - x(a)), \tag{9.7}$$

  where C is some normalization constant ($C = \psi_{\alpha,N}(a)/Ai(0)$). In other words, $\psi_{\alpha,N}$ is not singular near branch points.

- The only singularities of $\psi_{\alpha,N}$ are essential singularities at all the poles of $y\,dx$, with a singular part equal to $\exp\left(-N \int^\xi y\,dx\right)$.

This is why we call those formal functions Baker–Akhiezer functions (cf. [9]).

Remark 9.1. *In fact, those functions are exactly Baker–Akhiezer functions only when the curve has genus* $\mathfrak{g} = 0$. *In the general case, the Baker–Akhiezer functions must also have the property that they take the same value after going around a non-trivial cycle. It is not difficult to multiply* $\psi_{\alpha,N}$ *by the appropriate* θ *function in order to fulfill that property. However, if we do that, we destroy the* $1/N^2$ *expansion.*

This is why the $\psi_{\alpha,N}$ *defined above can be called a "formal Baker-Akhiezer function". This definition is sufficient to find a formal Hirota equation, valid only order-by-order in* $1/N^2$.

9.2. Sato relation

Given two points ξ and η on $\overline{\Sigma}$, and a complex number r, we define the curve:

$$\mathcal{E} + r[\xi, -\eta] = \left\{ \left(x(p), y(p) + r\frac{dS_{\xi,\eta}(p)}{dx(p)} \right) \quad , p \in \Sigma \right\}. \tag{9.8}$$

The differential $y\,dx + r\,dS_{\xi,\eta}$ has the same $\mathcal{A}$-contour integrals as $y\,dx$, the same poles with the same singular part, plus two additional simple poles, one located at $p = \xi$ with residue r, and one at $p = \eta$ with residue $-r$.

We have Sato's relation:

Theorem 9.1.

$$K_N(\xi, \eta) = \frac{Z_N(\mathcal{E} + (1/N)[\xi, -\eta])}{Z_N(\mathcal{E})}, \quad \psi_{\alpha,N}(\xi) = \frac{Z_N(\mathcal{E} + (1/N)[\xi, \alpha])}{Z_N(\mathcal{E})}. \tag{9.9}$$

Indeed, the definition of K_N is the formal Taylor expansion in powers of $r = 1/N$ of the RHS.

9.3. Hirota equation

Consider two algebraic curve $\mathcal{E}(x, y)$ and $\tilde{\mathcal{E}}(x, y)$ with the same conformal structure (i.e., the same compact Riemann surface $\overline{\Sigma}$), then we have an Hirota bilinear relation:

Theorem 9.2. *We have the bilinear relation:*

$$\operatorname*{Res}_{\eta \to \zeta} dx(\eta)\, K_N(\xi, \eta)\, \tilde{K}_{\tilde{N}}(\eta, \zeta) = K_N(\xi, \zeta) \tag{9.10}$$

and also

$$\operatorname*{Res}_{\xi \to \alpha} dx(\xi)\, \psi_{\alpha,N}(\xi)\, \tilde{\psi}_{\alpha,-\tilde{N}}(\xi) = 0 \quad \text{if } \tilde{N}\tilde{t}_\alpha > N t_\alpha + 1, \tag{9.11}$$

which takes exactly the form of the Hirota equation [9, 60].

It is important to notice that this Hirota equation makes sense only order-by-order in its $1/N^2$ expansion. In the way we have obtained it, it is meaningless for finite N (appart maybe from the genus zero case $\not{g} = 0$, under the condition that the $1/N^2$ series is convergent). Therefore, we have a "formal Hirota equation".

10. Application: topological expansion of matrix models

In this section, we show how the objects defined in Section 4.2.4 (i.e., $\kappa = 0$) correspond to the terms of the topological expansion of the free energy and correlation functions of various matrix models when one considers appropriate curves $\mathcal{E}(x, y)$. Notice that in all this section, we consider $\kappa = 0$.

10.1. Formal one-matrix model

The formal one-matrix model (cf. [41]), is known to be the generating function which enumerates maps of given topology since the work of [14], then [8, 25, 52]. Its topological expansion was computed in several steps. The authors of [7] introduced a recursive algorithm to compute the $F^{(g)}$'s, only in the one-cut case (i.e., $\not{g} = 0$), and then the method was extended to other cases [3, 4]. The computation of the subleading term $F^{(1)}$ was done in general by Chekhov [15]. The computation to all orders of the correlation functions was found in [33], and the free energies in [17].

10.1.1. Definition

Definition 10.1. Formal one-matrix model. *Consider a "semi-classical" potential $V(x)$, i.e., such that $V'(x)$ is a rational function. Let $D(x)$ be its denominator, i.e., $D(x)V'(x)$ is a polynomial of degree d, and let $X_1, \dots, X_d$ be its zeroes:*

$$D(x)V'(x) = \prod_{i=1}^{d}(x - X_i) \tag{10.1}$$

We write

$$\delta V_i(x) = V(x) - V(X_i) - \frac{1}{2}V''(X_i)(x - X_i)^2 \tag{10.2}$$

Choose an integer n, and a d-partition of n, $\vec{n} = \{n_1, \dots, n_d\}$, such that

$$\sum_{j=1}^{d} n_j = n. \tag{10.3}$$

The following gaussian integral (where each matrix M_i is of size n_i) is a polynomial in T of the form

$$\frac{(-1)^l n^l}{l!\,T^l}\, e^{-\frac{n}{T}\sum_i n_i V(X_i)} \int dM_1 \cdots dM_d \prod_{i=1}^{d} \exp\left(-\frac{nV''(X_i)}{2T}\,\mathrm{Tr}\,(M_i - X_i \mathbf{1}_{n_i})^2\right)$$

$$\prod_{i>j} \det(M_i \otimes \mathbf{1}_{n_j} - \mathbf{1}_{n_i} \otimes M_j)^2 \left(\sum_i \mathrm{Tr}\,\delta V_i(M_i)\right)^l$$

$$= \sum_{k=l/2}^{dl/2} A_{k,l} T^k.$$

(10.4)

We define the formal matrix integral as the formal power series in T:

$$Z_{1\mathrm{MM}} = \sum_{k=0}^{\infty} T^k \left(\sum_{j=0}^{2k} A_{k,j}\right). \tag{10.5}$$

One can also define its formal logarithm, i.e., the free energy

$$F_{1\mathrm{MM}} = -\ln Z_{1\mathrm{MM}} = \sum_{k=0}^{\infty} T^k B_k. \tag{10.6}$$

It is a standard computation discovered by 't Hooft ([41, 71]), *that, for fixed $\epsilon_i = \frac{Tn_i}{n}$, for every k, $n^{-2}B_k$ is a polynomial in $1/n^2$:*

$$B_k(n_1, \ldots, n_d) = \sum_{g=0}^{g_{\max}(k)} B_{k,g}(\epsilon_1, \ldots, \epsilon_d) \left(\frac{n}{T}\right)^{2-2g}. \tag{10.7}$$

Thus, we define the following formal power series in T

$$F^{(g)}_{1\mathrm{MM}} = \sum_{k=0}^{\infty} T^k B_{k,g}(\epsilon_1, \ldots, \epsilon_d). \tag{10.8}$$

Remark. Here, the question of convergence of those series is not relevant. It is well known that each $F^{(g)}_{1\mathrm{MM}}$ is a convergent series (because it is written in terms of algebraic functions of T), but $F_{1\mathrm{MM}}$ is not.

10.1.2. Loop equations and classical spectral curve It is easy to see (this property holds for each power of T because it holds for Gaussian integrals) that the formal matrix integral satisfies, order by order in powers of T, the loop equations (i.e., Virasoro constraints), which can be written (see [24, 26]):

$$y(x)^2 + \frac{T^2}{n^2}\omega_2(x,x) = \frac{V'(x)^2}{4} - \frac{T}{n}\left\langle \mathrm{Tr}\,\frac{V'(x)-V'(M)}{x-M}\right\rangle \tag{10.9}$$

where $y(x) = \frac{V'(x)}{2} - \frac{T}{n}\left\langle \mathrm{Tr}\,\frac{1}{x-M}\right\rangle$ and $\omega_2(x,x') = \left\langle \mathrm{Tr}\,\frac{1}{x-M}\,\mathrm{Tr}\,\frac{1}{x'-M}\right\rangle_c$, where the expectation value $< . >$ is defined in a formal way similar to F.

If one identifies the coefficients of n^0 in each side, one gets an algebraic equation (here hyperelliptical), which is called the “classical spectral curve”:

$$\mathcal{E}_{1\mathrm{MM}}(x,y) = D(x)^2\left(y^2 - \frac{1}{4}V'^2(x) + P(x)\right) \tag{10.10}$$

where $D(x)P(x)$ is a polynomial of degree at most $\deg(D(x)V'(x)) - 1$, and completely determined by the condition that the polynomial $P(x)$ is a formal power series in powers of T such that at $T = 0$:

$$P(x, T=0) = \sum_{i=1}^{d} \epsilon_i \frac{V'(x)}{x - X_i}. \tag{10.11}$$

It is such that there exist some contours $\mathcal{A}_i$, $i = 1, \dots, d$ such that:

$$\frac{1}{2i\pi}\oint_{\mathcal{A}_i} y\,dx = \epsilon_i \tag{10.12}$$

Notice that the genus $\not{g}$ of the curve $\mathcal{E}_{1\mathrm{MM}}$ is the number of non-vanishing ϵ_i’s minus one.

Most often in the litterature, V is chosen polynomial such that $V'(0) = 0$, and only the 1-cut case is considered, with only one non-vanishing filling fraction at $X = 0$. The resulting curve has genus $\not{g} = 0$. This is the case that is relevant for enumerating polygonal surfaces.

It was proved in [17, 33], in the case of polynomial potentials only (but it is clear that it can be extended to the semi-classical case), that one has:

Theorem 10.1.

$$\boxed{F^{(g)}_{1\mathrm{MM}} = \underline{F}^{(g)}(\mathcal{E}_{1\mathrm{MM}})} \tag{10.13}$$

This proves that $F^{(g)}$ (which we recall is a formal power series in T) generically has a finite radius of convergence $T < T_c$.

We also have

$$\left\langle \mathrm{Tr}\,\frac{1}{x(p_1)-M}\cdots\mathrm{Tr}\,\frac{1}{x(p_k)-M}\right\rangle_{\mathrm{c}} = \sum_{g=0}^{\infty} N^{2-k-2g}\,\frac{W_k^{(g)}(p_1,\ldots,p_k)}{dx(p_1)\cdots dx(p_k)} + \delta_{k,1}\,N\left(\frac{1}{2}V'(x(p_1)) - y(p_1)\right) - \frac{\delta_{k,2}}{(x(p_1)-x(p_2))^2}. \tag{10.14}$$

10.2. Two-matrix model

The formal two-matrix model is known to be the generating function which counts bicolored maps (let us say the two colors are $+$ or $-$, thus it is an Ising model on a random map), it was introduced by Kazakov [54]. The loop equations were first written in [69]. $F^{(0)}$ was computed in [12,13,38,63]. $F^{(1)}$ was first found in [34] for the $\not{g}=0$ case, the in [35] for $\not{g}=1$, then in [37] for arbitrary $\not{g}$. Then higher orders for the correlation functions were first derived in [40], and the $F^{(g)}$'s for $g \geq 2$ were first found in [19]. During the same time it became clear that matrix models topological expansion was closely related to algebraic geometry [21–23, 55].

10.2.1. Definition

Definition 10.2. *Similarly, consider V_1' and V_2' two rational functions with respective denominators $D_1(x)$ and $D_2(x)$. The equation*

$$\begin{cases} V_1'(X_i) = Y_i \\ V_2'(Y_i) = X_i \end{cases} \qquad i = 1,\ldots,d \tag{10.15}$$

has $d = \deg(V_1'D_1) * \deg(V_2'D_2)$ solutions. We then write

$$\delta V_{1,i}(x) = V_1(x) - V_1(X_i) - Y_i(x-X_i) - \frac{V_1''(X_i)}{2}(x-X_i)^2, \tag{10.16}$$

and

$$\delta V_{2,i}(y) = V_2(y) - V_2(Y_i) - X_i(y-Y_i) - \frac{V_2''(Y_i)}{2}(y-Y_i)^2. \tag{10.17}$$

We then choose an integer n, and a d-partition of n:

$$n = \sum_{i=1}^{d} n_i. \tag{10.18}$$

The following gaussian integral (where each matrix M_i or $\tilde{M}_i$ is of size n_i) is a polynomial in T of the form:

$$\begin{aligned}
&\frac{(-1)^l n^l}{l!\, T^l} \exp\left(-\frac{n}{T}\sum_i n_i \,\mathrm{Tr}\, \left(V_1(X_i) + V_2(Y_i) - X_i Y_i\right)\right) \int dM_1 \cdots dM_d d\tilde{M}_1 \cdots d\tilde{M}_d \\
&\quad \prod_{i=1}^{d} \exp\left(-\frac{n}{T}\,\mathrm{Tr}\, \left(\frac{V_1''(X_i)}{2}(M_i - X_i\,\mathbf{1}_{n_i})^2\right.\right. \\
&\quad \left.\left. + \ \frac{V_2''(Y_i)}{2}(\tilde{M}_i - Y_i\,\mathbf{1}_{n_i})^2 - (M_i - X_i\,\mathbf{1}_{n_i})(\tilde{M}_i - Y_i\,\mathbf{1}_{n_i})\right)\right) \\
&\quad \prod_{i>j} \det(M_i \otimes \mathbf{1}_{n_j} - \mathbf{1}_{n_i} \otimes M_j) \prod_{i>j} \det(\tilde{M}_i \otimes \mathbf{1}_{n_j} - \mathbf{1}_{n_i} \otimes \tilde{M}_j) \\
&\quad \left(\sum_i \ \mathrm{Tr}\, \delta V_{1,i}(M_i) + \delta V_{2,i}(\tilde{M}_i)\right)^l \\
&\qquad = \sum_{k=l/2}^{dl/2} A_{k,l} T^k.
\end{aligned} \tag{10.19}$$

Similarly to the one-matrix case, we can define the formal two-matrix model as a formal power series in powers of T (see [41]):

$$Z_{2\mathrm{MM}} = \sum_{k=0}^{\infty} T^k \left(\sum_{j=0}^{2k} A_{k,j}\right). \tag{10.20}$$

One can also define its formal logarithm, i.e., the free energy

$$F_{2\mathrm{MM}} = -\ln Z_{2\mathrm{MM}} = \sum_{k=0}^{\infty} T^k B_k. \tag{10.21}$$

Again, it is a standard computation ([41, 71]), that, for fixed $\epsilon_i = \frac{T n_i}{n}$, for every k, $n^{-2} B_k$ is a polynomial in $1/n^2$:

$$B_k(n_1, \ldots, n_d) = \sum_{g=0}^{g_{\max}(k)} B_{k,g}(\epsilon_1, \ldots, \epsilon_d) \left(\frac{n}{T}\right)^{2-2g}. \tag{10.22}$$

Thus, we define the following formal power series in T:

$$F^{(g)}_{2\mathrm{MM}} = \sum_{k=0}^{\infty} T^k B_{k,g}(\epsilon_1, \ldots, \epsilon_d). \tag{10.23}$$

10.2.2. Loop equations and classical spectral curve Again, the formal two-matrix integral satisfies, order by order in powers of T, the loop equations (i.e., W-algebra constraints), which can be written (see [24, 26])

$$(y - y(x))U(x,y) + \frac{T^2}{n^2} U(x,y;x) = E(x,y), \tag{10.24}$$

where $y(x) = V_1'(x) - \frac{T}{n}\left\langle \mathrm{Tr}\, \frac{1}{x-M_1}\right\rangle$, $U(x,y) = x - V_2'(y) + \frac{T}{n}\left\langle \mathrm{Tr}\, \frac{1}{x-M_1} \frac{V_2'(y)-V_2'(M_2)}{y-M_2}\right\rangle$ and $U(x,y;x') = \left\langle \mathrm{Tr}\, \frac{1}{x-M_1} \frac{V_2'(y)-V_2'(M_2)}{y-M_2}\, \mathrm{Tr}\, \frac{1}{x'-M_1}\right\rangle_c$, and $E(x,y) = (y - V_1'(x))\ (x - V_2'(y)) - \frac{T}{n}\langle \mathrm{Tr}\, \frac{V_1'(x)-V_1'(M_1)}{x-M_1}\ \frac{V_2'(y)-V_2'(M_2)}{y-M_2}\rangle + T$, and where the expectation value $< . >$ is defined in a formal way similar to F.

If one chooses $y = y(x)$ and identifies the coefficients of n^0 in each side, one gets an algebraic equation $\mathcal{E}_{2\mathrm{MM}}(x,y) = 0$, which is called the "classical spectral curve" [34, 35]:

$$\mathcal{E}_{2\mathrm{MM}}(x,y) = D_1(x)\, D_2(y)\, ((V_1'(x) - y)(V_2'(y) - x) - P(x,y) + T), \tag{10.25}$$

where $D_1(x)\, D_2(y) P(x,y)$ is a polynomial of degree $\leq \deg(D_1 V_1')$ in x and a polynomial of degree $\leq \deg(D_2 V_2')$ in y, and with fixed filling fractions:

$$\frac{1}{2i\pi} \oint_{\mathcal{A}_I} y\, dx = T \frac{n_I}{N} = \epsilon_I, \quad \sum_I \epsilon_I = T \tag{10.26}$$

and such that in the limit $T \to 0$ and $\forall I\ \epsilon_I \to 0$, one has

$$y \sim V_1'(x) - \sum_i \frac{\epsilon_i}{x - X_i} + O(T^2), \quad x \sim V_2'(y) - \sum_i \frac{\epsilon_i}{y - Y_i} + O(T^2). \tag{10.27}$$

Then one has the following.

Theorem 10.2.

$$\boxed{F^{(g)}_{2\mathrm{MM}} = \underline{F}^{(g)}(\mathcal{E}_{2\mathrm{MM}})} \tag{10.28}$$

Proof. This theorem was proved in [19, 40].

Again, this proves a posteriori, that $F^{(g)}_{2\mathrm{MM}}$ (which we recall is a formal power series in T) genericaly has a finite radius of convergence $T < T_c$.

We also have:

$$\left\langle \mathrm{Tr}\,\frac{1}{x(p_1)-M_1}\cdots \mathrm{Tr}\,\frac{1}{x(p_k)-M_1}\right\rangle_{\mathrm{c}} = \sum_{g=0}^{\infty} N^{2-k-2g}\,\frac{W_k^{(g)}(p_1,\dots,p_k)}{dx(p_1)\cdots dx(p_k)} + \delta_{k,1}N(V'(x(p_1))-y(p_1)) - \frac{\delta_{k,2}}{(x(p_1)-x(p_2))^2}. \tag{10.29}$$

Remark that the one-Matrix model is a special case of the two-matrix model with V_2 a quadratic polynomial.

10.3. Double scaling limits of matrix models, minimal CFT

It is well known that double scaling limits of the one-matrix model, or two-matrix models are in relationship with (p,q) minimal models of conformal field theory [20, 26, 59].

We have seen that as long as the curve is regular, all the $F^{(g)}$'s can be computed. This shows that the radius of convergence in T of $F^{(g)}(T)$ is reached for singular curves. So far, only rational singularities have been studied in detail.

Thus, consider the case where the potentials V_1 and V_2 are fine-tuned so that the curve $\mathcal{E}_{2\mathrm{MM}}$ has a p/q singularity at $T = T_c$ (notice that for the one-matrix model one necessarily has $q = 2$):

$$\begin{cases} T = T_c, \\ x(z) \sim_{p\to a} x(a) + (z - z(a))^q, \\ y(z) \sim_{p\to a} y(a) + (z - z(a))^p. \end{cases} \tag{10.30}$$

We can use the notation of Section 8 with $t = T - T_c$, and thus at $t \neq 0$, the singularity is resolved, and we have (the local parameter is now $\zeta = zt^{-\nu}$):

$$\begin{cases} x(z,t) \sim x(a) + t^{q\nu}\, Q(\zeta) + O(t^{q\nu}), \\ y(z,t) \sim y(a) + t^{p\nu}\, P(\zeta) + O(t^{p\nu}), \\ \zeta = zt^{-\nu}, \end{cases} \tag{10.31}$$

where Q is a polynomial of degree q and P is a polynomial of degree p.

The curve

$$\mathcal{E}_{\text{sing}}(\xi,\eta) = \begin{cases} \xi = Q(\zeta) \\ \eta = P(\zeta) \end{cases} = \text{Resultant}(Q-\xi, P-\eta) \tag{10.32}$$

is called the singular spectral curve.

The dependence on T of the two-Matrix model is such that

$$\left.\frac{d}{dT} y\,dx\right|_x = dS_{\infty_x,\infty_y}, \tag{10.33}$$

where ∞_x and ∞_y are the two common poles of x and y. They are far away from branch points, and in particular from the singularity. This means that in the vicinity of the singularity we have

$$\left.\frac{d}{dT} y\,dx\right|_x \sim C\,t^{\nu}\,d\zeta + O(t^{2\nu}), \tag{10.34}$$

where C is some constant of order 1. After substitution with the limit Equation (10.31) this implies the Poisson relation[7]

$$pP(\zeta)Q'(\zeta) - qQ(\zeta)P'(\zeta) = \frac{C}{\nu}\,t^{1-(p+q-1)\nu} \tag{10.35}$$

and therefore

$$\nu = \frac{1}{p+q-1}. \tag{10.36}$$

Thereom 8.1. proves that

$$F^{(g)}_{2\text{MM}}(T) \underset{T\to T_c}{\sim} (T-T_c)^{(2-2g)(p+q)/(p+q-1)} F^{(g)}(\mathcal{E}_{\text{sing}}), \quad \text{for } g \geq 2. \tag{10.37}$$

We also have

$$F^{(0)}_{2\text{MM}}(T) \underset{T\to T_c}{\sim} \frac{C^2}{2}\frac{(p+q-1)^2}{(p+q)(p+q+1)}(T-T_c)^{2+2\nu} + \text{reg}, \quad \text{for } g=0, \tag{10.38}$$

$$F^{(1)}_{2\text{MM}}(T) \underset{T\to T_c}{\sim} -\frac{1}{24}(p-1)(q-1)\nu \ln(T-T_c) + O(1), \quad \text{for } g=1 \tag{10.39}$$

[7]This Poisson relation is well known and can be found in [20, 26].

We thus have a way to compute explicitely the double scaling limit of $F^{(g)}_{2\mathrm{MM}}$.

10.3.1. (p,q) minimal models Let us study in more details the curve $\mathcal{E}_{(p,q)}$ (cf [20]).

As we have seen above, the curve for the (p,q) minimal model is of the form:

$$\mathcal{E}_{(p,q)}(x,y) = \begin{cases} x = Q(\zeta) \\ y = P(\zeta) \end{cases} = \mathrm{Resultant}(Q-x, P-y), \tag{10.40}$$

where P and Q are polynomials of respective degrees p and q, satisfying:

$$pPQ' - qQP' = \frac{t_1}{\nu} \tag{10.41}$$

The solution of which can be written [26]

$$P = (Q^{p/q})_+ \tag{10.42}$$

and

$$(Q^{p/q})_- = \frac{t_1}{q}\zeta^{1-q} + O(\zeta^{-q}), \tag{10.43}$$

where we have used the notations $f = f_+ + f_-$, with f_+ and f_- denoting respectively the positive and the negative part of the Laurent series of f. This last equation implies $q-2$ equations for the coefficients of Q.

The curve $\mathcal{E}_{(p,q)}$ has genus zero $\not{g} = 0$, and is such that x and y have only one pole $\alpha = \infty$. The Bergmann kernel is

$$B(\zeta_1,\zeta_2) = \frac{d\zeta_1 d\zeta_2}{(\zeta_1-\zeta_2)^2}. \tag{10.44}$$

The moduli (of the pole) of that curve are the Q_k and P_k such that:

$$Q(\zeta) = \sum_{k=0}^{q} Q_k \zeta^k, \quad P(\zeta) = \sum_{k=0}^{p} P_k \zeta^k \tag{10.45}$$

by a translation on ζ, we can assume that $Q_{q-1} = 0$, and by a rescaling of ζ we can assume that $Q_{q-2} = -qQ_q$, and the Poisson Equation (10.41) implies that $P_{p-1} = 0$ and $P_{p-2} = -pP_p$, thus

$$Q_{q-1} = P_{p-1} = 0, \quad \frac{Q_{q-2}}{Q_q} = -q, \quad \frac{P_{p-2}}{P_p} = -p. \tag{10.46}$$

We find

$$F^{(0)}(\mathcal{E}_{(p,q)}) = 0. \tag{10.47}$$

10.3.2. Other times More generaly [26], we can deform the (p, q) minimal model with $p + q - 2$ times $t_1, \ldots, t_{p+q-2}$. For this purpose, one considers $Q(\zeta)$ a degree q monic polynomial,

$$Q(\zeta) = \zeta^q + \sum_{j=0}^{q-2} u_{q-j}\zeta^j, \tag{10.48}$$

whose coefficients $u_2, \ldots, u_q$ are determined as functions of $q - 1$ parameters, $t_1, \ldots, t_{q-1}$, by the following requirement:

$$(Q^{p/q})_- = \sum_{j=1}^{q-2} \frac{q-j}{q} t_{q-j} Q^{-j/q} + \frac{t_1}{q}\zeta^{1-q} + O(\zeta^{-q}). \tag{10.49}$$

Then we define the degree p monic polynomial $P(\zeta)$ by

$$P = \zeta^p + \sum_{j=0}^{p-2} v_{p-j}\zeta^j = Q_+^{p/q} - \sum_{j=1}^{p-1} \frac{j+q}{q} t_{q+j-1} Q_+^{j/q}, \tag{10.50}$$

which depends on $p - 1$ other times $t_q, \ldots, t_{q+p-2}$.

The corresponding classical spectral curve is

$$\mathcal{E}_{(p,q)}(x, y) = \text{Resultant}(x - Q, y - P). \tag{10.51}$$

and depends on times $t_1, \ldots, t_{p+q-2}$. One can check that if $t_2 = t_3 = \cdots = t_{p+q-1} = 0$, one recovers the (p, q) minimal model.

It is well known that this curve is the spectral curve of the dispersionless Witham hierarchy [64].

10.3.3. Examples of minimal models

- **Airy curve** $(2,1)$
  The classical spectral curve for the $(2,1)$ minimal model is

$$\mathcal{E}_{(2,1)}(x,y) = y^2 + t_1 - x. \tag{10.52}$$

$$Q(\zeta) = \zeta^2 + t_1, \quad P(\zeta) = \zeta. \tag{10.53}$$

  It is studied with particular care in Section 10.5 since it describes the behavior of a generic curve around the branch points, and thus coincides with Tracy–Widom law [72].

- **Pure gravity** $(3,2)$

$$Q(\zeta) = \zeta^2 - 2v, \quad P(\zeta) = \zeta^3 - 3v\zeta, \quad t_1 = 3v^2. \tag{10.54}$$

  The classical spectral curve is

$$\mathcal{E}_{(3,2)}(x,y) = x^3 - 3v^2x - y^2 + 2v^3 \tag{10.55}$$

  and is studied in details in Section 10.6.

- **Ising model** $(4,3)$

$$Q(\zeta) = \zeta^3 - 3v\zeta - 3w, \quad P(\zeta) = \zeta^4 - 4v\zeta^2 - 4w\zeta + 2v^2 - \frac{5}{3}t_5(\zeta^2 - 2v) \tag{10.56}$$

  with

$$t_1 = 4v^3 + 6w^2, \quad t_2 = 6vw. \tag{10.57}$$

  The classical spectral curve is

$$\begin{aligned}
\mathcal{E}_{(4,3)}(x,y) = {} & x^4 - y^3 - 4v^3x^2 + 3v^4y + 2v^6 \\
& + 12wv(-xy + v^2x) + 6w^2(-x^2 + 2vy - 4v^3) + 8w^3x - 3w^4 \\
& + 5t_5(-x^2y - v^2x^2 + 2v^3y + 2v^5 \\
& - 2wyx + 2v^2wx + 3w^2y - 17v^2w^2) \\
& + \frac{25}{3}t_5^2(v^2y + 2v^4 - 4vwx - 12vw^2) \\
& + \frac{125}{27}t_5^3(-x^2 + 2v^3 - 6wx - 9w^2).
\end{aligned} \tag{10.58}$$

The variations with respect to the moduli t_5, t_2 and t_1 correspond, respectively, to the variations of the form $y\,dx$

$$\Omega_5 = -d\Lambda_5 = d(\zeta^5 - 5v\zeta^3 + 10v^2\zeta), \tag{10.59}$$

$$\Omega_1 = -d\Lambda_1 = d\left(\zeta + \frac{5}{12}\frac{t_5}{v^3 - w^2}\,(2v^2\zeta + 2vw - w\zeta^2)\right) \tag{10.60}$$

and

$$\Omega_2 = -d\Lambda_2 = d\left(\zeta^2 + \frac{5}{6}\frac{t_5}{v^3 - w^2}\,(v^2\zeta^2 - 2vw\zeta - 2w^2)\right) \tag{10.61}$$

in the notations of Section 5.

- **Unitary models** $(q+1, q)$

$$\mathcal{E}_{(q+1,q)}(x,y) = T_{q+1}(x) - T_q(y), \tag{10.62}$$

$$Q(\zeta) = T_q(\zeta), \quad P(\zeta) = T_{q+1}(\zeta) \tag{10.63}$$

where T_p is the pth Tchebychev's polynomial.

10.4. Matrix model with external field

Define the formal matrix model with external field [78]

$$Z_{\text{Mext}} = \int dM\ \mathrm{e}^{-N\,\mathrm{Tr}\,(V(M) - M\hat{\Lambda})}, \tag{10.64}$$

where $V'(x)$ is a rational function with denominator $D(x)$, and $\hat{\Lambda}$ is a fixed $N \times N$ matrix. Consider its topological expansion (in the sense of a formal integral as in the one-matrix case above [41]):

$$F_{\text{Mext}} = -\ln Z_{\text{Mext}} = \sum_{g=0}^{\infty} N^{2-2g}\, F^{(g)}_{\text{Mext}}. \tag{10.65}$$

Let us assume that $\hat{\Lambda}$ has s distinct eigenvalues $\hat{\lambda}_1, \ldots, \hat{\lambda}_s$ with respective multiplicities $m_1, \ldots, m_s$ such that $\sum_i m_i = N$. The minimal polynomial of $\hat{\Lambda}$ is:

$$S(y) = \prod_i (y - \hat{\lambda}_i). \tag{10.66}$$

Define the classical curve obtained by "removing" the $1/N^2$ connected term in the loop equations:

$$\mathcal{E}_{\mathrm{Mext}}(x,y) = ((V'(x) - y)S(y) - P(x,y))D(x) \tag{10.67}$$

where

$$P(x,y) = \frac{1}{N}\left\langle \mathrm{Tr}\, \frac{V'(x) - V'(M)}{x - M}\, \frac{S(y) - S(\hat{\Lambda})}{y - \hat{\Lambda}} \right\rangle \tag{10.68}$$

so that $P(x,y)D(x)$ is a polynomial in both x and y.

Then one has the following.

Theorem 10.3.

$$\boxed{F^{(g)}_{\mathrm{Mext}} = \underline{F}^{(g)}(\mathcal{E}_{\mathrm{Mext}})} \tag{10.69}$$

Proof. This theorem is proved in Appendix D, and the proof is very similar to that of the two-matrix case above, see [19]. □

10.4.1. Application to Kontsevitch integral The Kontsevitch integral is known to be the generating function which computes intersection numbers of moduli space of Riemann surfaces (see [21, 58]). It is defined as:

$$Z_{\mathrm{Kontsevitch}} = \int dM\, \exp\left(-N\,\mathrm{Tr}\,\left(\frac{M^3}{3} - M(\Lambda^2 + t_1)\right)\right), \quad t_1 = \frac{1}{N}\,\mathrm{Tr}\,\frac{1}{\Lambda} \tag{10.70}$$

where Λ has eigenvalues $\lambda_1, \ldots, \lambda_N$. Thus, it is Z_{Mext} with $V(x) = x^3/3$ and $\hat{\Lambda} = \Lambda^2 + t_1$. Its classical curve is

$$\mathcal{E}_{\mathrm{Kontsevitch}}(x,y) = (x^2 - y)S(y) - xS_1(y) - S_2(y), \tag{10.71}$$

where $S_1(y)$ and $S_2(y)$ are polynomials in y of degree at most $s - 1$.

If we assume that the curve has genus zero (which is the case if we want the $F^{(g)}_{\text{Kontsevitch}}$ to be the generating functions for intersection numbers), then we can find explicitely a rational parametrization:

$$\mathcal{E}_{\text{Kontsevitch}}(x,y) = \begin{cases} x(z) = z + \dfrac{1}{2N}\,\mathrm{Tr}\,\dfrac{1}{\Lambda}\dfrac{1}{z-\Lambda} \\ y(z) = z^2 + \dfrac{1}{N}\,\mathrm{Tr}\,\dfrac{1}{\Lambda} \end{cases} . \tag{10.72}$$

Using the symplectic invariance of Theorem 7.1, we may exchange the roles of x and y. There is a unique branch point in y, solution of $y'(z)=0$, located at $z=0$. Since the formulae for $F^{(g)}$ consist in taking residues of rational functions at the branch point, we may consider the Taylor expansion of $x(z)$ near $z=0$, i.e.,

$$\mathcal{E}_{\text{Kontsevitch}}(x,y) = \begin{cases} x(z) = z - \frac{1}{2}\sum_{k=0}^{\infty} t_{k+2} z^k, \\ y(z) = z^2 + t_1 \end{cases} \tag{10.73}$$

where we have defined the Kontsevitch times:

$$t_k = \frac{1}{N}\,\mathrm{Tr}\,\Lambda^{-k} \tag{10.74}$$

so that:

$$\boxed{F^{(g)}_{\text{Kontsevitch}} = F^{(g)}(\mathcal{E}_{\text{Kontsevitch}})} \tag{10.75}$$

Again, using symplectic invariance of Theorem 7.1, we may add to $x(z)$ any rational function of $y(z)$, i.e., we immediately get a one-line proof of the following well-known theorem:

Theorem 10.4. $F^{(g)}_{\text{Kontsevitch}}$ *depends only on the odd times* t_{2k+1}*, with* $k \leq 3g-2$:

$$\boxed{F^{(g)}_{\text{Kontsevitch}} = F^{(g)}_{\text{Kontsevitch}}(t_1, t_3, t_5, \ldots, t_{6g-3})} . \tag{10.76}$$

And, if we assume that $t_k = 0$ for $k \geq p+2$, the curve is:

$$\begin{cases} x(z) = z - \frac{1}{2}\sum_{k=0}^{p} t_{k+2} z^k \\ y(z) = z^2 + t_1 \end{cases} , \tag{10.77}$$

which is identical to the curve of the $(p,2)$ model of Section 10.3.1, and which is well known to satisfy KdV hierarchy. Thus, again we have a one-line proof of the well-known result:

Theorem 10.5. $Z_{\text{Kontsevitch}}$ *is a KdV hierarchy* τ*-function.*

This also allows to recover straightforwardly the equivalence of the double scaled limit of the hermitian one matrix model and the Kontsevich integral [66].

10.4.1.1. Examples: the first few correlation functions

For Kontsevitch's curve we have:

$$B(z,z') = \frac{dz\,dz'}{(z-z')^2}, \quad dE_z(z') = \frac{z\,dz'}{z^2 - z'^2},$$
$$\omega(z) = 2z^2 dz \left(2 - \sum_j t_{2j+3}\, z^{2j}\right), \tag{10.78}$$

and the only branch point is located at $z=0$.

From the Definition 4.2, we easily get the first correlation functions:

$$W_1^{(1)}(z) = -\frac{dz}{8(2-t_3)}\left(\frac{1}{z^4} + \frac{t_5}{(2-t_3)z^2}\right), \tag{10.79}$$

$$W_3^{(0)}(z_1,z_2,z_3) = -\frac{1}{2-t_3}\,\frac{dz_1\,dz_2\,dz_3}{z_1^2 z_2^2 z_3^2}, \tag{10.80}$$

$$W_2^{(1)}(z_1,z_2) = \frac{dz_1\,dz_2}{8(2-t_3)^4 z_1^6 z_2^6}\left[(2-t_3)^2(5z_1^4 + 5z_2^4 + 3z_1^2 z_2^2)\right.$$
$$\left. + 6t_5^2 z_1^4 z_2^4 + (2-t_3)(6t_5 z_1^4 z_2^2 + 6t_5 z_1^2 z_2^4 + 5t_7 z_1^4 z_2^4)\right] \tag{10.81}$$

and

$$W_1^{(2)}(z) = -\frac{dz}{128(2-t_3)^7 z^{10}} \left[252\, t_5^4 z^8 + 12\, t_5^2 z^6 (2-t_3)(50\, t_7 z^2 + 21\, t_5) \right.$$
$$+ z^4(2-t_3)^2(252\, t_5^2 + 348\, t_5 t_7 z^2 + 145\, t_7^2 z^4 + 308\, t_5 t_9 z^4)$$
$$+ z^2(2-t_3)(203\, t_5 + 145\, z^2 t_7 + 105\, z^4 t_9 + 105\, z^6 t_{11})$$
$$\left. + 105\,(2-t_3)^4 \right]. \tag{10.82}$$

The first and second order free energies are found:

$$F^{(1)}_{\text{Kontsevitch}} = -\frac{1}{24} \ln\left(1 - \frac{t_3}{2}\right) \tag{10.83}$$

and

$$F^{(2)}_{\text{Kontsevitch}} = \frac{1}{1920} \frac{252\, t_5^3 + 435\, t_5 t_7 (2-t_3) + 175\, t_9 (2-t_3)^2}{(2-t_3)^5}. \tag{10.84}$$

which coincide with expressions previously found in the literature [49].

10.5. Example: Airy curve

The curve $y = \sqrt{x}$ is particularly important, because it corresponds to the leading behavior of any generic curve near its branch points. It is also the minimal model $(1,2)$ (cf [26, 28]), also called Tracy–Widom law [72].

Consider the curve

$$\mathcal{E}(x,y) = y^2 - x. \tag{10.85}$$

We chose the uniformization $p = y$:

$$\begin{cases} x(p) = p^2 \\ y(p) = p. \end{cases} \tag{10.86}$$

There is only one pole $\alpha = \infty$, and there is only one branch-point located at $a = 0$, the conjugated point is $\overline{p} = -p$. The Bergmann kernel is the Bergmann kernel of the Riemann sphere:

$$B(p,q) = \frac{dp\, dq}{(p-q)^2}, \quad dE_q(p) = \frac{q\, dp}{q^2 - p^2}, \quad \omega(q) = 4q^2 dq \tag{10.87}$$

It is easy to see that all correlation functions with $2g+k \geq 3$ are of the form:

$$W_k^{(g)}(p_1, \dots, p_k) = \omega_k^{(g)}(p_1^2, \dots, p_k^2)\, dp_1 \dots dp_k. \tag{10.88}$$

Moreover, the diagrammatic rules are clearly homogenous, so that the function $W_1^{(g)}(p)$ must be a homogeneous function of p. It is easy to find that:

$$W_1^{(g)}(p) = \frac{c_g\, dp}{p^{6g-2}} \tag{10.89}$$

and the total 1−point function is

$$W_1(p, N) = -Ny\, dx + \sum_{g=1}^{\infty} N^{1-2g} W_1^{(g)}(p) = W_1(N^{1/3}p, 1). \tag{10.90}$$

Similarly, the total two-point function is

$$W_2(p, q, N) = \sum_{g=0}^{\infty} N^{-2g} W_2^{(g)}(p, q) = W_2(N^{1/3}p, N^{1/3}q, 1) \tag{10.91}$$

and in general

$$\begin{aligned} W_k(p_1, \dots, p_k, N) &= \sum_{g=0}^{\infty} N^{2-2g-k} W_k^{(g)}(p_1, \dots, p_k) \\ &= W_k(N^{1/3}p_1, \dots, N^{1/3}p_k, 1). \end{aligned} \tag{10.92}$$

The solution of the recursion Definition 4.2 can be found explicitely in terms of the Airy function.

Consider $g(x) = Ai'(x)/Ai(x)$ where $Ai(x)$ is the Airy function, i.e., $g'(x) + g^2(x) = x = p^2$. In terms of the variable p we write

$$f(p) = g(p^2), \quad f^2 + \frac{f'}{2p} = p^2. \tag{10.93}$$

It can be expanded for large p

$$f(p) = \sum_{k=0}^{\infty} f_k p^{1-3k} = p - \frac{1}{4p^2} - \frac{9}{32p^5} + \cdots, \tag{10.94}$$

where the coefficients in the expansion satisfy

$$\frac{4-3k}{2} f_{k-1} + \sum_{j=0}^{k} f_j f_{k-j} = \delta_{k,0}. \tag{10.95}$$

The solution of the recursion Definition 4.2 for the one-point function is

$$\begin{aligned} W_1(p,1) &= -2\frac{p^2 - f(p)f(-p)}{f(p)-f(-p)}\, p\, dp \\ &= -2p^2\, dp + \frac{dp}{(2p)^4} + \frac{9!!\, dp}{3^2\,(2p)^{10}} + \frac{15!!}{3^4\,(2p)^{16}} \cdots , \end{aligned} \tag{10.96}$$

$$W_2(p,p',1) = -4\frac{(f(p)-f(p'))\,(f(-p)-f(-p'))}{(p^2-p'^2)^2\,(f(p)-f(-p))\,(f(p')-f(-p'))}\, p\, dp\, p'\, dp'. \tag{10.97}$$

In particular

$$W_2(p,p,1) = \frac{f'(p)f'(-p)}{(f(p)-f(-p))^2}\, dp^2, \tag{10.98}$$

so that

$$W_2(p,p,1) + W_1(p,1)^2 = 4p^4\, dp^2 = x\, dx^2. \tag{10.99}$$

Similarly, we find for instance (with obvious cyclic conventions for the indices)

$$\begin{aligned} &W_3(p_1,p_2,p_3,1) \\ &\quad = \frac{dx_1 dx_2 dx_3}{(p_3^2-p_2^2)(p_3^2-p_1^2)(p_2^2-p_1^2)} \\ &\qquad \times \frac{\sum_{i=1}^3 f(p_i)f(-p_i)(f(p_{i-1})+f(-p_{i-1})-f(p_{i+1})-f(-p_{i+1}))}{(f(p_1)-f(-p_1))(f(p_2)-f(-p_2))(f(p_3)-f(-p_3))} \\ &\quad = \frac{dp_1\, dp_2\, dp_3}{2\, p_1^2\, p_2^2\, p_3^2} + \frac{dp_1\, dp_2\, dp_3}{2^6\, p_1^8\, p_2^8\, p_3^8} + \cdots \end{aligned} \tag{10.100}$$

and one can easily find similar expressions for all W_k's.

In fact, all correlation functions can be written with a determinantal formula [31, 32], with the Tracy–Widom kernel [72]:

$$K(x,x') = \frac{Ai(x)Ai'(x') - Ai'(x)Ai(x')}{x-x'} \tag{10.101}$$

The fact that the Baker–Akhiezer function is $Ai(x)$ and satisfies the differential equation $Ai'' = xAi$ can be seen as a consequence of the Hirota equation Theorem 9.2.

Remark 10.1. To large N leading order the first term $W_k^{(0)}$ can be written in terms of Ferrer diagrams (Young diagrams):

$$W_k^{(0)}(p_1, \dots, p_k) = \frac{k-3!}{2^{k-2}} \prod_j \frac{dp_j}{p_j^2} \sum_{|\lambda|=k-3} M_\lambda(1/p_i^2) \prod_j \frac{2\lambda_j + 1!!}{\lambda_j!} \frac{1}{n_j(\lambda)!}, \tag{10.102}$$

where λ is a Ferrer diagram, $n_i(\lambda) = \#\{j \,/\, \lambda_j = i\}$, and M_λ are the elementary monomial symetric polynomials:

$$M_\lambda(z_i) = \sum_{i_1 \neq i_2 \neq \cdots \neq i_{k-3}} z_{i_1}^{\lambda_1} \cdots z_{i_{k-3}}^{\lambda_{k-3}}. \tag{10.103}$$

For instance with $k = 4$ there is only one diagram (1), and $M_{(1)}(z_1, z_2, z_3, z_4) = z_1 + z_2 + z_3 + z_4$, and:

$$W_4^{(0)}(p_1, p_2, p_3, p_4) = \frac{3}{4} \frac{dp_1\, dp_2\, dp_3\, dp_4}{p_1^2\, p_2^2\, p_3^2\, p_4^2} \left(\frac{1}{p_1^2} + \frac{1}{p_2^2} + \frac{1}{p_3^2} + \frac{1}{p_4^2} \right). \tag{10.104}$$

Remark 10.2. The free energies are all vanishing for that curve:

$$\forall g \quad F^{(g)} = 0. \tag{10.105}$$

10.6. Example: pure gravity (3,2)

In this section, we study in details the $(3, 2)$ minimal model, also called pure gravity [26].

It corresponds to the curve

$$\mathcal{E}_{(3,2)} = \begin{cases} x(z) = z^2 - 2v, \\ y(z) = z^3 - 3vz, \\ t_1 = 3v^2. \end{cases} \tag{10.106}$$

We recognize Tchebychev's polynomials T_2 and T_3, which satisfy the Poisson relation Equation (10.41). Up to a rescaling $z = \sqrt{v}\,p$, the curve reads

$$\mathcal{E}_{(3,2)} = \begin{cases} x(p) = v(p^2 - 2), \\ y(z) = v^{3/2}(p^3 - 3p), \\ t_1 = 3v^2. \end{cases} \tag{10.107}$$

There is only one x-branch point at $p = 0$, and the conjugated point is $\overline{p} = -p$. The Bergmann kernel is the Bergmann kernel of the Riemann sphere:

$$B(p,q) = \frac{dp\,dq}{(p-q)^2}, \quad dE_q(p) = \frac{q\,dp}{q^2 - p^2} \tag{10.108}$$

$$\omega(q) = (y(q) - y(\overline{q}))dx(q) = 4v^{5/2}\,(q^2 - 3)\,q^2\,dq, \tag{10.109}$$

$$\Phi(q) = v^{5/2}\left(\frac{2\,q^5}{5} - 2q^3\right). \tag{10.110}$$

Under a variation of t_1 we have

$$\Omega_1(p) = -\frac{\partial y(p)dx(p)}{\partial t_1}\bigg|_{x(p)} = v^{1/2}\,dp = -v^{1/2}\,\operatorname*{Res}_{\infty}\,qB(p,q),$$

$$\Lambda_1(q) = -v^{1/2}\,q, \tag{10.111}$$

so the effect of $\partial/\partial t_1$ is equivalent to

$$\frac{\partial}{\partial t_1}W_k^{(g)}\bigg|_x = -v^{1/2}\,\operatorname*{Res}_{q\to\infty}\,qW_{k+1}^{(g)}. \tag{10.112}$$

10.6.1. Some correlation functions Using Definition 4.2, we find

$$W_3^{(0)}(p_1,p_2,p_3) = -\frac{v^{-5/2}}{6}\frac{dp_1dp_2dp_3}{p_1^2p_2^2p_3^2}, \tag{10.113}$$

$$W_1^{(1)}(p) = -\frac{v^{-5/2}}{(12)^2}\frac{p^2+3}{p^4}\,dp, \tag{10.114}$$

$$W_2^{(1)}(p,q) = v^{-5} \frac{\begin{array}{c}15q^4 + 15p^4 + 6p^4q^2 + 2p^4q^4 \\ + 9p^2q^2 + 6p^2q^4\end{array}}{2^5\, 3^3\ p^6\, q^6}\, dp\, dq, \tag{10.115}$$

$$W_1^{(2)}(p) = -v^{-15/2}\, 7 \frac{135 + 87p^2 + 36p^4 + 12p^6 + 4p^8}{2^{10}\, 3^5\, p^{10}}\, dp \tag{10.116}$$

$$W_4^{(0)}(p_1,p_2,p_3,p_4) = \frac{v^{-5}}{9p_1^2p_2^2p_3^2p_4^2}\left(1 + 3\sum_i \frac{1}{p_i^2}\right) dp_1 dp_2 dp_3 dp_4 \tag{10.117}$$

$$\frac{W_5^{(0)}(p_1,p_2,p_3,p_4,p_5)}{dp_1dp_2dp_3dp_4dp_5} = \frac{v^{-15/2}}{9p_1^2p_2^2p_3^2p_4^2p_5^2}\left(1 + 3\sum_i \frac{1}{p_i^2} + 6\sum_{i<j}\frac{1}{p_i^2p_j^2} + 5\sum_i \frac{1}{p_i^4}\right). \tag{10.118}$$

etc...

Using Definition 4.3, and Equation (10.112), we find from Equation (10.113):

$$\frac{\partial^3 F^{(0)}}{\partial t_1^3} = -\frac{1}{6v} = -\frac{1}{2\sqrt{3t_1}} \longrightarrow \frac{\partial^2 F^{(0)}}{\partial t_1^2} = -\frac{t_1^{1/2}}{\sqrt{3}} \tag{10.119}$$

and using Equation (10.114):

$$\frac{\partial F^{(1)}}{\partial t_1} = -\frac{1}{(12)^2 v^2} = -\frac{1}{48 t_1} \longrightarrow \frac{\partial^2 F^{(1)}}{\partial t_1^2} = \frac{1}{48\, t_1^2} \tag{10.120}$$

as well as using Equation (10.116):

$$\frac{\partial F^{(2)}}{\partial t_1} = -v^{-7}\, \frac{7}{2^8\, 3^5} = -\frac{7}{2^8\, 3^{3/2}\, t_1^{7/2}} \longrightarrow \frac{\partial^2 F^{(2)}}{\partial t_1^2} = \frac{49}{2^9\, 3^{3/2}\, t_1^{9/2}}. \tag{10.121}$$

We may thus verify that the second derivative of the free energy:

$$u = \frac{\partial^2 F}{\partial t_1^2} = \sum_{g=0}^{\infty} t_1^{(1-5g)/2} u^{(g)}, \quad F^{(g)} = \frac{4u^{(g)}}{5(1-g)(3-5g)}. \tag{10.122}$$

satisfies the Painlevé equation to the first orders:

$$u^2 + \frac{1}{6}u'' = \frac{1}{3}t_1. \tag{10.123}$$

It is well known that this equation is satisfied to all orders [26], and here this can be seen as a consequence of the Hirota equation Theorem 9.2.

11. Conclusion

In this paper, we have constructed an infinite family of invariants of algebraic curves. By construction, these invariants coincide with the topological expansion of matrix integrals in the special case where the algebraic curve is the large N part of the matrix integral's spectral curve. But we emphasize again that the construction presented here goes beyond matrix models.

Our invariants are defined only in terms of algebraic geometry, and they have many interesting properties, like homogeneity, and integrability (they obey some Hirota equation).

The problem of computing the $F^{(g)}$'s for matrix models is an old problem which was addressed many times, and which found more and more elaborate answers [7]. We claim that ours is more efficient, because it contains all multicut cases, and various types of matrix models at once. Also, even in the simplest cases (one-matrix model, 1 cut), our expressions are simpler than what existed before [7]. Our $F^{(g)}$'s are defined recursively, like those of [7], but the recursions are much easier to handle, and it is much easier to deduce properties to any order g from our construction.

The efficiency of our method becomes striking when one wants to compare different models (Kontsevitch and KdV for instance), or when one wants to take singular limits. Another important application of our method, is to prove that our $F^{(g)}$'s provide a solution to the holomorphic anomaly equations of [11] in topological string theory, thus confirming the Dijkgraaf Vafa correspondence. We claim that this can be proved easily from our work, and we present it in a coming paper [42]. It would also be interesting to compare our free energies with the D-Modules considered in [5, 6] as partition functions of a unified matrix M-theory by checking that they indeed satisfy the equations of [6].

One of the reasons our method is very efficient also, is that it can be represented diagrammatically, without equations, and thus very easy to remember.

11.1. Perspectives and generalizations

- The first thing one could think about is to understand what our $F^{(g)}$'s compute in algebraic geometry. There has been some attempts to recognize the first few of them $F^{(1)}$ as the Dedekind function [29, 30],

$F^{(2)}$ as the Eisenstein series [50], but the answer for higher g is still obscure. Also a combinatoric interpretation is missing.

- Beyond that, it would be important to understand the combinatorics behind our diagrammatic construction. Since all diagrams are obtained from trees, it seems to be related to Schaeffer's method for counting maps [70], but this issue needs to be investigated further.
- Double scaling limits of matrix models are in relationship with conformal field theory (CFT), and in particular minimal models. It would be interesting to compare our formulae with those obtained directely from CFT [28, 76, 77], and, since higher genus CFT is far less known, maybe our method can bring something new to CFT.
- Also, the link between formal matrix models (i.e., those defined as combinatorial generating functions, for which it makes sense to consider a topological expansion, cf. [41]), and actual convergent matrix integrals needs to be better understood. The difficulty lies beyond any order in perturbation, and integrability could play a key role in understanding the relationship in more details.
- It would be interesting to check that the topological expansion for the chain of matrices [38], is also given by the same $F^{(g)}$'s. This is an open question, but there are strong evidences that the answer is positive. For instance, it is easy to see from [38] that $F^{(0)}$ and some of the first few correlation functions are indeed the same.
- It would be interesting also to extend our construction to other types of matrix models, for instance non-hermitian (real symmetric or quaternionic). A first attempt was done in [18]. Another possible extension is toward the $O(n)$ matrix model, whose large N limit is known in terms of an agebraic curve [44, 45]. To leading order, the $O(n)$ matrix model solution looks very similar to that of the one-matrix model, except that correlation functions are no longer meromorphic functions on the curve. Instead they gain a phase shift after going around a non-trivial cycle. This would probably allow to define a twisted version of our construction.

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Appendix

A. Properties of correlation functions

In this section, we prove the theorems stated in Section 4.4.

We use very much the following obvious properties:

$$\sum_i \frac{B(p^i,q)}{dx(p^i)} = \frac{dx(q)}{(x(p)-x(q))^2}, \tag{A.1}$$

$$\frac{dE_{\overline{q}}(p)}{\omega(\overline{q})} = \frac{dE_q(p)}{\omega(q)}, \tag{A.2}$$

and

$$\lim_{q\to a_i} \frac{dE_q(p)}{\omega(q)}\, dx(q) = -\frac{1}{2} \lim_{q\to a_i} \frac{B(p,q)}{dy(q)}\, dx(q). \tag{A.3}$$

The third one is nothing but De L'Hôpital's rule.

Proof of Theorem 4.1. From the Definition 4.2, we have:

$$\begin{aligned}
W_3^{(0)}(p,p_1,p_2) &= \operatorname*{Res}_{q\to\mathbf{a}} \frac{dE_q(p)}{\omega(q)} (B(q,p_1)B(\overline{q},p_2) + B(q,p_2)B(\overline{q},p_1)) \\
&= 2 \operatorname*{Res}_{q\to\mathbf{a}} \frac{dE_q(p)}{\omega(q)}\, B(q,p_1)B(\overline{q},p_2) \\
&= 2 \operatorname*{Res}_{q\to\mathbf{a}} \operatorname*{Res}_{r\to\overline{q}} \frac{dE_q(p)}{(y(q)-y(\overline{q}))(x(r)-x(q))}\, B(q,p_1)B(r,p_2) \\
&= -2 \operatorname*{Res}_{q\to\mathbf{a}} \operatorname*{Res}_{r\to q} \frac{dE_q(p)}{(y(q)-y(\overline{q}))(x(r)-x(q))}\, B(q,p_1)B(r,p_2)
\end{aligned}$$

$$= -2 \operatorname*{Res}_{q\to \mathbf{a}} \frac{dE_q(p)}{\omega(q)} B(q,p_1)B(q,p_2)$$

$$= \operatorname*{Res}_{q\to \mathbf{a}} \frac{B(q,p)B(q,p_1)B(q,p_2)}{dx(q)dy(q)}, \tag{A.4}$$

where we have used Equations (A.2), (A.1) and (A.3). □

Proof of Theorem 4.2. It is obvious from the definition that if p is away from branch points, the residues are finite integrals, and $W_{k+1}^{(g)}$ is finite. The only poles can be obtained when p pinches an integration contour, i.e., at branch points.

Then, it is easy to see by recursion on k and g that it holds for any p_i. □

Proof of Theorem 4.3. The first one is a property of $dE_q(p)$, and the second one follows from recursion on k and g, and it holds for $W_2^{(0)} = B$. □

Proof of Theorem 4.4. The case $k = 1, g = 0$ comes from Equation (A.1), and by integration:

$$\sum_i \frac{dE_q(p^i)}{dx(p^i)} = 0, \tag{A.5}$$

which proves Equation (4.23). Then, it is clear from the iterative definition of $W_{k+1}^{(g)}$ for $k \geq 1$, that the dependance of $\frac{W_{k+1}^{(g)}(p_1,p,p_2,\dots,p_k)}{dx(p)}$ in p is a sum of integrals involving only the following quantities:

$$\sum_j \operatorname*{Res}_{q\to a_j} \frac{dE_{j,q}(q_1)}{\omega_j(q)} \frac{B(q,p)}{dx(p)} W_{l+1}^{(g)}(q,q_2,\dots,q_l), \tag{A.6}$$

for some $q_1,\dots,q_l$[8]. Using Equation (A.1) we have to compute:

$$\sum_i \sum_j \operatorname*{Res}_{q\to a_j} \frac{dE_{j,q}(q_1)}{\omega_j(q)} \frac{B(q,p^i)}{dx(p^i)} W_{l+1}^{(g)}(q,q_2,\dots,q_l)$$

$$= \sum_j \operatorname*{Res}_{q\to a_j} \frac{dE_{j,q}(q_1)}{\omega_j(q)} \frac{dx(q)}{(x(q)-x(p))^2} W_{l+1}^{(g)}(q,q_2,\dots,q_l)$$

[8] Since it may not be clear which branch point dE and ω refers to, one indicates it with an index.

$$= \frac{1}{2} \sum_j \mathop{\mathrm{Res}}_{q \to a_j} \frac{dE_{j,q}(q_1)}{\omega_j(q)} \frac{dx(q)}{(x(q)-x(p))^2} \Big(W_{l+1}^{(g)}(q, q_2, \dots, q_l) + W_{l+1}^{(g)}(\overline{q}, q_2, \dots, q_l) \Big)$$

$$= -\frac{1}{2} \sum_{q^i \neq q, \overline{q}} \sum_j \mathop{\mathrm{Res}}_{q \to a_j} \frac{dE_{j,q}(q_1)}{\omega_j(q)} \frac{(dx(q))^2}{(x(q)-x(p))^2} \frac{W_{l+1}^{(g)}(q^i, q_2, \dots, q_l)}{dx(q^i)}$$

$$= 0, \tag{A.7}$$

where the second equality holds due to (A.2), the third equality holds due to Equation (4.23), and the last equality holds because that last expression has no poles at the branch points. This proves Equation (4.24). □

Proof of Theorem 4.5. It is clearly a rational function of $x(p)$ because it is a symmetric sum on all sheets. From Theorem 4.2, the RHS may have poles at branch points, and/or at the poles of some $y(p^i)$, and/or when $x(p) = x(p_l)$ for some l. Let us prove that the poles at branch points actualy cancel.

Let us denote in this section:

$$U_k^{(g)}(q, q', p_K) = \sum_{m=0}^{g} \sum_{J \subset K} W_{j+1}^{(m)}(q, p_J) W_{k-j+1}^{(g-m)}(q', p_{K/J}) + W_{k+2}^{(g-1)}(q, q', p_K). \tag{A.8}$$

From Theorem 4.4, we have:

$$\begin{aligned} \sum_i U_k^{(g)}(q^i, q^i, p_K) &= -\sum_i \sum_{l \neq i} U_k^{(g)}(q^l, q^i, p_K) \\ &= -U_k^{(g)}(q, \overline{q}, p_K) - U_k^{(g)}(\overline{q}, q, p_K) \\ &\quad - \sum_{q^i \neq q, \overline{q}} (U_k^{(g)}(q, q^i, p_K) + U_k^{(g)}(\overline{q}, q^i, p_K)) \\ &\quad - \sum_{q^i \neq q, \overline{q}} (U_k^{(g)}(q^i, q, p_K) + U_k^{(g)}(q^i, \overline{q}, p_K)) \\ &\quad - \sum_{q^i \neq q, \overline{q}} \sum_{l \neq i, q^l \neq, q, \overline{q}} U_k^{(g)}(q^l, q^i, p_K) \end{aligned} \tag{A.9}$$

and from Theorems 4.4 and 4.2, only the first two terms have poles at branch-points, and thus,

$$\begin{aligned}
&-\sum_i \sum_j \mathop{\mathrm{Res}}_{q\to a_j} \frac{dE_{j,q}(p)}{\omega_j(q)}\, U_k^{(g)}(q^i, q^i, p_K) \\
&= \sum_j \mathop{\mathrm{Res}}_{q\to a_j} \frac{dE_{j,q}(p)}{\omega_j(q)} (U_k^{(g)}(q, \overline{q}, p_K) + U_k^{(g)}(\overline{q}, q, p_K)) \\
&= 2\sum_j \mathop{\mathrm{Res}}_{q\to a_j} \frac{dE_{j,q}(p)}{\omega_j(q)}\, U_k^{(g)}(q, \overline{q}, p_K) \\
&= 2W_{k+1}^{(g)}(p, p_K), \qquad \text{(A.10)}
\end{aligned}$$

where the last equality holds from the definition of $W_{k+1}^{(g)}$. Thus, we have

$$W_{k+1}^{(g)}(p, p_K) = -\frac{1}{2}\sum_j \mathop{\mathrm{Res}}_{q\to a_j} \frac{dE_{j,q}(p)}{\omega_j(q)} \left(\sum_i U_k^{(g)}(q^i, q^i, p_K)\right). \qquad \text{(A.11)}$$

Then, we rewrite it in terms of $P_k^{(g)}$ above

$$\begin{aligned}
W_{k+1}^{(g)}(p, p_K) &= -\frac{1}{2}\sum_j \mathop{\mathrm{Res}}_{q\to a_j} \frac{dE_{j,q}(p)}{\omega_j(q)} \Bigg(P_k^{(g)}(x(q), p_K)\, dx(q)^2 \\
&\quad +2\sum_i y(q^i)dx(q)W_{k+1}^{(g)}(q^i, p_K)\Bigg) \\
&= -\frac{1}{2}\sum_j \mathop{\mathrm{Res}}_{q\to a_j} \frac{dE_{j,q}(p)}{\omega_j(q)} \Big(P_k^{(g)}(x(q), p_K)\, dx(q)^2 \\
&\quad + 2y(q)dx(q)W_{k+1}^{(g)}(q, p_K) \\
&\quad + 2y(\overline{q})dx(q)W_{k+1}^{(g)}(\overline{q}, p_K)\Big) \\
&= -\frac{1}{2}\sum_j \mathop{\mathrm{Res}}_{q\to a_j} \frac{dE_{j,q}(p)}{\omega_j(q)} \Big(P_k^{(g)}(x(q), p_K)\, dx(q)^2 \\
&\quad + 2(y(q) - y(\overline{q}))dx(q)W_{k+1}^{(g)}(q, p_K)\Big), \qquad \text{(A.12)}
\end{aligned}$$

where we have used Theorem 4.4 again. That gives

$$W_{k+1}^{(g)}(p,p_K) = -\frac{1}{2}\sum_j \mathop{\mathrm{Res}}_{q\to a_j} \frac{dE_{j,q}(p)}{\omega_j(q)} P_k^{(g)}(x(q),p_K)\,dx(q)^2$$

$$- \sum_j \mathop{\mathrm{Res}}_{q\to a_j} dE_{j,q}(p)\,W_{k+1}^{(g)}(q,p_K). \tag{A.13}$$

Let us compute that last integral

$$\begin{aligned}
&\sum_j \mathop{\mathrm{Res}}_{q\to a_j} dE_{j,q}(p)\,W_{k+1}^{(g)}(q,p_K) \\
&\quad = -\mathop{\mathrm{Res}}_{q\to p} dE_{j,q}(p)\,W_{k+1}^{(g)}(q,p_K) + \frac{1}{2i\pi}\sum_i \oint_{q'\in\underline{\mathcal{B}}_i} B(p,q') \oint_{\underline{\mathcal{A}}_i} W_{k+1}^{(g)}(q,p_K) \\
&\qquad - \frac{1}{2i\pi}\sum_i \oint_{q'\in\underline{\mathcal{A}}_i} B(p,q') \oint_{\underline{\mathcal{B}}_i} W_{k+1}^{(g)}(q,p_K) \\
&\quad = -\mathop{\mathrm{Res}}_{q\to p} dE_{j,q}(p)\,W_{k+1}^{(g)}(q,p_K) + \sum_i du_i(p) \oint_{\mathcal{A}_i} W_{k+1}^{(g)}(q,p_K) \\
&\quad = -\mathop{\mathrm{Res}}_{q\to p} dE_{j,q}(p)\,W_{k+1}^{(g)}(q,p_K) \\
&\quad = -W_{k+1}^{(g)}(p,p_K),
\end{aligned} \tag{A.14}$$

where we have deformed the contour of integration using Riemann bilinear identity, and then we have used Theorem 4.3. Thus we have:

$$\sum_j \mathop{\mathrm{Res}}_{q\to a_j} \frac{dE_{j,q}(p)}{\omega_j(q)} P_k^{(g)}(x(q),p_K)\,dx(q)^2 = 0. \tag{A.15}$$

Since this holds for any p, we can write for any $m \geq 0$:

$$\begin{aligned}
0 = \mathop{\mathrm{Res}}_{p\to a_i} (y(p)-y(a_i))\,(x(p)& \\
- x(a_i))^m \sum_j \mathop{\mathrm{Res}}_{q\to a_j} &\frac{dE_{j,q}(p)}{\omega_j(q)} P_k^{(g)}(x(q),p_K)\,dx(q)^2
\end{aligned}$$

$$
\begin{aligned}
&= -\operatorname*{Res}_{q\to a_i}\operatorname*{Res}_{p\to q,\overline{q}} \frac{dE_{j,q}(p)}{\omega_j(q)}\,(y(p)-y(a_i))\,(x(p) \\
&\quad - x(a_i))^m\, P_k^{(g)}(x(q),p_K)\,dx(q)^2 \\
&= -\operatorname*{Res}_{q\to a_i} dx(q)\,(x(q)-x(a_i))^m\, P_k^{(g)}(x(q),p_K), \qquad \text{(A.16)}
\end{aligned}
$$

which proves that $P_k^{(g)}(x(q),p_K)$ can have no pole at $q=a_i$. □

Proof of Theorem 4.6. Assume this is proved for $h<g$, and at g, it is proved for $l<k$. Then we have:

$$
\begin{aligned}
W_2^{(g)}(p_1,p_2) &= \sum_i \operatorname*{Res}_{q\to a_i} \frac{dE_q(p_1)}{\omega(q)} \left[2\sum_{m=0}^{g} W_2^{(m)}(q,p_2)W_1^{(g-m)}(\overline{q}) \right.\\
&\qquad \left. + W_3^{(g-1)}(q,\overline{q},p_2) \right] \\
&= \sum_i \operatorname*{Res}_{q\to a_i} \frac{dE_q(p_1)}{\omega(q)}\, B(q,p_2)\, W_1^{(g)}(\overline{q}) \\
&\qquad + \sum_i \operatorname*{Res}_{q\to a_i} \sum_{i'} \operatorname*{Res}_{q'\to a_{i'}} \frac{dE_q(p_1)}{\omega(q)} \frac{dE_{i',q'}(p_2)}{\omega_{i'}(q')} \\
&\qquad \left[2\sum_{m'=0}^{g-1} W_2^{(m')}(q',q)W_2^{(g-m'-1)}(\overline{q}',\overline{q}) \right.\\
&\qquad + 2\sum_{m=1}^{g} W_1^{(g-m)}(\overline{q})W_3^{(m-1)}(q',\overline{q}',q) \\
&\qquad + 2\sum_{m'=0}^{g-1} W_3^{(m')}(q',q,\overline{q})W_1^{(g-m'-1)}(\overline{q}') \\
&\qquad + 4\sum_{m=1}^{g}\sum_{m'=0}^{m} W_2^{(m')}(q',q)W_1^{(m-m')}(\overline{q}')W_1^{(g-m)}(\overline{q}) \\
&\qquad \left. + W_4^{(g-2)}(q,q',\overline{q},\overline{q}') \right] \qquad \text{(A.17)}
\end{aligned}
$$

and $W_2^{(g)}(p_2,p_1)$ is given by the same integral except that the order for computing residues is reversed, the residue in q is computed before q'. The difference is thus obtained by pushing the contour of q' through the contour of q, and is obtained as the residue at $q=q'$. Notice that only $W_2^{(0)}(q,q')$ has a pole at $q=q'$, all the other $W_k^{(g)}$ have no poles at $q=q'$ from Theorem 4.2.

Thus, we have

$$\begin{aligned}
& W_2^{(g)}(p_1,p_2) - W_2^{(g)}(p_2,p_1) \\
& \quad - \sum_i \mathop{\rm Res}_{q\to a_i} \frac{dE_q(p_1)\,B(q,p_2) - dE_q(p_2)\,B(q,p_1)}{\omega(q)}\, W_1^{(g)}(\overline{q}) \\
& \quad = \sum_i \mathop{\rm Res}_{q\to a_i} \mathop{\rm Res}_{q'\to q} \frac{dE_q(p_1)}{\omega(q)} \frac{dE_{q'}(p_2)}{\omega_i(q')}\, B(q,q') \\
& \qquad \left[4\sum_{m=0}^{g} W_1^{(m)}(\overline{q}')W_1^{(g-m)}(\overline{q}) + 2W_2^{(g-1)}(\overline{q}',\overline{q}) \right] \\
& \quad = 2\sum_i \mathop{\rm Res}_{q\to a_i} \mathop{\rm Res}_{q'\to q} \frac{dE_q(p_1)}{\omega(q)} \frac{dE_{q'}(p_2)}{\omega_i(q')}\, B(q,q')\, U_0^{(g)}(\overline{q},\overline{q}') \\
& \quad = \sum_i \mathop{\rm Res}_{q\to a_i} \mathop{\rm Res}_{q'\to q} \frac{dE_q(p_1)}{\omega(q)} \frac{dE_{q'}(p_2)}{\omega_i(q')}\, B(q,q') \left(U_0^{(g)}(q,q') + U_0^{(g)}(\overline{q},\overline{q}') \right),
\end{aligned}$$

(A.18)

where we have used the notation of Theorem 4.5.

Similarly, for higher values of k, we find:

$$\begin{aligned}
& W_{2+k}^{(g)}(p_1,p_2,p_K) - W_{2+k}^{(g)}(p_2,p_1,p_K) \\
& \quad - \sum_i \mathop{\rm Res}_{q\to a_i} \frac{dE_q(p_1)\,B(q,p_2) - dE_q(p_2)\,B(q,p_1)}{\omega(q)}\, W_{k+1}^{(g)}(\overline{q},p_K) \\
& = \sum_i \mathop{\rm Res}_{q\to a_i} \mathop{\rm Res}_{q'\to q} \frac{dE_q(p_1)}{\omega(q)} \frac{dE_{q'}(p_2)}{\omega_i(q')}\, B(q,q') \Big(U_k^{(g)}(q,q',p_K) + U_k^{(g)}(\overline{q},\overline{q}',p_K) \Big)
\end{aligned}$$

(A.19)

Then it gives

$$\begin{aligned}
& W_{2+k}^{(g)}(p_1,p_2,p_K) - W_{2+k}^{(g)}(p_2,p_1,p_K) \\
& \quad - \sum_i \mathop{\rm Res}_{q\to a_i} \frac{dE_q(p_1)\,B(q,p_2) - dE_q(p_2)\,B(q,p_1)}{\omega(q)}\, W_{k+1}^{(g)}(\overline{q},p_K) \\
& = \sum_i \mathop{\rm Res}_{q\to a_i} \frac{dE_q(p_1)}{\omega(q)}\, d_{q'} \left(\frac{dE_{q'}(p_2)}{\omega_i(q')} \left(U_k^{(g)}(q,q',p_K) + U_k^{(g)}(\overline{q},\overline{q}',p_K) \right) \right)_{q'=q}
\end{aligned}$$

(A.20)

and by integrating half of it by parts we get

$$
\begin{aligned}
& W_{2+k}^{(g)}(p_1,p_2,p_K) - W_{2+k}^{(g)}(p_2,p_1,p_K) \\
& \quad = -\sum_i \operatorname*{Res}_{q\to a_i} \frac{B(q,p_1)dE_q(p_2) - B(q,p_2)dE_q(p_1)}{\omega(q)^2} \Big(U_k^{(g)}(q,q,p_K) \\
& \qquad + U_k^{(g)}(\overline{q},\overline{q},p_K)\Big) \\
& \qquad + \sum_i \operatorname*{Res}_{q\to a_i} \frac{dE_q(p_1)\,B(q,p_2) - dE_q(p_2)\,B(q,p_1)}{\omega(q)} W_{k+1}^{(g)}(\overline{q},p_K).
\end{aligned} \tag{A.21}
$$

Now we use the following Lemma

Lemma A.1. *If $f(q,q')$ is localy a bilinear differential near a branch point a_i, with no poles, and symmetric in q and $\overline{q}$, and in q' and $\overline{q}'$, then:*

$$
\operatorname*{Res}_{q\to a_i} \frac{B(q,p_1)dE_q(p_2) - B(q,p_2)dE_q(p_1)}{\omega(q)^2} f(q,q) = 0. \tag{A.22}
$$

Proof. The residue is a simple pole, and we can use formula (A.3), that gives:

$$
\begin{aligned}
& \operatorname*{Res}_{q\to a_i} \frac{B(q,p_1)dE_q(p_2) - B(q,p_2)dE_q(p_1)}{\omega(q)^2} f(q,q) \\
& \quad = \operatorname*{Res}_{q\to a_i} \frac{B(q,p_1)B(q,p_2) - B(q,p_2)B(q,p_1)}{\omega(q)dy(q)dx(q)} f(q,q) \\
& \quad = 0.
\end{aligned} \tag{A.23}
$$

□

Using this Lemma, as well as Theorem 4.5, we get:

$$
\begin{aligned}
& W_{2+k}^{(g)}(p_1,p_2,p_K) - W_{2+k}^{(g)}(p_2,p_1,p_K) \\
& \quad = -\sum_i \operatorname*{Res}_{q\to a_i} \frac{B(q,p_1)dE_q(p_2) - B(q,p_2)dE_q(p_1)}{\omega(q)^2} (y(q) \\
& \qquad - y(\overline{q}))dx(q)W_{k+1}^{(g)}(q,p_K) \\
& \qquad + \sum_i \operatorname*{Res}_{q\to a_i} \frac{dE_q(p_1)\,B(q,p_2) - dE_q(p_2)\,B(q,p_1)}{\omega(q)} W_{k+1}^{(g)}(\overline{q},p_K) \\
& \quad = 0.
\end{aligned} \tag{A.24}
$$

□

Proof of Corollary 4.1. For any rational function $R(x)$ with no pole at $x(a_i)$ we have

$$\begin{aligned}
&\operatorname*{Res}_{a_i} R(x(p)) W_{k+1}^{(g)}(p, p_1, \dots, p_k) \\
&\quad = \frac{1}{2} \operatorname*{Res}_{a_i} R(x(p)) \left(W_{k+1}^{(g)}(p, p_1, \dots, p_k) + W_{k+1}^{(g)}(\overline{p}, p_1, \dots, p_k) \right) \\
&\quad = 0 \qquad \text{(A.25)}
\end{aligned}$$

due to Theorem 4.4.

For $m = 0, 1$ compute

$$\begin{aligned}
&\sum_\alpha \operatorname*{Res}_{p\to\alpha} x(p)^m y(p) W_{k+1}^{(g)}(p, p_K) \\
&= -\frac{1}{2} \sum_\alpha \operatorname*{Res}_{p\to\alpha} \frac{x(p)^m}{dx(p)} \Bigg(-2y(p)dx(p) W_{k+1}^{(g)}(p, p_K) \\
&\quad + \sum_{h=0}^{g} \sum_{I\subset K} W_{|I|+1}^{(h)}(p, p_I) W_{k-|I|+1}^{(g-h)}(p, p_{K/I}) + W_{k+2}^{(g-1)}(p, p, p_K) \Bigg) \\
&= \frac{1}{2} \operatorname*{Res}_{p\to a_i, p_K} \frac{x(p)^m}{dx(p)} \Bigg(-2y(p)dx(p) W_{k+1}^{(g)}(p, p_K) \\
&\quad + \sum_{h=0}^{g} \sum_{I\subset K} W_{|I|+1}^{(h)}(p, p_I) W_{k-|I|+1}^{(g-h)}(p, p_{K/I}) + W_{k+2}^{(g-1)}(p, p, p_K) \Bigg) \\
&= \frac{1}{2} \sum_{j=1}^{k} \operatorname*{Res}_{p\to p_j} \frac{x(p)^m}{dx(p)} \left(B(p, p_j) W_k^{(g)}(p, p_{K/\{j\}}) \right) \\
&\quad + \frac{1}{2} \sum_i \operatorname*{Res}_{p\to a_i} \frac{x(p)^m}{dx(p)} \Bigg(-2y(p)dx(p) W_{k+1}^{(g)}(p, p_K) \\
&\quad + \sum_{h=0}^{g} \sum_{I\subset K} W_{|I|+1}^{(h)}(p, p_I) W_{k-|I|+1}^{(g-h)}(p, p_{K/I}) + W_{k+2}^{(g-1)}(p, p, p_K) \Bigg)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{j=1}^{k} d_{p_j}\left(\frac{x(p_j)^m\, W_k^{(g)}(p_K)}{dx(p_j)}\right) \\
&\quad + \frac{1}{4}\sum_i \mathop{\mathrm{Res}}_{p\to a_i} \frac{x(p)^m}{dx(p)}\Bigg(-2y(p)dx(p)W_{k+1}^{(g)}(p,p_K)+ \\
&\quad + \sum_{h=0}^{g}\sum_{I\subset K} W_{|I|+1s}^{(h)}(p,p_I)W_{k-|I|+1}^{(g-h)}(p,p_{K/I}) + W_{k+2}^{(g-1)}(p,p,p_K)\Bigg) \\
&\quad + \frac{1}{4}\sum_i \mathop{\mathrm{Res}}_{p\to a_i} \frac{x(p)^m}{dx(p)}\Bigg(-2y(\overline{p})dx(p)W_{k+1}^{(g)}(\overline{p},p_K)+ \\
&\quad + \sum_{h=0}^{g}\sum_{I\subset K} W_{|I|+1}^{(h)}(\overline{p},p_I)W_{k-|I|+1}^{(g-h)}(\overline{p},p_{K/I}) + W_{k+2}^{(g-1)}(\overline{p},\overline{p},p_K)\Bigg) \\
&= \frac{1}{2}\sum_{j=1}^{k} d_{p_j}\left(\frac{x(p_j)^m\, W_k^{(g)}(p_K)}{dx(p_j)}\right) \\
&\quad + \frac{1}{4}\sum_i \mathop{\mathrm{Res}}_{p\to a_i} \frac{x(p)^m}{dx(p)}\left(P_k^{(g)}(x(p),p_K)dx^2(p)\right) \\
&= \frac{1}{2}\sum_{j=1}^{k} d_{p_j}\left(\frac{x(p_j)^m\, W_k^{(g)}(p_K)}{dx(p_j)}\right)
\end{aligned}
$$

(A.26)

due to Theorem 4.5. □

Proof of Theorem 4.7. The case $k=1$, $g=0$ is easy:

$$\mathop{\mathrm{Res}}_{p_2\to p_1} \Phi(p)B(p_1,p_2) = d\Phi(p_2) = y(p_1)dx(p_1). \tag{A.27}$$

We prove the theorem by recursion on g and k. Suppose it is proved for all k' for $g'\leq g-1$, and for $k'\leq k-1$ if $g'=g$. We write $K=\{1,\dots,k\}$ and

$K' = \{1, \ldots, k-1\}$. Then we have from Equation (4.9):

$$\begin{aligned}
&\operatorname*{Res}_{p_k \to \mathbf{a}} \Phi(p_k) W_{k+1}^{(g)}(p, p_1, \ldots, p_k) \\
&\quad = \operatorname*{Res}_{p_k \to \mathbf{a}} \operatorname*{Res}_{q \to \mathbf{a}} \Phi(p_k) \frac{dE_q(p)}{\omega(q)} \left(\sum_{m=0}^{g} \sum_{J \subset K'} W_{j+2}^{(m)}(q, p_J, p_k) W_{k-j}^{(g-m)}(\overline{q}, p_{K'/J}) \right. \\
&\qquad \left. + \sum_{m=0}^{g} \sum_{J \subset K'} W_{j+1}^{(m)}(q, p_J) W_{k-j+1}^{(g-m)}(\overline{q}, p_{K'/J}, p_k) + W_{k+2}^{(g-1)}(q, \overline{q}, p_{K'}, p_k) \right).
\end{aligned}$$

(A.28)

Then we exchange the contours of integration

$$\operatorname*{Res}_{p_k \to \mathbf{a}} \operatorname*{Res}_{q \to \mathbf{a}} = \operatorname*{Res}_{q \to \mathbf{a}} \operatorname*{Res}_{p_k \to \mathbf{a}} + \operatorname*{Res}_{q \to \mathbf{a}} \operatorname*{Res}_{p_k \to q, \overline{q}}. \tag{A.29}$$

Thus

$$\begin{aligned}
&\operatorname*{Res}_{p_k \to \mathbf{a}} \Phi(p_k) W_{k+1}^{(g)}(p, p_1, \ldots, p_k) \\
&\quad = \operatorname*{Res}_{q \to \mathbf{a}} \operatorname*{Res}_{p_k \to \mathbf{a}} \Phi(p_k) \frac{dE_q(p)}{\omega(q)} \left(\sum_{m=0}^{g} \sum_{J \subset K'} W_{j+2}^{(m)}(q, p_J, p_k) W_{k-j}^{(g-m)}(\overline{q}, p_{K'/J}) \right. \\
&\qquad \left. + \sum_{m=0}^{g} \sum_{J \subset K'} W_{j+1}^{(m)}(q, p_J) W_{k-j+1}^{(g-m)}(\overline{q}, p_{K'/J}, p_k) + W_{k+2}^{(g-1)}(q, \overline{q}, p_{K'}, p_k) \right) \\
&\qquad + \operatorname*{Res}_{q \to \mathbf{a}} \operatorname*{Res}_{p_k \to q, \overline{q}} \Phi(p_k) \frac{dE_q(p)}{\omega(q)} \left(\sum_{m=0}^{g} \sum_{J \subset K'} W_{j+2}^{(m)}(q, p_J, p_k) W_{k-j}^{(g-m)}(\overline{q}, p_{K'/J}) \right. \\
&\qquad \left. + \sum_{m=0}^{g} \sum_{J \subset K'} W_{j+1}^{(m)}(q, p_J) W_{k-j+1}^{(g-m)}(\overline{q}, p_{K'/J}, p_k) + W_{k+2}^{(g-1)}(q, \overline{q}, p_{K'}, p_k) \right).
\end{aligned}$$

(A.30) □

The first term is computed from the recursion hypothesis, and the second term can exist only if the correlation function containing p_k has poles at $p_k = q$ or $p_k = \overline{q}$, and from Theorem 4.2, this can happen only if the correlation function containing p_k is a Bergmann kernel. That gives:

$$
\begin{aligned}
&\operatorname*{Res}_{p_k\to\mathbf{a}} \Phi(p_k) W_{k+1}^{(g)}(p,p_1,\dots,p_k)\\
&\quad= \operatorname*{Res}_{q\to\mathbf{a}} \frac{dE_q(p)}{\omega(q)} \left(\sum_{m=0}^{g}\sum_{J\subset K'} (2m+(j+1)\right.\\
&\qquad -2) W_{j+1}^{(m)}(q,p_J) W_{k-j}^{(g-m)}(\overline{q},p_{K'/J})\\
&\qquad + \sum_{m=0}^{g}\sum_{J\subset K'} (2(g-m)+(k-j)-2) W_{j+1}^{(m)}(q,p_J) W_{k-j}^{(g-m)}(\overline{q},p_{K'/J})\\
&\qquad \left. + (2(g-1)+k+1-2) W_{k+1}^{(g-1)}(q,\overline{q},p_{K'})\right)\\
&\qquad + \operatorname*{Res}_{q\to\mathbf{a}} \operatorname*{Res}_{p_k\to q,\overline{q}} \Phi(p_k) \frac{dE_q(p)}{\omega(q)} \big(B(q,p_k) W_{k+1}^{(g)}(\overline{q},p_{K'})\\
&\qquad + W_k^{(g)}(q,p_{K'}) B(\overline{q},p_k)\big)\\
&\quad= (2g+k-3) \operatorname*{Res}_{q\to\mathbf{a}} \frac{dE_q(p)}{\omega(q)} \left(\sum_{m=0}^{g}\sum_{J\subset K'} W_{j+1}^{(m)}(q,p_J) W_{k-j}^{(g-m)}(\overline{q},p_{K'/J})\right.\\
&\qquad \left. + W_{k+1}^{(g-1)}(q,\overline{q},p_{K'})\right)\\
&\qquad + \operatorname*{Res}_{q\to\mathbf{a}} \frac{dE_q(p)}{y(q)-y(\overline{q})} \Big(y(q) W_{k+1}^{(g)}(\overline{q},p_{K'}) + y(\overline{q}) W_k^{(g)}(q,p_{K'})\Big)\\
&\quad= (2g+k-3)\, W_k(p,p_1,\dots,p_{k-1})\\
&\qquad + \operatorname*{Res}_{q\to\mathbf{a}} \frac{dE_q(p)}{y(q)-y(\overline{q})} \big(y(q)(W_{k+1}^{(g)}(\overline{q},p_{K'})\\
&\qquad + W_k^{(g)}(q,p_{K'})) + (y(\overline{q})-y(q)) W_k^{(g)}(q,p_{K'})\big)\\
&\quad= (2g+k-3) W_k(p,p_1,\dots,p_{k-1})\\
&\qquad + \operatorname*{Res}_{q\to\mathbf{a}} \frac{dE_q(p)}{y(q)-y(\overline{q})} \, (y(\overline{q})-y(q)) W_k^{(g)}(q,p_{K'})\\
&\quad= (2g+k-3)\, W_k(p,p_1,\dots,p_{k-1}) - \operatorname*{Res}_{q\to\mathbf{a}} dE_q(p)\, W_k^{(g)}(q,p_{K'})
\end{aligned}
$$

$$
\begin{aligned}
&= (2g+k-3)\, W_k(p,p_1,\dots,p_{k-1}) + \frac{1}{2} \mathop{\mathrm{Res}}_{q\to\mathbf{a}} dS_{q,o}(p)\, W_k^{(g)}(q,p_{K'}) \\
&\quad - \frac{1}{2} \mathop{\mathrm{Res}}_{q\to\mathbf{a}} dS_{\overline{q},o}(p)\, W_k^{(g)}(q,p_{K'}) \\
&= (2g+k-3)\, W_k(p,p_1,\dots,p_{k-1}) + \frac{1}{2} \mathop{\mathrm{Res}}_{q\to\mathbf{a}} dS_{q,o}(p)\, W_k^{(g)}(q,p_{K'}) \\
&\quad - \frac{1}{2} \mathop{\mathrm{Res}}_{q\to\mathbf{a}} dS_{q,o}(p)\, W_k^{(g)}(\overline{q},p_{K'}) \\
&= (2g+k-3)\, W_k(p,p_1,\dots,p_{k-1}) + \mathop{\mathrm{Res}}_{q\to\mathbf{a}} dS_{q,o}(p)\, W_k^{(g)}(q,p_{K'}) \\
&= (2g+k-3)\, W_k(p,p_1,\dots,p_{k-1}) - \mathop{\mathrm{Res}}_{q\to p} dS_{q,o}(p)\, W_k^{(g)}(q,p_{K'}) \\
&\quad - \frac{1}{2i\pi} \sum_j \left(\oint_{\underline{\mathcal{B}}_j} B(q,p) \oint_{\underline{\mathcal{A}}_j} W_k^{(g)}(q,p_{K'}) \right. \\
&\quad \left. - \oint_{\underline{\mathcal{A}}_j} B(q,p) \oint_{\underline{\mathcal{B}}_j} W_k^{(g)}(q,p_{K'}) \right) \\
&= (2g+k-3)\, W_k(p,p_1,\dots,p_{k-1}) - \mathop{\mathrm{Res}}_{q\to p} dS_{q,o}(p)\, W_k^{(g)}(q,p_{K'}) \\
&\quad - du^t(p) \left((1+\kappa\tau) \oint_{\underline{\mathcal{A}}} W_k^{(g)}(q,p_{K'}) - \kappa \oint_{\underline{\mathcal{B}}} W_k^{(g)}(q,p_{K'}) \right) \\
&= (2g+k-3)\, W_k(p,p_1,\dots,p_{k-1}) - \mathop{\mathrm{Res}}_{q\to p} dS_{q,o}(p)\, W_k^{(g)}(q,p_{K'}) \\
&\quad - du^t(p) \oint_{\mathcal{A}} W_k^{(g)}(q,p_{K'}) \\
&= (2g+k-3)\, W_k(p,p_1,\dots,p_{k-1}) - \mathop{\mathrm{Res}}_{q\to p} dS_{q,o}(p)\, W_k^{(g)}(q,p_{K'}) \\
&= (2g+k-2)\, W_k(p,p_1,\dots,p_{k-1}).
\end{aligned}
\tag{A.31}
$$

B. Variation of the curve

In this appendix, we proove the theorems stated in Sections 5 and 6.1.

Proof of Lemma. 5.1

$$
\begin{aligned}
&D_\Omega\left(\sum_j \operatorname*{Res}_{q\to a_j} \frac{dE_q(p)}{\omega(q)}\, f(q,\overline{q})\right)_{x(p)}\\
&\quad= \sum_j \operatorname*{Res}_{q\to a_j} \frac{dE_q(p)}{\omega(q)}\, D_\Omega(f(q,\overline{q}))_{x(q)}\\
&\qquad - \sum_j \operatorname*{Res}_{q\to a_j} \frac{dE_q(p)}{(\omega(q))^2}\, (\Omega(q)-\Omega(\overline{q}))\, f(q,\overline{q})\\
&\qquad + 2\sum_j \operatorname*{Res}_{q\to a_j} \sum_i \operatorname*{Res}_{r\to a_i} \frac{dE_r(p)}{\omega(r)}\, \Omega(r)\, \frac{dE_q(r)}{\omega(q)}\, f(q,\overline{q})\\
&\quad= \sum_j \operatorname*{Res}_{q\to a_j} \frac{dE_q(p)}{\omega(q)}\, D_\Omega(f(q,\overline{q}))_{x(q)} - 2\sum_j \operatorname*{Res}_{q\to a_j} \frac{dE_q(p)}{(\omega(q))^2}\, \Omega(q)\, f(q,\overline{q})\\
&\quad + 2\sum_j \operatorname*{Res}_{q\to a_j} \sum_i \operatorname*{Res}_{r\to a_i} \frac{dE_r(p)}{\omega(r)}\, \Omega(r) \frac{dE_q(r)}{\omega(q)}\, f(q,\overline{q})\\
&\quad= \sum_j \operatorname*{Res}_{q\to a_j} \frac{dE_q(p)}{\omega(q)}\, D_\Omega(f(q,\overline{q}))_{x(q)}\\
&\qquad - 2\sum_j \operatorname*{Res}_{q\to a_j} \operatorname*{Res}_{r\to q} \frac{dE_q(r)dE_r(p)}{\omega(q)\omega(r)}\, \Omega(r)\, f(q,\overline{q})\\
&\qquad + 2\sum_j \operatorname*{Res}_{q\to a_j} \sum_i \operatorname*{Res}_{r\to a_i} \frac{dE_r(p)}{\omega(r)}\, \Omega(r) \frac{dE_q(r)}{\omega(q)}\, f(q,\overline{q})\\
&\quad= \sum_j \operatorname*{Res}_{q\to a_j} \frac{dE_q(p)}{\omega(q)}\, D_\Omega(f(q,\overline{q}))_{x(q)}\\
&\qquad + 2\sum_i \operatorname*{Res}_{r\to a_i} \sum_j \operatorname*{Res}_{q\to a_j} \frac{dE_r(p)}{\omega(r)}\, \Omega(r) \frac{dE_q(r)}{\omega(q)}\, f(q,\overline{q}).
\end{aligned}
\tag{B.1}
$$

□

Proof of Theorem 5.1. This theorem straightforwardly comes from the diagrammatic rules described in the preceding paragraph except for the variation of $F^{(1)}$.

Let us prove it for $F^{(1)}$ with $\Omega(p) = \int_{\mathcal{C}} B(p,q)\Lambda(q)$. One has

$$
\begin{aligned}
-\int_{\mathcal{C}} W_1^{(g)}(p)\Lambda(p) &= -\operatorname*{Res}_{q\to\mathbf{a}} \frac{\int_{\mathcal{C}} dE_q(p)\Lambda(p)}{\omega(q)} B(q,\overline{q}) \\
&= -\operatorname*{Res}_{q\to\mathbf{a}} \frac{\int_{\mathcal{C}} dE_q(p)\Lambda(p) dz_i(q) dz_i(\overline{q})}{\omega(q)} \left[\frac{1}{(z(q)-z(\overline{q}))^2}\right. \\
&\quad \left. + \frac{1}{6} S_B(q)\right],
\end{aligned}
\tag{B.2}
$$

where z_i is a local variable near the branch point a_i and S_B is the corresponding Bergmann projective connection. Since the last term has a simple pole at the branch point a_i, one can write

$$
\begin{aligned}
-\operatorname*{Res}_{q\to\mathbf{a}} \frac{\int_{\mathcal{C}} dE_q(p)\Lambda(p) dz_i(q) dz_i(\overline{q})}{\omega(q)} \frac{1}{6} S_B(q) &= -\frac{1}{2}\sum_i \Omega(a_i) dy(a_i) \operatorname*{Res}_{q\to a_i} \frac{B(q,\overline{q})}{dx(q)} \\
&= -\frac{1}{2}\delta_\Omega \ln \tau_{Bx}.
\end{aligned}
\tag{B.3}
$$

On the other hand, one can express the first term thanks to the local variables z_i and compute

$$
-\operatorname*{Res}_{q\to\mathbf{a}} \frac{\int_{\mathcal{C}} dE_q(p)\Lambda(p) dz_i(q) dz_i(\overline{q})}{\omega(q)(z(q)-z(\overline{q}))^2} = -\frac{1}{24}\delta_\Omega(y'(a_i))
\tag{B.4}
$$

provided that $z_i(q) - z_i(\overline{q}) = 2z_i(q)$. □

Proof of Theorem (5.3). Using Theorem 4.7, we have:

$$
(2-2g)F^{(g)} = -\operatorname*{Res}_{\mathbf{a}} \Phi W_1^{(g)}
\tag{B.5}
$$

and using Equation (3.61), we can choose for any arbitrary o':

$$
\Phi(p) = -\sum_\alpha \operatorname*{Res}_\alpha V_\alpha \, dS_{p,o'} + \sum_\alpha t_\alpha \int_o^\alpha dS_{p,o'} + \sum_i \epsilon_i \oint_{\mathcal{B}_i} dS_{p,o'},
\tag{B.6}
$$

which implies

$$
\begin{aligned}
(2-2g)F^{(g)} &= -\operatorname*{Res}_{\mathbf{a}} \Phi W_1^{(g)} \\
&= \sum_\alpha \operatorname*{Res}_{p\to\mathbf{a}} \operatorname*{Res}_{q\to\alpha} V_\alpha(q) dS_{p,o'}(q) W_1^{(g)}(p) \\
&\quad - \sum_\alpha t_\alpha \operatorname*{Res}_{p\to\mathbf{a}} \int_{q=o}^{\alpha} dS_{p,o'}(q) W_1^{(g)}(p) \\
&\quad - \sum_i \epsilon_i \operatorname*{Res}_{p\to\mathbf{a}} \oint_{q\in\mathcal{B}_i} dS_{p,o'}(q) W_1^{(g)}(p).
\end{aligned}
\tag{B.7}
$$

Since the poles α and the branch points a_i do not coincide, one can exchange the order of integration. Then, one can move the integration contours for p in order to integrate only around the last pole $p \to q$:

$$
\begin{aligned}
(2-2g)F^{(g)} &= \sum_\alpha t_\alpha \int_{q=o}^{\alpha} \operatorname*{Res}_{p\to q} dS_{p,o'}(q) W_1^{(g)}(p) \\
&\quad - \sum_\alpha \operatorname*{Res}_{q\to\alpha} \operatorname*{Res}_{p\to q} V_\alpha(q) dS_{p,o'}(q) W_1^{(g)}(p) \\
&\quad + \sum_i \epsilon_i \oint_{q\in\mathcal{B}_i} \operatorname*{Res}_{p\to q} dS_{p,o'}(q) W_1^{(g)}(p) \\
&= \sum_\alpha \operatorname*{Res}_{q\to\alpha} V_\alpha(q) W_1^{(g)}(q) - \sum_\alpha t_\alpha \int_{q=o}^{\alpha} W_1^{(g)}(q) \\
&\quad - \sum_i \epsilon_i \oint_{q\in\mathcal{B}_i} W_1^{(g)}(q).
\end{aligned}
\tag{B.8}
$$

□

Proof of Theorem (6.1). We have

$$
\begin{aligned}
2i\pi \frac{\partial}{\partial \kappa_{ij}} W_2^{(0)}(p_1,p_2) &= \frac{1}{2}(2i\pi)^2 (du_i(p_1)du_j(p_2) + du_i(p_2)du_j(p_1)) \\
&= \frac{1}{2} \oint_{r\in\mathcal{B}_j} \oint_{s\in\mathcal{B}_i} (B(p_1,r)B(p_2,s) + B(p_2,r)B(p_1,s)).
\end{aligned}
\tag{B.9}
$$

Thus the theorem holds for $k=2$ and $g=0$. And by integration we get

$$
\begin{aligned}
&-2i\pi \frac{\partial}{\partial \kappa_{ij}} dE_q(p) \\
&\quad = 2(i\pi)^2 (du_i(p)(u_j(q) - u_j(\overline{q})) + du_j(p)(u_i(q) - u_i(\overline{q}))) \\
&\quad = -\frac{1}{2} \oint_{r\in\mathcal{B}_i} \oint_{s\in\mathcal{B}_j} (B(p,r)dE_q(s) + B(p,s)dE_q(r)).
\end{aligned}
\tag{B.10}
$$

$$
\begin{aligned}
& 2i\pi \frac{\partial}{\partial \kappa_{ij}} W_{k+1}^{(g)}(p, p_K) \\
&= 2i\pi \frac{\partial}{\partial \kappa_{ij}} \mathop{\mathrm{Res}}_{q\to \mathbf{a}} \frac{dE_q(p)}{(y(q)-y(\overline{q}))dx(q)} \Big(W_{k+2}^{(g-1)}(q,\overline{q},p_K) \\
&\quad + \sum W_{j+1}^{(h)}(q,p_J) W_{k-j+1}^{(g-h)}(\overline{q},p_{K/J}) \Big) \\
&= \frac{1}{2} \oint_{r\in\mathcal{B}_i} \oint_{s\in\mathcal{B}_j} \mathop{\mathrm{Res}}_{q\to \mathbf{a}} \frac{B(p,r)dE_q(s)+B(p,s)dE_q(r)}{(y(q)-y(\overline{q}))dx(q)} (W_{k+2}^{(g-1)}(q,\overline{q},p_K) \\
&\quad + \sum W_{j+1}^{(h)}(q,p_J) W_{k-j+1}^{(g-h)}(\overline{q},p_{K/J}) \Big) \\
&\quad + \mathop{\mathrm{Res}}_{q\to \mathbf{a}} \frac{dE_q(p)}{(y(q)-y(\overline{q}))dx(q)} \left(2i\pi \frac{\partial}{\partial \kappa_{ij}} W_{k+2}^{(g-1)}(q,\overline{q},p_K) \right) \\
&\quad + \mathop{\mathrm{Res}}_{q\to \mathbf{a}} \frac{dE_q(p)}{(y(q)-y(\overline{q}))dx(q)} \left(\sum 2i\pi \frac{\partial}{\partial \kappa_{ij}} W_{j+1}^{(h)}(q,p_J) W_{k-j+1}^{(g-h)}(\overline{q},p_{K/J}) \right) \\
&\quad + \mathop{\mathrm{Res}}_{q\to \mathbf{a}} \frac{dE_q(p)}{(y(q)-y(\overline{q}))dx(q)} \left(\sum W_{j+1}^{(h)}(q,p_J) 2i\pi \frac{\partial}{\partial \kappa_{ij}} W_{k-j+1}^{(g-h)}(\overline{q},p_{K/J}) \right) \\
&= \frac{1}{2} \oint_{r\in\mathcal{B}_i} \oint_{s\in\mathcal{B}_j} (B(p,r) W_{k+1}^{(g)}(s,p_K) + B(p,s) W_{k+1}^{(g)}(r,p_K)) \\
&\quad + \frac{1}{2} \oint_{r\in\mathcal{B}_i} \oint_{s\in\mathcal{B}_j} \mathop{\mathrm{Res}}_{q\to \mathbf{a}} \frac{dE_q(p)}{(y(q)-y(\overline{q}))dx(q)} (W_{k+4}^{(g-2)}(q,\overline{q},p_K,r,s) \\
&\quad + 2 W_{j+3}^{(h)}(q,\overline{q},p_J,r) W_{k-j+1}^{(g-1-h)}(p_{K/J},s) \\
&\quad + 2 W_{j+s2}^{(h)}(q,p_J,r) W_{k-j+2}^{(g-1-h)}(\overline{q},p_{K/J},s)) \\
&\quad + \frac{1}{2} \oint_{r\in\mathcal{B}_i} \oint_{s\in\mathcal{B}_j} \mathop{\mathrm{Res}}_{q\to \mathbf{a}} \frac{dE_q(p)}{(y(q)-y(\overline{q}))dx(q)} \sum_J W_{k-j+1}^{(g-h)}(\overline{q},p_{K/J}) \times \\
&\quad\quad \times \left(W_{j+3}^{(h-1)}(q,p_J,r,s) + 2 \sum_L W_{l+2}^{(m)}(q,p_L,r) W_{k-j-l+1}^{(h-m)}(p_{K/(J\cup L)},s) \right) \\
&\quad + \frac{1}{2} \oint_{r\in\mathcal{B}_i} \oint_{s\in\mathcal{B}_j} \mathop{\mathrm{Res}}_{q\to \mathbf{a}} \frac{dE_q(p)}{(y(q)-y(\overline{q}))dx(q)} \sum_J W_{k-j+1}^{(g-h)}(q,p_{K/J}) \times \\
&\quad\quad \times \left(W_{j+3}^{(h-1)}(\overline{q},p_J,r,s) + 2 \sum_L W_{l+2}^{(m)}(\overline{q},p_L,r) W_{k-j-l+1}^{(h-m)}(p_{K/(J\cup L)},s) \right)
\end{aligned}
$$

(B.11)

We regroup together all terms with two W's and all terms with three W's:

$$
\begin{aligned}
&2i\pi \frac{\partial}{\partial \kappa_{ij}} W_{k+1}^{(g)}(p, p_K) \\
&\quad = \frac{1}{2} \oint_{r\in\mathcal{B}_i} \oint_{s\in\mathcal{B}_j} (B(p,r)W_{k+1}^{(g)}(s,p_K) + B(p,s)W_{k+1}^{(g)}(r,p_K)) \\
&\qquad + \oint_{r\in\mathcal{B}_i} \oint_{s\in\mathcal{B}_j} \mathop{\mathrm{Res}}_{q\to\mathbf{a}} \frac{dE_q(p)}{(y(q)-y(\overline{q}))dx(q)} \sum_J W_{j+3}^{(h-1)}(q, p_J, r, s) \\
&\qquad \times W_{k-j+1}^{(g-h)}(\overline{q}, p_{K/J}) \\
&\qquad + \oint_{r\in\mathcal{B}_i} \oint_{s\in\mathcal{B}_j} \mathop{\mathrm{Res}}_{q\to\mathbf{a}} \frac{dE_q(p)}{(y(q)-y(\overline{q}))dx(q)} (W_{j+2}^{(h)}(q, p_J, r) \\
&\qquad \times W_{k-j+2}^{(g-1-h)}(\overline{q}, p_{K/J}, s)) \\
&\qquad + \frac{1}{2} \oint_{r\in\mathcal{B}_i} \oint_{s\in\mathcal{B}_j} \mathop{\mathrm{Res}}_{q\to\mathbf{a}} \frac{dE_q(p)}{(y(q)-y(\overline{q}))dx(q)} W_{k+4}^{(g-2)}(q, \overline{q}, p_K, r, s) \\
&\qquad + \oint_{r\in\mathcal{B}_i} \oint_{s\in\mathcal{B}_j} \mathop{\mathrm{Res}}_{q\to\mathbf{a}} \frac{dE_q(p)}{(y(q)-y(\overline{q}))dx(q)} \\
&\qquad \times \left(\sum_J \sum_L W_{l+2}^{(m)}(\overline{q}, p_L, r) W_{k-j-l+1}^{(h-m)}(p_{K/(J\cup L)}, s) W_{k-j+1}^{(g-h)}(q, p_{K/J}) \right) \\
&\qquad + \oint_{r\in\mathcal{B}_i} \oint_{s\in\mathcal{B}_j} \mathop{\mathrm{Res}}_{q\to\mathbf{a}} \frac{dE_q(p)}{(y(q)-y(\overline{q}))dx(q)} \times \\
&\qquad \times \left(\sum_J \sum_L W_{l+2}^{(m)}(q, p_L, r) W_{k-j-l+1}^{(h-m)}(p_{K/(J\cup L)}, s) W_{k-j+1}^{(g-h)}(\overline{q}, p_{K/J}) \right) \\
&\qquad + \frac{1}{2} \oint_{r\in\mathcal{B}_i} \oint_{s\in\mathcal{B}_j} \mathop{\mathrm{Res}}_{q\to\mathbf{a}} \frac{dE_q(p)}{(y(q)-y(\overline{q}))dx(q)} \big(2W_{j+3}^{(h)}(q, \overline{q}, p_J, r) \\
&\qquad W_{k-j+1}^{(g-1-h)}(p_{K/J}, s)\big)
\end{aligned}
\tag{B.12}
$$

We recognize the recursion relation Equation (4.9) in lines 2–5, and in lines 6–8, and this gives the theorem.

The theorem for the free energies, is easy obtained using Theorem 4.7. □

C. Proof of the symplectic invariance of $F^{(0)}$ and $F^{(1)}$

C.1. $F^{(1)}$

Let us study how $F^{(1)}$ is changed under the exchange of the roles of x and y. For this purpose, we define the images of $F^{(1)}$ and $W_1^{(1)}(p)$ under this transormation, i.e.,

$$\widehat{W}_1^{(1)}(p) := -\operatorname*{Res}_{q\to\mathbf{b}} \frac{\int_{\tilde{q}}^{q} B(p,\xi)}{2(x(q)-x(\tilde{q}))dy(q)} B(q,\tilde{q}), \tag{C.1}$$

and

$$\widehat{F}^{(1)} = -\frac{1}{2}\ln \tau_{By} - \frac{1}{24}\ln \prod_j x'(b_j), \tag{C.2}$$

where $\mathbf{b}$ denotes the set of y-branch points, $\tilde{q}$ is the only point satisfying $y(q)=y(\tilde{q})$ and approaching a branch point b_j when $q\to b_j$, τ_{By} is the Bergmann τ-function associated to y and $x'(b_j)=\frac{dx(b_j)}{d\tilde{z}_j(b_j)}$. According to Theorem 5.1, for any variation Ω of the curve, the variation of the free energy reads

$$\delta_\Omega \widehat{F}^{(1)} = \int_{\mathcal{C}} \widehat{W}_1^{(1)}(p)\Lambda(p). \tag{C.3}$$

Thus, the variation of the difference between the two "free energies" reads

$$\delta_\Omega (F^{(1)} - \widehat{F}^{(1)}) = \int_{\mathcal{C}} (W_1^{(1)}(p) + \widehat{W}_1^{(1)}(p))\Lambda(p). \tag{C.4}$$

In order to evaluate this quantity, one needs the following lemma:

Lemma C.1. *For any choice of variable z:*

$$\begin{aligned} W_1^{(1)}(p) + \widehat{W}_1^{(1)}(p) = \frac{1}{24} d_p \Bigg[\frac{1}{x'y'}\Bigg(&2S_{Bz}(p) + \frac{x''y''}{x'y'} + \frac{x''^2}{x'^2} \\ &- \frac{x'''}{x'} + \frac{y''^2}{y'^2} - \frac{y'''}{y'}\Bigg)\Bigg], \end{aligned} \tag{C.5}$$

where the derivatives are taken with respect to the variable z and S_{Bz} denotes the Bergmann projective connection associated to z.

Proof. From the definitions, one easily derives that

$$W_1^{(1)}(p) + \widehat{W}_1^{(1)}(p) = \operatorname*{Res}_{q\to p} \frac{B(p,q)}{(x(p)-x(q))(y(p)-y(q))}. \tag{C.6}$$

Let us now expand the integrand in terms of an arbitrary local variable z when $p \to q$. The different factors read

$$\begin{aligned}\frac{B(p,q)}{dz(p)dz(q)} &= \frac{1}{(z(p)-z(q))^2} + \frac{1}{6}S_{Bz}(p) - \frac{z(p)-z(q)}{12}S'_{Bz}(p)\\ &\quad + O((z(p)-z(q))^2)\end{aligned} \tag{C.7}$$

and

$$\begin{aligned}\frac{1}{y(p)-y(q)} &= \frac{1}{(z(p)-z(qs))y'(p)}\left[1-(z(p)-z(q))\frac{y''}{2y'}\right.\\ &\quad + (z(p)-z(q))^2\left(\frac{y''^2}{4y'^2}-\frac{y'''}{6y'}\right)\\ &\quad + \frac{(z(p)-z(q))^3}{24}\left(-\frac{y''''}{y'}+4\frac{y'''y''}{y'^2}-\frac{3y''^3}{y'^3}\right)\\ &\quad \left.+O((z(p)-z(q))^4)\right].\end{aligned} \tag{C.8}$$

Inserting it altogether inside Equation (C.6), one can explicitly compute the residues and one recognizes the formula (C.5). □

We can now prove the following theorem stating the symmetry of $F^{(1)}$ under the exchange of $x \leftrightarrow y$:

Lemma C.2. *The two free energies transform in the same way under any variation of the moduli of the curve:*

$$\delta_\Omega F^{(1)} = \delta_\Omega \widehat{F}^{(1)}. \tag{C.9}$$

Proof. We already know that

$$\begin{aligned}\delta_\Omega\big(F^{(1)}-\widehat{F}^{(1)}\big) &= \int_{p\in\mathcal{C}} \frac{1}{24}d_p\left[\frac{1}{x'y'}\left(2S_{Bz}(p)+\frac{x''y''}{x'y'}+\frac{x''^2}{x'^2}\right.\right.\\ &\quad \left.\left.-\frac{x'''}{x'}+\frac{y''^2}{y'^2}-\frac{y'''}{y'}\right)\right]\Lambda(p)\end{aligned} \tag{C.10}$$

for an arbitrary variable z.

We just have to check that this quantity vanishes for the transformations corresponding to varying the moduli of the curve. Because the function inside the differential w.r.t. p vanishes at the poles of $y\,dx$, one can check that it is indeed the case. $\square$

Thus, the first correction to the free energy satisfies the following variation under symplectic transformations:

Theorem C.1. *$F^{(1)}$ does not change under the following transformations of $\mathcal{E}$:*

- *$y \to y + R(x)$ where R is a rational function.*
- *$y \to cy$ and $x \to \frac{1}{c}x$ where c is a non-zero complex number.*
- *$x \to x + R(y)$ where R is a rational function.*
- *$y \to x$ and $x \to y$.*

$F^{(1)}$ is shifted by a multiple of $i\pi/12$ when $\mathcal{E}$ is changed by $x \to -x$.

Proof. The first transformation is obvious from the definition since neither $\ln \tau_{Bx}$ nor $y'(a_i)$ changes.

The second one follows from Theorem 2 in [37] for $f = x$ and $g = \frac{x}{c}$ which shows that the variations of $\ln(\tau_{Bx})$ and $y'(a_i)$ compensate.

The fourth one is nothing but Lemma C.2 and it gives the third one when combined with the first.

The last transformation holds because $\ln \tau_{Bx}$ is left unchanged and $y'(a_i)$ changes sign. $\square$

C.2. $F^{(0)}$

Theorem C.2. *$F^{(0)}$ does not change under the following transformations of $\mathcal{E}$:*

- *$y \to y + \mathcal{P}(x)$, where $\mathcal{P}$ is a polynomial.*
- *$y \to y$ and $x \to -x$.*
- *$y \to cy$ and $x \to \frac{1}{c}x$, where c is a non-zero complex number.*
- *$x \to x + \mathcal{P}(y)$, where $\mathcal{P}$ is a polynomial.*
- *$y \to x$ and $x \to y$.*

Proof. The second and third transformations come from the structure of $F^{(0)}$ which is bilinear in objects proportionnal to $y\,dx$.

One can obtain the last one following the same method as for $F^{(1)}$: we compute the variation of the difference of the two free energies under the changes of the moduli of $\mathcal{E}$ and show that they vanish.

Let us now show the first invariance. In this case, the variation of the free energy is $\delta_{\mathcal{P}}(x)dxF^{(0)}$. Since $\mathcal{P}(x)$ is a polynomial one can write it

$$\mathcal{P}(x(p)) = \operatorname*{Res}_{q\to p} B(p,q)\mathcal{Q}(x(q)) = -\sum_{\alpha} \operatorname*{Res}_{q\to\alpha} B(p,q)\mathcal{Q}(x(q)), \tag{C.11}$$

where $\mathcal{Q}(x)$ is also a polynomial in x. Then, Theorem 5.1 implies the invariance of $F^{(0)}$ under this transformation.

The fourth transformation is a combination of the first and the last one. □

D. Matrix model with an external field

We consider the matrix model with external field defined in Section 10.4:

$$Z(\Lambda) := \int_{H_n} dM\, \mathrm{e}^{-N\mathrm{Tr}(V(M)-M\widehat{\Lambda})}, \tag{D.1}$$

where we assume that $\widetilde{\Lambda}$ is the diagonal matrix:

$$\widehat{\Lambda} = \mathrm{diag}\,(\overbrace{\widehat{\lambda}_1,\dots,\widehat{\lambda}_1}^{n_1}, \overbrace{\widehat{\lambda}_2,\dots,\widehat{\lambda}_2}^{n_2}, \dots, \overbrace{\widehat{\lambda}_s,\dots,\widehat{\lambda}_s}^{n_s}) \tag{D.2}$$

and $V'(x)$ is a rational fraction with denominator $D(x)$: $V'(x) = \frac{\sum_{k=0}^{d} g_k x^k}{D(x)}$.

In particular, the polynomial $S(y) := \prod_{i=1}^{s}(y-\widehat{\lambda}_i)$ is the minimal polynomial of $\widetilde{\Lambda}$.

We define the correlation functions $\overline{w}_k(x_1,\dots,x_k) := N^{k-2} \left\langle \prod_{i=1}^{k} \mathrm{Tr}\frac{1}{x_i-M}\right\rangle_c$ and their $1/N^2$ expansion

$$\bar{w}_k(\mathbf{x}_K) = \sum_{h=0}^{\infty} \frac{1}{N^{2h}}\, \overline{w}_k^{(h)}(\mathbf{x}_K). \tag{D.3}$$

We also define the auxiliary functions

$$\overline{u}_k(x,y;\mathbf{x}_K) := N^{|K|-1} \left\langle \mathrm{tr}\,\frac{1}{x-M}\frac{S(y)-S(\Lambda)}{y-\Lambda} \prod_{r=1}^{|K|} \mathrm{Tr}\frac{1}{x_{i_r}-M} \right\rangle_c \tag{D.4}$$

and

$$P_k(x,y;\mathbf{x}_K) := N^{|K|-1}\left\langle \operatorname{tr} \frac{V'(x)-V'(M_1)}{x-M}\frac{S(y)-S(\Lambda)}{y-\Lambda}\prod_{r=1}^{|K|}\operatorname{Tr}\frac{1}{x_{i_r}-M}\right\rangle_c . \tag{D.5}$$

Notice that $\overline{u}_{k,}(x,y;\mathbf{x}_K)$ is a polynomial in y of degree $s-1$, and $D(x)P_k$ $(x,y;\mathbf{x}_K)$ is a polynomial in x of degree $d-1$ and in y of degree $s-1$ (note that P_0 corresponds to P in Equation (10.68)).

It is convenient to renormalize those functions, and define:

$$u_k(x,y;\mathbf{x}_K) := \overline{u}_k(x,y;\mathbf{x}_K) - \delta_{k,0}S(y) \tag{D.6}$$

and

$$w_k(\mathbf{x}_K) := \overline{w}_k(\mathbf{x}_K) + \frac{\delta_{k,2}}{(x_1-x_2)^2}. \tag{D.7}$$

D.1. Loop equations

Consider the change of variables

$$\delta M = \frac{1}{x-M}\frac{S(y)-S(\Lambda)}{y-\Lambda}. \tag{D.8}$$

You get the loop equation

$$\begin{aligned}\overline{w}_1(x)\overline{u}_0(x,y) + \frac{1}{N^2}\overline{u}_1(x,y;x) &= V'(x)\overline{u}_0(x,y) - P_0(x,y) - y\overline{u}_0(x,y)\\ &\quad + S(y)\overline{w}_1(x),\end{aligned} \tag{D.9}$$

i.e.,

$$\begin{aligned}&(y+\overline{w}_1(x)-V'(x))(\overline{u}_0(x,y)-S(y)) + \frac{1}{N^2}\overline{u}_1(x,y;x)\\ &\quad = (V'(x)-y)S(y) - P_0(x,y).\end{aligned} \tag{D.10}$$

We define the polynomial both in x and y

$$E_{\text{Mext}}(x,y) := ((V'(x)-y)S(y) - P_0(x,y))D(x) \tag{D.11}$$

and

$$Y(x) := V'(x) - \overline{w}_1(x). \tag{D.12}$$

The loop equation thus implies

$$(y - Y(x))u_0(x, y)D(x) + \frac{1}{N^2}u_1(x, y; x)D(x) = E_{\mathrm{Mext}}(x, y) \tag{D.13}$$

and in particular

$$E_{\mathrm{Mext}}(x, Y(x)) = \frac{1}{N^2}u_1(x, Y(x); x)D(x). \tag{D.14}$$

The leading order of the topological expansion reads

$$E^{(0)}_{\mathrm{Mext}}(x, Y(x)) = 0, \tag{D.15}$$

which defines an algebraic curve.

D.2. Leading order algebraic curve

Let us study the curve $\mathcal{E}_{\mathrm{Mext}}(x, y) = E^{(0)}_{\mathrm{Mext}}(x, y) = 0$ defining a compact Riemann surface $\overline{\Sigma}$ and two functions x and y defined on it.

Because y is a solution of a degree $s + 1$ equation, $\mathcal{E}_{\mathrm{Mext}}(x, y)$ has $s + 1$ x-sheets. The sheets can be identified by their large x behavior:

- in the physical sheet, we have $Y(x) \sim V'(x) - 1/x + O(1/x^2)$
- in the other sheets, $Y(x) \sim \widehat{\lambda}_i + \frac{n_i}{N}\frac{1}{x} + O(1/x^2)$

Let us note by $p^i \in \overline{\Sigma}$ with $i = 0 \dots s$ the different points of the curve whose x-projection are $x(p)$, i.e.,

$$\forall i, j \quad x(p^i) = x(p^j). \tag{D.16}$$

The superscript 0 corresponds to the point in the physical sheet.

From the correlation functions previously defined on the x and y projections, one defines the corresponding meromorphic one-forms on the curve as follows:

$$W_k(\mathbf{p_K}) := w_k(\mathbf{x}(\mathbf{p_K}))\, dx(p_1) \cdots dx(p_k) \tag{D.17}$$

and

$$U_k(p, y; \mathbf{p_K}) := u_k(x(p), y; \mathbf{x}(\mathbf{p_K}))\, dx(p)dx(p_1) \cdots dx(p_k) \tag{D.18}$$

as well as there to pological expansions

$$W_k(\mathbf{p_K}) = \sum_{h=0}^{\infty} N^{2-2h}\, W_k^{(h)}(\mathbf{p_K}) \quad \text{and}$$

$$U_k(p, y; \mathbf{p_K}) = \sum_{h=0}^{\infty} N^{2-2h}\, U_k^{(h)}(p, y; \mathbf{p_K}). \tag{D.19}$$

D.2.1. Filling fractions and genus The curve has a genus $g \leq ds - 1$ and we work with fixed filling fractions

$$\epsilon_I := 12 i\pi \oint_{\mathcal{A}_I} y\, dx. \tag{D.20}$$

D.2.2. Subleading loop equations Consider the topological expansion of the loop (D.14). It reads, for $h \geq 1$:

$$\begin{aligned} E^{(h)}(x, y) &= D(x)(y - Y(x)) u_0^{(h)}(x, y) + D(x) w_{1,0}^{(h)}(x) u_0^{(0)}(x, y) \\ &\quad + D(x) \sum_{m=1}^{h-1} w_{1,0}^{(m)}(x) u_0^{(h-m)}(x, y) + D(x) u_1^{(h-1)}(x, y; x), \end{aligned} \tag{D.21}$$

where $E^{(h)}(x, y)$ is the hth term in the $\hbar^2$-expansion of the spectral curve.

D.3 Diagrammatic rules for the correlation functions and the free energy

In this section, one proves that the correlation functions' and the free energy's topological expansion of this model do coincide with the $\underline{W}_k^{(h)}$'s and $\underline{F}^{(h)}$'s defined following the definitions of Equations (4.15) and (4.16) for the classical spectral curve $\mathcal{E}_{\mathrm{Mext}}(x, y) = 0$.

D.3.1 The semi-classical spectral curve Let us re-express the semi-classical spectral curve (i.e., the whole formal series $E_{\mathrm{Mext}}(x, y)$) in terms of the classical one $E_{\mathrm{Mext}}^{(0)}(x, y)$.

Theorem D.3.

$$\begin{aligned} E_{\mathrm{Mext}}(x, y) &= -D(x)'' \left\langle \prod_{i=0}^{s} \left(y - V'(x(p)) + \frac{1}{N} \operatorname{Tr} \frac{1}{x(p^{(i)}) - M} \right) \right\rangle'' \\ &= D(x) \left[(V'(x) - y) S(y) - P_0(x, y) \right] \end{aligned} \tag{D.22}$$

and

$$U_0(p,y) = -'' \left\langle \prod_{i=1}^{s} \left(y - V'(x(p)) + \frac{1}{N} \operatorname{Tr} \frac{1}{x(p^{(i)}) - M} \right) \right\rangle'' , \tag{D.23}$$

where $'' < . >''$ means that one replace $\overline{w}_2$ by w_2 in the expansion.

Proof. One proves that the $1/N^2$-expansions of

$$\widetilde{E}(x,y) = -D(x) \left\langle \prod_{i=0}^{s} \left(y - V'(x(p)) + \frac{1}{N} \operatorname{Tr} \frac{1}{x(p^{(i)}) - M} \right) \right\rangle \tag{D.24}$$

and

$$\widetilde{U}(p,y) = -D(x) \left\langle \prod_{i=1}^{s} \left(y - V'(x(p)) + \frac{1}{N} \operatorname{Tr} \frac{1}{x(p^{(i)}) - M} \right) \right\rangle \tag{D.25}$$

coincide with the expansion of $E_{\mathrm{Mext}}(x,y)$ and $U_0(p,y)$.

Let the topological expansions be

$$\widetilde{E}(x,y) = \sum_g N^{-2g} \widetilde{E}^{(g)}(x,y), \quad \widetilde{U}(p,y) = \sum_g N^{-2g} \widetilde{U}^{(g)}(p,y). \tag{D.26}$$

Expanding the expressions of $\widetilde{E}(x,y)$ and $\widetilde{U}(p,y)$ into cumulants, one recovers

$$\begin{aligned} \widetilde{E}^{(h)}(x,y) &= (y - Y(x)) D(x) \widetilde{U}_0^{(h)}(x,y) + D(x) w_1^{(h)}(x) \widetilde{U}_0^{(0)}(x,y) \\ &\quad + D(x) \sum_{m=1}^{h-1} w_1^{(m)}(x) \widetilde{U}_0^{(h-m)}(x,y) + D(x) \widetilde{U}_1^{(h-1)}(x,y;x), \end{aligned} \tag{D.27}$$

which coincides with Equation (D.21).

One easily proves that this system of equations admits a unique solution thanks to the polynomial properties of $\widetilde{U}(p,y)$ and that the leading orders $h = 0$ coincide. The proof is extremely similar to that for the two-matrix model (cf. Theorem 1 in [19]). □

D.3.2. Diagrammatic solution One has

$$W_2^{(0)}(p_1,p_2) = \underline{B}(p_1,p_2), \tag{D.28}$$

where $\underline{B}$ is the Bergmann kernel of the algebraic curve $\mathcal{E}_{\mathrm{Mext}}$.

The coefficient of y^s of Equation (D.22), divided by $D(x)$, is

$$V'(x) + \sum_i \Lambda_i = \sum_i Y(p^i). \tag{D.29}$$

It implies that

$$\frac{dx(p)dx(q)}{(x(p)-x(q))^2} = \sum_i W_2^{(0)}(p^i,q) \quad \text{and} \quad \forall h > 1, \quad \sum_i W_2^{(h)}(p^i,q) = 0, \tag{D.30}$$

i.e.,

$$\overline{\omega}_2(p,q) + \sum_{i=1}^{s} \omega_2(p^i,q) = 0. \tag{D.31}$$

The coefficient of y^{s-1} is

$$\begin{aligned}\sum_{i<j} Y(p^i)Y(p^j) + \frac{1}{N^2}\omega_2(p^i,p^j) = V'(x)\sum_i \Lambda_i + \sum_{i<j}\Lambda_i\Lambda_j \\ + \frac{1}{N}\left\langle \mathrm{Tr}\, \frac{V'(x)-V'(M_1)}{x-M_1} \right\rangle.\end{aligned} \tag{D.32}$$

Notice that

$$\begin{aligned}&\sum_{i<j}\left(Y(p^i)Y(p^j) + \frac{1}{N^2}\omega_2(p^i,p^j)\right) \\ &= \frac{1}{2}\sum_i \left(Y(p^i)(V'(x) + \sum_j \Lambda_j - Y(p^i)) - \frac{1}{N^2}\overline{\omega}_2(p^i,p^i)\right) \\ &= \frac{1}{2}\left(V'(x) + \sum_j \Lambda_j\right)^2 - 12\sum_i \left(Y(p^i)^2 + \frac{1}{N^2}\overline{\omega}_2(p^i,p^i)\right).\end{aligned} \tag{D.33}$$

Thus,

$$\begin{aligned}&V'(x)^2 + \sum_i \Lambda_i^2 - \frac{2}{N}\left\langle \mathrm{Tr}\, \frac{V'(x)-V'(M_1)}{x-M_1} \right\rangle \\ &= \sum_i \left(Y(p^i)^2 + \frac{1}{N^2}\overline{\omega}_2(p^i,p^i)\right).\end{aligned} \tag{D.34}$$

Notice that the LHS is the ratio of a polynomial in x and $D(x)$: $Q(x)/D(x) = V'(x)^2 + \sum_i \Lambda_i^2 - 2N\langle\, \mathrm{Tr}\,(V'(x) - V'(M_1))/x - M_1\rangle$.

The topological expansion of this equation reads for $h \geq 1$

$$\boxed{\begin{aligned} & 2\sum_{i=0}^{d_2} y(p^i) W_{1,0}^{(h)}(p^i)dx(p) \\ & \quad = \sum_{i=0}^{d_2}\sum_{m=1}^{h-1} W_{1,0}^{(m)}(p^i) W_{1,0}^{(h-m)}(p^i) \\ & \qquad + \sum_{i=0}^{d_2} \overline{W}_{2,0}^{(h-1)}(p^i,p^i) + 2\frac{Q^{(h)}(x(p))dx(p)^2}{D(x(p))}. \end{aligned}} \tag{D.35}$$

From now on, following the lines of [19], one multiplies these equations by $1/2(dE_{p,\overline{p}}(q))/(y(p)-y(\overline{p}))$, takes the residues when $p \to \mu_\alpha$ and sums over all the branch-points and obtains:

$$W_{1,0}^{(h)}(q) = \sum_\alpha \mathop{\mathrm{Res}}_{p\to\mu_\alpha} \frac{(1/2)dE_{p,\overline{p}}(q)(W_{2,0}^{(h-1)}(p,\overline{p}) + \sum_{m=1}^{h-1} W_{1,0}^{(m)}(p)W_{1,0}^{(h-m)}(\overline{p}))}{(y(p)-y(\overline{p}))dx(p)}. \tag{D.36}$$

Differentiating wrt the potential $V(x_i)$, one can finally write down an expression for the correlation functions:

$$\boxed{\begin{aligned} W_{k+1,0}^{(h)}(q,p_K) \quad = \quad & \sum_\alpha \mathop{\mathrm{Res}}_{p\to\mu_\alpha} \frac{(1/2)dE_{p,\overline{p}}(q)}{(y(p)-y(\overline{p}))\,dx(p)} \Bigg(W_{k+1,0}^{(h-1)}(p,\overline{p},p_K) \\ & + \sum_{j,m} W_{j+1,0}^{(m)}(p,p_J)\, W_{k+1-j,0}^{(h-m)}(\overline{p},p_{K-J}) \Bigg). \end{aligned}} \tag{D.37}$$

This coincides with the reccursive Definition 4.15 and ensures the equality of the correlation functions with the former defined special "loop functions".

Keeping on following [19], one finds that the topological expansion of the free energy also coincides with the special free energies defined on (4.16), that is the τ-function of the algebraic curve.

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